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Qualitative Analysis of The Burgers-Huxley Equation

Jing Tian

University of Texas-Pan American

QUALITATIVE ANALYSIS OF THE BURGERS-HUXLEY EQUATION

A Thesis

by

JING TIAN

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COMMITTEE MEMBERS

Zhaosheng Feng
Chair of Committee

Andras Balogh
Committee Member

Paul Bracken
Committee Member

Zhijun Qiao
Committee Member

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ABSTRACT

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There are many well-known techniques for obtaining exact solutions of differential equations, but some of them only work for a very limited class of problems and are merely special cases of a few power symmetry methods. These approaches can be applied to nonlinear differential of unfamiliar type; they do not rely on special “tricks." Instead, a given differential equation can be made to reveal its symmetries, which are then used to construct exact solutions.

In this thesis, we briefly present the theory of the Lie symmetry method for finding exact solutions of nonlinear differential equations, then apply it to the study of the generalized Burgers-Huxley equation. Through analyzing the linearized symmetry condition and the associated determining system, we find two nontrivial infinitesimal generators, and obtain exact solutions by solving the reduced differential equation under certain parametric conditions. An approximate solution of the generalized Burgers-Huxley equation is established by means of the Adomian decomposition method.

DEDICATION

I would like to dedicate my thesis to My mother, Xiaomei Liu, my father, Tongbin Tian for their love and support. Also, my grandfather, Zhigao Liu, whom I lost last year, I will always keep his love in my heart.

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CHAPTER I

INTRODUCTION

There is the widespread existence of wave phenomena in physics, chemistry and biology [1, 15]. There has been a correspondingly rich development of mathematical concepts and techniques to understand wave phenomena from the theoretical standpoint and to solve the problems that arise. It is well recognized that a great number of such kind of problems can be modeled as nonlinear differential equations and systems. Finding innovative methods to solve and analyze these equations has been an interesting subject in the field of differential equations and dynamical systems. In these problems, it is not always possible and sometimes not even advantageous to express various wave solutions of nonlinear partial equations explicitly in terms of elementary functions, but it is possible to find and prove the existence of traveling wave solutions by the qualitative theory of differential equations and dynamical systems [2, 5, 19]. Traveling wave solutions usually can be characterized as solutions invariant with respect to translation in space, and determine the behavior of the solutions of the Cauchy-type problems. From the physical point of view, traveling waves usually describe transition processes. Transition from one equilibrium to another is a typical case although more complicated situations can arise. These transition processes in many cases ignore their initial conditions and reflect the properties of the medium itself. Different types of traveling waves are of fundamental importance to our understanding of physical and biological phenomena modeled differential equations. In fact, for any physical or biological system where the dynamics is driven by, and mainly determined by, phase coherence of the individual waves, it has applications and consequences [3, 15, 18].

In this thesis, we consider the following equation of the form

$$\frac{\partial u}{\partial t} + \alpha u^{\delta}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u^{\delta})(u^{\delta}-\gamma), \tag{1}$$

where α, β, γ and δ are real constants. This equation is referred to as the generalized Burgers-Huxley equation [9, 20]. Here the choices of $\alpha = 0$ and $\delta = 1$ leads equation (1) to the reduced Huxley equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u)(u-\gamma). \tag{2}$$

Equation (2) is originally reduced from the Hodgkin-Huxley model that explains the ionic mechanisms underlying the initiation and propagation of action potentials in the squid giant axon [12]. The model initially consists of a set of nonlinear ordinary differential equations that approximates the electrical characteristics of excitable cells such as neurons and cardiac myocytes. Solitary wave solutions of equation (2) have been investigated widely by the Hirota method [17], by the particular ansatz [20], and by the Painlevé analysis [9] et al.

The choices of $\beta = 0$ and $\delta = 1$ leads equation (1) to the Burgers equation [6, 7]

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{3}$$

Equation (3) describes the far field of wave propagation in nonlinear dissipative systems, and has been chosen as a simplified form of the Navier-Stokes equations [6] and as the simplest one to combine both nonlinear propagation effects and diffusive effects [7]. Exact solution of equation (3) can be found directly through the Cole-Hopf transformation [7].

In the past two decades, there is considerable attention devoted to the study of solitary wave solutions of equation (1). In [11], the Adomian decomposition method was applied to explore an analytical solution to equation (1) in the form of a convergent series. Solitary traveling wave solutions of equation (1) were presented explicitly by the particular ansatz [20] and the Painlevé analysis [9], respectively. Recently, traveling wave solutions to equation (1) were studied by using the first-integral method [8], which is based on the ring theory of commutative algebra [10], and a class of traveling solitary wave solutions for the generalized Burgers-Huxley equation was described in a straightforward manner. However, for various kinds of nonlinear differential equations and systems, the particular ansatz method only works for a very limited class of problems. When the first integral method [8] and the Painlevé analysis [9] are applied to the generalized Burgers-Huxley equation (1), they work only under the assumption that δ is a positive integer. In this paper, we extend the study of traveling wave solutions of equation (1) to a more general case with δ being an arbitrary positive number through the Lie symmetry reduction method [21, 22].

The rest of the thesis is organized as follows. In Section 2, we give a short introduction of the Lie symmetry method to the Ordinary differential equations and the Partial differential equations. In Section 3, Lie point symmetries for the Burgers-Huxley equation are found by differentiating the symmetry condition. Two nontrivial infinitesimal generators are obtained through analyzing the determining system. A class of traveling wave solutions of equation is presented. Section 4 is the process of using the Lie symmetry method to a special equation. Section 5 is the case for the generalised Huxley equation. Section 6 is using the Adomian decomposition method to obtain the approximate solutions.

CHAPTER II

PRELIMINARIES

Following references [4, 21, 22] , I will give the brief introduction of Lie symmetry.

2.1 The Partial Differential Equations

In this section, in order to present our results in a straightforward way, we start our attention by briefly reviewing the basic concepts about the partial differential equations.

There are three types of PDEs: the heat equation/diffusion equation: $u_t = u_{xx}$; the wave equation: $u_{tt} = u_{xx}$; the Laplace's equation/potential equation: $u_{xx} + u_{yy} = 0$. We consider about the heat equation: $u_t = a^2 u_{xx}$.

Given the linear PDE $L[u] = f$, if $f \equiv 0$, we say the PDE is homogeneous, otherwise, the PDE is nonhomogeneous.

The standard form of the PDE([23]) is $au_{xx} + bu_{xy} + cu_{yy} + du_x + fu_y + gu = 0.$

Case 1: if $b^2 - 4ac > 0$, it is Hyperbolic. The typical Hyperbolic PDE is the wave equation.

Case 2: if $b^2 - 4ac = 0$, it is Parabolic. The typical Parabolic PDE is the heat equation.

Case 3: if $b^2 - 4ac < 0$, it is Elliptic. The typical Elliptlic PDE is the Laplace's equation.

Since we are more concern about the Heat equation. First, let us look at the simplest case of the heat equation([23]):$u_t = u_{xx}$.

let

$$u(x,t) = X(x)T(t),$$

then

$$X(x)T'(t) = X''(x)T(t),$$

so

$$T'(t)/T(t) = X''(x)/X(t) = \text{constant},$$

let

$$T'(t)/T(t) = X''(x)/X(t) = -\lambda,$$

or

$$X''(x) + \lambda X(x) = 0,$$

and

$$T'(t) + \lambda T(t) = 0.$$

Until now we have three cases to consider:

Case 1: $\lambda > 0$, $x = c\cos(\sqrt{\lambda}x) + d\sin(\sqrt{\lambda}x)$, $T = e^{-\lambda t}$

and $u = e^{-\lambda t}[c\cos(\sqrt{\lambda}x) + d\sin(\sqrt{\lambda}x)]$.

Case 2: $a = 0$, $x = cx + d$, $T = 1$
and $u = cx + d$.

Case 3: $\lambda < 0$, $x = ce^{\sqrt{-\lambda}t} + de^{\sqrt{-\lambda}t}$,$T = e^{-\lambda t}$
and $u = e^{-\lambda t}[ce^{\sqrt{-\lambda}t} + de^{\sqrt{-\lambda}t}]$.

So from this example we can see that same simple heat equations we can use some method to solve them. But in most cases, it is very hard to use the normal method to solve. We are trying to use the Lie-method to solve the PDEs.

2.2 The Lie Symmetry Methods for Ordinary Differential Equations

In order to understand symmetries of ordinary differential equations, we give a brief introduction of prolonged infinitesimal generators and determining equations for the Lie point symmetries[21, 22] . Consider an ODE of the form

$$y^{(n)} = \omega\left(x, y, y', \cdots, y^{(n-1)}\right), \qquad y^{(k)} \equiv \frac{d^k y}{dx^k}, \tag{4}$$

where ω is (locally) a smooth function of all of its arguments. A symmetry of equation (4) is a diffeomorphism that maps the set of solutions of the ODE to itself. For a diffeomorphism:

$$\Gamma : (x, y) \mapsto (\hat{x}, \hat{y}),$$

it maps smooth planar curves to smooth planar curves. The diffeomorphism Γ on the plane induces an action on the derivatives $y^{(k)}$ which is the mapping as

$$\Gamma : \left(x, y, y', \cdots, y^{(n)}\right) \mapsto \left(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n)}\right),$$

where

$$\hat{y}^{(k)} = \frac{d^k \hat{y}}{d\hat{x}^k}, \qquad k = 1,\, 2, \cdots, n.$$

This mapping is usually called the nth prolongation of the diffeomorphism Γ. The functions $\hat{y}^{(k)}$ can be found recursively through the chain rule

$$\hat{y}^{(k)} = \frac{d\hat{y}^{(k-1)}}{d\hat{x}} = \frac{D_x \hat{y}^{k-1}}{D_x \hat{x}}, \qquad \hat{y}^{(0)} \equiv \hat{y}, \tag{5}$$

where D_x is the total derivative with respect to x

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots.$$

The symmetry condition for the ODE (4) is given by

$$\hat{y}^{(n)} = \omega\left(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n-1)}\right), \tag{6}$$

where the functions $\hat{y}^{(k)}$ $(k = 1,\, 2, \cdots, n)$ is given by formula (5).

For almost all ODEs, the symmetry condition (6) is nonlinear. Lie symmetries are obtained by linearizing (6) when $\varepsilon = 0$. No such linearization is possible for discrete symmetries, which makes them hard to find. However, it is usually easy to find out whether or not a given diffeomorphism is a symmetry of a particular ODE. The trivial symmetry corresponding to $\varepsilon = 0$ leaves every point unchanged. Thus, for ε sufficiently close to zero, the prolonged Lie symmetries take the form

$$\begin{aligned}
&\hat{x} = x + \varepsilon\xi + \mathbb{O}(\varepsilon^2),\\
&\hat{y} = y + \varepsilon\eta + \mathbb{O}(\varepsilon^2),\\
&\hat{y}^{(k)} = y^{(k)} + \varepsilon\eta^{(k)} + \mathbb{O}(\varepsilon^2), \quad k \geq 1.
\end{aligned} \tag{7}$$

After inserting (7) into the symmetry condition (6), the $\mathbb{O}(\varepsilon)$ terms give the linearized symmetry condition

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \cdots + \eta^{(n-1)}\omega_{y^{(n-1)}}, \tag{8}$$

where the functions $\hat{y}^{(k)}$ and $\eta^{(k)}$ $(k = 1,\, 2, \cdots, n)$ can be derived recursively from formula (5). That is,

$$\begin{aligned}
\hat{y}^{(1)} &= \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{y' + \varepsilon D_x\eta + \mathbb{O}(\varepsilon^2)}{1 + \varepsilon D_x\xi + \mathbb{O}(\varepsilon^2)},\\
&= y' + \varepsilon(D_x\eta - y' D_x\xi) + \mathbb{O}(\varepsilon^2).
\end{aligned}$$

$$\hat{y}^{(k)} = \frac{y^{(k)} + \varepsilon D_x\eta^{(k-1)} + \mathbb{O}(\varepsilon^2)}{1 + \varepsilon D_x\xi + \mathbb{O}(\varepsilon^2)},$$

where from (7) we have

$$\eta^{(1)} = D_x\eta - y' D_x\xi, \tag{9}$$

$$\eta^{(k)}\left(x, y, y', \cdots, y^{(k)}\right) = D_x\eta^{(k-1)} - y^{(k)} D_x\xi. \tag{10}$$

In order to find the symmetry group G admitted by a differential equation with infinitesimal operator

$$X = \xi\partial_x + \eta\partial_y,$$

we introduce the prolonged infinitesimal generator

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}}. \tag{11}$$

This can be used to express the linearized symmetry condition (8) in a compact form:

$$X^{(n)}\left(y^{(n)} - \omega(x, y, y', \cdots, y^{(n-1)})\right) = 0 \text{ when equation (4) holds.}$$

Consider a diffeomorphism of the form:

$$(\hat{x}, \hat{y}) = \left(\hat{x}(x, y),\ \hat{y}(x, y)\right).$$

This type of diffeomorphism is called a point transformation. Any point transformation that is also a symmetry is called a point symmetry. To find the Lie point symmetries of an ODE (4), one needs to find $\eta^{(k)}$ $(k = 1,\ 2,\ \cdots, n)$. The functions ξ and η depend upon x and y only. It follows from (9) and (10) that

$$\begin{aligned}
\eta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \\
\eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\
&\quad +\{\eta_y - 2\xi_x - 3\xi_y y'\}y'', \qquad (12) \\
\eta^{(3)} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 \\
&\quad -\xi_{yyy}y'^4 + 3\{\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2\}y'' \\
&\quad -3\xi_y y''^2 + \{\eta_y - 3\xi_x - 4\xi_y y'\}y'''.
\end{aligned}$$

The number of terms in $\eta^{(k)}$ increases exponentially with k. Hence, to study the high-order ODEs, computer algebra or symbolic package is recommended.

Now we consider about:

$$y'' = F(x, y, y').$$

The linearized symmetry condition can be deduced by plugging (12) into (8) and replacing y'' by $F(x, y, y')$. This yields

$$\begin{aligned}
&\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + \{\eta_y - 2\xi_x - 3\xi_y y'\}F \\
= \;& \xi F_x + \eta F_y + \{\eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2\}F_{y'}. \qquad (13)
\end{aligned}$$

Although equation (13) looks complicated, in some cases it can be solved without much trouble. Both ξ and η are independent of y', and therefore (13) can be decomposed into a system of PDEs, which are called the determining equations for the Lie point symmetries.

An example using the Lie-method in ordinary differential equations:

Considering the equation: $y'' = \frac{y'^2}{y} - y^2$.

The linearized symmetry condition is

$$
\begin{aligned}
& \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + \{\eta_y - 2\xi_x - 3\xi_y y'\}(\frac{y'^2}{y} - y^2) \\
= \; & \xi\left((-\frac{y'^2}{y^2} - 2y) + \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2)\right)\frac{2y'}{y}.
\end{aligned}
$$

By comparing powers of y', we obtain the determining equations:

$$
\begin{aligned}
& \xi_{yy} + \frac{1}{y}\xi_y = 0, \\
& \eta_{yy} - 2\xi_{xy} - \frac{1}{y}\eta_y + \frac{1}{y^2}\eta = 0, \\
& 2\eta_{xy} - 2\xi_{xx} + 3y^2\eta_y - \frac{2}{y}\eta_x = 0, \\
& \eta_{xx} - y^2(\eta_y - 2\xi_x) + 2y\eta = 0.
\end{aligned} \tag{14}
$$

The first of (14) is integrated to give

$$\xi = A(x)\ln|y| + B(x),$$

then the second of (14) yields

$$\eta = A'(x)y(\ln|y|)^2 + C(x)\ln|y| + D(x)y.$$

Here A, B, C, and D are unknown functions. After substituting and computing, we have:

$$\xi = c_1 + c_2 x, \qquad \eta = -2c_2 y.$$

2.3 The Lie Symmetry Methods for Partial Differential Equations

Point symmetries of PDEs[21, 22] are defined in much the same way as those of ODEs. For simplicity, let us start by considering PDEs with one dependent variable, u ,and two independent variables, x and t. A point transformation is a diffeomorphism:

$$\Gamma : (x, t, u) \mapsto (\hat{x}(x, t, u), \hat{t}(x, t, u), \hat{u}(x, t, u)). \tag{15}$$

This transformation maps the surface $u = u(x, t)$ to the following surface (which is parametrized by x and t):

$$\begin{aligned} \hat{x} &= \hat{x}(x, t, u(x, t)), \\ \hat{t} &= \hat{t}(x, t, u(x, t)), \\ \hat{u} &= \hat{u}(x, t, u(x, t)). \end{aligned} \tag{16}$$

To calculate the prolongation of a given transformation,we need to differentiate(16) with respect to each of the parameters x and t. To do this, we introduce the following total derivatives:

$$\begin{aligned} D_x &= \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xt} \partial_{u_t} + \cdots, \\ D_t &= \partial_t + u_t \partial_u + u_{xt} \partial_{u_x} + u_{tt} \partial_{u_t} + \cdots. \end{aligned} \tag{17}$$

(Total derivatives treat the dependent variable u and its derivatives as functions of the independent variables.)

The first two equation of (16) may be inverted(locally) to give x and t in terms of $\hat{x}$ and $\hat{t}$, provided that the Jacobian is nonzero, that is,

$$H \equiv \begin{vmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{vmatrix} \neq 0 \qquad \text{when} \qquad \text{u=u(x,t)} \tag{18}$$

If (18) is satisfied, then the last equation of (16) can be rewritten as

$$\hat{u} = \hat{u}(\hat{x}, \hat{t}) \tag{19}$$

Applying the chain rule to (19), we obtain

$$\begin{bmatrix} D_x\hat{u} \\ D_t\hat{u} \end{bmatrix} = \begin{bmatrix} D_x\hat{x} & D_x\hat{t} \\ D_t\hat{x} & D_t\hat{t} \end{bmatrix} \begin{bmatrix} \hat{u}_{\hat{u}} \\ \hat{u}_{\hat{t}} \end{bmatrix}$$

and therefore(by Cramer's rule)

$$\hat{u}_{\hat{x}} = \frac{1}{H}\begin{vmatrix} D_x\hat{u} & D_x\hat{t} \\ D_t\hat{u} & D_t\hat{t} \end{vmatrix}, \qquad \hat{u}_{\hat{t}} = \frac{1}{H}\begin{vmatrix} D_x\hat{x} & D_x\hat{u} \\ D_t\hat{x} & D_t\hat{u} \end{vmatrix} \tag{20}$$

Higher-order prolongations are obtained recursively by repreating the above argument. If $\hat{u}_J$ is any derivative of $\hat{u}$ with respect to $\hat{x}$ and $\hat{t}$, then

$$\hat{u}_{J\hat{x}} = \partial \hat{u}_J / \partial \hat{x} = \frac{1}{H} \begin{vmatrix} D_x \hat{u}_J & D_x \hat{t} \\ D_t \hat{u}_J & D_t \hat{t} \end{vmatrix},$$

$$\hat{u}_{J\hat{t}} = \partial \hat{u}_J / \partial \hat{t} = \frac{1}{H} \begin{vmatrix} D_x \hat{x} & D_x \hat{u}_J \\ D_t \hat{x} & D_t \hat{u}_J \end{vmatrix}$$

For example, the transformation is prolonged to second derivatives as follows:

$$\hat{u}_{\hat{x}\hat{x}} = \frac{1}{H} \begin{vmatrix} D_x \hat{u}_{\hat{x}} & D_x \hat{t} \\ D_t \hat{u}_{\hat{x}} & D_t \hat{t} \end{vmatrix}, \qquad \hat{u}_{\hat{t}\hat{t}} = \frac{1}{H} \begin{vmatrix} D_x \hat{x} & D_x \hat{u}_{\hat{t}} \\ D_t \hat{x} & D_t \hat{u}_{\hat{t}} \end{vmatrix},$$

$$\hat{u}_{\hat{x}\hat{t}} = \frac{1}{H} \begin{vmatrix} D_x \hat{u}_{\hat{t}} & D_x \hat{t} \\ D_t \hat{u}_{\hat{t}} & D_t \hat{t} \end{vmatrix} = \frac{1}{H} \begin{vmatrix} D_x \hat{x} & D_x \hat{u}_{\hat{x}} \\ D_t \hat{x} & D_t \hat{u}_{\hat{x}} \end{vmatrix}.$$

We are now in a position to define point symmetries of an nth order PDE:

$$\Delta(x, t, u, u_x, u_t, ...) = 0$$

For simplicity, we shall restrict attention to PDEs of the form

$$\Delta = u_\sigma - \omega(x, t, u, u_x, u_t, ...) = 0 \tag{21}$$

where u_σ is one of the nth order derivatives of u and ω is independent of u_σ.(More generally, u_σ could be of order $k < n$ provided that ω does not depend upon u_σ or any derivatives of u_σ.)

The point transformation Γ is a point symmetry of (21) if

$$\Delta(\hat{x}, \hat{t}, \hat{u}, \hat{u}_{\hat{x}}, \hat{u}_{\hat{t}}, ...) = 0, \quad \text{when(19) holds.} \tag{22}$$

Typically, the symmetry condition is extremely complicated, so we shall not try to solve it directly. Nevertheless, it is quite easy to check whether or not a given point transformation is a symmetry of a particular PDE.

Generally speaking,we do not know a priori what form the point symmetries of a given PDE will take. However, it is usually possible to carry out a systematic search for one-parameter Lie groups of point symmetries. The technique is essentially the same as for ODEs. We seek point symmetries of the form

$$\hat{x} = x + \varepsilon\xi(x, t, u) + \mathbb{O}(\varepsilon^2),$$
$$\hat{t} = t + \varepsilon\tau(x, t, u) + \mathbb{O}(\varepsilon^2),$$
$$\hat{u} = u + \varepsilon\eta(x, t, u) + \mathbb{O}(\varepsilon^2).$$

Just as for Lie point transformations of the plane, each one-parameter(local)Lie group of point transformations is obtained by exponentiating its infinitesimal generator, which is

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u.$$

Equivalently,we can obtain $(\hat{x}, \hat{t}, \hat{u})$ by solving

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{x}, \hat{t}, \hat{u}), \qquad \frac{d\hat{t}}{d\varepsilon} = \tau(\hat{x}, \hat{t}, \hat{u}), \qquad \frac{d\hat{u}}{d\varepsilon} = \eta(\hat{x}, \hat{t}, \hat{u}),$$

subject to the initial conditions

$$(\hat{x}, \hat{t}, \hat{u})_{\varepsilon=0} = (x, t, u).$$

A surface $u = u(x, t)$ is mapped to itself by the group of transformations generated by X if

$$X(u - u(x, t)) = 0 \text{ when u=u(x, t).} \tag{23}$$

This condition can be expressed neatly by using the characteristic of the group which is

$$Q = \eta - \xi u_x - \tau u_t$$

The surface $u = u(x, t)$ is invariant provided that

$$Q = 0 \text{ when u=u(x, t).} \tag{24}$$

Equation(24) is called the invariant surface condition; it is central to some of the main techniques for finding exact solutions of PDEs.

The prolongation of the point transformation to first derivatives is

$$
\begin{aligned}
\hat{u}_{\hat{x}} &= u_x + \varepsilon \eta^x(x, t, u, u_x, u_t) + \mathbb{O}(\varepsilon^2), \\
\hat{u}_{\hat{t}} &= u_t + \varepsilon \eta^t(x, t, u, u_x, u_t) + \mathbb{O}(\varepsilon^2),
\end{aligned}
$$

where,

$$
\begin{aligned}
\eta^x(x, t, u, u_x, u_t) &= D_x \eta - u_x D_x \xi - u_t D_x \tau, \\
\eta^t(x, t, u, u_x, u_t) &= D_t \eta - u_x D_t \xi - u_t D_t \tau
\end{aligned}
$$

The transformation is prolonged to higher-order derivatives recursively, using (21). Suppose that

$$
\hat{u}_J = u_J + \varepsilon \eta^J + \mathbb{O}(\varepsilon^2),
$$

where

$$
u_J \equiv \frac{\partial^{j_1 + j_2} u}{\partial x^{j_1} \partial t^{j_2}}, \qquad \hat{u}_J \equiv \frac{\partial^{j_1 + j_2} \hat{u}}{\partial \hat{x}^{j_1} \partial \hat{t}^{j_2}},
$$

for some numbers j_1 and j_2. The (21) yields

$$
\begin{aligned}
\hat{u}_{J\hat{x}} &= u_{Jx} + \varepsilon \eta^{Jx} + \mathbb{O}(\varepsilon^2), \\
\hat{u}_{J\hat{t}} &= u_{Jt} + \varepsilon \eta^{Jt} + \mathbb{O}(\varepsilon^2),
\end{aligned}
$$

where

$$\eta^{Jx} = D_x\eta^J - u_{Jx}D_x\xi - u_{Jt}D_x\tau,$$
$$\eta^{Jt} = D_t\eta^J - u_{Jx}D_t\xi - u_{Jt}D_t\tau. \tag{25}$$

Alternatively, we can express the functions η^J in terms of the characteristic, for example,

$$\eta^x = D_xQ + \xi u_{xx} + \tau u_{xt},$$
$$\eta^t = D_tQ + \xi u_{xt} + \tau u_{tt}.$$

The higher-order terms are obtained by induction on j_1 andj_2:

$$\eta^J = D_JQ + \xi D_Ju_x + \tau D_Ju_t,$$

where

$$D_J \equiv D_x^{j_1}D_t^{j_2},$$

The infinitesimal generator is prolonged to derivatives by adding all terms of the form $\eta^J\partial_{u_J}$ up to the desired order. For example,

$$X^{(1)} = \xi\partial_x + \tau\partial_t + \eta\partial_u + \eta^x\partial_{u_x} + \eta^t\partial_{u_t} = X + \eta^x\partial_{u_x} + \eta^t\partial_{u_t},$$
$$X^{(2)} = X^{(1)} + \eta^{xx}\partial_{u_{xx}} + \eta^{xt}\partial_{u_{xt}} + \eta^{tt}\partial_{u_{tt}}.$$

From now on, we adopt the convention that the generator is prolonged as many times as is needed to describe the group's action on all variables. (We shall not usually refer explicitly to the order of prolongation.) To find the Lie point symmetries, we need explicit expressions for (25). Here are some:

$$\eta^x = \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u {u_x}^2 - \tau_u u_x u_t,$$

$$\eta^t = \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u {u_t}^2.$$

$$
\begin{aligned}
\eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu}){u_x}^2 \\
&\quad -2\tau_{xu}u_xu_t - \xi_{uu}{u_x}^3 - \tau_{uu}{u_x}^2u_t + (\eta_u - 2\xi_x)u_{xx} \\
&\quad -2\tau_xu_{xt} - 3\xi_uu_xu_{xx} - \tau_uu_tu_{xx} - 2\tau_uu_xu_{xt} \\
\eta^{xt} &= \eta_{xt} + (\eta_{tu} - \xi_{xt})u_x + (\eta_{xu} - \tau_{xt})u_t - \xi_{tu}{u_x}^2 \\
&\quad +(\eta_{uu} - \xi_{xu} - \tau_{tu})u_xu_t - \tau_{xu}{u_t}^2 - \xi_{uu}{u_x}^2u_t - \tau_{uu}u_x{u_t}^2 \\
&\quad -\xi_tu_{xx} - \xi_uu_tu_{xx} + (\eta_u - \xi_x - \tau_t)u_{xt} - 2\xi_uu_xu_{xt} \\
&\quad -\tau_uu_tu_{xt} - \tau_uu_xu_{tt} \\
\eta^{tt} &= \eta_{tt} - \xi_{tt}u_x + (2\eta_{tu} - \tau_{tt})u_x - 2\xi_{tu}u_xu_t \\
&\quad +(\eta_{uu} - 2\tau_{tu}){u_t}^2 - \xi_{uu}u_x{u_t}^2 - \tau_{uu}{u_t}^3 - 2\xi_tu_{xt} \\
&\quad -2\xi_uu_tu_{xt} + (\eta_u - 2\tau_t)u_{tt} - \xi_uu_xu_{tt} - 3\tau_uu_tu_{tt}.
\end{aligned}
$$

Lie point symmetries are obtained by differentiating the symmetry condition with respect to $\varepsilon = 0$. We obtain the linearized symmetry condition

$$X\Delta = 0 \text{ when } \Delta = 0\,. \tag{26}$$

The restriction (21) enables us to eliminate u_σ from (25); then we split the remaining terms (according to their dependence on derivatives of u) to obtain a linear system of determining equations for ξ, τ,and η. The vector space L of all Lie point symmetry generators of a given PDE is a Lie algebra, although it may not be finite dimensional.

For a given PDE

$$\Delta = 0. \tag{27}$$

For now, let us restrict attention to scalar PDEs with two independent variables. Recall that a solution $u = u(x,t)$ is invariant under the group generated by

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u.$$

if and only if the characteristic vanishes on the solution. In other words, every invariant solution satisfies the invariant surface condition

$$Q = \eta - \xi u_x - \tau u_t = 0. \tag{28}$$

Usually (28) is much easier to solve than the original PDE.Having solved (28), we can find out which solutions also satisfy. For example, the characteristic

$$Q = -cu_x - u_t. \tag{29}$$

The travelling wave ansatz $u = F(x - ct)$ is the general solution of the invariant surface condition $Q = 0$.

For now, suppose that ξ and τ are not both zero. Then the invariant surface condition is a first-order quasilinear PDE that can be solved by the method of characteristics. The characteristic equations are

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{30}$$

If $r(x,t,u)$ and $v(x,t,u)$ are two functionally independent first integrals of (30), every invariant of the group is a function of r and v. Usually, it is convenient to let one invariant play the role of a dependent variable. Suppose(without lose of generality) that $v_u \neq 0$; then the general solution of the invariant surface condition is

$$v = F(r). \tag{31}$$

This solution is now substituted into the PDE (27) to determine the function F.
If r and v both depend on u, it is necessary to find out whether the PDE has any solution of the form

$$r = c. \tag{32}$$

These are the only solutions of the invariant surface condition that are not (locally) of the form (31). If r is a function of the independent variables x and t only, then (32) cannnot yield a solution $u = u(x,t)$.

CHAPTER III

LIE-METHOD FOR THE BURGERS-HUXLEY EQUATION

In this chapter, we will consider the Burgers-Huxley equation.

3.1 Using the Lie-method to solve the Burgers-Huxley equation when $\delta = 2$

The famous generalised Burgers-Huxley equation is as follows:

$$\frac{\partial u}{\partial t} + \alpha u^{\delta} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u^{\delta})(u^{\delta} - \gamma),$$

where α, β, γ and δ are parameters, $\beta \geq 0, \delta > 0, \gamma \in (0, 1)$. Firstly, we consider the case $\delta = 2$. So our equation is:

$$\frac{\partial u}{\partial t} + \alpha u^{2} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u^{2})(u^{2} - \gamma).$$

Considering the Lie-method: a surface $u = u(x, t)$ is mapped to itself by the group of transformations generated by X if

$$X(u - u(x,t)) = 0 \text{ when u=u(x, t).} \tag{33}$$

This condition can be expressed neatly by using the characteristic of the group which is

$$Q = \eta - \xi u_x - \tau u_t$$

The surface $u = u(x,t)$ is invariant provided that

$$Q = 0 \text{ when u=u(x, t).} \tag{34}$$

For simplicity, we let $\tau = 1$. So

$$u_t = \eta - \xi u_x.$$

And,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \alpha u^2 \frac{\partial u}{\partial x} - \beta(1+\gamma)u^3 + \beta u^5 + \beta\gamma u.$$

So,we have the linearized symmetry condition for these non-classical symmetries:

$$\eta^t + 2\alpha u \eta u_x + \alpha u^2 \eta^x - \eta^{xx} = 3\beta(1+\gamma)u^2\eta - 5\beta u^4 \eta - \beta\gamma\eta$$

Substituting these formulas, we get:

$$\begin{aligned}
&\eta_t - \xi_t u_x + \eta_u(\eta - \xi u_x) - \xi_u u_x(\eta - \xi u_x) + 2\alpha u \eta u_x + \alpha u^2(\eta_u - \xi_x)u_x \\
&+\alpha u^2 \eta_x - \alpha u^2 \xi_u u_x^2 - \eta_{xx} - (2\eta_{xu} - \xi_{xx})u_x + \xi_{uu}u_x^3 - (\eta_{uu} - 2\xi_{xu})u_x^2 \\
&-(\eta_u - 2\xi_x)(\eta - \xi u_x + \alpha u^2 u_x - \beta(1+\gamma)u^3 + \beta u^5 + \beta u \gamma) + 3\xi_u u_x \eta \\
&+3\xi_u u_x((\alpha u^2 - \xi)u_x - \beta u(1-u^2)(u^2-\gamma)) = 3\beta(1+\gamma)u^2\eta - (5\beta u^4 + \beta\gamma)\eta.
\end{aligned}$$

and then splitting the resulting equation by equating powers of u_x, we obtain the determining equations for ξ and η as follows:

$$
\begin{aligned}
&\xi_{uu} = 0,\\
&\eta_{uu} - 2\xi_{xu} + 2\xi_u\xi - 2\alpha u^2\xi_u = 0,\\
&\xi_t - 2\alpha u\eta + 2\eta_{xu} - \xi_{xx} + 2\xi_x\xi - \alpha u^2\xi_x - 2\xi_u\eta + 3\beta(1+\gamma)u^3\xi_u - 3\xi_u\beta u^5 - 3\xi_u\beta u\gamma = 0,\\
&\eta_t + \alpha u^2\eta_x - \eta_{xx} + \eta_u\beta(1+\gamma)u^3 - \eta_u\beta u^5 - \eta_u\beta u\gamma + 2\xi_x\eta - 2\xi_x\beta(1+\gamma)u^3\\
&+2\xi_x\beta u^5 + 2\xi_x\beta u\gamma = 3\beta(1+\gamma)u^2\eta - 5\beta u^4\eta - \beta\gamma\eta
\end{aligned}
\tag{35}
$$

Although the system(35) contains some non-linear equations and looks complicated, it is easily solved since it happens to be inatriangle form. The general solution of the first equation is:

$$\xi = H(x,t)u + V(x,t),$$

which leads to the solution of the second equation as:

$$\eta = H_x u^2 - \frac{1}{3}H^2u^3 - HVu^2 + \frac{1}{6}\alpha H u^4 + Au + B.$$

Substituting those two equations into the third and fourth equations and equating powers of u, we obtain:

$$H = 0; B = 0$$

so

$$\xi = V(x,t); \eta = A(x,t)u,$$

and $V(x,t)$,$A(x,t)$ should satisfy this system:

$$
\begin{aligned}
&V_x = -2A,\\
&V_t + 2A_x - V_{xx} + 2V_xV = 0,\\
&\alpha A_x + 2A\beta(1+\gamma) = 0,\\
&A_t - A_{xx} + 2V_xA + 2V_x\beta\gamma = 0.
\end{aligned}
\tag{36}
$$

From the third equation of system (36),we get:

$$A = C(t)e^{-\frac{2\beta(1+\gamma)x}{\alpha}}.$$

And then,from the first equation of system (36),we get:

$$V = C(t)e^{-\frac{2\beta(1+\gamma)x}{\alpha}}\frac{\alpha}{\beta(1+\gamma)} + \varphi(t).$$

Substituting into the second and last equation of system (36), we found that only if $C(t) = 0, \varphi(t) = Const$, the equation can hold. So $\xi = const, \eta = 0$.

We draw the conclusion that the Burgers-Huxley equation only has the travelling wave solution.

Now we are going to search for the travelling wave solution:

Firstly, we make the transformation:

$$u = v^{\frac{1}{2}}$$

and let

$$v = v(x - ct) = v(z).$$

So the equation becomes:

$$-cv' + \alpha vv' - v'' + \frac{1}{2}v^{-1}(v')^2 = 2\beta(1+\gamma)v^2 - 2\beta v^3 - 2\beta\gamma v \tag{37}$$

The linearized symmetry condition is

$$\begin{aligned}
&\eta_{zz} + (2\eta_{zv} - \xi_{zz})v' + (\eta_{vv} - 2\xi_{zv})v'^2 - \xi_{vv}v'^3 + \{\eta_v - 2\xi_z - 3\xi_v v'\} \\
&\left(-cv' + \alpha vv' + \frac{1}{2}v^{-1}(v')^2 - 2\beta(1+\gamma)v^2 + 2\beta v^3 + 2\beta\gamma v\right) = \eta\alpha v' \\
&-\eta\left(\frac{1}{2}v^{-2}(v')^2 + (4(1+\gamma)v - 6v^2 - 2\gamma)\beta\right) + \left(\eta_z + (\eta_v - \xi_z)v' - \xi_v v'^2\right)(-c + \alpha v + v^{-1}v').
\end{aligned} \tag{38}$$

By comparing powers of v', we obtain the determining equations:

$$\begin{aligned}
&\xi_{vv} + \frac{1}{2}v^{-1}\xi_v = 0, \\
&\eta_{vv} - 2\xi_{zv} - \frac{1}{2}v^{-1}\eta_v - 2\xi_v(\alpha v - c) + \frac{1}{2}v^{-2}\eta = 0, \\
&2\eta_{zv} - \xi_{zz} - \xi_z(\alpha v - c) - 3\xi_v\left(-2\beta(1+\gamma)v^2 + 2\beta v^3 + 2\beta\gamma v\right) - \alpha\eta - \eta_z v^{-1} = 0, \\
&\eta_{zz} + (\eta_v - 2\xi_z)\left(-2\beta(1+\gamma)v^2 + 2\beta(v^3 + \gamma v)\right) - \eta\left(-4\beta(1+\gamma)v + 6\beta v^2 + 2\beta\gamma\right) - \eta_z(-c + \alpha v) = 0.
\end{aligned} \tag{39}$$

The first equation of system (39) is integrated to give

$$\xi = A(z)|v|^{\frac{1}{2}} + B(z).$$

Case 1: if $|v| = v$,we have $\xi = A(z)v^{\frac{1}{2}} + B(z)$ then the second euqation of systme (39) yields

$$\eta = 2A'(x)v^{\frac{3}{2}} - 2Acv^{\frac{3}{2}} + \frac{\alpha A}{3}v^{\frac{5}{2}} + \varphi v^{\frac{1}{2}} + \psi v,$$

here A, B, φ, and ψ are unknown functions.

Substituting ξ and η into the fourth equation of system (39), we get: $A = 0; \varphi = 0$, and after substituting and computing, and by comparing powers of v, we obtain the determining equations:

$$\begin{aligned} &\psi' - B'' + B'c = 0, \\ &\psi = -B', \\ &\psi'' - 4B'\beta\gamma + \psi'c = 0, \\ &4B'\beta(1+\gamma) + 2\psi\beta(1+\gamma) - \alpha\psi' = 0. \end{aligned} \tag{40}$$

From the second and fourth equations of system (40), we get:

$$B = c_1 \frac{\alpha}{2\beta(1+\gamma)} e^{-\frac{2\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \psi = c_1 e^{-\frac{2\beta(1+\gamma)z}{\alpha}}.$$

Substituting into the first and third equations of system (40),we found the condition:

$$c = -\frac{4\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{3\beta(1+\gamma)^2}{\alpha^2}.$$

So under the condition: $c = -\frac{4\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{3\beta(1+\gamma)^2}{\alpha^2}$,we have:

$$\xi = c_1 \frac{\alpha}{2\beta(1+\gamma)} e^{-\frac{2\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \eta = c_1 e^{-\frac{2\beta(1+\gamma)z}{\alpha}} v.$$

Case 2: if $|v| = -v$, we have $\xi = A(z)(-v)^{\frac{1}{2}} + B(z)$ then the second euqation of systme (39) yields

$$\eta = -2A'(x)(-v)^{\frac{3}{2}} + 2Ac(-v)^{\frac{3}{2}} + \frac{\alpha A}{3}(-v)^{\frac{5}{2}} + \varphi(-v)^{\frac{1}{2}} + \psi(-v),$$

here A, B, φ, and ψ are unknown functions.

Substituting ξ and η into the fourth equation of systme (39), we get: $A = 0; \varphi = 0$, and after substituting and computing, and by comparing powers of v, we obtain the determining equations:

$$
\begin{aligned}
&\psi' + B'' - B'c = 0,\\
&\psi = B',\\
&\psi'' + 4B'\beta\gamma + \psi'c = 0,\\
&4B'\beta(1+\gamma) - 2\psi\beta(1+\gamma) + \alpha\psi' = 0.
\end{aligned}
\tag{41}
$$

From the second and fourth equations of system (41), we get:

$$
B = -c_1 \frac{\alpha}{2\beta(1+\gamma)} e^{-\frac{2\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \psi = c_1 e^{-\frac{2\beta(1+\gamma)z}{\alpha}}.
$$

Substituting into the first and third equations of system (41),we found the condition:

$$
c = -\frac{4\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{3\beta(1+\gamma)^2}{\alpha^2}.
$$

So under the condition: $c = -\frac{4\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{3\beta(1+\gamma)^2}{\alpha^2}$,we have:

$$
\xi = -c_1 \frac{\alpha}{2\beta(1+\gamma)} e^{-\frac{2\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \eta = -c_1 e^{-\frac{2\beta(1+\gamma)z}{\alpha}} v.
$$

The invariant curve condition is :

$$
Q = \eta - v'\xi = 0
$$

Case 1: if $|v| = v$:

$$
v' = \frac{v}{\frac{\alpha}{2\beta(1+\gamma)} + \frac{c_2}{c_1} e^{\frac{2\beta(1+\gamma)z}{\alpha}}}
$$

Setting $B = \frac{\alpha}{2\beta(1+\gamma)} + \frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}}$, so $v' = \frac{v}{B}$, and

$$v'' = \frac{v'}{B} - v\frac{\frac{c_2}{c_1}\frac{2\beta(1+\gamma)}{\alpha}e^{\frac{2\beta(1+\gamma)z}{\alpha}}}{B^2}$$

Substiting into the equation (37) ,we get:

$$2\beta B^2 v^2 + (B\alpha - 2\beta B^2(1+\gamma))v + \left(2\beta B^2\gamma - \frac{1}{2} - Bc + \frac{c_2}{c_1}\frac{2\beta(1+r)}{\alpha}e^{\frac{2\beta(1+\gamma)z}{\alpha}}\right) = 0$$

So we get:

$$v = \frac{2\beta B^2(1+\gamma) - \alpha B \pm \sqrt{(B\alpha - 2\beta B^2(1+\gamma))^2 - 8\beta B^2\left(2\beta B^2\gamma - \frac{1}{2} - Bc + \frac{c_2}{c_1}\frac{2\beta(1+r)}{\alpha}e^{\frac{2\beta(1+\gamma)z}{\alpha}}\right)}}{4\beta B^2}$$

Substituting into the condition :$c = -\frac{4\beta(1+\gamma)}{\alpha}, \quad -\gamma = \frac{3\beta(1+\gamma)^2}{\alpha^2}$, and simplify, we get:

$$v = \frac{2\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}} \pm \sqrt{4\beta^2(\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}})^2(1-\gamma)^2}}{4\beta B}$$

Since $\gamma < 1$,so

$$v = \frac{2\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}} \pm 2\beta(\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}})(1-\gamma)}{4\beta B}$$

Substituting into $B = \frac{\alpha}{2\beta(1+\gamma)} + \frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}}$, we have

$$v = \frac{1}{1 + \frac{\alpha}{2\beta(1+\gamma)}(\frac{c_1}{c_2}e^{-\frac{2\beta(1+\gamma)z}{\alpha}})}$$

Or

$$v = \frac{\gamma}{1 + \frac{\alpha}{2\beta(1+\gamma)}(\frac{c_1}{c_2}e^{-\frac{2\beta(1+\gamma)z}{\alpha}})}$$

Similarly,

Case 2: if $|v| = -v$:

$$v' = \frac{v}{\frac{\alpha}{2\beta(1+\gamma)} - \frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}}}$$

Setting $C = \frac{\alpha}{2\beta(1+\gamma)} - \frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}}$, so $v' = \frac{v}{C}$, and

$$v'' = \frac{v'}{C} + v\frac{\frac{c_2}{c_1}\frac{2\beta(1+\gamma)}{\alpha}e^{\frac{2\beta(1+\gamma)z}{\alpha}}}{C^2}$$

Substiting into the equation (37),we get:

$$2\beta C^2 v^2 + \left(C\alpha - 2\beta C^2(1+\gamma)\right)v + \left(2\beta C^2\gamma - \frac{1}{2} - Cc - \frac{c_2}{c_1}\frac{2\beta(1+r)}{\alpha}e^{\frac{2\beta(1+\gamma)z}{\alpha}}\right) = 0$$

So we get:

$$v = \frac{2\beta C^2(1+\gamma) - \alpha C \pm \sqrt{(C\alpha - 2\beta C^2(1+\gamma))^2 - 8\beta C^2\left(2\beta C^2\gamma - \frac{1}{2} - Cc - \frac{c_2}{c_1}\frac{2\beta(1+r)}{\alpha}e^{\frac{2\beta(1+\gamma)z}{\alpha}}\right)}}{4\beta C^2}$$

Substituting into the condition :$c = -\frac{4\beta(1+\gamma)}{\alpha}, \quad -\gamma = \frac{3\beta(1+\gamma)^2}{\alpha^2}$, and simplify, we get:

$$v = \frac{-2\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}} \pm \sqrt{4\beta^2(\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}})^2(1-\gamma)^2}}{4\beta C}$$

Since $\gamma < 1$,so

$$v = \frac{-2\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}} \pm 2\beta(\frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}})(1-\gamma)}{4\beta C}$$

Substituting into $C = \frac{\alpha}{2\beta(1+\gamma)} - \frac{c_2}{c_1}e^{\frac{2\beta(1+\gamma)z}{\alpha}}$, we have

$$v = \frac{1}{1 - \frac{\alpha}{2\beta(1+\gamma)}(\frac{c_1}{c_2}e^{-\frac{2\beta(1+\gamma)z}{\alpha}})}$$

Or

$$v = \frac{\gamma}{1 - \frac{\alpha}{2\beta(1+\gamma)}(\frac{c_1}{c_2}e^{-\frac{2\beta(1+\gamma)z}{\alpha}})}$$

So in conclusion: under the condition:

$$c = -\frac{4\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{3\beta(1+\gamma)^2}{\alpha^2}.$$

$$v = \frac{1}{1 \mp \frac{3(1+\gamma)c_1}{2c_2\gamma\alpha}e^{\frac{2\gamma\alpha z}{3(1+\gamma)}}}$$

Or

$$v = \frac{\gamma}{1 \mp \frac{3(1+\gamma)c_1}{2c_2\gamma\alpha}e^{\frac{2\gamma\alpha z}{3(1+\gamma)}}}$$

Since

$$u = v^{\frac{1}{2}}$$

and let

$$v = v(x - ct) = v(z).$$

So we get:

$$u = \left(\frac{1}{1 \mp \frac{3(1+\gamma)c_1}{2c_2\gamma\alpha} e^{\frac{2\gamma\alpha(x - \frac{4\gamma\alpha t}{3(1+\gamma)})}{3(1+\gamma)}}} \right)^{\frac{1}{2}}$$

Since

$$\frac{e^{2t}}{1 + e^{2t}} = \frac{1}{2}\tanh t + \frac{1}{2},$$

and

$$\frac{e^{2t}}{e^{2t} - 1} = \frac{1}{2}\coth t + \frac{1}{2}.$$

Let $\frac{c_1}{c_2} = \frac{2\gamma\alpha}{3(1+\gamma)}$,we get,

$$u = \left(\frac{1}{2}\tanh(-\frac{\gamma\alpha}{3(1+\gamma)}(x - \frac{4\gamma\alpha t}{3(1+\gamma)})) + \frac{1}{2} \right)^{\frac{1}{2}}$$

Or

$$u = \left(\frac{1}{2}\coth(-\frac{\gamma\alpha}{3(1+\gamma)}(x - \frac{4\gamma\alpha t}{3(1+\gamma)})) + \frac{1}{2} \right)^{\frac{1}{2}}$$

Or

$$u = \left(\frac{\gamma}{1 \mp \frac{3(1+\gamma)c_1}{2c_2\gamma\alpha} e^{\frac{2\gamma\alpha(x - \frac{4\gamma\alpha t}{3(1+\gamma)})}{3(1+\gamma)}}} \right)^{\frac{1}{2}}$$

Similarly, we have

$$u = \left(\frac{\gamma}{2}\tanh(-\frac{\gamma\alpha}{3(1+\gamma)}(x - \frac{4\gamma\alpha t}{3(1+\gamma)})) + \frac{\gamma}{2}\right)^{\frac{1}{2}}$$

Or

$$u = \left(\frac{\gamma}{2}\coth(-\frac{\gamma\alpha}{3(1+\gamma)}(x - \frac{4\gamma\alpha t}{3(1+\gamma)})) + \frac{\gamma}{2}\right)^{\frac{1}{2}}$$

3.2 The Lie-method to solve the generalized Burgers-Huxley equation

The famous generalized Burgers-Huxley equation is as follows:

$$\frac{\partial u}{\partial t} + \alpha u^{\delta}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1-u^{\delta})(u^{\delta}-\gamma),$$

where α, β, γ and δ are parameters, $\beta \geq 0, \delta > 0, \gamma \in (0,1)$. Firstly, we make the transformation:

$$u = v^{\frac{1}{\delta}}$$

So the equation becomes:

$$v_t + \alpha v v_x - v_{xx} - (\frac{1}{\delta} - 1)v^{-1}(v_x)^2 = \delta\beta(1+\gamma)v^2 - \delta\beta v^3 - \delta\beta\gamma v \qquad (42)$$

Considering the Lie-method: a surface $v = v(x,t)$ is mapped to itself by the group of transformations generated by X if

$$X(v - v(x,t)) = 0 \text{ when v=v(x, t).} \quad (43)$$

This condition can be expressed neatly by using the characteristic of the group which is

$$Q = \eta - \xi v_x - \tau v_t$$

The surface $v = v(x,t)$ is invariant provided that

$$Q = 0 \text{ when v=v(x, t).} \quad (44)$$

For simplicity, we let $\tau = 1$. So

$$v_t = \eta - \xi v_x.$$

And, So,we have the linearized symmetry condition for these non-classical symmetries:

$$\eta^t + \alpha\eta v_x + \alpha v\eta^x - \eta^{xx} + (\frac{1}{\delta} - 1)v^{-2}\eta(v_x)^2 - (\frac{1}{\delta} - 1)v^{-1}\eta^x 2v_x = 2\delta\beta(1+\gamma)v\eta - 3\delta\beta v^2\eta - \delta\beta\gamma\eta$$

Substituting these formulas, we get:

$$
\begin{aligned}
&\eta_t - \xi_t v_x + \eta_v(\eta - \xi v_x) - \xi_v v_x(\eta - \xi v_x) + \alpha\eta_x v + \alpha\eta v_x + \alpha v(\eta_v - \xi_x)v_x \\
&-\alpha v {v_x}^2 \xi_v - \eta_{xx} - (2\eta_{xv} - \xi_{xx})v_x + \xi_{vv}v_x^3 - (\eta_{vv} - 2\xi_{xv})v_x^2 - (\eta_v - 2\xi_x)\eta \\
&-(\eta_v - 2\xi_x - 3\xi_v v_x)\left(-\xi v_x + \alpha v v_x - \delta\beta(1+\gamma)v^2 + \delta\beta v^3 + \delta\beta v\gamma - (\frac{1}{\delta} - 1)v^{-1}(v_x)^2\right) \\
&+3\xi_v v_x \eta + (\frac{1}{\delta} - 1)v^{-2}\eta {v_x}^2 - (\frac{1}{\delta} - 1)v^{-1}2v_x\left(\eta_x + (\eta_v - \xi_x)v_x - \xi_v {v_x}^2\right) \\
&= 2\delta\beta(1+\gamma)v\eta - (3\delta\beta v^2 + \beta\delta\gamma)\eta.
\end{aligned}
$$

and then splitting the resulting equation by equating powers of v_x, we obtain the determining equations for ξ and η as follows:

$$
\begin{aligned}
&\xi_{vv} - (\frac{1}{\delta} - 1)v^{-1}\xi_v = 0, \\
&\eta_{vv} - 2\xi_{xv} + (\frac{1}{\delta} - 1)v^{-1}\eta_v - 2\xi_v(\alpha v - \xi) - (\frac{1}{\delta} - 1)v^{-2}\eta = 0, \\
&2\eta_{xv} - \xi_{xx} - \xi_x(\alpha v - 2\xi) - 3\xi_v\beta(-\delta(1+\gamma)v^2 + \delta v^3 + \delta\gamma v) - \alpha\eta + 2(\frac{1}{\delta} - 1)\eta_x v^{-1} + \xi_t - 2\xi_v\eta = 0, \\
&\eta_{xx} + (\eta_v - 2\xi_x)\beta\delta(-(1+\gamma)v^2 + v^3 + \gamma v) - \eta\delta\beta(-2(1+\gamma)v + 3v^2 + \gamma) - \eta_x v\alpha - \eta_t - 2\xi_x\eta = 0.
\end{aligned} \tag{45}
$$

Although the system contains some non-linear equations and looks complicated, it is easily solved since it happens to be inatriangle form. The general solution of the first equation of system (45) is:

$$
\xi = A(x,t)|v|^{\frac{1}{\delta}} + B(x,t).
$$

Case 1: if $|v| = v$,we have $\xi = A(x,t)v^{\frac{1}{\delta}} + B(x,t)$ then the second euqation of system (45) yields

$$
\eta = \delta A_x v^{(\frac{1}{\delta}+1)} - \delta AB v^{(\frac{1}{\delta}+1)} - \frac{\delta A^2}{3}v^{(\frac{2}{\delta}+1)} + \frac{2\alpha\delta A}{(1+\delta)(2+\delta)}v^{(\frac{1}{\delta}+2)} + \varphi v^{(1-\frac{1}{\delta})} + \psi v,
$$

here A, B, φ, and ψ are unknown functions.

Substituting ξ and η into the fourth equation of system (45) , we get: $A = 0; \varphi = 0$, and after substituting and computing, and by comparing powers of v, we obtain the determining equations:

$$
\begin{aligned}
&2\frac{1}{\delta}\psi_x - B_{xx} + B_t + 2BB_x = 0,\\
&\psi = -B_x,\\
&\psi_{xx} - 2B_x\delta\beta\gamma - 2B_x\psi - \psi_t = 0,\\
&B_x\delta\beta(1+\gamma) - \alpha\psi_x = 0.
\end{aligned}
\tag{46}
$$

From the second and fourth equations of system (46) , we get:

$$
B = b(t)\frac{\alpha}{\delta\beta(1+\gamma)}e^{-\frac{\delta\beta(1+\gamma)x}{\alpha}} + f(t), \qquad \psi = b(t)e^{-\frac{\delta\beta(1+\gamma)x}{\alpha}}.
$$

Substituting into the first and third equations of system (46) ,we found that only if $b(t) = 0, \varphi(t) = Const$, the equation can hold. So $\xi = const, \eta = 0$.

Similarily,

Case 2: we have : $\xi = const, \eta = 0$ So we draw the conclusion that the Burgers-Huxley equation only has the travelling wave solution.

Now we are going to search for the travelling wave solution:

Firstly, we make the transformation:

$$
u = v^{\frac{1}{\delta}}
$$

So the equation becomes:

$$
v_t + \alpha v v_x - v_{xx} - (\frac{1}{\delta} - 1)v^{-1}(v_x)^2 = \delta\beta(1+\gamma)v^2 - \delta\beta v^3 - \delta\beta\gamma v \tag{47}
$$

and let

$$v = v(x - ct) = v(z).$$

So the equation becomes:

$$-cv' + \alpha vv' - v'' - (\frac{1}{\delta} - 1)v^{-1}(v')^2 = \delta\beta(1 + \gamma)v^2 - \delta\beta v^3 - \delta\beta\gamma v \tag{48}$$

The linearized symmetry condition is

$$\begin{aligned}
&\eta_{zz} + (2\eta_{zv} - \xi_{zz})v' + (\eta_{vv} - 2\xi_{zv})v'^2 - \xi_{vv}{v'}^3 + (\eta_v - 2\xi_z - 3\xi_v v') \\
&\left(-cv' + \alpha vv' - (\frac{1}{\delta} - 1)v^{-1}(v')^2 - \delta\beta(1 + \gamma)v^2 + \delta\beta v^3 + \delta\beta\gamma v\right) \\
&= \eta\left(\alpha v' + (\frac{1}{\delta} - 1)v^{-2}(v')^2 + -\delta\beta(2(1 + \gamma)v - 3v^2 - \gamma)\right) \\
&+ \left(\eta_z + (\eta_v - \xi_z)v' - \xi_v(v')^2\right)\left(-c + \alpha v - 2(\frac{1}{\delta} - 1)v^{-1}v'\right).
\end{aligned}$$

By comparing powers of v', we obtain the determining equations:

$$\begin{aligned}
&\xi_{vv} - (\frac{1}{\delta} - 1)v^{-1}\xi_v = 0, \\
&\eta_{vv} - 2\xi_{zv} + (\frac{1}{\delta} - 1)v^{-1}\eta_v - 2\xi_v(\alpha v - c) - (\frac{1}{\delta} - 1)v^{-2}\eta = 0, \\
&2\eta_{zv} - \xi_{zz} - \xi_z(\alpha v - c) - 3\xi_v\beta(-\delta(1 + \gamma)v^2 + \delta v^3 + \delta\gamma v) - \alpha\eta + 2(\frac{1}{\delta} - 1)\eta_z v^{-1} = 0, \\
&\eta_{zz} + (\eta_v - 2\xi_z)\beta\delta(-(1 + \gamma)v^2 + v^3 + \gamma v) - \eta\delta\beta(-2(1 + \gamma)v + 3v^2 + \gamma) - \eta_z(\alpha v - c) = 0.
\end{aligned} \tag{49}$$

When δ is even,the first equation of system (49) is integrated to give

$$\xi = A(z)|v|^{\frac{1}{\delta}} + B(z).$$

When δ is odd,the first equation of system (49) is integrated to give

$$\xi = A(z)v^{\frac{1}{\delta}} + B(z).$$

Firstly, let us consider about δ is even

Case 1: if $|v| = v$,we have $\xi = A(z)v^{\frac{1}{\delta}} + B(z)$ then the second euqation of system (49) yields

$$\eta = \delta A_x v^{(\frac{1}{\delta}+1)} - \delta A c v^{(\frac{1}{\delta}+1)} + \frac{2\alpha\delta A}{(1+\delta)(2+\delta)} v^{(\frac{1}{\delta}+2)} + \varphi v^{(1-\frac{1}{\delta})} + \psi v,$$

here A, B, φ, and ψ are unknown functions.

Substituting ξ and η into the fourth equation of the system, we get: $A = 0; \varphi = 0$, and after substituting and computing, and by comparing powers of v, we obtain the determining equations:

$$\begin{aligned}
&2\frac{1}{\delta}\psi' - B'' + B'c = 0,\\
&\psi = -B',\\
&\psi'' - 2\delta B'\beta\gamma + \psi'c = 0,\\
&\psi\delta\beta(1+\gamma) + \alpha\psi' = 0.
\end{aligned} \tag{50}$$

From the second and fourth equations of system (50) , we get:

$$B = c_1 \frac{\alpha}{\delta\beta(1+\gamma)} e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \psi = c_1 e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}}.$$

Substituting into the first and third equations of system (50) ,we found the condition:

$$c = -\frac{(2+\delta)\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{(1+\delta)\beta(1+\gamma)^2}{\alpha^2}.$$

So under the condition: $c = -\frac{(2+\delta)\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{(1+\delta)\beta(1+\gamma)^2}{\alpha^2}$,we have:

$$\xi = c_1 \frac{\alpha}{\delta\beta(1+\gamma)} e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \eta = c_1 e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}} v.$$

Case 2: if $|v| = -v$,we have $\xi = A(z)(-v)^{\frac{1}{\delta}} + B(z)$ then the second euqation of system (49) yields

$$\eta = -\delta A_x(-v)^{(\frac{1}{\delta}+1)} + \delta Ac(-v)^{(\frac{1}{\delta}+1)} + \frac{2\alpha\delta A}{(1+\delta)(2+\delta)}(-v)^{(\frac{1}{\delta}+2)} + \varphi(-v)^{(1-\frac{1}{\delta})} + \psi(-v),$$

here A, B, φ, and ψ are unknown functions.

Substituting ξ and η into the fourth equation of system (49), we get: $A = 0; \varphi = 0$, and after substituting and computing, and by comparing powers of v, we obtain the determining equations:

$$\begin{aligned} &2\frac{1}{\delta}\psi' + B'' - B'c = 0, \\ &\psi = B', \\ &\psi'' + 2\delta B'\beta\gamma + \psi' c = 0, \\ &\psi\delta\beta(1+\gamma) + \alpha\psi' = 0. \end{aligned} \tag{51}$$

From the second and fourth equations of system (51) , we get:

$$B = -c_1 \frac{\alpha}{\delta\beta(1+\gamma)} e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \psi = c_1 e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}}.$$

Substituting into the first and third equations of system (51),we found the condition:

$$c = -\frac{(2+\delta)\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{(1+\delta)\beta(1+\gamma)^2}{\alpha^2}.$$

So under the condition: $c = -\frac{(2+\delta)\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{(1+\delta)\beta(1+\gamma)^2}{\alpha^2}$,we have:

$$\xi = -c_1 \frac{\alpha}{\delta\beta(1+\gamma)} e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}} + c_2, \qquad \eta = c_1 e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}}(-v).$$

The invariant curve condition is :

$$Q = \eta - v'\xi = 0$$

Case 1: if $|v| = v$:

$$v' = \frac{v}{\frac{\alpha}{\delta\beta(1+\gamma)} + \frac{c_2}{c_1} e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}}$$

Setting $B = \frac{\alpha}{\delta\beta(1+\gamma)} + \frac{c_2}{c_1} e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}$, so $v' = \frac{v}{B}$, and

$$v'' = \frac{v'}{B} - v \frac{\frac{c_2}{c_1}\frac{\delta\beta(1+\gamma)}{\alpha} e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}}{B^2}$$

Substiting into the equation (48) ,we get:

$$\delta\beta B^2 v^2 + (B\alpha - \delta\beta B^2(1+\gamma))v + \left(\delta\beta B^2\gamma - \frac{1}{\delta} - Bc + \frac{c_2}{c_1}\frac{\delta\beta(1+r)}{\alpha} e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}\right) = 0$$

So we get:

$$v = \frac{\delta\beta B^2(1+\gamma) - \alpha B \pm \sqrt{\left(B\alpha - \delta\beta B^2(1+\gamma)\right)^2 - 4\delta\beta B^2\left(\delta\beta B^2\gamma - \frac{1}{\delta} - Bc + \frac{c_2}{c_1}\frac{\delta\beta(1+r)}{\alpha}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}\right)}}{2\delta\beta B^2}$$

Substituting into the condition :$c = -\frac{(2+\delta)\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{(1+\delta)\beta(1+\gamma)^2}{\alpha^2}$, and simplify, we get:

$$v = \frac{\delta\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}} \pm \sqrt{\delta^2\beta^2\left(\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}\right)^2(1-\gamma)^2}}{2\delta\beta B}$$

Since $\gamma < 1$,so

$$v = \frac{\delta\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}} \pm \delta\beta(\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}})(1-\gamma)}{2\delta\beta B}$$

Substituting into $B = \frac{\alpha}{\delta\beta(1+\gamma)} + \frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}$, we have

$$v = \frac{1}{1 + \frac{\alpha}{\delta\beta(1+\gamma)}(\frac{c_1}{c_2}e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}})}$$

Or

$$v = \frac{\gamma}{1 + \frac{\alpha}{\delta\beta(1+\gamma)}(\frac{c_1}{c_2}e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}})}$$

Similarly,

Case 2: if $|v| = -v$:

$$v' = \frac{v}{\frac{\alpha}{\delta\beta(1+\gamma)} - \frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}}$$

Setting $C = \frac{\alpha}{\delta\beta(1+\gamma)} - \frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}$, so $v' = \frac{v}{C}$, and

$$v'' = \frac{v'}{C} + v\frac{\frac{c_2}{c_1}\frac{\delta\beta(1+\gamma)}{\alpha}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}}{C^2}$$

Substiting into the equation (48) ,we get:

$$\delta\beta C^2 v^2 + \left(C\alpha - \delta\beta C^2(1+\gamma)\right)v + \left(\delta\beta C^2\gamma - \frac{1}{\delta} - Cc - \frac{c_2}{c_1}\frac{\delta\beta(1+r)}{\alpha}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}\right) = 0$$

So we get:

$$v = \frac{\delta\beta C^2(1+\gamma) - \alpha C \pm \sqrt{(C\alpha - \delta\beta C^2(1+\gamma))^2 - 4\delta\beta C^2\left(\delta\beta C^2\gamma - \frac{1}{2} - Cc - \frac{c_2}{c_1}\frac{\delta\beta(1+r)}{\alpha}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}\right)}}{2\delta\beta C^2}$$

Substituting into the condition :$c = -\frac{(2+\delta)\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{(1+\delta)\beta(1+\gamma)^2}{\alpha^2}$, and simplify, we get:

$$v = \frac{-\delta\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}} \pm \sqrt{\delta^2\beta^2\left(\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}\right)^2(1-\gamma)^2}}{2\delta\beta C}$$

Since $\gamma < 1$,so

$$v = \frac{-\delta\beta(1+\gamma)\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}} \pm 2\beta(\frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}})(1-\gamma)}{2\delta\beta C}$$

Substituting into $C = \frac{\alpha}{\delta\beta(1+\gamma)} - \frac{c_2}{c_1}e^{\frac{\delta\beta(1+\gamma)z}{\alpha}}$, we have

$$v = \frac{1}{1 - \frac{\alpha}{\delta\beta(1+\gamma)}(\frac{c_1}{c_2}e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}})}$$

Or

$$v = \frac{\gamma}{1 - \frac{\alpha}{\delta\beta(1+\gamma)}\left(\frac{c_1}{c_2}e^{-\frac{\delta\beta(1+\gamma)z}{\alpha}}\right)}$$

So in conclusion: When δ is even,under the condition:

$$c = -\frac{(2+\delta)\beta(1+\gamma)}{\alpha}, \qquad -\gamma = \frac{(1+\delta)\beta(1+\gamma)^2}{\alpha^2}.$$

$$v = \frac{1}{1 \mp \frac{(1+\delta)(1+\gamma)c_1}{\delta c_2 \gamma\alpha} e^{\frac{\delta\gamma\alpha z}{(1+\delta)(1+\gamma)}}}$$

Or

$$v = \frac{\gamma}{1 \mp \frac{(1+\delta)(1+\gamma)c_1}{\delta c_2 \gamma\alpha} e^{\frac{\delta\gamma\alpha z}{(1+\delta)(1+\gamma)}}}$$

Since

$$u = v^{\frac{1}{\delta}}$$

and let

$$v = v(x - ct) = v(z).$$

So we get:

$$u = \left(\frac{1}{1 \mp \frac{(1+\delta)(1+\gamma)c_1}{\delta c_2 \gamma\alpha} e^{\frac{\delta\gamma\alpha z}{(1+\delta)(1+\gamma)}}}\right)^{\frac{1}{\delta}}$$

Since

$$\frac{e^{2t}}{1+e^{2t}} = \frac{1}{2}\tanh t + \frac{1}{2},$$

and

$$\frac{e^{2t}}{e^{2t}-1} = \frac{1}{2}\coth t + \frac{1}{2}.$$

Let $\frac{c_1}{c_2} = \frac{\delta\gamma\alpha}{(1+\delta)(1+\gamma)}$,we get,

$$u = \left(\frac{1}{2}\tanh(-\frac{\delta\gamma\alpha}{2(1+\delta)(1+\gamma)}(x - \frac{(2+\delta)\gamma\alpha t}{(1+\delta)(1+\gamma)})) + \frac{1}{2}\right)^{\frac{1}{\delta}}$$

Or

$$u = \left(\frac{1}{2}\coth(-\frac{\delta\gamma\alpha}{2(1+\delta)(1+\gamma)}(x - \frac{(2+\delta)\gamma\alpha t}{(1+\delta)(1+\gamma)})) + \frac{1}{2}\right)^{\frac{1}{\delta}}$$

Or

$$u = \left(\frac{\gamma}{1 \mp \frac{(1+\delta)(1+\gamma)c_1}{\delta c_2\gamma\alpha} e^{\frac{\delta\gamma\alpha(x - \frac{(2+\delta)\gamma\alpha t}{(1+\delta)(1+\gamma)})}{(1+\delta)(1+\gamma)}}}\right)^{\frac{1}{\delta}}$$

Similarly, we have

$$u = \left(\frac{\gamma}{2}\tanh(-\frac{\delta\gamma\alpha}{2(1+\delta)(1+\gamma)}(x - \frac{(2+\delta)\gamma\alpha t}{(1+\delta)(1+\gamma)})) + \frac{\gamma}{2}\right)^{\frac{1}{\delta}}$$

Or

$$u = \left(\frac{\gamma}{2}\coth(-\frac{\delta\gamma\alpha}{2(1+\delta)(1+\gamma)}(x - \frac{(2+\delta)\gamma\alpha t}{(1+\delta)(1+\gamma)})) + \frac{\gamma}{2}\right)^{\frac{1}{\delta}}$$

Following the same procedure, when δ is odd , it is the sames as Case 1, so we have the same conclusion.

CHAPTER IV

A SPECIAL CASE SYSTEM

In this chapter, we are going to find the solution of a special equation by the method of Lie symmetry theory.

4.1 The Lie-method for the equation: $\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = Pu(1-u^a)(u^a+1)$

Consider:

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = Pu(1-u^a)(u^a+1),$$

where D, P, and a are parameters, $D \geq 0, P \geq 0, a \geq 0$.
Considering the Lie-method: a surface $u = u(x,t)$ is mapped to itself by the group of transformations generated by X if

$$X(u - u(x,t)) = 0 \text{ when u=u(x, t).} \tag{52}$$

This condition can be expressed neatly by using the characteristic of the group which is

$$Q = \eta - \xi u_x - \tau u_t$$

The surface $u = u(x,t)$ is invariant provided that

$$Q = 0 \text{ when u=u(x, t).} \tag{53}$$

And,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{D}\left(\frac{\partial u}{\partial t} + Pu^{2a+1} - Pu\right). \tag{54}$$

So,we have the linearized symmetry condition for these non-classical symmetries:

$$\eta^t - D\eta^{xx} = -(2a+1)Pu^{2a}\eta + P\eta \tag{55}$$

Substituting these formulas, we get:

$$\begin{aligned}
&\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u {u_t}^2 + 2D\tau_{xu}u_x u_t + D\tau_{uu}{u_x}^2 u_t \\
&-D\eta_{xx} - D(2\eta_{xu} - \xi_{xx})u_x + D\tau_{xx}u_t + D\xi_{uu}u_x^3 - D(\eta_{uu} - 2\xi_{xu})u_x^2 \\
&-(\eta_u - 2\xi_x - \tau_u u_t - 3\xi_u u_x)(u_t + Pu^{2a+1} - Pu) = -P\left((2a+1)u^{2a} - 1\right)\eta.
\end{aligned}$$

Once u_{xx} has been replaced by the right-hand side of (54), the highest-order derivative terms in (55) have a factor u_{xt}. We start by writing down those terms alone:

$$0 = -2D\tau_x u_{xt} - 2D\tau_u u_x u_{xt}$$

This leads to:

$$\tau_x = \tau_u = 0,$$

which removes many terms from the linearized symmetry condition; the remaining terms are:

$$\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - D\eta_{xx} - D(2\eta_{xu} - \xi_{xx})u_x + D\xi_{uu}{u_x}^3$$
$$-D(\eta_{uu} - 2\xi_{xu})u_x^2 - (\eta_u - 2\xi_x - 3\xi_u u_x)(u_t + Pu^{2a+1} - Pu) = -P\left((2a+1)u^{2a} - 1\right)\eta.$$

In particular,the terms multiplied by u_t are:

$$(\eta_u - \tau_t)u_t - \xi_u u_x u_t - (\eta_u - 2\xi_x - 3\xi_u u_x)u_t = 0.$$

This yields two determining equations:

$$\xi_u = 0, \xi_x = \frac{1}{2}\tau'(t)$$

Hence

$$\xi = \frac{1}{2}\tau'(t)x + \alpha(t),$$

for some function α, the remaining linearized symmety condition becomes:

$$\eta_t - \xi_t u_x - D\eta_{xx} - D(2\eta_{xu} - \xi_{xx})u_x + D\xi_{uu}{u_x}^3 - D(\eta_{uu} - 2\xi_{xu})u_x^2$$
$$-(\eta_u - 2\xi_x - 3\xi_u u_x)(Pu^{2a+1} - Pu) = -P\left((2a+1)u^{2a} - 1\right)\eta.$$

And then splitting the resulting equation by equating powers of u_x, we obtain the determining equations for ξ and η as follows:

$$
\begin{aligned}
&D\xi_{uu} = 0, \\
&D(\eta_{uu} - 2\xi_{xu}) = 0, \\
&\xi_t + 2D\eta_{xu} - D\xi_{xx}3P\xi_u u^{2a+1} + 3\xi_u Pu = 0, \qquad (56) \\
&\eta_t - D\eta_{xx} - Pu^{2a+1}\eta_u + \eta_u Pu + 2\xi_x Pu + 2Pu^{2a+1}\xi_x - 2P\xi_x u = -(2a+1)Pu^{2a}\eta + P\eta
\end{aligned}
$$

For the first and second equation of system(56), we get:

$$\eta_{uu} = 0$$

From the third equation , we get:

$$-\frac{1}{2}\tau''x - \alpha' - 2D\eta_{xu} = 0$$

So we have:

$$\eta = \left(-\frac{1}{8D}\tau''x - \frac{1}{2D}\alpha'x + C_1\right) + C_2$$

Substituting those into the fourth equations and equating powers of u, we obtain:

$$\tau'' = 0; \alpha' = 0, C_1 = 0, C_2 = 0$$

So $\xi = C_4, \eta = 0, \tau = C_3$, where C_3, C_4 are arbitrary constants.

So we have:

$$Q = -C_4u_x - C_3u_t$$

It is the same to:

$$u_t = -cu_x.$$

Substituting into the equation, we get:

$$-c\frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} = Pu(1-u^a)(u^a+1),$$

So it turns to search for the travelling wave solution.

Now we are going to search for the travelling wave solution:

Let

$$v = v(x-ct) = v(z).$$

So the equation becomes:

$$-cv' - Dv'' - D(\frac{1}{a}-1)v^{-1}(v')^2 = -Pav^3 + Pav \tag{57}$$

The linearized symmetry condition is

$$
\begin{aligned}
&\eta_{zz} + (2\eta_{zv} - \xi_{zz})v' + (\eta_{vv} - 2\xi_{zv})v'^2 - \xi_{vv}v'^3 + (\eta_v - 2\xi_z - 3\xi_v v') \\
&\frac{1}{D}\left(-cv' - D(\frac{1}{a} - 1)v^{-1}(v')^2 + Pav^3 - Pav\right) \\
&= \eta\frac{1}{D}\left(D(\frac{1}{a} - 1)v^{-2}(v')^2 - Pa(-3v^2 + 1)\right) \\
&+ \left(\eta_z + (\eta_v - \xi_z)v' - \xi_v(v')^2\right)\frac{1}{D}\left(-c - 2D(\frac{1}{a} - 1)v^{-1}v'\right).
\end{aligned}
$$

By comparing powers of v', we obtain the determining equations:

$$
\begin{aligned}
&\eta_{vv} - (\frac{1}{a} - 1)v^{-1}\xi_v = 0, \\
&\eta_{vv} - 2\xi_{zv} + (\frac{1}{a} - 1)v^{-1}\eta_v + 2\frac{c}{D}\xi_v - (\frac{1}{a} - 1)v^{-2}\eta = 0 \\
&2\eta_{zv} - \xi_{zz} + \frac{c}{D}\xi_z - 3\xi_v\frac{P}{D}(av^3 - av) + 2(\frac{1}{a} - 1)v^{-1}\eta_z = 0, \\
&\eta_{zz} + (\eta_v - 2\xi_z)\frac{P}{D}a(v^3 - v) - \eta\frac{P}{D}a(3v^2 - 1) + \frac{c}{D}\eta_z = 0.
\end{aligned}
\tag{58}
$$

When a is even,the first equation is integrated to give

$$\xi = A(z)|v|^{\frac{1}{a}} + B(z).$$

When a is odd,the first equation is integrated to give

$$\xi = A(z)v^{\frac{1}{a}} + B(z).$$

Firstly, let us consider about a is even

Case 1: if $|v| = v$,we have $\xi = A(z)v^{\frac{1}{a}} + B(z)$ then the second euqation yields

$$\eta = aA_x v^{(\frac{1}{a}+1)} - aA\frac{c}{D}v^{(\frac{1}{a}+1)} + \varphi v^{(1-\frac{1}{a})} + \psi v,$$

here A, B, φ, and ψ are unknown functions.

Substituting ξ and η into the fourth equation of system (58), we get: $A = 0; \varphi = 0$, and after substituting and computing, and by comparing powers of v, we obtain the determining equations:

$$\begin{aligned} &2\frac{1}{a}\psi' - B'' + \frac{c}{D}B' = 0, \\ &\psi = -B', \\ &\psi'' + 2\frac{P}{D}aB' + \frac{c}{D}\psi' = 0, \end{aligned} \tag{59}$$

From the first and second equations of system (59), we get:

$$B = -c_1\frac{a+2}{a\frac{c}{D}}e^{\frac{a\frac{c}{D}}{a+2}z} + c_2, \qquad \psi = c_1 e^{\frac{a\frac{c}{D}}{a+2}z}.$$

Substituting into the third equations of system (59),we found the condition:

$$c = \pm(a+2)\sqrt{\frac{PD}{a+1}}.$$

So

$$\frac{a\frac{c}{D}}{a+2} = \pm a\sqrt{\frac{P}{D(a+1)}}.$$

Setting,

$$s = \frac{a\frac{c}{D}}{a+2} = \pm a\sqrt{\frac{P}{D(a+1)}}.$$

So, we get

$$B = -c_1 \frac{1}{s} e^{sz} + c_2, \qquad \psi = c_1 e^{sz}.$$

And,

$$\xi = -c_1 \frac{1}{s} e^{sz} + c_2, \qquad \eta = c_1 e^{sz} v.$$

The invariant curve condition is :

$$Q = \eta - v'\xi = 0$$

$$v' = \frac{v}{-\frac{1}{s} + \frac{c_2}{c_1} e^{-sz}}$$

Setting $H = -\frac{1}{s} + \frac{c_2}{c_1} e^{-sz}$, so $v' = \frac{v}{H}$, and

$$v'' = \frac{v'}{H} + v \frac{\frac{c_2}{c_1} s e^{-sz}}{H^2}$$

Substiting into equation (57),we get:

$$aPH^2 v^2 - (PaH^2 + D\frac{1}{a} + Hc + D\frac{c_2}{c_1} s e^{-sz}) = 0$$

So we get:

$$v^2 = \frac{PaH^2 + D\frac{1}{a} + Hc + D\frac{c_2}{c_1} s e^{-sz}}{PaH^2}$$

Substituting into the condition :$c = \pm(a+2)\sqrt{\frac{PD}{a+1}}$, and simplify, we get:

$$D\frac{1}{a} + Hc + D\frac{c_2}{c_1}se^{-sz} = -D\frac{a+1}{a}(1 - 2s\frac{c_2}{c_1}se^{-sz})$$

So,

$$v^2 = \frac{(\frac{c_2}{c_1}se^{-sz})^2}{(1 - \frac{c_2}{c_1}se^{-sz})^2}$$

So, we have

$$v = \pm\frac{\frac{c_2}{c_1}se^{-sz}}{1 - \frac{c_2}{c_1}se^{-sz}}$$

Substiting : $s = \pm a\sqrt{\frac{P}{D(a+1)}}$

We get,

$$v = \frac{\frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}}{1 \mp \frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}}$$

Since $u = v^{\frac{1}{a}}$,so

$$u = \frac{\left(\frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}\right)^{\frac{1}{a}} e^{\mp\sqrt{\frac{P}{D(a+1)}}z}}{\left(1 \mp \frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}\right)^{\frac{1}{a}}}$$

Let $\frac{c_2}{c_1} = \frac{1}{a}\sqrt{\frac{D(a+1)}{P}}$,

We have,

$$u = \frac{e^{\mp\sqrt{\frac{P}{D(a+1)}}z}}{\left(1 \mp e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}\right)^{\frac{1}{a}}}$$

Case 2: if $|v| = -v$,we have $\xi = A(z)(-v)^{\frac{1}{a}} + B(z)$ then the second euqation yields

$$\eta = -aA_x(-v)^{(\frac{1}{a}+1)} + aA\frac{c}{D}(-v)^{(\frac{1}{a}+1)} + \varphi(-v)^{(1-\frac{1}{a})} + \psi(-v),$$

here A, B, φ, and ψ are unknown functions.

Substituting ξ and η into the fourth equation of system , we get: $A = 0; \varphi = 0$, and after substituting and computing, and by comparing powers of v, we obtain the determining equations:

$$\begin{aligned} &-2\frac{1}{a}\psi' - B'' + \frac{c}{D}B' = 0, \\ &\psi = B', \\ &\psi'' - 2\frac{P}{D}aB' + \frac{c}{D}\psi' = 0, \end{aligned} \tag{60}$$

From the first and second equations of system (60), we get:

$$B = c_1\frac{a+2}{a\frac{c}{D}}e^{\frac{a\frac{c}{D}}{a+2}z} + c_2, \qquad \psi = c_1 e^{\frac{a\frac{c}{D}}{a+2}z}.$$

Substituting into the third equations of system (60),we found the condition:

$$c = \pm(a+2)\sqrt{\frac{PD}{a+1}}.$$

So

$$\frac{a\frac{c}{D}}{a+2} = \pm a\sqrt{\frac{P}{D(a+1)}}.$$

Setting,

$$s = \frac{a\frac{c}{D}}{a+2} = \pm a\sqrt{\frac{P}{D(a+1)}}.$$

So, we get

$$B = c_1\frac{1}{s}e^{sz} + c_2, \qquad \psi = c_1 e^{sz}.$$

And,

$$\xi = c_1\frac{1}{s}e^{sz} + c_2, \qquad \eta = c_1 e^{sz}(-v).$$

The invariant curve condition is :

$$Q = \eta - v'\xi = 0$$

$$v' = \frac{-v}{\frac{1}{s} + \frac{c_2}{c_1}e^{-sz}}$$

Setting $C = \frac{1}{s} + \frac{c_2}{c_1}e^{-sz}$, so $v' = \frac{v}{C}$, and

$$v'' = -\left(\frac{v'}{C} + v\frac{\frac{c_2}{c_1}se^{-sz}}{C^2}\right)$$

Substiting into equation (57),we get:

$$aPcC^2v^2 - (PaC^2 + D\frac{1}{a} - Cc - D\frac{c_2}{c_1}se^{-sz}) = 0$$

So we get:

$$v^2 = \frac{PaC^2 + D\frac{1}{a} - Cc - D\frac{c_2}{c_1}se^{-sz}}{PaC^2}$$

Substituting into the condition :$c = \pm(a+2)\sqrt{\frac{PD}{a+1}}$, and simplify, we get:

$$D\frac{1}{a} - Cc - D\frac{c_2}{c_1}se^{-sz} = -D\frac{a+1}{a}(1 + 2s\frac{c_2}{c_1}se^{-sz})$$

So,

$$v^2 = \frac{(\frac{c_2}{c_1}se^{-sz})^2}{(1 + \frac{c_2}{c_1}se^{-sz})^2}$$

So, we have

$$v = \pm\frac{\frac{c_2}{c_1}se^{-sz}}{1 + \frac{c_2}{c_1}se^{-sz}}$$

Substiting : $s = \pm a\sqrt{\frac{P}{D(a+1)}}$

We get,

$$v = \frac{\frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}}{1 \pm \frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}}$$

Since $u = v^{\frac{1}{a}}$,so

$$u = \frac{\left(\frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}\right)^{\frac{1}{a}} e^{\mp\sqrt{\frac{P}{D(a+1)}}z}}{\left(1 \pm \frac{c_2}{c_1}a\sqrt{\frac{P}{D(a+1)}}e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}\right)^{\frac{1}{a}}}$$

Let $\frac{c_2}{c_1} = \frac{1}{a}\sqrt{\frac{D(a+1)}{P}}$, We have,

$$u = \frac{e^{\mp\sqrt{\frac{P}{D(a+1)}}z}}{\left(1 \pm e^{\mp a\sqrt{\frac{P}{D(a+1)}}z}\right)^{\frac{1}{a}}}$$

Also, we can change it into:

$$u = \left(\frac{1}{2} + \frac{1}{2}tanh(\mp\frac{a}{2}\sqrt{\frac{P}{D(a+1)}}(x \mp \frac{a+2}{a+1}\sqrt{PD(a+1)}t)\right)^{\frac{1}{a}}$$

CHAPTER V

GENERALIZED HUXLEY EQUATION

Following the previous procedure, we can apply this Lie symmetry method to solve the generalized Huxley equation. In this section, we consider the generalized Huxley equation :

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = Pu^2(1-u^a)(u^a+1), \tag{61}$$

with the parametric assumption

$$D > 0, \quad P > 0, \quad a > 0.$$

Lie point symmetries are obtained by differentiating the symmetry condition with respect to $\varepsilon = 0$. We obtain the linearized symmetry condition

$$X^{(2)}\Delta = 0 \text{ when } \Delta = 0, \tag{62}$$

where

$$X^{(2)} = \xi\partial_x + \eta\partial_u + \eta^x\partial_{u_x} + \eta^t\partial_{u_t} + \eta^{xx}\partial_{u_{xx}} + \eta^{xt}\partial_{u_{xt}} + \eta^{tt}\partial_{u_{tt}},$$

So the surface $u = u(x,t)$ is invariant provided that

$$\eta - \xi u_x - \tau u_t = 0 \text{ when } u = u(x,t). \tag{63}$$

Also, we have:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{D}\left(\frac{\partial u}{\partial t} + Pu^{2a+2} - Pu^2\right). \tag{64}$$

So,we have the linearized symmetry condition:

$$\eta^t - D\eta^{xx} = -(2a+2)Pu^{2a+1}\eta + 2Pu\eta \tag{65}$$

5.1 Classical symmetries

If X generates classical symmetries of the PDE, it satisfies the linearized symmetry condition (62).

Also, we have the formulas:

$$\eta^t = \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u {u_t}^2$$

$$\begin{aligned}\eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu}){u_x}^2 \\ &\quad -2\tau_{xu}u_x u_t - \xi_{uu}{u_x}^3 - \tau_{uu}{u_x}^2 u_t + (\eta_u - 2\xi_x)u_{xx} \\ &\quad -2\tau_x u_{xt} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}\end{aligned}$$

Substituting these formulas into the linearized condition, we get:

$$\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u {u_t}^2 + 2D\tau_{xu}u_x u_t + D\tau_{uu}{u_x}^2 u_t + 2D\tau_x u_{xt}$$
$$-D\eta_{xx} - D(2\eta_{xu} - \xi_{xx})u_x + D\tau_{xx}u_t + D\xi_{uu}u_x^3 - D(\eta_{uu} - 2\xi_{xu})u_x^2 + 2D\tau_u u_x u_{xt}$$
$$-(\eta_u - 2\xi_x - \tau_u u_t - 3\xi_u u_x)(u_t + Pu^{2a+2} - Pu^2) = -P\left((2a+2)u^{2a+1} - 2u\right)\eta.$$

Once u_{xx} has been replaced by the right-hand side of (64), the highest-order derivative terms in (65) have a factor u_{xt}. We start by writing down those terms alone:

$$0 = -2D\tau_x u_{xt} - 2D\tau_u u_x u_{xt}$$

This leads to:

$$\tau_x = \tau_u = 0,$$

which removes many terms from the linearized symmetry condition; the remaining terms are:

$$\eta_t - \xi_t u_x + (\eta_u - \tau_t)u_t - \xi_u u_x u_t - D\eta_{xx} - D(2\eta_{xu} - \xi_{xx})u_x + D\xi_{uu}{u_x}^3 \tag{66}$$
$$-D(\eta_{uu} - 2\xi_{xu})u_x^2 - (\eta_u - 2\xi_x - 3\xi_u u_x)(u_t + Pu^{2a+2} - Pu^2) = -P\left((2a+2)u^{2a+1} - 2u\right)\eta$$

In the equation(66),the terms multiplied by u_t are:

$$(\eta_u - \tau_t)u_t - \xi_u u_x u_t - (\eta_u - 2\xi_x - 3\xi_u u_x)u_t = 0.$$

This yields two determining equations:

$$\xi_u = 0, \xi_x = \frac{1}{2}\tau'(t)$$

Hence

$$\xi = \frac{1}{2}\tau'(t)x + \alpha(t),$$

for some function α, the remaining linearized symmety condition becomes:

$$\begin{aligned} &\eta_t - \xi_t u_x - D\eta_{xx} - 2D\eta_{xu}u_x - D\eta_{uu}u_x^2 \\ &-(\eta_u - 2\xi_x)(Pu^{2a+2} - Pu^2) = -P\left((2a+2)u^{2a+1} - 2u\right)\eta. \end{aligned}$$

And then splitting the resulting equation by equating powers of u_x, we obtain the determining equations for ξ and η as follows:

$$\begin{aligned} &[u_x]^2: \quad D\eta_{uu} = 0, \\ &[u_x]^1: \quad \xi_t + 2D\eta_{xu} = 0, \\ &[u_x]^0: \quad \eta_t - D\eta_{xx} - Pu^{2a+2}\eta_u + \eta_u Pu^2 + 2Pu^{2a+2}\xi_x - 2P\xi_x u^2 = -(2a+2)Pu^{2a+1}\eta + 2Pu\eta \end{aligned} \tag{67}$$

For the first and second equation of system(67), we get:

$$\eta_{uu} = 0$$

From the second equation , we get:

$$\frac{1}{2}\tau'' x + \alpha' + 2D\eta_{xu} = 0$$

So we have:

$$\eta = \left(-\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t)\right)u + C_2(x,t)$$

Substituting those into the third equation of system(67) and equating powers of u, we obtain:

$$\begin{aligned}
[u]^0: &\quad C_{2t} - DC_{2xx} = 0,\\
[u]^1: &\quad -\frac{1}{8D}\tau(t)'''x^2 - \frac{1}{2D}\alpha(t)''x + C_{1t}(t) + \frac{1}{4}\tau(t)'' = 2PC_2(x,t),\\
[u]^2: &\quad \left(-\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t)\right) - \tau(t)' = 2\left(-\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t)\right),\\
[u]^{2a+1}: &\quad -(2a+2)C_2 = 0,\\
[u]^{2a+2}: &\quad \frac{1}{8D}\tau(t)''x^2 + \frac{1}{2D}\alpha(t)'x - C_1(t) + \tau(t)' = -(2a+2)\left(-\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t)\right).
\end{aligned} \tag{68}$$

Since if $a = \frac{1}{2}$, we can combine the third and fourth equations together, so we have two cases:

Case 1: when $a = \frac{1}{2}$, the system(68) becomes:

$$\begin{aligned}
[u]^0: &\quad C_{2t} - DC_{2xx} = 0,\\
[u]^1: &\quad -\frac{1}{8D}\tau(t)'''x^2 - \frac{1}{2D}\alpha(t)''x + C_{1t}(t) + \frac{1}{4}\tau(t)'' = 2PC_2(x,t),\\
[u]^2: &\quad -\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t) + \tau(t)' - 3C_2(x,t) = 0,\\
[u]^3: &\quad 2\left(\frac{1}{8D}\tau(t)''x^2 + \frac{1}{2D}\alpha(t)'x - C_1(t)\right) - \tau(t)' = 0.
\end{aligned}$$

Solving this system, we obtain:

$$\tau' = 0; \alpha' = 0, C_1 = 0, C_2 = 0$$

So $\xi = C_4, \eta = 0, \tau = C_3$, where C_3, C_4 are arbitrary constants.

Case 2: when $a \neq \frac{1}{2}$, we have the system(68).

So from the fourth equation of system(68), we get $C_2 = 0$. The system turns to:

$$\begin{aligned}
[u]^1: \quad & -\frac{1}{8D}\tau(t)'''x^2 - \frac{1}{2D}\alpha(t)''x + C_{1t}(t) + \frac{1}{4}\tau(t)'' = 0, \\
[u]^2: \quad & -\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t) - \tau(t)' = 2\left(-\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t)\right), \\
[u]^{2a+2}: \quad & \left(-\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t) - \tau(t)\right) = (2a+2)\left(-\frac{1}{8D}\tau(t)''x^2 - \frac{1}{2D}\alpha(t)'x + C_1(t)\right).
\end{aligned} \tag{69}$$

Solving this system, we obtain:

$$\tau' = 0; \alpha' = 0, C_1 = 0, C_2 = 0$$

So $\xi = C_4, \eta = 0, \tau = C_3$, where C_3, C_4 are arbitrary constants.

So we have:

$$Q = -C_4 u_x - C_3 u_t$$

It is the same to:

$$u_t = -cu_x.$$

Substituting into the equation, we get:

$$-c\frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} = Pu^2(1-u^a)(u^a+1),$$

So it turns to search for the traveling wave solution.

Also, we can find that in the system(68),if $a = -\frac{1}{2}$, we can combine the first and fourth equations together, second and fifth equations togeter. The equation(61) becomes the generalized Fisher's equation, which is:

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = Pu(u-1).$$

Following the same procedure, it turns to search for the traveling wave solution for it.

5.2 Nonclassical symmetries

If X generates the nonclassical symmetries of the PDE, it satisfies the linearized symmetry condition (62) and (63).

There are two sorts of nonclassical symmetries , those where the infinitesimal τ is zero, and those where it is non-zero.

Case 1: $\tau = 0$, without loss of generality, we can assume that $\xi \equiv 1$.

Now , we have $\xi = 1; \tau = 0$.Since

$$Q = \eta - \xi u_x - \tau u_t = 0$$

$$u_x = \eta.$$

Also the linearized condition becomes:

$$\eta_t - D\eta_{xx} - 2D\eta_{xu}\eta - D\eta_{uu}\eta^2 - \eta_u(Pu^{2a+2} - Pu^2) = -P\left((2a+2)u^{2a+1} - 2u\right)\eta.$$

In order to solve for η, we can try the ansatze that $\eta = P_2u^2 + P_1u + P_0$. Substituting into the above equation, and comparing the powers of u.

Subcase 1: when $a \neq \frac{1}{2}$,we have:

$$\begin{aligned}
[u]^4: &\quad -2DP_2{}^3 = 0,\\
[u]^3: &\quad -4DP_1P_2{}^2 = 0,\\
[u]^2: &\quad -2DP_2P_1{}^2 - 4DP_0P_2{}^2 = PP_1,\\
[u]^1: &\quad -4DP_0P_2P_1 = 2P_0P,\\
[u]^0: &\quad -2DP_0{}^2P_2 = 0,\\
[u]^{2a+3}: &\quad -2PP_2 = -(2a+2)PP_2,\\
[u]^{2a+2}: &\quad -PP_1 = -(2a+2)PP_1,\\
[u]^{2a+1}: &\quad -(2a+2)PP_0 = 0.
\end{aligned}$$

Solving this system, we obtain:

$$P_0 = 0; P_1 = 0; P_2 = 0.$$

So $\eta = 0$, $u = u(t)$. Equation(61) turns to:

$$\frac{du}{dt} = Pu^2(1 - u^a)(u^a + 1).$$

Subcase 2: when $a = \frac{1}{2}$, we have :

$$
\begin{aligned}
[u]^4: &\quad -2DP_2{}^3 - 2PP_2 = -3PP_2, \\
[u]^3: &\quad -4DP_1P_2{}^2 - PP_1 = -3PP_1, \\
[u]^2: &\quad -2DP_2P_1{}^2 - 4DP_0P_2{}^2 = PP_1 - 3P_0P, \\
[u]^1: &\quad -4DP_0P_2P_1 = 2P_0P, \\
[u]^0: &\quad -2DP_0{}^2P_2 = 0.
\end{aligned}
$$

Solving this system, we get:

1: $P_0 = 0; P_1 = 0; P_2 = 0$, the equation(61) becomes:

$$\frac{du}{dt} = Pu^2(1-u).$$

Soving it , we get the implicit solution:

$$\frac{1}{u} + \ln\frac{-Pu+P}{u} = -P(t+c_1),$$

where c_1 is an arbitrary number.

2: $P_0 = 0; P_1 = 0; P_2 = \pm\sqrt{\frac{P}{2D}}$, so $\eta = \pm\sqrt{\frac{P}{2D}}u^2$. We have the system:

$$u_x = \pm\sqrt{\frac{P}{2D}}u^2,$$
$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = Pu^2(1-u).$$

Solving the system:

$$u = \frac{1}{-Pt \pm \sqrt{\frac{P}{2D}}x + c_1}$$

3: $P_0 = 0; P_1 = \mp\sqrt{\frac{P}{2D}}; P_2 = \pm\sqrt{\frac{P}{2D}}$, so $\eta = \pm\sqrt{\frac{P}{2D}}u(u-1)$. We have the system:

$$u_x = \pm\sqrt{\frac{P}{2D}}u(u-1),$$
$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = Pu^2(1-u).$$

Solving the system:

$$u = \frac{1}{1 + c_1 e^{-\frac{P}{2}t \mp \sqrt{\frac{P}{2D}}x}}$$

Case 2:τ equals a nonzero constant. For simplicity, we can set $\tau = 1$.

$$u_t = \eta - \xi u_x.$$

We have the formulas:

$$
\begin{aligned}
\eta^x &= \eta_x + (\eta_u - \xi_x)u_x - \xi_u(u_x)^2, \\
\eta^t &= \eta_t - \xi_t u_x + \eta_u u_t - \xi_u u_x u_t, \\
\eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x + (\eta_{uu} - 2\xi_{xu})(u_x)^2 \\
&\quad -\xi_{uu}(u_x)^3 + (\eta_u - 2\xi_x)u_{xx} - 3\xi_u u_x u_{xx}, \\
\eta^{xt} &= \eta_{xt} + (\eta_{tu} - \xi_{xt})u_x + \eta_{xu}u_t - \xi_{tu}(u_x)^2 \\
&\quad +(\eta_{uu} - \xi_{xu})u_x u_t - \xi_{uu}(u_x)^2 u_t - \xi_t u_{xx} \\
&\quad -\xi_u u_t u_{xx} + (\eta_u - \xi_x)u_{xt} - 2\xi_u u_x u_{xt}, \\
\eta^{tt} &= \eta_{tt} - \xi_{tt}u_x + 2\eta_{tu}u_t - 2\xi_{tu}u_x u_t \\
&\quad +\eta_{uu}(u_t)^2 - \xi_{uu}u_x(u_t)^2 - 2\xi_t u_{xt} \\
&\quad -2\xi_u u_t u_{xt} + \eta_u u_{tt} - \xi_u u_x u_{tt}.
\end{aligned}
$$

After substitution, rewriting equation (62) gives

$$\eta^t - D\eta^{xx} = -(2a+2)Pu^{2a+1}\eta + 2Pu\eta$$

Substituting these formulas, we get:

$$
\begin{aligned}
&\eta_t - \xi_t u_x + \eta_u(\eta - \xi u_x) - \xi_u u_x(\eta - \xi u_x) \\
&-D\eta_{xx} - D(2\eta_{xu} - \xi_{xx})u_x + D\xi_{uu}u_x^3 - D(\eta_{uu} - 2\xi_{xu})u_x^2 \\
&-(\eta_u - 2\xi_x)(\eta - \xi u_x + Pu^{2a+2} - Pu^2) + 3\xi_u u_x \eta \\
&+3\xi_u u_x(-\xi u_x + Pu^{2a+2} - Pu^2) = -P(2a+2)u^{2a+1}\eta + 2Pu\eta.
\end{aligned}
$$

Splitting the resulting equation by equating powers of u_x, we obtain the determining equations for ξ and η as follows:

$$
\begin{aligned}
&[u_x]^3: \quad && D\eta_{uu} = 0,\\
&[u_x]^2: \quad && D(\eta_{uu} - 2\xi_{xu}) + 2\xi_u\xi = 0,\\
&[u_x]^1: \quad && \xi_t + 2D\eta_{xu} - D\xi_{xx} + 2\xi_x\xi - 2\xi_u\eta - 3P\xi_u u^{2a+2} + 3\xi_u P u^2 = 0,\\
&[u_x]^0: \quad && \eta_t - D\eta_{xx} - Pu^{2a+2}\eta_u + \eta_u P u^2 + 2\xi_x\eta + 2Pu^{2a+2}\xi_x - 2P\xi_x u^2 = -(2a+2)Pu^{2a+1}\eta + 2Pu\eta
\end{aligned}
\tag{70}
$$

Although the above system contains some nonlinear equations and looks complicated, it can be solved since it happens to be of a triangle form. Solving the first equation of system (70) gives

$$\xi = A(x,t)u + B(x,t). \tag{71}$$

Substituting (71) into the second equation of system (70) yields

$$\eta = A_x u^2 - \frac{1}{3D}A^2u^3 - \frac{1}{D}ABu^2 + Eu + F, \tag{72}$$

where $A,\ B,\ E$, and F are functions of x and t to be determined.

Subcase 1: if $a \neq \frac{1}{2}$, substituting (71) and (72) into the third equation of system (70),after equating the coefficients of the powers of u, we obtain :

$$
\begin{aligned}
[u]^0: &\quad -B_t - 2DE_x + DB_{xx} + 2AF - 2BB_x = 0,\\
[u]^1: &\quad -A_t - 4D(A_{xx} - \frac{1}{D}A_xB - \frac{1}{D}AB_x) + DA_{xx} + 2AE - 2BA_x - 2AB_x = 0,\\
[u]^2: &\quad 4AA_x - 2\frac{1}{D}A^2B - 3AP = 0,\\
[u]^3: &\quad -\frac{2}{3D}A^3 = 0,\\
[u]^{2a+2}: &\quad 3AP = 0.
\end{aligned}
$$

Solving this system, we get:$A = 0$,so:

$$\xi = B(x,t), \quad \eta = Eu + F.$$

Substituting ξ and η into the fourth equation of system (70),we obtain :

$$
\begin{aligned}
&E_t - DE_{xx} + 2EB_x = 2PF,\\
&F_t - DF_{xx} + 2FB_x = 0,\\
&-2B_xP = PE,\\
&-(E - 2B_x)P = -2PE(a+1),\\
&-2PF(a+1) = 0,\\
&-B_t - 2DE_x + DB_{xx} - 2BB_x = 0.
\end{aligned}
$$

Solving this system, we get,

$$E = 0; \quad F = 0; \quad B_x = 0; \quad B_t = 0.$$

so

$$\xi = Const, \quad \eta = 0.$$

So we have:

$$u_t = -cu_x.$$

Substituting into the equation, we get:

$$-c\frac{\partial u}{\partial x} - D\frac{\partial^2 u}{\partial x^2} = Pu(1-u^a)(u^a+1),$$

So it turns to search for the traveling wave solution.

Subcase 2: if $a = \frac{1}{2}$, substituting (71) and (72) into the third equation of system (70),after equating the coefficients of the powers of u, we obtain :

$$\begin{aligned}
[u]^0 : \quad & -B_t + DB_{xx} - 2DE_x + 2AF - 2BB_x = 0, \\
[u]^1 : \quad & -A_t + 2BA_x - 3DA_{xx} + 2AB_x + 2AE = 0, \\
[u]^2 : \quad & 4AA_x - \frac{2A}{D}AB - 3AP = 0, \\
[u]^3 : \quad & -2\frac{A^3}{3D} + 3AP = 0.
\end{aligned} \tag{73}$$

Solving this system:

If $A = 0$,substituting ξ and η into the fourth equation of system (70),we obtain :

$$
\begin{aligned}
&E_t - DE_{xx} + 2EB_x = 2PF,\\
&-B_t + DB_{xx} - 2DE_x - 2BB_x = 0,\\
&F_t - DF_{xx} + 2FB_x = 0,\\
&-2B_xP = PE - 3PF,\\
&-(E - 2B_x)P = -3PE.
\end{aligned}
$$

we get that: $E = 0, \quad F = 0, \quad B_x = 0, \quad B_t = 0$

So :

$$\xi = Const, \quad \eta = 0.$$

It turns to search for the traveling wave solution.

If $A \neq 0$,solving the system(73), we get that:

$E = 0, \quad A = \pm 3\sqrt{\frac{DP}{2}}, \quad B = \mp\sqrt{\frac{DP}{2}}$

substituting the fourth equation of system (70),we obtain :

$$F_t - DF_{xx} = -3PFu^2 + 2PFu.$$

So $F = 0$.

We have:

$$\xi = \pm(3\sqrt{\frac{DP}{2}}u - \sqrt{\frac{DP}{2}}), \quad \eta = \frac{3}{2}Pu^2(1-u).$$

It gives the nonclassical symmetry generators

$$X = \xi\partial_x + \tau\partial_t + \eta\partial_u = \pm(3\sqrt{\frac{DP}{2}}u - \sqrt{\frac{DP}{2}})\partial_x + \partial_t + \frac{3}{2}Pu^2(1-u)\partial_u$$

The invariant surface condition for the nonclassical symmetries is

$$u_t \pm (3\sqrt{\frac{DP}{2}}u - \sqrt{\frac{DP}{2}})u_x = \frac{3}{2}Pu^2(1-u)$$

Solving it by the method of characteristics. The characteristic equation is:

$$\frac{dx}{\pm\sqrt{\frac{DP}{2}}(3u-1)} = \frac{dt}{1} = \frac{du}{\frac{3}{2}Pu^2(1-u)}.$$

The two functionally independent invariants are:

$$r = (\frac{1}{u} - 1)e^{\frac{P}{2}t \pm \sqrt{\frac{P}{2D}}x}, \qquad v = \frac{1}{u} + Pt \mp \sqrt{\frac{P}{2D}}x$$

Now we substitute $v = F(r)$ into the equation (61), which reduces to

$$F'' = 0$$

Therefore $F(r) = c_1 r + c_2$; writing this in terms of the original variables, we obtain

$$u = \frac{1 - c_1 e^{\frac{P}{2}t \pm \sqrt{\frac{P}{2D}}x}}{-Pt \pm \sqrt{\frac{P}{2D}}x - c_1 e^{\frac{P}{2}t \pm \sqrt{\frac{P}{2D}}x} + c_2}$$

The solution with $c_1 \neq 0$ are not obtainable by any classical reduction. If $c_1 = 0$, the solution

$v = c_2$ is a traveling wave.That is:

$$u = \frac{1}{-Pt \pm \sqrt{\frac{P}{2D}}x + c_2}$$

There is also a traveling wave solution is when $r = c_3$, that is:

$$u = \frac{1}{1 + c_3 e^{-\frac{P}{2}t \mp \sqrt{\frac{P}{2D}}x}}$$

In fact, we have already obtained these two traveling wave solution in the case $\tau \equiv 0, \xi \equiv 1$.

Similarily,when $a = -\frac{1}{2}$,following the same procedure, we get the travling wave solution for the Fisher's equation. The result matches with the result that has been derived in the literature. Ablowitz and Zeppetella.

Also, for the general case

$$u_t = u_{xx} + f(u),$$

using the Lie symmetry method, we can obtain the generators and exact solutions for different $f(u)$.

$f(u)$	ξ	τ	η
$-u^3$	$\frac{3\sqrt{2}u}{2}$	1	$-\frac{3}{2}u^3$
$-u^3 - u$	$\frac{3\sqrt{2}u}{2}$	1	$-\frac{3}{2}(u^3 + u)$
$-u^3 - bu^2$	$\frac{3\sqrt{2}u}{2} + \frac{\sqrt{2}b}{2}$	1	$-\frac{3}{2}(u^3 + bu^2)$
$-u^3 + 2u^2 - 2u$	$\frac{3\sqrt{2}u}{2} - \sqrt{2}$	1	$-\frac{3}{2}(u^3 - 2u^2 + 2u)$

$f(u)$	$u(x,t)$
$-u^3$	$\frac{\sqrt{2}(2x+k_1)}{x^2+k_1x+6t+k_2}$
$-u^3-u$	$\frac{k_2\sin(\frac{\sqrt{2}x}{2})}{k_1\exp(\frac{3t}{2})+k_2\cos(\frac{\sqrt{2}x}{2})}$
$-u^3-bu^2$	$-\frac{bk_1\exp[\frac{1}{2}(\sqrt{2}bx+b^2t)]+\sqrt{2}k_2}{k_1\exp[\frac{1}{2}(\sqrt{2}bx+b^2t)]+k_2(x-\sqrt{2}bt)}$
$-u^3+2u^2-2u$	$\frac{k_2[\cos(\frac{\sqrt{2}x}{2}-t)+\sin(\frac{\sqrt{2}x}{2}-t)]}{k_1\exp(\frac{\sqrt{2}x}{2}+2t)+k_2\cos(\frac{\sqrt{2}x}{2}-t)}$

CHAPTER VI

ADOMIAN DECOMPOSITION METHOD FOR GENERALISED HUXLEY EQUATION

In this chapter, we will use the adomian decomposition method to get the approximate solutions for the generalised Huxley equation.

6.1 Introduction of the Adomian decomposition method

We begin with the equation

$$Lu + R(u) + N(u) = g(t), \tag{74}$$

where L is the operator of the highest-ordered derivatives of t and R is the remainder of the linear operator. The nonlinear term is represented by $N(u)$.
Thus we get

$$Lu = g(t) - R(u) - N(u). \tag{75}$$

The inverse,

$$L^{-1} = \int_0^t (.)dt, \tag{76}$$

operating with the operator L^{-1} on both sides of Eq. (74) we have

$$u = f_0 + L^{-1}(g(t) - R(u) - N(u)), \tag{77}$$

where f_0 is the solution of homogeneous equation

$$Lu = 0$$

The Adomian decomposition method assume that the unknown function $u(x,t)$ can be expressed by an infinite series of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{78}$$

and the nonlinear operator $N(u)$ can be decomposed by an infinite series of polynomials given by

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{79}$$

where $u_n(x,t)$ will be determined recurrently, and A_n are the so-called polynomianls of $u_0, u_1, ..., u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda}\left[F\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, ... \tag{80}$$

The solution of the nonlinear PDEs in the form (74) with the initial $u(x,0) = f(x)$ can be determined by the series (78) with the iterative

$$u_0(x,t) = f(x);$$

$$u_{n+1}(x,t) = -L^{-1}(-R(u_n) - A_n), n \geq 0,$$

In this method, the definition of the L operator avoids difficult integrations involving Green's functions. The use of a finite approximation in series form for the excitation term, and calculation only to necessary accuracy simplifies integrations still further.

6.2 Adomian decomposition method for generalised Huxley equation

Applying the inverse operator L^{-1} on both sides of Eq. (58) and using the initial condition we find

$$u(x,t) = f(x) - L^{-1}(-Du_{xx} - Pu^2(1 - u^{2a})) \tag{81}$$

Substituting (78) and (79) into the functional eqation (nov 8) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) - L^{-1}\left(-D(\sum u_n)_{xx} + P\sum_{n=0}^{\infty} A_n\right) \tag{82}$$

Identifying the zeros component $u_0(x,t)$ by $f(x)$, the remaining components $n \geq 1$ can be determined by using the recurrence relation

$$u_0(x,t) = f(x);$$

$$u_{n+1}(x,t) = -L^{-1}\left(-D(u_n)_{xx} + PA_n\right), n \geq 0,$$

where A_n are Adomian polynomials that represent the nonlinear term($u^2(1-u^{2a})$) and given by

$$A_0 = u_0^2 - u_0^{2a+2},$$

$$A_1 = 2u_1u_0 - (2a+2)u_1u_0^{2a+1},$$

$$A_2 = 2u_2u_0 + {u_1}^2 - (2a+2)u_2u_0^{2a+1} - \frac{1}{2}(2a+2)(2a+1){u_1}^2{u_0}^{2a}.$$

Other polynomials can be generated in a similar way. The first few components of $u_n(x,t)$ follows immediately upon setting

$$u_0(x,t) = f(x),$$

$$u_1(x,t) = -L^{-1}\left(-D(u_0)_{xx} + PA_0\right),$$

$$u_2(x,t) = -L^{-1}\left(-D(u_1)_{xx} + PA_1\right).$$

$$u_3(x,t) = -L^{-1}\left(-D(u_2)_{xx} + PA_2\right).$$

So:

$$u_{Approximation}(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t).$$

Consider two cases:

Case (i) $u_0(x,t) = f(x) = \frac{1}{\sqrt{\frac{P}{2D}}x}$.

Then

$$u_1(x,t) = \int_0^t \left(D(u_0)_{xx} - P{u_0}^2 + P{u_0}^{2a+2}\right) dt.$$

Substituting u_0, we obtain:

$$u_1(x,t) = 2Dt[\frac{2D}{P}\sqrt{\frac{P}{2D}}x^{-3} - x^{-2} + (\frac{2D}{P})^a x^{-2-2a}].$$

Following the fomulas:

$$u_2(x,t) = \int_0^t \left(D(u_1)_{xx} - 2Pu_1u_0 + P(2a+2)u_1{u_0}^{2a+1}\right) dt.$$

Substituting u_0 and u_1, we obtain:

$$\begin{aligned} u_2(x,t) &= t^2D^2[\frac{24D}{P}\sqrt{\frac{P}{2D}}x^{-5} - 10x^{-4} + 4\sqrt{\frac{P}{2D}}x^{-3} \\ &\quad +(6a+10)(a+1)(\frac{2D}{P})^a x^{-2a-4} - 4(a+2)\sqrt{\frac{P}{2D}}(\frac{2D}{P})^a x^{-2a-3} \\ &\quad +4(\frac{2D}{P})^{2a}\sqrt{\frac{P}{2D}}(a+1)x^{-3-4a}] \end{aligned}$$

Similarly,

$$u_3(x,t) = \int_0^t \left(D(u_2)_{xx} - 2Pu_2u_0 - P{u_1}^2 + P(2a+2)u_2{u_0}^{2a+1} + P(a+1)(2a+1){u_1}^2{u_0}^{2a} \right) dt.$$

Substituting u_0 , u_1 and u_2, we obtain:

$$\begin{aligned} u_3(x,t) &= \frac{t^3}{3}D^2[720\frac{D^2}{P}\sqrt{\frac{P}{2D}}x^{-7} - 256Dx^{-6} + 104D\sqrt{\frac{P}{2D}}x^{-5} - 12Px^{-4} \\ &+D(a+1)(\frac{2D}{P})^a[(6a+10)(2a+4)(2a+5) + 16a + 56]x^{-2a-6} \\ &+[(-16a^3 - 112a^2 - 264a - 192)D\sqrt{\frac{P}{2D}} + 4P(a+1)(2a+1)(\frac{2D}{P})^{2a}]x^{-2a-5} \\ &+8D(11a^2 + 26a + 13)(a+1)(\frac{2D}{P})^{2a}\sqrt{\frac{P}{2D}}x^{-5-4a} \\ &+4P(\frac{2D}{P})^a(4Pa + 8P + 2a^2 + 3a + 1)x^{-2a-4} - 4P(6a^2 + 14a + 9)(\frac{2D}{P})^{2a}x^{-4-4a} \\ &+4P(a+1)(4a+3)(\frac{2D}{P})^{3a}x^{-6a-4}] \end{aligned}$$

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t).$

Case (ii) $u_0(x,t) = f(x) = \frac{1}{1+e^{\sqrt{\frac{P}{2D}}x}}$.

Then

$$u_1(x,t) = \int_0^t \left(D(u_0)_{xx} - P{u_0}^2 + P{u_0}^{2a+2} \right) dt.$$

Substituting u_0, we obtain:

$$u_1(x,t) = Pt\left[\frac{e^{2\sqrt{\frac{P}{2D}}x} - e^{\sqrt{\frac{P}{2D}}x}}{2(1+e^{\sqrt{\frac{P}{2D}}x})^3} - \frac{1}{(1+e^{\sqrt{\frac{P}{2D}}x})^2} + \frac{1}{(1+e^{\sqrt{\frac{P}{2D}}x})^{2a+2}}\right].$$

Following the fomulas:

$$u_2(x,t) = \int_0^t \left(D(u_1)_{xx} - 2Pu_1u_0 + P(2a+2)u_1{u_0}^{2a+1}\right) dt.$$

Substituting u_0 and u_1, we obtain:

$$\begin{aligned} u_2(x,t) &= \frac{t^2}{2}P^2[(-4e^{3\sqrt{\frac{P}{2D}}x} + e^{2\sqrt{\frac{P}{2D}}x} + e^{4\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-5} \\ &\quad +(-2e^{2\sqrt{\frac{P}{2D}}x} - \frac{3}{4}e^{3\sqrt{\frac{P}{2D}}x} + \frac{3}{4}e^{\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-4} + (2+e^{\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-3} \\ &\quad -\left((a+1)e^{\sqrt{\frac{P}{2D}}x} + 2a+4\right)(1+e^{\sqrt{\frac{P}{2D}}x})^{-2a-3} \\ &\quad +(a+1)\left((2a+4)e^{2\sqrt{\frac{P}{2D}}x} - e^{\sqrt{\frac{P}{2D}}x}\right)(1+e^{\sqrt{\frac{P}{2D}}x})^{-2a-4} + 2(a+1)(1+e^{\sqrt{\frac{P}{2D}}x})^{-4a-3}] \end{aligned}$$

Similarly,

$$u_3(x,t) = \int_0^t \left(D(u_2)_{xx} - 2Pu_2u_0 - P{u_1}^2 + P(2a+2)u_2{u_0}^{2a+1} + P(a+1)(2a+1){u_1}^2{u_0}^{2a}\right) dt.$$

Substituting u_0 , u_1 and u_2, we obtain:

$$
\begin{aligned}
u_3(x,t) &= \frac{t^3}{6}P^3[(20e^{2\sqrt{\frac{P}{2D}}x}+\frac{37}{2}e^{4\sqrt{\frac{P}{2D}}x}-\frac{9}{2}e^{3\sqrt{\frac{P}{2D}}x}-\frac{7}{2}e^{\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-5}\\
&+(-60e^{5\sqrt{\frac{P}{2D}}x}+15e^{4\sqrt{\frac{P}{2D}}x}+15e^{6\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-7}\\
&+(\frac{175}{2}e^{4\sqrt{\frac{P}{2D}}x}-11e^{3\sqrt{\frac{P}{2D}}x}-15e^{5\sqrt{\frac{P}{2D}}x}-\frac{5}{2}e^{2\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-6}\\
&+(-\frac{27}{8}e^{3\sqrt{\frac{P}{2D}}x}-\frac{17}{2}e^{2\sqrt{\frac{P}{2D}}x}-23e^{\sqrt{\frac{P}{2D}}x}-6)(1+e^{\sqrt{\frac{P}{2D}}x})^{-4}\\
&+\frac{1}{2}e^{\sqrt{\frac{P}{2D}}x}(1+e^{\sqrt{\frac{P}{2D}}x})^{-3}+(-\frac{a}{2}e^{\sqrt{\frac{P}{2D}}x}-\frac{1}{2}e^{\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-2a-3}\\
&+((7a^2+\frac{39}{2}a+\frac{25}{2})e^{2\sqrt{\frac{P}{2D}}x}+(2a^2+\frac{21}{2}a+\frac{19}{2})e^{\sqrt{\frac{P}{2D}}x}\\
&+4a^2+14a+18)(1+e^{\sqrt{\frac{P}{2D}}x})^{-2a-4}\\
&+(-(a+1)(12a^2+47a+\frac{95}{2})e^{3\sqrt{\frac{P}{2D}}x}-(4a^3+27a^2+51a+33)e^{2\sqrt{\frac{P}{2D}}x}\\
&+(4a^2+\frac{19}{2}a+\frac{15}{2})e^{\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-2a-5}\\
&+((a+1)((a+2)(2a+5)(2a+4)+a+\frac{5}{2})e^{4\sqrt{\frac{P}{2D}}x}-(a+1)(2a^2+7a+1)e^{3\sqrt{\frac{P}{2D}}x}\\
&+(a+1)(a+\frac{5}{2})e^{2\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-2a-6}\\
&+((a+1)(20a^2+44a+22)e^{2\sqrt{\frac{P}{2D}}x}-6(a+1)^2e^{\sqrt{\frac{P}{2D}}x})(1+e^{\sqrt{\frac{P}{2D}}x})^{-4a-5}\\
&+(-(a+1)(6a+5)e^{\sqrt{\frac{P}{2D}}x}-12a^2-28a-18)(1+e^{\sqrt{\frac{P}{2D}}x})^{-4a-4}\\
&2(a+1)(4a+3)(1+e^{\sqrt{\frac{P}{2D}}x})^{-6a-4}]
\end{aligned}
$$

$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t).$

CONCLUSION

In this work, we presented the theory of the Lie symmetry method, then apply it to the study of the generalized Burgers-Huxley equation. Through analyzing the linearized symmetry condition and the associated determining system, we find two nontrivial infinitesimal generators, and obtain exact solutions by solving the reduced differential equation under certain parametric conditions. An approximate solution of the generalized Burgers-Huxley equation was described by means of the Adomian decomposition method.

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BIOGRAPHICAL SKETCH

Jing Tian, the daughter of Tongbin Tian and Xiaomei Liu, was born in China in 1989. She received her bachelor degree in Mathematics from China University of Mining and Technology, Xuzhou, Jiangsu, China in June of 2010. In August of 2010, she joined the Mathematical Master's Program at the University of Texas-Pan American, Edinburg, Texas. Her main research interests were in Differential Equations and Dynamic Systems. Her permanent mailing address is: No16. Dangxiao Rd, Shishou city, Hu Bei province, P.R.China, 434400.