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2-Hyponormality on Unilateral Weighted Shifts

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2-HYPONORMALITY ON UNILATERAL WEIGHTED SHIFTS

A Thesis

by

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ABSTRACT

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Given the concept of a normal operator, several weaker notions have been proposed in order to extend the properties of normal operators to a wider range of operators. One such notion is that of k-hyponormal operators. In this document, we focus our attention on the 2-hyponormality of weighted shift operators over a discrete Hilbert space. It will be shown that if a certain relation between the weights $\alpha = (\alpha_0, \alpha_1, \dots)$ of a weighted shift W_α is satisfied, then the 2-hyponormality of W_α implies the hyponormality of W_α^m for any $m = 2, 3, \dots$.

DEDICATION

To my family, for their lovingly patience and support along the way.

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CHAPTER I

INTRODUCTION

Bounded normal operators are important and interesting operators; for instance, the spectral theorem applies to them. Finding characterizations for these operators is a task that continues under development, via weaker notions than normality. One such notion is that of subnormality, which is to say that an operator is subnormal if it can be extended into a normal operator. In turn, subnormality is characterized by means of k-hyponormality, which will be defined in the next chapter together with other preliminary definitions. Suffice it to say that an operator is subnormal if and only if it is k-hyponormal for all $k = 1, 2, \dots$.

The contents of this document originates from the following open problem, posed as Problem 1.1 in *When is hyponormality for 2-variable weighted shifts invariant under powers?* by R. Curto and J. Yoon in the Indiana University Mathematics Journal [2]:

Let T be a bounded operator on a separable Hilbert space $\mathscr{H}$. If T is k-hyponormal, is it true that T^m is also k-hyponormal, for $m = 1, 2, \dots$?

Looking to find either confirmation or a counterexample to the question above, in this document we study the 2-hyponormality of the unilateral weighted shift W_α with weights denoted by $\alpha = \{\alpha_0, \alpha_1, \dots\}$, and find a characterization that shows how to construct a 2-hyponormal weighted shift given any three consecutive weights $\alpha_i, \alpha_{i+1}, \alpha_{i+2}$. It will be shown that under a special case, the 2-hyponormality of W_α implies that W_α^2 also is 2-hyponormal. We conjecture that the original question has a positive answer in the case of a weighted shift. The proof of this last statement will be a topic for future research.

CHAPTER II

PRELIMINARIES

Normal operators are linear operators that commute with their adjoint, that is, T is normal if and only if $T^*T = TT^*$. To characterize and/or extend these operators, concepts such as subnormality and hyponormality have been introduced relaxing the requirement of normality, in the hope of extending the results for normal operators to operators that are almost normal in some sense.

Definition 2.0.1. A Hilbert space $\mathscr{H}$ is *separable* if it has a countable basis $\{e_0, e_1, \ldots\}$.

Definition 2.0.2. An operator $T : \mathscr{H} \to \mathscr{H}$ on a Hilbert space $\mathscr{H}$ is *bounded* if there exists a constant $c \geq 0$ such that

$$||Tx|| \leq c||x|| \quad \text{for all } x \in H.$$

The space of all bounded operators on a separable Hilbert space $\mathscr{H}$ is denoted $\mathscr{B}(\mathscr{H})$.

Definition 2.0.3. $T \in \mathscr{B}(\mathscr{H})$ is *subnormal* on a Hilbert space $\mathscr{H}$ if T has a normal extension, that is, T is subnormal if there exists a Hilbert space $\mathscr{K}$ such that $\mathscr{H}$ can be embedded in $\mathscr{K}$ and there exists a normal operator S of the form

$$S = \begin{pmatrix} T & A \\ 0 & B \end{pmatrix}$$

for some bounded operators $A : H^\perp \to H$ and $B : H^\perp \to H^\perp$.

It is clear that every normal operator is subnormal.

For $S, T \in \mathscr{B}(\mathscr{H})$ let $[S,T] := ST - TS$. Notice that T is normal if and only if $[T^*, T] = T^*T - TT^* = 0$.

Definition 2.0.4. A bounded linear operator T on a complex Hilbert space $\mathscr{H}$ is said to be *k-hyponormal* if

$$\begin{pmatrix} [\mathrm{T}^*,\mathrm{T}] & [(\mathrm{T}^*)^2,\mathrm{T}] & \dots & [(\mathrm{T}^*)^{\mathrm{k}},\mathrm{T}] \\ [\mathrm{T}^*,\mathrm{T}^2] & [(\mathrm{T}^*)^2,\mathrm{T}^2] & \dots & [(\mathrm{T}^*)^{\mathrm{k}},\mathrm{T}^2] \\ \vdots & \vdots & \ddots & \vdots \\ [\mathrm{T}^*,\mathrm{T}^{\mathrm{k}}] & [(\mathrm{T}^*)^2,\mathrm{T}^{\mathrm{k}}] & \dots & [(\mathrm{T}^*)^{\mathrm{k}},\mathrm{T}^{\mathrm{k}}] \end{pmatrix} \geq 0 \tag{2.1}$$

on the direct sum of k copies of $\mathscr{H}$.

In particular, when $k = 1$ we say that the operator T is *hyponormal* and in that case we only need $[T^*,T] = T^*T - TT^* \geq 0$. It follows from Definition (2.0.3) that every subnormal operator is hyponormal and, more in general, k-hyponormal for $k = 1,2,\dots$ since all the entries in matrix (2.1) would be 0. In this paper we will focus all of our attention on 2-hyponormal weighted shifts, which will be defined below.

Definition 2.0.5. For $\alpha \equiv \{\alpha_n\}_{n=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n = 0,1,\dots$ where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$.

It is easy to see that W_α is never normal, and that it is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n = 0,1,\dots$.

Next, some basic concepts will be defined in order to use a known characterization of 2-hyponormality for unilateral weighted shifts in terms of 3×3 Hankel matrices.

The *moments* of α are given by

$$\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{k-1}^2 & \text{if } k > 0 \end{cases} .$$

and they are important because of the following result, proved in Theorem 4 in [1]:

Theorem 2.0.1. *W_α is k-hyponormal if and only for all $n = 0, 1, \ldots$ the $(k+1) \times (k+1)$ matrices*

$$\Gamma_{(\alpha,k,n)} := \left(\gamma_{n+i+j-2}\right)_{i,j=1}^{k+1} \tag{2.2}$$

are positive.

Notice that in the theorem above, the matrices $\Gamma_{(\alpha,k,n)}$ corresponding to the characterization of k-hyponormality are submatrices of the corresponding matrices $\Gamma_{(\alpha,k+1,n)}$ characterizing $(k+1)$-hyponormality. Hence $(k+1)$-hyponormality implies k-hyponormality.

CHAPTER III

CHARACTERIZATION OF 2-HYPONORMAL WEIGHTED SHIFTS

In order to characterize 2-hyponormal weighted shifts, let us first do a simple calculation. Remember that a principal minor of an $n \times n$ matrix M is a submatrix obtained by deleting any k rows and the correspond k columns of M $(0 \leq k < n)$.

Lemma 3.0.1. *Given a matrix of the form*

$$M = \begin{pmatrix} 1 & a & ab \\ 1 & b & bc \\ 1 & c & cd \end{pmatrix}$$

where $0 < a < b < c < d$*, we have that all principal minors of* M *are non-negative if and only if*

$$\frac{d-c}{c-b} \geq \frac{a(c-b)}{c(b-a)} \iff d \geq \frac{a(c-b)^2}{c(b-a)} + c \tag{3.1}$$

Proof: Indeed, it is clear that all the 1×1 principal minors $1, b, cd$ are positive. Also, the 2×2 principal minors satisfy $b - a > 0$, $cd - ab > 0$ and $bcd - bc^2 > 0$. Finally, the 3×3 principal minor is the determinant of the matrix M, and it is non-negative if and only if

$$\begin{aligned} \det(M) &= bcd - bc^2 - a(cd - bc) + ab(c-b) \\ &= bc(d-c) - ac(d-c) - ac(c-b) + ab(c-b) \\ &= c(b-a)(d-c) + a(b-c)(c-b) \geq 0 \end{aligned}$$

$$\iff \quad \frac{d-c}{c-b} \geq \frac{a(c-b)}{c(b-a)}$$

$$\iff \quad d \geq \frac{a(c-b)^2}{c(b-a)} + c$$

which finishes the proof of this simple lemma. □

In fact, a separate calculation for $0 < a = b \leq c \leq d$ under the assumption $M \geq 0$ yields:

$$\begin{aligned} a = b, \quad \det(M) &= c(b-a)(d-c) + a(b-c)(c-b) \geq 0 \\ &\implies a(b-c)(c-b) \geq 0 \\ &\implies b = c \end{aligned} \tag{3.2}$$

which is a result with the following implication:

Lemma 3.0.2. *If W_α is a 2-hyponormal weighted shift with $\alpha_j = \alpha_{j+1}$ for some $j = 0, 1, \ldots,$ then $\alpha_n = \alpha_{n+1}$, for all $n = j, j+1, \ldots$.*

Proof: Theorem 2.0.1 with $k = 2$ says that if W_α is 2-hyponormal then

$$\Gamma_{(\alpha,2,n)} := \begin{pmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} \\ \gamma_{n+1} & \gamma_{n+2} & \gamma_{n+3} \\ \gamma_{n+2} & \gamma_{n+3} & \gamma_{n+4} \end{pmatrix} = \begin{pmatrix} \alpha_0^2 \cdots \alpha_{n-1}^2 & \alpha_0^2 \cdots \alpha_n^2 & \alpha_0^2 \cdots \alpha_{n+1}^2 \\ \alpha_0^2 \cdots \alpha_n^2 & \alpha_0^2 \cdots \alpha_{n+1}^2 & \alpha_0^2 \cdots \alpha_{n+2}^2 \\ \alpha_0^2 \cdots \alpha_{n+1}^2 & \alpha_0^2 \cdots \alpha_{n+2}^2 & \alpha_0^2 \cdots \alpha_{n+3}^2 \end{pmatrix} \geq 0$$

for all $n = 0, 1, \ldots$.

If $j = 0$, consider

$$\Gamma_{(\alpha,2,0)} = \begin{pmatrix} 1 & \alpha_1^2 & \alpha_1^4 \\ \alpha_1^2 & \alpha_1^4 & \alpha_1^4\alpha_2^2 \\ \alpha_1^4 & \alpha_1^4\alpha_2^2 & \alpha_1^4\alpha_2^2\alpha_3^2 \end{pmatrix}$$

which has determinant $-\alpha_1^8\left(\alpha_2^2 - \alpha_1^2\right)^2 \geq 0$ since W_α is 2-hyponormal. Hence $\alpha_1 = \alpha_2$. This proves the base case and now we can proceed by induction.

For the determinant of any matrix M, dividing a row of M by a positive constant does not change the sign of the determinant, and it does not change the sign of any of its minors either. If $n \geq 1$, in each matrix $\Gamma_{(\alpha,2,n)}$ if we divide the first row by $\alpha_0^2 \cdots \alpha_{n-1}^2$, the second row by $\alpha_0^2 \cdots \alpha_n^2$, and the third row by $\alpha_0^2 \cdots \alpha_{n+1}^2$, the sign of each principal minor of $\Gamma_{(\alpha,2,n)}$ is the same as the sign of the corresponding minor in:

$$\begin{pmatrix} 1 & \alpha_n^2 & \alpha_n^2 \alpha_{n+1}^2 \\ 1 & \alpha_{n+1}^2 & \alpha_{n+1}^2 \alpha_{n+2}^2 \\ 1 & \alpha_{n+2}^2 & \alpha_{n+2}^2 \alpha_{n+3}^2 \end{pmatrix}$$

which is of the form $\left(\begin{smallmatrix} 1 & a & ab \\ 1 & b & bc \\ 1 & c & cd \end{smallmatrix}\right)$.

The hypothesis that W_α is 2-hyponormal implies that the last matrix above has a non-negative determinant. Suppose $\alpha_j = \alpha_{j+1}$ for some $j \geq 1$; then from $n = j$ in the matrix above, together with (3.2) we obtain

$$\alpha_n^2 = \alpha_{n+1}^2 \implies \alpha_{n+1}^2 = \alpha_{n+2}^2$$

and the proof is finished. □

Theorem 3.0.3. *If W_α is 2-hyponormal, then exactly one of the following is true:*

1. $\alpha_0 = \alpha_1 = \ldots$;
2. $\alpha_0 < \alpha_1 = \alpha_2 = \ldots$; *or*
3. $\alpha_0 < \alpha_1 < \alpha_3 < \ldots$.

Proof: If $\alpha_0 = \alpha_1$, the previous lemma implies $\alpha_j = \alpha_{j+1}$ for all $j = 0, 1, \ldots$. On the other hand, if $\alpha_0 < \alpha_1$ and $\alpha_1 = \alpha_2$, then again the previous lemma implies *(2)*. Finally, if $\alpha_0 < \alpha_1 < \alpha_2$ then the determinant of $\Gamma_{(\alpha,2,0)}$ shows that

$$\alpha_3^2 \geq \frac{\alpha_0^2}{\alpha_2^2} \cdot \frac{(\alpha_2^2 - \alpha_1^2)^2}{\alpha_1^2 - \alpha_0^2} + \alpha_2^2 > \alpha_2^2$$

and therefore $\alpha_1 < \alpha_2 < \alpha_3$.

Next, suppose that for some index j we have $\alpha_j < \alpha_{j+1} < \alpha_{j+2}$. From (3.1) we have

$$\alpha_{n+3}^2 \geq \alpha_{n+2}^2 + \frac{\alpha_n^2}{\alpha_{n+2}^2} \frac{(\alpha_{n+2}^2 - \alpha_{n+1}^2)^2}{\alpha_{n+1}^2 - \alpha_n^2} > \alpha_{n+2}^2$$

so that $\alpha_{j+1} < \alpha_{j+2} < \alpha_{j+3}$. This finishes the proof by induction. □

Given the result in the theorem above, the weighted shifts W_α that are of interest in this discussion are those with α satisfying $\alpha_i < \alpha_{i+1}$, $i \in \mathbb{N}$. From now on this will be part of the hypothesis.

As we saw in the proof of Lemma (3.0.2), each minor of $\Gamma_{(\alpha,2,n)}$ has the same sign as the corresponding minor of a matrix of the form $\begin{pmatrix} 1 & a & ab \\ 1 & b & bc \\ 1 & c & cd \end{pmatrix}$. Hence we have the following result:

$$\det(\Gamma(\alpha,2,n)) \geq 0 \iff \frac{\alpha_{n+3}^2 - \alpha_{n+2}^2}{\alpha_{n+2}^2 - \alpha_{n+1}^2} \geq \frac{\alpha_n^2}{\alpha_{n+2}^2} \frac{\alpha_{n+2}^2 - \alpha_{n+1}^2}{\alpha_{n+1}^2 - \alpha_n^2} \tag{3.3}$$

$$\iff \alpha_{n+3}^2 \geq \alpha_{n+2}^2 + \frac{\alpha_n^2}{\alpha_{n+2}^2} \frac{(\alpha_{n+2}^2 - \alpha_{n+1}^2)^2}{\alpha_{n+1}^2 - \alpha_n^2} \tag{3.4}$$

and we state the above as the following theorem:

Theorem 3.0.4. *W_α is 2-hyponormal if and only if*

$$\frac{\alpha_{n+3}^2 - \alpha_{n+2}^2}{\alpha_{n+2}^2 - \alpha_{n+1}^2} \geq \frac{\alpha_n^2}{\alpha_{n+2}^2} \frac{\alpha_{n+2}^2 - \alpha_{n+1}^2}{\alpha_{n+1}^2 - \alpha_n^2}, \quad n = 0, 1, \ldots$$

or equivalently

$$\alpha_{n+3} \geq \alpha_{n+2} + \frac{\alpha_n}{\alpha_{n+2}} \frac{(\alpha_{n+2} - \alpha_{n+1})^2}{\alpha_{n+1} - \alpha_n}, \quad n = 0, 1, \ldots$$

Proof: Apply Lemma 3.0.1. □

J. G. Stampfli proves in [3] the following result, which makes the importance of (3.3) apparent:

Theorem 3.0.5. *Given $\alpha_0, \alpha_1, \alpha_2$ where $0 < |\alpha_0| < |\alpha_1| < |\alpha_2|$, then there exists a subnormal completion of $\alpha_0, \alpha_1, \alpha_2 \ldots$. Moreover, if S is any subnormal completion of $\alpha_0, \alpha_1, \alpha_2$, then*

$$\|S\|^2 \geqq \frac{1}{2}|\alpha_1|^2 \frac{|\alpha_2|^2 - |\alpha_0|^2}{|\alpha_1|^2 - |\alpha_0|^2} + \frac{1}{2}\sqrt{\left[|\alpha_1|^2 \frac{|\alpha_2|^2 - |\alpha_0|^2}{|\alpha_0|^2}\right]^2 - 4|\alpha_0|^2|\alpha_1|^2 \frac{|\alpha_2|^2 - |\alpha_1|^2}{|\alpha_1|^2 - |\alpha_0|^2}}$$

Further, there is (up to unitary equivalence) exactly one subnormal completion for which equality holds.

Stampfli arrives to this result by assuming there is a normal extension of the weighted shift W_α, that is, extending the operator

$$\begin{pmatrix} 0 & & & \\ \alpha_1 & 0 & & \\ & \alpha_2 & 0 & \\ & & \alpha_3 & \ddots \\ & & & \ddots \end{pmatrix}$$

first into

$$\left(\begin{array}{cccccc|cccccc} 0 & & & & & & b_1^{(2)} & & & & & \\ \alpha_1^{(1)} & 0 & & & & & & b_2^{(2)} & & & & \\ & \alpha_2^{(1)} & 0 & & & & & & b_3^{(2)} & & & \\ & & \alpha_3^{(1)} & \ddots & & & & & & \ddots & & \\ & & & \ddots & \ddots & & & & & & \ddots & \\ & & & & \ddots & \ddots & & & & & & \ddots \\ \hline & & & & & & 0 & & & & & \\ & & & & & & \alpha_1^{(2)} & 0 & & & & \\ & & & & & & & \alpha_2^{(2)} & 0 & & & \\ & & & & & & & & \alpha_3^{(2)} & \ddots & & \\ & & & & & & & & & \ddots & \ddots & \\ & & & & & & & & & & \ddots & \ddots \end{array}\right)$$

with $\alpha_n^{(1)} = \alpha_n$, and then he continues extending the operator recursively. Depending on the specific value of the weights, this process might end after a finite number of steps, or require a countable number of extensions.

It turns out that the subnormal completion which satisfies the equality in Theorem 3.0.5 is given by the relation

$$\frac{\alpha_{n+3}^2-\alpha_{n+2}^2}{\alpha_{n+2}^2-\alpha_{n+1}^2}=\frac{\alpha_0^2}{\alpha_{n+2}^2}\frac{\alpha_{n+2}^2-\alpha_{n+1}^2}{\alpha_{n+1}^2-\alpha_0^2}, \quad n=0,1,\dots$$

hence if W_α is a 2-hyponormal weighted shift that satisfies the equality in (3.3) then W_α is also subnormal. Since subnormality implies 2-hyponormality, this result seems trivial. But the fact that we arrived to the same expression for hyponormality as Stampfli did in Theorem 3.0.5 brings us to the question of how to relate k-hyponormality with Stampfli's characterization of subnormality.

Back to the concept of 2-hyponormality, a little more can be said about W_α in this circumstance. First, $W_\alpha^2 = W_\alpha \circ W_\alpha$ can be decomposed as the direct sum of two weighted shifts as described next.

Split the Hilbert space $\mathscr{H}$ as the direct sum of the two orthogonal subspaces

$$\mathscr{H}_1=\operatorname{span}\langle e_0,e_2,\dots\rangle, \quad \mathscr{H}_2=\operatorname{span}\langle e_1,e_3,\dots\rangle.$$

Then, from

$$W_\alpha^2(e_n)=W_\alpha(W_\alpha(e_n))=W_\alpha(\alpha_k e_{n+1})=\alpha_n\alpha_{n+1}e_{n+2}$$

it is clear that $W_\alpha^2(\mathscr{H}_1)\subseteq\mathscr{H}_1$, $W_\alpha^2(\mathscr{H}_2)\subseteq\mathscr{H}_2$, and thus $W_{\alpha(2:0)} := W_\alpha^2|_{\mathscr{H}_1}$ and $W_{\alpha(2:1)} := W_\alpha^2|_{\mathscr{H}_2}$ satisfy

$$W_\alpha^2 = W_{\alpha(2:0)}\oplus W_{\alpha(2:1)} \,:\, \mathscr{H}_1\oplus\mathscr{H}_2 \longrightarrow \mathscr{H}_1\oplus\mathscr{H}_2\,.$$

It will be shown next that $W_{\alpha(2:0)}$ and $W_{\alpha(2:1)}$ are also 2-hyponormal when (3.3) is an equality.

Let $e_n' = e_{2n},\ n=0,1,\dots$ be a renaming of the basis for $\mathscr{H}_1$, and similarly let $\alpha_n' = \alpha_{2n}\alpha_{2n+1}$.

Then by definition of $W_{\alpha(2:0)}$ we have

$$W_{\alpha(2:0)}(e'_n) = W_\alpha^2(e_{2n}) = W_\alpha(\alpha_{2n}e_{2n+1}) = \alpha_{2n}\alpha_{2n+1}e_{2n+2} = \alpha'_n e'_{n+1}$$

making it clear that the weight sequence denoted $\alpha(2:0)$ for the weighted shift $W_{\alpha(2:0)}$ is given by

$$\alpha(2:0) = (\alpha'_0, \alpha'_1, \dots) = (\alpha_0\alpha_1, \alpha_2\alpha_3, \dots)$$

and a similar calculation shows that the weight sequence for $W_{\alpha(2:1)}$ with weights denoted by $\alpha(2:1)$ is

$$\alpha(2:1) = (\alpha_1\alpha_2, \alpha_3\alpha_4, \dots).$$

Since $\alpha_n < \alpha_{n+1}$ then $\alpha'_n < \alpha'_{n+1}$, and formula (3.3) in Theorem 3.0.4 tells us that $W_{\alpha(2:0)}$ is 2-hyponormal if and only if for all $n = 0, 1, \dots,$

$$\frac{\alpha_{2n+6}^2\alpha_{2n+7}^2 - \alpha_{2n+4}^2\alpha_{2n+5}^2}{\alpha_{2n+4}^2\alpha_{2n+5}^2 - \alpha_{2n+2}^2\alpha_{2n+3}^2} \geq \frac{\alpha_{2n}^2\alpha_{2n+1}^2}{\alpha_{2n+4}^2\alpha_{2n+5}^2} \frac{\alpha_{2n+4}^2\alpha_{2n+5}^2 - \alpha_{2n+2}^2\alpha_{2n+3}^2}{\alpha_{2n+2}^2\alpha_{2n+3}^2 - \alpha_{2n}^2\alpha_{2n+1}^2}. \tag{3.5}$$

We want to show that when equality holds for W_α in 3.3, then equality also holds in formula (3.5) for $W_{\alpha(2:0)}$. That is

$$\frac{\alpha_{n+3}^2 - \alpha_{n+2}^2}{\alpha_{n+2}^2 - \alpha_{n+1}^2} = \frac{\alpha_n^2}{\alpha_{n+2}^2} \frac{\alpha_{n+2}^2 - \alpha_{n+1}^2}{\alpha_{n+1}^2 - \alpha_n^2} \quad \text{for all } n = 0, 1, \dots \tag{3.6}$$

$$\implies \frac{\alpha_{2n+6}^2\alpha_{2n+7}^2 - \alpha_{2n+4}^2\alpha_{2n+5}^2}{\alpha_{2n+4}^2\alpha_{2n+5}^2 - \alpha_{2n+2}^2\alpha_{2n+3}^2} = \frac{\alpha_{2n}^2\alpha_{2n+1}^2}{\alpha_{2n+4}^2\alpha_{2n+5}^2} \frac{\alpha_{2n+4}^2\alpha_{2n+5}^2 - \alpha_{2n+2}^2\alpha_{2n+3}^2}{\alpha_{2n+2}^2\alpha_{2n+3}^2 - \alpha_{2n}^2\alpha_{2n+1}^2} \quad \text{for all } n = 0, 1, \dots \tag{3.7}$$

In fact, (3.6) will only be needed for n even, that is

$$\frac{\alpha_{2n+3}^2 - \alpha_{2n+2}^2}{\alpha_{2n+2}^2 - \alpha_{2n+1}^2} = \frac{\alpha_{2n}^2}{\alpha_{2n+2}^2} \frac{\alpha_{2n+2}^2 - \alpha_{2n+1}^2}{\alpha_{2n+1}^2 - \alpha_{2n}^2} \quad \text{for all } n = 0, 1, \dots \tag{3.8}$$

The non-linear nature of the recurrence relation makes the calculations extremely long. Using (3.8) and making $C_n = \alpha_{2n}^2 \alpha_{2n+1}^2 \frac{\alpha_{2n+2}^2 - \alpha_{2n+1}^2}{\alpha_{2n+1}^2 - \alpha_{2n}^2}$ it is possible to rewrite each term α_{2n+3}^2, $\alpha_{2n+4}^2, \ldots,$ α_{2n+7}^2 as follows:

$$\begin{aligned}
\frac{\alpha_{2n+3}^2 - \alpha_{2n+2}^2}{\alpha_{2n+2}^2 - \alpha_{2n+1}^2} &= \frac{\alpha_{2n}^2}{\alpha_{2n+2}^2} \frac{\alpha_{2n+2}^2 - \alpha_{2n+1}^2}{\alpha_{2n+1}^2 - \alpha_{2n}^2} = \frac{C_n}{\alpha_{2n+1}^2 \alpha_{2n+2}^2}; \\
\frac{\alpha_{2n+4}^2 - \alpha_{2n+3}^2}{\alpha_{2n+3}^2 - \alpha_{2n+2}^2} &= \frac{\alpha_{2n+1}^2}{\alpha_{2n+3}^2} \frac{\alpha_{2n+3}^2 - \alpha_{2n+2}^2}{\alpha_{2n+2}^2 - \alpha_{2n+1}^2} = \frac{\alpha_{2n+1}^2}{\alpha_{2n+3}^2} \frac{C_n}{\alpha_{2n+1}^2 \alpha_{2n+2}^2} = \frac{C_n}{\alpha_{2n+2}^2 \alpha_{2n+3}^2}; \\
\frac{\alpha_{2n+5}^2 - \alpha_{2n+4}^2}{\alpha_{2n+4}^2 - \alpha_{2n+3}^2} &= \frac{\alpha_{2n+2}^2}{\alpha_{2n+4}^2} \frac{\alpha_{2n+4}^2 - \alpha_{2n+3}^2}{\alpha_{2n+3}^2 - \alpha_{2n+2}^2} = \frac{\alpha_{2n+2}^2}{\alpha_{2n+4}^2} \frac{C_n}{\alpha_{2n+2}^2 \alpha_{2n+3}^2} = \frac{C_n}{\alpha_{2n+3}^2 \alpha_{2n+4}^2}; \\
\frac{\alpha_{2n+6}^2 - \alpha_{2n+5}^2}{\alpha_{2n+5}^2 - \alpha_{2n+4}^2} &= \frac{\alpha_{2n+3}^2}{\alpha_{2n+5}^2} \frac{\alpha_{2n+5}^2 - \alpha_{2n+4}^2}{\alpha_{2n+4}^2 - \alpha_{2n+3}^2} = \frac{\alpha_{2n+3}^2}{\alpha_{2n+5}^2} \frac{C_n}{\alpha_{2n+3}^2 \alpha_{2n+4}^2} = \frac{C_n}{\alpha_{2n+4}^2 \alpha_{2n+5}^2}; \\
\frac{\alpha_{2n+7}^2 - \alpha_{2n+6}^2}{\alpha_{2n+6}^2 - \alpha_{2n+5}^2} &= \frac{\alpha_{2n+4}^2}{\alpha_{2n+6}^2} \frac{\alpha_{2n+6}^2 - \alpha_{2n+5}^2}{\alpha_{2n+5}^2 - \alpha_{2n+4}^2} = \frac{\alpha_{2n+4}^2}{\alpha_{2n+6}^2} \frac{C_n}{\alpha_{2n+4}^2 \alpha_{2n+5}^2} = \frac{C_n}{\alpha_{2n+5}^2 \alpha_{2n+6}^2};
\end{aligned} \tag{3.9}$$

that is

$$\begin{aligned}
&\frac{\alpha_{2n+k}^2 - \alpha_{2n+k-1}^2}{\alpha_{2n+k-1}^2 - \alpha_{2n+k-2}^2} = \frac{C_n}{\alpha_{2n+k-1}^2 \alpha_{2n+k-2}^2}, \quad n = 3,4,\ldots,7 \\
&\Longrightarrow \quad \alpha_{2n+k}^2 = \alpha_{2n+k-1}^2 + \left(\frac{1}{\alpha_{2n+k-2}^2} - \frac{1}{\alpha_{2n+k-1}^2} \right) C_n, \quad n = 3,4,\ldots,7
\end{aligned} \tag{3.10}$$

which means that to obtain an expression for α_{2n+k}^2 we need to substitute recursively α_{2n+k-1}^2 and α_{2n+k-2}^2. We can do a little better replacing recursively the last expression above in (3.9) after solving there for the highest index, resulting in

$$\begin{aligned}
\alpha_{2n+3}^2 &= \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+2}^2} \right) C_n; \\
\alpha_{2n+4}^2 &= \alpha_{2n+3}^2 + \left(\frac{1}{\alpha_{2n+2}^2} - \frac{1}{\alpha_{2n+3}^2} \right) C_n
\end{aligned}$$

$$= \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+2}^2}\right) C_n + \left(\frac{1}{\alpha_{2n+2}^2} - \frac{1}{\alpha_{2n+3}^2}\right) C_n$$

$$= \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+3}^2}\right) C_n; \tag{3.11}$$

$$\alpha_{2n+5}^2 = \alpha_{2n+4}^2 + \left(\frac{1}{\alpha_{2n+3}^2} - \frac{1}{\alpha_{2n+4}^2}\right) C_n$$

$$= \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+3}^2}\right) C_n + \left(\frac{1}{\alpha_{2n+3}^2} - \frac{1}{\alpha_{2n+4}^2}\right) C_n$$

$$= \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+4}^2}\right) C_n; \tag{3.12}$$

$$\alpha_{2n+6}^2 = \cdots = \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+5}^2}\right) C_n;$$

$$\alpha_{2n+7}^2 = \cdots = \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+6}^2}\right) C_n.$$

or simply put,

$$\alpha_{2n+k}^2 = \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+k-1}^2}\right) C_n, \quad \text{for all } k = 3, 4, \ldots, 7 \tag{3.13}$$

It is clear now that substituting α_{2n+3}^2 in α_{2n+4}^2 as given by (3.11) makes α_{2n+4}^2 into an expression of only $\alpha_{2n}^2, \alpha_{2n+1}^2$ and α_{2n+2}^2; subsequently, substituting α_{2n+4}^2 in α_{2n+5}^2 as given by (3.12) makes α_{2n+5}^2 into an expression of the same $\alpha_{2n}^2, \alpha_{2n+1}^2$ and α_{2n+2}^2; repeating this process, the same can be said of α_{2n+6}^2 and α_{2n+7}^2. We can see the expressions for α_{2n+3}^2 and α_{2n+4}^2 in terms of $\alpha_{2n}^2, \alpha_{2n+1}^2$ and α_{2n+2}^2 below:

$$\alpha_{2n+3}^2 = \frac{\alpha_{2n+1}^2 \left(\alpha_{2n+2}^4 - 2\alpha_{2n}^2\alpha_{2n+2}^2 + \alpha_{2n}^2\alpha_{2n+1}^2\right)}{\left(\alpha_{2n+1}^2 - \alpha_{2n}^2\right)\alpha_{2n+2}^2};$$

$$\alpha_{2n+4}^2 = \frac{\alpha_{2n+1}^2(\alpha_{2n+2}^6 - 4\alpha_{2n}^2\alpha_{2n+2}^4 + 2\alpha_{2n}^2\alpha_{2n+1}^2\alpha_{2n+2}^2 + \alpha_{2n}^4\alpha_{2n+2}^2 - \alpha_{2n}^4\alpha_{2n+1}^2) + \alpha_{2n}^4\alpha_{2n+2}^4}{\left(\alpha_{2n+1}^2 - \alpha_{2n}^2\right)\left(\alpha_{2n+2}^4 - 2\alpha_{2n}^2\alpha_{2n+2}^2 + \alpha_{2n}^2\alpha_{2n+1}^2\right)};$$

but clearly the calculations become long very fast. In Appendix A the corresponding formulas for $\alpha_{2n+5}^2, \alpha_{2n+6}^2$ and α_{2n+7}^2 will be given.

Substituting these expressions in terms of α_{2n}^2, α_{2n+1}^2 and α_{2n+2}^2 on the left and right-hand sides of (3.7) shows that equality holds. The calculations are substantially long, hence we ask the reader to take our word for it at the moment. In Appendix B, code to verify (3.7) using a computer algebra system will be provided.

All of the calculations made for $W_{\alpha(2:0)}$ are valid for $W_{\alpha(2:1)}$; an increment in the indexes by +1 is sufficient. Hence if W_α satisfies (3.6) then both $W_{\alpha(2:0)}$ and $W_{\alpha(2:1)}$ also do. In summary, applying Theorem 3.0.5 we have the following result:

Theorem 3.0.6. *If the weighted shift W_α satisfies (3.6), then W_α^2, $W_{\alpha(2:0)}$ and $W_{\alpha(2:1)}$ are subnormal.*

As a corollary we obtain a partial answer to the open problem given in the introduction:

Corollary. *If W_α is 2-hyponormal and satisfies (3.6), then $W_\alpha^2 = W_{\alpha(2:0)} \oplus W_{\alpha(2:1)}$ is also 2-hyponormal.*

CHAPTER IV

FUTURE WORK

To address the original question posed in the introduction, the work made in the previous chapter can be expanded as follows:

- We found that when formula (3.3) is satisfied as an equality, then the weighted shift in question is subnormal. There is still the possibility that if (3.3) is met with $>$ instead, then although 2-hyponormality is guaranteed, k-hyponormality is not given for $k = 3, 4, \ldots$.

- We have conjectured that 2-hyponormality of W_α implies that of W_α^m for any natural m. Evidence towards this was obtained by generating several 2-hyponormal weighted shifts with different rates of growth among its weights. Whenever (3.3) was met with $>$ sign, the same was true for formula (3.5). We are optimistic that using formula (3.3) it is possible to prove formula (3.5). The fact that (3.13) was derived in our calculations is encouraging in that sense.

- It remains to be proved that the above is also true for k-hyponormality in general. Possibly a characterization from the case $k = 2$ will emerge once a proof is found; perhaps the calculations can be extended in such a way that the general case is confirmed. We are inclined to think this could be the case.

- If the general case is proved, it means we could not use weighted shifts as a source to generate counterexamples to the original question proposed in [2]. To attack this question, it would be necessary to look at other operators in search for confirmation or counterexamples.

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APPENDIX A

APPENDIX A

EXPLICIT EXPRESSIONS FOR THE WEIGHTS

The weights α_{2n}, α_{2n+1} and α_{2n+2} are the weights used to express all subsequent weights in terms of them by applying formula (3.13) recursively:

$$\alpha_{2n+k}^2 = \alpha_{2n+2}^2 + \left(\frac{1}{\alpha_{2n+1}^2} - \frac{1}{\alpha_{2n+k-1}^2} \right) C_n, \quad \text{for all } k = 3, 4, \ldots, 7$$

where $C_n = \alpha_{2n}^2 \alpha_{2n+1}^2 \frac{\alpha_{2n+2}^2 - \alpha_{2n+1}^2}{\alpha_{2n+1}^2 - \alpha_{2n}^2}$.

After factoring as polynomials in the integers, the corresponding expressions are as follows:

$$\alpha_{2n+3}^2 = \frac{\alpha_{2n+1}^2 \left(\alpha_{2n+2}^4 - 2\alpha_{2n}^2 \alpha_{2n+2}^2 + \alpha_{2n}^2 \alpha_{2n+1}^2 \right)}{\left(\alpha_{2n+1}^2 - \alpha_{2n}^2 \right) \alpha_{2n+2}^2};$$

$$\alpha_{2n+4}^2 = \frac{\alpha_{2n+1}^2 (\alpha_{2n+2}^6 - 4\alpha_{2n}^2 \alpha_{2n+2}^4 + 2\alpha_{2n}^2 \alpha_{2n+1}^2 \alpha_{2n+2}^2 + \alpha_{2n}^4 \alpha_{2n+2}^2 - \alpha_{2n}^4 \alpha_{2n+1}^2) + \alpha_{2n}^4 \alpha_{2n+2}^4}{\left(\alpha_{2n+1}^2 - \alpha_{2n}^2 \right) \left(\alpha_{2n+2}^4 - 2\alpha_{2n}^2 \alpha_{2n+2}^2 + \alpha_{2n}^2 \alpha_{2n+1}^2 \right)};$$

$$\alpha_{2n+5}^2 = \frac{\alpha_{2n+1}^2}{\alpha_{2n+1}^2 - \alpha_{2n}^2} \cdot \frac{P_n}{Q_n},$$

$$\begin{aligned} P_n = {} & \alpha_{2n+1}^2 \alpha_{2n+2}^8 - 6\alpha_{2n}^2 \alpha_{2n+1}^2 \alpha_{2n+2}^6 + 2\alpha_{2n}^4 \alpha_{2n+2}^6 + 3\alpha_{2n}^2 \alpha_{2n+1}^4 \alpha_{2n+2}^4 + 6\alpha_{2n}^4 \alpha_{2n+1}^2 \alpha_{2n+2}^4 \\ & - 3\alpha_{2n}^6 \alpha_{2n+2}^4 - 6\alpha_{2n}^4 \alpha_{2n+1}^4 \alpha_{2n+2}^2 + 2\alpha_{2n}^6 \alpha_{2n+1}^2 \alpha_{2n+2}^2 + \alpha_{2n}^4 \alpha_{2n+1}^6, \end{aligned}$$

$$\begin{aligned} Q_n = {} & \alpha_{2n+1}^2 \alpha_{2n+2}^6 - 4\alpha_{2n}^2 \alpha_{2n+1}^2 \alpha_{2n+2}^4 + \alpha_{2n}^4 \alpha_{2n+2}^4 \\ & + 2\alpha_{2n}^2 \alpha_{2n+1}^4 \alpha_{2n+2}^2 + \alpha_{2n}^4 \alpha_{2n+1}^2 \alpha_{2n+2}^2 - \alpha_{2n}^4 \alpha_{2n+1}^4; \end{aligned}$$

$$\alpha_{2n+6}^2 = \frac{1}{\alpha_{2n+1}^2 - \alpha_{2n}^2} \cdot \frac{R_n}{P_n},$$

$$
\begin{aligned}
R_n =& \alpha_{2n+1}^4\alpha_{2n+2}^{10} - 8\alpha_{2n}^2\alpha_{2n+1}^4\alpha_{2n+2}^8 + 3\alpha_{2n}^4\alpha_{2n+1}^2\alpha_{2n+2}^8 + 4\alpha_{2n}^2\alpha_{2n+1}^6\alpha_{2n+2}^6 \\
&+ 15\alpha_{2n}^4\alpha_{2n+1}^4\alpha_{2n+2}^6 - 10\alpha_{2n}^6\alpha_{2n+1}^2\alpha_{2n+2}^6 + \alpha_{2n}^8\alpha_{2n+2}^6 - 15\alpha_{2n}^4\alpha_{2n+1}^6\alpha_{2n+2}^4 \\
&+ 2\alpha_{2n}^6\alpha_{2n+1}^4\alpha_{2n+2}^4 + 3\alpha_{2n}^8\alpha_{2n+1}^2\alpha_{2n+2}^4 + 3\alpha_{2n}^4\alpha_{2n+1}^8\alpha_{2n+2}^2 + 6\alpha_{2n}^6\alpha_{2n+1}^6\alpha_{2n+2}^2 \\
&- 4\alpha_{2n}^8\alpha_{2n+1}^4\alpha_{2n+2}^2 - 2\alpha_{2n}^6\alpha_{2n+1}^8 + \alpha_{2n}^8\alpha_{2n+1}^6;
\end{aligned}
$$

$$
\alpha_{2n+7}^2 = \frac{\alpha_{2n+1}^2}{\alpha_{2n+1}^2 - \alpha_{2n}^2} \cdot \frac{S_n}{R_n},
$$

$$
\begin{aligned}
S_n =& \alpha_{2n+1}^4\alpha_{2n+2}^{12} - 10\alpha_{2n}^2\alpha_{2n+1}^4\alpha_{2n+2}^{10} + 4\alpha_{2n}^4\alpha_{2n+1}^2\alpha_{2n+2}^{10} + 5\alpha_{2n}^2\alpha_{2n+1}^6\alpha_{2n+2}^8 \\
&+ 28\alpha_{2n}^4\alpha_{2n+1}^4\alpha_{2n+2}^8 - 21\alpha_{2n}^6\alpha_{2n+1}^2\alpha_{2n+2}^8 + 3\alpha_{2n}^8\alpha_{2n+2}^8 - 28\alpha_{2n}^4\alpha_{2n+1}^6\alpha_{2n+2}^6 \\
&- 8\alpha_{2n}^6\alpha_{2n+1}^4\alpha_{2n+2}^6 + 20\alpha_{2n}^8\alpha_{2n+1}^2\alpha_{2n+2}^6 - 4\alpha_{2n}^{10}\alpha_{2n+2}^6 + 6\alpha_{2n}^4\alpha_{2n+1}^8\alpha_{2n+2}^4 \\
&+ 30\alpha_{2n}^6\alpha_{2n+1}^6\alpha_{2n+2}^4 - 23\alpha_{2n}^8\alpha_{2n+1}^4\alpha_{2n+2}^4 + 2\alpha_{2n}^{10}\alpha_{2n+1}^2\alpha_{2n+2}^4 - 12\alpha_{2n}^6\alpha_{2n+1}^8\alpha_{2n+2}^2 \\
&+ 4\alpha_{2n}^8\alpha_{2n+1}^6\alpha_{2n+2}^2 + 2\alpha_{2n}^{10}\alpha_{2n+1}^4\alpha_{2n+2}^2 + \alpha_{2n}^6\alpha_{2n+1}^{10} + \alpha_{2n}^8\alpha_{2n+1}^8 - \alpha_{2n}^{10}\alpha_{2n+1}^6.
\end{aligned}
$$

APPENDIX B

APPENDIX B

COMPUTER ALGEBRA SYSTEM CODE

The following follows the syntax of the computer algebra system Maxima, and can be used in Maxima to verify all calculations.

The function `a(x)` defined below calculates α^2_{2n+k} in terms of α^2_{2n}, α^2_{2n+1} and α^2_{2n+2} recursively, as derived from the hypothesis (3.6) in (3.10). To make the code simpler, we let `a_j` represent α^2_{2n+j}.

(% i2)

```
C_n :  a_0 * a_1 * (a_2 - a_1) / (a_1 - a_0);
a(x) :=
block (
if x = 0
then return(a_0)
else
if x = 1
then return(a_1)
else
if x = 2
then return(a_2)
else
return((a_1 + C_n * (1/a_0 - 1/a(x-1) ) ))
);
```

$$\frac{a_0 a_1 (a_2 - a_1)}{a_1 - a_0} \qquad \text{(C\_ n)}$$

$$\mathrm{a}(x) := \mathrm{block}(\mathrm{if}\, x = 0\, \mathrm{then\, return}\,(a_0)\, \mathrm{else\, if}\, x = 1\, \mathrm{then\, return}\,(a_1)\, \mathrm{else\, if}\, x = 2 \ldots \tag{\% o2}$$

(% i3) `factor(a(3));`

$$\frac{a_1\left(a_2{}^2 - 2a_0a_2 + a_0a_1\right)}{(a_1 - a_0)\, a_2} \tag{\% o3}$$

(% i4) `factor(a(4));`

$$\frac{a_1a_2{}^3 - 4a_0a_1a_2{}^2 + a_0{}^2a_2{}^2 + 2a_0a_1{}^2a_2 + a_0{}^2a_1a_2 - a_0{}^2a_1{}^2}{(a_1 - a_0)\left(a_2{}^2 - 2a_0a_2 + a_0a_1\right)} \tag{\% o4}$$

(% i5) `factor(a(5));`

$$(a_1(a_1a_2{}^4 - 6a_0a_1a_2{}^3 + 2a_0{}^2a_2{}^3 + 3a_0a_1{}^2a_2{}^2 + 6a_0{}^2a_1a_2{}^2 - 3a_0{}^3a_2{}^2 - 6a_0{}^2a_1{}^2a_2 + 2a_0{}^3a_1a_2 + \ldots \tag{\% o5}$$

(% i6) `factor(a(6));`

$$(a_1{}^2a_2{}^5 - 8a_0a_1{}^2a_2{}^4 + 3a_0{}^2a_1a_2{}^4 + 4a_0a_1{}^3a_2{}^3 + 15a_0{}^2a_1{}^2a_2{}^3 - 10a_0{}^3a_1a_2{}^3 + a_0{}^4a_2{}^3 - 15a_0{}^2a_1{}^3 \ldots \tag{\% o6}$$

(% i7) `factor(a(7));`

$$(a_1(a_1{}^2a_2{}^6 - 10a_0a_1{}^2a_2{}^5 + 4a_0{}^2a_1a_2{}^5 + 5a_0a_1{}^3a_2{}^4 + 28a_0{}^2a_1{}^2a_2{}^4 - 21a_0{}^3a_1a_2{}^4 + 3a_0{}^4a_2{}^4 - 28\ldots \tag{\% o7}$$

If formula (3.7) holds, solving for α^2_{2n+7} should give us an exact representation of α^2_{2n+7} in terms of $\alpha^2_{2n}, \ldots, \alpha^2_{2n+6}$ and the following difference should be equal to 0.

(% i8)
```
difference :  a(7) - 1/a(6) * (a(4)*a(5) + a(0)*a(1)/(a(4)*a(5))
( (a(4)*a(5) - a(2)*a(3))^2 / (a(2)*a(3) - a(0)*a(1)) )) ;
```

(% i9) `ratsimp(expression);`

$$0 \quad (\% o9)$$

BIOGRAPHICAL SKETCH

Juan Benitez was born in Bogota, Colombia were he received his degree in math from the Universidad Nacional de Colombia. He received a Masters in Mathematics at the University of Texas - Rio Grande Valley in August 2019.

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