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Identities for Partitions of N with Parts from A Finite Set

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IDENTITIES FOR PARTITIONS OF *N* WITH PARTS FROM A FINITE SET

A Thesis

by

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ABSTRACT

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We show for a prime power number of parts m that the first differences of partitions into at most m parts can be expressed as a non-negative linear combination of partitions into at most $m-1$ parts. To show this relationship, we combine a quasipolynomial construction of $p(n,m)$ with a new partition identity for a finite number of parts. We prove these results by providing combinatorial interpretations of the quasipolynomial of $p(n,m)$ and the new partition identity. We extend these results by establishing conditions for when partitions of n with parts coming from a finite set A can be expressed as a non-negative linear combination of partitions with parts coming from a finite set B. We extend these results to Gaussian Polynomials and show how our techniques can be used to reprove asymptotic formulas for partitions of n into parts from a finite set A.

DEDICATION

To Gauss, the cat and Gerald, the (fictional) lion cub.

In memory of Tinka Larsen. I miss you grandma.

ACKNOWLEDGMENTS

Family, faculty, and friends have made me the learner that I am today and will be tomorrow.

Family. You all have supported, encouraged, and enabled me to peruse my passion far way from the sunshine of California. Grammie, we'll get lunch when I come home and Dyami we'll climb mountains like when we were little. Dad, I'll make you proud and don't worry Mom, I'll be home for the holidays.

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CHAPTER I

A (BRIEF) INTRODUCTION TO THE THEORY OF PARTITIONS

1.1 Prelude

The first examples of integers partitions, the number of unordered ways to sum to an integer n, can be traced back to a letter from Leibniz to Bernoulli. Neither provided much insight into the study of partitions. Euler gave the first examples of partition identities, expressing one kind of partitions as another, which have been extensively studied by mathematicians since. This thesis aims to establish new partition identities for partitions with parts from a finite set. To start, we invite the reader to join us in a (brief) introduction to the theory of partitions.

1.2 A (Brief) Introduction to the Theory of Partitions

Definition 1. A **partition**, λ, of a non-negative integer n is a non-increasing sequence of parts $\lambda_1,...,\lambda_k$ such that the parts sum to n. We denote this $\lambda \vdash n$ and is read "λ is a partition of n". The notation $\lambda = (a_1,a_2,...,a_n)$ where the i^{th} component denotes the number of parts of size i is used throughout. The partition function of n, $p(n)$, enumerates all partitions of n.

Example 2. Let $n = 5$, then $p(5) = 7$.

$$\begin{aligned}5 = \lambda &= (a_1, a_2, a_3, a_4, a_5)\\ &= 5 = (0,0,0,0,1)\\ &= 4+1 = (1,0,0,1,0)\\ &= 3+2 = (0,1,1,0,0)\\ &= 3+1+1 = (2,0,1,0,0)\\ &= 2+2+1 = (1,2,0,0,0)\\ &= 2+1+1+1 = (3,1,0,0,0)\\ &= 1+1+1+1+1 = (5,0,0,0,0).\end{aligned}$$

Definition 3. The q-rising factorial is defined as

$$(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \tag{1.1}$$

$$\text{and } (a;q)_\infty = \lim_{n\to\infty}(a;q)_n. \tag{1.2}$$

Theorem 4 (Euler)**.** *The generating function for $p(n)$ is*

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} q^{ij}\right) = \prod_{i=1}^{\infty}\frac{1}{1-q^i} = \frac{1}{(q;q)_\infty}. \tag{1.3}$$

Proof. Let $|q| < 1$, and recall that $\sum_{j=0}^{\infty} q^{ij} = \frac{1}{1-q^i}$. For a given i and j, q^{ij} represents j parts of size i. After multiplication of the infinite series, the coefficient of q^n is all such unordered ways to sum to n, that is the partitions of n. The multiplication as described,

$$
\begin{aligned}
\prod_{i=1}^{\infty}\Big(\sum_{j=0}^{\infty} q^{ij}\Big) &= (1+q^1+q^{1+1}+q^{1+1+1}+q^{1+1+1+1}+q^{1+1+1+1+1}+)\cdots \\
&\times(1+q^2+q^{2+2}+q^{2+2+2}+q^{2+2+2+2}+q^{2+2+2+2+2}+\cdots) \\
&\times(1+q^3+q^{3+3}+q^{3+3+3}+q^{3+3+3+3}+q^{3+3+3+3+3}+\cdots) \\
&\times(1+q^4+q^{4+4}+q^{4+4+4}+q^{4+4+4+4}+q^{4+4+4+4+4}+\cdots) \\
&\vdots \\
&\times(1+q^i+q^{i+i}+q^{i+i+i}+q^{i+i+i+i}+q^{i+i+i+i+i}+\cdots) \\
&\vdots \\
&= 1+q^1+q^2+q^{1+1}+q^3+q^{2+1}+q^{1+1+1}+q^4+q^{3+1} \\
&+q^{2+2}+q^{2+1+1}+q^{1+1+1+1}+\cdots \\
&= p(0)q^0+p(1)q^1+p(2)q^2+p(3)q^3+p(4)q^4+\cdots.
\end{aligned}
$$

■

Definition 5. A **restricted partition** is any rule placed on the parts of a partition. We use $'$ to denote an arbitrary restricted partition and $p'(n)$ to be the function which enumerates the number of partitions of n which obey that rule.

We now give two examples of restricted partitions in a partition identity proved by Euler.

Theorem 6 (Euler)**.** *The partitions of n such that all parts are odd is equinumerous to the number of partitions of n such that all parts are distinct.*

Definition 7. Let $p_o(n)$ be the function which enumerates the partitions of n such that all parts are odd and the generating functions for $p_o(n)$ is

$$\sum_{n=0}^{\infty} p_o(n)q^n = \frac{1}{(q;q^2)_\infty}. \tag{1.4}$$

Definition 8. Let $p_d(n)$ be the function which enumerates the partitions of n such that all parts are distinct and the generating function for $p_d(n)$

$$\sum_{n=0}^{\infty} p_d(n)q^n = (-q;q)_\infty. \tag{1.5}$$

We now prove Theorem 6.

Proof.

$$\sum_{n=0}^{\infty} p_o(n)q^n = \frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty} = \frac{(-q;q)_\infty(q;q)_\infty}{(q;q)_\infty} = (-q;q)_\infty = \sum_{n=0}^{\infty} p_d(n)q^n. \tag{1.6}$$

■

The generating function proof, while simple and short, does not provide much intuition as to how partitions of n into odd parts can be mapped to partitions of n into distinct parts. We offer a bijective proof of Theorem 6 which following closely to the proof found in [2].

Proof. For a partition of n into only odd parts, λ_o, we merge (combine into one part) two parts of the same size into a new part of twice the original size until no more merges can be performed. This produces as partition of n into only distinct parts as the process of merging creates an even part of largest size and divisible by $i = 2k+1$. If there are an odd number of parts of size i, then only one part of size i is left.

Likewise, for a partition of n into only distinct parts, λ_d, we split (divide into two equal parts) parts of even size until no more splits can be preformed. This produces a partition into only odd parts. ■

Manipulating generating functions shows equality between $p_o(n)$ and $p_d(n)$. Thus there exists a bijection between the set of partitions of n into distinct parts and the set of partitions of n into odd parts. Likewise, since we have shown a bijection by manipulating partitions of n into odd parts and showing that they map uniquely to partitions of n into only distinct parts and vice verse.

A bijective proof type can relay underlying bits of generating function manipulation as partition mappings. Likewise, generating function manipulations can relay bits of information as to what mapping might be used. In the generating function proof of Theorem 6, a convenient re-indexing of the infinite product shows $\left(\frac{1}{(q;q^2)_\infty}\right)\left(\frac{1}{(q^2;q^2)_\infty}\right) = \frac{1}{(q;q)_\infty}$. Implicitly, this describes the fact that partitions of n can be 'separated' into even and odd parts. This is observed by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \left(\frac{1}{(q;q^2)_\infty}\right)\left(\frac{1}{(q^2;q^2)_\infty}\right) = \left(\sum_{n=0}^{\infty} p_o(n)q^n\right)\left(\sum_{n=0}^{\infty} p_e(n)q^n\right) \tag{1.7}$$

where $p_e(n)$ denotes the partitions of n into only even parts. Looking at even and odd parts is found in the bijective proof of Theorem 6. The 'separation' of even and odd parts motivates the following definitions.

Definition 9. A **subpartition** of λ is a partition such that $a_i' \leqslant a_i$ for all parts of size i and is denoted $\lambda' = (a_1', a_2', \ldots, a_k')$.

Definition 10. We say that two partitions λ' and λ'' **sum** to a partition λ if for all i, $a_i = a_i' + a_i''$ and is denoted $\lambda = \lambda' + \lambda''$.

Choliy and Sills [7] refer to the sum as the *union* of two partitions. We chose to use the term *sum* as a large part of this work considers dividing partitions into quotient and remainders. Equation (1.7) implies that for any partition λ of n, there is a unique way to express λ as the sum of subpartitions of only even parts and only odd parts. The product of the generating functions for $p_o(n)$ and $p_e(n)$, we observe that it must be the same as the generating function for $p(n)$. We phrase this in a complicated, but useful manner.

Proposition 11. *Let $'$,$''$, and $'''$ be restrictions on partitions such that for any λ'' and λ''' obeying $''$ and $'''$ respectively, that the sum of λ'' and λ''' is a partition, λ', obeying $'$. Then, there are unique partitions λ'' and λ''' such that $\lambda' = \lambda'' + \lambda'''$ if an only if*

$$\sum_{n\geqslant 0} p'(n)q^n = \left(\sum_{n\geqslant 0} p''(n)q^n\right)\left(\sum_{n\geqslant 0} p'''(n)q^n\right).$$

Proof. Suppose that there are unique partitions λ'' and λ''' such that $\lambda' = \lambda'' + \lambda'''$. Let n be given, we work on counting the number of partitions of n such that they obey the restriction $'$. First, for any $0 \leqslant i \leqslant n$, $p''(i)p'''(n-i)$ is the number of ways such that if $\lambda'' \vdash i$ and $\lambda''' \vdash n-i$ then $\lambda'' + \lambda''' \vdash n$ where $\lambda'' + \lambda'''$ is a partition that satisfies the restrictions $'$. Then over all possibilities of i, this yields $p'(n) = \sum_{i=0}^{n} p''(i)p'''(n-i)$ as the sums are assumed to be unique, no partition is counted twice and there is never a partition satisfying the restriction $'$ such that there are no λ'' and λ''' that sum to it. We now observe the product of the generating functions of partitions of n satisfying $''$ and $'''$ respectively. Then,

$$\left(\sum_{n\geqslant 0} p''(n)q^n\right)\left(\sum_{n\geqslant 0} p'''(n)q^n\right) = \sum_{n\geqslant 0}\left(\sum_{i=0}^{n} p''(i)p'''(n-i)\right)q^n = \sum_{n\geqslant 0} p'(n)q^n. \tag{1.8}$$

Conversely, suppose that $\sum_{n\geqslant 0} p'(n)q^n = \left(\sum_{n\geqslant 0} p''(n)q^n\right)\left(\sum_{n\geqslant 0} p'''(n)q^n\right)$. We have that for any $n \geqslant 0$, $p'(n)q^n = \sum_{i=0}^{n} p''(i)q^i p'''(n-i)q^{n-i}$ by multiplication of the generating functions. We further re-express this equality as $\sum_{\lambda'\vdash n} q^{\lambda'} = \sum_{i=0}^{n}\left(\sum_{\lambda''\vdash i} q^{\lambda''}\right)\left(\sum_{\lambda'''\vdash n-i} q^{\lambda'''}\right) = \sum_{i=0}^{n}\left(\sum_{\lambda''\vdash i}\left(\sum_{\lambda'''\vdash n-i} q^{\lambda''+\lambda'''}\right)\right)$. By assumption we have, $\lambda'' + \lambda''' = \lambda'$, and by the equality there are the same number of $q^{\lambda'}$ on the left hand side as $q^{\lambda''+\lambda'''}$ on there right hand side. Then the mapping, $q^{\lambda''+\lambda'''} \to q^{\lambda'}$ by $\lambda'' + \lambda''' = \lambda'$ is a bijection by equality so sets hence the sum must be unique as desired. ∎

Proposition 39 explains how to 'count' partitions in regards to a well behaved sum. It is certainly not an unexpected proposition considering it mimics the behavior of sums of even and odd partitions. We are gifted with a powerful tool to tell how a generating functions of partitions factor. Proposition 39 serves as a key point in producing proofs for new partition identities for partitions of n into at most m parts.

Definition 12. **Singly restricted partitions** are partitions of n such that there are at most m parts. The function $p(n,m)$ enumerates the number of singly restricted partitions of n into at most m

parts and its generating function is

$$\sum_{n=0}^{\infty} p(n,m)q^n = \frac{1}{(q;q)_m}. \tag{1.9}$$

We can provide our second example of a partition identity, a recurrence relationship for $p(n,m)$.

Proposition 13 (Euler). *For all non-negative integers n and all positive integers m,*

$$p(n,m) = p(n,m-1) + p(n-m,m). \tag{1.10}$$

Proof. We present a straightforward manipulation of generating function,

$$\begin{aligned}\sum_{i=1}^{\infty} \left(p(n,m-1) + p(n-m,m)\right) q^n = \frac{1}{(q;q)_{m-1}} + \frac{q^m}{(q;q)_m} = \frac{1-q^m}{(q;q)_m} + \frac{q^m}{(q;q)_m} = \frac{1}{(q;q)_m} \\ = \sum_{n=0}^{\infty} p(n,m)q^n. \end{aligned} \tag{1.11}$$

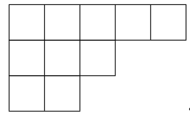
■

Euler used Proposition 13 as a method to compute $p(n)$ by observing that $p(n,m) = p(n)$ when $n \leqslant m$.

Inherent in the discussion of partitions of n into at most m parts is the fact that these partitions are also partitions of n into no part larger than m. Frequently, partitions of n into at most m parts is used in texts. To show this, we define a classic graphic representation of partitions, the Ferrer's Diagram (Ferrer's graph, Ferrer's board, or Young Tabalux).

Definition 14. The **Ferrer's diagram** of a partition $\lambda \vdash n$ is an upper left justified diagram of n dots or boxes such that λ_i is the i^{th} row of dots.

Example 15. Let $\lambda \vdash 10$ and $\lambda = 5+3+2$. The Ferrer's diagram of λ is



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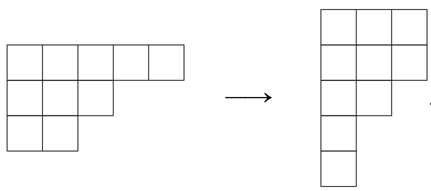
Definition 16. Let $\lambda \vdash n$. The **conjugate** partition of λ (simply the conjuate) is the partition that interchanges the rows and columns of the Ferrer's diagram of λ and is denoted λ^t.

Example 17. Let $\lambda \vdash 10$ and $\lambda = 5+3+2$. The conjugate partition of λ is $\lambda^t = 3+3+2+1+1$, as



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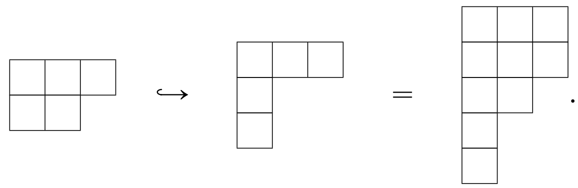
Proposition 18. *The number of partitions into at most m parts is equinumerous to the number of partitions of n into parts not than m.*

Proof. Given any partition of n into at most m parts, it's Ferrer's diagram has at most m rows, hence the conjugate partition will have at most m columns and this is a partition of n into no part larger than m. Similarly, for a partition of n into no part larger than m, it's Ferrer's diagram has at most m columns, hence the conjugate partition will have at most m rows, a partition of n into at most m parts. ■

Frequently, we examine $p(n,m)$ thorough Proposition 18 and make arguments in regards to the number of parts of partitions of n into parts not than m. Lastly, we wish to describe sums of partitions in terms of Ferrer's diagrams. To do this, we define an operation called **insertion**.

Definition 19. A partition λ_2 is **inserted** into a partition λ_1 by placing the number of parts of size i in λ_2 below the last occurrence of a part of that size in λ_1. This is denoted by $\lambda_2 \hookrightarrow \lambda_1$.

Example 20. Let $\lambda_1 = 3+1+1$ and $\lambda_2 = 3+2$. Then $\lambda_2 \hookrightarrow \lambda_1 = 3+3+2+1+1$ as



Insertion is a graphical description of the sum operation in regards to Ferrer's diagrams. While not a fundamental portion of the proceeding work, proof with the sum operation have a natural correspondence to a proof with Ferrer's diagrams with the operation of insertion.

By examining the connection of partitions of n into at most m parts and partitions of n with no part larger than m, we have given the first example of partitions with parts from a finite set. That is partitions of n with no part larger that m is exactly partitions n with parts from the set $\{1,2,...,m\}$.

Definition 21. Let A be a finite set of positive integers not necessarily distinct. We denote the number of partitions of n with parts from A as $p(n,A)$. Its generating function is given by

$$\sum_{n=0}^{\infty} p(n,A)q^n = \prod_{i\in A} \frac{1}{1-q^i}. \tag{1.12}$$

We provide the following combinatorial interpretation for a partition from a finite set A. Let α_i denote the number of copies of an integer $i \in A$. For a partition λ into parts from A, a part of size i can have colors $1,2,...,\alpha_i$ and we use the notation $\lambda = (a_{i_1,1}, a_{i_1,2}, ..., a_{i_1,\alpha_{i_1}}, ..., a_{i_k,1}, a_{i_k,2}, ..., a_{i_k,\alpha_{i_k}})$ $= (a_{1,1}, a_{1,2}, ..., a_{1,\alpha_1}, ..., a_{k,1}, ...a_{k,\alpha_k})$ where $i_1, i_2, ..., i_k$ are the distinct elements of A in order of least to greatest and $a_{t,s}$ is the number of parts of size i_t in color s. We remark that $p(n,m) = p(n,\{1,2,...,m\})$ and it is assumed throughout that m in place of A is the set of positive integers 1 through m.

1.3 Outline

This thesis is organized around three theorems for $p(n,A)$ each with a special case for $p(n,m)$ which will serve as an example. Each of these theorems are based around an object called a quasipolynomial. Chapter 2 defines and establishes a quasipolynomial for $p(n,A)$ providing both generating function and combinatorial proofs for this. In Chapter 3, we start by examining the first differences of partitions $p(n,m) - p(n-1,m)$ for prime power m. We capture several known partition identities found in [3] and [6] as well as previously unknown partition identities.

Example 22. For $k \geqslant 0$,

$$p(6k+3,3) - p(6k+2,3) = p(2k+1,2) \tag{1.13}$$

$$p(12k+5,4) - p(12k+4,4) = p(6k+1,3) \tag{1.14}$$

$$\begin{aligned} p(60k+0,5) - p(60k-1,5) &= p(12k+0,4) + p(12k-1,4) + 4p(12k-2,4) + 5p(12k-3,4) \\ &\quad + 7p(12k-4,4) + 4p(12k-5,4) + 3p(12k-6,4). \end{aligned} \tag{1.15}$$

Lines (1.13) and (1.14)are known ([3], [6]) while line (1.15) was previously unknown. These identities are special cases of the following theorem.

Theorem 23. *Let s be a prime. If $m = s^x$ where x is a positive integer and for $k \geqslant 0$, $0 \leqslant j < lcm([m])$, then*

$$p(lcm([m])k + j, m) - p(lcm([m])k + j - 1, m) = \sum_{i \geqslant 0} g_{r+is}\, p(lcm(m-1)k + l' - i, m-1) \tag{1.16}$$

where l' and r satisfy $j = l's + r$ with $0 \leqslant r < s$, $lcm([m])$ is the least common multiple of the numbers 1 through m, and g_{r+is} are the coefficients of some polynomial $G(q)$.

$G(q)$ is given in Chapter 3 in the statement of Lemma 36. Chapter 4 generalizes Theorem 23 to $p(n,A)$ by establishing a group structure on a finite set of partitions. Together, Chapters 2,3 and 4 represent the work in a [16] forthcoming paper by the author. Chapter 5 takes the group structure established in Chapter 4 and the quasipolynomial construction to provide a 'new' proof of the asymptotic formula for $p(n,A)$ and highlights several facts about the quasipolynomial of $p(n,A)$.

Theorem 24. *Let $d = |A|$, then*

$$p(n,A) \sim \frac{n^{d-1}}{(d-1)!\prod_{i\in A} i} + \mathscr{O}(n^{d-2}). \tag{1.17}$$

Chapter 6 reproves some of the work in [6] in regards to Gaussian Polynomials and builds on it by utilizing Theorem 23.

CHAPTER II

ESTABLISHING A QUASIPOLYNOMIAL FOR $P(N,A)$

2.1 Quasipolynomials of $p(n,A)$

Definition 25. A **quasipolynomial** is a piecewise function, f, such that there are polynomials $f_0(k), f_1(k), ..., f_{j-1}(k)$ with rational coefficients, called **constituents**, such that

$$f(k) = \begin{cases} f_0(k) & \text{if } k \equiv 0 \pmod{j} \\ f_1(k) & \text{if } k \equiv 1 \pmod{j} \\ \vdots & \\ f_{j-1}(k) & \text{if } k \equiv j-1 \pmod{j}. \end{cases} \tag{2.1}$$

The number of constituents, j, of the quasipolynomial is called the period.

We refer the reader to [20] for further reading about quasipolynomials. The particular technique in Theorem 26 that we show has been presented a number of times, notably a integral aspect of the geometry of Ehrhart in [8], as a consequence of generating functions in [11] and [22], and more recently in [4], [5], [6], and [18]. Other methods for creating quasipolynomials of $p(n,m)$ rely on partial fraction decomposition of rational functions such as in [17] and [19].

We start by generalizing the notation of $\text{lcm}([m])$ in the previous section. For a finite set of positive integers, A, we define $\text{lcm}(A)$ to be the least common multiple of all the elements of A. For example $\text{lcm}([3]) = \text{lcm}(\{1,2,3\}) = 6$. Let d count the number of elements in A. We define the polynomial $E_A(q)$ by

$$E_A(q) = \prod_{i\in A}\left(\sum_{j=0}^{\frac{\mathrm{lcm}(A)}{i}-1} q^{ij}\right) = \sum_{x=0}^{d\,\mathrm{lcm}(A)-\sum_{i\in A} i} h_x q^x. \tag{2.2}$$

Furthermore, $E_A(q)$ has the following property,

$$\prod_{i\in A}(1-q^i)E_A(q) = (1-q^{\mathrm{lcm}(A)})^d. \tag{2.3}$$

Lastly, we remark that it is known that for d a non-negative integer,

$$\frac{1}{(1-q)^d} = \sum_{k=0}^{\infty}\binom{k+d-1}{d-1}q^k. \tag{2.4}$$

Theorem 26. *For all $k \geqslant 0$ and for $0 \leqslant j < lcm(A)$,*

$$p(lcm(A)k+j,A) = \sum_{t\geqslant 0} h_{j+lcm(A)t}\binom{k-t+(d-1)}{d-1}. \tag{2.5}$$

Proof. We begin by manipulating the generating function for $p(n,A)$,

$$\sum_{n=0}^{\infty} p(n,A)q^n = \prod_{i\in A}\frac{1}{(1-q^i)}\cdot\frac{E_A(q)}{E_A(q)} = \frac{E_A(q)}{(1-q^{\mathrm{lcm}(A)})^d} = E_A(q)\sum_{k=0}^{\infty}\binom{k+(d-1)}{d-1}q^{\mathrm{lcm}(A)k} \tag{2.6}$$

by line (2.3) and substituting q with $q^{\mathrm{lcm}(A)}$ in (2.4). Consider $n \pmod{\mathrm{lcm}(A)}$, that is $n = \mathrm{lcm}(A)k + j$ for some positive integers k and j. We consider the possibilities for arriving at an exponent of q which is $\mathrm{lcm}(A)k + j$ in

$$E_A(q)\sum_{k=0}^{\infty}\binom{k+(d-1)}{d-1}q^{\mathrm{lcm}(A)k} = \left(\sum_{x=0}^{d\,\mathrm{lcm}(A)-\sum_{i\in A} i} h_x q^x\right)\left(\sum_{k=0}^{\infty}\binom{k+(d-1)}{d-1}q^{\mathrm{lcm}(A)k}\right) \tag{2.7}$$

which is exactly when $x+\operatorname{lcm}(A)(k-t)=\operatorname{lcm}(A)k+j$ and hence we have

$$p(\operatorname{lcm}(A)k+j,A)=\sum_{t\geqslant 0}h_{j+\operatorname{lcm}(A)t}\binom{k-t+(d-1)}{d-1}. \tag{2.8}$$

■

We arrive at a quasipolynomial for $p(n,A)$ when all values possible values of j are considered in Theorem 26 are considered. In the case of $p(n,m)$, $E_m(q)$ has a nice form $E_m(q)=\frac{(1-q^{\operatorname{lcm}([m])})^m}{(q;q)_m}$. For example, let $m=3$ then,

$$\begin{aligned}E_3(q)=\frac{(1-q^6)^3}{(q;q)_3}&=(1+q+q^2+q^3+q^4+q^5)(1+q^2+q^4)(1+q^3)\\&=1+q+2q^2+3q^3+4q^4+5q^5+4q^6+5q^7+4q^8+3q^9+2q^{10}+q^{11}+q^{12}.\end{aligned} \tag{2.9}$$

Therefore, as a consequence of Theorem 26, for $k\geqslant 0$,

$$p(n,3)=\begin{cases}p(6k+0,3)=1\binom{k+2}{2}+4\binom{k+1}{2}+1\binom{k}{2} & =3k^2+3k+1\\ p(6k+1,3)=1\binom{k+2}{2}+5\binom{k+1}{2} & =3k^2+4k+1\\ p(6k+2,3)=2\binom{k+2}{2}+4\binom{k+1}{2} & =3k^2+5k+2\\ p(6k+3,3)=3\binom{k+2}{2}+3\binom{k+1}{2} & =3k^2+6k+3\\ p(6k+4,3)=4\binom{k+2}{2}+2\binom{k+1}{2} & =3k^2+7k+4\\ p(6k+5,3)=5\binom{k+2}{2}+1\binom{k+1}{2} & =3k^2+8k+5.\end{cases} \tag{2.10}$$

There are other quasipolynomoial expressions for $p(n,3)$ such as $\left\|\frac{1}{12}(n+3)^2\right\|$, where $\|\cdot\|$ denotes the nearest integer. Further explicit quasipolynomials for $p(n,4),p(n,5)$ and $p(n,6)$ can be found in [15].

2.2 A combinatorial interpretation of a quasipolynomial for $p(n,m)$

Choliy and Sills in [7] provide a formula for $p(n)$ that "counts" using Durfee squares. Following their work, we aim to provide an analogous proof for the quasipolynomial formula of $p(n,m)$ in Theorem 26 that "counts". We start with essential definitions.

Recall that the notation $\lambda = (a_1, a_2, ..., a_n)$ where the i^{th} component denotes the number of parts of size i is used throughout. A **subpartition** of λ is a partition such that $a_i' \leqslant a_i$ for all parts of size i and is denoted $\lambda' = (a_1', a_2', ..., a_k')$. We say that two partitions λ' and λ'' **sum** to a partition λ if for all i, $a_i = a_i' + a_i''$ and is denoted $\lambda = \lambda' + \lambda''$. Choliy and Sills [7] refer to this sum as the *union* of two partitions. We define remainder-like objects for $p(n,m)$.

Definition 27. A $\mathrm{lcm}([m])$-remainder partition is a partition such that there are no parts larger than m and for any part of size i, there are less than $\frac{\mathrm{lcm}([m])}{i}$ copies of that part. Let E_m be the collection of all $\mathrm{lcm}([m])$-remainder partitions. The generating function for $\mathrm{lcm}([m])$-remainder partitions is $E_m(q)$.

Example 28. A $\mathrm{lcm}(3)$-remainder partition is $3+2+1 = (1,1,1)$ but $3+3+1 = (1,0,2)$ is not an $\mathrm{lcm}(3)$-remainder partition.

Definition 29. Let the set of partitions E_{m_j} be the partitions in E_m such that they partition numbers that when divided by $\mathrm{lcm}([m])$ have remainder j.

Example 30. We give the example for $m = 3$ and $j = 0$. We have the set of partitions

$$E_{3_0} = \{(0,0,0), (1,1,1), (2,2,0), (3,0,1), (4,1,0), (5,2,1)\}.$$

Definition 31. Let x be a non-negative integer, we say h_x is the number of partitions in E_m such that they partition x.

We note that h_x is the x^{th} coefficient of $E_m(q)$. We now present a proof that "counts" the case of Theorem 26 when $A = \{1, 2, ..., m\}$.

Theorem 32. *For all $k \geqslant 0$ and $0 \leqslant j < lcm([m])$,*

$$p(lcm([m])k+j,m) = \sum_{t\geqslant 0} a_{lcm([m])t+j}\binom{k-t+m-1}{m-1}. \tag{2.11}$$

Proof. It is sufficient to consider partitions of n into parts no larger than m. Let $n = \mathrm{lcm}([m])k + j$ and consider a partition $\lambda = (a_1, a_2, ..., a_m)$. Let r_i be the remainder of a_i when divided by $\frac{\mathrm{lcm}([m])}{i}$. Define $\lambda_r = (r_1, ..., r_m)$ and $\lambda_q = (a_1 - r_1, ..., a_m - r_m)$. Then we note first, $\lambda = \lambda_q + \lambda_r$. Second, $\lambda_r \in E_m$, and third, for any i, $\frac{\mathrm{lcm}([m])}{i}$ divides $a_i - r_i$, that is, $\frac{\mathrm{lcm}([m])}{i}k_i = a_i - r_i$ for some positive integer k_i. We will show that $\lambda_r \in E_{m_j}$ and that $k - t = \sum_{i=1}^{m} k_i$ for some non-negative integer t.

We start by showing $\lambda_r \in E_{m_j}$. Since $\lambda \vdash \mathrm{lcm}([m])k + j$, we have

$$\mathrm{lcm}([m])k + j = \sum_{i=1}^{m} ia_i = \sum_{i=1}^{m} i(a_i - r_i) + \sum_{i=1}^{m} ir_i = \sum_{i=1}^{m} i\frac{\mathrm{lcm}([m])}{i}k_i + \sum_{i=1}^{m} ir_i. \tag{2.12}$$

Therefore,

$$\mathrm{lcm}([m])k + j \equiv \sum_{i=1}^{m} ir_i \equiv j \pmod{\mathrm{lcm}([m])}. \tag{2.13}$$

As $\sum_{i=1}^{m} ir_i = \lambda_r$, and is a partition of a number that when divided by $\mathrm{lcm}([m])$ has remainder j, then $\lambda_r \in E_{m_r}$ as desired. Express $\sum_{i=1}^{m} ir_i$ as $\mathrm{lcm}([m])t + j$ for some non-negative integer t. Furthermore, $\lambda_q \vdash \mathrm{lcm}([m])(k-t)$ by combining lines (2.12) and (2.13). Hence, by canceling $\mathrm{lcm}([m])$, we have $k - t = \sum_{i=1}^{m} k_i$ as desired.

Now, we count the partitions of $\mathrm{lcm}([m])k + j$ into parts no larger than m in the following manner. We count the number of partitions such that $\lambda = \lambda_q + \lambda_r$ where $\lambda_r \vdash \mathrm{lcm}([m])t + j$ and $\lambda_q \vdash \mathrm{lcm}([m])(k-t)$. The number of choices for λ_r is $h_{\mathrm{lcm}([m])t+j}$ since this is the number of partitions in E_m and in E_{m_j} such that they partition $\mathrm{lcm}([m])t + j$. By line (2.12) and Stars and Bars counting, the number of choices for $\lambda_q \vdash \mathrm{lcm}([m])(k-t)$ which is $\binom{k-t+m-1}{m-1}$. Thus the number of choices for λ is the product of the number of choices for λ_q and λ_r. Accounting for every possibility of t, we have

$$p(\text{lcm}([m])k+j,m) = \sum_{t \geqslant 0} h_{\text{lcm}([m])t+j} \binom{k-t+m-1}{m-1}. \tag{2.14}$$

■

We should remark that the proof of Theorem 32 could easily be Theorem 26. We replace every instance of $\text{lcm}([m])$ with $\text{lcm}(A)$, each m in an index of a sum with d. We consider $\text{lcm}(A)$-remainder partitions as defined in Definition 38 rather than $\text{lcm}([m])$-remainder the partitions. Next, rather than i as the index variable, we have i_v for color v. We should also note that we we consider the insertion operation and conjugation, we can quickly recover the proposed l-box decomposition of [4].

CHAPTER III

FIRST DIFFERENCES OF $P(N,M)$

With a quasipolynomial for $p(n,m)$ in hand, we now aim to prove Theorem 23 and a new proof of Proposition 13.

Definition 33. For $k \leqslant m$, we define the k^{th} difference of $p(n,m)$, denoted $\Delta_k(n,m)$, is

$$\sum_{n=0}^{\infty} \Delta_k(n,m)q^n = \sum_{n=0}^{\infty} \left(p(n,m) - p(n-k,m)\right) q^n = \frac{1-q^k}{(q;q)_m}. \tag{3.1}$$

$\Delta_k(n,m)$ can be interpreted as partitions of n with at most m parts without any parts of size k. In particular, when $k = 1$, these are twin partitions following [3] and we omit the subscript. Furthermore, when $k = m$, $\Delta_m(n,m) = p(n,m-1)$. When $k \leqslant m$, it follows that if $A = \{1,2,...m\}\backslash\{k\}$, then $\Delta_k(n,m) = p(n,A)$. In this case we say $p(n,A) = p(n,m\backslash k)$ and use the notation $m\backslash k$ in regards to E_A. Naturally, by Theorem 26, we can produce a quasipolynomial for $\Delta_k(n,m)$. Proposition 13 studies the m^{th} difference of $p(n,m)$ and Theorem 23 studies the first differences of $p(n,m)$.

The case of $m = 3,4$ of Theorem 23 occurs in [3] and [6]. We provide a new proof for this via generating functions distinguishing the proof from the direct computation of differences of quasipolynomials as in [6] and the recursion used in [3]. It illustrates a necessary lemma to prove Theorem 23. Then we provide an example of the case of $m = 5$ to build intuition for the arithmetic of Theorem 23. Results similar to the case of $m = 3$ are found in articles regarding Gaussian polynomials such as in [6], [12], [13], and [14].

3.1 The case of $m = 3$ and $m = 4$

We start by showing the proof of the case for $m = 3$. This follows the proof in [6] and the result is found in both [3] and [6].

Theorem 34. *For $\ell \geqslant 0$,*

$$\Delta(3\ell+0,3) = \ell+1 \tag{3.2}$$

$$\Delta(3\ell+1,3) = \ell \tag{3.3}$$

$$\Delta(3\ell+2,3) = \ell+1. \tag{3.4}$$

Proof. We begin by computing the quasipolynomial for $p(n,2)$ via Theorem 26. We see that for all $k \geqslant 0$,

$$p(n,2) = \begin{cases} p(2k,2) = k+1 \\ p(2k+1,2) = k+1 \end{cases} \tag{3.5}$$

We set $\ell = 2k+1$. Then we examine the constituent $\Delta(3(2k+1))+0) = \Delta(6k+3,3)$. We have

$$\Delta(6k+3,3) = p(6k+3,3) - p(6k+2,3) = (3k^2+6k+3) - (3k^2+5k+2) = k+1 = p(2k+1,2). \tag{3.6}$$

The remaining 5 cases follow by setting $\ell = 2k$ or $\ell = 2k+1$ in lines (3.2), (3.3), and (3.4) and then preforming similar computations using lines (2.10) and (3.5). ■

This proof, while making use of a clever observations, fails to give any insight to how this statement might generalize. Furthermore, it relies on having computed all constituents of a quasipolynomial which the number of which depends on $\mathrm{lcm}([m])$. This process quickly becomes unfeasible trying by hand or computer with large enough m. We now provide the proof for the case of $m = 4$ which will illustrate the intuition for Theorem 23.

Theorem 35. *For $\ell \geqslant 0$,*

$$p(2\ell-3,4)-p(2\ell-4,4)=p(\ell-3,3) \tag{3.7}$$

$$p(2\ell-4,4)-p(2\ell-5,4)=p(\ell-2,3). \tag{3.8}$$

Proof. By considering ℓ modulo 6 in lines (3.7) and (3.8), we will prove an equivalent statement. For $k \geqslant 0$,

$$\left.\begin{array}{r} p(12k,4)-p(12k-1,4) \\ p(12k+3,4)-p(12k+2,4) \end{array}\right\} = p(6k,3)$$
$$\left.\begin{array}{r} p(12k+2,4)-p(12k+1,4) \\ p(12k+5,4)-p(12k+4,4) \end{array}\right\} = p(6k+1,3)$$
$$\left.\begin{array}{r} p(12k+4,4)-p(12k+3,4) \\ p(12k+7,4)-p(12k+6,4) \end{array}\right\} = p(6k+2,3)$$
$$\left.\begin{array}{r} p(12k+6,4)-p(12k+5,4) \\ p(12k+9,4)-p(12k+8,4) \end{array}\right\} = p(6k+3,3)$$
$$\left.\begin{array}{l} p(12k+8,4)-p(12k+7,4) \\ p(12k+11,4)-p(12k+10,4) \end{array}\right\} = p(6k+4,3)$$
$$\left.\begin{array}{l} p(12k+10,4)-p(12k+9,4) \\ p(12k+13,4)-p(12k+12,4) \end{array}\right\} = p(6k+5,3).$$

We following the proof of Theorem 26 we produce a quasipolynomial for $\Delta(n,4)$ and $p(n,3)$. Line (2.9) gives $E_3(q)$. Next, we must compute $E_{4\setminus 1}(q)$ which is,

$$\begin{aligned} E_{4\setminus 1}(q) = {} & 1+q^2+q^3+2q^4+q^5+3q^6+2q^7+4q^8+3q^9+5q^{10}+4q^{11}+4q^{12}+5q^{13}+5q^{14} \\ & +4q^{15}+4q^{16}+5q^{17}+3q^{18}+4q^{19}+2q^{20}+3q^{21}+q^{22}+2q^{23}+q^{24}+q^{25}+q^{27}. \end{aligned} \tag{3.9}$$

We highlight that,

$$E_{4\setminus 1}(q) = (1+q^3)(1+q^2+2q^4+3q^6+4q^8+5q^{10}+4q^{12}+5q^{14}+4q^{16}+3q^{18}+2q^{20}+q^{22}+q^{24})$$
$$= (1+q^3)E_3(q^2). \quad (3.10)$$

Next, generating function arithmetic yields

$$\begin{aligned}
\sum_{n=0}^{\infty}\Delta(n,4)q^n &= \frac{1}{(q^2;q)_3} = \frac{E_{4\setminus 1}(q)}{E_{4\setminus 1}(q)(q^2;q)_3} = \frac{E_{4\setminus 1}(q)}{(1-q^{12})^3} = E_{4\setminus 1}(q)\sum_{k=0}^{\infty}\binom{k+2}{2}q^{12k} \\
&= (1+q^3)E_3(q^2)\sum_{k=0}^{\infty}\binom{k+2}{2}q^{12k} \\
&= (1+q^3)\Bigg(\sum_{k=0}^{\infty}\left(\binom{k+2}{2}+4\binom{k+1}{2}+\binom{k}{2}\right)q^{12k} \\
&+ \sum_{k=0}^{\infty}\left(\binom{k+2}{2}+5\binom{k+1}{2}\right)q^{12k+2} \\
&+ \sum_{k=0}^{\infty}\left(2\binom{k+2}{2}+4\binom{k+1}{2}\right)q^{12k+4} + \sum_{k=0}^{\infty}\left(3\binom{k+2}{2}+3\binom{k+1}{2}\right)q^{12k+6} \\
&+ \sum_{k=0}^{\infty}\left(4\binom{k+2}{2}+2\binom{k+1}{2}\right)q^{12k+8} + \sum_{k=0}^{\infty}\left(5\binom{k+2}{2}+\binom{k+1}{2}\right)q^{12k+10}\Bigg) \\
&= (1+q^3)\Bigg(\sum_{k=0}^{\infty}p(6k,3)q^{12k} + \sum_{k=0}^{\infty}p(6k+1,3)q^{12k+2} + \sum_{k=0}^{\infty}p(6k+2,3)q^{12k+4} \\
&+ \sum_{k=0}^{\infty}p(6k+3,3)q^{12k+6} + \sum_{k=0}^{\infty}p(6k+4,3)q^{12k+8} + \sum_{k=0}^{\infty}p(6k+4,3)q^{12k+10}\Bigg).
\end{aligned} \tag{3.11}$$

by lines (2.9), (2.10), and (3.9). Comparing powers of q modulo 12 in the preceding equation, the result is as desired. ■

3.2 A Lemma for First Differences of Partitions

The ability to factor $E_{4\setminus 1}(q)$ into $(1+q^3)E_3(q^2)$ was essential in proving the claim. Fortunately, in this specific case a computer can handle this factorization with ease as $E_{4\setminus 1}(q)$ is a polynomial of degree 27. This leads to the following lemma which states under what conditions $E_{m\setminus 1}(q)$ factors. We will provide three proofs for. The first two proofs will occur in this section; one following the generating function arithmetic of Theorem 26 and another following the counting of Theorem 32. The third occurs in the following chapter.

Lemma 36. *Let $m = s^k$ where s is prime and k is a positive integer, then $E_{m\setminus 1}(q) = G(q)E_{m-1}(q^s)$ where $G(q) = \prod_{i=2, i\neq s^a, \forall a\in\mathbb{N}}^{m} \sum_{j=0}^{s-1} q^{ij}$.*

Remark 37. Strictly speaking, $G(q)$ is the generating function for the collection of $\mathrm{lcm}([m])$-remainder partitions which have no parts of size 1, any positive integer power of s, and parts of any size occur less than s times. The nature of $G(q)$ is seemly mysterious. In a broader context, we can also think of $G(q)$ as the generating function for a collection of "remainder partitions" when the collection of partitions generated by $E_{m\setminus 1}(q)$ are considered modulo $\mathrm{lcm}([m-1])$-remainder partitions. This arises naturally from defining a group structure on partitions generated by $E_A(q)$ which will be treated a later chapter. For now, we can experience $G(q)$ as a consequence of arithmetic and as a property of the sum of partitions and some particular combinatorial map.

Proof. We show that $\frac{E_{m\setminus 1}(q)}{E_{m-1}(q)} = G(q)$ by simplifying the expression,

$$\frac{E_{m\setminus 1}(q)}{E_{m-1}(q^s)} = \frac{\frac{(1-q^{\mathrm{lcm}([m])})^{m-1}}{(q^2;q)_{m-1}}}{\frac{(1-q^{\mathrm{lcm}([m])})^{m-1}}{(q^s;q^s)_{m-1}}} = \frac{(q^s;q^s)_{m-1}}{(q^2;q)_{m-1}} = \frac{\prod_{r=1}^{k}\left((1-q^{s^r})\prod_{i=s^{r-1}+1}^{s^r-1}(1-q^{si})\right)}{\prod_{r=1}^{k}\left((1-q^{s^r})\prod_{i=s^{r-1}+1}^{s^r-1}(1-q^{i})\right)}$$

$$= \frac{\prod_{r=1}^{k}\prod_{i=s^{r-1}+1}^{s^r-1}\left(\sum_{j=0}^{s-1} q^{ij}\right)(1-q^i)}{\prod_{r=1}^{k}\prod_{i=s^{r-1}+1}^{s^r-1}(1-q^{i})} = G(q). \quad (3.12)$$

■

For the combinatorial proof of Lemma 36, we introduce a generalization of $\mathrm{lcm}([m])$-remainder partitions. Furthermore, we discuss how the sums of restricted partitions behave in regards to their generating functions.

Definition 38. A lcm(A)-remainder partition is a partition of n into parts from A such that for each $i \in A$, the number of parts of size i is less than $\frac{\text{lcm}(A)}{i}$. The generating function for lcm(A)-remainder partitions $E_A(q)$ and the collection of all lcm(A)-remainder partitions is denoted E_A.

Proposition 39. *Let $'$,$''$, and $'''$ be restrictions on partitions such that for any λ'' and λ''' obeying $''$ and $'''$ respectively, that the sum of λ'' and λ''' is a partition, λ', obeying $'$. Then, there are unique partitions λ'' and λ''' such that $\lambda' = \lambda'' + \lambda'''$ if an only if*

$$\sum_{n\geqslant 0} p'(n)q^n = \left(\sum_{n\geqslant 0} p''(n)q^n\right)\left(\sum_{n\geqslant 0} p'''(n)q^n\right).$$

Proof. Suppose that there are unique partitions λ'' and λ''' such that $\lambda' = \lambda'' + \lambda'''$. Let n be given, we work on counting the number of partitions of n such that they obey the restriction $'$. First, for any $0 \leqslant i \leqslant n$, $p''(i)p'''(n-i)$ is the number of ways such that if $\lambda'' \vdash i$ and $\lambda''' \vdash n-i$ then $\lambda'' + \lambda''' \vdash n$ where $\lambda'' + \lambda'''$ is a partition that satisfies the restrictions $'$. Then over all possibilities of i, this yields $p'(n) = \sum\limits_{i=0}^{n} p''(i)p'''(n-i)$ as the sums are assumed to be unique, no partition is counted twice and there is never a partition satisfying the restriction $'$ such that there are no λ'' and λ''' that sum to it. We now observe the product of the generating functions of partitions of n satisfying $''$ and $'''$ respectively. We arrive at

$$\left(\sum_{n\geqslant 0} p''(n)q^n\right)\left(\sum_{n\geqslant 0} p'''(n)q^n\right) = \sum_{n\geqslant 0}\left(\sum_{i=0}^{n} p''(i)p'''(n-i)\right)q^n = \sum_{n\geqslant 0} p'(n)q^n. \tag{3.13}$$

Conversely, suppose that $\sum\limits_{n\geqslant 0} p'(n)q^n = \left(\sum\limits_{n\geqslant 0} p''(n)q^n\right)\left(\sum\limits_{n\geqslant 0} p'''(n)q^n\right)$. We have that for any $n \geqslant 0$, $p'(n)q^n = \sum\limits_{i=0}^{n} p''(i)q^i p'''(n-i)q^{n-i}$ by multiplication of the generating functions. We further re-express this equality as $\sum\limits_{\lambda'\vdash n} q^{\lambda'} = \sum\limits_{i=0}^{n}\left(\sum\limits_{\lambda''\vdash i} q^{\lambda''}\right)\left(\sum\limits_{\lambda'''\vdash n-i} q^{\lambda'''}\right) = \sum\limits_{i=0}^{n}\left(\sum\limits_{\lambda''\vdash i}\left(\sum\limits_{\lambda'''\vdash n-i} q^{\lambda''+\lambda'''}\right)\right)$.
By assumption we have, $\lambda'' + \lambda''' = \lambda'$, and by the equality there are the same number of $q^{\lambda'}$ on the left hand side as $q^{\lambda''+\lambda'''}$ on there right hand side. Then the mapping, $q^{\lambda''+\lambda'''} \to q^{\lambda'}$ by $\lambda'' + \lambda''' = \lambda'$ is a bijection by equality so sets hence the sum must be unique as desired. ■

We now provide a combinatorial proof of Lemma 36.

Proof. Let $\lambda_{\backslash 1} = (0, a_2, ..., a_m) \in E_{m\backslash 1}$. Let $\lambda_{\backslash 1}^{(r)} = (0, r_2, ..., r_m)$ be a sub partition of $\lambda_{\backslash 1}$. $\lambda_{\backslash 1}^{(r)}$ is defined by two cases, parts that are powers of s and parts that are not powers of s. If a part of size i is a power of s, then the number of parts r_i, is zero. Otherwise, we define the number of parts, r_i, to be the remainder of a_i when divided by s. Let $\lambda_{\backslash 1}^{(q)}$ be $(0, a_2 - r_2, ..., a_m - r_m)$. The number $a_i - r_i$, unless i is a power of s, is divisible by s. We have $\lambda_{\backslash 1} = \lambda_{\backslash 1}^{(r)} + \lambda_{\backslash 1}^{(q)}$. Furthermore, by the uniqueness of quotient and remainders of non-negative integer division, this sum is unique.

$\lambda_{\backslash 1}^{(r)}$ is a partition generated $G(q)$. That is, $\lambda_{\backslash 1}^{(r)}$ is a partition such that no part has a power of s and for any part of size i there are less than s copies. The generating function $E_{m-1}(q^s)$ describes partitions such parts of size i that are powers of s occur no more than $\mathrm{lcm}([m-1]) = \frac{\mathrm{lcm}([m])}{s}$ times and parts of size i that are not powers of s occur a multiple of s number of times, with no more than $\frac{s \cdot \mathrm{lcm}([m-1])}{i}$ copies of a part of size i. Any subpartition $\lambda_{\backslash 1}^{(q)}$ if $\lambda_{\backslash 1}$ is generated by $E_{m-1}(q^s)$ as its construction satisfies the description of partitions generated by $E_{m-1}(q^s)$.

Let $'$ be the restriction describing partitions in $E_{m\backslash 1}$. Let $''$ be the restriction that is described by $\lambda_{\backslash 1}^{(r)}$ in the first paragraph of this proof and $'''$ be the restriction that is described by $\lambda_{\backslash 1}^{(q)}$ in the first paragraph. The restrictions $', '', '''$ satisfy the conditions of Proposition 39, and the lemma is proven.

■

With Lemma 36 in hand, deducing if a constituent of $\Delta(n,m)$ can be expressed as a non-negative linear combination of constituents of $p(n, m-1)$ is straightforward.

Example 40. Let $m = 5$, then by Lemma 36, $E_{5\backslash 1}(q) = G(q)E_4(q^5) = \left(\prod_{i=2}^{4} \sum_{j=0}^{4} q^{ij} \right) E_4(q^5)$. Now, construct the respective quasipolynomials of $\Delta(n,5)$ and $p(n,4)$,

$$\sum_{n=0}^{\infty} \Delta(n,5) q^n = \frac{1}{(q^2;q)_4} = \frac{E_{5\backslash 1}(q)}{(1-q^{60})^4} = E_{5\backslash 1}(q) \sum_{k=0}^{\infty} \binom{k+3}{3} q^{60k} \tag{3.14}$$

$$= G(q)E_4(q^5) \sum_{k=0}^{\infty} \binom{k+3}{3} q^{60k} = \left(\prod_{i=2}^{4} \sum_{j=0}^{4} q^{ij} \right) E_4(q^5) \sum_{k=0}^{\infty} \binom{k+3}{3} q^{60k} \tag{3.15}$$

$$\text{and } \sum_{n=0}^{\infty} p(n,4)q^n = \frac{1}{(q;q)_4} = \frac{E_4(q)}{(1-q^{12})^4} = E_4(q)\sum_{k=0}^{\infty}\binom{k+3}{3}q^{12k}. \tag{3.16}$$

Where

$$G(q) = \left(\prod_{i=2}^{4}\sum_{j=0}^{4} q^{ij}\right) = 1+q^2+q^3+2q^4+q^5+3q^6+2q^7+4q^8+3q^9+4q^{10}+4q^{11}+6q^{12}$$
$$+4q^{13}+6q^{14}+5q^{15}+7q^{16}+5q^{17}+7q^{18}+5q^{19}+7q^{20}+5q^{21}+6q^{22}+4q^{23}+6q^{24}+4q^{25}$$
$$+4q^{26}+3q^{27}+4q^{28}+2q^{29}+3q^{30}+q^{31}+2q^{32}+q^{33}+q^{34}+q^{36}, \tag{3.17}$$

$$E_4(q) = 1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10}+27q^{11}+30q^{12}$$
$$+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19}+48q^{20}+48q^{21}+44q^{22}+42q^{23}$$
$$+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28}+18q^{29}+15q^{30}+11q^{31}+9q^{32}+6q^{33}+5q^{34}$$
$$+3q^{35}+2q^{36}+q^{37}+q^{38}, \text{ and} \tag{3.18}$$

$$E_4(q^5) = 1+q^5+2q^{10}+3q^{15}+5q^{20}+6q^{25}+9q^{30}+11q^{35}+15q^{40}+18q^{45}+23q^{50}+27q^{55}$$
$$+30q^{60}+35q^{65}+39q^{70}+42q^{75}+44q^{80}+48q^{85}+48q^{90}+50q^{95}+48q^{100}+48q^{105}$$
$$+44q^{110}+42q^{115}+39q^{120}+35q^{125}+30q^{130}+27q^{135}+23q^{140}+18q^{145}+15q^{150}+11q^{155}$$
$$+9q^{160}+6q^{165}+5q^{170}+3q^{175}+2q^{180}+q^{185}+q^{190}. \tag{3.19}$$

For example, $\Delta(60k+0,5) = p(12k+0,4)+p(12k-1,4)+4p(12k-2,4)+5p(12k-3,4)+7p(12k-4,4)+4p(12k-5,4)+3p(12k-6,4)$.

While further arithmetic could explicitly show the partition identity between $\Delta(n,5)$ and $p(n,4)$, it is tedious and the observed relationship above motivates the proof of the Theorem 23.

3.3 Proof of a Theorem on First Differences of $p(n,m)$

We now prove Theorem 23.

Proof. We manipulate the generating function of $\Delta(n,m)$ following the proof for Theorem 26 and apply Lemma 36,

$$\sum_{n=0}^{\infty}\Delta(n,m)q^n = \frac{1}{(q^2;q)_{m-1}} = \frac{E_{m\backslash 1}(q)}{E_{m\backslash 1}(q)(q^2;q)_{m-1}} = \frac{E_{m\backslash 1}(q)}{(1-q^{\mathrm{lcm}([m])})^{m-1}} \tag{3.20}$$

$$= E_{m\backslash 1}(q)\sum_{k=0}^{\infty}\binom{k+m-2}{m-2}q^{\mathrm{lcm}([m])k} = G(q)E_{m-1}(q^s)\sum_{k=0}^{\infty}\binom{k+m-2}{m-2}q^{\mathrm{lcm}([m])k}. \tag{3.21}$$

with $G(q) = \prod_{i=2, i\neq s^a, \forall a\in\mathbb{N}}^{m}\sum_{j=0}^{s-1}q^{ij}$. Next, we construct the quasipolynomial for $p(n,m-1)$,

$$\sum_{n=0}^{\infty}p(n,m-1)q^n = \frac{1}{(q;q)_{m-1}} = \frac{E_{m-1}(q)}{(1-q^{\mathrm{lcm}([m-1])})^{m-1}} = E_{m-1}(q)\sum_{k=0}^{\infty}\binom{k+m-2}{m-2}q^{\mathrm{lcm}([m-1])k}. \tag{3.22}$$

Express the polynomials $E_{m\backslash 1}(q)$, $E_{m-1}(q)$, and $G(q)$ as

$$E_{m\backslash 1}(q) = \sum_{i\geqslant 0}h_iq^i,\ E_{m-1}(q) = \sum_{i\geqslant 0}f_iq^i,\text{ and } G(q) = \sum_{i\geqslant 0}g_iq^i. \tag{3.23}$$

Notice if the index of any coefficient of $E_{m\backslash 1}(q)$, $E_{m-1}(q)$, and $G(q)$ is less than zero, then the coefficient is zero. For any constituent

$$\Delta(\mathrm{lcm}([m])k+j,m) = \sum_{t\geqslant 0}h_{j+\mathrm{lcm}([m])t}\binom{k+m-2-t}{m-2} \tag{3.24}$$

of the quasipolynomial of $\Delta(n,m)$, let l and r be integers such that $j = l+r$ where r is the remainder of j when divided by s with $0\leqslant r < s$. Since $G(q)E_{m-1}(q^s) = E_{m\backslash 1}(q)$ by Lemma 36, this implies by polynomial multiplication that the coefficient $h_{j+\mathrm{lcm}([m])t}$ in line (3.24) is

$$h_{j+\mathrm{lcm}([m])t} = \sum_{i\geqslant 0} g_{is+r} f_{\mathrm{lcm}([m])t+l-is}. \tag{3.25}$$

Therefore, we apply line (3.25) to line (3.24) and rearrange the sum to see

$$\sum_{t\geqslant 0} h_{j+\mathrm{lcm}([m])t}\binom{k+m-2-t}{m-2} = \sum_{t\geqslant 0}\sum_{i\geqslant 0} g_{is+r} f_{\mathrm{lcm}([m])t+l-is}\binom{k+m-2-t}{m-2} \tag{3.26}$$

$$= g_r \sum_{t\geqslant 0} f_{\mathrm{lcm}([m])t+l}\binom{k+m-2-t}{m-2} + \sum_{i\geqslant 1}\sum_{t\geqslant 0} g_{is+r} f_{\mathrm{lcm}([m])t+l-is}\binom{k+m-2-t}{m-2} \tag{3.27}$$

$$= g_r \sum_{t\geqslant 0} f_{\mathrm{lcm}([m])t+l}\binom{k+m-2-t}{m-2} + \sum_{i\geqslant 1}\sum_{t\geqslant 1} g_{is+r} f_{\mathrm{lcm}([m])t+l-is}\binom{k+m-2-t}{m-2} \tag{3.28}$$

$$= g_r \sum_{t\geqslant 0} f_{\mathrm{lcm}([m])t+l}\binom{k+m-2-t}{m-2} + \sum_{i\geqslant 1} g_{is+r} \sum_{t\geqslant 0} f_{\mathrm{lcm}([m])(t-1)+l-is}\binom{k+m-2-(t+1)}{m-2}. \tag{3.29}$$

Set $l' = \frac{l}{s}$. We now compare line (3.29) to constituents of $p(n, m-1)$. In particular,

$$p(\mathrm{lcm}([m-1])k + l' + (\mathrm{lcm}([m-1]) - i), m-1) = \sum_{t\geqslant 0} f_{\mathrm{lcm}([m-1])t+l'+(\mathrm{lcm}([m-1])-i)}\binom{k+m-2-t}{m-2}. \tag{3.30}$$

Taking $k = k-1$ we have,

$$p(\mathrm{lcm}([m-1])(k-1) + l' + (\mathrm{lcm}([m-1]) - i), m-1) = p(\mathrm{lcm}([m-1])k + l' - i, m-1)$$
$$= \sum_{t\geqslant 0} f_{\mathrm{lcm}([m-1])t+l'+(\mathrm{lcm}([m-1])-i)}\binom{k+m-2-(t+1)}{m-2} \tag{3.31}$$

By substituting q for q^s in $E_{m-1}(q)$, we have $f_{\mathrm{lcm}([m])t+l+is} = f_{\mathrm{lcm}([m-1])t+l'+i}$. Furthermore, since $\mathrm{lcm}([m]) = s \cdot \mathrm{lcm}([m-1])$, the following holds by combining lines (3.29) and (3.31),

$$\Delta(\mathrm{lcm}([m])k + j, m) = \sum_{i\geqslant 0} g_{r+si}\, p(\mathrm{lcm}([m-1])k + l' - i, m-1). \tag{3.32}$$

∎

We now prove Proposition 13.

Proof. We start by creating the quasipolynomial for $\Delta(n,m)$ and apply properties of $E_m(q)$ and $E_{m\backslash 1}(q)$,

$$\sum_{n=0}^{\infty}\Delta(n,m)q^n = \frac{1}{(q^2;q)_{m-1}} = \frac{1-q}{(q;q)_m} = \frac{(1-q)E_m(q)}{(1-q^{\mathrm{lcm}([m])})^m} \tag{3.33}$$

$$= \frac{E_{m\backslash 1}(q)}{(1-q^{\mathrm{lcm}([m])})^{m-1}} = E_{m\backslash 1}(q)\sum_{k=0}^{\infty}\binom{k+m-2}{m-2}q^{\mathrm{lcm}([m])k}. \tag{3.34}$$

Next, we address $p(n,m-1)$.

$$\sum_{n=0}^{\infty}p(n,m-1)q^n = \frac{1}{(q;q)_{m-1}} = \frac{1-q^m}{(q;q)_m} = \frac{(1-q^m)E_m(q)}{(1-q^{\mathrm{lcm}([m])})^m} \tag{3.35}$$

$$= \frac{(1-q)\left(\sum_{i=0}^{m-1}q^i\right)E_m(q)}{(1-q^{\mathrm{lcm}([m])})^m} = \frac{\left(\sum_{i=0}^{m-1}q^i\right)E_{m\backslash 1}(q)}{(1-q^{\mathrm{lcm}([m])})^{m-1}} = \left(\sum_{i=0}^{m-1}q^i\right)E_{m\backslash 1}(q)\sum_{k=0}^{\infty}\binom{k+m-2}{m-2}q^{\mathrm{lcm}([m])k}. \tag{3.36}$$

The arguments in the proof of Theorem 23 yield,

$$p(\mathrm{lcm}([m-1])k+j,m-1) = \sum_{i=0}^{m-1}\Delta(\mathrm{lcm}([m])k+j-i,m) = \Delta_m(\mathrm{lcm}([m])k+j,m). \tag{3.37}$$

Thus $p(n,m-1)$ can always be written as a linear combination of constituents of $\Delta(n,m)$ in a manner that is equivalent to the familiar partition identity, $p(n,m-1) = p(n,m)-p(n-m,m)$. ∎

CHAPTER IV

THE GROUP OF REMAINDER PARTITIONS

We now aim to generalize the previous section into partitions with parts from finite sets A and B.

4.1 lcm(A)-Remainder Partitions

First, lcm(A)-remainder partitions are equipped with a natural group structure as an operation on the number of parts.

Definition 41. Let A be a finite set of positive integers and consider the set E_A consisting of all lcm(A)-remainder partitions. We define an operation, $\oplus$, on E_A. Let $\lambda_1, \lambda_2 \in E_A$, and $\lambda_1 = (a_{i_1,1}, ..., a_{i_k,\alpha_{i_k}})$ and $\lambda_2 = (b_{i_1,1}, ..., b_{i_k,\alpha_{i_k}})$. Then $\lambda_1 \oplus \lambda_2 = \lambda_3 = (c_{i_1,1}, ..., c_{i_k,\alpha_{i_k}})$ by defining $c_{i_j,\ell}$ with $\ell \in \{1, 2, ..., \alpha_{i_j}\}$ to be the smallest non-negative remainder of $a_{i_j,\ell} + b_{i_j,\ell}$ divided by $\frac{\text{lcm}(A)}{i_j}$. That is, $a_{i_j,\ell} + b_{i_j,\ell} \equiv c_{i_j,\ell} \pmod{\frac{\text{lcm}(A)}{i_j}}$. The operation $\oplus$ is called **piecewise modular addition.**

Proposition 42. *The operation $\oplus$ on E_A is a abeilan group.*

Proof. A routine check of axioms follows. We note the operation is closed by the choice of the smallest non-negative remainder. The identity of E_A is the empty partition, $\lambda_e = (0, 0, ..., 0)$ as for any $\lambda \in E_A$, $\lambda_e \oplus \lambda = \lambda$ as $0 + a_{i_j,\ell} = a_{i_j,\ell}$ which is less than $\frac{\text{lcm}(A)}{i_j}$ but greater than or equal to 0. Next, we show $\oplus$ is associative. For any $\lambda_1, \lambda_2, \lambda_3 \in E_A$, we observe $(\lambda_1 \oplus \lambda_2) \oplus \lambda_3$ component wise, we have $(a_{i_j,\ell} + b_{i_j,\ell}) + c_{i_j,\ell} \equiv a_{i_j,\ell} + (b_{i_j,\ell} + c_{i_j,\ell}) \pmod{\frac{\text{lcm}(A)}{i_j}}$ which implies that $\lambda_1 \oplus (\lambda_2 \oplus \lambda_3)$. We follow this by showing the existence of inverses. For $\lambda_1 \in E_A$, we pick λ_2 by $b_{i_j,\ell} = \frac{\text{lcm}(A)}{i_j} - a_{i_j,\ell}$ unless $a_{i_j,\ell} = 0$, then we set $b_{i_j,\ell} = 0$. We see that $\lambda_1 \oplus \lambda_2 = \lambda_e$. If $a_{i_j,\ell} = 0$, then $a_{i_j,\ell} + b_{i_j,\ell} = 0 + 0 = 0 \pmod{\frac{\text{lcm}(A)}{i_j}}$ and if not then, $a_{i_j,\ell} + b_{i_j,\ell} = a_{i_j,\ell} + \frac{\text{lcm}(A)}{i_j} - a_{i_j,\ell} \equiv 0 \pmod{\frac{\text{lcm}(A)}{i_j}}$.

Lastly, the commutative property follows as for observing $\lambda_1 \oplus \lambda_2$ component wise. We have $a_{i_j,\ell} + b_{i_j,\ell} = b_{i_j,\ell} + a_{i_j,\ell} \pmod{\frac{\text{lcm}(A)}{i_j}}$ which implies $\lambda_1 \oplus \lambda_2 = \lambda_2 \oplus \lambda_1$. ■

The notation E_A will be used for both the group $(E_A, \oplus)$ and the set of partitions lcm(A)-remainder partitions. We now give several important properties of E_A, the group of lcm(A)-remainder partitions.

Proposition 43. *The group E_A is isomorphic to $\prod_{i\in A} \mathbb{Z}_{\frac{lcm(A)}{i}}$ where the product is the usual direct product of groups and $\mathbb{Z}_n$ is a cyclic group of order n.*

The proof is left to the reader. The group E_A has order $\prod_{i\in A} \frac{\text{lcm}(A)}{i}$ by Proposition 43 or by counting and the order of any element in E_A is at most lcm(A).

Proposition 44. *Let $\lambda_1, \lambda_2 \in E_A$ and suppose that $\lambda_1 \vdash n$ and $\lambda_2 \vdash m$. Then $\lambda_1 \oplus \lambda_2 \vdash (n+m) \pmod{lcm(A)}$.*

Proof. We first note that $n = \sum_{s=1}^{k} \sum_{t=1}^{\alpha_{i_s}} i_j \cdot a_{i_s,t}$ and $m = \sum_{s=1}^{k} \sum_{t=1}^{\alpha_{i_s}} i_j \cdot b_{i_s,t}$. Furthermore, it must be the case that $\lambda_1 \oplus \lambda_2$ implies that $(a_{i_s,t} + b_{i_s,t}) = c_{i_s,t} + x_{i_s,t}\left(\frac{\text{lcm}(A)}{i_s}\right)$ for some positive integer $x_{i_s,t}$ by definition. The following arithmetic shows the result,

$$\begin{aligned} n+m = \sum_{s=1}^{k}\sum_{t=1}^{\alpha_{i_s}} i_j\,(a_{i_s,t}+b_{i_s,t}) &= \sum_{s=1}^{k}\sum_{t=1}^{\alpha_{i_s}} i_j\left(c_{i_s,t}+x_{i_s,t}\left(\frac{\text{lcm}(A)}{i_s}\right)\right) = \sum_{s=1}^{k}\sum_{t=1}^{\alpha_{i_s}}\left(i_j c_{i_s,t}+x_{i_s,t}\,\text{lcm}(A)\right) \\ &= \lambda_1\oplus\lambda_2 + \text{lcm}(A)\sum_{s=1}^{k}\sum_{t=1}^{\alpha_{i_s}} x_{i_s,t} \equiv \lambda_1\oplus\lambda_2 \pmod{\text{lcm}(A)}. \end{aligned} \tag{4.1}$$

■

The proof of Proposition 44 closely follows the strategy employed in Theorem 32 applied strictly to lcm(A)-remainder partitions. We are now ready to state and prove the generalization of Lemma 36.

Lemma 45. *Let A and B be two sets of positive integers with $|B| \leqslant |A|$. If there is an onto homomorphism from E_A to E_B, then $E_A(q) = G(q)E_B(q^s)$ for $G(q)$ a generating function of some collection of lcm(A)-remainder partitions and $s = \frac{lcm(A)}{lcm(B)}$.*

Proof. Suppose that there is an onto homomorphism φ from E_A to E_B. By the first isomorphism theorem, $Im(\varphi) \cong E_A/\ker(\varphi)$ and as the mapping is onto, $Im(\varphi) \cong E_B$. We essentially have two tasks; first, show that there is a collection of partitions $H \subseteq E_A$ that is generated by $E_B(q^s)$; second, to find a collection, G, of $\mathrm{lcm}(A)$-remainder partitions such that H and G are restrictions of $\mathrm{lcm}(A)$-remainder partitions which satisfy the conditions of Proposition 39.

To satisfy the first task, we make a few notes about E_B in relationship to E_A. Let $\lambda^{(A)}_{i_{v,w}} \in E_A$ be be a partition that has one part of size i_v in color w. $\lambda^{(A)}_{i_{v,w}}$ is a generator of group E_A. Likewise, let $\lambda^{(B)}_{i_{t,u}} \in E_B$ be a partition that has one part of size i_t in color u. Likewise $\lambda^{(B)}_{i_{t,u}}$ is a generator of the group E_B. As φ can be viewed as an onto homomorphism for direct products of cyclic groups, it is the case that $\varphi(x \cdot \lambda^{(A)}_{i_{v,w}}) = \lambda^{(B)}_{i_{t,u}}$ where $x \cdot \lambda^{(A)}_{i_{v,w}} := \bigoplus_{k=1}^{x} \lambda^{(A)}_{i_{v,w}}$ is piecewise modular addition of $\lambda^{(A)}_{i_{v,w}}$ x times and $\gcd(x, \frac{\mathrm{lcm}(A)}{i_v}) = 1$. Since the order of $x \cdot \lambda^{(A)}_{i_{v,w}}$ is $\frac{\mathrm{lcm}(A)}{i_{v,w}}$, we have

$$\sum_{k=1}^{\frac{\mathrm{lcm}(A)}{i_{v,w}}} \left(\sum_{j=1}^{x} \lambda^{(A)}_{i_{v,w}} \right) \vdash x \cdot i_{v,w} \cdot \frac{\mathrm{lcm}(A)}{i_{v,w}} = x \cdot \mathrm{lcm}(A) \tag{4.2}$$

and in the group E_A, it is the case that $\frac{\mathrm{lcm}(A)}{i_{v,w}} \cdot (x \cdot \lambda^{(A)}_{i_{v,w}}) = \lambda_e$. We apply φ, which yields

$$\frac{\mathrm{lcm}(A)}{i_{v,w}} \cdot \varphi(x \cdot \lambda^{(A)}_{i_{v,w}}) = \frac{\mathrm{lcm}(A)}{i_{v,w}} \cdot \lambda^{(B)}_{i_{t,u}} = \lambda^{(B)}_e. \tag{4.3}$$

Applying Proposition 44 to (4.3), we see that

$$\lambda^{(B)}_e = \bigoplus_{r=1}^{\frac{\mathrm{lcm}(A)}{i_{v,w}}} \lambda^{(B)}_{i_{t,u}} \vdash \sum_{r=1}^{\frac{\mathrm{lcm}(A)}{i_{v,w}}} i_{t,u} = \frac{\mathrm{lcm}(A)}{i_{v,w}} \cdot i_{t,u} \equiv 0 \pmod{\mathrm{lcm}(B)}. \tag{4.4}$$

This implies that there is a smallest positive integer y such that

$$y \cdot \mathrm{lcm}(B) = \frac{\mathrm{lcm}(A)}{i_{v,w}} \cdot i_{t,u} \text{ which is equivalent to } y \cdot i_{v,w} = \frac{\mathrm{lcm}(A)}{\mathrm{lcm}(B)} \cdot i_{t,u} = s \cdot i_{t,u}. \tag{4.5}$$

If $s \mid y$ in the previous line, there is a positive integer z such that $z \cdot i_{v,w} = i_{t,u}$. Furthermore, $z \cdot$

$i_{v,w} \cdot \frac{\mathrm{lcm}(B)}{i_{t,u}} = \mathrm{lcm}(A)$. If $s \nmid y$ in line (4.5), then $s \mid i_{v,w}$. For each generator $\lambda_{i_{t,u}}^{(B)}$ of E_B, we name a partition $\lambda_{i_{t,u}}^{(A)} \in E_A$. Let $\lambda_{i_{t,u}}^{(A)} = y \cdot \lambda_{i_{v,w}}^{(A)} \in E_A$ such that $y \cdot \lambda_{i_{v,w}}^{(A)} \vdash s \cdot i_{t,u}$. We define the set H by

$$H = \left\{ \bigoplus_{i_{t,u} \in B} k_{i_{t,u}} \cdot \lambda_{i_{t,u}}^{(A)} \;\middle|\; 0 \leqslant k_{i_{t,u}} < \frac{\mathrm{lcm}(B)}{i_{t,u}} \right\}. \tag{4.6}$$

We have a natural bijection between H and E_B by mapping $\lambda_{i_{t,u}}^{(A)}$ to $\lambda_{i_{t,u}}^{(B)}$. Next, if $\lambda \vdash n$ in E_B, the corresponding partition by the natural bijection in H must partition $s \cdot n$. By this correspondence, the generating function for H must be $E_B(q^s)$.

With our first task complete, we turn to the second. Let $\lambda \in E_A$. We will define λ_q and λ_r such that $\lambda_q + \lambda_r = \lambda$ and that $\lambda_q \in H$, $\lambda_r \in G$. We start by defining λ_r. For the number $a_{i_{v,w}}$ of parts of size i_v and color w, if there is a $\lambda_{i_{t,u}}^{(A)} \in H$ such that $\lambda_{i_{t,u}}^{(A)} = y \cdot \lambda_{i_{v,w}}^{(A)} \vdash s \cdot i_{t,u}$ then we have two cases: if $y \mid s$ and if $y \nmid s$. For the first case, we divide $a_{i_{v,w}}$ by y letting the remainder be defined as $r_{i_{v,w}}$. In the second case, we set $r_{i_{v,w}} = 0$. In all other cases, we set $r_{i_{v,w}}$ to $a_{i_{v,w}}$. Let λ_r be the partition with the number $r_{i_{v,w}}$ of parts of size i_v and color w. Let λ_q be the partition with the number $q_{i_{v,w}} = a_{i_{v,w}} - r_{i_{v,w}}$ of parts of size i_v and color w. Then we define G be the set of all λ_r partitions. As G is a finite set, $G(q)$ is the polynomial generating function for the partitions in G. Furthermore, by uniqueness of non-negative integer division, we the unique sum $\lambda_q + \lambda_r = \lambda$.

We verify that $\lambda_q \in H$. For the number $q_{i_{v,w}}$ of parts of size i_v and color w, if there is a $\lambda_{i_{t,u}}^{(A)} \in H$ such that $\lambda_{i_{t,u}}^{(A)} = y \cdot \lambda_{i_{v,w}}^{(A)} \vdash s \cdot i_{t,u}$, then we have two cases. If $s \mid y$, then $y \mid q_{i_{v,w}}$ by definition and hence $q_{i_{v,w}} = k_{i_{t,u}} \cdot y$ for some $0 \leqslant k_{i_{t,u}} < \frac{\mathrm{lcm}(B)}{i_{t,u}}$. If $s \nmid y$, then $\gcd(y, \frac{\mathrm{lcm}(B)}{i_{t,u}}) = 1$ and there is some $0 \leqslant k_{i_{t,u}} < \frac{\mathrm{lcm}(B)}{i_{t,u}}$ such that $y \cdot k_{i_{t,u}} \equiv q_{i_{v,w}} \pmod{\frac{\mathrm{lcm}(A)}{i_{v,w}}}$. In all other cases, $q_{i_{v,w}} = 0$ and hence implicitly is in H. Since $\lambda_q + \lambda_r = \lambda$ is unique, we have satisfied the conditions of Proposition 39, proving the lemma. ■

Remark 46. The reader is encouraged to show that H is isomorphic to E_B (hence a subgroup of E_A) and that $E_A/H \cong G \cong \ker(\varphi)$. This fact is not necessary in the proof but gives an idea how to describe G. We can think of the set G as the “remainder” of E_A when considered modulo partitions from E_B.

When $A=\{2,...,m\}$ and $B=\{1,...,m-1\}$, we construct an explicit mapping in the second proof of Lemma 36 between $E_{m\backslash 1}$ and E_{m-1}. This mapping inspires the more general proof. Since both $E_{m\backslash 1}$ and E_{m-1} are isomorphic to direct products of cyclic groups, we simply need to describe a homomorphism between these groups. We now sketch the proof Lemma 36 using Lemma 45. This exhibits how Lemma 45 can be used to establish infinite families for partition identities.

Proof. Let $m=s^k$ where s is prime and k is a positive integer. By Proposition 43, it is sufficient to define a homomorphism $\varphi : \prod_{i=2}^{m} \mathbb{Z}_{\frac{\mathrm{lcm}([m])}{i}} \to \prod_{i=1}^{m-1} \mathbb{Z}_{\frac{\mathrm{lcm}(m-1)}{i}}$. Let $(x_2,x_3,...,x_m)=x\in \prod_{i=2}^{m} \mathbb{Z}_{\frac{\mathrm{lcm}([m])}{i}}$ and $(y_1,y_2,...,y_{m-1})=y\in \prod_{i=1}^{m-1} \mathbb{Z}_{\frac{\mathrm{lcm}(m-1)}{i}}$. The generators of $\prod_{i=2}^{m} \mathbb{Z}_{\frac{\mathrm{lcm}([m])}{i}}$ are $e_2=(1,0,0,...,0), e_3=(0,1,0,...,0),...,e_m=(0,0,0,...,1)$ and the generators of $\prod_{i=1}^{m-1} \mathbb{Z}_{\frac{\mathrm{lcm}(m-1)}{i}}$ are $f_1=(1,0,0,...,0), f_2=(0,1,0,...,0),...,f_{m-1}=(0,0,0,...,1)$. We define φ based on generators. If $j=s^r$ for any $r=1,...,k$, let $\varphi(e_j)=f_{s^{r-1}}$ and $\varphi(e_j)=f_j$ otherwise. With φ defined, the completion of the proof is routine using $\mathrm{lcm}([m])=s\cdot \mathrm{lcm}(m-1)$ and Lemma 45 and it is left to the reader. ■

With Lemma 45, we can now generalize Theorem 23.

Theorem 47. *Let A and B be two sets of positive integers such that $|A|=|B|$. If there is an onto homomorphism from E_A to E_B, then any constituent of $p(n,A)$ can be expressed as a non-negative linear combination of constituents $p(n,B)$.*

Proof. First, by Lemma 45, $E_A(q)=G(q)E_B(q^s)$ for $s=\frac{lcm(A)}{lcm(B)}$. That is, $G(q)=\frac{E_A(q)}{B_E q^s}$ and can be computed because $E_A(q)$ and $E_B(q^s)$ are known. We proceed to manipulate the generating function of $p(n,A)$ using the facts noted above,

$$\sum_{n=0}^{\infty} p(n,A)q^n = \prod_{i\in A}\frac{1}{(1-q^i)} = \frac{E_A(q)}{(1-q^{lcm(A)})^d} = E_A(q)\sum_{k=0}^{\infty}\binom{k+d-1}{d-1}q^{lcm(A)k} \tag{4.7}$$

$$= G(q)E_B(q^s)\sum_{k=0}^{\infty}\binom{k+d-1}{d-1}q^{lcm(A)k}. \tag{4.8}$$

Next, we construct the quasipolynomial for $p(n,B)$,

$$\sum_{n=0}^{\infty} p(n,B)q^n = \prod_{i\in B} \frac{1}{(1-q^i)} = \frac{E_B(q)}{(1-q^{lcm(B)})^d} = E_B(q)\sum_{k=0}^{\infty}\binom{k+d-1}{d-1} q^{lcm(B)k}. \tag{4.9}$$

The remainder of the proof follows the proof of Theorem 23 by using the same diligent arithmetic. In particular considering j when divided by s, that is $j = l's + r$, we have,

$$p(\mathrm{lcm}(A)k + j, A) = \sum_{i\geqslant 0} g_{r+si}\, p(\mathrm{lcm}(B)k + l' - i, B). \tag{4.10}$$

■

The third proof of Lemma 36 using Lemma 45 in conjunction with Theorem 47 proves Theorem 23.

4.2 Notes for Future Projects

We have given three proofs of Theorem 23. First, though classic generating function arithmetic. Second, by showing a combinatorial map between 1-free partitions into at most m parts and partitions with no more than $m-1$ parts via conjugation and the sum operation in Chapter 3. Third, by proving the more general theorem, Theorem 47, which applies algebraic structure to $\mathrm{lcm}(A)$-remainder partitions and the sum operation to find a 'universal' mapping. Unifying these proofs is the fact that $p(n,A)$ can be expressed as a quasipolynomial with a finite number of constituents.

The existence of an onto homomoprhism from E_A to E_B gives a sufficient condition for when a constituent of the quasipolynomial of $p(n,A)$ can be expressed as a non-negative linear combination of constituents of the quasipolynomial of $p(n,B)$. The converse is in general not true. For example, Proposition 13 implies that any a constituent of $p(n,m-1)$ can be written as a non-negative linear combination of constituents of $\Delta(n,m)$ by line (3.37). When $m = 5$, there is no onto homomorphism from $E_4 \to E_{5\setminus 1}$ as $\left|E_{5\setminus 1}\right| > |E_4|$. Given Theorem 23, we pose the following questions.

Question 48. *If m is not a prime power, can any constituent of $\Delta(n,m)$ be expressed as a non-negative linear combination of constituents of $p(n,m-1)$ in accordance to Theorem 26?*

Question 49. *What other infinite families of partition identities can be established using Theorem 47?*

CHAPTER V

AYSMPTOTICS OF $P(N,A)$

We have utilized the group E_A to establish certain partition identities of $p(n,m)$. Now we will use the group E_A to prove well known asymptotic formula of $p(n,A)$. These The is work in this chapter is joint with Alexander Swarzes-Beard and Dr. Brandt Kronholm [21].

Definition 50. Let the set of partition E_{A_j} be the partitions in E_A such that they partition numbers when divided by $\mathrm{lcm}(A)$ have remainder j.

Proposition 51. *E_{A_0} is a subgroup of E_A.*

Proof. Let λ_1 and λ_2 be in E_{A_0}. By Proposition 44, $\lambda_2^{-1} \in E_{A_0}$ and again by Proposition 44 $\lambda_1 \oplus \lambda_2^{-1} \vdash (g+h) \equiv 0+0 \pmod{\mathrm{lcm}(A)}$ and hence $\lambda_1 \oplus \lambda_2^{-1} \in E_{A_0}$ ∎

We now can prove Theorem 24 which appears in some form [18], [10], and [] . We utilize several properties of abstract algebra to quickly provide a proof of the asymptotic formula. We make the simplifying assumption that the elements of A are relatively prime.

Proof. From Theorem 26 we have,

$$p(\mathrm{lcm}(A)k+j,A) = \sum_{t=0}^{d} h_{j+\mathrm{lcm}(A)t}\binom{k-t+(d-1)}{d-1} = \sum_{t=0}^{d} \frac{a_{j+\mathrm{lcm}(A)t}\prod_{i=0}^{d-2}(k+d-1-i-t)}{(d-1)!}$$
$$= \left(\frac{1}{(d-1)!}\sum_{t=0}^{d} h_{j+\mathrm{lcm}(A)t}k^{d-1}\right) + f_j(k). \quad (5.1)$$

where $f_j(k)$ is a polynomial of degree $d-2$. Since E_{A_0} is a normal subgroup of E_A, we have $\mathrm{lcm}(A)$ distinct cosets each corresponding with representative elements that which partitions of numbers with remainders 0 to $\text{lcm}(A)-1$ and thus $\text{lcm}(A) = \frac{|E_A|}{E_{A_0}}$ which implies that $\left|E_{A_0}\right| = \frac{\text{lcm}(A)^{d-1}}{\prod_{i\in A} i}$. Now we have $\left|E_{A_0}\right| = \left|E_{A_j}\right| = \frac{\text{lcm}(A)^{d-1}}{\prod_{i\in A} i}$. We further relate E_{A_j} to the coefficients $h_{j+\text{lcm}(A)t}$ by noting that $\left|E_{A_j}\right| = \sum_{t=0}^{d} h_{j+\text{lcm}(A)t}$. Line (5.1) becomes

$$p(\text{lcm}(A)k+j,A) = \left(\frac{1}{(d-1)!}\sum_{s\geqslant 0} h_{j+lcm(A)s}k^{d-1}\right) + f_j(k) = \left(\frac{1}{(d-1)!}\frac{\text{lcm}(A)^{d-1}}{\prod_{i\in A} i}k^{d-1}\right) + f_j(k). \tag{5.2}$$

Lastly, we take $k = \frac{n-j}{\text{lcm}(A)}$,

$$p(n,A) \sim \frac{n^{d-1}}{(d-1)!\prod_{i\in A} i} + \mathscr{O}(n^{d-2}) \tag{5.3}$$

as desired. ■

Remark 52. Though this proof we have shown that the sum of coefficients of any constituent of $p(n,A)$ is $\frac{\text{lcm}(A)^{d-1}}{\prod_{i\in A} i}$ with the assumption that $\gcd(A) = 1$. If this is not the case, a simple adjustment in formula and proof shows that the sum of any constituent is $\frac{\gcd(A)\,\text{lcm}(A)^{d-1}}{\prod_{i\in A} i}$. This comes from the fact that the number of cosets when examining E_A/E_{A_0} will be $\frac{\text{lcm}(A)}{\gcd(A)}$.

We can refine this process to recover part of another result of [18] and [9]. We start with the example for $p(n,m)$ to illustrate the proof and then generalize the statement and proof for $p(n,A)$.

Theorem 53. *For any $m \geqslant 3$,*

$$p(lcm([m])k+j,m) = \alpha_{1,j}k^{m-1} + \alpha_{2,j}k^{m-2} + \mathscr{O}(k^{m-3})$$

where

$$\alpha_{1,j} = \frac{lcm([m])^{m-1}}{m!(m-1)!} \text{ and } \alpha_{2,j} = \frac{lcm([m])^{m-2}}{m!(m-2)!}\left(j + \frac{m(m+1)}{4}\right).$$

Proof. The proof of Theorem 24, we immediately have $\alpha_{1,j} = \frac{\text{lcm}([m])^{m-1}}{m!(m-1)!}$. Next, we consider the first differences of $p(n,m)$, $\Delta(n,m)$. We have that

$$\Delta(\operatorname{lcm}([m])k+j,m) = p(\operatorname{lcm}([m])k+j,m) - p(\operatorname{lcm}([m])k+j-1,m)$$
$$= \sum_{t=1}^{m} \alpha_{t,j}k^{m-t} - \sum_{t=1}^{m} \alpha_{t,j-1}k^{m-t} = \sum_{t=2}^{m} \left(\alpha_{t,j} - \alpha_{t,j-1}\right) k^{m-t}. \quad (5.4)$$

By Theorem 24 and that $\Delta(n,m) = p(n,m\backslash 1)$, we have $\alpha_{2,j} - \alpha_{2,j-1} = \frac{\operatorname{lcm}([m])^{m-2}}{m!(m-2)!}$ hence

$$\alpha_{2,j+r} - \alpha_{2,j} = \frac{\operatorname{lcm}([m])^{m-2}}{m!(m-2)!} \cdot r. \quad (5.5)$$

Furthermore, by changing a constituent of $p(n,m)$ from the binomial basis to a monnomial in k, we have,

$$p(\operatorname{lcm}([m])k+j,m) = \frac{1}{(m-1)!}\left(\sum_{t=1}^{m-1} h_{j+\operatorname{lcm}([m])t}k^{m-1}\right) +$$
$$\frac{1}{(m-1)!}\left(\sum_{t=0}^{m-1} h_{j+\operatorname{lcm}([m])t}\frac{(m-2t)(m-1)}{2}\right) + f_j(k) \quad (5.6)$$

where $f_j(k)$ is a polynomial of degree $m-3$. We wish to evaluate $\alpha_{2,-\frac{m(m+1)}{4}}$ for the purpose of computation. $E_m(q)$ has the property that $h_{t\operatorname{lcm}([m])-\frac{m(m+1)}{4}} = h_{(m-t)\operatorname{lcm}([m])-\frac{m(m+1)}{4}}$ and $h_{t\operatorname{lcm}([m])-\lfloor\frac{m(m+1)}{4}\rfloor}$

$= h_{(m-t)\,\mathrm{lcm}([m])-\lceil\frac{m(m+1)}{4}\rceil}$. We compute

$$
\begin{aligned}
&\alpha_{2,\lceil-\frac{m(m+1)}{4}\rceil}+\alpha_{2,\lfloor-\frac{m(m+1)}{4}\rfloor} = \\
&\quad \left(\sum_{t=0}^{m-1} h_{\lceil-\frac{m(m+1)}{4}\rceil+\mathrm{lcm}([m])t}\frac{(m-2t)(m-1)}{2}\right)+\left(\sum_{t=0}^{m-1} h_{\lfloor-\frac{m(m+1)}{4}\rfloor+\mathrm{lcm}([m])t}\frac{(m-2t)(m-1)}{2}\right)\\
&= \left(\sum_{t=0}^{m-1} h_{\lceil-\frac{m(m+1)}{4}\rceil+\mathrm{lcm}([m])t}\frac{(m-2t)(m-1)}{2}\right)+\left(\sum_{t=0}^{m-1} h_{(m-t)\,\mathrm{lcm}([m])-\lceil\frac{m(m+1)}{4}\rceil}\frac{(m-2t)(m-1)}{2}\right)\\
&= \left(\sum_{t=0}^{m-1} h_{\lceil-\frac{m(m+1)}{4}\rceil+\mathrm{lcm}([m])t}\frac{(m-2t)(m-1)}{2}\right)+\left(\sum_{t=0}^{m-1} h_{t\,\mathrm{lcm}([m])-\lceil\frac{m(m+1)}{4}\rceil}\frac{(m-2(m-t))(m-1)}{2}\right)\\
&= \sum_{t=0}^{m-1}\left(h_{\lceil-\frac{m(m+1)}{4}\rceil+\mathrm{lcm}([m])t}\frac{(m-2t)(m-1)}{2}+h_{t\,\mathrm{lcm}([m])-\lceil\frac{m(m+1)}{4}\rceil}\frac{(m-2(m-t))(m-1)}{2}\right)\\
&= \sum_{t=0}^{m-1}\left(h_{\lceil-\frac{m(m+1)}{4}\rceil+\mathrm{lcm}([m])t}\frac{(m-2t)(m-1)}{2}+h_{t\,\mathrm{lcm}([m])-\lceil\frac{m(m+1)}{4}\rceil}\frac{(m-2(m-t))(m-1)}{2}\right)\\
&= \sum_{t=0}^{m-1}\left(h_{\lceil-\frac{m(m+1)}{4}\rceil+\mathrm{lcm}([m])t}\frac{(m-2t)(m-1)}{2}-h_{t\,\mathrm{lcm}([m])-\lceil\frac{m(m+1)}{4}\rceil}\frac{(m-2t)(m-1)}{2}\right)\\
&= 0. \qquad (5.7)
\end{aligned}
$$

Line (5.7) and (5.5) implies that

$$\alpha_{2,-\frac{m(m+1)}{4}} = \alpha_{2,\lceil-\frac{m(m+1)}{4}\rceil}-\alpha_{2,\lfloor-\frac{m(m+1)}{4}\rfloor} = 0. \tag{5.8}$$

Therefore,

$$\alpha_{2,j} = \alpha_{2,(j+\frac{m(m+1)}{4})-\frac{m(m+1)}{4}}-\alpha_{2,-\frac{m(m+1)}{4}} = \frac{\mathrm{lcm}([m])^{m-2}}{m!(m-2)!}\cdot\left(j+\frac{m(m+1)}{4}\right). \tag{5.9}$$

■

We define a similar difference for $p(n,A)$ as for $p(n,m)$. This allows us to generalize the statement for $p(n,m)$ to $p(n,A)$.

Definition 54. Let x be the smallest integer in A, then we define the smallest difference of A as

$\Delta(n,A) = p(n,A) - p(n-x,A)$.

Theorem 55. *For any $d \geqslant 3$, let $y = \sum_{i \in A} i$ and x be the smallest number in A then,*

$$p(lcm(A)k + j, m) = \alpha_{1,j}k^{d-1} + \alpha_{2,j}k^{d-2} + \mathscr{O}(k^{d-3})$$

where

$$\alpha_{1,j} = \frac{lcm(A)^{d-1}}{(d-1)!\prod_{i \in A} i} \text{ and } \alpha_{2,j} = \frac{lcm(A \backslash \{x\})^{d-2}}{(d-2)!\prod_{i \in A \backslash \{x\}} i}\left(j + \frac{y}{2}\right).$$

The proof is exactly as in the previous theorem except m is replaced with d in any index, $\mathrm{lcm}([m])$ is replaced with $\mathrm{lcm}(A)$, $\frac{m(m+1)}{4}$ is replaced with $\frac{y}{2}$. Lastly, we should see the difference we must observe is

$$\alpha_{2,j+xr} - \alpha_{2,j} = \frac{\mathrm{lcm}(A \backslash \{x\})^{d-2}}{(d-2)!\prod_{i \in A \backslash \{x\}} i} \cdot r. \tag{5.10}$$

CHAPTER VI

EXTENSIONS TO GAUSSIAN POLYNOMIALS

We work to extend the results of Chapter 3 to Gaussian Polynomials. We will arrive at generalization of an identify derived in [6].

Definition 56. The function $p(n,m,N)$ denotes the number of partitions of **doubly restricted partitions** of n into at most m part with no part larger than N and its generating function is

$$\sum_{n=0}^{\infty} p(n,m,N)q^n = \frac{(q;q)_{N+m}}{(q;q)_N(q;q)_m} = \frac{(q^{N+1};q)_m}{(q;q)_m}. \tag{6.1}$$

The notation $\begin{bmatrix} N+m \\ m \end{bmatrix} = \frac{(q;q)_{N+m}}{(q;q)_N(q;q)_m}$ is used.

Andrews [1] gives justification for the generating functions of $p(n,m)$ and $p(n,m,N)$ as well as proof that $\frac{(q^{N+1};q)_m}{(q;q)_m}$ is a polynomial.

Theorem 57.

$$(z;q)_m = \sum_{h=0}^{m} \begin{bmatrix} m \\ h \end{bmatrix} (-1)^h q^{\frac{h(h-1)}{2}} z^h. \tag{6.2}$$

The reader may find a proof of the q-Binomial Theorem in [1]. We provide an alternate proof of a Theorem occurring [6].

Theorem 58. *If $n < 2N+2$, then $\Delta(n,m,N) = \Delta(n,m)$.*

Proof. The proof is split into two parts. First, we manipulate the generating function of $\Delta(n,m,N)$ to show $\Delta(n,m) = \Delta(n,m,N)$ agree up to some n, then we show that this happens when $n < 2N+2$.

The generating function of $\Delta(n,m,N)$ is

$$\sum_{n=0}^{\infty}\Delta(n,m,N)q^n = \frac{(q^{N+1};q)_m}{(q;q)_m} - q\frac{(q^N;q)_m}{(q;q)_m}$$

$$= \frac{(q^{N+1};q)_{m-1}((1-q^{m+N})-q(1-q^N))}{(q;q)_m}$$

$$= \frac{(q^{N+1};q)_{m-1}(1-q)\left(\sum_{i=0}^{m+N-1} q^i - \sum_{j=1}^{N} q^j\right)}{(q;q)_m}$$

$$= \left(\sum_{n=0}^{\infty}\Delta(n,m)q^n\right)(q^{N+1};q)_{m-1}\left(1+\sum_{i=1}^{m-1} q^{N+i}\right). \quad (6.3)$$

If the smallest non-zero power of q of $(q^{N+1};q)_{m-1}\left(1+q^N\sum_{i=1}^{m-1} q^i\right)$ in line (6.3) is $2N+2$, then the statement is shown. To show this, we will carefully multiply $(q^{N+1};q)_{m-1}\left(1+q^N\sum_{i=1}^{m-1} q^i\right)$. Prior to multiplying, we define a recursive function, $A_j(q)$, such that each $A_j(q)$ has the property that the smallest non-zero power of q is $2N+2$. For an index j with $2\leqslant j\leqslant m-1$, let

$$A_1(q) = -q^{N+1}\sum_{i=1}^{m-1} q^{N+i} \text{ and}$$

$\mathrm{A}_j(q) = A_{j-1}(q) - q^{N+j}A_{j-1}(q) - q^{N+j}\sum_{i=j}^{m-1} q^{N+i}$.(6.4)Inspection of $A_1(q)$ reveals that $A_1(q)$ has the property that the smallest non-zero power of q is $2N+2$. Let $2\leqslant j\leqslant m-1$ be given and assume that $A_{j-1}(q)$ has the property that the smallest non-negative power of q is $2N+2$. By assumption $A_{j-1}(q)$; assumption and multiplying by a non-zero power of q, $-q^{N+j}A_{j-1}(q)$; and inspection $-q^{N+j}\sum_{i=j}^{m-1} q^{N+i}$, respectively have the property that the smallest non-zero power of q is $2N+2$. Therefore, $A_j(q)$ must have a smallest non-zero power of q is $2N+2$.

Now, to show that careful multiplication of $(q^{N+1};q)_{m-1}\left(1+q^N\sum_{i=1}^{m-1} q^i\right)$ yields the smallest non-zero power of q is $2N+2$. First, notice that

$$(1-q^{N+1})\left(1+q^N\sum_{i=1}^{m-1}q^i\right)=1-q^{N+1}+q^N\sum_{i=1}^{m-1}q^i-q^{N+1}\sum_{i=1}^{m-1}q^{i+N}=1+q^N\sum_{i=2}^{m-1}q^i+A_1(q). \tag{6.5}$$

Furthermore, note

$$\begin{aligned}(1-q^{N+j})\left(1+q^N\sum_{i=j}^{m-1}q^i+A_{j-1}(q)\right)&=1-q^{N+j}+q^N\sum_{i=j}^{m-1}q^i-q^{N+j}\sum_{i=j}^{m-1}q^{i+N}+A_{j-1}(q)\\&+q^{N+j}A_{j-1}(q)=1+q^N\sum_{i=j+1}^{m-1}q^i+A_j(q).\end{aligned} \tag{6.6}$$

We multiply $(q^{N+1};q)_{m-1}\left(1+q^N\sum_{i=1}^{m-1}q^i\right)$ sequentially starting with the factor $(q^{N+1};q)_{m-1}$ that contains the smallest power of q to the largest by the either $1+q^N\sum_{i=1}^{m-1}q^i$ or the resulting product of the previous computation. Line (6.5) is the result of the first multiplication. In general on the j^{th} multiplication, the q^{N+j} term is zero by line (6.6). As there are $m-2$ factors in $(q^{N+1};q)_{m-1}$ and $m-2$ terms in $q^N\sum_{i=1}^{m-1}q^i$, we see that

$$(q^{N+1};q)_{m-1}\left(1+q^N\sum_{i=1}^{m-1}q^i\right)=1+A_{m-1}(q) \tag{6.7}$$

which indeed has a smallest non-zero power of q that is $2N+2$, hence the lemma is proven. ■

Rather than defining an recursive function, we offer a second proof utilizing the the q-Binomial Theorem. This is how the authors in [6] proved Theorem 58.

Proof. We start from line (6.3). We apply the q-binomial theorem to re-express $(q^{N+1};q)_{m-1}$ in

(6.3)) in the following way.

$$
\begin{aligned}
(q^{N+1};q)_{m-1} &= \sum_{h=0}^{m-1} \begin{bmatrix} m-1 \\ h \end{bmatrix} q^{\frac{h(h-1)}{2}+h(N+1)} \\
&= \sum_{h=0}^{m-1} \sum_{i=0}^{h(m-1-h)} (-1)^h p(i,h,m-1-h) q^{\frac{h(h-1)}{2}+h(N+1)+i} \\
&= 1 - \sum_{j=0}^{m-2} q^{N+1+j} + \sum_{h=2}^{m-1} \sum_{i=0}^{h(m-1-h)} (-1)^h p(i,h,m-1-h) q^{\frac{h(h-1)}{2}+h(N+1)+i}. \qquad (6.8)
\end{aligned}
$$

Note that the following polynomial $A(q)$ has degree strictly greater than to $2N+2$,

$$
A(q) = \sum_{h=2}^{m-1} \sum_{i=0}^{h(m-1-h)} (-1)^h p(i,h,m-1-h) q^{\frac{h(h-1)}{2}+h(N+1)+i}.
$$

Now we have

$$
\begin{aligned}
(q^{N+1};q)_{m-1} \left(1 + \sum_{i=1}^{m-1} q^{N+i}\right) \\
&= \left(1 - q^{N+1} \sum_{j=0}^{m-2} q^j + A(q)\right)\left(1 + q^{N+1} \sum_{i=0}^{m-2} q^i\right) \\
&= 1 - q^{2N+2} \left(\sum_{j=0}^{m-2} q^j\right)^2 + A(q) + q^{N+1} \sum_{i=0}^{m-2} q^i A(q). \qquad (6.9)
\end{aligned}
$$

Therefore, the smallest non-zero power of q in line (6.9) is $2N+2$, and Theorem 58 is proved. ■

The first proof provides intuition into how the second proof with the q-Binomial Theorem ought to function. We now reprove two results from [6].

Theorem 59 (Castillo et al.). *For $k \geqslant 0$ and $k < \frac{N-1}{3}$, then*

$$p(2k,2) = \begin{cases} p(6k,3,N) - p(6k-1,3,N-1) \\ p(6k+2,3,N) - p(6k+1,3,N-1) \\ p(6k+4,3,N) - p(6k+3,3,N-1) \end{cases}, \tag{6.10}$$

$$p(2k+1,2) = \begin{cases} p(6k+3,3,N) - p(6k+2,3,N-1) \\ p(6k+5,3,N) - p(6k+4,3,N-1) \\ p(6k+7,3,N) - p(6k+6,3,N-1) \end{cases}. \tag{6.11}$$

For $k \geqslant 0$ and $k < \frac{N-1}{6}$,

$$p(2k-3,4,N) - p(2k-4,4,N-1) = p(k-3,3), \tag{6.12}$$

$$p(2k-4,4,N) - p(2k-5,4,N-1) = p(k-2,3). \tag{6.13}$$

Proof. We apply Theorem 58 to Theorem 23 for the cases $m = 3,4$. In the first case, we express n as $6k + j$ for a positive integer j such that $0 \leqslant j \leqslant 5$, then $k < \frac{N-\lfloor j/2 \rfloor+1}{3}$ by the condition of Theorem 58. When $j = 5$ gives the smallest bound, we recover lines (6.10) and (6.11). To find lines (6.12) and (6.13), we express $n = 12k + j$ for a positive integer j such that $0 \leqslant j \leqslant 11$, we see that $0 \leqslant k < \frac{N-\lfloor j/2 \rfloor+1}{6}$. ■

We now state the generalization of Theorem 23 to Gaussian Polynomials.

Theorem 60. *Let s be a prime. If $m = s^x$ where x is a positive integer and for $0 \leqslant j < lcm([m])$, $0 \leqslant k < \frac{2N-j+2}{lcm([m])}$ then*

$$p(lcm([m])k + j, m, N) - p(lcm([m])k + j - 1, m, N-1) = \sum_{i \geqslant 0} g_{r+si} p(lcm(m-1)k + l' - i, m-1) \tag{6.14}$$

where l' and r satisfy $j = l's + r$ with $0 \leqslant r < s$, $lcm([m])$ is the least common multiple of the num-

bers 1 through m, and g_{r+si} *are the coefficients of some polynomial* $G(q)$ *as defined by Lemma 36.*

Proof. We apply Theorem 58 Theorem to 23 proving the result. ■

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BIOGRAPHICAL SKETCH

Acadia was born in Walnut Creek, California to Paul and Deborah. They raised him in Pleasant Hill, California where he played soccer, ran, and biked. Acadia attended Whittier College in Southern California where he earned his Bachelors of Arts in Mathematics earning the Pyle Prize for Outstanding Graduate in Mathematics and the prestigious Mellon Mays Undergraduate Fellowship. From there, Acadia moved to Denver on an Americorps award to mentor, teach, and tutor high school students in Mathematics. After that he took odd jobs while studying as a non-degree student at CU:Denver until he was admitted as a Dean's Fellow at UTRGV where he has earned his Master of Science in Mathematics in December 2018.

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