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TRAVELING WAVE SOLUTION TO TWO-DIMENSIONAL BURGERS-KORTEWEG-DE VRIES EQUATION

A Thesis

by

XIAOQIAN GONG

Submitted to the Graduate School of the
University of Texas-Pan American
In partial fulfillment of the requirements for the degree of

MASTERS OF SCIENCE

August 2013

Major Subject: Mathematics

TRAVELING WAVE SOLUTION TO TWO-DIMENSIONAL

BURGERS-KORTEWEG-DE VRIES EQUATION

A Thesis
by
XIAOQIAN GONG

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August 2013

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ABSTRACT

Gong, Xiaoqian, Traveling Wave Solution to Two-Dimensional Burgers-Korteweg-de Vries Equation. Master of Science (MS), August, 2013, 27pp, 14 references, 20 titles.

In this thesis, we study the Two-Dimensional Burgers-Korteweg-de Vries (2D-BKdV) equation by analyzing the equivalent Abel equation, which indicates that under some particular conditions, the 2D-BKdV equation has a unique bounded traveling wave solution. By using the theorem of contractive mapping, a traveling wave solution to the 2D-BKdV equation is expressed explicitly. In the end, the behavior of the proper solution of the 2D-BKdV equation is established by applying the comparison theorem of differential equations.

DEDICATION

The completion of my masters studies would not have been possible without the love and support of my family and professor. My father, Zhizhong Gong, my mother, Lanzhi Cui, my husband, Yinhai Zhao, and my adviser, Dr. Zhaosheng Feng, who inspired, motivated and supported me by all means to accomplish all my dreams until now. Thank you for your love and support.

ACKNOWLEDGMENTS

I would like to thank my advisor Dr. Zhaosheng Feng for his patient guidance and tremendous help in making this work possible. I would like to thank my thesis committee members: Dr. Andras Balogh, Dr. Zhijun Qiao, and Dr. Cristina Villalobos for their kind service and constructive comments on this thesis.

I will also be grateful to the Department of Mathematics of University of Texas–Pan American for providing me with a generous teaching assistantship, which greatly supported my study and research work.

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CHAPTER I

INTRODUCTION

Searching for explicit solutions of nonlinear equations by using various different methods have been more and more important for many researchers in the last few decades. This has been driven by the fact that an enormous growth in the applicability of nonlinear models and in the development of related nonlinear concepts have been seen.And many new mathematical techniques has been discovered, for example, the theory of dynamical systems and the theory of integrable systems and so on. But not all the systems from the physical phenomena are integrable, for example, the two-dimensional Burgers-Korteweg-de Vries (2D-BKdV) equation. Therefore, a direct method together with qualitative analysis for treating such nonlinear systems appears to be more powerful. Applications of nonlinear models range from atmospheric science to condensed matter physics and to biology, from smallest scales of theoretical particle physics up to the largest scales of cosmic structure.

Consider the 2D-BKdV equation

$$(U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx})_x + \gamma U_{yy} = 0 \tag{1}$$

where α, β, s and γ are real constants and $\alpha\beta s\gamma \neq 0$. Equation (1) is a two-dimensional generalization of the Burgers-Korteweg-de Vries equation

$$U_t + \alpha U U_x + \beta U_{xx} + s U_{xxx} = 0 \tag{2}$$

which arise from many different physical contexts as a nonlinear model equation incorporating the effects of dispersion, dissipation and nonlinearity. Johnson derived (3) as the governing equation for waves propagating in a liquid-filled elastic tube [1] and Wijngaarden and Gao used it as a nonlinear model in the flow of liquids containing gas bubbles [2] and turbulence [3]. Grad and Hu used a steady state version of (3) to describe a weak shock profile in plasmas [4].

During the last few decades, many theoretical issues concerning the exact solutions of 2D-BKdV equation have received considerable attention. Barrera and Brugarino applied Lie group analysis to study the similarity solutions of (1) and examined some features of these invariant solutions, but the explicit traveling wave solution was not shown in [5]. Li and Wang used the Hopf-Cole transformation and a computer algebra system to study (1) and found an exact traveling wave solution to (1) [6]. In the mean time, Ma proposed a bounded traveling wave solution to (1) by applying a special solution of square Hopf-Code type to an ordinary differential equation [7]. These two methods were compared to each other, and the solution are proved to be equivalent by Parkes [8]. In papers $[10-12]$, Feng studied equation (1) by utilizing the first integral method and the Painleve analysis, respectively, and obtained a more general traveling wave solution in terms of elliptic functions and in paper [13], Feng studied equation (1) by analyzing an equivalent two-dimensional autonomous system and a traveling solitary wave solution to the 2D-BKdV equation is expressed explicitly.

In the present thesis, our purpose is to apply the contractive mapping theory and comparison theory in differential equations to the studies of traveling wave solutions and proper solutions of the 2D-BKdV equation. A traveling wave solution is obtained more efficiently by a direct method and the asymptotic behavior of proper solution is presented.

The rest of the thesis is organized as follows. In Chapter 2, we give a short introduction of the contractive mapping theorem, Abel equation and the comparison theorem. In Chapter 3, we apply the contractive mapping theorem to study the solution of the Abel equation. A proper solution is presented. Chapter 4 is the process to find the traveling wave solution of 2D-KdV equation and analyze its asymptotic behavior by comparison theorem. Chapter 5 is the conclusions we made.

CHAPTER II

PRELIMINARIES

Contraction Mappings and the Banach Fixed Point Theorem

We will give a brief introduction of the Contraction Mapping and the Banach Fixed Point Theorem.

The name, fixed point theorem, is usually given to a result which says that, if a mapping f satisfies certain conditions, then there is a point z such that $f(z) = z$. Such a point z is called a fixed point of f.

Definition 1. *(Vector space) By a vector space we mean a nonempty set E with two operations:*
$(x, y) \to x + y$ *from* $E \times E$ *into E called addition,*
$(\lambda, x) \to \lambda x$ *from* $\mathcal{F} \times E$ *called multiplication by scalars,*
such that the following conditions are satisfied for all x, y, $z \in E$ *and* α, $\beta \in \mathcal{F}$*:*

(a) $x + y = y + x$;

(b) $(x + y) + z = x + (y + z)$;

(c) *For every* $x, y \in E$ *there exists a* $z \in E$ *such that* $x + z = y$;

(d) $\alpha(\beta x) = (\alpha\beta)x$;

(e) $(\alpha + \beta)x = \alpha x + \beta x$;

(f) $\alpha(x + y) = \alpha x + \alpha y$;

(g) $1x = x$.

Definition 2. *(Norm) A function* $x \to \|x\|$ *from a vector space E into* $\mathcal{R}$ *is called a norm if it satisfies the following conditions:*

(a) $\|x\| = 0$ *implies* $x = 0$;

(b) $\|\lambda x\| = \|\lambda\|\|x\|$ *for every* $x \in E$ *and* $\lambda \in \mathcal{F}$;

(c) $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x, y \in E$.

Definition 3. *(Normed space) A vector space with a norm is called a normed space.*

Definition 4. *(Convergence in a normed space) Let* $(E, \|\dot{\|})$ *be a normed space. We say that a sequence* (x_n) *of elements of E converges to some* $x \in E$*, if for every* $\varepsilon > 0$ *there exists a number M such that for every* $n \geq M$ *we have* $\|x_n - x\| < \varepsilon$*. In such a case we write* $\lim_{n\to\infty} x_n = x$ *or simply* $x_n \to x$.

Definition 5. *(Cauchy sequence) A sequence of vectors* x_n *in a normed space is called a Cauchy sequence if for every* $\varepsilon > 0$ *there exists a number* M *such that* $\|x_m - x_n\| < \varepsilon$ *for all* m*,* $n > M$.

Definition 6. *A normed space E is called complete if every Cauchy sequence in E converges to an element of E. A complete normed space is called a Banach space.*

Definition 7. *(contraction mapping) A mapping f from a subset A of a normed space E into E is called a contraction mapping (or simply a contraction) if there exists a positive number* $\alpha < 1$ *such that*

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|$$

for all x, $y \in A$.

Theorem 1. *Let F be a closed subset of a Banach space E and let f be a contraction mapping from F to F. Then there exists a unique fix point* $z \in F$ *of* f.

Proof. Let $0 < \alpha < 1$ be such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|$$

for all x, $y \in F$. Let x_0 be an arbitrary point in F and let $x_n = f(x_{n-1})$ for $n =$ 1,2, $\cdots$. We will show that (x_n) is a Cauchy sequence. First observe that, for any $n, m \in \mathcal{N}$,

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_{m+1} - x_m\| \\ &\leq (\alpha^{n-1} + \alpha^{n-2} + \cdots \alpha^m)\|x_1 - x_0\| \\ &\leq \frac{\|x_1 - x_0\|}{1 - \alpha}\alpha^m \to 0 \text{ as } m \to \infty. \end{aligned}$$

Thus, (x_n) is a Cauchy sequence. Since F is a closed subset of a complete space, there exists a $z \in F$ such that $x_n \to z$ as $n \to \infty$. We are going to show that z is the unique point such that

$f(z) = z$. Indeed, since

$$\begin{aligned} \|f(z) - z\| &\leq \|f(z) - x_n\| + \|x_n - z\| \\ &= \|f(z) - f(x_{n-1})\| + \|x_n - z\| \\ &\leq \alpha\|z - x_{n-1}\| + \|x_n - z\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

we have $\|f(z) - z\| = 0$, and thus $f(z) = z$. Suppose now $f(w) = w$ for some $w \in F$. Then

$$\|z - w\| = \|f(z) - f(w)\| \leq \|z - w\|.$$

Since $0 < \alpha < 1$, we must have $\|z - w\| = 0$, which implies $z = w$. □

Abel Equation

Abel Equation of the First Kind

Based on reference [14], we will give a brief introduction of the Abel Equation of the First Kind.

An Abel equation of the first kind is an equation of the form,

$$y' = f_3(x)y^3 + f_2(x)y^2 + f_1(x)y + f_0(x)$$

where $f_3(x) \neq 0$. If $f_3(x) = 0$ and $f_0(x) = 0$, or $f_2(x) = 0$ and $f_0(x) = 0$, the equation reduces to a Bernoulli equation, while if $f_3(x) = 0$ the equation reduces to a Riccati equation.

(a). If f_1 is continuous, f_2 and f_3 are differentiable, $f_3 \neq 0$, then the substitution

$$y = w(x)\eta(\xi) - \frac{f_2}{3f_3}$$

where

$$\xi = \int f_3 w^2 dx,$$

$$w(x) = \exp \int \left(f_1 - \frac{f_2^2}{3f_3} \right) dx$$

brings the Abel equation of the first kind to the canonical form

$$\eta^{'} = \eta^3 + I(x),$$

where

$$f_3 w^3 I = f_0 + \frac{d}{dx}\frac{f_2}{3f_3} - \frac{f_1 f_2}{3f_3} + \frac{2f_2^3}{27f_3^2}.$$

Panayotounakos and Zarmpoutis discovered an analytic method to solve the above equation generally.

(b). The substitution $y = \frac{1}{u}$ brings the Abel equation of the first kind to the Abel equation of the second kind of the form

$$uu^{'} = -f_0(x)u^3 - f_1(x)u^2 - f_2(x)u - f_3(x).$$

Abel Equation of the Second Kind

Based on reference [4], we will give a brief introduction of the Abel Equation of the Second Kind. An Abel equation of the second kind is an equation of the form,

$$yy^{'} = f(x)y^2 + g(x)y + h(x).$$

where f, g, h are continuous functions. (a). The substitution

$$y = E(x)w,$$

where $E(x) = \exp\left(\int f(x)dx\right)$, brings this equation to the simper form,

$$ww^{'} = F_1(x)w + F_0(x), \tag{3}$$

where

$$F_1(x) = g(x)/E(x),$$

$$F_0(x) = h(x)/E^2(x).$$

(b). By introducing the new independent variable

$$z = \int F_1(x)dx,$$

the Abel equation of the second kind can be reduced to the canonical form,

$$ww^{'} - w = \Phi(z). \tag{4}$$

Here the function $\Phi(z)$ is defined parametrically (x is the parameter) by the relations

$$\Phi = \frac{F_0(x)}{F_1(x)},$$

$$z = \int F_1(x)dx.$$

The books by Zaitsev & Polyanin (1994) and Polyanin & Zaitsev (2003) preset a large number of solutions to the Abel equations of the form (3) and (4).

Comparison Theorem

Differential Form of Gronwall's Inequality

Theorem 2. *Let I denote an interval of the real line of the form* $[a, b]$ *with* $a < b$*. let* β *and* u *be real-valued continuous functions defined on I. If* u *is differentiable in the interior* I^0 *of I and satisfies the differential inequality*

$$u^{'}(t) \leq \beta(t)u(t)$$

with $t \in I^0$*, then*

$$u(t) \leq u(a) \exp\left(\int_a^t \beta(s)ds\right)$$

for all $t \in I$.

Proof. Define the function

$$v(t) = \exp\left(\int_a^t \beta(s)ds\right), t \in I.$$

Note that v satisfies

$$v^{'}(t) = \beta(t)v(t), t \in I^0,$$

with $v(a) = 1$ and $v(t) > 0$ for all $t \in I$. By the quotient rule,

$$\frac{d}{dt}\frac{u(t)}{v(t)} = \frac{u^{'}(t)v(t) - v^{'}(t)u(t)}{v^2(t)} \leq \frac{\beta u(t)v(t) - \beta v(t)u(t)}{v^2(t)} = 0, t \in I^0$$

so,

$$\frac{u(t)}{v(t)} \leq \frac{u(a)}{v(a)} = u(a), t \in I,$$

which is Gronwall's inequality. $\square$

First Comparison Theorem

Theorem 3. *Let $f(t,x)$ and $F(t,x)$ be continuous functions in $G \subset \mathcal{R}^2$ and $f(t,x) < F(t,x)$. Let $(t_0, x_0) \in G$ and let $\varphi(t)$ and $\phi(t)$ be the solutions, respectively, of*

$$x^{'} = f(t,x), x(t_0) = x_0$$

and

$$x^{'} = F(t,x), x(t_0) = x_0,$$

then

$$\varphi(t) < \phi(t), \forall t \in (t_0, b).$$

Proof. By contradiction. If $\varphi(t) > \phi(t)$ for some $T > t_0$, then set

$$t_1 = \sup\{t : t_0 \leq t < T \text{ and } \varphi(t) \leq \phi(t)\}.$$

Then $t_0 \leq t_1 < T$, $\varphi(t_1) = \phi(t_1)$, and $\varphi(t) > \phi(t)$ for $t > t_1$ (using the continuity of $\varphi - \phi$). For $t_1 \leq t \leq T$, $|\varphi(t) - \phi(t)| = \varphi(t) - \phi(t)$, and since we can see that$f(t,\varphi) - F(t,\phi)$ is continuous and bounded when $t_1 \leq t < T$ so, there exists $L \in \mathcal{R}$ such that

$$(\varphi - \phi)^{'} = \varphi^{'}(t) - \phi^{'}(t) = f(t,\varphi) - F(t,\phi) \leq L|\varphi(t) - \phi(t)| = L(\varphi(t) - \phi(t)).$$

But by Gronwall's inequality (applied to $\varphi - \phi$ on $[t_1, T]$, with $(\varphi - \phi)(t_1) = 0$, $\beta(t) = L$), $(\varphi - \phi)(t) \leq 0$ on $[t_1, T]$. A contradiction. □

CHAPTER III

PROPER SOLUTION TO ABEL EQUATION

Integral Form of Abel Equations

In this section, we present a simple integral form of Abel equations which plays a fundamental role in the discussion. Let us consider the Abel equation

$$r' = a(t)r^2 + b(t)r^n, \qquad t \in [t_0, t_1], \quad n \geq 3. \tag{5}$$

Dividing equation (5) by r^2 where $r \neq 0$ on both sides,

$$\frac{r'}{r^2} = a(t) + b(t)r^{n-2}. \tag{6}$$

Integrating equation (6) from t_0 to t, where $t \in [t_0, t_1]$, we obtain

$$\int_{t_0}^{t} \frac{r'}{r^2} d\tau = \int_{t_0}^{t} \left(a(\tau) + b(\tau)r^{n-2}\right) d\tau + c_1,$$

where c_1 is an integral constant.

so we have,

$$\int_{r(t_0)}^{r(t)} \frac{1}{r^2} dr = \int_{t_0}^{t} a(\tau)d\tau + \int_{t_0}^{t} b(\tau)r^{n-2}d\tau + c_1$$
$$-\frac{1}{r(t)} - \left(-\frac{1}{r(t_0)}\right) = A(t) + \int_{t_0}^{t} b(\tau)r^{n-2}d\tau + c_1$$

where $A(t) = \int_{t_0}^{t} a(\tau)d\tau$

$$-\frac{1}{r(t)} = A(t) + \int_{t_0}^{t} b(\tau)r^{n-2}d\tau + c_2$$

where $c_2 = c_1 - \frac{1}{r(t_0)}$

$$\begin{aligned} r(t) &= \frac{1}{-A(t) - \int_{t_0}^{t} b(\tau)r^{n-2}d\tau - c_2} \\ r(t) &= \frac{-\frac{1}{c_2}}{\frac{1}{c_2}A(t) + \frac{1}{c_2}\int_{t_0}^{t} b(\tau)r^{n-2}d\tau + 1} \\ r(t) &= \frac{c}{1 - cA(t) - c\int_{t_0}^{t} b(\tau)r^{n-2}d\tau} \end{aligned}$$

where $c = -\frac{1}{c_2}$.

So

$$r(t)\left(1 - cA(t) - c\int_{t_0}^{t} b(\tau)r^{n-2}d\tau\right) = c$$

or equivalently,

$$r(t) = c\left(1 + r(t)A(t) + r(t)\int_{t_0}^{t} b(\tau)r^{n-2}d\tau\right) \tag{7}$$

Proposition 1. *A continuous function $r(\cdot)$ on the closed interval $[t_0, t_1]$ satisfies the integral equation (7) if and only if it is continuously differentiable on the open interval (t_0, t_1) and satisfies the Abel equation (5) with the initial condition $r(t_0) = c$.*

Proof. The conclusion is obvious. $\square$

Solutions to Abel Equation

In this section we first define a nonlinear operator T_c for given continuous functions a and b and a constant c. Then we prove that T_c is contractive and that an iterated sequence $\{T_c^n(f)\}$ with a suitable function f converges to the solution of the Abel equation (5) .

For convenience we take $[t_0, t_1]$ to be $[0, 1]$. Let $\mathcal{C}[0, 1]$ denote the Banach space of all continuous functions on the interval $[0, 1]$ with the norm $\|f\| = \max_{0\le t\le 1} |f(t)|$. With the equation (7)in mind we define a nonlinear operator,

$$T_c : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1],$$

$$T_c(f)(t) \stackrel{def}{=} \frac{c + cf(t)\int_0^t b(\tau)f^{n-2}d\tau}{1 - cA(t)}$$

for given $a, b \in \mathcal{C}[0, 1]$, $c \in \mathcal{R}$, and $A(t) = \int_0^t a(\tau)d\tau$, $c \neq \frac{1}{A(t)}$ for any $t \in [0, 1]$.

Lemma 1. *If* $\|f\| \le 1$ *and* $\left\|\frac{c}{1-cA(t)}\right\|(1 + \|b\|) \le 1$, *then* $\|T_c f\| \le 1$.

Proof.

$$\begin{aligned}
\left\|T_c(f)(t)\right\| &= \left\|\frac{c + cf(t)\int_0^t b(\tau)f^{n-2}d\tau}{1 - cA(t)}\right\| \\
&= \left\|\frac{c}{1 - cA(t)}\right\|\left\|1 + f(t)\int_0^t b(\tau)f^{n-2}d\tau\right\| \\
&\le \left\|\frac{c}{1 - cA(t)}\right\|(1 + \|b\|) \le 1.
\end{aligned}$$

Lemma 2. *If* $\left\|\frac{c}{1-cA(t)}\right\|(1 + \|b\|) \le 1$ *and* $\left\|\frac{c}{1-cA(t)}\right\|\|b\|(n-1) \le 1$, *then* T_c *is a contractive mapping on the closed unit ball* $\mathcal{B}_1 = \{f \in \mathcal{C}[0, 1] | \, \|f\| \le 1\} of \mathcal{C}[0, 1]$.

Proof. From lemma (1), we have for any $f, g \in \mathcal{B}_1$, $T_c(f), T_c(g) \in \mathcal{B}_1$.

Moreover,

$$\begin{aligned}
\left\| T_c(f)(t) - T_c(g)(t) \right\| &= \left\| \frac{c}{1-cA(t)} \left(f(t) \int_0^t b(\tau) f^{n-2} d\tau - g(t) \int_0^t b(\tau) g^{n-2} d\tau \right) \right\| \\
&= \left\| \frac{c}{1-cA(t)} \right\| \left\| (f-g) \int_0^t b(\tau) f^{n-2} d\tau + g \int_0^t b(\tau)(f^{n-2} - g^{n-2} d\tau) \right\| \\
&\leq \left\| \frac{c}{1-cA(t)} \right\| \left(\|f-g\| \|b\| + \|b\| \left\| f^{n-2} - g^{n-2} \right\| \right).
\end{aligned}$$

Note that,

$$\begin{aligned}
\left\| f^{n-2} - g^{n-2} \right\| &= \left\| (f-g)(f^{n-3} + f^{n-4} g + \cdots f g^{n-4} + g^{n-3}) \right\| \\
&\leq \|f-g\| (n-2),
\end{aligned}$$

so,

$$\begin{aligned}
\|T_c(f) - T_c(g)\| &\leq \left\| \frac{c}{1-cA(t)} \right\| \|f-g\| \|b\| (1+n-2) \\
&= \left\| \frac{c}{1-cA(t)} \right\| \|b\| (n-1) \|f-g\|.
\end{aligned}$$

Therefore, T_c is contractive on $\mathcal{B}_1$.

□

According to the well known Banach contraction principle, an iterated sequence $T_c^n(f)$ with $f \in \mathcal{B}_1$ converges uniformly for $t \in [0,1]$ to the unique fixed point of T_c in $\mathcal{B}_1$. Proposition (1) shows that the fixed point is no other than the solution $r(t)$ of the equation (5). □

Theorem 4. *For given $a, b \in \mathcal{C}[0,1]$, $c \in \mathcal{R}$, $c \neq \frac{1}{A(t)}$ for any $t \in [0,1]$ with $\left\| \frac{c}{1-cA(t)} \right\| (1+\|b\|) \leq 1$*

and $\left\| \frac{c}{1-cA(t)} \right\| \|b\| (n-1) \leq 1$ *, the solution* $r(t)$ *of the equation* (1) *with* $r(0) = c$ *can be uniformly approximated by an iterated sequence* $\{T_c^n(f)(t)\}$*,i.e.,*

$$r(t) = \lim_{n\to\infty} T_c^n(f)(t), \quad 0 \leq t \leq 1 \tag{8}$$

for arbitrary $f \in \mathcal{C}[0,1]$ *with* $\|f\| \leq 1$.

Proof. As we mentioned above, the conclusion follows from the Banach contraction principle. □

If we set $n = 3$, and for $r_i(t) \in \mathcal{B}_1 (i = 1, 2, \cdots)$, let $h_i(t) = \int_0^t b(\tau) r_i(\tau) d\tau$ with $b \in \mathcal{C}[0,1]$ and $\|b\| < 1$, then it gives

$$\begin{aligned}
r_2(t) &= \frac{c}{1-cA(t)}(1 + r_1(t)h_1(t)) \\
r_3(t) &= \frac{c}{1-cA(t)}(1 + r_2(t)h_2(t)) = \frac{c}{1-cA(t)} + \left(\frac{c}{1-cA(t)}\right)^2 (1 + r_1(t)h_1(t))h_2(t) \\
r_4(t) &= \frac{c}{1-cA(t)}(1 + r_3(t)h_3(t)) = \frac{c}{1-cA(t)} + \left(\frac{c}{1-cA(t)}\right)^2 h_3(t) \\
&+ \left(\frac{c}{1-cA(t)}\right)^3 (1 + r_1(t)h_1(t))h_2(t)h_3(t).
\end{aligned}$$

$\vdots$

so,

$$\begin{aligned}
r_n &= \frac{c}{1-cA(t)} + \left(\frac{c}{1-cA(t)}\right)^2 h_{n-1} + \left(\frac{c}{1-cA(t)}\right)^3 h_{n-2}h_{n-1} \\
&+ \cdots + \left(\frac{c}{1-cA(t)}\right)^{n-1} (1 + r_1 h_1) h_2 \cdots h_{n-1}.
\end{aligned}$$

Note, since $\left\| \frac{c}{1-cA(t)} \right\| (1 + \|b\|) \leq 1$, $\left\| \frac{c}{1-cA(t)} \right\| \leq 1$ and since $\|h_i(t)\| = \left\| \int_0^t b(\tau) r_i(\tau) d\tau \right\| \leq \|b\| < 1$, $h_j h_{j+1} h_{j+2} \cdots h_{j+k}$ is bounded for any $j \geq 1$, $k \geq 0$. Note that $\frac{c}{1-cA(t)} \to 0$ as

$c \to 0$,thus, if c is sufficiently small and $n \geq 3$, we have

$$r_n \approx \frac{c}{1-cA(t)} + \left(\frac{c}{1-cA(t)}\right)^2 h_{n-1}.$$

Take limit $n \to +\infty$ of both sides of the above equation, then

$$r(t) \approx \frac{c}{1-cA(t)} + \left(\frac{c}{1-cA(t)}\right)^2 \int_0^t b(\tau)r(\tau)d\tau$$

where $t \in [0,1]$.

Furthermore, $r(t)$ is between

$$\frac{c}{1-cA(t)}+\left(\frac{c}{1-cA(t)}\right)^2 = \frac{c(1-cA(t))+c^2}{(1-cA(t))^2} \text{ and } \frac{c}{1-cA(t)}-\left(\frac{c}{1-cA(t)}\right)^2 = \frac{c(1-cA(t))-c^2}{(1-cA(t))^2},$$

that is,

$$\frac{c(1-cA(t))-c^2}{(1-cA(t))^2} < r(t) < \frac{c(1-cA(t))+c^2}{(1-cA(t))^2}.$$

CHAPTER IV

TRAVELING WAVE SOLUTION TO 2D-KdV EQUATION

From PDE to ODE

In this section, we will transform the 2D-BKdV Equation to a second order nonlinear ODE. Consider the 2D-BKdV equation

$$(U_t + \alpha UU_x + \beta U_{xx} + sU_{xxx})_x + \gamma U_{yy} = 0$$

where $\alpha, \beta, s,$ and γ are constants and $\alpha\beta s\gamma \neq 0$.
Assume that equation (1) has an exact solution in the form

$$U(x, y, t) \equiv U(\xi) \quad \xi = hx + ly - wt \tag{9}$$

where h, l, w are real constants to be determined. Substitution of (9) into equation (1) yields

$$-whU_{\xi\xi} + \alpha h^2 (UU_\xi)_\xi + \beta h^3 U_{\xi\xi\xi} + sh^4 U_{\xi\xi\xi\xi} + \gamma l^2 U_{\xi\xi} = 0$$

Integrating the above equation twice with respect to ξ, we have

$$sh^4 U_{\xi\xi} + \beta h^3 U_\xi + \frac{\alpha}{2} h^2 U^2 + \gamma l^2 U - whU = C$$

where we set the first integration constant to zero and set the second one as$C$. Rewrite this second-order ordinary differential equation as

$$U''(\xi) + \lambda U'(\xi) + aU^2(\xi) + bU(\xi) + d = 0 \tag{10}$$

where $\lambda = \frac{\beta}{sh}$, $a = \frac{\alpha}{2sh^2}$, $b = \frac{\gamma l^2 - wh}{sh^4}$ and $d = -\frac{C}{sh^4}$.

From Second Order ODE to Abel Equation

In this section, we will transform the second order nonlinear ODE to an Abel Equation.

From equation (10), let $v = U(\xi)$ and $y = U'(\xi)$, then,

$$U''(\xi) = \frac{dU'(\xi)}{d\xi} = \frac{dy}{d\xi} = \frac{dy}{dv}\frac{dv}{d\xi} = \frac{dy}{dv}y.$$

So we can rewrite the equation (10) into

$$\frac{dy}{dv}y + \lambda y + av^2 + bv + d = 0. \tag{11}$$

Solving for $\frac{dy}{dv}$ we get,

$$\frac{dy}{dv} = -\lambda - (av^2 + bv + d)y^{-1}. \tag{12}$$

Let $z = \frac{1}{y}$, then $y = \frac{1}{z}$, $\quad \frac{dy}{dv} = \frac{dy}{dz}\frac{dz}{dv} = -\frac{1}{z^2}\frac{dz}{dv}$ so,

$$-\frac{1}{z^2}\frac{dz}{dv} = -\lambda - (av^2 + bv + d)z.$$

By multiplying $-z^2$ both sides, we have

$$\frac{dz}{dv} = \lambda z^2 + (av^2 + bv + d)z^3. \tag{13}$$

Let $f(v) = \lambda$, $g(v) = av^2 + bv + d$, then

$$z^{'} = f(v)z^2 + g(v)z^3$$

where $v = U(\xi) \in [v_0, v_1]$. Also $U(\xi_0) = v_0$, $U^{'}(\xi_0) = \frac{1}{c}$, $z(v_0) = \frac{1}{U^{'}(\xi_0)} = c$ and c is a real constant.

Let $\eta = \frac{v-v_0}{v_1-v_0}$, then $\eta \in [0, 1]$ and $v = v_0 + (v_1 - v_0)\eta$, so let

$$r(\eta) = z(v),$$

$$h(\eta) = (v_1 - v_0)f(v) = (v_1 - v_0)\lambda,$$

$$k(\eta) = (v_1 - v_0)g(v),$$

then we have

$$r^{'} = h(\eta)r^2 + k(\eta)r^3 \text{ with initial condition } r(0) = c \tag{14}$$

where $h(\eta), k(\eta) \in C[0, 1]$.

By Theorem 1, if $c \in \mathcal{R}$,$c \neq \frac{1}{H(\eta)}$ for any $\eta \in [0, 1]$ with $\left\| \frac{c}{1-cH(\eta)} \right\| (1 + \|k\|) \leq 1$ and $\left\| \frac{c}{1-cH(\eta)} \right\| \|k\| (n - 1) \leq 1$, the solution to the equation (14) is

$$r(\eta) = \lim_{n \to +\infty} T_C^n(w)(\eta) \tag{15}$$

where $0 \leq \eta \leq 1$ for any $w \in C[0, 1]$ with $\|w\| \leq 1$ and

$$T_c(w)(\eta) = \frac{c + cw(\eta) \int_0^{\eta} k(x)wdx}{1 - cH(\eta)}$$

with

$$\begin{aligned} H(\eta) &= \int_0^\eta h(x)\,dx = \int_0^\eta (v_1 - v_0)\lambda\,dx = (v_1 - v_0)\lambda\eta, \\ k(x) &= (v_1 - v_0)\left(a(v_0 + (v_1 - v_0)x)^2 + b(v_0 + (v_1 - v_0)x) + d\right). \end{aligned}$$

Let $v_2 = v_1 - v_0$, then

$$\begin{aligned} k(x) &= v_2(a(v_0 + v_2 x)^2 + b(v_0 + v_2 x) + d) \\ &= v_2(a v_2^2 x^2 + (2a v_2 v_0 + b v_2)x + a v_0^2 + b v_0 + d). \end{aligned}$$

Let $\bar{\alpha} = a v_2^3$, $\bar{\beta} = v_2(2a v_2 v_0 + b v_2)$, $\bar{\mu} = v_2(a v_0^2 + b v_0 + d)$, then

$$k(x) = \bar{\alpha}x^2 + \bar{\beta}x + \bar{\mu}.$$

Application of the Contraction Mapping

In this section, we will use the contraction Mapping theorem to find the boundary of the solution to the Abel Equation.

Recall that $r = z = \frac{1}{y}$, $y = U^{'}(\xi)$, $\eta = \frac{v - v_0}{v_1 - v_0}$, $v = U(\xi)$,

so when $c \in \mathcal{R}$,$c \neq \frac{1}{H(\eta)}$ for any $\eta \in [0, 1]$ with $2\left\|\frac{c}{1 - cH(\eta)}\right\| \leq 1$ and $\|k\| < 1$, $U^{'}(\xi) = \frac{1}{r(\eta)}$ is between $F(\xi)$ and $G(\xi)$,

where $F(\xi) = \frac{(1 - c\lambda(U(\xi) - v_0))^2}{c(1 - c\lambda(U(\xi) - v_0)) + c^2}$ and $G(\xi) = \frac{(1 - c\lambda(U(\xi) - v_0))^2}{c(1 - c\lambda(U(\xi) - v_0)) - c^2}$.

That's, if we let $U^{'}(\xi) = \hat{H}(\xi)$, where $\hat{H}(\xi)$ is continuous on $[\xi_0, +\infty)$, then $F(\xi) < \hat{H}(\xi) < G(\xi)$.

Integral of the Boundary Functions

In this section, we will look into the solution to an Ordinary Differential Equation.

Note,

$$
\begin{aligned}
& \frac{dU}{d\xi} = U^{'}(\xi) = \frac{(1-c\lambda(U-v_0))^2}{c(1-c\lambda(U-v_0)) \pm c^2} \\
\Rightarrow \quad & \frac{d(U-v_0)}{d\xi} = \frac{(1-c\lambda(U-v_0))^2}{c(1-c\lambda(U-v_0)) \pm c^2}.
\end{aligned}
$$

Let $v = U - v_0$, then

$$
\begin{aligned}
& \frac{dv}{d\xi} = \frac{(1-c\lambda v)^2}{c(1-c\lambda v) \pm c^2} \\
\Rightarrow \quad & \frac{-1}{c\lambda}\frac{d(1-c\lambda v)}{d\xi} = \frac{(1-c\lambda v)^2}{c(1-c\lambda v) \pm c^2}.
\end{aligned}
$$

Let $\bar{v} = 1 - c\lambda v$, then

$$
\begin{aligned}
& \frac{-1}{c\lambda}\frac{d\bar{v}}{d\xi} = \frac{\bar{v}^2}{c\bar{v} \pm c^2} \\
\Rightarrow \quad & \frac{d\bar{v}}{d\xi} = \frac{-\lambda\bar{v}^2}{\bar{v} \pm c} \\
\Rightarrow \quad & \frac{\bar{v} \pm c}{\lambda\bar{v}^2} d\bar{v} = -d\xi \\
\Rightarrow \quad & \int \left(\frac{1}{\lambda\bar{v}} \pm \frac{c}{\lambda\bar{v}^2} \right) d\bar{v} = \int -1 d\xi \\
\Rightarrow \quad & \frac{1}{\lambda}\ln|\bar{v}| \mp \frac{c}{\lambda}\frac{1}{\bar{v}} = -\xi + c^* \\
\Rightarrow \quad & \ln|\bar{v}| \mp c\frac{1}{\bar{v}} = -\lambda\xi + \bar{c}
\end{aligned}
$$

where $c^*$is a constant and $\bar{c} = \lambda c^*$. Plug $\bar{v} = 1 - c\lambda(U - v_0)$ into the above result, we have

$$
|1 - c\lambda(U - v_0)| = \exp\left\{ \pm\frac{c}{1 - c\lambda(U - v_0)} - \lambda\xi + \bar{c} \right\}.
$$

Implicit Function Theorem

Note that $U \in [v_0, v_1]$, if c is positive and sufficiently small, then $1 - c\lambda(U - v_0) > 0$, so $|1 - c\lambda(U - v_0)| = 1 - c\lambda(U - v_0)$.

Let $\hat{F}(\xi, U) = \exp\left\{\frac{c}{1-c\lambda(U-v_0)} - \lambda\xi + \bar{c}\right\} - 1 + c\lambda(U - v_0)$,

$D = \{(\xi, U)|\xi \in (-\infty, +\infty), U \in (v_0 - 1, v_1 + 1)\}$, $\xi_0 = 0, U_0 = v_0$,

and $\bar{c} = -c \neq -\frac{1}{\lambda(U-v_0)}$ for any $U \in (v_0 - 1, v_1 + 1)$. Then we have

(1). $(\xi_0, U_0) \in D$ and $\hat{F}(\xi, U)$ is continuous in D;

(2). $\hat{F}(\xi_0, U_0) = 0$;

(3). Since,

$$\hat{F}_U = \exp\left\{\frac{c}{1 - c\lambda(U - v_0)} - \lambda\xi - c\right\} \frac{c^2\lambda}{(1 - c\lambda(U - v_0))^2} + c\lambda$$

So, $\hat{F}_U$ is continuous in D.

(4). $\hat{F}_U(\xi_0, U_0) = c^2\lambda + c\lambda = c\lambda(1 + c) \neq 0$. According to the Implicit Function Theorem, there exists an unique continuous function $U_1(\xi)$,$\xi \in [0, +\infty)$, such that $U_1(0) = v_0$ and $\hat{F}(\xi, U_1(\xi)) = 0$.

Similarly,we can prove that under the parametric condition, there exists an unique continuous function $U_2(\xi)$,$\xi \in [0, +\infty)$, such that $U_2(0) = v_0$ and $\hat{G}(\xi, U_2(\xi)) = 0$ with

$$\hat{G}(\xi, U) = \exp\left\{-\frac{c}{1 - c\lambda(U - v_0)} - \lambda\xi + \bar{c}\right\} - 1 + c\lambda(U - v_0).$$

First Comparison Theorem

In this section, we will apply the First Comparison Theorem to find the boundary of the solution to the 2D-BKdV equation.

Theorem 5. *Let $f(t, x)$, $F(t, x)$ be two continuous functions in $G \in \mathcal{R}^2$ with $f(t, x) < F(t, x)$.*

Let $(t_0, x_0) \in G$, and for $t \in (a, b)$, $\psi(t)$ and $\phi(t)$ are respectively the solutions of the initial value problems,

$$\dot{x} = f(t, x), x(t_0) = x_0$$

and

$$\dot{x} = F(t, x), x(t_0) = x_0,$$

then

$$\psi(t) < \phi(t), \forall t \in (t_0, b),$$

$$\psi(t) > \phi(t), \forall t \in (a, t_0).$$

Note that if $v_1 < v_0 + \frac{1 \pm c}{c\lambda}$, $F(\xi)$ and $G(\xi)$ is continuous where $\xi \in [\xi_0, +\infty)$. According to the Comparison Theorem, we have,$U_1(\xi) \leq U(\xi) \leq U_2(\xi)$, where $\xi \in [\xi_0, +\infty)$ and
$U_1^{'}(\xi) = F(\xi), U_1(\xi_0) = v_0$;$U_2^{'}(\xi) = G(\xi), U_2(\xi_0) = v_0$.
Plug $U_1(\xi_0) = v_0$ into equation

$$|1 - c\lambda(U_1 - v_0)| = \exp\left\{\frac{c}{1 - c\lambda(U_1 - v_0)} - \lambda\xi + \bar{c}\right\},$$

we have $\bar{c} = \lambda\xi_0 - c$, so,

$$|1 - c\lambda(U_1 - v_0)| = \exp\left\{\frac{c}{1 - c\lambda(U_1 - v_0)} - \lambda\xi + \lambda\xi_0 - c\right\}.$$

Similarly,

Plug $U_2(\xi_0) = v_0$ into equation

$$|1 - c\lambda(U_2 - v_0)| = \exp\left\{-\frac{c}{1 - c\lambda(U_2 - v_0)} - \lambda\xi + \bar{c}\right\},$$

we have $\bar{c} = \lambda\xi_0 + c$, so,

$$|1 - c\lambda(U_2 - v_0)| = \exp\left\{-\frac{c}{1 - c\lambda(U_2 - v_0)} - \lambda\xi + \lambda\xi_0 + c\right\}.$$

CHAPTER V

CONCLUSION

In this thesis, we studied the Two-Dimensional Burgers-Korteweg-de Vries (2D-BKdV) equation by analyzing the equivalent Abel equation, which indicates that under some particular conditions, the 2D-BKdV equation has a unique bounded traveling wave solution. By using the theorem of contractive mapping, a traveling wave solution to the 2D-BKdV equation is expressed explicitly. In the end, the behavior of the proper solution of the 2D-BKdV equation is established by applying the comparison theorem of differential equations.

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BIOGRAPHICAL SKETCH

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