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Lax Pairs for Some Nonlinear Equations

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LAX PAIRS FOR NONLINEAR EQUATIONS

A Thesis

by

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ABSTRACT

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Several methods have been proposed to approach the topic of integrable systems of nonlinear partial differential equations. One of these methods is called the Lax pair. The Lax pair is a pair of matrices or operators, that depend on time and satisfy the Lax equation. Based on the inverse scattering method introduced by Gardner, Greene, Kruskal and Miura (1967), Peter Lax introduced the Lax pair to derive soliton equations from the Lax equation. This thesis provides with a brief background on soliton theory, inverse scattering theory, and Lax pairs. The details missing in the work published by Ablowitz, Kaup, Newell, and Segur (AKNS) and Ablowitz-Ladik to derive nonlinear evolution equations for the 2×2 Zakharov-Shabat continuous and discrete cases are exposited here. The ideas of AKNS and Ablowitz-Ladik are extended to derive nonlinear evolution equations for both 3×3 continuous and discrete cases. Mathematica is used to obtain the nonlinear evolution equations.

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CHAPTER I

Introduction

Systems of integrable nonlinear partial differential equations has been a major research area in the past couple of years. The fact that there have been several approaches taken to solve such systems has made this field gained such popularity. Of many schemes considered to solve such systems one idea remains in common to some of these methods. This concept can be called the Lax pair, and it has appeared to be of great help when solving nonlinear partial differential equations. This combination of nonlinear partial differential equations and Lax pairs is just a fabulous formulation. It is based on reintegrating the main system of nonlinear differential equations as the compatibility conditions for a system of linear equations [11]. The history of Lax pairs began in 1968. It was then that Peter Lax introduced the famous Lax pairs to find a solution to the Korteweg-de Vries (KdV) equation, which modeled solitary waves (solitons) and other waves in plasma physics[6]. What Lax used were two matrices or operators L and M, which would satisfy the Lax equation $L_t + [L,M] = 0$, where $[L,M] := LM - ML$ [9]. Mathematicians in this field were eager to apply Lax pairs to their work. In 1972, Zakharov and Shabat [14] used Lax's ideas to relate to the nonlinear Schödinger equation. Then, in 1973, Ablowitz, Kaup, Newell and Segur [8] (subsequently referred to by AKNS) solved the Sine-Gordon equation using the inverse scattering transform (IST) involving Lax pairs. They considered a 2×2 scattering problem, meaning the problem consisted of two linear equations, where the matrices defined in these equations would satisfy Lax equation. A full account of the history of Lax pairs is given in [6]. The reader should also consult the foundational papers [1, 2, 3, 4, 5, 7, 8, 10, 12, 13, 14].

Just as IST and Lax pairs were used to solve systems of these equations, the IST and Lax pairs were also be applied to discrete types of nonlinear evolution equations. The Toda lattice was one example of this scheme. Toda, Flaschka and Malatov solved this equation and discovered a number of solutions for this lattice [9]. Around 1975, Ablowtiz and Ladik proposed a new discrete scattering problem [3]. This was a discrete version of the 2×2 Zakharov-Shabat problem, AKNS had considered before and proposed to solve the Toda lattice and other discrete nonlinear partial difference differential equations [6]. Ablowitz and Ladik considered the discrete Schödinger scattering problem and its relationship with the Toda lattice [9]. They followed the work of Flaschka and then discussed the evolution equations associated with the 2×2 Zakharov-Shabat scattering problem [14].

To arrive to the evolution equations for both discrete and continuous cases, AKNS and Ablowitz and Ladik carried out extensive calculations, but omitted many details in their final published versions of their work. In [9], the work by Ablowitz and Ladik showed how to arrive to the evolution equations for the 2×2 continuous case [3]. However, as before, many elementary calculations were omitted from their book. Likewise, in [6], a concise overview is given of the calculations necessary to arrive to the evolution equations of the 2×2 discrete case. The material covered in these books did not extend to the application of a 3×3 either for a discrete and continuous case.

The purpose of this thesis is to show the details of the calculations that were omitted to arrive to the evolution equations. Then, given the fact that corresponding methods were not applied to a 3×3 discrete and continuous case, this thesis will provide complete details for the derivation of the evolution equations in the 3×3 case for both the discrete and continuous case. Mathematica will be used to arrive to the results.

This thesis will be broken up into several chapters to discuss the material in an appropriate manner. Chapter 1 will introduce the problem and summarize the motivation that led to consideration of this problem for discussion. Chapter 2 will give an ample background on soliton theory, the inverse scattering method, and Lax pairs. Chapter 3 will provide the missing details of the 2×2 continuous case that Ablowitz, Kaup, Newell, and Segur (AKNS) applied to obtain the evolution equations. Chapter 4 will add the omitted information on the discrete case that Ablowitz and Ladik proposed in 1975. In Chapter 5, Mathematica code will be used to derive the appropriate evolution equations for the 3×3 continuous case. Results will be summarized. Lastly, Chapter 6, a similar approach is taken using Mathematica to acquire the desired evolution equations in the 3×3 non-dsicrete case and conclusions will be analyzed thoroughly. Chapter 7 will summarize the material covered in this thesis. The Appendix will provide the Mathematica code used throughout this thesis.

CHAPTER II

Solitons, Inverse Scattering Transform, Lax Pairs

When solving nonlinear equations, a variety of themes need to be considered to arrive to the final result. In this chapter, a brief background is given on the solutions to some of these nonlinear equations, which in this case are referred to solitons. Mathematicians derived equations that modeled solution behavior. One method that applied in a large number of cases is the Inverse Scattering Transform (IST) combined with Lax Pairs, which will be introduced in the subsequent sections of this chapter.

2.1 The Discovery of the Soliton

Solitons are waves that are able to keep the same form and velocity after they collide with other waves. It was Scottish Engineer named John Scott Russell, who first became acquainted with Solitons. In 1834, he was doing experiments on the Union Canal (near Edinburgh). He was trying to measure the relationship between the speed of a boat and its propelling force aiming to look for parameters that would help him convert from horsepower to steam. One day, the cord that was attached to his measurements apparatus parted in such a way that the boat completely stopped, not the mass of water that had made the boat moved. The water moved violently and forward with great velocity, having the form of a solitary elevation and following a course without altering its speed. Russell did not ignore this unexpected phenomenon and continued investigating. He noticed that the water conserved its original figure, moved at about eight to nine miles per hour and had a height of about one and a half foot and a length of about thirty feet. He moved on analyzing this phenomenon in tanks and other canals. He discovered that this was a dynamic identity with constant shape and speed. In water tanks, he was able to prove four facts. These were that solitary waves have the shape of $\text{sech}^2 k[(x\text{-}vt)]$ a large mass of water produces two or more independent waves, solitary waves cross each other without changing and a wave of height and traveling in a channel of depth has a velocity given by v=$\sqrt{g(d+h)}$, where g is the acceleration of gravity [6].

After Russell's death, other scientists carried on research in this area. Airy, Stokes, Boussinesq and Rayleigh were able to continue investigating this phenomenon in order to understand it better. Boussinesq and Rayleigh were able to obtain approximations of this wave phenomena, and then Rayleigh was able to model it with a nonlinear evolution equation. Since there was too much controversy as whether these water waves exhibit solitary wave solutions, Korteweg and de Vries were able to finally resolved this issue. They derived a nonlinear evolution equation that could be simplified to non-dimensional form by making a transformation. This equation was named the KdV equation, and has the form

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The application of this equation was only set to be used to this phenomena. However, other researchers introduced new ideas and obtained new results as they performed a variety of experiments. In 1939, Frenkel and Kontorova presented a problem in the field of solid-state physics. They also obtained a wave solution to the model they presented. Then, in 1940, Enrico Fermi, John Pasta, and Stan Ulam introduced a different topic. They were studying how energy flows in a one-dimensional lattice consisting of equal masses connected by nonlinear springs. Gardner and Morikawa discovered that this equation had other applications. For example, they found that the KdV equation would appeared in the study of collision-free hydro-magnetic waves, and later, on other contexts such as in stratified internal waves, ion-acoustic waves, plasma physics and lattice dynamics. Thus, it is known that the KdV equation possesses the solitary wave solution,

$$u(x,t) = 2\kappa^2 \text{sech}^2 \kappa(x - 4\kappa^2 t - x_0), \tag{2}$$

where κ and x_0 are constants [6]. The results they obtained in this experiment did not meet their expectations, so in 1960, Norman Zabusky and Martin Kruskal took over this project. Their attempt was to go back and study (1), finding that the solutions to this equation were pulse-like waves. These waves would collide with each other and keep the same shape and velocity. This outcome made Zabusky and Kruskal to name such waves solitons [10]. All the details in the prior discussion have been taken from [6, 10].

2.2 The Inverse Scattering Method

There are many ways to solve linear partial differential equations, and one of these ways is by the Fourier Transform. The way this method works is by having the initial data transformed to a Fourier space. Once transformed to a Fourier space, the time evolution in the Fourier space is simple since it satisfies the transformed linear partial differential equation. Finally, the inverse Fourier Transform is applied to recover the solution of the equation. If recalled, the Fourier transform applies only to linearized cases. The KdV equation might be a case that might not work. However, if linearized, the KdV in the form of $u_t + u_{xxx} = 0$ has a solution. Hence, when the Fourier transform fails, the Inverse Scattering method is the alternative to follow. The inverse scattering method was discovered by Gardner, Greene, Kruskal and Miura as a way to solve initial value problems for the KdV equation [10]. Many other one-dimensional equations were solved using this method, which later was referred as the Inverse Scattering Transform (IST). Some famous problems approached by this method are the nonlinear Schrödinger, Sine-Gordon, three-wave interaction, Modified KdV and Boussinesq equations. Integro-differential equations such as the ILW and BO also used the IST. scheme. This was derived by Kodama, Ablowitz, Satsuma and Fokas in the 1980's. The inverse scattering method works similarly to the Fourier Transform Method, except for the last step, so to obtain the solution of the KdV equation, a similar process takes place. The main idea of this method is to relate the KdV equation to the time-independent Schrödinger scattering problem given by

$$\begin{aligned} Lv &:= v_{xx} + u(x,t)v \\ &= \lambda v. \end{aligned} \tag{3}$$

The main idea of this came from Miura transformation relating solutions of the KdV and mKdV equations. For instance, first the initial data is given at a time, t, for which $t = 0$. This step can be called the direct problem step, where the data is stated. At this step, the direct scattering problem is solved and the scattering data $S(\lambda,t)$ is found. Then, the time evolution of the scattering data is found. Finally, the inverse scattering problem is reconstructed, given the scattering data, $S(\lambda,t)$ and the potential $u(x,t)$. In summary, the eigenvalues and the behavior of the eigenfuctions as x determines the scattering data, $S(\lambda,t)$, which mainly depends on the potential $u(x,t)$. The direct scattering problem is to map the potential into the scattering data, while the inverse scattering problems is to reconstruct the potential from the scattering data. All details from the prior discussion can be found in [6].

2.3 Lax's Generalization of the Inverse Scattering Method

In 1968, Lax used the inverse scattering method for solving the KdV equation [12]. Lax considered two operators L and M, where L is the operator of the spectral problem and M is the operator governing the associated time evolution of the eigenfunctions. In this case

$$Lv = \lambda v, \tag{4}$$

$$v_t = Mv. \tag{5}$$

If the partial derivative of (4) is taken with respect to time, it gives,

$$L_t v + L v_t = \lambda_t v + \lambda v_t. \tag{6}$$

Then, if (5) is replaced in (4), it is found that,

$$L_t v + LMv = \lambda_t v + MLv. \tag{7}$$

After some simplification, this implies that,

$$[L_t + (LM - ML)]v = \lambda_t v. \tag{8}$$

In order to solve for nontrivial eigenfunctions $v(x,t)$, Lax Equation,

$$L_t + [L,M] = 0, \tag{9}$$

where $[L,M] := LM - ML$, if and only if $\lambda_t = 0$. Lax's equation contains a nonlinear evolution equation, for which L and M are chosen suitably. This means that if L and M are assigned a value,

$$L := \frac{\partial^2}{\partial x^2} + u, \tag{10}$$

$$M := (\gamma + u_x) - (4\lambda + 2u)\frac{\partial}{\partial x}, \tag{11}$$

these should satisfy Lax's Equation given that u satisfies the KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0. \tag{12}$$

All details added in this section have been taken from [6].

CHAPTER III

Aplications of the Lax Pairs

The research done by Korteweg and de-Vries and Lax insightful inspired AKNS to consider problems for further study. In [8], AKNS considered a 2×2 scattering problem following the ideas of Lax pairs. However, AKNS omitted significant but elementary details of how to obtain important results that led to solve the 2×2 problem. The calculations not mentioned in [9] will be carried out in this paper included as part of the work done by AKNS.

For a while it was thought that the method of inverse scattering was an oddity and not generally useful in other contexts. However, in 1972, Zakharov and Shabat proved that the method was worthwhile [9, 14]. They confirmed this by extending Lax's ideas by relating it to the nonlinear Schrödinger equation,

$$iu_t + u_{xx} + ku^2u^* = 0. \tag{13}$$

Here, the $*$ signifies the complex conjugate and k is a constant to linear scattering problem. They showed that given if the operators defined earlier by Lax, L and M and k were given a value would satisfy Lax's equation, then $u(x,t)$ would be a solution to the Schröndiger Equation. Using assigned values for the operators, Zakharov and Shabat were able to solve the nonlinear Schrödinger equation. Soon after, Wadati solved the Modified KdV equation,

$$u_t - 6u^2u_x + u_{xxx} = 0. \tag{14}$$

As well, Ablowitz, Kaup, Newell and Segur made contributions by solving the Sine-Gordon Equation,

$$u_{xt} = \sin u, \tag{15}$$

and then, came up with the scheme to solve nonlinear evolution equations [1]. This scheme, as mentioned before, it is called the Inverse Scattering Transform (IST). This was to the similarity between the Fourier transform and the inverse scattering method of solving initial value problems. The process that Ablowitz, Kaup, Newell and Segur took was similar to the process that has been talked about in this paper before. They considered two linear equations

$$v_x = Xv, \tag{16}$$

$$v_t = Tv, \tag{17}$$

where v is an n-dimensional vector and X and T are $n \times n$ matrices and X and T are $n \times n$ matrices. To show that these two equations are compatible, it is required that v_{xt}=v_{tx}, and so, X and T should satisfy

$$X_t - T_x + [X,T] = 0. \tag{18}$$

They noticed that this equation and Lax's Equation were very similar. Hence, they decided to define values for v_x, v_t, X and T. Named by them, the 2×2 scattering problem, they defined, v_x as,

$$v_{1,x} = -ikv_1 + q(x,t)v_2, \tag{19}$$

$$v_{2,x} = ikv_2 + r(x,t)v_1, \tag{20}$$

and the linear time dependence was given by

$$v_{1,t} = Av_1 + Bv_2, \tag{21}$$

$$v_{2,t} = Cv_1 + Dv_2. \tag{22}$$

Then, X and T were specified as follows

$$X = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix}, \tag{23}$$

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{24}$$

They noticed that if $r = -1$, then v_x would be the Schrödinger scattering problem. Letting $r = -1$, in $v_{2,x}$, then, it was observed that $v_{2,x}$=$ikv_2 - v_1$. Taking the derivative with respect to x of (20) gives,

$$v_{2,xx} = ikv_{2,x} - v_{1,x}, \tag{25}$$

and replacing (19) and (20) into (25) respectively gives,

$$\begin{aligned} v_{2,xx} &= ikv_{2,x} - v_{1,x} \\ &= ik(ikv_2 + r(x,t)v_1) - (-ikv_1 + q(x,t)v_2) \\ &= i^2k^2v_2 - ikv_1 + ikv_1 - q(x,t)v_2 \\ &= -k^2v_2 - qv_2 \end{aligned} \tag{26}$$

and then moving everything to one side gives,

$$v_{2,xx} + k^2 v_2 + q v_2 = 0, \tag{27}$$

$$v_{2,xx} + (k^2 + q) v_2 = 0, \tag{28}$$

(28) being the Schrödinger scattering equation, which is similar to the equation derived earlier in the Inverse Scattering Method section of this paper. Again, all the details in the discussion above have been taken from [6].

Now, to obtain the nonlinear evolution equations that that are expressed by (18), they required that

$$v_{j,xt} = v_{j,tx}, \tag{29}$$

when $j = 1, 2$. This imposes a condition for A, B, C and D and must satisfy the equation, assuming, $\frac{dk}{dt} = 0$. Hence, letting

$$v_{1,xt} = v_{1,tx}, \tag{30}$$

$$v_{2,xt} = v_{2,tx}, \tag{31}$$

and finding $v_{1,xt}$ followed by $v_{1,tx}$ gives

$$v_{1,x} = -ikv_1 + qv_2 \tag{32}$$

$$v_{1,xt} = -ikv_{1,t} + q_t v_2 + q v_{2,t}. \tag{33}$$

Replacing (21) results in (33),

$$v_{1,xt} = -ikv_{1,t} + q_t v_2 + qv_{2,t} \tag{34}$$

$$= -ik(Av_1 + Bv_2) + q_t v_2 + q(Cv_1 + Dqv_2) \tag{35}$$

$$= -ikAv_1 - ikBv_2 + q_t v_2 + qCv_1 + Dqv_2. \tag{36}$$

Similarly, $v_{1,tx}$, it is obtained by taking the derivative of (21) with respect to x. This is done by,

$$v_{1,t} = Av_1 + Bv_2 \tag{37}$$

$$v_{1,tx} = A_x v_1 + Av_{1,x} + B_x v_2 + Bv_{2,x}. \tag{38}$$

Replacing the values of (19) and (20) results in (33),

$$v_{1,tx} = A_x v_1 + Av_{1,x} + B_x v_2 + Bv_{2,x} \tag{39}$$

$$= A_x v_1 + A(-ikv_1 + qv_2) + B_x v_2 + B(ikv_2 + rv_1) \tag{40}$$

$$= A_x v_1 - Aikv_1 + Aqv_2 + B_x v_2 + Bikv_2 + Brv_1. \tag{41}$$

Having (30) gives A_x and B_x. This is shown as,

$$v_{1,xt} = v_{1,tx} \tag{42}$$

$$-ikAv_1 - ikBv_2 + q_t v_2 + qCv_1 + Dqv_2 = A_x v_1 - Aikv_1 - Aqv_2 + B_x v_2 + Bikv_2 + Brv_1 \tag{43}$$

Rearranging some of the terms in the equation gives, and solving for A_x gives,

$$v_{1,xt} = v_{1,tx}$$
$$-ikAv_1 + qCv_1 - ikBv_2 + q_t v_2 + Dqv_2 = A_x v_1 - Aikv_1 + Brv_1 + Aqv_2 + B_x v_2 + Bikv_2$$
$$v_1(-ikA + qC) + v_2(-ikB + q_t + Dq) = v_1(A_x - Aik + Br) + v_2(Aq + B_x + Bik).$$

Hence, equating the coefficients of v_1 from each side of the equation gives,

$$v_1(-ikA + qC) = v_1(A_x - Aik + Br), \tag{44}$$
$$-ikA + qC = A_x - Aik + Br, \tag{45}$$

$$0 = A_x - Aik + ikA + rB - qC, \tag{46}$$
$$0 = A_x + rB - qC, \tag{47}$$

$$A_x = qC - rB, \tag{48}$$
$$\tag{49}$$

Carrying the same procedure for v_2 gives,

$$v_2(-ikB + q_t + Dq) = v_2(Aq + B_x + Bik), \tag{50}$$
$$-ikB + q_t + Dq = Aq + B_x + Bik, \tag{51}$$

$$0 = Aq + B_x + Bik + Bik - q_t - Dq, \tag{52}$$

$$0 = Aq + B_x - q_t - Dq, \tag{53}$$

$$0 = B_x + 2ikB - q_t - Dq + Aq, \tag{54}$$

$$0 = B_x + 2ikB - q_t - q(D - A), \tag{55}$$

$$B_x + 2ikB = q_t + q(D - A), \tag{56}$$

$$B_x + 2ikB = q_t - q(-D + A), \tag{57}$$

$$B_x + 2ikB = q_t - q(A - D). \tag{58}$$

The same was done for (31) to find C_x and D_x. Thus,

$$\begin{aligned} v_{2,x} &= ikv_2 + rv_{1,} \\ v_{2,xt} &= ikv_{2,t} + r_t v_1 + rv_{1,t}. \end{aligned} \tag{59}$$

Replacing the value of (22) in (3.47) results in,

$$\begin{aligned} v_{2,xt} &= ikv_{2,t} + r_t v_1 + rv_{1,t} \\ &= ik(Cv_1 + Dv_2) + r_t v_1 + r(Av_1 + Bv_2) \quad (60) \\ &= ikCv_1 + ikDv_2 + r_t v_1 + rAv_1 + rBv_2. \quad (61) \end{aligned}$$

Similarly, $v_{2,tx}$, it is obtained by taking the derivative of (22) with respect to x. This is done by,

$$v_{2,t} = Cv_1 + Dv_2 \tag{62}$$

$$v_{2,tx} = C_x v_1 + Cv_{1,x} + D_x v_2 + Dv_{2,x}. \tag{63}$$

Replacing the values of (19) and (20) results in,

$$v_{2,tx} = C_x v_1 + C v_{1,x} + D_x v_2 + D v_{2,x} \tag{64}$$

$$= C_x v_1 + C(-ikv_1 + qv_2) + D_x v_2 + D(ikv_2 + rv_1) \tag{65}$$

$$= C_x v_1 - Cikv_1 + Cqv_2 + D_x v_2 + Dikv_2 + Drv_1. \tag{66}$$

Having (31) gives C_x and D_x. This is shown as,

$$v_{2,xt} = v_{2,tx} \tag{67}$$

$$ikCv_1 + ikDv_2 + r_t v_1 + rAv_1 + rBv_2 = C_x v_1 - Cikv_1 + Cqv_2 + D_x v_2 + Dikv_2 + Drv_1 \tag{68}$$

Rearranging some of the terms in the equation gives,

$$v_{2,xt} = v_{2,tx}$$

$$ikCv_1 + r_t v_1 + rAv_1 + ikDv_2 + rBv_2 = C_x v_1 - Cikv_1 + Drv_1 + Cqv_2 + D_x v_2 + Dikv_2$$

$$v_1(ikC + r_t + rA) + v_2(ikD + rB) = v_1(C_x - Cik + Dr) + v_2(Cq + D_x + Dik)$$

Hence, equating the coefficients of v_1 from each side of the equation and solving for C_x and D_x gives,

$$v_1(ikC + r_t + rA) = v_1(C_x - Cik + Dr), \tag{69}$$

$$ikC + r_t + rA = C_x - Cik + Dr, \tag{70}$$

$$0 = C_x - Cik + Dr - ikC - r_t - rA, \tag{71}$$

$$0 = C_x - 2Cik + Dr - r_t - rA, \tag{72}$$

$$0 = C_x - 2ikC - r_t - Dr + Ar, \tag{73}$$

$$0 = C_x - 2ikC - r_t - r(D - A), \tag{74}$$

$$C_x - 2ikC = r_t + 2Ar. \tag{75}$$

For v_2, a similar procedure is followed,

$$v_2(ikD + rB) = v_2(Cq + D_x + Dik), \tag{76}$$

$$ikD + rB = Cq + D_x + Dik, \tag{77}$$

$$0 = Cq + D_x + Dik - ikD - rB, \tag{78}$$

$$0 = Cq + D_x - rB, \tag{79}$$

$$D_x = -qC + rB, \tag{80}$$

$$D_x = -qC + rB, \tag{81}$$

$$D_x = -(qC - rB), \tag{82}$$

$$D_x = -A_x. \tag{83}$$

Hence,

$$A_x = qC - rB, \tag{84}$$

$$B_x = -2ikB + q_t - q(-D + A), \tag{85}$$

$$= -2ikB + q_t - q(-(-A) + A), \tag{86}$$

$$= -2ikB + q_t - q(A + A), \tag{87}$$

$$= -2ikB + q_t - 2Aq, \tag{88}$$

$$C_x = 2ikC + r_t + Dr - Ar, \tag{89}$$

$$D_x = -qC + rB, \tag{90}$$

$$= -A_x. \tag{91}$$

It is straightforward to noticed that $D_x = -A_x$, and so, A, B, and C satisfy the compatibility conditions of A_x, B_x, and C_x. Now, A, B, and C are found by requiring another condition. Because k is an eigenvalue and free parameter, solvable evolution equations finding finite series were found. These would be finite power series about A, B_x, and C_x, which are defined [6] as follows,

$$A = \sum_{j=0}^{n} A_j k^j, \tag{92}$$

$$B = \sum_{j=0}^{n} B_j k^j, \tag{93}$$

$$C = \sum_{j=0}^{n} C_j k^j. \tag{94}$$

If A, B, and C are substituted into (84), (85), and (89) and the coefficients of the powers of k are set equal to each other, then two evolution equations for r and q can be obtained. Ablowitz considered examples for this case for when $n = 2$, $n = 3$ and $n = -1$. The example for which $n = 2$ is provided in this paper. If $n = 2$, then the values of A, B, and C are as follows,

$$A = A_2 k^2 + A_1 k + A_0 \tag{95}$$

$$B = B_2 k^2 + B_1 k + B_0 \tag{96}$$

$$C = C_2 k^2 + C_1 k + C_0. \tag{97}$$

As mentioned, (95), (96), and (97) are substituted into (84), (85), and (89), and then the coefficients of k are set equal to each other. First, the equations for each, (84), (85), and (89) are considered. Replacing the values of (97) and (96) into (84) gives, and taking the derivative of (95) with respect to x, gives

$$A_x = qC - rB,$$

$$(A_2 k^2 + A_1 k + A_0)_x = q(C_2 k^2 + C_1 k + C_0) - r(B_2 k^2 + B_1 k + B_0), \tag{98}$$

$$A_{2,x} k^2 + A_{1,x} k + A_{0,x} = qC_2 k^2 + qC_1 k + qC_0 - rB_2 k^2 - rB_1 k - rB_0, \tag{99}$$

$$A_{2,x} k^2 + A_{1,x} k + A_{0,x} = qC_2 k^2 - rB_2 k^2 + qC_1 k - rB_1 k + qC_0 - rB_0, \tag{100}$$

$$A_{2,x} k^2 + A_{1,x} k + A_{0,x} = k^2 (qC_2 - rB_2) + k(qC_1 - rB_1) + qC_0 - rB_0. \tag{101}$$

From these sets of equations, the following can be determined,

$$A_{2,x} k^2 = k^2 (qC_2 - rB_2), \tag{102}$$

$$A_{2,x} = qC_2 - rB_2, \tag{103}$$

$$A_{2,x} = qC_2, \tag{104}$$

$$A_{2,x} = 0, \tag{105}$$

Integrating $A_{2,x}$ gives

$$A_2 = a. \tag{106}$$

Now we consider

$$A_{1,x}k = k(qC_1 - rB_1), \tag{107}$$

$$A_{1,x} = qC_1 - rB_1,$$

$$A_{1,x} = q(irA_2) - r(iqA_2), \tag{108}$$

$$A_{1,x} = qirA_2 - qirA_2, \tag{109}$$

$$A_{1,x} = 0. \tag{110}$$

Integrating $A_{1,x}$ gives

$$A_1 = b, \tag{111}$$

$$b = 0. \tag{112}$$

$$A_{0,x} = qC_0 - rB_0, \tag{113}$$

$$= q(\frac{a}{2})r_x - r(\frac{a}{2})q_x, \tag{114}$$

$$= \frac{a}{2}(rq)_x. \tag{115}$$

Similarly, a similar procedure for B_x and C_x is followed,

$$B_x = -2ikB + q_t - 2Aq,$$

$$(B_2k^2 + B_1k + B_0)_x = -2ik(B_2k^2 + B_1k + B_0) + q_t - 2q(A_2k^2 + A_1k + A_0), \tag{116}$$

$$B_{2,x}k^2 + B_{1,x}k + B_{0,x} = -2ikB_2k^2 - 2iB_1kk - 2ikB_0 + q_t - 2qA_2k^2 - 2qA_1k - 2qA_0, \tag{117}$$

$$B_{2,x}k^2 + B_{1,x}k + B_{0,x} = -2iB_2k^3 - 2B_1ik^2 - 2ikB_0 + q_t - 2qA_2k^2 - 2qA_1k - 2qA_0, \tag{118}$$

$$B_{2,x}k^2 + B_{1,x}k + B_{0,x} = -2iB_2k^3 - 2iB_1k^2 - 2qA_2k^2 - 2qA_1k - 2ikB_0 + q_t - 2qA_0, \tag{119}$$

$$B_{2,x}k^2 + B_{1,x}k + B_{0,x} = -2iB_2k^3 + k^2(-2iB_1 - 2qA_2) + k(-2qA_1 - 2iB_0) + q_t - 2qA_0. \tag{120}$$

Then, the values for $B_{2,x}$, $B_{1,x}$, and $B_{0,x}$ are obtained, as the coefficients of k^3, k^2, and k are set equal to each term of the other side of the equation [6]. Considering the term k^3 gives

$$-2iB_2k^3 = 0 \tag{121}$$

$$B_2 = 0. \tag{122}$$

Finding $B_{2,x}$, we get

$$B_{2,x} = 0. \tag{123}$$

Consider

$$B_{2,x}k^2 = k^2(-2iB_1 - 2qA_2), \tag{124}$$

$$B_{2,x} = -2iB_1 - 2qA_2. \tag{125}$$

Then we find B_1 as

$$-2iB_1 - 2qA_2 = 0, \tag{126}$$

$$-2iB_1 = 2qA_2, \tag{127}$$

$$-iB_1 = qA_2, \tag{128}$$

$$B_1 = iqA_2. \tag{129}$$

Since we know that $A_2 = a$, then we may solve for B and find $B_{1,x}$ as

$$B_1 = iaq, \tag{130}$$

$$B_{1,x} = iqA_{2,x}, \tag{131}$$

$$B_{1,x} = iq(qC_2), \tag{132}$$

$$B_{1,x} = iq^2C_2, \tag{133}$$

$$B_{1,x} = 0. \tag{134}$$

Consider

$$B_{1,x}k = k(-2qA_1 - 2iB_0), \tag{135}$$

$$B_{1,x} = -2qA_1 - 2iB_0. \tag{136}$$

Then we find B_0

$$-2qA_1 - 2iB_0 = 0 \tag{137}$$

$$-2qA_1 = 2iB_0, \tag{138}$$

$$-qA_1 = iB_0, \tag{139}$$

$$iqA_1 = B_0. \tag{140}$$

Finding $B_{0,x}$ and replacing the value of $A_{1,x}$ gives

$$B_0 = iqA_1, \tag{141}$$

$$B_{0,x} = iqA_{1,x}, \tag{142}$$

$$B_{0,x} = iq(qC_1 - rB_1), \tag{143}$$

$$B_{0,x} = iq^2C_1 - iqrB_1, \tag{144}$$

$$B_{0,x} = iq^2(irA_2) - iqr(iqA_2), \tag{145}$$

$$B_{0,x} = i^2q^2rA_2 - i^2q^2rA_2, \tag{146}$$

$$B_{0,x} = 0. \tag{147}$$

Knowing the value for $B_{0,x}$, we can find q_t as

$$B_{0,x} = q_t - 2qA_0, \tag{148}$$

$$0 = q_t - 2qA_0, \tag{149}$$

$$q_t = 2qA_0. \tag{150}$$

The equations for C_x follow as

$$C_x = 2ikC + r_t + 2Ar,$$

$$(C_2k^2 + C_1k + C_0)_x = 2ik(C_2k^2 + C_1k + C_0) + r_t + 2(A_2k^2 + A_1k + A_0)r, \tag{151}$$

$$C_{2,x}k^2 + C_{1,x}k + C_{0,x} = 2ikC_2k^2 + 2ikkC_1 + 2ikC_0 + r_t + 2rA_2k^2 + 2rA_1k + 2rA_0, \tag{152}$$

$$C_{2,x}k^2 + C_{1,x}k + C_{0,x} = 2iC_2k^3 + 2iC_1k^2 + 2ikC_0 + r_t + 2rA_2k^2 + 2rA_1k + 2rA_0, \tag{153}$$

$$C_{2,x}k^2 + C_{1,x}k + C_{0,x} = 2iC_2k^3 + 2iC_1k^2 + 2rA_2k^2 + 2ikC_0 + 2rA_1k + r_t + 2rA_0, \tag{154}$$

$$C_{2,x}k^2 + C_{1,x}k + C_{0,x} = 2iC_2k^3 + k^2(2iC_1 + 2rA_2) + k(2iC_0 + 2rA_1) + r_t + 2rA_0. \tag{155}$$

Then, the values for $C_{2,x}$, $C_{1,x}$, and $C_{0,x}$ are obtained, as the coefficients of k^3, k^2, and k are set equal to each term of the other side of the equation. We consider the term k^3 to find C_2 and $C_{2,x}$. This yields

$$2iC_2k^3 = 0 \tag{156}$$

$$C_2 = 0 \tag{157}$$

$$C_{2,x} = 0. \tag{158}$$

Likewise, considering the term k^2, we find

$$C_{2,x}k^2 = k^2(2iC_1 + 2rA_2), \tag{159}$$

$$C_{2,x} = 2iC_1 + 2rA_2. \tag{160}$$

We know that $C_{2,x}$ is zero, so we find the value of C

$$2iC_1 + 2rA_2 = 0, \tag{161}$$

$$2iC_1 = -2rA_2, \tag{162}$$

$$iC_1 = -rA_2, \tag{163}$$

$$C_1 = irA_2, \tag{164}$$

$$C_1 = iar. \tag{165}$$

We can now find the value of $C_{1,x}$. We obtain

$$C_{1,x} = rA_{2,x}, \tag{166}$$

$$C_{1,x} = r(qC_2), \tag{167}$$

$$C_{1,x} = rqC_2, \tag{168}$$

$$C_{1,x} = 0. \tag{169}$$

We now consider the term k to find $C_{1,x}$

$$C_{1,x}k = k(2iC_0 + 2rA_1), \tag{170}$$

$$C_{1,x} = 2iC_0 + 2rA_1. \tag{171}$$

Since we know that $C_{!,x}$ is zero, we find an equation for C_0 as

$$2iC_0 + 2rA_1 = 0, \tag{172}$$

$$2iC_0 = -2rA_1, \tag{173}$$

$$C_0 = irA_1. \tag{174}$$

Taking the derivative of C_0 with respect to x allows us to find $C_{0,x}$ as

$$C_{0,x} = irA_{1,x}, \tag{175}$$

$$C_{0,x} = ir(qC_1 - rB_1), \tag{176}$$

$$C_{0,x} = irqC_1 - irrB_1, \tag{177}$$

$$C_{0,x} = irqC_1 - ir^2B_1, \tag{178}$$

$$C_{0,x} = irq(irA_2) - ir^2(iqA_2), \tag{179}$$

$$C_{0,x} = i^2r^2qA_2 - i^2r^2qA_2, \tag{180}$$

$$C_{0,x} = 0. \tag{181}$$

Given that $C_{0,x}$ is zero, we can find r_t as

$$C_{0,x} = r_t + 2rA_0, \tag{182}$$

$$r_t + 2rA_0 = 0, \tag{183}$$

$$r_t = -2rA_0,. \tag{184}$$

From the results above, the values each of the coefficients of A, B, and C are following,

$$A_1 = b, \tag{185}$$

$$A_2 = a, \tag{186}$$

$$B_1 = iaq, \tag{187}$$

$$B_2 = 0, \tag{188}$$

$$C_1 = iar, \tag{189}$$

$$C_2 = 0. \tag{190}$$

However, the values of A_0, B_0, and C_0 are not given. From the values above, it can be determined that if $b = 0$, then

$$B_0 = \frac{-1}{2} a q_x, \tag{191}$$

and,

$$C_0 = \frac{1}{2} a r_x. \tag{192}$$

If the value of b is nonzero, another evolution equation can be obtained. From (191) and (192), the value of A_0 can be obtained. We can now find a general formula for $A_{0,x}$

$$A_{0,x} = qC_0 - rB_0, \tag{193}$$

$$A_{0,x} = q(\frac{1}{2} a r_x) - r(\frac{-1}{2} a q_x), \tag{194}$$

$$A_{0,x} = \frac{1}{2} a q r_x + 12 a r q_x, \tag{195}$$

$$A_{0,x} = \frac{1}{2} a (q r_x + r q_x), \tag{196}$$

$$A_{0,x} = \frac{1}{2} a (rq)_x. \tag{197}$$

Integrating both sides, gives,

$$A_0 = \frac{1}{2} a r q + c, \tag{198}$$

where c is a constant, and $c = 0$. In this case,

$$A_0 = \frac{1}{2} a r q. \tag{199}$$

The evolution equations are found by taking the derivative of (191) and (192) and replacing into the (148) and (182) equation.

$$B_{0,x} = q_t - 2qA_0, \tag{200}$$

$$B_{0,x} = q_t - 2q(\frac{1}{2}arq), \tag{201}$$

$$B_{0,x} = q_t - aq^2r. \tag{202}$$

It is known that

$$B_0 = \frac{-1}{2}aq_x, \tag{203}$$

so $B_{0,xx}$ is found.

$$B_0 = \frac{-1}{2}aq_x, \tag{204}$$

$$B_{0,x} = \frac{-1}{2}aq_{xx}. \tag{205}$$

Hence,

$$\frac{-1}{2}aq_{xx} = q_t - aq^2r. \tag{206}$$

Similarly, for (192), the following is obtained.

$$C_{0,x} = r_t - 2rA_0, \tag{207}$$

$$C_{0,x} = r_t - 2r(\frac{1}{2}arq), \tag{208}$$

$$C_{0,x} = r_t - ar^2q. \tag{209}$$

It is known that

$$C_0 = \frac{-1}{2} a r_x, \tag{210}$$

so $C_{0,xx}$ is found.

$$C_0 = \frac{-1}{2} a r_x, \tag{211}$$

$$C_{0,x} = \frac{-1}{2} a r_{xx}. \tag{212}$$

Hence,

$$\frac{-1}{2} a r_{xx} = r_t - a q r^2. \tag{213}$$

The two evolution equations are

$$\frac{-1}{2} a q_{xx} = q_t - a q^2 r, \tag{214}$$

$$\frac{1}{2} a r_{xx} = r_t - a q r^2. \tag{215}$$

If $r = \mp q^*$ and $a = 2i$, and substituting these values into

$$\frac{-1}{2} a q_{xx} = q_t - a q^2 r, \tag{216}$$

the nonlinear Schrödinger equation,

$$i q_t = q_{xx} \pm 2 q^2 q*, \tag{217}$$

is found. Given (216), we derive

$$\frac{-1}{2}aq_{xx} = q_t - aq^2r, \tag{218}$$

$$\frac{-1}{2}(2i)q_{xx} = q_t - (2i)q^2(\mp q^*), \tag{219}$$

$$-iq_{xx} = q_t - 2iq^2(\mp q^*). \tag{220}$$

Dividing both sides by $-i$,

$$-iq_{xx} = q_t - 2iq^2(\mp q^*), \tag{221}$$

$$q_{xx} = iq_t \pm 2q^2q*, \tag{222}$$

$$iq_t = q_{xx} \pm 2q^2q*. \tag{223}$$

It can be determined that if the scattering problem its associated time dependence are given, then these are compatible given that (84), (85), and(89) hold. For this case, consider $r = -q^*$ and $a = 2i$, then the values of A,B,C are determined to satisfy (84), (85), and(89), knowing that $q(x,t)$ fit the nonlinear Schrödinger equation. Replacing values of (199), (185), (186), (191), (187), (188), (192), (189), (190) into (95), (96), (97) respectively gives, for A,

$$A = A_2k^2 + A_1k + A_0, \tag{224}$$

$$A = (a)k^2 + (0)k + (\frac{1}{2}arq), \tag{225}$$

$$A = (2i)k^2 + (\frac{1}{2}(2i)(-q^*)q), \tag{226}$$

$$A = 2ik^2 \pm iqq^*. \tag{227}$$

For B,

$$B = B_2k^2 + B_1k + B_0, \tag{228}$$

$$B = (0)k^2 + (iaq)k + (\frac{-1}{2}aq_x), \tag{229}$$

$$B = i(2i)qk + (\frac{-1}{2}(2i)q_x), \tag{230}$$

$$B = 2i^2qk - iq_x, \tag{231}$$

$$B = 2(1)qk - iq_x, \tag{232}$$

$$B = 2qk - iq_x. \tag{233}$$

For C, we obtain

$$C = C_2k^2 + C_1k + C_0, \tag{234}$$

$$C = (0)k^2 + (iar)k + (\frac{1}{2}ar_x), \tag{235}$$

$$C = i(2i)(-q^*)k + \frac{1}{2}(2i)(-q_x^*), \tag{236}$$

$$C = 2i^2(-q^*)k \pm iq_x*, \tag{237}$$

$$C = \pm 2q^*k \pm iq_x^*. \tag{238}$$

Thus, the equations for (95), (96), (97) are

$$A = 2ik^2 \pm iqq^* \tag{239}$$

$$B = 2qk - iq_x \tag{240}$$

$$C = \pm 2q^*k \pm iq_x^*. \tag{241}$$

CHAPTER IV

The Ablowitz-Ladik Discrete Scattering Problem

4.1 Discrete Problems and the Inverse Scattering Transform

There are a variety of discrete problems modeled by nonlinear discrete equations. These examples involve the Toda Lattice (vibration of particles in 1D), ladder type electric circuits, collapse of Languimuir waves in plasma physics, growth of conflicting populations in biological science, difference simulations of differential equations and many more. The surprising aspect is that the inverse scattering transform apply to certain types of these discrete evolution equations [9].

One of the examples that will be considered as an example is the Toda lattice. The Toda Lattice is a system of unit masses connected by nonlinear springs whose restoring force is exponential. Sometimes, this system is called the the exponential lattice. The equations of motion that modeled this system are,

$$Q_{n,tt} = e^{(Q_n - Q_{n-1})} - e^{-(Q_{n+1} - Q_n)}, \tag{242}$$

and can be derived by the Hamiltonian,

$$H = \sum_{j=-\infty}^{\infty} \frac{1}{2} P_j^2 + (e^{-(-Q_n - Q_{n-1})} - 1) \tag{243}$$

where $P_j = Q_{j_t}$. This lattice was studied by Toda. He discovered a number of explicit solutions for both the periodic and infinite lattice [9]. Then, Flaschka was able to solve the lattice by the inverse scattering transform. He used the discrete inverse scattering theory of Case and Kac to be able to solve the lattice by considering the discrete Schrödinger equation [9]. Later, similar results were obtained by Manakov [9]. Next, Ablowitz and Ladik proposed a new discrete scattering problem [2, 3, 4, 5, 9].

The main ideas that have been discussed above have been taken from [6].

4.2 Deriving Evolution Equations for the Schrödinger Scattering Problem

The problem that Ablowitz and Ladik proposed was a modified version of the 2×2 Zakharov-Shabat problem and served as a basis for generating solvable discrete equations, like the Toda lattice, a nonlinear self-dual network and more [9]. These ideas were extended to nonlinear partial differential equations, which it is what will be discussed next [9]. The discrete Schrödinger equation will be considered [9].

The famous Schrödinger Scattering Problem can be represented as

$$\psi_{xx} + (\lambda_c + q)\psi, \tag{244}$$

where λ_c is the eigenvalue of the continuous problem. More information about this case can be found by seeing Case's and Kac's work that was performed in 1973 [9]. If the discrete case of this problem is considered, then, the following corresponds to

$$\frac{\psi_{n+1} + \psi_{n-1} - 2\psi_n}{h^2} + (\lambda_c + q_n)\psi_n = 0, \tag{245}$$

Before considering the question of discrete inverse scattering let us consider the discretization associated with the Zakharov-Shabat eigenvalue problem [9]. Consider the discrete equation,

$$(v_i)_x = \frac{v_{i,n+1} - v_{i,n}}{h}, \tag{246}$$

and using the above equations to define the discrete value of v_1 and v_2 gives

$$v_{1,n+1} = v_{1,n}(1 - i\zeta h) + q_n h v_{2,n}, \tag{247}$$

$$v_{2,n+1} = v_{2,n}(1 + i\zeta h) + r_n h v_{1,n}, \tag{248}$$

where $v_{i,n} \equiv v_i(nh), q_n = q(nh)$, and so on. If q_n, r_n were zero it would be natural to define $z = e^{-i\zeta h}$, so that the continuous solution v_1 goes to the discrete limit: $v_1 = e^{-i\zeta x} = e^{-i\zeta nh} = z^{-n}$ and similarly for v_2. Hence here we take $z = e^{-i\zeta h} \sim 1 - i\zeta h, 1/z \equiv e^{i\zeta h} \sim 1 + i\zeta h$ and we define $Q_n = q_n h, R_n = r_n h$ we find

$$v_{1,n+1} = z v_{1,n} + Q_n v_{2,n}, \tag{249}$$

$$v_{2,n+1} = \frac{1}{z} v_{2,n} + R_n v_{1,n}. \tag{250}$$

The generalization of the above equations is

$$v_{1,n+1} = z v_{1,n} + Q_n v_{2,n} + S_n v_{2,n+1}, \tag{251}$$

$$v_{2,n+1} = \frac{1}{z} v_{2,n} + R_n v_{1,n} + T_n v_{1,n+1} \tag{252}$$

noting that the equations relax to

$$v_{1_x} = -i\zeta v_1 + q v_2, \tag{253}$$

$$v_{2_x} = i\zeta v_2 + r v_1, \tag{254}$$

The following discussion is paraphrased from [9]. First we shall discuss the nonlinear differential-difference equations. Associated with either equations.

$$v_{1,n+1} = zv_{1,n} + Q_n v_{2,n}, \tag{255}$$

$$v_{2,n+1} = \frac{1}{z} v_{2,n} + R_n v_{1,n}, \tag{256}$$

or with the equations

$$v_{1,n+1} = zv_{1,n} + Q_n v_{2,n} + S_n v_{2,n+1}, \tag{257}$$

$$v_{2,n+1} = \frac{1}{z} v_{2,n} + R_n v_{1,n} + T_n v_{1,n+1}, \tag{258}$$

we write the time evolution equations of v_i as

$$\frac{\partial}{\partial t} v_{1,n} = A_n v_{1,n} + B_n v_{2,n}, \tag{259}$$

$$\frac{\partial}{\partial t} v_{2,n} = C_n v_{1,n} + D_n v_{2,n}. \tag{260}$$

A similar approach is taken as in the continuous case. Here,

$$\frac{\partial}{\partial t}(Ev_{i,n}) = E(\frac{\partial v_{i,n}}{\partial t}), \qquad i = 1, 2, \tag{261}$$

where E is the shift operator $E(v_{i,n}) \equiv v_{i,n+1}$. In what follows we take, for simplicity of representation, $T_n = S_n = 0$. We have for example,

$$v_{1,n+1} = zv_{1,n} + Q_n v_{2,n} + S_n v_{2,n+1},$$

$$v_{1,n+1} = zv_{1,n} + Q_n v_{2,n}.$$

We consider the case $\frac{\partial}{\partial t}(E_{vi,n}) = E(\frac{\partial}{\partial t}v_{1,n})$. We define $\frac{\partial}{\partial t}(Ev_{1,n})$ as follows:

$$Ev_{1,n+1} = zv_{1,n} + Q_n v_{2,n}, \tag{262}$$
$$\frac{\partial}{\partial t}(Ev_{1,n}) = \frac{\partial}{\partial t}(zv_{1,n} + Q_n v_{2,n}), \tag{263}$$
$$(Ev_{1,n})_t = (zv_{1,n} + Q_n v_{2,n})_t. \tag{264}$$

Consider (264) to obtain

$$\begin{aligned}
(Ev_{1,n})_t =& (zv_{1,n} + Q_n v_{2,n})_t && (265)\\
=& z\frac{\partial}{\partial t}(v_{1,n}) + \frac{\partial}{\partial t}(Q_n)v_{2,n} + Q_n\frac{\partial}{\partial t}(v_{2,n}), && (266)\\
=& z\frac{\partial}{\partial t}(v_{1,n}) + Q_{n,t}v_{2,n} + Q_n\frac{\partial}{\partial t}(v_{2,n}), && (267)\\
=& z(A_n v_{1,n} + B_n v_{2,n}) + Q_{n,t}v_{2,n} + Q_n(C_n v_{1,n} + D_n v_{2,n}), && (268)\\
=& zA_n v_{1,n} + zB_n v_{2,n} + Q_{n,t}v_{2,n} + Q_n C_n v_{1,n} + Q_n D_n v_{2,n}, && (269)\\
=& zA_n v_{1,n} + Q_n C_n v_{1,n} + zB_n v_{2,n} + Q_{n,t}v_{2,n} + Q_n D_n v_{2,n}, && (270)\\
=& v_{1,n}(zA_n + Q_n C_n) + v_{2,n}(zB_n + Q_{n,t} + Q_n D_n). && (271)
\end{aligned}$$

Now we define $E(\frac{\partial}{\partial t}v_{1,n}) = A_{n+1}v_{1,n+1} + B_{n+1}v_{2,n+1}$. We find that

$$\begin{aligned}
E(\frac{\partial}{\partial t}v_{1,n}) =& A_{n+1}v_{1,n+1} + B_{n+1}v_{2,n+1}, && (272)\\
=& A_{n+1}(zv_{1,n} + Q_n v_{2,n}) + B_{n+1}(\frac{1}{z}v_{2,n} + R_n v_{1,n}), && (273)\\
=& A_{n+1}zv_{1,n} + A_{n+1}Q_n v_{2,n} + B_{n+1}\frac{1}{z}v_{2,n} + B_{n+1}R_n v_{1,n}, && (274)\\
=& A_{n+1}zv_{1,n} + B_{n+1}R_n v_{1,n} + A_{n+1}Q_n v_{2,n} + B_{n+1}\frac{1}{z}v_{2,n}, && (275)\\
=& v_{1,n}(A_{n+1}z + B_{n+1}R_n) + v_{2,n}(A_{n+1}Q_n + \frac{1}{z}B_{n+1}). && (276)
\end{aligned}$$

Doing the same for $v_{2,n}$, it is found that

$$v_{2,n+1} = \frac{1}{z}v_{2,n} + R_n v_{1,n} + T_n v_{1,n+1}, \tag{277}$$

In this case, $T_n = 0$, so this equation becomes, $v_{2,n+1} = \frac{1}{z}v_{2,n} + R_n v_{1,n}$. Hence,

$$v_{2,n+1} = \frac{1}{z}v_{2,n} + R_n v_{1,n} + T_n v_{1,n+1}, \tag{278}$$

$$v_{2,n+1} = \frac{1}{z}v_{2,n} + R_n v_{1,n}. \tag{279}$$

If we apply the operator to (279), we get

$$Ev_{2,n+1} = \frac{1}{z}v_{2,n} + R_n v_{1,n}, \tag{280}$$

$$\frac{\partial}{\partial t}(Ev_{2,n}) = \frac{\partial}{\partial t}(\frac{1}{z}v_{2,n} + R_n v_{1,n}), \tag{281}$$

$$(Ev_{2,n})_t = (\frac{1}{z}v_{2,n} + R_n v_{1,n})_t. \tag{282}$$

Considering (282), we get

$$(Ev_{2,n})_t =(\frac{1}{z}v_{2,n} + R_n v_{1,n})_t, \tag{283}$$

$$=\frac{1}{z}\frac{\partial}{\partial t}(v_{2,n}) + \frac{\partial}{\partial t}(R_n)v_{1,n} + R_n\frac{\partial}{\partial t}(v_{1,n}), \tag{284}$$

$$=\frac{1}{z}\frac{\partial}{\partial t}(v_{2,n}) + R_{n,t}v_{1,n} + R_n\frac{\partial}{\partial t}(v_{1,n}), \tag{285}$$

$$=\frac{1}{z}(C_n v_{1,n} + D_n v_{2,n}) + R_{n,t}v_{1,n} + R_n(A_n v_{1,n} + B_n v_{2,n}), \tag{286}$$

$$=\frac{1}{z}C_n v_{1,n} + \frac{1}{z}D_n v_{2,n} + R_{n,t}v_{1,n} + R_n A_n v_{1,n} + R_n B_n v_{2,n}, \tag{287}$$

$$=\frac{1}{z}C_n v_{1,n} + R_n A_n v_{1,n} + R_{n,t}v_{1,n} + \frac{1}{z}D_n v_{2,n} + R_n B_n v_{2,n}, \tag{288}$$

$$=v_{1,n}(\frac{1}{z}C_n + R_{n,t} + R_n A_n) + v_{2,n}(\frac{1}{z}D_n + R_n B_n). \tag{289}$$

Now, consider $E(\frac{\partial}{\partial t}v_{1,n}) = C_{n+1}v_{1,n+1} + D_{n+1}v_{2,n+1}$. We obtain

$$E(\frac{\partial}{\partial t}v_{2,n}) = C_{n+1}v_{1,n+1} + D_{n+1}v_{2,n+1} \tag{290}$$

$$= C_{n+1}(zv_{1,n} + Q_n v_{2,n}) + D_{n+1}(\frac{1}{z}v_{2,n} + R_n v_{1,n}) \tag{291}$$

$$= C_{n+1}zv_{1,n} + C_{n+1}Q_n v_{2,n} + D_{n+1}\frac{1}{z}v_{2,n} + D_{n+1}R_n v_{1,n} \tag{292}$$

$$= C_{n+1}zv_{1,n} + D_{n+1}R_n v_{1,n} + C_{n+1}Q_n v_{2,n} + D_{n+1}\frac{1}{z}v_{2,n} \tag{293}$$

$$= v_{1,n}(C_{n+1}z + D_{n+1}R_n) + v_{2,n}(C_{n+1}Q_n + \frac{1}{z}D_{n+1}) \tag{294}$$

Hence,

$$\frac{\partial}{\partial t}(Ev_{1,n}) = E(\frac{\partial}{\partial t}v_{1,n}) \tag{295}$$

translates to

$$v_{1,n}(zA_n + Q_nC_n) + v_{2,n}(zB_n + Q_{n,t} + Q_nD_n) = \tag{296}$$

$$v_{1,n}(A_{n+1}z + B_{n+1}R_n) + v_{2,n}(A_{n+1}Q_n + \frac{1}{z}B_{n+1}). \tag{297}$$

For $v_{1,n}$, we have,

$$v_{1,n}(zA_n + Q_nC_n) = v_{1,n}(A_{n+1}z + B_{n+1}R_n), \tag{298}$$

$$zA_n + Q_nC_n = A_{n+1}z + B_{n+1}R_n, \tag{299}$$

$$zA_n - A_{n+1}z = B_{n+1}R_n - Q_nC_n, \tag{300}$$

$$A_{n+1}z - zA_n = Q_nC_n - B_{n+1}R_n, \tag{301}$$

$$z(A_{n+1} - A_n) = Q_nC_n - B_{n+1}R_n, \tag{302}$$

$$z\Delta_n A_n = Q_nC_n - R_nB_{n+1}. \tag{303}$$

For $v_{2,n}$, we have,

$$v_{2,n}(zB_n + Q_{n,t} + Q_nD_n) = v_{2,n}(A_{n+1}Q_n + \frac{1}{z}B_{n+1}), \tag{304}$$

$$zB_n + Q_{n,t} + Q_nD_n = A_{n+1}Q_n + \frac{1}{z}B_{n+1}, \tag{305}$$

$$zB_n - \frac{1}{z}B_{n+1} = A_{n+1}Q_n - Q_{n,t} - Q_nD_n, \tag{306}$$

$$\frac{1}{z}B_{n+1} - zB_n = Q_{n,t} - A_{n+1}Q_n + D_nQ_n. \tag{307}$$

Consider the condition s $\frac{\partial}{\partial t}(Ev_{2,n}) = E(\frac{\partial}{\partial t}v_{2,n})$. This translates to

$$\frac{\partial}{\partial t}(Ev_{2,n}) = E(\frac{\partial}{\partial t}v_{2,n}), \tag{308}$$

$$v_{1,n}(\frac{1}{z}C_n + R_{n,t} + R_nA_n) + v_{2,n}(\frac{1}{z}D_n + R_nB_n), \tag{309}$$

$$= v_{1,n}(C_{n+1}z + D_{n+1}R_n) + v_{2,n}(C_{n+1}Q_n + \frac{1}{z}D_{n+1}). \tag{310}$$

For $v_{1,n}$, we have

$$v_{1,n}(\frac{1}{z}C_n + R_{n,t} + R_nA_n) = v_{1,n}(C_{n+1}z + D_{n+1}R_n), \tag{311}$$

$$\frac{1}{z}C_n + R_{n,t} + R_nA_n = C_{n+1}z + D_{n+1}R_n, \tag{312}$$

$$\frac{1}{z}C_n - C_{n+1}z = D_{n+1}R_n - R_{n,t} - R_nA_n, \tag{313}$$

$$C_{n+1}z - \frac{1}{z}C_n = -D_{n+1}R_n + R_{n,t} + R_nA_n, \tag{314}$$

$$zC_{n+1} - \frac{1}{z}C_n = R_{n,t} + R_n(A_n - D_{n+1}). \tag{315}$$

For $v_{2,n}$, we have

$$v_{2,n}(\frac{1}{z}D_n + R_nB_n) = v_{2,n}(C_{n+1}Q_n + \frac{1}{z}D_{n+1}) \tag{316}$$

$$\frac{1}{z}D_n + R_nB_n = C_{n+1}Q_n + \frac{1}{z}D_{n+1}, \tag{317}$$

$$\frac{1}{z}D_n - \frac{1}{z}D_{n+1} = C_{n+1}Q_n - R_nB_n, \tag{318}$$

$$\frac{1}{z}D_n - \frac{1}{z}D_{n+1} = R_nB_n - C_{n+1}Q_n, \tag{319}$$

$$\frac{1}{z}(D_{n+1} - D_n) = R_nB_n - C_{n+1}Q_n, \tag{320}$$

$$\frac{1}{z}(\Delta_nD_n) = R_nB_n - C_{n+1}Q_n, \tag{321}$$

$$\frac{1}{z}(\Delta_nD_n) = -(Q_nC_{n+1} - R_nB_n). \tag{322}$$

In summary, the equations obtained are

$$z\Delta_nA_n = Q_nC_n - R_nB_{n+1}, \tag{323}$$

$$\frac{1}{z}B_{n+1} - zB_n = Q_{n,t} - A_{n+1}Q_n + D_nQ_n, \tag{324}$$

$$zC_{n+1} - \frac{1}{z}C_n = R_{n,t} + R_n(A_n - D_{n+1}), \tag{325}$$

$$\frac{1}{z}\Delta_nD_n = -(Q_nC_{n+1} - R_nB_n), \tag{326}$$

where $\Delta_nA_n = A_{n+1} - A_n$, and $\Delta_nD_n = D_{n+1} - D_n$.

We now solve (323)–(326). Expand A_n, B_n, C_n, D_n as follows:

$$A_n = z^2A_n^{(2)} + A_n^{(0)}, \tag{327}$$

$$B_n = zB_n^{(1)} + \frac{1}{z}B_n^{(-1)}, \tag{328}$$

$$C_n = zC_n^{(1)} + \frac{1}{z}C_n^{(-1)}, \tag{329}$$

$$D_n = D_n^{(0)} + \frac{1}{z^2} D_n^{(-2)}. \tag{330}$$

If we replace (327)-(330) into (323)-(326), we can find the coefficients for (323)-(326). If we take (323) and perform the corresponding substitution, we get

$$z\Delta_n A_n = Q_n C_n - R_n B_{n+1},$$

$$z(A_{n+1} - A_n) = Q_n C_n - R_n B_{n+1}, \tag{331}$$

$$z(z^2 A_{n+1}^{(2)} + A_{n+1}^{(0)} - z^2 A_n^{(2)} - A_n^{(0)}) = Q_n (z C_n^{(1)} + \frac{1}{z} C_n^{(-1)}) - R_n (z B_{n+1}^{(1)} + \frac{1}{z} B_{n+1}^{(-1)}), \tag{332}$$

$$z^3 A_{n+1}^{(2)} + z A_{n+1}^{(0)} - z^3 A_n^{(2)} - z A_n^{(0)} = z Q_n C_n^{(1)} + Q_n \frac{1}{z} C_n^{(-1)} - z R_n B_{n+1}^{(1)} - \frac{1}{z} R_n B_{n+1}^{(-1)}. \tag{333}$$

Take (333), and equate the coefficients of z^3 to get

$$z^3 A_{n+1}^{(2)} - z^3 A_n^{(2)} = 0, \tag{334}$$

$$z^3 (A_{n+1}^{(2)} - A_n^{(2)}) = 0, \tag{335}$$

$$A_{n+1}^{(2)} - A_n^{(2)} = 0, \tag{336}$$

$$A_{n+1}^{(2)} = A_n^{(2)}, \tag{337}$$

$$A_n^{(2)} = A_\infty^{(2)} = constant. \tag{338}$$

Considering the coefficients of z in (333) gives

$$z A_{n+1}^{(0)} - z A_n^{(0)} = z Q_n C_n^{(1)} - z R_n B_{n+1}^{(1)}, \tag{339}$$

$$z(A_{n+1}^{(0)} - A_n^{(0)}) = z(Q_n C_n^{(1)} - R_n B_{n+1}^{(1)}), \tag{340}$$

$$A_{n+1}^{(0)} - A_n^{(0)} = Q_n C_n^{(1)} - R_n B_{n+1}^{(1)}, \tag{341}$$

$$A_{n+1}^{(0)} - A_n^{(0)} = Q_n R_{n-1} A_\infty^{(2)} - R_n A_\infty^{(2)} Q_{n+1}, \tag{342}$$

$$A_{n+1}^{(0)} - A_n^{(0)} = A_\infty^{(2)} (Q_n R_{n-1} - Q_{n+1} R_n). \tag{343}$$

Considering the coefficients of $\frac{1}{z}$ in (333), we obtain

$$Q_n \frac{1}{z} C_n^{(-1)} - \frac{1}{z} R_n B_{n+1}^{(-1)} = 0, \tag{344}$$

$$\frac{1}{z}(Q_n C_n^{(-1)} - R_n B_{n+1}^{(-1)}) = 0, \tag{345}$$

$$Q_n C_n^{(-1)} - R_n B_{n+1}^{(-1)}, \tag{346}$$

$$Q_n C_n^{(-1)} = R_n B_{n+1}^{(-1)}. \tag{347}$$

Consider the expansions,

$$A_{n+1} = z^2 A_{n+1}^{(2)} + A_{n+1}^{(0)}, \tag{348}$$

$$B_{n+1} = z B_{n+1}^{(1)} + \frac{1}{z} B_{n+1}^{(-1)}. \tag{349}$$

Using these and (324) gives

$$\frac{1}{z} B_{n+1} - z B_n = Q_{n,t} - A_{n+1} Q_n + D_n Q_n, \tag{350}$$

$$\frac{1}{z}(z B_{n+1}^{(2)} + \frac{1}{z} B_{n+1}^{(-1)}) - z(z B_n^{(2)} + \frac{1}{z} B_n^{(-1)}) = Q_{n,t} - (z^2 A_{n+1}^{(2)} + A_{n+1}^{(0)}) Q_n + (D_n^{(0)} + \frac{1}{z^2} D_n^{(-2)}) Q_n, \tag{351}$$

$$\frac{1}{z} z B_{n+1}^{(1)} + \frac{1}{z}\frac{1}{z} B_{n+1}^{(-1)} - z^2 B_n^{(1)} - z \frac{1}{z} B_n^{(-1)} = Q_{n,t} - z^2 A_{n+1}^{(2)} Q_n - A_{n+1}^{(0)} Q_n + Q_n D_n^{(0)} + \frac{1}{z^2} D_n^{(-2)} Q_n, \tag{352}$$

$$B_{n+1}^{(1)} + \frac{1}{z^2} B_{n+1}^{(-1)} - z^2 B_n^{(2)} - B_n^{(-1)} = Q_{n,t} - z^2 A_{n+1}^{(2)} Q_n - A_{n+1}^{(0)} Q_n + Q_n D_n^{(0)} + \frac{1}{z^2} D_n^{(-2)} Q_n. \tag{353}$$

In a similar manner, if we consider the coefficients of $\frac{1}{z^2}$ in (353), we obtain a value for $B_{n+1}^{(-1)}$ as follows:

$$B_{n+1}^{(1)} + \frac{1}{z^2} B_{n+1}^{(-1)} - z^2 B_n^{(1)} - B_n^{(-1)} = Q_{n,t} - z^2 A_{n+1}^{(2)} Q_n - A_{n+1}^{(0)} Q_n + Q_n D_n^{(0)} + \frac{1}{z^2} D_n^{(-2)} Q_n, \tag{354}$$

$$\frac{1}{z^2}B_{n+1}^{(-1)} = \frac{1}{z^2}D_n^{(-2)}Q_n, \tag{355}$$

$$B_{n+1}^{(-1)} = D_n^{(-2)}Q_n. \tag{356}$$

If we use (353) to get the coefficients of z^2, we get

$$-z^2B_n^{(1)} = -z^2A_{n+1}^{(2)}Q_n, \tag{357}$$

$$B_n^{(1)} = A_{n+1}^{(2)}Q_n, \tag{358}$$

$$B_n^{(1)} = A_\infty^{(2)}Q_n, \tag{359}$$

where

$$B_{n+1}^{(1)} = A_\infty^{(2)}Q_{n+1}. \tag{360}$$

Lastly, taking the coefficients of z^0 in (353), we obtain

$$B_{n+1}^{(1)} - B_n^{(-1)} = Q_{n,t} - A_{n+1}^{(0)}Q_n + Q_nD_n^{(0)}, \tag{361}$$

$$Q_{n,t} = A_{n+1}^{(0)}Q_n - Q_nD_n^{(0)} + B_{n+1}^{(1)} - B_n^{(-1)}. \tag{362}$$

Now we consider the expansions for C_{n+1} and D_{n+1}. These are given by

$$C_{n+1} = zC_{n+1}^{(1)} + \frac{1}{z}C_{n+1}^{(-1)}, \tag{363}$$

$$D_{n+1} = D_{n+1}^{(0)} + \frac{1}{z^2}D_{n+1}^{(-2)}. \tag{364}$$

We substitute these values into (329). This yields the following expansions

$$z(zC_{n+1}^{(1)} + \frac{1}{z}C_{n+1}^{(-1)}) - \frac{1}{z}(zC_n^{(1)} + \frac{1}{z}C_n^{(-1)}) = R_{n,t} + R_n(z^2A_n^{(2)} + A_n^{(0)} - (D_{n+1}^{(0)} + \frac{1}{z^2}D_{n+1}^{(-2)})), \tag{365}$$

$$zzC_{n+1}^{(1)}+z\frac{1}{z}C_{n+1}^{(-1)}-\frac{1}{z}zC_n^{(1)}-\frac{1}{z}\frac{1}{z}C_n^{(-1)}=R_{n,t}+R_nz^2A_n^{(2)}+R_nA_n^{(0)}-R_nD_{n+1}^{(0)}-R_n\frac{1}{z^2}D_{n+1}^{(-2)}, \tag{366}$$

$$z^2C_{n+1}^{(1)}+C_{n+1}^{(-1)}-C_n^{(1)}-\frac{1}{z^2}C_n^{(-1)}=R_{n,t}+z^2R_nA_n^{(2)}+R_nA_n^{(0)}-R_nD_{n+1}^{(0)}-\frac{1}{z^2}R_nD_{n+1}^{(-2)}. \tag{367}$$

If the coefficients of z^2 are considered in (367), then we obtain, for $C_{n+1}^{(1)}$

$$z^2C_{n+1}^{(1)}=z^2R_nA_n^{(2)}, \tag{368}$$

$$C_{n+1}^{(1)}=R_nA_n^{(2)}, \tag{369}$$

$$C_{n+1}^{(1)}=R_nA_\infty^{(2)}. \tag{370}$$

Next we consider the coefficients of $\frac{1}{z^2}$ in (367). For this case, we get

$$-\frac{1}{z^2}C_n^{(-1)}=-\frac{1}{z^2}R_nD_{n+1}^{(-2)}, \tag{371}$$

$$C_n^{(-1)}=R_nD_{n+1}^{(-2)}. \tag{372}$$

For the coefficients of z^0, we obtain

$$C_{n+1}^{(-1)}-C_n^{(1)}=R_{n,t}+R_nA_n^{(0)}-R_nD_{n+1}^{(0)}, \tag{373}$$

$$R_{n,t}=C_{n+1}^{(-1)}-C_n^{(1)}-R_nA_n^{(0)}+R_nD_{n+1}^{(0)}. \tag{374}$$

Finally, we consider (363) and (364). These values are replaced in (330). The results we get are summarized below.

$$\frac{1}{z}(D_{n+1}-D_n)=-(Q_nC_{n+1}-R_nB_n), \tag{375}$$

$$\frac{1}{z}(D_{n+1}^{(0)}+\frac{1}{z^2}D_{n+1}^{(-2)}-(D_n^{(0)}+\frac{1}{z^2}D_n^{(-2)}))=-(Q_n(zC_{n+1}^{(1)}+\frac{1}{z}C_{n+1}^{(-1)})-R_n(zB_n^{(1)}+\frac{1}{z}B_n^{(-1)})), \tag{376}$$

$$\frac{1}{z}(D_{n+1}^{(0)}+\frac{1}{z^2}D_{n+1}^{(-2)}-D_n^{(0)}-\frac{1}{z^2}D_n^{(-2)})=-(zQ_nC_{n+1}^{(1)}+\frac{1}{z}Q_nC_{n+1}^{(-1)}-zR_nB_n^{(1)}-\frac{1}{z}R_nB_n^{(-1)}), \tag{377}$$

$$\frac{1}{z}D_{n+1}^{(0)}+\frac{1}{z}\frac{1}{z^2}D_{n+1}^{(-2)}-\frac{1}{z}D_n^{(0)}-\frac{1}{z}\frac{1}{z^2}D_n^{(-2)}=-zQ_nC_{n+1}^{(1)}-\frac{1}{z}Q_nC_{n+1}^{(-1)}+zR_nB_n^{(1)}+\frac{1}{z}R_nB_n^{(-1)}, \tag{378}$$

$$\frac{1}{z}D_{n+1}^{(0)}+\frac{1}{z^3}D_{n+1}^{(-2)}-\frac{1}{z}D_n^{(0)}-\frac{1}{z^3}D_n^{(-2)}=-zQ_nC_{n+1}^{(1)}-\frac{1}{z}Q_nC_{n+1}^{(-1)}+zR_nB_n^{(1)}+\frac{1}{z}R_nB_n^{(-1)}. \tag{379}$$

The first step is to consider the coefficients of $\frac{1}{z^3}$ in (379). We get

$$\frac{1}{z^3}D_{n+1}^{(-2)}-\frac{1}{z^3}D_n^{(-2)}=0, \tag{380}$$

$$D_{n+1}^{(-2)}-D_n^{(-2)}=0, \tag{381}$$

$$D_{n+1}^{(-2)}=D_n^{(-2)}, \tag{382}$$

$$D_n^{(-2)}=D_\infty^{(-2)}, \tag{383}$$

$$D_n^{(-2)}=D_\infty^{(-2)}=constant. \tag{384}$$

Next we consider the coefficients of $\frac{1}{z}$ in (379). We get

$$\frac{1}{z}D_{n+1}^{(0)}-\frac{1}{z}D_n^{(0)}=-\frac{1}{z}Q_nC_{n+1}^{(-1)}+\frac{1}{z}R_nB_n^{(-1)}, \tag{385}$$

$$\frac{1}{z}(D_{n+1}^{(0)}-D_n^{(0)})=\frac{1}{z}(-Q_nC_{n+1}^{(-1)}+R_nB_n^{(-1)}), \tag{386}$$

$$D_{n+1}^{(0)}-D_n^{(0)}=-Q_nC_{n+1}^{(-1)}+R_nB_n^{(-1)}. \tag{387}$$

When we consider the coefficients of z, we find a value for $C_{n+1}^{(1)}$ as follows:

$$0=-zQ_nC_{n+1}^{(1)}+zR_nB_n^{(1)}, \tag{388}$$

$$-z(Q_nC_{n+1}^{(1)}-R_nB_n^{(1)})=0, \tag{389}$$

$$Q_nC_{n+1}^{(1)}-R_nB_n^{(1)}=0, \tag{390}$$

$$Q_nC_{n+1}^{(1)}=R_nB_n^{(1)}, \tag{391}$$

$$Q_nC_{n+1}^{(1)}=R_nA_\infty^{(2)}Q_n, \tag{392}$$

$$C_{n+1}^{(1)} = R_n A_\infty^{(2)}. \tag{393}$$

The evolution equations are obtained from (362) and (374). These are again,

$$Q_{n,t} = A_{n+1}^{(0)} Q_n - Q_n D_n^{(0)} + B_{n+1}^{(1)} - B_n^{(-1)}, \tag{394}$$

$$R_{n,t} = C_{n+1}^{(-1)} - C_n^{(1)} - R_n A_n^{(0)} + R_n D_{n+1}^{(0)}. \tag{395}$$

These equations must be satisfied since with $R_n = \pm Q_n^*$ if $D_-^{(-2)} = A_-^{(-2)^*}$ and $(A_\infty^{(0)} - D_\infty^{(0)}) = -(A_\infty^{(0)} - D_\infty^{(0)})^*$. Taking $A_-^{(2)} = -i/h^2$ and $A_\infty^{(0)} - D_\infty^{(0)}) = 2i/h^2$, we find

$$Q_{n,t} = (\frac{-i}{h^2})(Q_{n+1} + Q_{n-1} - 2Q_n) \pm Q_n Q_n^* (Q_{n+1} + Q_{n-1})(\frac{-i}{h^2}), \tag{396}$$

or, if $Q_n = hq_n$,

$$iq_{n,t} = \frac{q_{n+1} + q_{n-1} - 2q_n}{h^2} \pm q_n q_n^* (q_{n+1} + q_{n-1}). \tag{397}$$

We refer to this as the differential-difference nonlinear Schrödinger equation. The equations for A_n, B_n, C_n and D_n are obtain after replacing the correponding coefficients [9]. These appear below:

$$A_n = (\frac{i}{h^2})(1 - z^2 \mp Q_n Q_{n-1}^*), \tag{398}$$

$$B_n = (\frac{i}{h^2})(-Q_n z + \frac{Q_{n-1}}{z}), \tag{399}$$

$$C_n = (\frac{\pm i}{h^2})(Q_{n-1}^* z - \frac{Q_n^*}{z}), \tag{400}$$

$$D_n = (\frac{-i}{h^2})(1 - \frac{1}{z^2} \mp Q_{n-1} Q_n^*). \tag{401}$$

Again, the main ideas discussed above have been taken from [2, 3, 4, 5, 9].

CHAPTER V

Applications of Lax pairs to a 3×3 continuous case

A similar approach to the 2×2 continuous case is taken. In [7], Ablowitz describes a method for $n \times n$ cases. This approach is similar to the approach mentioned in Chapter 3 of this thesis. We begin by defining

$$v_x = i\zeta Dv + Nv, \tag{402}$$

$$v_t = Qv, \tag{403}$$

where v is an n-dimensional vector and D, N, Q are $n \times n$ matrices. To show that these two equations are compatible, it is required that v_{xt}=v_{tx}, and this implies

$$Q_x = N_t + i\zeta[D,Q] + [N,Q], \tag{404}$$

where $[D,Q] = DQ - QD$ and $[N,Q] = NQ - QN$ [7]. Values for v_x are defined as,

$$v_{1,x} = id_1\zeta v_1 + n_{12}v_2 + n_{13}v_3, \tag{405}$$

$$v_{2,x} = id_2\zeta v_1 + n_{21}v_2 + n_{31}v_3, \tag{406}$$

$$v_{3,x} = id_3\zeta v_1 + n_{31}v_2 + n_{32}v_3. \tag{407}$$

and the linear time dependence is given by

$$v_{1,t} = Av_1 + Bv_2 + Cv_3, \tag{408}$$

$$v_{2,t} = Dv_1 + Ev_2 + Fv_3, \tag{409}$$

$$v_{3,t} = Gv_1 + Hv_2 + Iv_3. \tag{410}$$

Then, N, D, Q are specified as follows

$$N = \begin{pmatrix} 0 & n_{1,2} & n_{1,3} \\ n_{2,1} & 0 & n_{2,3} \\ n_{3,1} & n_{3,2} & 0 \end{pmatrix}, \tag{411}$$

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \tag{412}$$

$$Q = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}. \tag{413}$$

Letting

$$v_{1,xt} = v_{1,tx}, \tag{414}$$

$$v_{2,xt} = v_{2,tx}, \tag{415}$$

$$v_{3,xt} = v_{3,tx}, \tag{416}$$

and finding $v_{1,xt}$ followed by $v_{1,tx}$ gives

$$v_{1,x} = id_1\zeta v_1 + n_{1,2}v_2 + n_{1,3}v_3, \tag{417}$$

$$v_{1,xt} = id_1\zeta v_{1,t} + n_{1,2}v_2 + n_{1,2}v_{2,t} + n_{1,3}v_3 + n_{1,3}v_{2,t}. \tag{418}$$

Replacing (408) results in (418) gives

$$v_{1,xt} = id_1\zeta v_{1,t} + n_{1,2}v_2 + n_{1,2}v_{2,t} + n_{1,3}v_3 + n_{1,3}v_{2,t}, \tag{419}$$

$$= id_1\zeta(Av_1 + Bv_2 + Cv_3) + n_{1,2}v_2 + n_{1,2}(Dv_1 + Ev_2 + Fv_3) \tag{420}$$

$$+ n_{1,3}v_3 + n_{1,3}(Gv_1 + Hv_2 + Iv_3). \tag{421}$$

Similarly, $v_{1,tx}$, is obtained by taking the derivative of (408) with respect to x. This implies

$$v_{1,t} = Av_1 + Bv_2 + Cv_3 \tag{422}$$

$$v_{1,tx} = A_x v_1 + Av_{1,x} + B_x v_2 + Bv_{2,x} + C_x v_3 + Cv_{3,x}. \tag{423}$$

Replacing the values of (405) and (406) and (407) into (423) gives

$$v_{1,tx} = A_x v_1 + Av_{1,x} + B_x v_2 + Bv_{2,x} + C_x v_3 + Cv_{3,x} \tag{424}$$

$$= A_x v_1 + A(id_1\zeta v_1 + n_{12}v_2 + n_{13}v_3) + B_x v_2 + B(id_2\zeta v_1 + n_{21}v_2 + n_{31}v_3) + C_x v_3$$
$$+ C(id_3\zeta v_1 + n_{31}v_2 + n_{32}v_3). \tag{425}$$

Hence, for $v_{1xt} = v_{1tx}$, the following equation is obtained:

$$id_1\zeta(Av_1 + Bv_2 + Cv_3) + n_{1,2}v_2 + n_{1,2}(Dv_1 + Ev_2 + Fv_3) \tag{426}$$

$$+ n_{1,3}v_3 + n_{1,3}(Gv_1 + Hv_2 + Iv_3) \tag{427}$$

$$= A_x v_1 + A(id_1\zeta v_1 + n_{12}v_2 + n_{13}v_3) + B_x v_2 + B(id_2\zeta v_1 + n_{21}v_2 + n_{31}v_3) + C_x v_3$$
$$+ C(id_3\zeta v_1 + n_{31}v_2 + n_{32}v_3). \tag{428}$$

Then, after collecting all terms with v_1, v_2, and v_3, we have

$$v_1(Dn_{12} + Gn_{13} - Bn_{21} - Cn_{31} - A_x) \tag{429}$$

$$+v_2(i(d_1-d_2)\zeta B-An_{12}+En_{12}+Hn_{13}-Cn_{32}+n_{12t}-B_x) \tag{430}$$

$$+v_3(i(d_{1x}-d_3)\zeta C+Fn_{12}-An_{13}+I_xn_{13}-Bn_{23}+n_{13t}-C_x)=0. \tag{431}$$

Setting the coefficients belonging to v_1, v_2, v_3 to zero yields for v_1,

$$v_1(Dn_{12}+Gn_{13}-Bn_{21}-Cn_{31}-A_x)=0, \tag{432}$$

$$Dn_{12}+Gn_{13}-Bn_{21}-Cn_{31}-A_x=0, \tag{433}$$

$$A_x=Dn_{12}+Gn_{13}-Bn_{21}-Cn_{31}, \tag{434}$$

for v_2,

$$v_2(i(d_1-d_2)\zeta B-An_{12}+En_{12}+Hn_{13}-Cn_{32}+n_{12t}-B_x)=0, \tag{435}$$

$$i(d_1-d_2)\zeta B-An_{12}+En_{12}+Hn_{13}-Cn_{32}+n_{12t}-B_x=0, \tag{436}$$

$$B_x=i(d_1-d_2)\zeta B-An_{12}+En_{12}+Hn_{13}-Cn_{32}+n_{12t}-B_x, \tag{437}$$

for v_3,

$$v_3(i(d_{1x}-d_3)\zeta C+Fn_{12}-An_{13}+I_xn_{13}-Bn_{23}+n_{13t}-C_x)=0, \tag{438}$$

$$i(d_{1x}-d_3)\zeta C+Fn_{12}-An_{13}+I_xn_{13}-Bn_{23}+n_{13t}-C_x=0, \tag{439}$$

$$C_x=i(d_{1x}-d_3)\zeta C+Fn_{12}-An_{13}+I_xn_{13}-Bn_{23}+n_{13t}. \tag{440}$$

Carrying the same procedure as above and finding $v_{2,xt}$ followed by $v_{2,tx}$ gives

$$v_{2,x}=id_2\zeta v_1+n_{2,1}v_2+n_{3,1}v_3, \tag{441}$$

$$v_{2,xt}=id_1\zeta v_{1,t}+n_{2,1}v_2+n_{2,1}v_{2,t}+n_{3,1}v_3+n_{3,1}v_{3,t}. \tag{442}$$

Replacing (408) results in (442) gives

$$v_{2,xt} = id_1\zeta v_{1,t} + n_{2,1}v_2 + n_{2,1}v_{2,t} + n_{3,1}v_3 + n_{3,1}v_{3,t}, \tag{443}$$

$$= id_1\zeta(Av_1 + Bv_2 + Cv_3) + n_{1,2}v_2 + n_{1,2}(Dv_1 + Ev_2 + Fv_3) \tag{444}$$

$$+ n_{1,3}v_3 + n_{1,3}(Gv_1 + Hv_2 + Iv_3). \tag{445}$$

Finding $v_{2,tx}$ yields

$$v_{2,t} = Dv_1 + Ev_2 + Fv_3 \tag{446}$$

$$v_{2,tx} = D_xv_1 + Dv_{1,x} + E_xv_2 + Ev_{2,x} + F_xv_3 + Fv_{3,x}. \tag{447}$$

Replacing the values of (405) and (406) and (407) into (447), we obtain

$$v_{2,tx} = D_xv_1 + Dv_{1,x} + E_xv_2 + Ev_{2,x} + F_xv_3 + Fv_{3,x} \tag{448}$$

$$= D_xv_1 + D(id_1\zeta v_1 + n_{12}v_2 + n_{13}v_3) + E_xv_2 + E(id_2\zeta v_1 + n_{21}v_2 + n_{31}v_3) + F_xv_3$$

$$+ F(id_3\zeta v_1 + n_{31}v_2 + n_{32}v_3), \tag{449}$$

From the expressions for each parameter above, the condition $v_{2,xt} = v_{2,tx}$, translates to

$$v_1(i(d_2 - d_1)\zeta D + An_{21} - En_{21} + Gn_{23} - Fn_{31} + n_{21t} - D_x) \tag{450}$$

$$+ v_2(-Dn_{12} + Bn_{21} + Hn_{23} - Fn_{32} - E_x) \tag{451}$$

$$+ v_3(i(d_2 - d_3)\zeta F - Dn_{13} + Cn_{21} - En_{23} + In_{23} + n_{23t} - F_x) = 0. \tag{452}$$

Taking the equation belonging to v_1, v_2, v_3 and equating it to zero yields, for v_1,

$$v_1(i(d_2 - d_1)\zeta D + An_{21} - En_{21} + Gn_{23} - Fn_{31} + n_{21t} - D_x) = 0, \tag{453}$$

$$i(d_2 - d_1)\zeta D + An_{21} - En_{21} + Gn_{23} - Fn_{31} + n_{21t} - D_x = 0, \tag{454}$$

$$D_x = i(d_2 - d_1)\zeta D + An_{21} - En_{21} + Gn_{23} - Fn_{31} + n_{21t}, \tag{455}$$

for v_2,

$$v_2(-Dn_{12} + Bn_{21} + Hn_{23} - Fn_{32} - E_x) = 0, \tag{456}$$

$$-Dn_{12} + Bn_{21} + Hn_{23} - Fn_{32} - E_x = 0, \tag{457}$$

$$E_x = -Dn_{12} + Bn_{21} + Hn_{23} - Fn_{32}, \tag{458}$$

and for v_3,

$$v_3(i(d_2 - d_3)\zeta F - Dn_{13} + Cn_{21} - En_{23} + In_{23} + n_{23t} - F_x) = 0, \tag{459}$$

$$i(d_2 - d_3)\zeta F - Dn_{13} + Cn_{21} - En_{23} + In_{23} + n_{23t} - F_x = 0, \tag{460}$$

$$F_x = i(d_2 - d_3)\zeta F - Dn_{13} + Cn_{21} - En_{23} + In_{23} + n_{23t}. \tag{461}$$

Carrying the same procedure as above and finding $v_{3,xt}$ followed by $v_{3,tx}$ gives

$$v_{3,x} = id_3\zeta v_1 + n_{3,1}v_2 + n_{3,2}v_3, \tag{462}$$

$$v_{3,xt} = id_3\zeta v_{1,t} + n_{3,1}v_2 + n_{3,1}v_{2,t} + n_{3,2}v_3 + n_{3,2}v_{3,t}. \tag{463}$$

Substituting (408) into (463) results in

$$v_{3,xt} = id_3\zeta v_{1,t} + n_{3,1}v_2 + n_{3,1}v_{2,t} + n_{3,2}v_3 + n_{3,2}v_{3,t}, \tag{464}$$

$$= id_3\zeta(Av_1 + Bv_2 + Cv_3) + n_{3,1}v_2 + n_{3,1}(Dv_1 + Ev_2 + Fv_3) \tag{465}$$

$$+ n_{3,2}v_3 + n_{3,2}(Gv_1 + Hv_2 + Iv_3). \tag{466}$$

Solving for $v_{3,tx}$ yields

$$v_{3,t} = Gv_1 + Hv_2 + Iv_3 \tag{467}$$

$$v_{3,tx} = G_x v_1 + Gv_{1,x} + H_x v_2 + Hv_{2,x} + I_x v_3 + Iv_{3,x}. \tag{468}$$

Substituting the values of (405) and (406) and (407) into (468), we obtain

$$v_{3,tx} = G_x v_1 + Gv_{1,x} + H_x v_2 + Hv_{2,x} + I_x v_3 + Iv_{3,x} \tag{469}$$

$$= G_x v_1 + G(id_1\zeta v_1 + n_{1,2}v_2 + n_{1,3}v_3) + H_x v_2 + H(id_2\zeta v_1 + n_{21}v_2 + n_{31}v_3) + I_x v_3$$
$$+ I(id_3\zeta v_1 + n_{31}v_2 + n_{32}v_3), \tag{470}$$

Then, from the above expressions, the condition $v_{3xt} = v_{3tx}$ translates to

$$v_1(i(d_3 - d_1)\zeta G - Hn_{21} + An_{31} - In_{31} + Dn_{32} + n_{31t} - G_x) \tag{471}$$

$$+ v_2(i(d_3 - d_2)\zeta H - Gn_{21} + Bn_{31} - En_{31} + In_{32} + n_{32t} - H_x) \tag{472}$$

$$+ v_3(-Gn_{13} - Hn_{23} + Cn_{31} + Fn_{32} - I_x) = 0. \tag{473}$$

Taking the equation belonging to v_1, v_2, v_3 and equating it to zero yields, for v_1,

$$v_1(i(d_3 - d_1)\zeta G - Hn_{2,1} + An_{31} - In_{3,1} + Dn_{3,2} + n_{3,1t} - G_x) = 0, \tag{474}$$

$$i(d_3 - d_1)\zeta G - Hn_{21} + An_{3,1} - In_{31} + Dn_{32} + n_{3,1t} - G_x = 0, \tag{475}$$

$$G_x = i(d_3 - d_1)\zeta G - Hn_{2,1} + An_{3,1} - In_{31} + Dn_{3,2} + n_{3,1t}, \tag{476}$$

for v_2,

$$v_2(i(d_3 - d_2)\zeta H - Gn_{2,1} + Bn_{3,1} - En_{3,1} + In_{3,2} + n_{3,2t} - H_x) = 0, \tag{477}$$

$$i(d_3 - d_2)\zeta H - Gn_{21} + Bn_{3,1} - En_{3,1} + In_{3,2} + n_{3,2t} - H_x = 0, \tag{478}$$

$$H_x = i(d_3 - d_2)\zeta H - Gn_{2,1} + Bn_{3,1} - En_{3,1} + In_{3,2} + n_{3,2t}, \tag{479}$$

and for v_3,

$$v_3(-Gn_{13} - Hn_{23} + Cn_{31} + Fn_{32} - I_x) = 0, \tag{480}$$

$$-Gn_{13} - Hn_{23} + Cn_{31} + Fn_{32} - I_x = 0, \tag{481}$$

$$I_x = -Gn_{13} - Hn_{23} + Cn_{31} + Fn_{32}. \tag{482}$$

The following equations are obtained for $A_x, B_x, C_x, D_x, E_x, F_x, G_x, H_x, I_x$.

$$A_x = Dn_{12} + Gn_{13} - Bn_{21} - Cn_{31}, \tag{483}$$

$$B_x = i(d_1 - d_2)\zeta B - An_{12} + En_{12} + Hn_{13} - Cn_{32} + n_{12t}, \tag{484}$$

$$C_x = i(d_{1x} - d_3)\zeta C + Fn_{12} - An_{13} + I_x n_{13} - Bn_{23} + n_{13t}, \tag{485}$$

$$D_x = i(d_2 - d_1)\zeta D + An_{21} - En_{21} + Gn_{23} - Fn_{31} + n_{21t}, \tag{486}$$

$$E_x = -Dn_{12} + Bn_{21} + Hn_{23} - Fn_{32}, \tag{487}$$

$$F_x = i(d_2 - d_3)\zeta F - Dn_{13} + Cn_{21} - En_{23} + In_{23} + n_{23t}, \tag{488}$$

$$G_x = i(d_3 - d_1)\zeta G - Hn_{2,1} + An_{3,1} - In_{31} + Dn_{3,2} + n_{3,1t}, \tag{489}$$

$$H_x = i(d_3 - d_2)\zeta H - Gn_{2,1} + Bn_{3,1} - En_{3,1} + In_{3,2} + n_{3,2t}, \tag{490}$$

$$I_x = -Gn_{13} - Hn_{23} + Cn_{31} + Fn_{32}. \tag{491}$$

The parameters A, B, C, D, E, G, H, I satisfy the compatibility conditions of $A_x, B_x, C_x, D_x, E_x, F_x, G_x, H_x, I_x$. Now, A, B, C, D, E, G, H, I are found by assuming that they are finite power series in ζ, given by

$$A = \sum_{j=0}^{n} A_j \zeta^j, \tag{492}$$

$$B = \sum_{j=0}^{n} B_j \zeta^j, \tag{493}$$

$$C = \sum_{j=0}^{n} C_j \zeta^j, \tag{494}$$

$$D = \sum_{j=0}^{n} D_j \zeta^j, \tag{495}$$

$$E = \sum_{j=0}^{n} E_j \zeta^j, \tag{496}$$

$$F = \sum_{j=0}^{n} F_j \zeta^j, \tag{497}$$

$$G = \sum_{j=0}^{n} G_j \zeta^j, \tag{498}$$

$$H = \sum_{j=0}^{n} H_j \zeta^j, \tag{499}$$

$$I = \sum_{j=0}^{n} I_j \zeta^j, \tag{500}$$

Letting $j = 1$, we obtain the following polynomials for A, B, C, D, E, G, H, I,

$$A = A_1 \zeta + A_0, \tag{501}$$

$$B = B_1 \zeta + B_0, \tag{502}$$

$$C = C_1 \zeta + C_0, \tag{503}$$

$$D = D_1 \zeta + D_0, \tag{504}$$

$$E = E_1 \zeta + E_0, \tag{505}$$

$$F = F_1 \zeta + F_0, \tag{506}$$

$$G = G_1 \zeta + G_0, \tag{507}$$

$$H = H_1 \zeta + H_0, \tag{508}$$

$$I = I_1 \zeta + I_0. \tag{509}$$

Proceeding as before, the values of (492)–(500)) are substituted into (483)-(491), and then the coefficients are found. After substituting the corresponding values, we get

$$v_1(D_0n_{12}+G_0n_{13}-B_0n_{21}-C_0n_{31}-A_{0x}+\zeta(D_1n_{12}+G_1n_{13} \tag{510}$$

$$-B_1n_{21}-C_1n_{31}-A_{1x}) \tag{511}$$

$$+v_2(i(d_1-d_2)\zeta^2B_1-A_0n_{12}+E_0n_{12}+H_0n_{13}-C_0n_{32}+n_{12t}-B_{0x} \tag{512}$$

$$+\zeta(i(d_1-d_2)B_0 \tag{513}$$

$$-A_1n_{12}+E_1n_{12}+H_1n_{13}-C_1n_{32}-B_{1x})) \tag{514}$$

$$+v_3(i(d_1-d_3\zeta^2C_1+F_0n_{12}-A_0n_{13}+I_0n_{13}-B_0n_{23}+n_{13t}-C_{0x} \tag{515}$$

$$+\zeta(i(d_1-d_3)C_0+F_1n_{12} \tag{516}$$

$$-A_1n_{13}+I_1n_{13}-B_1n_{23}-C_{1x}))=0. \tag{517}$$

To find the coefficients, one side of the equation is considered, and each v_1, v_2, v_3 part of the equation is set to zero. This is demonstrated for v_1 as

$$v_1(D_0n_{12}+G_0n_{13}-B_0n_{21}-C_0n_{31}-A_{0x}+\zeta(D_1n_{12}+G_1n_{13}-B_1n_{21}-C_1n_{31}-A_{1x})=0, \tag{518}$$

$$D_0n_{12}+G_0n_{13}-B_0n_{21}-C_0n_{31}-A_{0x}+\zeta(D_1n_{12}+G_1n_{13}-B_1n_{21}-C_1n_{31}-A_{1x})=0. \tag{519}$$

This shows that $B_1=0, B_{1,x}=0, D_1=0, G_1=0, C_1=0$, and

$$A_{1x}=0, \tag{520}$$

$$A_1=a_1. \tag{521}$$

A value for A_{0x} is also obtained

$$A_{0x}=D_0n_{12}+G_0n_{13}-B_0n_{21}-C_0n_{31}. \tag{522}$$

For v_2, we obtain

$$+v_2(i(d_1-d_2)\zeta^2 B_1 - A_0 n_{12} + E_0 n_{12} + H_0 n_{13} - C_0 n_{32} + n_{12t} - B_{0x} + \zeta(i(d_1-d_2)B_0 - A_1 n_{12} + E_1 n_{12} + H_1 n_{13} - C_1 n_{32} - B_{1x})) = 0, \quad (523)$$

$$i(d_1-d_2)\zeta^2 B_1 - A_0 n_{12} + E_0 n_{12} + H_0 n_{13} - C_0 n_{32} + n_{12t} - B_{0x} + \zeta(i(d_1-d_2)B_0 - A_1 n_{12} + E_1 n_{12} + H_1 n_{13} - C_1 n_{32} - B_{1x}) = 0. \quad (524)$$

This shows that $B_1 = 0$ and B_{0x} is obtained as

$$B_{0x} = -A_0 n_{12} + E_0 n_{12} + H_0 n_{13} - C_0 n_{32} + n_{12t}, \quad (525)$$

and

$$\zeta(i(d_1-d_2)B_0 - A_1 n_{12} + E_1 n_{12} + H_1 n_{13} - C_1 n_{32} - B_{1x}) = 0, \quad (526)$$

$$\zeta(i(d_1-d_2)B_0 - a_1 n_{12} + E_1 n_{12} + H_1 n_{13} - B_{1x}) = 0, \quad (527)$$

$$i(d_1-d_2)B_0 = a_1 n_{12} - E_1 n_{12} - H_1 n_{13} + B_{1x}, \quad (528)$$

$$B_0 = \frac{a_1 n_{12} - E_1 n_{12} - H_1 n_{13}}{i(d_1-d_2)}. \quad (529)$$

For v_3, we obtain

$$v_3(i(d_1 - d_3\zeta^2 C_1 + F_0 n_{12} - A_0 n_{13} + I_0 n_{13} - B_0 n_{23} + n_{13t} - C_{0x} + \zeta(i(d_1-d_3)C_0 + F_1 n_{12} - A_1 n_{13} + I_1 n_{13} - B_1 n_{23} - C_{1x})) = 0, \quad (530)$$

$$i(d_1 - d_3\zeta^2 C_1 + F_0 n_{12} - A_0 n_{13} + I_0 n_{13} - B_0 n_{23} + n_{13t} - C_{0x} + \zeta(i(d_1-d_3)C_0 + F_1 n_{12} - A_1 n_{13} + I_1 n_{13} - B_1 n_{23} - C_{1x}) = 0. \quad (531)$$

This shows that $C_1 = 0$,

$$i(d_1 - d_3\zeta^2 C_1 + F_0 n_{12} - A_0 n_{13} + I_0 n_{13} - B_0 n_{23} + n_{13t} - C_{0x}) = 0, \tag{532}$$

$$d_1 - d_3\zeta^2 C_1 + F_0 n_{12} - A_0 n_{13} + I_0 n_{13} - B_0 n_{23} + n_{13t} - C_{0x} = 0, \tag{533}$$

$$C_{0x} = d_1 + F_0 n_{12} - A_0 n_{13} + I_0 n_{13} - B_0 n_{23} + n_{13t}. \tag{534}$$

and

$$\zeta(i(d_1 - d_3)C_0 + F_1 n_{12} - A_1 n_{13} + I_1 n_{13} - B_1 n_{23} - C_{1x}) = 0,$$

$$i(d_1 - d_3)C_0 + F_1 n_{12} - A_1 n_{13} + I_1 n_{13} - B_1 n_{23} - C_{1x} = 0,$$

$$C_{1x} = i(d_1 - d_3)C_0 + F_1 n_{12} - A_1 n_{13} + I_1 n_{13} - B_1 n_{23}.$$

Inserting the corresponding values into (484) gives

$$\begin{aligned}
&v_1(i(-d_1 + d_2)\zeta^2 D_1 + A_0 n_{21} - E_0 n_{21} + G_0 n_{23} - F_0 n_{31} + n_{21t} - D_{0x} + \zeta(i(-d_1 + d_2)D_0 \\
&+ A_1 n_{21} - E_1 n_{21} + G_1 n_{23} - F_1 n_{31} - D_{1x})), \\
&+ v_2(-D_0 n_{12} + B_0 n_{21} + H_0 n_{23} - F_0 n_{32} - E_{0x} + \zeta(-D_1 n_{12} + B_1 n_{21} + H_1 n_{23} - F_1 n_{32} - E_{1x})) \\
&+ v_3(i(d_2 - d_3)\zeta^2 F_1 - D_0 n_{13} + C_0 n_{21} - E_0 n_{23} + I_0 n_{23} + n_{23t} - F_{0x} + \zeta(i(d_2 - d_3)F_0 \\
&- D_1 n_{13} + C_1 n_{21} - E_1 n_{23} + I_1 n_{23} - F_{1x})) = 0.
\end{aligned} \tag{535}$$

For v_1, we get

$$\begin{aligned}
&v_1(i(-d_1 + d_2)\zeta^2 D_1 + A_0 n_{21} - E_0 n_{21} + G_0 n_{23} - F_0 n_{31} + n_{21t} - D_{0x} + \zeta(i(-d_1 + d_2)D_0 \\
&+ A_1 n_{21} - E_1 n_{21} + G_1 n_{23} - F_1 n_{31} - D_{1x})) = 0, \\
&i(-d_1 + d_2)\zeta^2 D_1 + A_0 n_{21} - E_0 n_{21} + G_0 n_{23} - F_0 n_{31} + n_{21t} - D_{0x} + \zeta(i(-d_1 + d_2)D_0
\end{aligned} \tag{536}$$

$$+A_1 n_{21} - E_1 n_{21} + G_1 n_{23} - F_1 n_{31} - D_{1x}) = 0.$$

This tells us that $D_1 = 0$,

$$D_{0x} = -d_1 + d_2)\zeta^2 D_1 + A_0 n_{21} - E_0 n_{21} + G_0 n_{23} - F_0 n_{31} + n_{21t} = 0, \tag{537}$$

and

$$\zeta(i(-d_1 + d_2)D_0 + A_1 n_{21} - E_1 n_{21} + G_1 n_{23} - F_1 n_{31} - D_{1x}) = 0, \tag{538}$$

$$D_{1x} = \zeta(i(-d_1 + d_2)D_0 + A_1 n_{21} - E_1 n_{21} + G_1 n_{23} - F_1 n_{31}, \tag{539}$$

$$D_{1x} = \zeta(i(-d_1 + d_2)D_0 + a_1 n_{21} - E_1 n_{21} - F_1 n_{31}. \tag{540}$$

For v_2, we obtain

$$v_2(-D_0 n_{12} + B_0 n_{21} + H_0 n_{23} - F_0 n_{32} - E_{0x}$$
$$+ \zeta(-D_1 n_{12} + B_1 n_{21} + H_1 n_{23} - F_1 n_{32} - E_{1x})) = 0, \tag{541}$$
$$-D_0 n_{12} + B_0 n_{21} + H_0 n_{23} - F_0 n_{32} - E_{0x}$$
$$+ \zeta(-D_1 n_{12} + B_1 n_{21} + H_1 n_{23} - F_1 n_{32} - E_{1x}) = 0. \tag{542}$$

This demonstrates that $B_1 = D_1 = H_1 = F_1 = 0$, which gives, E_1 and E_{0x},

$$\zeta(-D_1 n_{12} + B_1 n_{21} + H_1 n_{23} - F_1 n_{32} - E_{1x}) = 0, \tag{543}$$
$$-E_{1x} = 0, \tag{544}$$
$$E_1 = e_1, \tag{545}$$

$$-D_0 n_{12} + B_0 n_{21} + H_0 n_{23} - F_0 n_{32} - E_{0x} = 0, \tag{546}$$

$$E_{0x} = -D_0 n_{12} + B_0 n_{21} + H_0 n_{23} - F_0 n_{32}. \tag{547}$$

For v_3, we get

$$+v_3(i(d_2-d_3)\zeta^2 F_1 - D_0 n_{13} + C_0 n_{21} - E_0 n_{23} + I_0 n_{23} + n_{23t} - F_{0x} + \zeta(i(d_2-d_3)F_0$$
$$-D_1 n_{13} + C_1 n_{21} - E_1 n_{23} + I_1 n_{23} - F_{1x})) = 0, \tag{548}$$
$$i(d_2-d_3)\zeta^2 F_1 - D_0 n_{13} + C_0 n_{21} - E_0 n_{23} + I_0 n_{23} + n_{23t} - F_{0x} + \zeta(i(d_2-d_3)F_0$$
$$-D_1 n_{13} + C_1 n_{21} - E_1 n_{23} + I_1 n_{23} - F_{1x}) = 0. \tag{549}$$

This shows that $F_1 = 0$, and gives a value for F_{0x} and F_{1x}. For F_{0x}, the value is

$$F_{0x} = i(d_2-d_3)\zeta^2 F_1 - D_0 n_{13} + C_0 n_{21} - E_0 n_{23} + I_0 n_{23} + n_{23t}, \tag{550}$$

and

$$\zeta(i(d_2-d_3)F_0 - D_1 n_{13} + C_1 n_{21} - E_1 n_{23} + I_1 n_{23} - F_{1x}) = 0, \tag{551}$$
$$i(d_2-d_3)F_0 - D_1 n_{13} + C_1 n_{21} - E_1 n_{23} + I_1 n_{23} - F_{1x} = 0, \tag{552}$$
$$F_{1x} = i(d_2-d_3)F_0 - D_1 n_{13} + C_1 n_{21} - E_1 n_{23} + I_1 n_{23}. \tag{553}$$

Inserting corresponding values into (485) gives

$$v_1(i(-d_1+d_3)\zeta^2 G_1 - H_0 n_{21} + A_0 n_{31} - I_0 n_{31} + D_0 n_{32} + n_{31x} - G_{0x}$$
$$+\zeta(i(-d_1+d_3)G_0 - H_1 n_{21} + A_1 n_{31} - I_1 n_{31} + D_1 n_{32} - G_{1x})),$$
$$+v_2(i(-d_2+d_3)\zeta^2 H_1 - G_0 n_{12} + B_0 n_{31} + E_0 n_{32} - I_0 n_{32}$$
$$+n_{32t} - H_{0x} + \zeta(i(-d_2+d_3)H_0 - G_1 n_{12} + B_1 n_{31} + E_1 n_{32} - I_1 n_{32} - H_{1x})),$$
$$+v_3(-G_0 n_{13} - H_0 n_{23} + C_0 n_{31} + F_0 n_{32} - I_{0x}$$

$$+\zeta(-G_1n_{13}-H_1n_{23}+C_1n_{31}+F_1n_{32}-I_{1x}))=0.$$

Now consider the equation corresponding to v_1:

$$v_1(i(-d_1+d_3)\zeta^2G_1-H_0n_{21}+A_0n_{31}-I_0n_{31}+D_0n_{32}+n_{31x}-G_{0x}$$
$$+\zeta(i(-d_1+d_3)G_0-H_1n_{21}+A_1n_{31}-I_1n_{31}+D_1n_{32}-G_{1x}))=0 \quad (554)$$
$$ci(-d_1+d_3)\zeta^2G_1-H_0n_{21}+A_0n_{31}-I_0n_{31}+D_0n_{32}+n_{31x}$$
$$-G_{0x}+\zeta(i(-d_1+d_3)G_0-H_1n_{21}+A_1n_{31}-I_1n_{31}+D_1n_{32}-G_{1x})=0. \quad (555)$$

This shows that $G_1=H_1=0$. A value for G_{0x} and G_{1x} is found as

$$i(-d_1+d_3)\zeta^2G_1-H_0n_{21}+A_0n_{31}-I_0n_{31}+D_0n_{32}+n_{31x}-G_{0x}=0, \quad (556)$$
$$-H_0n_{21}+A_0n_{31}-I_0n_{31}+D_0n_{32}+n_{31x}-G_{0x}=0, \quad (557)$$
$$G_{0x}=-H_0n_{21}+A_0n_{31}-I_0n_{31}+D_0n_{32}+n_{31x}, \quad (558)$$

and

$$\zeta(i(-d_1+d_3)G_0-H_1n_{21}+A_1n_{31}-I_1n_{31}+D_1n_{32}-G_{1x})=0, \quad (559)$$
$$i(-d_1+d_3)G_0-H_1n_{21}+A_1n_{31}-I_1n_{31}+D_1n_{32}-G_{1x}=0, \quad (560)$$
$$G_{1x}=i(-d_1+d_3)G_0-H_1n_{21}+A_1n_{31}-I_1n_{31}+D_1n_{32}. \quad (561)$$

For v_2, we get

$$+v_2(i(-d_2+d_3)\zeta^2H_1-G_0n_{12}+B_0n_{31}+E_0n_{32}-I_0n_{32}+n_{32t}-H_{0x}+\zeta(i(-d_2+d_3)H_0$$
$$-G_1n_{12}+B_1n_{31}+E_1n_{32}-I_1n_{32}-H_{1x})), \quad (562)$$
$$i(-d_2+d_3)\zeta^2H_1-G_0n_{12}+B_0n_{31}+E_0n_{32}-I_0n_{32}+n_{32t}-H_{0x}+\zeta(i(-d_2+d_3)H_0$$

$$-G_1n_{12}+B_1n_{31}+E_1n_{32}-I_1n_{32}-H_{1x}). \tag{563}$$

This provides a value for $H_1=0$, H_{0x}, and H_{1x}.

$$i(-d_2+d_3)\zeta^2H_1-G_0n_{12}+B_0n_{31}+E_0n_{32}-I_0n_{32}+n_{32t}-H_{0x}=0, \tag{564}$$

$$H_{0x}=i(-d_2+d_3)\zeta^2H_1-G_0n_{12}+B_0n_{31}+E_0n_{32}-I_0n_{32}+n_{32t}, \tag{565}$$

and,

$$\zeta(i(-d_2+d_3)H_0-G_1n_{12}+B_1n_{31}+E_1n_{32}-I_1n_{32}-H_{1x})=0, \tag{566}$$

$$i(-d_2+d_3)H_0-G_1n_{12}+B_1n_{31}+E_1n_{32}-I_1n_{32}-H_{1x}=0, \tag{567}$$

$$H_{1x}=i(-d_2+d_3)H_0-G_1n_{12}+B_1n_{31}+E_1n_{32}-I_1n_{32}. \tag{568}$$

For v_3, we obtain

$$v_3(-G_0n_{13}-H_0n_{23}+C_0n_{31}+F_0n_{32}-I_{0x}+\zeta(-G_1n_{13}-H_1n_{23}+C_1n_{31}+F_1n_{32}-I_{1x}))=0, \tag{569}$$

$$-G_0n_{13}-H_0n_{23}+C_0n_{31}+F_0n_{32}-I_{0x}+\zeta(-G_1n_{13}-H_1n_{23}+C_1n_{31}+F_1n_{32}-I_{1x})=0. \tag{570}$$

In this case, $G_1=H_1=C_1=F_1=0$. As well, I_x is found and I_{0x}. This is,

$$\zeta(-G_1n_{13}-H_1n_{23}+C_1n_{31}+F_1n_{32}-I_{1x})=0, \tag{571}$$

$$-G_1n_{13}-H_1n_{23}+C_1n_{31}+F_1n_{32}-I_{1x}=0, \tag{572}$$

$$I_{1x}=0, \tag{573}$$

$$I_1=i_1, \tag{574}$$

and,

$$-G_0n_{13}-H_0n_{23}+C_0n_{31}+F_0n_{32}-I_{0x}, \tag{575}$$

$$I_{0x}=-G_0n_{13}-H_0n_{23}+C_0n_{31}+F_0n_{32}. \tag{576}$$

Then the coefficients of $A_1,A_0,B_1,B_0,C_1,C_0,D_1,D_0,E_1,E_0,F_1,F_0,G_1,G_0,H_1,H_0,I_1$, and I_0 are found. The following are the values found.

$$B_1=0 \tag{577}$$

$$C_1=0 \tag{578}$$

$$D_1=0 \tag{579}$$

$$F_1=0 \tag{580}$$

$$G_1=0 \tag{581}$$

$$H_1=0 \tag{582}$$

$$A_1=a_1 \tag{583}$$

$$E_1=e_1 \tag{584}$$

$$I_1=i_1 \tag{585}$$

$$B_0=\frac{(a_1-e_1)n_{12}}{I(d_1-d_2)} \tag{586}$$

$$C_0=\frac{(a_1-i_1)n_{13}}{I(d_1-d_3)} \tag{587}$$

$$D_0=\frac{(a_1-e_1)n_{21}}{I(d_1-d_2)} \tag{588}$$

$$F_0=\frac{(e_1-i_1)n_{23}}{I(d_2-d_3)} \tag{589}$$

$$G_0=\frac{(a_1-i_1)n_{31}}{I(d_1-d_3)} \tag{590}$$

$$H_0=\frac{(e_1-i_1)n_{32}}{I(d_2-d_3)} \tag{591}$$

$$A_0 = a_0 \tag{592}$$

$$E_0 = e_0 \tag{593}$$

$$I_0 = i_0 \tag{594}$$

The equations for $A, B, C, D, E, F, G, H, I$ are

$$A = a_1\zeta + a_0, \tag{595}$$

$$B = \frac{(a_1 - e_1)n_{12}}{I(d_1 - d_2)}, \tag{596}$$

$$C = \frac{(a_1 - i_1)n_{13}}{I(d_1 - d_3)}, \tag{597}$$

$$D = \frac{(a_1 - e_1)n_{21}}{I(d_1 - d_2)}, \tag{598}$$

$$E = e_1\zeta + e_0, \tag{599}$$

$$F = \frac{(e_1 - i_1)n_{23}}{I(d_2 - d_3)}, \tag{600}$$

$$G = \frac{(a_1 - i_1)n_{31}}{I(d_1 - d_3)}, \tag{601}$$

$$H = \frac{(e_1 - i_1)n_{32}}{I(d_2 - d_3)}, \tag{602}$$

$$I = i_1\zeta + i_0. \tag{603}$$

The evolution equations are found from the coefficients of A_0, E_0, I_0.

CHAPTER VI

Applications of Lax pairs to a 3×3 discrete case

Consider the equations,

$$v_{1,n+1} = z_1(\zeta + \frac{1}{\zeta})v_{1,n} + q_{1,2}v_{2,n} + n_{1,2}v_{3,n}, \tag{604}$$

$$v_{2,n+1} = z_2(\zeta + \frac{1}{\zeta})v_{2,n} + n_{2,1}v_{1,n} + n_{2,3}v_{3,n}, \tag{605}$$

$$v_{3,n+1} = n_{3,1}v_{1,n} + n_{3,2}v_{2,n} + z_3(\zeta + \frac{1}{\zeta})v_{3,n}, \tag{606}$$

associated with the time evolution equations,

$$v_{1n,t} = A_n v_{1,n} + B_n v_{2,n} + C_n v_{3,n}, \tag{607}$$

$$v_{2n,t} = D_n v_{1,n} + E_n v_{2,n} + F_n v_{3,n}, \tag{608}$$

$$v_{3n,t} = G_n v_{1,n} + H_n v_{2,n} + I_n v_{3,n}. \tag{609}$$

Consider,

$$\frac{\partial}{\partial t}(Ev_{i,n}) = E(\frac{\partial v_{i,n}}{\partial t}), \qquad i = 1,2,3 \tag{610}$$

where E is the shift operator $E(v_{i,n}) \equiv v_{i,n+1}$. This process is intiated similarly to the one in Chapter 4. The operator is applied to $v_{1,n}$ first. Thus,

$$Ev_{1,n} = z_1(\zeta + \frac{1}{\zeta})v_{1,n-1} + q_{1,2}v_{2,n-1} + n_{1,2}v_{3,n-1}, \tag{611}$$

$$
\begin{aligned}
(Ev_{1,n})_t &= (z_1(\zeta+\frac{1}{\zeta})v_{1,n-1}+q_{1,2}v_{2,n-1}+n_{1,2}v_{3,n-1})_t, \\
&= z_1(\zeta+\frac{1}{\zeta})\frac{\partial}{\partial t}(v_{1,n-1})+\frac{\partial}{\partial t}(q_{1,2})v_{2,n-1}+q_{1,2}\frac{\partial}{\partial t}(v_{2,n-1}) \\
&+\frac{\partial}{\partial t}(n_{1,2})v_{3,n-1}+n_{1,2}\frac{\partial}{\partial t}(v_{3,n-1}), \\
&= z_1(\zeta+\frac{1}{\zeta})(A_n v_{1,n}+B_{n-1}v_{2,n}+C_{n-1}v_{3,n})+q_{12,_{n,t}}v_{2,n-1}+q_{1,2_{n,t}}(D_n v_{1,n}+E_n v_{2,n}+F_n v_{3,n}), \\
&+n_{1,2_{n,t}}v_{1,n-1}+n_{1,2_{n,t}}(G_n v_{1,n}+H_n v_{2,n}+I_n v_{3,n})
\end{aligned} \tag{612, 613}
$$

$$
= v_{1,n}(B_{n+1}n_{2,1}+C_{n+1}n_{3,1})+v_{2,n}(A_{n+1}n_{1,2}+C_{n+1}n_{3,2})+v_{3,n}(A_{n+1}n_{1,3}B_{n+1}n_{2,3}) \tag{614}
$$

Now, apply the operator

$$
\frac{\partial}{\partial t}v_{1,n}=A_n v_{1,n}+B_n v_{2,n}+C_n v_{3,n}, \tag{615}
$$

$$
\begin{aligned}
E(\frac{\partial}{\partial t}) &= A_{1,n+1}v_{1,n+1}+B_{n+1}v_{2,n+1}+C_{n+1}v_{3,n+1}, \\
&= A_{n+1}(z_1(\zeta+\frac{1}{\zeta})v_{1,n}+q_{1,2}v_{2,n}+n_{1,2}v_{3,n})+B_{n+1}(z_2(\zeta+\frac{1}{\zeta}v_{2,n}+n_{2,1}v_{2,n}) \\
&+C_{n+1}(n_{3,1}v_n+n_{3,2}v_{2,n}+z_3(\zeta+\frac{1}{\zeta})v_{3,n}) \\
&= v_{1,n}(z_1(\zeta+\frac{1}{\zeta})A_n+z_1(-\zeta-\frac{1}{\zeta})A_{n+1,t}+D_n n_{1,2}+G_n n_{1,3}) \\
&+v_{2,n}(z_1(\zeta+\frac{1}{\zeta})+z_2(\zeta+\frac{1}{\zeta})B_{n+1}+E_n n_{1,2}+H_n n_{1,3}+n_{1,2_t}) \\
&+v_{3,n}(z_1(\zeta+\frac{1}{\zeta})C_n+z_3(-\zeta-\frac{1}{\zeta})C_{n+1}+F_{n_{1,2}}+I_n n_{1,3}+n_{1,3_t}).
\end{aligned} \tag{616}
$$

Doing the same for $v_{2,n}$, it is found that

$$
\begin{aligned}
(Ev_{2,n})_t &= (z_2(\zeta+\frac{1}{\zeta})v_{2,n}+n_{2,1}v_{1,n}+n_{2,3}v_{3,n})_t \\
&= z_2(\zeta+\frac{1}{\zeta})\frac{\partial}{\partial t}(v_{2,n})+n_{2,1}\frac{\partial}{\partial t}(v_{1,n})+v_{1,n}\frac{\partial}{\partial t}(n_{2,1})
\end{aligned} \tag{617}
$$

$$+n_{2,3}\frac{\partial}{\partial t}(v_{3,n})+v_{3,n}\frac{\partial}{\partial t}(n_{2,3})$$
$$=z_2(\zeta+\frac{1}{\zeta})(D_nv_{1,n}+E_nv_{2,n}+F_nv_{3,n})+n_{2,1}(A_nv_{1,n}+B_nv_{2,n}+C_nv_{3,n})+v_{1,n}n_{2,1_t}$$
$$+n_{2,3}(G_nv_{1,n}+H_nv_{2,n}+I_nv_{3,n})+v_{3,n}n_{2,3_t}. \tag{618}$$

Apply the operator to

$$\begin{aligned}E(\frac{\partial}{\partial t}v_{2,n}) &= D_{n+1}+E_{n+1}v_{2,n+1}+F_{n+1}v_{3,n+1}\\ &= D_{n+1}(z_1(\zeta+\frac{1}{\zeta})v_{1,n}+q_{1,2}v_{2,n}+n_{1,2}v_{3,n})\\ &+E_{n+1}(z_2(\zeta+\frac{1}{\zeta})v_{2,n}+n_{2,1}v_{1,n}+n_{2,3}v_{3,n})\\ &+F_{n+1}(n_{3,1}v_{1,n}+n_{3,2}v_{2,n}+z_3(\zeta+\frac{1}{\zeta})v_{3,n})\end{aligned} \tag{619}$$

Performing the corresponding calculations for $v_{3,n}$ gives

$$\begin{aligned}(Ev_{3,n})_t &= (n_{3,1}v_{1,n}+n_{3,2}v_{2,n}+z_3(\zeta+\frac{1}{\zeta})v_{3,n})_t\\ &= n_{3,1}\frac{\partial}{\partial t}(v_{1,n})+v_{1,n}\frac{\partial}{\partial t}(n_{3,1})\\ &+n_{3,2}\frac{\partial}{\partial t}(v_{2,n})+v_{2,n}\frac{\partial}{\partial t}(n_{3,2})\\ &+\frac{\partial}{\partial t}(v_{3,n})\\ &n_{3,1}(A_nv_{1,n}+B_nv_{2,n}+C_nv_{3,n})+v_{1,n}n_{3,1_t}\\ &+n_{3,2}(D_nv_{1,n}+E_nv_{2,n}+F_nv_{3,n})+v_{2,n}n_{3,2_t}\\ &+(\zeta+\frac{1}{\zeta})(G_nv_{1,n}+H_nv_{2,n}+I_nv_{3,n})\end{aligned} \tag{620}$$

$$=v_1(z_3(\frac{1}{\zeta}+\zeta)G_n+A_nn_{3,1}+D_nn_{3,2}+n_{3,1_t}) \tag{621}$$
$$+v_2(z_3(\frac{1}{\zeta})H_n+B_nn_{3,1}+E_nn_{3,2}+n_{3,2_t}) \tag{622}$$

$$+v_3(z_3(\zeta+\frac{1}{\zeta})I_n+C_n n_{3,1}+F_n n_{3,2}). \tag{623}$$

Apply the operator to $E(\frac{\partial}{\partial t}v_{1,n})=G_{n+1}v_{n+1}+H_{n+1}v_{n+2}+In+1v_{n+1}$.

$$\begin{aligned}E(\frac{\partial}{\partial t}v_{2,n}) &= G_{n+1}v_{n+1}+H_{n+1}v_{n+2}+In+1v_{n+1}\\ &G_{n+1}(z_1(\zeta+\frac{1}{\zeta})v_{1,n}+q_{1,2}v_{2,n}+n_{1,2}v_{3,n})\\ &+H_{n+1}(z_2(\zeta+\frac{1}{\zeta})v_{2,n}+n_{2,1}v_{1,n}+n_{2,3}v_{3,n})\\ &+I_{n+1}(n_{3,1}v_{1,n}+n_{3,2}v_{2,n}+z_3(\zeta+\frac{1}{\zeta})v_{3,n})\\ &=v_{1,n}(z_1(\zeta+\frac{1}{\zeta})G_{n+1}+H_{n_{n+1}}+I_{n+1}n_{3,1})\\ &+v_{2,n}(z_2(\zeta+\frac{1}{\zeta})H_{n+1}+G_{n+1}n_{1,3}+H_{n+1}n_{2,3})\\ &+v_{3,n}(z_3(\zeta+\frac{1}{\zeta})+G_{n+1}n_{1,3}+H_{n+1}n_{2,3})\end{aligned} \tag{624}$$

Considering $(Ev_{3,n})_t=E(\frac{\partial}{\partial t}v_{2,n})$ gives

$$\begin{aligned}&(Ev_{3,n})_t=E(\frac{\partial}{\partial t}v_{3,n})\\ &v_1(z_3(\frac{1}{\zeta}+\zeta)G_n+A_n n_{3,1}+D_n n_{3,2}+n_{3,1_t})\\ &+v_2(z_3(\frac{1}{\zeta})H_n+B_n n_{3,1}+E_n n_{3,2}+n_{3,2_t})\\ &+v_3(z_3(\zeta+\frac{1}{\zeta})I_n+C_n n_{3,1}+F_n n_{3,2})=v_{1,n}(z_1(\zeta+\frac{1}{}\zeta)G_{n+1}+H_{n_{n+1}}+I_{n+1}n_{3,1})\\ &+v_{2,n}(z_2(\zeta+\frac{1}{\zeta})H_{n+1}+G_{n+1}n_{1,3}+H_{n+1}n_{2,3})\\ &+v_{3,n}(z_3(\zeta+\frac{1}{\zeta})+G_{n+1}n_{1,3}+H_{n+1}n_{2,3})\end{aligned} \tag{625}$$

This gives the equations,

$$(z_3 - z_1)(\zeta + \frac{1}{\zeta})\Delta G_n = A_n n_{3,1} + D_n n_{3,2} + n_{3,1_t} + H_{n_{n+1}} + I_{n+1} n_{3,1} \tag{626}$$

$$(z_2 - z_3)(\frac{1}{\zeta})\Delta H_n = B_n n_{3,1} + E_n n_{3,2} + n_{3,2_t}) + G_{n+1} n_{1,3} + H_{n+1} n_{2,3} \tag{627}$$

$$z_3(\zeta + \frac{1}{\zeta})\Delta I_n = C_n n_{3,1} + F_n n_{3,2}) + G_{n+1} n_{1,3} + H_{n+1} n_{2,3}. \tag{628}$$

If $A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n, I_n$ are expanded, the following is assumed

$$A_n = A_1^{(1)}\zeta + A_0^{(0)} + \frac{1}{\zeta}A_1^{(-1)}, \tag{629}$$

$$B_n = B_1^{(1)}\zeta + B_0^{(0)} + \frac{1}{\zeta}B_1^{(-1)}, \tag{630}$$

$$C_n = C_1^{(1)}\zeta + C_0^{(0)} + \frac{1}{\zeta}C_1^{(-1)}, \tag{631}$$

$$D_n = D_1^{(1)}\zeta + D_0^{(0)} + \frac{1}{\zeta}D_1^{(-1)}, \tag{632}$$

$$E_n = E_1^{(1)}\zeta + E_0^{(0)} + \frac{1}{\zeta}E_1^{(-1)}, \tag{633}$$

$$F_n = F_1^{(1)}\zeta + F_0^{(0)} + \frac{1}{\zeta}F_1^{(-1)}, \tag{634}$$

$$G_n = G_1^{(1)}\zeta + G_0^{(0)} + \frac{1}{\zeta}G_1^{(-1)}, \tag{635}$$

$$H_n = H_1^{(1)}\zeta + H_0^{(0)} + \frac{1}{\zeta}H_1^{(-1)}, \tag{636}$$

$$I_n = I_1^{(1)}\zeta + I_0^{(0)} + \frac{1}{\zeta}I_1^{(-1)}. \tag{637}$$

Since this is a tedious work by hand, Mathematica was used to obtain the coefficients of A_0, B_0, C_0, D_0, E_0, F_0, G_0, H_0, I_0, A_1, B_1, C_1, D_1, E_1, F_1, $G_1, H_1, I_1, A_0^{(-1)}$, $B_0^{(-1)}, C_0^{(-1)}, D_0^{(-1)}$, $E_0^{(-1)}, F_0^{(-1)}$, $G_0^{(-1)}, H_0^{(-1)}, I_0^{(-1)}$. For $A_1^{(1)}$, the value was found to be

$$A_1^{(1)} = a_1, \tag{638}$$

$$B_1^{(1)} = b_1, \tag{639}$$

$$C_1^{(1)} = c_1, \tag{640}$$

$$D_1^{(1)} = d_1, \tag{641}$$

$$E_1^{(1)} = e_1, \tag{642}$$

$$F_1^{(1)} = f_1, \tag{643}$$

$$G_1^{(1)} = g_1, \tag{644}$$

$$H_1^{(1)} = h_1, \tag{645}$$

$$I_1^{(1)} = i_1, \tag{646}$$

For $A_0^{(-1)}$, it gives

$$A_0^{(-1)} = a_{-1}, \tag{647}$$

$$B_0^{(-1)} = b_{-1}, \tag{648}$$

$$C_0^{(-1)} = c_{-1}, \tag{649}$$

$$D_0^{(-1)} = d_{-1}, \tag{650}$$

$$E_0^{(-1)} = e_{-1}, \tag{651}$$

$$F_0^{(-1)} = f_{-1}, \tag{652}$$

$$G_0^{(-1)} = g_{-1}, \tag{653}$$

$$H_0^{(-1)} = h_{-1}, \tag{654}$$

$$I_0^{(-1)} = i_{-1}, \tag{655}$$

The evolution equations are found from the coefficients of A_0, E_0, I_0.

$$A_0 = D_0 n_{1,2} + G_0 n_{1,3} - B_0 n_{2,1} + a_1 n_{1,2} n_{2,1} - e_1 n_{1,2} n_{2,1}$$

$$- C_0 n_{3,1} + a_1 n_{1,3} n_{3,1} - i_1 n_{1,3} n_{3,1}$$

$$E_0 = B_0 n_{2,1} - n_{1,2} D_0 + (a_1 - e_1) n_{2,1} + H_0 n_{2,3}$$

$$- (F_0 + (-e_1 + i_1)n_{2,3})n_{3,2}$$

$$I_0 = C_0 n_{3,1} - n_{1,3}(G_0 + (a_1 - i_1)n_{3,1}) + F_0 n_{3,2}$$

$$- n_{2,3}(H_0 + (e_1 - i_1)n_{3,2}).$$

CHAPTER VII

Conclusion

Many contributions have been made in the field of integrable systems of nonlinear partial differential equations. This has popularized the field in the past few decades. Finding solutions from Lax pairs, first introduced by Peter Lax, has been fruitful in some contexts. The Lax pair strategy consists of finding two operators or matrices L and M, which satisfy the Lax equation $L_t + [L,M] = 0$, where $[L,M] := LM - ML$ [9]. Many mathematicians have used Lax pairs to solve numerous equations. Zakharov and Shabat, and AKNS [6] used Lax's ideas to solve the nonlinear Schödinger equation and Sine-Gordon equation, by considering a 2×2 continuous problem.

Lax pairs have also been applied to discrete equations [9]. Toda, Flaschka and Malatov solved the Toda lattice equation and discovered a number of solutions. Years later, Ablowtiz and Ladik proposed a discrete scattering problem, which was similar to the 2×2 Zakharov-Shabat continuous case to show the relation between the Toda lattice and nonlinear Schödinger equation [9].

AKNS and Ablowitz and Ladik indicate relevant calculations in their work but they did show many of the details of their work in their publications. Also, the material covered in their publications did not give, the special case of a 3×3 discrete and continuous case. In this thesis, we have shown in detail for both discrete and continuous cases the calculations not explicitly given in the work of AKNS and Ablowitz-Ladik. Moreover, we have used Mathematica to obtain the coefficients for solutions to the AKNS equations and those corresponding to the 3×3 discrete and continuous cases. See the Appendix for the relevant Mathematica code.

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APPENDIX

APPENDIX

In this Appendix we include the Mathematica code used to perform a significant number of calculations. We begin with the AKNS nonlinear Schrödinger equation. The code was developed in close collaboration with the advisor of this thesis.

```
/* AKNS Lax, NLS equation*/

/* v1x: derivative of v1, v1t: time derivative of v1 */

v1x:=-I \[Xi] v1[x,t]+q[x,t] v2[x,t]

v1t:=A[x,t] v1[x,t]+B[x,t] v2[x,t]

/* v2x: derivative of v2, v2t: time derivative of v2 */

v2x:=I \[Xi] v2[x,t]+r[x,t] v1[x,t]

v2t:=CC[x,t] v1[x,t]+DD[x,t] v2[x,t]

/*Compatibility Condition for v1 */

Collect[FullSimplify[D[v1x,t]-D[v1t,x]/.{(v1^(1,0))[x,t]->

v1x,(v1^(0,1))[x,t]
```

```
->v1t,(v2^(1,0))[x,t]-> v2x,(v2^(0,1))[x,t]-> v2t} ],v2[x,t]]

CC[x,t] q[x,t] v1[x,t]-B[x,t] r[x,t] v1[x,t]-v1[x,t]

(A^(1,0))[x,t]+v2[x,t] (-2 I \[Xi] B[x,t]+(-A[x,t]+DD[x,t]) q[x,t]

+(q^(0,1))[x,t]-(B^(1,0))[x,t])

eq1:=CC[x,t] q[x,t]-B[x,t] r[x,t]-D[A[x,t],x]

eq2:=-2 I \[Xi] B[x,t]+(-A[x,t]

+DD[x,t]) q[x,t]+D[q[x,t],t]-D[B[x,t],x]

/* Compatibility Condition for v2*/

Collect[FullSimplify[D[v2x,t]-D[v2t,x]

/.{(v1^(1,0))[x,t]->v1x,(v1^(0,1))[x,t]->v1t,(v2^(1,0))[x,t]

-> v2x,(v2^(0,1))[x,t]-> v2t} ],v1[x,t]]

-CC[x,t] q[x,t] v2[x,t]+B[x,t] r[x,t] v2[x,t]

+v1[x,t] (2 I \[Xi] CC[x,t]+A[x,t] r[x,t]-DD[x,t] r[x,t]

+(r^(0,1))[x,t]-(CC^(1,0))[x,t])-v2[x,t] (DD^(1,0))[x,t]

eq3:=2 I \[Xi] CC[x,t]+A[x,t] r[x,t]-DD[x,t]

 r[x,t]+D[r[x,t],t]-D[CC[x,t],x]

eq4:=-CC[x,t] q[x,t]+B[x,t] r[x,t]- D[DD[x,t],x]
```

```
eq1

CC[x,t] q[x,t]-B[x,t] r[x,t]-(A^(1,0))[x,t]

eq2

-2 I \[Xi] B[x,t]+(-A[x,t]+DD[x,t]) q[x,t]

+(q^(0,1))[x,t]-(B^(1,0))[x,t]

eq3

2 I \[Xi] CC[x,t]+A[x,t] r[x,t]-DD[x,t] r[x,t]

+(r^(0,1))[x,t]-(CC^(1,0))[x,t]

eq4

-CC[x,t] q[x,t]+B[x,t] r[x,t]-(DD^(1,0))[x,t]

DD[x,t]=-A[x,t]

-A[x,t]

eq1

CC[x,t] q[x,t]-B[x,t] r[x,t]-(A^(1,0))[x,t]

eq2

-2 I \[Xi] B[x,t]-2 A[x,t] q[x,t]+(q^(0,1))[x,t]-(B^(1,0))[x,t]

eq3
```

```
2 I \[Xi] CC[x,t]+2 A[x,t] r[x,t]+(r^(0,1))[x,t]-(CC^(1,0))[x,t]

eq4

-CC[x,t] q[x,t]+B[x,t] r[x,t]+(A^(1,0))[x,t]

A[x_,t_]:=A2[x,t] \[Xi]^2+A1[x,t] \[Xi]+A0[x,t]

B[x_,t_]:=B2[x,t] \[Xi]^2+B1[x,t] \[Xi]+B0[x,t]

CC[x_,t_]:=C2[x,t] \[Xi]^2+C1[x,t] \[Xi]+C0[x,t]

Collect[Simplify[eq1],\[Xi]]

0

Collect[Simplify[eq2],\[Xi]]

-a2 q[x,t]^2 r[x,t]+(q^(0,1))[x,t]-I a1 (q^(1,0))[x,t]

+1/2 a2 (q^(2,0))[x,t]

Collect[Simplify[eq3],\[Xi]]

a2 q[x,t] r[x,t]^2+(r^(0,1))[x,t]-I a1 (r^(1,0))[x,t]

-1/2 a2 (r^(2,0))[x,t]

Collect[Simplify[eq4],\[Xi]]

0

B2[x,t]=0;
```

```
C2[x,t]=0;

A2[x,t]=a2;

B1[x,t]=I a2 q[x,t];

C1[x,t]=I a2 r[x,t];

A1[x,t]=a1;

B0[x,t]=I a1 q[x,t]-a2/2 D[q[x,t],x];

C0[x,t]=I a1 r[x,t]+a2/2 (r^(1,0))[x,t];

A0[x,t]=1/2 a2 r[x,t] q[x,t];

/* Nonlinear Schrodinger equation */

Simplify[eq2]

-a2 q[x,t]^2 r[x,t]+(q^(0,1))[x,t]-I a1 (q^(1,0))[x,t]

+1/2 a2 (q^(2,0))[x,t]

Simplify[eq3]

a2 q[x,t] r[x,t]^2+(r^(0,1))[x,t]-I a1 (r^(1,0))[x,t]

-1/2 a2 (r^(2,0))[x,t]
```

The following code corresponds to the Ablowitz-Ladik 2×2 discrete case.

```
/* Ablowitz-Ladik Lax, dNLS equation*/
```

```
/* v1t[n]: time derivative of v1[n], v2t[n]:
time derivative of v2[n] */
v1[n_]:=\[Xi] v1[n-1,t]+Q[n-1,t] v2[n-1,t]
v1t[n_]:=A[n,t] v1[n,t]+B[n,t] v2[n,t]
v2[n_]:=1/\[Xi] v2[n-1,t]+R[n-1,t] v1[n-1,t]
v2t[n_]:=CC[n,t] v1[n,t]+DD[n,t] v2[n,t]
/* Compatibility Condition for v1[n] */
Collect[FullSimplify[D[v1[n+1],t]-v1t[n+1]
 /.{(v1^(0,1))[n,t]->v1t[n],(v2^(0,1))[n,t]
->v2t[n],v1[1+n,t]->v1[n+1],v2[1+n,t
]->v2[n+1]}],v2[n,t]]
\[Xi] A[n,t] v1[n,t]-\[Xi] A[1+n,t] v1[n,t]
+CC[n,t] Q[n,t] v1[n,t]-B[1+n,t] R[n,t] v1[n,t]+v2[n,t]
 (\[Xi] B[n,t]-B[1+n,t]/\[Xi]-A[1+n,t] Q[n,t]+DD[n,t]
 Q[n,t]+(Q^(0,1))[n,t])
Factor[\[Xi] A[n,t] v1[n,t]-\[Xi] A[1+n,t] v1[n,t]
+CC[n,t] Q[n,t] v1[n,t]-B[1+n,t] R[n,t] v1[n,t]]
```

```
(\[Xi] A[n,t]-\[Xi] A[1+n,t]+CC[n,t] Q[n,t]-B[1+n,t]

R[n,t]) v1[n,t]

eq1:=\[Xi] A[n,t]-\[Xi] A[1+n,t]+CC[n,t] Q[n,t]

-B[1+n,t] R[n,t]

eq2:=\[Xi] B[n,t]-B[1+n,t]/\[Xi]-A[1+n,t] Q[n,t]

+DD[n,t] Q[n,t]+D[Q[n,t],t]

/* Compatibility Condition for v2[n] */

Collect[FullSimplify[D[v2[n+1],t]-v2t[n+1]

/.{(v1^(0,1))[n,t]->v1t[n],(v2^(0,1))[n,t]

->v2t[n],v1[1+n,t]->v1[n+1],v2[1+n,t]->v2[n+1]}],v1[n,t]]

(DD[n,t] v2[n,t])/\[Xi]-(DD[1+n,t] v2[n,t])

/\[Xi]-CC[1+n,t] Q[n,t] v2[n,t]+B[n,t] R[n,t]

v2[n,t]+v1[n,t] (CC[n,t]/\[Xi]-\[Xi] CC[1+n,t]+A[n,t]

 R[n,t]-DD[1+n,t] R[n,t]+(R^(0,1))[n,t])

Factor[(DD[n,t] v2[n,t])/\[Xi]-(DD[1+n,t]

 v2[n,t])/\[Xi]-CC[1+n,t] Q[n,t] v2[n,t]+B[n,t]

 R[n,t] v2[n,t]] -(((-DD[n,t]+DD[1+n,t]+\[Xi] CC[1+n,t]
```

```
Q[n,t]-\[Xi] B[n,t] R[n,t]) v2[n,t])/\[Xi])

DD[n,t]/\[Xi]-DD[1+n,t]/\[Xi]-CC[1+n,t] Q[n,t]

+B[n,t] R[n,t] DD[n,t]/\[Xi]-DD[1+n,t]/\[Xi]-CC[1+n,t]

 Q[n,t]+B[n,t] R[n,t]

eq3:=CC[n,t]/\[Xi]-\[Xi] CC[1+n,t]+A[n,t] R[n,t]

-DD[1+n,t] R[n,t]+D[R[n,t],t]

eq4:=DD[n,t]/\[Xi]-DD[1+n,t]/\[Xi]

-CC[1+n,t] Q[n,t]+B[n,t] R[n,t]

eq1

\[Xi] A[n,t]-\[Xi] A[1+n,t]+CC[n,t] Q[n,t]

-B[1+n,t] R[n,t]

eq2

\[Xi] B[n,t]-B[1+n,t]/\[Xi]-A[1+n,t] Q[n,t]

+DD[n,t] Q[n,t]+(Q^(0,1))[n,t]

eq3

CC[n,t]/\[Xi]-\[Xi] CC[1+n,t]+A[n,t] R[n,t]

-DD[1+n,t] R[n,t]+(R^(0,1))[n,t]
```

```
eq4

DD[n,t]/\[Xi]-DD[1+n,t]/\[Xi]-CC[1+n,t] Q[n,t]

+B[n,t] R[n,t]

A[n_,t_]:=A2[n,t] \[Xi]^2+A1[n,t] \[Xi]+A0[n,t]

B[n_,t_]:=B1[n,t] \[Xi]+Bm1[n,t] 1/\[Xi]

CC[n_,t_]:=C1[n,t] \[Xi]+Cm1[n,t] 1/\[Xi]

DD[n_,t_]:=Dm2[n,t] 1/\[Xi]^2+Dm1[n,t] 1/\[Xi]

+D0[n,t]

Collect[Simplify[eq1],\[Xi]]

\[Xi]^2 (A1[n,t]-A1[1+n,t])+\[Xi]^3 (A2[n,t]

-A2[1+n,t])+\[Xi] (A0[n,t]-A0[1+n,t]+C1[n,t] Q[n,t]

-B1[1+n,t] R[n,t])+(Cm1[n,t] Q[n,t]

-Bm1[1+n,t] R[n,t])/\[Xi]

Collect[Simplify[eq2],\[Xi]]

-B1[1+n,t]+Bm1[n,t]-A0[1+n,t] Q[n,t]

-\[Xi] A1[1+n,t] Q[n,t]+D0[n,t] Q[n,t]

+(Dm1[n,t] Q[n,t])/\[Xi]+\[Xi]^2 (B1[n,t]
```

```
-A2[1+n,t] Q[n,t])+(-Bm1[1+n,t]

+Dm2[n,t] Q[n,t])/\[Xi]^2+(Q^(0,1))[n,t]

Collect[Simplify[eq3],\[Xi]]

C1[n,t]-Cm1[1+n,t]+A0[n,t] R[n,t]

+\[Xi] A1[n,t] R[n,t]-D0[1+n,t] R[n,t]

-(Dm1[1+n,t] R[n,t])/\[Xi]+\[Xi]^2

 (-C1[1+n,t]+A2[n,t] R[n,t])+(Cm1[n,t]

-Dm2[1+n,t] R[n,t])/\[Xi]^2+(R^(0,1))[n,t]

Collect[Simplify[eq4],\[Xi]]

(Dm1[n,t]-Dm1[1+n,t])/\[Xi]^2+(Dm2[n,t]

-Dm2[1+n,t])/\[Xi]^3+\[Xi] (-C1[1+n,t] Q[n,t]+B1[n,t]

 R[n,t])+(D0[n,t]-D0[1+n,t]-Cm1[1+n,t] Q[n,t]

+Bm1[n,t] R[n,t])/\[Xi]

A1[n_,t]=0;

Dm1[n_,t]=0;

A2[n_,t]=a2;

Bm1[n_,t]:=Dm2[n-1,t] Q[n-1,t]
```

```
Cm1[n_,t]:=Dm2[n+1,t] R[n,t]

B1[n_,t]:=a2 Q[n,t]

C1[n_,t]:=a2 R[n-1,t]

A0[n_,t]:=-a2 Q[n,t] R[n-1,t]

D0[n_,t]:=-d2 Q[n-1,t] R[n,t]

Dm2[n_,t]:=d2

MatrixForm[{{AA[n,t],BB[n,t]},{CC[n,t],DD[n,t]}}]

(AA[n,t] BB[n,t]

a2 \[Xi] R[-1+n,t]+(d2 R[n,t])/\[Xi]

d2/\[Xi]^2-d2 Q[-1+n,t] R[n,t])

/* Ablowitz-Ladik lattice

(discrete nonlinear Schrodinger equation) */

Simplify[eq2]

a2 Q[1+n,t] (-1+Q[n,t] R[n,t])+Q[-1+n,t]

 (d2-d2 Q[n,t] R[n,t])+(Q^(0,1))[n,t]

Simplify[eq3]

R[-1+n,t] (a2-a2 Q[n,t] R[n,t])+d2
```

```
(-1+Q[n,t] R[n,t]) R[1+n,t]+(R^(0,1))[n,t]
```

The following code corresponds to the 3×3 continuous case.

```
/* Lax, 3x3, 3 Wave Interaction System, See Ablowitz & Segur */

/* v1x: derivative of v1, v1t: time derivative of v1 */

v1x:=I d1 \[Xi] v1[x,t]+N12[x,t] v2[x,t]+N13[x,t] v3[x,t]

v1t:=AA[x,t] v1[x,t]+BB[x,t] v2[x,t]+CC[x,t] v3[x,t]

/* v2x: derivative of v2, v2t: time derivative of v2 */

v2x:=I d2 \[Xi] v2[x,t]+N21[x,t] v1[x,t]+N23[x,t] v3[x,t]

v2t:=DD[x,t] v1[x,t]+EE[x,t] v2[x,t]+FF[x,t] v3[x,t]

/* v3x: derivative of v3, v3t: time derivative of v3 */

v3x:=I d3 \[Xi] v3[x,t]+N31[x,t] v1[x,t]+N32[x,t] v2[x,t]

v3t:=GG[x,t] v1[x,t]+HH[x,t] v2[x,t]+II[x,t] v3[x,t]

/*Compatibility Condition for v1 */

Collect[FullSimplify[D[v1x,t]-D[v1t,x]/.{(v1^(1,0))[x,t]

->v1x,(v1^(0,1))[x,t]->v1t,(v2^(1,0))[x,t]

-> v2x,(v2^(0,1))[x,t]-> v2t,(v3^(1,0))[x,t]

-> v3x,(v3^(0,1))[x,t]-> v3t} ],v3[x,t]]
```

```
DD[x,t] N12[x,t] v1[x,t]+GG[x,t] N13[x,t] v1[x,t]

-BB[x,t] N21[x,t] v1[x,t]-CC[x,t] N31[x,t] v1[x,t]

+I d1 \[Xi] BB[x,t] v2[x,t]-I d2 \[Xi] BB[x,t] v2[x,t]

-AA[x,t] N12[x,t] v2[x,t]+EE[x,t] N12[x,t] v2[x,t]

+HH[x,t] N13[x,t] v2[x,t]-CC[x,t] N32[x,t] v2[x,t]

+v2[x,t] (N12^(0,1))[x,t]-v1[x,t] (AA^(1,0))[x,t]

-v2[x,t] (BB^(1,0))[x,t]+v3[x,t] (I d1 \[Xi] CC[x,t]

-I d3 \[Xi] CC[x,t]+FF[x,t] N12[x,t]-AA[x,t] N13[x,t]

+II[x,t] N13[x,t]-BB[x,t] N23[x,t]+(N13^(0,1))[x,t]

-(CC^(1,0))[x,t])

eq1:=DD[x,t] N12[x,t]+GG[x,t] N13[x,t]-BB[x,t] N21[x,t]

-CC[x,t] N31[x,t]-D[AA[x,t],x]

eq2:=I (d1-d2) \[Xi] BB[x,t]-AA[x,t] N12[x,t]

+EE[x,t] N12[x,t]+HH[x,t] N13[x,t]-CC[x,t] N32[x,t]

+D[N12[x,t],t]-D[BB[x,t],x]

eq3:=I (d1-d3) \[Xi] CC[x,t]+FF[x,t] N12[x,t]

-AA[x,t] N13[x,t]+II[x,t] N13[x,t]-BB[x,t] N23[x,t]
```

```
+D[N13[x,t],t]-D[CC[x,t],x]

/* Compatibility Condition for v2*/

Collect[FullSimplify[D[v2x,t]-D[v2t,x]/.{(v1^(1,0))[x,t]

->v1x,(v1^(0,1))[x,t]->v1t,(v2^(1,0))[x,t]

-> v2x,(v2^(0,1))[x,t]-> v2t,(v3^(1,0))[x,t]

-> v3x,(v3^(0,1))[x,t]-> v3t}  ],v3[x,t]]

-I d1 \[Xi] DD[x,t] v1[x,t]+I d2

\[Xi] DD[x,t] v1[x,t]+AA[x,t] N21[x,t] v1[x,t]

-EE[x,t] N21[x,t] v1[x,t]+GG[x,t] N23[x,t] v1[x,t]

-FF[x,t] N31[x,t] v1[x,t]-DD[x,t] N12[x,t] v2[x,t]

+BB[x,t] N21[x,t] v2[x,t]+HH[x,t] N23[x,t] v2[x,t]

-FF[x,t] N32[x,t] v2[x,t]+v1[x,t] (N21^(0,1))[x,t]

-v1[x,t] (DD^(1,0))[x,t]-v2[x,t] (EE^(1,0))[x,t]

+v3[x,t] (I d2 \[Xi] FF[x,t]-I d3 \[Xi] FF[x,t]

-DD[x,t] N13[x,t]+CC[x,t] N21[x,t]-EE[x,t] N23[x,t]

+II[x,t] N23[x,t]+(N23^(0,1))[x,t]-(FF^(1,0))[x,t])

eq4:=I (d2-d1) \[Xi] DD[x,t]+AA[x,t] N21[x,t]
```

```
-EE[x,t] N21[x,t]+GG[x,t] N23[x,t]-FF[x,t] N31[x,t]

+D[N21[x,t],t]-D[DD[x,t],x]

eq5:=-DD[x,t] N12[x,t]+BB[x,t] N21[x,t]

+HH[x,t] N23[x,t]-FF[x,t] N32[x,t]-D[EE[x,t],x]

eq6:=I (d2-d3) \[Xi] FF[x,t]-DD[x,t] N13[x,t]

+CC[x,t] N21[x,t]-EE[x,t] N23[x,t]+II[x,t] N23[x,t]

+D[N23[x,t],t]-D[FF[x,t],x]

/* Compatibility Condition for v3*/

Collect[FullSimplify[D[v3x,t]-D[v3t,x]

/.{(v1^(1,0))[x,t]->v1x,(v1^(0,1))[x,t]

->v1t,(v2^(1,0))[x,t]-> v2x,(v2^(0,1))[x,t]

-> v2t,(v3^(1,0))[x,t]-> v3x,(v3^(0,1))[x,t]-> v3t}  ],v3[x,t]]

-I (d1-d3) \[Xi] GG[x,t] v1[x,t]-HH[x,t] N21[x,t]

 v1[x,t]+AA[x,t] N31[x,t] v1[x,t]-II[x,t] N31[x,t]

 v1[x,t]+DD[x,t] N32[x,t] v1[x,t]-I (d2-d3) \[Xi]

 HH[x,t] v2[x,t]-GG[x,t] N12[x,t] v2[x,t]+BB[x,t]

 N31[x,t] v2[x,t]+EE[x,t] N32[x,t] v2[x,t]-II[x,t]
```

```
 N32[x,t] v2[x,t]+v1[x,t] (N31^(0,1))[x,t]+v2[x,t]
(N32^(0,1))[x,t]-v1[x,t] (GG^(1,0))[x,t]-v2[x,t]
 (HH^(1,0))[x,t]+v3[x,t] (-GG[x,t] N13[x,t]-HH[x,t]
 N23[x,t]+CC[x,t] N31[x,t]+FF[x,t] N32[x,t]-(II^(1,0))[x,t])
eq7:=I (d3-d1) \[Xi] GG[x,t]-HH[x,t] N21[x,t]+AA[x,t]
N31[x,t]-II[x,t] N31[x,t]+DD[x,t] N32[x,t]+D[N31[x,t],t]
-D[GG[x,t],x]
eq8:=I (d3-d2) \[Xi] HH[x,t]-GG[x,t] N12[x,t]+BB[x,t]
 N31[x,t]+EE[x,t] N32[x,t]-II[x,t] N32[x,t]+D[N32[x,t],t
]-D[HH[x,t],x]
eq9:=-GG[x,t] N13[x,t]-HH[x,t] N23[x,t]+CC[x,t]
 N31[x,t]+FF[x,t] N32[x,t]-D[II[x,t],x]
eq1
DD[x,t] N12[x,t]+GG[x,t] N13[x,t]-BB[x,t] N21[x,t]
-CC[x,t] N31[x,t]-(AA^(1,0))[x,t]
eq2
I (d1-d2) \[Xi] BB[x,t]-AA[x,t] N12[x,t]+EE[x,t] N12[x,t]
```

```
+HH[x,t] N13[x,t]-CC[x,t] N32[x,t]+(N12^(0,1))[x,t]-(BB^(1,0))[x,t]

eq3

I (d1-d3) \[Xi] CC[x,t]+FF[x,t] N12[x,t]-AA[x,t] N13[x,t]

+II[x,t] N13[x,t]-BB[x,t] N23[x,t]+(N13^(0,1))[x,t]-(CC^(1,0))[x,t]

eq4

I (-d1+d2) \[Xi] DD[x,t]+AA[x,t] N21[x,t]-EE[x,t] N21[x,t]

+GG[x,t] N23[x,t]-FF[x,t] N31[x,t]+(N21^(0,1))[x,t]-(DD^(1,0))[x,t]

eq5

-DD[x,t] N12[x,t]+BB[x,t] N21[x,t]+HH[x,t] N23[x,t]

-FF[x,t] N32[x,t]-(EE^(1,0))[x,t]

eq6

I (d2-d3) \[Xi] FF[x,t]-DD[x,t] N13[x,t]+CC[x,t] N21[x,t]

-EE[x,t] N23[x,t]+II[x,t] N23[x,t]+(N23^(0,1))[x,t]-(FF^(1,0))[x,t]

eq7

I (-d1+d3) \[Xi] GG[x,t]-HH[x,t] N21[x,t]+AA[x,t] N31[x,t]

-II[x,t] N31[x,t]+DD[x,t] N32[x,t]+(N31^(0,1))[x,t]-(GG^(1,0))[x,t]

eq8
```

```
I (-d2+d3) \[Xi] HH[x,t]-GG[x,t] N12[x,t]+BB[x,t] N31[x,t]

+EE[x,t] N32[x,t]-II[x,t] N32[x,t]+(N32^(0,1))[x,t]-(HH^(1,0))[x,t]

eq9

-GG[x,t] N13[x,t]-HH[x,t] N23[x,t]+CC[x,t] N31[x,t]

+FF[x,t] N32[x,t]-(II^(1,0))[x,t]

AA[x_,t_]:=A1[x,t] \[Xi]+A0[x,t]

BB[x_,t_]:=B1[x,t] \[Xi]+B0[x,t]

CC[x_,t_]:=C1[x,t] \[Xi]+C0[x,t]

DD[x_,t_]:=D1[x,t] \[Xi]+D0[x,t]

EE[x_,t_]:=E1[x,t] \[Xi]+E0[x,t]

FF[x_,t_]:=F1[x,t] \[Xi]+F0[x,t]

GG[x_,t_]:=G1[x,t] \[Xi]+G0[x,t]

HH[x_,t_]:=H1[x,t] \[Xi]+H0[x,t]

II[x_,t_]:=I1[x,t] \[Xi]+I0[x,t]

Collect[Simplify[eq1],\[Xi]]

0

Collect[Simplify[eq2],\[Xi]]
```

```
(-a0+e0) N12[x,t]+(I (a1 (-d2+d3)-d3 e1+d1 (e1-i1)+d2 i1)
 N13[x,t] N32[x,t])/((d1-d3) (-d2+d3))+(N12^(0,1))[x,t]
+(I a1 (N12^(1,0))[x,t])/(d1-d2)-(I e1 (N12^(1,0))[x,t])/(d1-d2)
Collect[Simplify[eq3],\[Xi]]
(-a0+i0) N13[x,t]+(I (a1 (d2-d3)+d3 e1-d2 i1+d1 (-e1+i1))
 N12[x,t] N23[x,t])/((d1-d2) (d2-d3))+(N13^(0,1))[x,t
]+(I a1 (N13^(1,0))[x,t])/(d1-d3)-(I i1 (N13^(1,0))[x,t])/(d1-d3)
Collect[Simplify[eq4],\[Xi]]
(a0-e0) N21[x,t]+(I (a1 (d2-d3)+d3 e1-d2 i1+d1 (-e1+i1))
 N23[x,t] N31[x,t])/((d1-d3) (-d2+d3))+(N21^(0,1))[x,t]
+(I a1 (N21^(1,0))[x,t])/(d1-d2)-(I e1 (N21^(1,0))[x,t])/(d1-d2)
Collect[Simplify[eq5],\[Xi]]
0
Collect[Simplify[eq6],\[Xi]]
(I (a1 (d2-d3)+d3 e1-d2 i1+d1 (-e1+i1)) N13[x,t]
 N21[x,t])/((d1-d2) (d1-d3))+(-e0+i0) N23[x,t]
+(N23^(0,1))[x,t]+(I e1 (N23^(1,0))[x,t])
```

```
/(d2-d3)-(I i1 (N23^(1,0))[x,t])/(d2-d3)

Collect[Simplify[eq7],\[Xi]]

(a0-i0) N31[x,t]+(I (a1 (-d2+d3)-d3 e1

+d1 (e1-i1)+d2 i1) N21[x,t] N32[x,t])/((d1-d2) (d2-d3))

+(N31^(0,1))[x,t]+(I a1 (N31^(1,0))[x,t])

/(d1-d3)-(I i1 (N31^(1,0))[x,t])/(d1-d3)

Collect[Simplify[eq8],\[Xi]]

(I (a1 (-d2+d3)-d3 e1+d1 (e1-i1)+d2 i1) N12[x,t] N31[x,t])

/((d1-d2) (d1-d3))+(e0-i0) N32[x,t]+(N32^(0,1))[x,t]

+(I e1 (N32^(1,0))[x,t])/(d2-d3)-(I i1 (N32^(1,0))[x,t])/(d2-d3)

Collect[Simplify[eq9],\[Xi]]

0

B1[x,t]=0;

C1[x,t]=0;

D1[x,t]=0;

F1[x,t]=0;

G1[x,t]=0;
```

```
H1[x,t]=0;

A1[x,t]=a1;

E1[x,t]=e1;

I1[x,t]=i1;

B0[x,t]=(a1-e1) N12[x,t]/(I (d1-d2));

C0[x,t]=(a1-i1) N13[x,t]/(I (d1-d3));

D0[x,t]=(a1-e1) N21[x,t]/(I (d1-d2));

F0[x,t]=(e1-i1) N23[x,t]/(I (d2-d3));

G0[x,t]=(a1-i1) N31[x,t]/(I (d1-d3));

H0[x,t]=(e1-i1) N32[x,t]/(I (d2-d3));

A0[x,t]=a0;

E0[x,t]=e0;

I0[x,t]=i0;

/* 3 wave interaction */

Simplify[eq1]

0

Simplify[eq2]
```

```
(-a0+e0) N12[x,t]+(I (a1 (-d2+d3)-d3 e1+d1 (e1-i1)+d2 i1)

 N13[x,t] N32[x,t])/((d1-d3) (-d2+d3))+(N12^(0,1))[x,t]

+(I a1 (N12^(1,0))[x,t])/(d1-d2)-(I e1 (N12^(1,0))[x,t])

/(d1-d2)

Simplify[eq3]

(-a0+i0) N13[x,t]+(I (a1 (d2-d3)+d3 e1-d2 i1+d1

(-e1+i1)) N12[x,t] N23[x,t])/((d1-d2) (d2-d3))

+(N13^(0,1))[x,t]+(I a1 (N13^(1,0))[x,t])/(d1-d3)

-(I i1 (N13^(1,0))[x,t])/(d1-d3)

Simplify[eq4]

(a0-e0) N21[x,t]+(I (a1 (d2-d3)+d3 e1-d2 i1+d1

(-e1+i1)) N23[x,t] N31[x,t])/((d1-d3) (-d2+d3)

)+(N21^(0,1))[x,t]+(I a1 (N21^(1,0))[x,t])

/(d1-d2)-(I e1 (N21^(1,0))[x,t])/(d1-d2)

Simplify[eq5]

0

Simplify[eq6]
```

```
(I (a1 (d2-d3)+d3 e1-d2 i1+d1 (-e1+i1)) N13[x,t]

 N21[x,t])/((d1-d2) (d1-d3))+(-e0+i0) N23[x,t]

+(N23^(0,1))[x,t]+(I e1 (N23^(1,0))[x,t])/(d2-d3)

-(I i1 (N23^(1,0))[x,t])/(d2-d3)

Simplify[eq7]

(a0-i0) N31[x,t]+(I (a1 (-d2+d3)-d3 e1+d1 (e1-i1)

+d2 i1) N21[x,t] N32[x,t])/((d1-d2) (d2-d3))

+(N31^(0,1))[x,t]+(I a1 (N31^(1,0))[x,t])/(d1-d3)

-(I i1 (N31^(1,0))[x,t])/(d1-d3)

Simplify[eq8]

(I (a1 (-d2+d3)-d3 e1+d1 (e1-i1)+d2 i1) N12[x,t]

N31[x,t])/((d1-d2) (d1-d3))+(e0-i0) N32[x,t]

+(N32^(0,1))[x,t]+(I e1 (N32^(1,0))[x,t])/(d2-d3)

-(I i1 (N32^(1,0))[x,t])/(d2-d3)

Simplify[eq9]

0

/* Matrix of time evolution */
```

```
MatrixForm[{{AA[x,t],BB[x,t],CC[x,t]},{DD[x,t],EE[x,t],FF[x,t]}
,{GG[x,t],HH[x,t],II[x,t]}}]
(a0+a1 \[Xi] -((I (a1-e1) N12[x,t])/(d1-d2))
-((I (a1-i1) N13[x,t])/(d1-d3))
-((I (a1-e1) N21[x,t])/(d1-d2)) e0+e1 \[Xi]
-((I (e1-i1) N23[x,t])/(d2-d3))
-((I (a1-i1) N31[x,t])/(d1-d3)) -((I (e1-i1) N32[x,t])
/(d2-d3)) i0+i1 \[Xi]


)
```

We conclude the Appendix with the code for the 3×3 discrete case.

```
/* 3x3 discrete Lax, discrete equation, discrete 3 wave
 interaction, nonlocal equation with condition,.*/
/* v1t[n]: time derivative of v1[n],
 v2t[n]: time derivative of v2[n],
v3t[n]: time derivative of v3[n] */
v1[n_]:=z1 (\[Xi]+1/\[Xi]) v1[n-1,t]+N12[n-1,t]
```

```
v2[n-1,t]+N13[n-1,t] v3[n-1,t]

v1t[n_]:=AA[n,t] v1[n,t]+BB[n,t] v2[n,t]+CC[n,t] v3[n,t]

v2[n_]:=z2 (\[Xi]+1/\[Xi]) v2[n-1,t]+N21[n-1,t] v1[n-1,t]

+N23[n-1,t] v3[n-1,t]

v2t[n_]:=DD[n,t] v1[n,t]+EE[n,t] v2[n,t]+FF[n,t] v3[n,t]

v3[n_]:=N31[n-1,t] v1[n-1,t]+N32[n-1,t] v2[n-1,t]+z3

(\[Xi]+1/\[Xi]) v3[n-1,t]

v3t[n_]:=GG[n,t] v1[n,t]+HH[n,t] v2[n,t]+II[n,t] v3[n,t]

sub:={(v1^(0,1))[n,t]->v1t[n],(v2^(0,1))[n,t]

->v2t[n],v1[1+n,t]->v1[n+1],v2[1+n,t]->v2[n+1],

(v3^(0,1))[n,t]->v3t[n],v3[1+n,t]->v3[n+1]}

/* Compatibility Condition for v1[n] */

Collect[FullSimplify[D[v1[n+1],t]-v1t[n+1] /.sub],v1[n,t]]

(z1 (1/\[Xi]+\[Xi]) AA[n,t]-z1 (1/\[Xi]+\[Xi])

AA[1+n,t]+DD[n,t] N12[n,t]+GG[n,t] N13[n,t]

-BB[1+n,t] N21[n,t]-CC[1+n,t] N31[n,t]) v1[n,t]+z1

(1/\[Xi]+\[Xi]) BB[n,t] v2[n,t]-z2 (1/\[Xi]+\[Xi])
```

```
BB[1+n,t] v2[n,t]-AA[1+n,t] N12[n,t] v2[n,t]

+EE[n,t] N12[n,t] v2[n,t]+HH[n,t] N13[n,t] v2[n,t]

-CC[1+n,t] N32[n,t] v2[n,t]+z1 (1/\[Xi]+\[Xi])

CC[n,t] v3[n,t]-z3 (1/\[Xi]+\[Xi]) CC[1+n,t] v3[n,t]

+FF[n,t] N12[n,t] v3[n,t]-AA[1+n,t] N13[n,t] v3[n,t]

+II[n,t] N13[n,t] v3[n,t]-BB[1+n,t] N23[n,t] v3[n,t]

+v2[n,t] (N12^(0,1))[n,t]+v3[n,t] (N13^(0,1))[n,t]

eq1:=z1 (1/\[Xi]+\[Xi]) AA[n,t]+z1 (-(1/\[Xi])-\[Xi])

AA[1+n,t]+DD[n,t] N12[n,t]+GG[n,t] N13[n,t]-BB[1+n,t]

N21[n,t]-CC[1+n,t] N31[n,t]

eq2:=z1 (1/\[Xi]+\[Xi]) BB[n,t]+z2 (-(1/\[Xi])-\[Xi])

BB[1+n,t]-AA[1+n,t] N12[n,t]+EE[n,t] N12[n,t]+HH[n,t]

 N13[n,t]-CC[1+n,t] N32[n,t]+D[N12[n,t],t]

eq3:=z1 (1/\[Xi]+\[Xi]) CC[n,t]+z3 (-(1/\[Xi])-\[Xi])

 CC[1+n,t]+FF[n,t] N12[n,t]-AA[1+n,t] N13[n,t]+II[n,t]

 N13[n,t]-BB[1+n,t] N23[n,t]+D[N13[n,t],t]

/* Compatibility Condition for v2[n] */
```

```
Collect[FullSimplify[D[v2[n+1],t]-v2t[n+1]

/.sub],v3[n,t]]

z2 (1/\[Xi]+\[Xi]) DD[n,t] v1[n,t]-z1 (1/\[Xi]+\[Xi])

DD[1+n,t] v1[n,t]+AA[n,t] N21[n,t] v1[n,t]-EE[1+n,t]

 N21[n,t] v1[n,t]+GG[n,t] N23[n,t] v1[n,t]-FF[1+n,t]

N31[n,t] v1[n,t]+z2 (1/\[Xi]+\[Xi]) EE[n,t] v2[n,t]-z2

 (1/\[Xi]+\[Xi]) EE[1+n,t] v2[n,t]-DD[1+n,t] N12[n,t]

v2[n,t]+BB[n,t] N21[n,t] v2[n,t]+HH[n,t] N23[n,t]

v2[n,t]-FF[1+n,t] N32[n,t] v2[n,t]+v1[n,t]

 (N21^(0,1))[n,t]+v3[n,t] (z2 (1/\[Xi]+\[Xi])

 FF[n,t]-z3 (1/\[Xi]+\[Xi]) FF[1+n,t]-DD[1+n,t]

 N13[n,t]+CC[n,t] N21[n,t]-EE[1+n,t] N23[n,t]

+II[n,t] N23[n,t]+(N23^(0,1))[n,t])

eq4:=z2 (1/\[Xi]+\[Xi]) DD[n,t]-z1 (1/\[Xi]+\[Xi])

DD[1+n,t]+AA[n,t] N21[n,t]-EE[1+n,t] N21[n,t]

+GG[n,t] N23[n,t]-FF[1+n,t] N31[n,t]+D[N21[n,t],t]

eq5:=z2 (1/\[Xi]+\[Xi]) EE[n,t]-z2 (1/\[Xi]+\[Xi]) EE[1+n,t]
```

```
-DD[1+n,t] N12[n,t]+BB[n,t] N21[n,t]+HH[n,t]

N23[n,t]-FF[1+n,t] N32[n,t]

eq6:=z2 (1/\[Xi]+\[Xi]) FF[n,t]-z3 (1/\[Xi]+\[Xi])

FF[1+n,t]-DD[1+n,t] N13[n,t]+CC[n,t] N21[n,t]

-EE[1+n,t] N23[n,t]+II[n,t] N23[n,t]+D[N23[n,t],t]

/* Compatibility Condition for v3[n] */

Collect[FullSimplify[D[v3[n+1],t]-v3t[n+1]

/.sub],v3[n,t]]

z3 (1/\[Xi]+\[Xi]) GG[n,t] v1[n,t]-z1 (1/\[Xi]

+\[Xi]) GG[1+n,t] v1[n,t]-HH[1+n,t] N21[n,t] v1[n,t]

+AA[n,t] N31[n,t] v1[n,t]-II[1+n,t] N31[n,t]

 v1[n,t]+DD[n,t] N32[n,t] v1[n,t]+z3 (1/\[Xi]+\[Xi])

 HH[n,t] v2[n,t]-z2 (1/\[Xi]+\[Xi]) HH[1+n,t]

 v2[n,t]-GG[1+n,t] N12[n,t] v2[n,t]+BB[n,t] N31[n,t]

 v2[n,t]+EE[n,t] N32[n,t] v2[n,t]-II[1+n,t] N32[n,t]

v2[n,t]+(z3 (1/\[Xi]+\[Xi]) II[n,t]-z3 (1/\[Xi]+\[Xi])

 II[1+n,t]-GG[1+n,t] N13[n,t]-HH[1+n,t] N23[n,t]
```

```
+CC[n,t] N31[n,t]+FF[n,t] N32[n,t]) v3[n,t]+v1[n,t]
 (N31^(0,1))[n,t]+v2[n,t] (N32^(0,1))[n,t]
eq7:=z3 (1/\[Xi]+\[Xi]) GG[n,t]-z1 (1/\[Xi]+\[Xi])
 GG[1+n,t]-HH[1+n,t] N21[n,t]+AA[n,t] N31[n,t]
-II[1+n,t] N31[n,t]+DD[n,t] N32[n,t]+D[N31[n,t],t]
eq8:=z3 (1/\[Xi]+\[Xi]) HH[n,t]-z2 (1/\[Xi]+\[Xi])
 HH[1+n,t]-GG[1+n,t] N12[n,t]+BB[n,t] N31[n,t]
+EE[n,t] N32[n,t]-II[1+n,t] N32[n,t]+D[N32[n,t],t]
eq9:=z3 (1/\[Xi]+\[Xi]) II[n,t]-z3 (1/\[Xi]+\[Xi])
II[1+n,t]-GG[1+n,t] N13[n,t]-HH[1+n,t] N23[n,t]
+CC[n,t] N31[n,t]+FF[n,t] N32[n,t]
AA[n_,t_]:=A1[n,t] \[Xi]+A0[n,t]+Am1[n,t]/\[Xi]
BB[n_,t_]:=B1[n,t] \[Xi]+B0[n,t]+Bm1[n,t]/\[Xi]
CC[n_,t_]:=C1[n,t] \[Xi]+C0[n,t]+Cm1[n,t]/\[Xi]
DD[n_,t_]:=D1[n,t] \[Xi]+D0[n,t]+Dm1[n,t]/\[Xi]
EE[n_,t_]:=E1[n,t] \[Xi]+E0[n,t]+Em1[n,t]/\[Xi]
FF[n_,t_]:=F1[n,t] \[Xi]+F0[n,t]+Fm1[n,t]/\[Xi]
```

```
GG[n_,t_]:=G1[n,t] \[Xi]+G0[n,t]+Gm1[n,t]/\[Xi]

HH[n_,t_]:=H1[n,t] \[Xi]+H0[n,t]+Hm1[n,t]/\[Xi]

II[n_,t_]:=I1[n,t] \[Xi]+I0[n,t]+Im1[n,t]/\[Xi]

Collect[Simplify[eq1],\[Xi]]

z1 A1[n,t]-z1 A1[1+n,t]+\[Xi]^2 (z1 A1[n,t]-z1
 A1[1+n,t])+z1 Am1[n,t]-z1 Am1[1+n,t]+(z1 Am1[n,t]
-z1 Am1[1+n,t])/\[Xi]^2+D0[n,t] N12[n,t]+G0[n,t]
N13[n,t]-B0[1+n,t] N21[n,t]-C0[1+n,t] N31[n,t]
+\[Xi] (z1 A0[n,t]-z1 A0[1+n,t]+D1[n,t] N12[n,t]
+G1[n,t] N13[n,t]-B1[1+n,t] N21[n,t]-C1[1+n,t]
 N31[n,t])+(z1 A0[n,t]-z1 A0[1+n,t]+Dm1[n,t]
N12[n,t]+Gm1[n,t] N13[n,t]-Bm1[1+n,t]
N21[n,t]-Cm1[1+n,t] N31[n,t])/\[Xi]

Collect[Simplify[eq2],\[Xi]]

z1 B1[n,t]-z2 B1[1+n,t]+\[Xi]^2 (z1 B1[n,t]
-z2 B1[1+n,t])+z1 Bm1[n,t]-z2 Bm1[1+n,t]
+(z1 Bm1[n,t]-z2 Bm1[1+n,t])/\[Xi]^2-A0[1+n,t] N12[n,t]
```

```
+E0[n,t] N12[n,t]+H0[n,t] N13[n,t]-C0[1+n,t] N32[n,t]

+\[Xi] (z1 B0[n,t]-z2 B0[1+n,t]-A1[1+n,t] N12[n,t]

+E1[n,t] N12[n,t]+H1[n,t] N13[n,t]-C1[1+n,t] N32[n,t])

+(z1 B0[n,t]-z2 B0[1+n,t]-Am1[1+n,t] N12[n,t]+Em1[n,t]

N12[n,t]+Hm1[n,t] N13[n,t]-Cm1[1+n,t] N32[n,t])

/\[Xi]+(N12^(0,1))[n,t]

Collect[Simplify[eq3],\[Xi]]

z1 C1[n,t]-z3 C1[1+n,t]+\[Xi]^2 (z1 C1[n,t]-z3 C1[1+n,t])

+z1 Cm1[n,t]-z3 Cm1[1+n,t]+(z1 Cm1[n,t]

-z3 Cm1[1+n,t])/\[Xi]^2+F0[n,t] N12[n,t]

-A0[1+n,t] N13[n,t]+I0[n,t] N13[n,t]-B0[1+n,t]

N23[n,t]+\[Xi] (z1 C0[n,t]-z3 C0[1+n,t]+F1[n,t]

 N12[n,t]-A1[1+n,t] N13[n,t]+I1[n,t] N13[n,t]

-B1[1+n,t] N23[n,t])+(z1 C0[n,t]-z3 C0[1+n,t]

+Fm1[n,t] N12[n,t]-Am1[1+n,t] N13[n,t]+Im1[n,t]

 N13[n,t]-Bm1[1+n,t] N23[n,t])/\[Xi]+(N13^(0,1))[n,t]

Collect[Simplify[eq4],\[Xi]]
```

```
z2 D1[n,t]-z1 D1[1+n,t]+\[Xi]^2 (z2 D1[n,t]-z1 D1[1+n,t])

+z2 Dm1[n,t]-z1 Dm1[1+n,t]+(z2 Dm1[n,t]-z1 Dm1[1+n,t])

/\[Xi]^2+A0[n,t] N21[n,t]-E0[1+n,t] N21[n,t]+G0[n,t]

N23[n,t]-F0[1+n,t] N31[n,t]+\[Xi] (z2 D0[n,t]-z1 D0[1+n,t]

+A1[n,t] N21[n,t]-E1[1+n,t] N21[n,t]+G1[n,t] N23[n,t]

-F1[1+n,t] N31[n,t])+(z2 D0[n,t]-z1 D0[1+n,t]+Am1[n,t]

 N21[n,t]-Em1[1+n,t] N21[n,t]+Gm1[n,t] N23[n,t]

-Fm1[1+n,t] N31[n,t])/\[Xi]+(N21^(0,1))[n,t]

Collect[Simplify[eq5],\[Xi]]

z2 E1[n,t]-z2 E1[1+n,t]+\[Xi]^2 (z2 E1[n,t]

-z2 E1[1+n,t])+z2 Em1[n,t]-z2 Em1[1+n,t]+(z2 Em1[n,t]

-z2 Em1[1+n,t])/\[Xi]^2-D0[1+n,t] N12[n,t]+B0[n,t]

N21[n,t]+H0[n,t] N23[n,t]-F0[1+n,t] N32[n,t]+\[Xi]

(z2 E0[n,t]-z2 E0[1+n,t]-D1[1+n,t] N12[n,t]+B1[n,t]

 N21[n,t]+H1[n,t] N23[n,t]-F1[1+n,t] N32[n,t])

+(z2 E0[n,t]-z2 E0[1+n,t]-Dm1[1+n,t] N12[n,t]

+Bm1[n,t] N21[n,t]+Hm1[n,t] N23[n,t]-Fm1[1+n,t] N32[n,t])/\[Xi]
```

```
Collect[Simplify[eq6],\[Xi]]

z2 F1[n,t]-z3 F1[1+n,t]+\[Xi]^2 (z2 F1[n,t]

-z3 F1[1+n,t])+z2 Fm1[n,t]-z3 Fm1[1+n,t]+(z2 Fm1[n,t]

-z3 Fm1[1+n,t])/\[Xi]^2-D0[1+n,t] N13[n,t]+C0[n,t]

 N21[n,t]-E0[1+n,t] N23[n,t]+I0[n,t] N23[n,t]+\[Xi]

(z2 F0[n,t]-z3 F0[1+n,t]-D1[1+n,t] N13[n,t]+C1[n,t]

N21[n,t]-E1[1+n,t] N23[n,t]+I1[n,t] N23[n,t])

+(z2 F0[n,t]-z3 F0[1+n,t]-Dm1[1+n,t] N13[n,t]

+Cm1[n,t] N21[n,t]-Em1[1+n,t] N23[n,t]+Im1[n,t]

 N23[n,t])/\[Xi]+(N23^(0,1))[n,t]

Collect[Simplify[eq7],\[Xi]]

z3 G1[n,t]-z1 G1[1+n,t]+\[Xi]^2 (z3 G1[n,t]

-z1 G1[1+n,t])+z3 Gm1[n,t]-z1 Gm1[1+n,t]

+(z3 Gm1[n,t]-z1 Gm1[1+n,t])/\[Xi]^2-H0[1+n,t]

N21[n,t]+A0[n,t] N31[n,t]-I0[1+n,t] N31[n,t]

+D0[n,t] N32[n,t]+\[Xi] (z3 G0[n,t]-z1 G0[1+n,t]

-H1[1+n,t] N21[n,t]+A1[n,t] N31[n,t]-I1[1+n,t]
```

```
 N31[n,t]+D1[n,t] N32[n,t])+(z3 G0[n,t]
-z1 G0[1+n,t]-Hm1[1+n,t] N21[n,t]+Am1[n,t] N31[n,t]
-Im1[1+n,t] N31[n,t]+Dm1[n,t] N32[n,t])/\[Xi
]+(N31^(0,1))[n,t]
Collect[Simplify[eq8],\[Xi]]
z3 H1[n,t]-z2 H1[1+n,t]+\[Xi]^2 (z3 H1[n,t]
-z2 H1[1+n,t])+z3 Hm1[n,t]-z2 Hm1[1+n,t]
+(z3 Hm1[n,t]-z2 Hm1[1+n,t])/\[Xi]^2-G0[1+n,t]
 N12[n,t]+B0[n,t] N31[n,t]+E0[n,t
] N32[n,t]-I0[1+n,t] N32[n,t]+\[Xi]
(z3 H0[n,t]-z2 H0[1+n,t]-G1[1+n,t] N12[n,t]
+B1[n,t] N31[n,t]+E1[n,t] N32[n,t]-I1[1+n,t]
N32[n,t])+(z3 H0[n,t]-z2 H0[1+n,t]-Gm1[1+n,t]
 N12[n,t]+Bm1[n,t] N31[n,t]+Em1[n,t] N32[n,t]
-Im1[1+n,t] N32[n,t])/\[Xi]+(N32^(0,1))[n,t]
Collect[Simplify[eq9],\[Xi]]
z3 I1[n,t]-z3 I1[1+n,t]+\[Xi]^2 (z3 I1[n,t]
```

```
-z3 I1[1+n,t])+z3 Im1[n,t]-z3 Im1[1+n,t]

+(z3 Im1[n,t]-z3 Im1[1+n,t])/\[Xi]^2-G0[1+n,t]

 N13[n,t]-H0[1+n,t] N23[n,t]+C0[n,t]

 N31[n,t]+F0[n,t] N32[n,t]+\[Xi] (z3 I0[n,t]

-z3 I0[1+n,t]-G1[1+n,t] N13[n,t]

-H1[1+n,t] N23[n,t]+C1[n,t] N31[n,t]+F1[n,t]

 N32[n,t])+(z3 I0[n,t]-z3 I0[1+n,t]-Gm1[1+n,t]

 N13[n,t]-Hm1[1+n,t] N23[n,t]+Cm1[n,t] N31[n,t]

+Fm1[n,t] N32[n,t])/\[Xi]

z1:=z;z2:=z;z3:=z;

A1[n_,t]=a1;

B1[n_,t]=b1;

C1[n_,t]=c1;

D1[n_,t]=d1;

E1[n_,t]=e1;

F1[n_,t]=f1;

G1[n_,t]=g1;
```

```
H1[n_,t]=h1;

I1[n_,t]=i1;

Am1[n_,t]=am1;

Bm1[n_,t]=bm1;

Cm1[n_,t]=cm1;

Dm1[n_,t]=dm1;

Em1[n_,t]=em1;

Fm1[n_,t]=fm1;

Gm1[n_,t]=gm1;

Hm1[n_,t]=hm1;

Im1[n_,t]=im1;

am1:=a1;bm1:=b1;cm1:=c1;dm1:=d1;em1:=e1

;fm1:=f1;gm1:=g1;hm1:=h1;im1:=i1;

MatrixForm[{{AA[n,t],BB[n,t],CC[n,t]},

{DD[n,t],EE[n,t],FF[n,t]},{GG[n,t],HH[n,t],II[n,t]}}]

(a0+a1/\[Xi]+a1 \[Xi] B0[n,t] C0[n,t]

D0[n,t] e0+e1/\[Xi]+e1 \[Xi] F0[n,t]
```

```
G0[n,t] H0[n,t] i0+i1/\[Xi]+i1 \[Xi]


)

Collect[Simplify[eq9],\[Xi]]

-G0[1+n,t] N13[n,t]-H0[1+n,t] N23[n,t]+C0[n,t]

 N31[n,t]+F0[n,t] N32[n,t]+(z I0[n,t]

-z I0[1+n,t]-g1 N13[n,t]-h1 N23[n,t]+c1 N31[n,t]

+f1 N32[n,t])/\[Xi]+\[Xi] (z I0[n,t]-z I0[1+n,t]

-g1 N13[n,t]-h1 N23[n,t]+c1 N31[n,t]+f1 N32[n,t])

A0[n+1,t]:=(z A0[n,t]+d1 N12[n,t]+g1 N13[n,t]

-b1 N21[n,t]-c1 N31[n,t])/z

B0[n+1,t]:=(z B0[n,t]-a1 N12[n,t]+e1 N12[n,t]

+h1 N13[n,t]-c1 N32[n,t])/z

C0[n+1,t]:=(z C0[n,t]+f1 N12[n,t]-a1 N13[n,t]

+i1 N13[n,t]-b1 N23[n,t])/z

D0[n+1,t]:=(z D0[n,t]+a1 N21[n,t]-e1 N21[n,t]

+g1 N23[n,t]-f1 N31[n,t])/z
```

```
E0[n+1,t]:=(z E0[n,t]-d1 N12[n,t]+b1 N21[n,t]

+h1 N23[n,t]-f1 N32[n,t])/z

F0[n+1,t]:=(z F0[n,t]-d1 N13[n,t]+c1 N21[n,t]

-e1 N23[n,t]+i1 N23[n,t])/z

G0[n+1,t]:=(z G0[n,t]-h1 N21[n,t]+a1 N31[n,t]

-i1 N31[n,t]+d1 N32[n,t])/z

H0[n+1,t]:=(z H0[n,t]-g1 N12[n,t]+b1 N31[n,t]

+e1 N32[n,t]-i1 N32[n,t])/z

I0[1+n,t]:=(z I0[n,t]-g1 N13[n,t]-h1 N23[n,t]

+c1 N31[n,t]+f1 N32[n,t])/z

b1:=0;c1:=0;d1:=0;f1:=0;g1:=0;h1:=0;

z:=1;

A0[n_,t]=a0;

E0[n_,t]=e0;

I0[n_,t]=i0;

/* A0[n+1,t], B0[n+1,t], ...., I0[n+1],

 Solve recurrence equations,
```

```
 you can express using Summation*/

A0[n+1,t]

a0

B0[n+1,t]

B0[n,t]-a1 N12[n,t]+e1 N12[n,t]

C0[n+1,t]

C0[n,t]-a1 N13[n,t]+i1 N13[n,t]

D0[n+1,t]

D0[n,t]+a1 N21[n,t]-e1 N21[n,t]

E0[n+1,t]

e0

F0[n+1,t]

F0[n,t]-e1 N23[n,t]+i1 N23[n,t]

G0[n+1,t]

G0[n,t]+a1 N31[n,t]-i1 N31[n,t]

H0[n+1,t]

H0[n,t]+e1 N32[n,t]-i1 N32[n,t]
```

```
I0[n+1,t]

i0

/* B0, C0, D0, F0, G0, H0 satisfy

the following conditions ( equation=0)*/

Simplify[eq1]

D0[n,t] N12[n,t]+G0[n,t] N13[n,t]-B0[n,t] N21[n,t]

+a1 N12[n,t] N21[n,t]-e1 N12[n,t] N21[n,t]-C0[n,t]

N31[n,t]+a1 N13[n,t] N31[n,t]-i1 N13[n,t] N31[n,t]

Simplify[eq5]

B0[n,t] N21[n,t]-N12[n,t] (D0[n,t]+(a1-e1)

 N21[n,t])+H0[n,t] N23[n,t]-(F0[n,t]+(-e1+i1)

 N23[n,t]) N32[n,t]

Simplify[eq9]

C0[n,t] N31[n,t]-N13[n,t] (G0[n,t]+(a1-i1)

 N31[n,t])+F0[n,t] N32[n,t]-N23[n,t]

(H0[n,t]+(e1-i1) N32[n,t])

/* final equation */
```

```
Simplify[eq2]

(-a0+e0) N12[n,t]+H0[n,t] N13[n,t]-C0[n,t]

 N32[n,t]+a1 N13[n,t] N32[n,t]-i1 N13[n,t]

 N32[n,t]+(N12^(0,1))[n,t]

Simplify[eq3]

F0[n,t] N12[n,t]+(-a0+i0) N13[n,t]-B0[n,t]

 N23[n,t]+a1 N12[n,t] N23[n,t]-e1 N12[n,t]

 N23[n,t]+(N13^(0,1))[n,t]

Simplify[eq4]

(a0-e0) N21[n,t]+G0[n,t] N23[n,t]-F0[n,t]

 N31[n,t]+e1 N23[n,t] N31[n,t]-i1 N23[n,t]

 N31[n,t]+(N21^(0,1))[n,t]

Simplify[eq6]

-D0[n,t] N13[n,t]+C0[n,t] N21[n,t]

-a1 N13[n,t] N21[n,t]+e1 N13[n,t] N21[n,t]

-e0 N23[n,t]+i0 N23[n,t]+(N23^(0,1))[n,t]

Simplify[eq7]
```

```
-H0[n,t] N21[n,t]+(a0-i0) N31[n,t]+D0[n,t]

 N32[n,t]-e1 N21[n,t] N32[n,t]+i1 N21[n,t]

 N32[n,t]+(N31^(0,1))[n,t]

Simplify[eq8]

-G0[n,t] N12[n,t]+B0[n,t] N31[n,t]-a1 N12[n,t] N31[n,t]

+i1 N12[n,t] N31[n,t]+e0 N32[n,t]-i0 N32[n,t]

+(N32^(0,1))[n,t]
```

BIOGRAPHICAL SKETCH

Ana T. Castillo was born in Rio Bravo, Tamaulipas Mexico. She attended elementary and middle school in Mexico and completed high school in the Rio Grande Valley. She received a bachelor's degree in Electrical Engineering, with an additional major in Spanish from the University of Texas-Pan American in 2004. She joined the Master's Program in Mathematics at the University of Texas-Pan American on July 2010. Her main research interests are soliton solutions to non-linear systems of differential equations for both discrete and continuous cases. She is an enthusiastic certified secondary mathematics and spanish teacher in the Rio Grande Valley. In her spare time, she enjoys running, listening to classical music, traveling, writing short stories, and reading classical and non-classical literature. Currently, she lives in Edinburg, Texas. She can be emailed at ana.t.castillo.rivas@gmail.com.