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The Mathematical Aspects of Theoretical Physics

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THE MATHEMATICAL ASPECTS OF THEORETICAL PHYSICS

A Thesis

by

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ABSTRACT

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The aim of this thesis is to outline the mathematical machinery of general relativity, quantum gravity, cosmology and an introduction to string theory under one body of work. We will flesh out tensor algebra and the formalism of differential geometry. After deriving the Einstein field equation, we will outline its traditional applications. We then linearize the field equation by a perturbation method and describe the mathematics of gravitational waves and their spherical harmonic analysis. We then transition into the derivation of the Schwarzschild metric and the Kruskal coordinate transformation, in order to set the stage for quantum gravity. This sets the background in order to segway into the principles of cosmology. We then introduce the formalism of quantum mechanics and derive the Hawking radiation formula of a non-spinning blackhole. We describe the phenomenology of quantum scattering, Regge theory and its mathematical underpinnings. This allows us to introduce string theory by studying infinite momentum boosts and strings in two dimensions.

DEDICATION

I dedicate this thesis to my dear wife Diana whose indefatigable love and support made it possible for me to complete this work. Her infinite patience with me juggling between professional life, academic studies and raising a family of five children made this thesis possible. I also want to thank my children Ali, Adam, Zein, Lia and Aya for not complaining about my missing many hours of family activity.

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CHAPTER I

INTRODUCTION

1.1 Motivation

The aim of this thesis was to flesh out the mathematical and phenomenological machinery of general relativity in its full glory and to stitch together fundamental areas of theoretical physics including quantum mechanics, quantum gravity and string theory. This body of physics and applied mathematics, especially general relativity, is notorious for its laborious and complicated calculations, despite the fact that the ideas and principles are intrinsically relatively simple. This thesis is unique in that for the first time, this body of knowledge has been teased out and honeycombed under one edifice. Special attention was taken to outline some of the historical perspectives, the underlying physical mechanisms i.e. phenomenology, and the basic applied mathematical methods.

Hence, a whole swath of mathematics was covered in some detail; including tensor algebra, traditional and modern differential geometry, vector calculus, variational calculus, Hamiltonian mechanics, partial differential equations, perturbation theory, Green's functions. the formalism of quantum mechanics (bra-ket notation, linear algebra, operators and basic scalar field theory), classical Newtonian physics, spherical harmonics, symplectic geometry, Fourier transforms and complex analysis. This is a truncated list of the major areas covered. The body of this thesis covers general relativity and its applications, quantum mechanics and Hawking radiation, the basic principles of cosmology and an introduction into string theory. An extensive appendix was provided in order to fill in any gaps and enhance certain methods in more detail. Some areas were repetitive but in a different guise. For example, we contrast the parallel ideas of vector calculus and p-forms, the contrasting roles of Poisson brackets in classical physics versus commutators in quantum mechanics and the bra-ket notation of quantum mechanics, which is just the multiplication of a vector with its dual.

While writing this thesis, two major events unfolded; Cambridge University released Stephen Hawkings' 1965 dissertation on the properties of the cosmos and blackhole radiation and the 2017 Nobel prize for physics was awarded for the final discovery of gravitational waves. Naturally, these are two areas that we explored in some detail and we approached both subjects from first principles. While writing, we were acutely aware of the fact that physical principles are rooted in phenomenology. As a contrasting example, let us look at the insolubility of the quintic in group theory. Here we start with an axiomatic approach; we begin with a field, extend the field, permute the roots of a polynomial whose coefficients lie in the field, obtain an automorphism group and then break the group into normal subgroups. Since, for polynomials of degree 5 or above, the automorphism groups are insoluble (Jordan - Holder theorem), hence we cannot arrive at an equation for the roots of a polynomial of degree 5 or above. This axiomatic approach is not available here, since the mathematical principles are rooted in empirical facts. As illustrating examples; the momentum operator $p = - i\hbar\frac{\partial}{\partial x}$, has its inception in the duality of waves and particles (an experimental finding); Regge field theory and quantum scattering have their origins in the empirical observation that the angular momentum of sub-atomic particles is proportional to the mass squared. General relativity was catapulted into the consciousness of physicists by explaining age old phenomena, such as the precession of the perihelion of the planet Mercury, which could not be explained by Newtonian gravity. These ideas are embedded in the applied mathematics of these topics, which we have exposed in some detail.

In order to cover this vast subject in a comprehensive manner, the thesis is long. Extensive use of the literature was made. Many standard textbooks in general relativity, cosmology, quantum mechanics and string theory were consulted. Several research articles from dissertations, established physics and mathematical journals were studied. Unfortunately, most treatises in general relativity are relatively advanced and difficult to follow. This thesis' aim was to circumvent that and allow one to segway naturally into quantum mechanics and string theory.The calculations here are computed step by step. It it is assumed that the reader is well versed in linear algebra and standard calculus. Nevertheless, the computations can be difficult to follow as the calculations of the Christoffel symbols (connections) are laborious, and require a concentrated effort. A word of caution; we frequently equate the Planck's constant, $\hbar$, the speed of light, c, and the gravitational constant, G, as unity, in order to minimize clutter in the calculations. No effort is spared in trying to adumbrate and explicitly explain the physical and mathematical reasoning behind ideas, in order to better one's understanding. There was a deliberate attempt to repeat calculations and outline mathematical methods in a different guise or format. All of this in an effort to better one's deep understanding of these interwining disciplines.

1.2 Historical Perspective and Preamble of Mathematical Aspects

The theory of general relativity is one of the most sublime intellectual creations of human kind. It was a breakthrough that shattered previously held notions of space and time. It placed space and time at an equal footing, the spacetime manifold. Its consequences were far reaching from re-formulating the structure of the universe to practical applications in global satellite positioning systems, GPS. It provided a new theory of the large scale structure of the universe and fused mathematics (differential geometry) and physics (energy-momentum tensor) into one equation. It basically described the whole universe in one simple equation; the field equation;

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$$

where the left hand side describes the curvature of spacetime and the right hand side describes the energy density of matter. As we outline in this thesis, by computing the connections in empty space, this simple equation gave birth to the concept of blackholes , and to the emergence of gravitational waves as a solution of the field equation, by linearizing through perturbation theory. In the first four chapters, we will flesh out the formalism of tensor algebra and the basics of differential geometry. After all, the language of general relativity is the language of tensor algebra. We will derive Einstein's field equation by the Hilbert action. The action of a physical system is the integral over time of a Lagrangian function, from which the behavior of a physical system can be determined by the principle of least action. Tensors are the mathematical nuts and bolts of general relativity. A tensor is invariant under coordinate transformation. A useful test of tensor character is the idea that tensors contract into a scalar, which is indeed invariant under coordinate transformations. For example, the Riemann curvature tensor contracts inro the Ricci tensor, which contracts into the Ricci scalar;

$$R^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\alpha\delta} = R_{\beta\delta} = S$$

Many Nobel prizes in physics have been awarded for contributions and discoveries in this field. In fact, the very first Nobel prize in physics to an American was awarded to Albert Michelson in 1907 for the Michelson - Morley experiment for disproving the ether hypothesis. Many more were to follow; In 1983, Chandrasekhar was awarded the prize for his work on stellar evolution and the mathematics of blackholes. Blackholes are amongst the most fantastic structures in the universe, yet their mathematics is relatively simple; being described by two parameters, their mass and angular momentum. In 1993, Hulse and Taylor received the prize for the discovery of a new type of pulsar whose orbital characteristics provided indirect evidence for the existence of gravitational waves. Perlmutter and colleagues, in 2011, received the prize for the discovery of the accelerating expansion of the universe through the observation of distant supernovae. And finally, in 2017, Thorne and colleagues received the prize for the decisive observation of gravitational waves. It should also be noted that two other prizes were awarded for the discovery of cosmic rays and the cosmic microwave background radiation, an area which is important, but not a direct area of study of this thesis.

The same year that Einstein introduced the general theory of relativity, in 1915, Karl Schwarzschild provided the first exact solution to the Einstein field equation, for the limited case of a single spherical, non-rotating blackhole. In June 8, 1989, Arthur Eddington provided direct experimental evidence for the general theory of relativity by observing the solar eclipse on that day, and demonstrating the gravitational bending of light by the Sun, by oberving the stars of the constellation of Taurus, in the island of Principe, off the coast of West Africa. At the time, this was a major discovery and catapulted general relativity to the frontlines of physics. Pari-passu with these developments, quantum theory was in its birth pangs. Between 1900 and 1918, Max Planck put forward his theory on energy quanta. It was Kirchoff, who postulated that the intensity of radiation from a blackbody is dependent upon the wavelength of the radiation and the temperature of the radiating body. Using the current theory at the time, the radiation in the high frequency area of the spectrum becomes infinite; according to the Rayleigh-Jeans law

$$B_\lambda(T) = \frac{2ck_bT}{\lambda^4}$$

at very low wavelengths of λ, the spectral radiance, B_λ(T), becomes infinitely large; not in accordance with observation. This is the so called ultraviolet catastrophe. Max Planck was able to remedy this aberration by theorising that the energy of radiation is quantized

$$E = \hbar\nu$$

and the black body spectrum becomes

$$B_\lambda(T) = \frac{2\hbar c^2}{\lambda^5}\frac{1}{e^{\frac{\hbar c}{\lambda k_B T}} - 1}$$

where k_B is Boltzmann's constant, c is the speed of light, T, is temperature and, ν, is the frequency of radiation. In this form, Planck's law avoids the ultraviolet catastrophe. Incredibly, Einstein promulgated this hypothesis by explaining the photoelectric effect. He was able to show that only light of a certain frequency, no matter how high the intensity, can displace an electron from a metal, according to Planck's law

$$E = h\nu$$

If this limit is exceeded, the effect is proportional to the light intensity at constant frequency.

For this work, and not for general relativity, Einstein received the Nobel prize for physics in 1921. It was Paul Dirac and Erwin Schrodinger who formally mathematized the wave nature of particles. Why is quantum mechanics quantized ? Another way to look at this is via De Broglie's conception of waves in a circle, for wavelength λ, as explained later in this introduction, the wave number k is expressed as

$$k = \frac{2\pi}{\lambda}$$

and the momentum is expressed in terms of the Planck's constant $\hbar$ as

$$p = \hbar k$$

As we will show later, the wave function $\psi(x)$ for a free particle is

$$\psi(x) = ae^{i\frac{p}{\hbar}x} = ae^{ikx}$$

When normalized, after all, the particle has to exist somewhere, we have

$$\int_0^{2\pi R} dx\psi^*(x)\psi(x) = a^2 \int_0^{2\pi R} dxe^{-ikx}e^{ikx} = 1 = 2\pi Ra^2$$

Therefore,

$$\psi(x) = \frac{1}{\sqrt{2\pi R}}e^{i\frac{p}{\hbar}x}$$

where a is a constant and R is the radius of the circle. We get

$$e^{i\frac{p}{\hbar}x} = e^{i\frac{p}{\hbar}(x+2\pi R)} \Longrightarrow e^{i\frac{p}{\hbar}2\pi R} = 1$$

$$\Longrightarrow \frac{pR}{\hbar} = n \Longrightarrow p = \frac{n}{R}\hbar$$

Note, the angular momentum L is expressed as

$$L = pR = R\frac{n}{R}\hbar \Longrightarrow L = n\hbar$$

Therefor n has integer values; angular momentum is quantized.

In relativistic quantum mechanics

$$\frac{W^2}{c^2} - {p_r}^2 - m^2c^2 = 0$$

where W is the kinetic energy of the particle, $p_r (r = 1, 2, 3)$ is the momentum. Paul Dirac defined the operators

$$W = i\frac{\partial\hbar}{\partial t}$$

and

$$p_r = -i\hbar\frac{\partial}{\partial x}$$

and demanded the left hand side act on the wave function ψ . However, for the theory of quantum mechanics to work, this equation needs to be linear in the operators, so four new variables were introduced; α_r and α_σ, operators that operate on ψ; such that

$$(\frac{W}{c} + \alpha_r p_r + \alpha_\sigma mc)(\frac{W}{c} - \alpha_r p_r - \alpha_\sigma mc) = 0$$

with the conditions that ${\alpha_r}^2 = 1$ and $\alpha_r\alpha_\sigma + \alpha_\sigma\alpha_r = 0$. Furthermore, the α' s commute with the $p's$ and W. The new variables α give rise to the spin of the electron, the spin angular momentum of half quantum. An astute observation by Dirac led to the prediction of the positron. It is to be noted that these equations allow for a positive W greater than mc^2 or a negative W less than mc^2. The negative W gives rise to electrons with a strange property; the faster they travel, the less energy they have, and one must inject energy into them to bring them to rest. It turns out experimentally that these negative energies correspond to electrons with a positive charge, positrons. However, in nature, there is no such thing as negative energy. So Dirac invoked the Pauli exclusion principle, in which no two electrons can exist in the same state. The negative energy state is interpreted as a hole with positive energy. This hole is the positron with opposite charge to the electron. When an electron with positive energy falls into a hole and occupies it, electromagnetic radiation is released. In this way, an electron and positron annihilate each other. The converse process with creation of an electron and a positron from electromagnetic radiation should also be able to take place, a process which has been documented experimentally. These commutation relationships in quantum mechanics are known as first quantization, and are exemplified by the non-commutativity of the position $\hat{x}$ and momentum $\hat{p}$ operators;

$$[\hat{x}, \hat{p}] = i\hbar$$

This is a very poweful identity. It tells us that the position and momentum operators cannot be simultaneously zero, leading to the Heisenberg uncertainty principle, which we shall prove in chapter 14. In chapter 16, we will appreciate the underlying algebraic structure of these operators in the context of Lie groups and their associated Lie algebras.

Having laid out the mathematics of general relativity and quantum theory, we are now well poised to combine the physics of the very large with the physics of the very small. These two meet at the event horizon of a blackhole. Hence, the birth of quantum gravity. Indeed, in 1965, Stephen Hawkings was able to apply quantum mechanics to particle creation near the event horizon to show that blackholes emit black body radiation. We use a different approach in this thesis, by applying scalar field theory and number operators to derive the equation of black body radiation of a blackhole. The formalism of quantum theory is introduced in its full glory in chapter 13 and in the appendix. We also make a brief excursion into scalar field theory and introduce the machinery of creation and annihilation operators.

During the first half of the twentieth century, equally stunning developments occured in the field of cosmology. Through the work of Slipher, Friedmann and Hubble, theoretical and experimental findings had suggested that the universe is expanding;

$$v = \dot{a}(t)H(t)D$$

where, v, is the velocity of the receding galaxy, $a(t)$, is the scaling factor (an artificial flexible coordinate grid), $H(t)$ is Hubble's "constant", and, D, is the distance between galaxies. Through simple algebra and calculus, Friedman was able to show that in a matter dominated universe ,

$$a(t) = ct^{\frac{2}{3}}$$

where c is a constant and t is time. This says as time increases, the universe's grid expands. In an energy dominated universe, the equation becomes

$$a(t) = ct^{\frac{1}{2}}$$

Einstein entertained this idea by adding the so called cosmological constant,Λ , to his field equation

$$\Lambda g_{\mu\nu} + R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}$$

By setting $T_{\mu\nu} = 0$ and taking the trace of this equation, we get

$$4\Lambda = S$$

where S is the Ricci scalar. This tells us that empty space has curvature. Einstein dropped this term, only to regret it later, and stating that this was his greatest blunder.

In 1964, Fred Hoyle and Jayant Narlikar proposed an action at a distance cosmology, by applying Mach's principle that the inertia of a particle is due to the rest of the particles in the universe. They applied the concept of retarded and advanced waves of a moving charge, Green's functions (appendix J) and an action principle to describe their cosmology. Stephen Hawking opposed this view by showing that the advanced mass diverges to infinity in an expanding universe. However, Hawkings was not armed with what we know now, that the universe is accelerating with a cosmic event horizon. His views were challenged by Heidi Fearn in 2014, who showed that the advanced mass does not diverge to infinity if one factors in the cosmic event horizon, and so the upper limit of the advanced wave integral should be finite and not infinite.

String theory is a byproduct of S-matrix theory, which describes how incoming particles convert into outgoing ones; quantum scattering. The S-matrix describes this process, where the numerical entries of the matrix are the scattering amplitudes. An example is an electron and a positron annihilating each other to produce two photons. Poles of the S-matrix in the complex energy plane are identified with the bound states (virtual states, resonances). Branch cuts are associated with the opening of a scattering channel. In 1959, Tullio Regge, studied the analytic properties of scattering as a function of complex angular momentum. As a simple example, let us explore the quantum mechanical treatment of the Coulomb potential, $V(r)$;

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}$$

where ε_0 is the permissivity of the vacuum, r is the distance from the point source and e is the charge. By computing the binding energy, E, of an electron to a proton and obtaining the solution of the radial Schrodinger equation, we obtain the quantum number, l, of the orbital angular momentum. The equation of l in terms of E is a complex function, known as the Regge trajectory. These Regge trajectories exist in the complex plane and can be obtained for many other potentials, the most important being the Yukawa potential, as explained in chapter 13. Regge trajectories appear as poles of the scattering amplitudes of the S-matrix. It was Gabrielle Veneziano, who in the 1968, attempted to glue together the various observations. He noted that the beta function has poles at non positive integers. These are identified as the particle energies, resonances. The gamma function is defined as

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx (Re(z) > 0)$$

It can be easily shown that

$$\Gamma(z+1) = z\Gamma(z)$$

The gamma function $\Gamma(z)$, has an analytic continuation to the domain $\mathfrak{C}$ - { 0, -1, -2, ...}. $\Gamma(z)$ has simple poles at $z =$ 0, -1, -2, ..., and no zeros. Its cousin, the beta function, is defined as

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1} dx, (Re(r), Re(s) > 0)$$

Like the gamma function, this integral converges absolutely and uniformly on a neighbourhood of any (r,s), and defines a function analytic in each variable. By change of variables $u = x+y$, (r,s) can be written as

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

It was noted that the plot of angular momentum, J, of hadrons is proportional to the energy, E , squared

$$R = \alpha J^2$$

As noted above, these are the Regge trajectories. This relationship is between J and E^2, emerges naturally from a rotating classical open string. Veneziano proposed that the beta function closely approximates the behavior of quantum scattering and the Regge trajectories. Leonard Susskind interpreted the cross-sectional invariance of these trajectories as due to the binding together of the heavy hadrons at a finer level by strings of quarks.

A central theme of this thesis is to understand the basic concepts of wave-particle duality. Mass, m, and energy, E, are equivalent and interconvertible,

$$E = mc^2$$

In electromagnetism, energy is related to the wavelength, λ, and frequency, ν, by

$$E = \hbar\nu = \frac{c}{\lambda} = mc^2$$

where $\hbar$ is Planck's constant. Hence mass, energy and radiation can be unified as

$$m = \frac{\hbar\nu}{c^2} = \frac{1}{\lambda c}$$

This simple but powerful equation suggests that when radiation reaches very high frequencies or very short wavelengths , it is expected to exhibit mass properties. It is assumed that the reader has some idea about special relatvity. However, of vital import is the total energy of a relativistic particle, which is well worth deriving in the introduction, in order to provide a flavor of the mood of this thesis. It should be noted from the special theory of relativity, for a relativistic particle, of rest mass $m_{0,}$ the total energy, E, is expressed as

Total Energy = Kinetic Energy, E_k + Rest Mass Energy;

$$mc^2 = E_k + m_0c^2$$

Hence,

$$E_k = (m - m_0)c^2$$

We know from special relativity that the relativistic mass, with $\beta = \frac{v}{c}$, can be written as

$$m = \frac{m_0}{\sqrt{1-\beta^2}}$$

Substituting into E_k, we get

$$E_k = (\frac{1}{\sqrt{1-\beta^2}} - 1)m_0c^2$$

By Taylor expanding, we obtain

$$E_k = (1 + \frac{1}{2}\beta^2 + ... - 1)m_0c^2$$

When v <<c;

$$E_k = \frac{1}{2}(\frac{v}{c})^2 m_0c^2 = \frac{1}{2}m_0v^2$$

This means that for velocities far less than the speed of light, we recover classic mechanics, the kinetic energy of a particle. Since the momentum , p, is

$$p = \frac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}^2}}$$

$$p^2 = \frac{m_0^2v^2}{1-\frac{v^2}{c^2}^2}$$

$$p^2c^2 = \frac{m_0^2v^2c^2}{1-\frac{v^2}{c^2}} = \frac{m_0^2\frac{v^2}{c^2}c^4}{1-\frac{v^2}{c}}$$

By adding and subtracting a term;

$$p^2c^2 = \frac{m_0^2c^4(\frac{v^2}{c^2}-1)}{1-\frac{v^2}{c^2}} + \frac{m_0^2c^4}{1-\frac{v^2}{c}} = -m_0^2c^4 + (mc^2)^2$$

We arrive at the famous relativistic equation

$$E^2 = p^2c^2 + (m_0^2c^2)^2$$

We will use this identity to derive the Klein-Gordon equation, which is the quantized version of the relativistic energy-momentum relation. This is done by simply substituting the momentum and Hamiltonian operator. Many of the equations we use are of the form

$$\nabla^2 f(x) = g(x)$$

for functions $f(x)$ and $g(x)$. These are second order linear elliptic differential equations. The simplest being Poisson's equation

$$\nabla^2 f(x) = 0$$

Another interesting example is the Helmholtz equation for a vibrating drum with Dirichlet boundary conditions; where the velocities at the edge $u(t) = 0$.

$$(\nabla^2 + k^2) f = g$$

$$(\nabla^2 + k^2) u = 0$$

$$\nabla^2 u = -k^2 u$$

where k is the wave number described below. The three dimensional wave equation is generalized as

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2})\psi = 0$$

where $\square$ is the d'Alembertian operator ; ∇^2- $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$. The plane and spherical waves are special solutions. At each point of the wavefront, a normal vector k is assigned, where

$$|k| = \frac{2\pi}{\lambda}$$

where λ is the wavelength. The equation is solved by the ansatz

$$ae^{-i(k.r-\omega t)}$$

where r is the position vector. From this, we obtain

$$|k|^2 = \frac{\omega^2}{c^2} \Longrightarrow |k| = \frac{\omega}{c} = \frac{2\pi n}{k} \equiv k$$

wkere k is the wave number. The wavefront is the surface

$$k.r = constant$$

This is the locus of points on the wave that have the same phase, modulo 2π, after propagating by the same time . The wave equation for the electromagnetic field in a vacuum is

$$\Box A^{\mu} = 0$$

where A^{μ} is the electromagnetic four-potential. The Klein-Gordon equation has the form

$$(\Box + m^2)\psi = 0$$

As will be explained below, the Green's function for the d'Alembertian is defined as

$$\Box G(x, x^{'}) = \delta(x - x^{'})$$

where $\delta(x - x^{'})$is the multi-dimensional Dirac delta function and $x, x^{'}$ are two points in space.

Plane wave solutions of the generalized wave equation are of the form

$$\psi = \psi_0 e^{-ik.r}$$

By assuming that the wave equation is separable;

$$\psi(r,t) = A(r)T(t)$$

and substituting into the generalized wave equation, we get

$$\frac{\nabla^2 A(r)}{A(r)} = \frac{1}{c^2 T(t)} \frac{d^2 T(t)}{dt^2} = -k^2$$

The left hand side depends on r only, the right hand side depends on t only. This can only be so, if both sides are equal to a constant, which is set as $-k^2$. Hence, we get Helmholtz's equation

$$(\nabla^2 + k^2)A = 0$$

The two dimensional solution to this equation in cylindrical coordinates gives us the radial and angular harmonics of a vibrating drum and the two dimensional solutions to the Schrodinger equation. The three dimensional solutions in polar coordinates gives us the spherical harmonics of the wave equation in angular and radial components, as outlined in chapter 11.

A central feature of this thesis is the deployment of applied mathematical methods. An important method in quantum mechanics is Green's functions. In the thesis, we will introduce Green's functions to help solve second order linear differential equations. The method simply involves inverting a differential operator. In this introduction, I will utilize the bra-ket notation of quantum mechanics, which we will outline in detail in both the body of the thesis and in the appendix. Let us define the differential operator $\mathbb{D}$

$$\mathbb{D} f(x) = g(x)$$

In bra-ket notation;

$$D|f >= |g >$$

$$\Longrightarrow |f>= D^{-1}|g> + \sum_i c_i|h_i>$$

where the first term is the complementary function and the second term is the homogeneous solution, such that $D|h_i>= 0$. Notimg that

$$f(x) =< x|f>$$

Hence

$$< x|f>= f(x) =< x|D^{-1}|g> + \sum_i c_i < x|h_i>$$

$$< x|D^{-1}|g>=< x|D^{-1}|x^{'} >< x^{'}|g>= \int dx^{'} G(x,x^{'})g(x^{'})$$

where the Green function, $G(x,x^{'})$,for $\mathbb{D}$ is

$$G(x,x^{'}) =< x|D^{-1}|x^{'} >$$

Since $DD^{-1} = 1$

$$\mathbb{D}G(x,x^{'}) =< x|x^{'} >= \delta(x-x')$$

Due to its importance, I will illustrate with a simple example. Let us look at a self-adjoint differential operator

$$\frac{d^2}{dx^2} f(x) = g(x), x \in [0,1], f(0) = 1, f(1) = b$$

The Green's function should satisfy

$$\frac{d^2}{dx^2} G(x,x^{'}) = \delta(x-x^{'})$$

This tells us that the second derivative is 0, when $x \neq x^{'}$. When $0 \leq x \leq x^{'}$, the second derivative is 0; therefore

$$G(x,x^{'}) = A_1 x + A_2$$

where A_1 and A_2 could depend on $x^{'}$. When $x^{'} \leq x \leq 1$,

$$G(x,x^{'}) = A_3 x + A_4$$

where the A_i's are constants. We use the boundary conditions, continuity of $G(x,x^{'})$ at $x^{'}$ and the fact that the integral of $\frac{d^2G}{dx^2}$ = 1 to obtains the constants.

Albert Einstein was inspired by Poisson's equation, the left hand side expressing the properties of space and the right hand side, the mass density, in order to derive the field equation of general relativity. We will discuss solutions to this second order linear equation, which plays a fundamental part in later chapters. Let us analyze the general Poisson equation

$$\nabla^2 f(\overrightarrow{r}) = g(\overrightarrow{r})$$

In terms of Green's function, ignoring the complementary function

$$f(\overrightarrow{r}) = \int d^3\overrightarrow{r}\, G(\overrightarrow{r}\,\overrightarrow{r}^{\prime})g(\overrightarrow{r}^{\prime})$$

By construction of Green's function

$$\nabla^2 G(\vec{r}\,\overrightarrow{r}^{\prime}) = \delta^{(3)}(\overrightarrow{r} - \overrightarrow{r}^{\prime})$$

Since the boundary is asymptotically flat, letting $\overrightarrow{R}$ = $\overrightarrow{r}$ - $\overrightarrow{r}^{\prime}$,

$$\nabla_R^2 = \nabla_r^2$$

Hence,

$$\nabla_R^2 G(\overrightarrow{R}) = \delta^{(3)}(\overrightarrow{R})$$

We next take the Fourier transform

$$G(\overrightarrow{R}) = (\frac{1}{2\pi})^3 \int d^3k e^{i\overrightarrow{k}\overrightarrow{R}} \widetilde{G}(\overrightarrow{k})$$

where

$$\nabla^2 e^{i\overrightarrow{k}.\overrightarrow{R}} = -k^2 e^{i\overrightarrow{k}.\overrightarrow{R}}$$

Hence,

$$G(\overrightarrow{R}) = (\frac{1}{2\pi})^3 \int d^3k e^{i\overrightarrow{k}\overrightarrow{R}} [-k^2 \widetilde{G}(\overrightarrow{k})] = (\frac{1}{2\pi})^3 \int d^3k e^{i\overrightarrow{k}.\overrightarrow{R}}$$

The equality on the right is the three dimensional delta function

$$\Longrightarrow \widetilde{G}(k) = -\frac{1}{k^2}$$

Note that $e^{i\overrightarrow{k}\overrightarrow{R}}$ are the unit vectors. Next, we integrate

$$G(\overrightarrow{R}) = -\frac{1}{(2\pi)^3} \int dk^3 e^{i\overrightarrow{k}\overrightarrow{R}}$$

Choose polar axis in $k-$space in direction of $\overrightarrow{R}$ and noting $G(\overrightarrow{R})$ is a scalar;

$$G(\overrightarrow{R}) = -\frac{1}{(2\pi)^3} \int_0^\infty dk \int_0^{2\pi} d\phi \int_{-1}^{1} d(cos\theta) e^{ikRcos\theta}$$

$$G(\overrightarrow{R}) = -\frac{1}{(2\pi)^2} \int_0^\infty dk \frac{2isinkR}{ikR} = \frac{1}{4\pi^2} \int_0^\infty \frac{2sink}{kR} = \frac{1}{4\pi R}$$

Note that the last integral is the Dirichlet integral and we arrive at the Coulomb potential. So we can define the Green function to be the Coulomb potential or the inverse Fourier transform. So

$$f(\overrightarrow{r}) = -\frac{1}{4\pi}\int d^3r' \frac{g(r')}{|\overrightarrow{r} - \overrightarrow{r}'|}$$

If the angle between $\overrightarrow{r}$ and $\overrightarrow{r}'$ is γ, $\frac{1}{\overrightarrow{r} - \overrightarrow{r}'}$ can be expanded , if $r > r'$

$$\frac{1}{\overrightarrow{r} - r'} = \frac{1}{r}\sum_{l=0}^{\infty}(\frac{r'}{r})P_l(cos\gamma)$$

This is the multipole expansion. In polar coordinates, we get the spherical harmonics

$$\frac{1}{\overrightarrow{r} - r'} = \frac{1}{r}\sum_{l=0}^{\infty}(\frac{r'}{r})\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)Y_{lm}^{*}(\theta'\phi')$$

Next, we shift gear to Cosmology. In order to explain the cosmic inflation model of the universe, we invoke the concept of quantum fluctutations. This is the temporary appearance of energetic particles out of empty space. It makes intuitive sense when viewed through the prism of Heisenbeg's uncertainity principle, as we explain in chapter 9. This allows for the creation of particle-antiparticle pairs of virtual particles. This is the vacuum or zero-point energy underlying the cosmological constant, Λ, which explains the accelerating expansion of the universe. Since the particle number operator does not commute with the field's Hamiltonian or energy operator, the ground state is not empty, see below, but a quantum superposition of particle number eigenstates. A key idea is one of density of states, If a particle is described by a wave function $\phi(r)$, then the probability, w, to find it in a small element of space $d\tau = dxdydz$ around r is

$$w(r)d\tau = \phi^*(r)\phi(r)d\tau$$

where, * , is the complex conjugate. Since the particle has to be found somewhere, it has to be normalized

$$\int w(r)d\tau = \int \phi^*(r)\phi(r)d\tau = 1$$

$\phi(r)$ is of the form

$$e^{ikr}$$

In quantum mechanics, as introduced by Paul Dirac and John Von Neumann, the functions exist in Hilbert space. A Hilbert space is an infinite-dimensional inner-product space. The inner product of two vectors $f(x)$ and $g(x)$ is defined as

$$(f(x), g(x)) = \int f^*(x)g(x)dx$$

For orthogonal functions, for integers j and k,

$$(e^{ijx}, e^{ikx}) = \int_0^{2\pi} dxe^{-jx}e^{ikx}dx = 2\pi\delta_{jk}$$

These functions can be used as basis vectors, $\{e_k\}$=$\{e^{ikx}\}$. A vector function $f(x)$ in the Hilbert space can be expressed as

$$f(x) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} F_k e^{ikx}$$

where

$$F_k = (e^{ikx}, f(x)) = \int_0^{2\pi} dxf(x)e^{-ikx}$$

The coefficients F_k are the inner product $(e^{ikx}, f(x))$ in a manner analogous to that seen in Fourier series. When the interval over which functions are defined stretches from -∞ to +∞, the relations become Fourier transforms. The basis becomes

$$e^{i\omega x}$$

where ω is a continuous variable. The components of the vector are given by the inner product of f with basis vectors, the Fourier transform

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx f(x) e^{-i\omega x}$$

The inverse Fourier transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx \hat{f}(x) e^{i\omega x}$$

For non-periodic functions, the Fourier transform is used to decompose a function of time into its frequency components. Time and frequency are conjugate variables. Position and momentum are conjugate variables. There are many ways of dissecting the Fourier transforms. One can think of winding the frequency function around the origin at a variable frequency and at resonant frequency, compute the displacement of the center of mass from the origin. The greater the energy density at a particular frequency, the greater the displacement from the origin. We can also think of abstractly dividing the frequency function by the complex unit circle. And in this way decomposing the frequency function. An interesting consequence is an analogue of the Heisenberg uncertainty principle. If the duration of analysis of the frequency function is short, then the Fourier transform at the resonant frequency is broad, and vice versa. A more apt term for the Uncertanty Principle is the Unsharpness Principle, as the frequency band is splayed out rather than standing out sharply.

In the general form above, x and ω are conjugate variables. The position, x, and momentum, p, pair can be expressed as

$$\hat{f}(p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx f(x) e^{-ipx}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dp f(p) e^{ipx}$$

Why is $-i\hbar\frac{\partial}{\partial x}$ the momentum operator ? As outlined above, the wave number k is

$$k = \frac{2\pi}{\lambda}$$

The momentum p is

$$p = \hbar k$$

Since the operator $\frac{\partial}{\partial x}$ has the basis vectors e^{ikx} as its basis vectors, using k as our continuous variable, and setting $\hbar$=1

$$-i\frac{\partial}{\partial x}(e^{ikx}) = k(e^{ikx})$$

One can see by inspection that

$$p = -i\frac{\partial}{\partial x}$$

Next, we examine how Fourier transforms appear in quantum mechanics. It arises naturally from the definitions of the momentum operator and the bra-ket formalism of the vector-dual vector spaces. We know that for a wave function ψ(x),

$$-i\hbar\frac{\partial}{\partial x}\psi(x) = p\psi(x) =< p|\psi(x) >$$

This is the momentum operator in position basis, We will set $\hbar$= 1, and by separation of variables, and integrating, we arrive at

$$\psi(x) = ae^{-ipx}$$

where a is a constant. The eigenfunctions of the momentum operator in the x basis is

$$< x|p >= ae^{ipx}$$

From the bra-ket formalism

$$< x|\psi >= \int_{-\infty}^{+\infty} dp < x|p >< p|\psi >= \int_{-\infty}^{+\infty} dpe^{ipx} < p|\psi >$$

Therefore,

$$< x|\psi >= \psi(x) = \int_{-\infty}^{+\infty} dpe^{ipx} \widetilde{\psi}(p)$$

Hence, the momentum-basis wave function is the inverse Fourier transform of the position basis wave function.

Whereas in classical mechanics, we deploy Poisson brackets to compute with the Hamiltonian, in quantum mechanics, we deploy the commutator. John Von Neumann introduced commutator rules for quantum mechanics and Paul Dirac built on it in a huge way. Dirac loved to commute conjugate variables. Commutator relationships provide much more than algebraic structure. They have deep physical meaning. For example, non commutativity in a scalar field means that both variables cannot be measured simultaneously. For two positions x and y in a scalar field, if the creation, $\Psi^{+}$and annihilation operators, Ψ^{-} commute

$$[\Psi^{+}(x), \Psi^{-}(y)] = 0$$

then there is no interference when we measure the fields at x and y. Furthermore, a symmetry is a unitary operator, U, that commutes with the Hamiltonian. Even more, a symmetry that commutes with the Hamiltonian is conserved. The Hamiltonian commutes with itself, hence it is conserved. The mathematical basis of these ideas will be explained in Chapter 11 and Appendices L and O.

In this introduction, we hope to demonstrate the parallel themes in the concepts of the classical and non-classical physics and the thread that runs through the different disciplines. Analyzing the time-dependent Schrodinger wave equation,

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{-i\hbar}{2m}\nabla^2\psi + V(r)\psi$$

Taking the complex conjugate

$$-i\hbar\frac{\partial\psi^*}{\partial t} = \frac{i\hbar}{2m}\nabla^2\psi^* + V(r)\psi^*$$

Multiplying the first equation by $i\psi^*$ and the second equation by $i\psi$ and subtracting the second from the first

$$\hbar(\psi\frac{\partial\psi^*}{\partial t} + \psi\frac{\partial\psi}{\partial t}) = -\frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)$$

The left hand equality is nothing but

$$\hbar\frac{\partial}{\partial t}(\psi^*\psi)$$

where

$$\psi^*\psi = |\psi|^2$$

is the density of the waveform, the probability of finding a particle. Consider the case of a stream of electrons of charge density, ρ. The right hand side is the divergence

$$\nabla.J$$

where J is the probability current. Hence, the Schrodinger equation is nothing but the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla . J = 0$$

Let us next juxtapose the three great equations; Poisson's Equation, Einstein's Field Equation and the Friedmann-Robertson-Walker equation;

$$\nabla^2 \phi = \rho$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$$

$$\frac{\ddot{a}}{a} = -\frac{4}{3} \pi G (\rho + 3p)$$

A common thread runs through these equations of classical physics, general relativity and cosmology; the left hand side represents geometry (curvature) and the right hand side, matter/energy density. It is hoped that this thesis runs a thread through the various concepts of theoretical physics, and is able to stitch together the mathematical underpinnings of general relativity and quantum mechanics. A very useful concept that we will deploy in this thesis frequently is the idea that the total derivative of a scalar function vanishes at infinity in a spacetime manifold. Here, we apply the divergence theorem, which we will prove in its full glory by utilizing the machinery of p-cubes, pull-backs, meaure theory and Fubini theorem. We begin with a vector field $\omega(x,y,z)$, such as velocity, and a scalar field $\psi(x,y,z,t)$, such as density, pressure or temperature. By the chain and product rule, we get

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + (\omega . \nabla) \psi$$

where ω= ($\frac{dx}{dt}$,$\frac{dy}{dt}$, $\frac{dz}{dt}$) and ∇= ($\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{,\partial}{\partial z}$). At spatial infinity, $\frac{\partial \psi}{\partial t}$ = 0 and for the second divergence term, we invoke Stoke's theorem, for a p-form $d\omega$ and a smooth manifold M

$$\int_{\partial M} \omega = \int_{M} d\omega$$

The left hand equality is zero at spatial infinity, therefore

$$\int_{\partial M} \omega = \int_{M} d\omega \Longrightarrow \omega.\nabla = 0$$

This identity is very useful when we derive the field equation of general relativity via the Hilbert action and when we derive the wave equation of a scalar field. This is a brief outline of some of the important physics and mathematical machinery deployed throughout the thesis. It is hoped that at the end of this thesis, one can see the parallel ideas between covariant/contravariant tensors, vectors/ dual vectors of a vector space and the bra/ket formalism of quantum mechanics. To add the icing on the cake, we will outline the machinery of advanced quantum mechanics, that is, propagators and the second quantization formalism. Finally, I will adumbrate on the symmetries of motion and conservation laws, as illustrated by Emily Noether. As a simple example, if a Lagrangian is rotationally symmetric, Noether's theorem dictates that the angular momentum of the system is conserved. If the Lagrangian is symmetric under continuous under translations of space, then linear momentum is conserved. Under translation of time, energy is preserved. The conserved quantities are known as invariants. It should be noted that ordinary differential equations are usually deployed to describe distinct particles, and partial differential equations are used to describe fields. Meanwhile, the conservation laws are usually expressed in the form of a continuity equation. And the symmetry is usually equivalent to the covariance of the equations in a Lie group of transformations(chapter 16, section 65). An invariant X in the evolution of a system is such that

$$\frac{dX}{dt} = 0$$

X is conserved, also known as a constant of motion. The action S of a system, with coordinates q

and $\dot{q}$, with Lagrangian L. is defined as

$$S = \int L(q, \dot{q}, t) dt$$

The Euler-Lagrange equation, where the particle takes a path, with no variation in S is

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q}) = 0$$

This is easily derived by setting boundary conditions, integrating by parts and and applying the fundamental lemma of variational calculus. For a general system, we set q as q_i. Noting that the momentum p_k is

$$p_k = \frac{\partial L}{\partial \dot{q}}$$

If one of the coordinates q_k does not appear in the Lagrangian, we get

$$\frac{d}{dt} p_k = 0$$

Then the Lagrangian is invariant under a transformation of q_k; that is, a symmetry under this transformation, p_k is conserved. The beauty of Noether's theorem is that symmetries lead to conservation laws, which lead to a simplication of equations of motion, as the right hand side of the equations are set equal to a constant. We will address Noether's theorem in more detail in section 52 of chapter 14. In the last chapter, ideas coalesce, when we outline Lie groups and their Lie algebras, a common thread will emerge between seemingly unrelated areas. We will also make a brief excursion into gauge transfomations and the seeds of the Yang-Mills theory. With gauge theory under our belt, we will conclude with a brief outline of the Standard Model of particle physics.

CHAPTER II

OVERVIEW OF TENSORS AND MANIFOLDS

2.1 Tensors on a Vector Space

We will begin with basic definitions and concepts,[27]. It is assumed that the reader is familar with the basic tenets of linear algebra and vector spaces (linearity and multiplication of vectors by scalars) linear maps, bilinear maps, basic vector calculus, multi-variable calculus and some measure theory. Let V be a finite dimensional vector space and $V*$ be the dual space of V. The dual space is the space of covectors or real valued linear functionals. Hence, we define the mapping;

A covariant k-tensor on V is a multilinear map

$$F : V \times \ldots \times V \to \mathbb{R}$$

with k copies of V. A contravariant l- tensor is a multilinear map

$$: V^* \times \ldots \times V^* \to \mathbb{R}$$

with l copies of V^*.

A mixed tensor of type $\binom{k}{l}$ is a multilinear map

$$F : V^* \times \ldots V^* \times V \times \ldots \times V \to \mathbb{R}$$

with k copies of V^* and l copies of V. The tensor product linking the various tensor spaces over V is as follows ; If $F \in T^k{}_l(V)$ and $G \in T^p{}_q(\ V)$, the tensor product $F \otimes G \in T^{k+p}{}_{l+q}$ is defined as

$$F \otimes G(\omega^l, \ldots, \omega^{l+q}, X_l, \ldots, X_{k+p}) = F(\omega^1, \ldots, w^l, X_1, \ldots, X_k)G(\omega^{l+1}, \ldots, \omega^{l+q}, X_{k+1}, \ldots, X_{k+p})$$

, where , $\omega \in V^*$ and $X \in V$. Let $(E_1,\ldots, E_n)$ be a basis for V and $(\varphi^1,\ldots,\varphi^n)$ be the dual basis for V^*. Then , by construction , we define

$$\varphi^i(E_j) = \delta^i_j \tag{2.1.1}$$

where δ^i_j is the Kronecker delta. A basis for ${T^k}_l$is given by the set of all tensors of the form

$$E_{j1} \otimes \ldots \otimes E_{jl} \otimes \varphi^{i1} \otimes \ldots \otimes \varphi^{ik}$$

where the i's and j's run from l to n. Any tensor $F \in {T^k}_l(V)$ can be written in terms of this basis. Note: components of vectors have upper indices and components of covectors have lower indices.There are obvious identities : ${T^k}_0(V) = T^k(V)$, ${T^0}_l(V) = T_l(V)$, $T^1(V) = T^*(V)$, $T_1(V) = V^{**} = V$. An important identity is ${T^1}_1(V)$= End (V)= $\mathbb{R}$

Theorem . If V is a finite - dimensional vector space , then there is a natural , basis independent isomorphism between ${T^k}_{l+1}(V)$ and the space of multi - linear maps

$$V^* \times \ldots \times V^* \times \ldots \times V \to V$$

l terms of V^* and k term of V. We define the trace tr: ${T^{k+1}}_{l+1}\ (V) \to {T^k}_l(V)$ by letting tr $F(\omega^1,\ldots,\omega^l,V_1,\ldots,V_k)$ be the trace of the endomorphism

$$F(\omega^1, \ldots, \omega^l, V_1, \ldots, V_k) \in T^1_1(V)$$

Any pair of indices can be contracted as long as one is contravariant and the other is covariant , i.e. contraction of tensors. For example ; $A_k = B^i_{ik}$

Testing for Tensor Character . A map

$$\tau : T^*(M) \times ... \times T^*(M) \times T(M). \times ... \times T(M) \to C^\infty(M)$$

is induced by a $\binom{k}{l}$-tensor field if and only if it is multi-linear over C^∞. Similarly, a map

$$\tau : T^*(M) \times ...T^*(M) \times T(M) \times ... \times T(M) \to T(M)$$

is induced by a $\binom{k}{l+1}$-tensor field if and only if it is multi-linear over $C^\infty(M)$ .Note ; in the first case , we have an equal number of vectors and covectors and we produce a function. In the second instance, we have one more vector that than covector and we produce a vector.

2.2 Index - based Approach

At this point , we will outline the traditional approach to tensors using local coordinates and indices,[30]. A set of basis vector {e} is chosen so that any vector , V, can be expressed as

$$V = V^i e_i$$

Given the basis set {e} , a basis set of dual vectors is {e^j} defined by

$$e^i e_j = \delta^i_j \tag{2.2.1}$$

The dual basis vectors are perpendicular to all basis vectors with a different index, and the scalar product of the dual basis vecor with the basis vector of the same index is unity. A vector V can be expressed in terms of the dual basis vector as

$$V = V_i e^i$$

The metric tensor g_{ij} is defined by the basis vectors

$$g_{ij} = e_i e_j \tag{2.2.2}$$

The metric tensor provides the scalar product of a pair of vectors A and B by

$$A.B = g_{ij} V^i V^j \tag{2.2.3}$$

where V^i and V^j are the contravariant components of A and B

Contravariant Vectors.Vectors are best understood by the way they transform under different coordinate systems. The components of a vector V in one coordinate system may be transformed into a vector $V^{'}$ in another coordinate system by the transformatiom matrix, the Jacobian matrix, as follows

$$V^{'\alpha} = \frac{\partial x^{'\alpha}}{\partial x^{\beta}} V^{\beta}$$

The upper index is the row , the lower index is the column of the matri**x.**

Covariant Vectors (1-forms). The covariant components of a vector transform as

$$V^{'}_{\alpha} = \frac{\partial x^{\beta}}{\partial x^{'\alpha}} V_{\beta}$$

Here β is the row , α is the column . The contravariant and covariant components are constructed such that

$$V^{\alpha} V_{\beta} = \delta^{\alpha}_{\beta} \tag{2.2.4}$$

is the identity matrix , where δ is the Kronecker Delta

Tensor Transformation Rules**.** These follow the rules for vector transformation. For example , for a tensor of contravariant rank 2 and a tensor of covariant rank 1 ;

$$T'^{\alpha\beta}_{\gamma} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\gamma}} T^{\mu\nu}_{\rho}$$

where the prime symbol indicates the new coordinate system and the transformed tensor. We will discuss the metric tensor later , but I will list a few important rules. The metric tensor $g_{\mu\nu}$ is defined by

$$g_{\mu\nu} = e_\mu . e_\nu \tag{2.2.5}$$

The infinitesimal displacement vector dx

$$dx = dx^\mu e_\mu$$

$$dx^2 = dx^\mu e_\mu dx^\nu e_\nu = g_{\mu\nu} dx^\mu dx^\nu \tag{2.2.6}$$

The metric tensor $g_{\mu\mu}$ and its inverse $g^{\mu\nu}$ are used to raise and lower indices

$$T_\nu = g_{\mu\nu} T^\mu$$

$$T^\nu = g^{\mu\nu} T_\mu$$

This process also works for higher order tensors. Tensors are invariant under coordinate transformations. Let us explore this idea further. For example , let us examine a $\binom{0}{2}$ tensor under a coordinate transformation. Asume $X_{ab} = Y_{ab}$ in a coordinate frame , $\mathfrak{X}$. What does it look like in a coordinate frame, $\mathfrak{X}'$? . Applying tensor transformation rules; multiplying both sides by $\frac{\partial x^a}{\partial x'^c} . \frac{\partial x^b}{\partial x'^d}$, we obtain

$$\frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d} X_{ab} = \frac{\partial x^a}{\partial x'^a} \frac{\partial x^b}{\partial x'^d} Y_{ab}$$

Therefore, $X'_{ab} = Y'_{ab}$, i.e. tensors hold in any coordinate system.

2.3 Manifolds

A manifold is a topological space , that is smooth (infinitely differentiable), Hausdorff and second countable,[17], [31]. The space is locally Euclidean. The passage from one coordinate system to another is smooth. Some important topological concepts ;

Hausdorff Spaces - a topological space X is a Hausdorff space, if for every $x, y \in X$, x$\neq$ y, there are neighbourhoods U, V of x, y respectively , such that $U \cap V = \emptyset$. In a Hausdorff space , singleton sets $\{x\}$ are always closed. A metric topology is always Hausdorff.

Connectedness **-** A topological space X is connected if the only subsets of X which are both closed and open are $\emptyset$ and X itself. Conversely , a topological space X is not connected if and only if there are nonempty sets G and H , such that G $\cap$ H = $\emptyset$, G $\cup$ H = X.

Compactness - If A $\subset$ X , a covering of A is a family { C_α| $\alpha \in$ J) in the power set of X , such that A $\subset \cup_{\alpha \in J}$ C_α. An open covering is one for which the family consists of open sets. A subset A of X is compact if every open covering of A has a finite subcover . Conversely, the closed bounded set of $\mathbb{R}$ are compact. This result generalises to $\mathbb{R}^n$**-** the compact subsets of $\mathbb{R}^n$are exactly those which are closed and bounded.

If X is a topological space , a chart at $p \in X$ is a function

$$\mu : U \to \mathbb{R}^d$$

where U is an open set containing p and μ is a homeomorphism onto an open set of $\mathbb{R}^d$. The dimension of the chart μ: U $\to \mathbb{R}^d$ is d. Two chart μ: U $\to \mathbb{R}^d$ and τ: V $\to \mathbb{R}^e$ on a topological space X are C^∞ , related if $d = e$ and either U $\cap$ V = $\emptyset$ or $\mu \circ \tau^{-1}$ and $\tau \circ \mu^{-1}$ are C^∞ maps . The domain of $\mu \circ \tau^{-1}$ is τ(U $\cap$ V), an open set in $\mathbb{R}^d$. See Figure **2.1** for an illustration.

Homeomorphism. A homeomorphism f: X $\to Y$ is a 1-1 onto function such that f and f^{-1}: $Y \to X$ are both continuous.

Diffeomorphism. A diffeomorphism ϕ : M $\to$ N is a smooth map that has an inverse map that is also smooth. It is said that M and N are diffeomorphic under ϕ. A simple example , any open interval (a, b) in $\mathbb{R}^1$ is diffeomorphic to (-1,1) under a suitable linear map.

Tangent Space. For any point $p \in$ M , the tangent space T_pM is the set of all tangent vectors, the directional derivatives. The local coordinates (x^i) give a basis for T_pM consisting of the partial derivative operators $\frac{\partial}{\partial x^i}$.

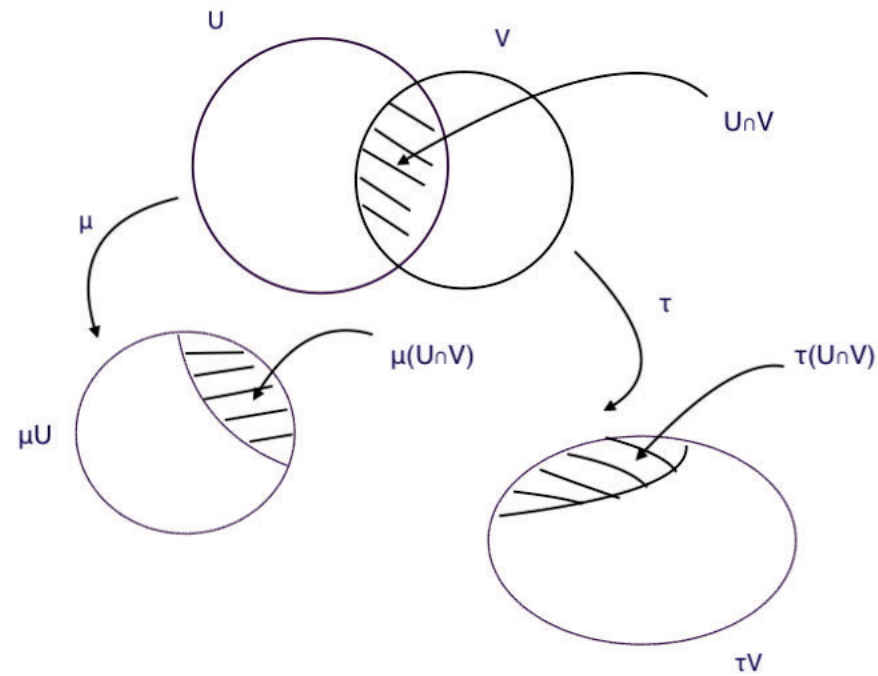

Figure 2.1: Homomorphism between charts

Pick a vector $X \in$ V , a finite dimensional vector space , with a smooth manifold structure. The directional derivative is defined as ;

$$Xf = \frac{d}{dt} |_{t=0} f(p+tX) \tag{2.3.1}$$

This is the usual identification $(x^1,...,x^n) \mapsto x^i \frac{\partial}{\partial x^i}$

Submanifold. If M’ is a smooth manifold , a submanifold (or immersed submanifold) of $\mathrm{M}^{'}$ is a smooth M together with an injective immersion ι;

$$\iota : M \to M^{'},$$

where ι(M) $\in$ $\mathrm{M}^{'}$ .Whereas M is a subset of $\mathrm{M}^{'}$, the topology of M may be different. An embedding is where the inclusion map is an onto homeomorphism. Let M be an embedded submanifold (n-dimensional) of a manifold $\mathrm{M}^{'}$(m-dimensional). For every point $p \in$ M , there exists slice coordinates $(x^1,...,x^n)$ on a neighbourhood $\mathrm{U}^{'}$ of p in M , such that $\mathrm{U}^{'} \cap$ M is given by

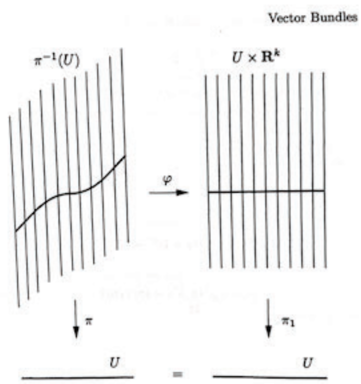

Figure 2.2: Local trivialization

$(x^{n+1},\ldots,x^m = 0)$ and $(x^1,\ldots,x^n)$ form local coordinates for M. At each $q \in U' \cap M$, T_qM can be identified as the subspace of T_qM spanned by the vectors $(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n})$.

Vector Bundles. When the tangent space at all points on a manifold M are glued together, a vector bundle is constructed. It is a union of vector spaces and a manifold. A vector bundle is defined as a combination of ;

(a) the manifold , M , the base

(b) a smooth manifold , E , the total space

(c) a projection , an onto map π: E $\to$ M, with the following properties ;

(1) each set E_p, called the fiber of E over p , $E_p := \pi^{-1}(p)$

(2) for each $p \in$ M , $\exists$ a neighbourhood U of p and a diffeomorphism φ ; such that

$$\varphi : \pi^{-1}(U) \to U \times R^k$$

This is known as a local trivialization . See Figure 2.2 for an illustration.

In simple words, for any small enough region U in M, the manifold E is a product U $\times$ $\mathbb{R}^k$. Noting that in the case of the tangent bundle TM , each coordinate system U $\in$ M gives rise to a coordinate system ∂_i at p on π^{-1}(U) $\subset$ TM . These function are bundle charts making TM an n vector bundle over M^n. A vector field X on M is a section X : M $\to$ TM of the tangent bundle . Having built our topological space and manifold, we later build upon this theme to reach the

Riemannian and Semi-Riemannian manifold as follows;

Set $\xrightarrow[topology]{}$ Topological Space $\xrightarrow[locally R^n]{}$ Manifold $\xrightarrow[connection]{}$ Manifold with Connection $\xrightarrow[metric]{}$ Riemannian Manifold

CHAPTER III

CALCULUS ON MANIFOLDS

3.1 Differential Forms

Differential forms are a critical branch of differential geometry, as they allow complex theoretical constructs and bridge together multi-variable calculus , vector calculus and operations on manifolds,[9]. I will first illustrate this in the 3-dimensional case. The differential or exterior derivative, df, of a function $f(x,y,z)$ is defined to be

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

The expression df is called a one-form. More generally , a 1-form ϕ on $\mathbb{R}^n$ is a real valued function on the set of all tangent vectors to $\mathbb{R}^n$**,** that is

$$\phi : T\mathbb{R}^{\mathbf{n}} \to \mathbb{R}$$

where ϕ is linear on the tangent space $T_x\mathbb{R}^{\mathbf{n}}$**,** for every $x \in \mathbb{R}^n$ and for any smooth vector field $v = v(x)$, the function ϕ(v) : $\mathbb{R}^n \to \mathbb{R}$ is smooth. The 1-forms ϕ , for each $x \in \mathbb{R}^n$**,** are elements of the dual space $(T\mathbb{R}^n)^*$**.** The space of 1- forms on $\mathbb{R}^n$ is dual to the space of vector fields on $\mathbb{R}^n$. For any vector v $\in \mathbb{R}^n$, the 1-forms dx^i pick out the *i*-th coefficient

$$dx^i(v) = v^i$$

The dx^i's form a basis for the 1-forms , so a 1-form ϕ can be expressed as

$$\phi = f_i(x)dx^i$$

For any vector $v \in \mathbb{R}^n$, where $v = v^i$;

$$\phi(v) = f_i v^i$$

Algebra of Differential Forms. There are at least three operations on p-forms. The most basic is the wedge product. Multiplication is known as a wedge product and is skew-symmetric.

$$dx^i \wedge dx^j = -dx^j \wedge dx^i \tag{3.1.1}$$

Hence $dx^i \wedge dx^j = 0$, if $i = j$. If ϕ contains pdx^i's , then we have a $p-$ form. More generally , if ζ is a $k-$form and η is a l - form , then

$$\zeta \wedge \eta = (-1)^{kl} \zeta \wedge \eta \tag{3.1.2}$$

and the derivation rule for differentiation, using Leibniz rule and skew-symmetry property , becomes

$$d(\zeta \wedge \eta) = d\zeta \wedge \eta + (-1)^{kl} \zeta \wedge d\eta$$

Furthermore , the differential operator d is nilpotent .

$$d(d\phi) = 0$$

This is an important property that is easily proven by using the commutativity of mixed partial derivatives and the skew symmetry property.

0-forms can be identified with scalar functions. With the differentiation operator $\omega \longmapsto d\omega$, this is identifiable to the gradient operation $f \mapsto \nabla f$

$$\begin{array}{ccc} M_1 & \xrightarrow{\alpha} & M_2 \\ f \circ \alpha \searrow & & \swarrow f \\ & M_3 & \end{array}$$

Figure 3.1: Pullback

1-forms can be identified with vector fields, and to the curl operation X$\rightarrow \nabla \times X$

2- forms can be identified with vector fields via the right hand rule and to the divergence operation X$\rightarrow \nabla . X$

3- forms can be identified with scalar functions

This is an example of Hodge duality , which we will address later.

When we apply the nilpotent property , $d(dw)$= 0 , we arrive at

$$\nabla \times \nabla f = 0$$

$$\nabla . (\nabla \times X) = 0$$

for a smooth scalar function f and a vector field X .

Next, we describe a very important mapping, the pull back , which will help us jump across maps without coordinate transformation.

Pull Back. Suppose M_1, M_2 and M_3 are smooth manifolds. Let α : $M_1 \rightarrow M_2$ to be a differentiable map . Then, given a function f: $M_2 \rightarrow M_3$, we can define a pullback of f under α, denoted by

This is equivalent to substituting the coordinates of α into the formula of f, see Figure 3.1 for an illustration. A useful theorem , which we will not prove here, but is useful for later proving Stoke's theorem, will be shown next.

Theorem. If φ: M $\rightarrow$ N is a C^∞ map and θ is a $p-$ form on N , then $d\varphi^*\theta = \varphi^* d\theta$

3.2 Integration of Forms

Integration over p-forms is done by integrating over C$^\infty$ p-cubes and sums of p-cubes (chains),[26].

Rectilinear p-Cube. A rectilinear p- cube in $\mathbb{R}^p$ is a closed cubical neighbourhood in Cartesian coordinates

$$U = (u^1, ..., u^p)|b^i \leq u^i \leq b^i + c^i, i = 1, ..., p$$

where b^i and c^i are constants , so U is closed and bounded , hence compact.

p- Cube. A C$^\infty$ $p-$cube α in a manifold M is a C$^\infty$ map

$$\alpha : U \to M$$

, where U is a rectilinear p- cube

Oriented p - Cube. An oriented p- cube is a pair (α,ω), where α is a p-cube and ω is an orientation of $\mathbb{R}^p$. If the coordinates x^i and y^i are related by the Jacobian determinant $J = det(\frac{\partial x^i}{\partial y^j})$, then $dx^1 ... dx^p = J dy^1 ... dy^p$. This way we compare coordinate volume elements $dx^i$and dy^j, noting that one of the two Cartesion systems $u^1, ..., u^p$ or $-u^1, u^2, ..., u^p$ must be consistently oriented with respect to ω. Another theorem we need to prove Stoke's theorem is Fubini's theorem .

Theorem. Fubini's Thorem . If f is continuous on U , then the definite integrals

$$f_p(u_1, ..., u^{p-1}) = \int_{a^p}^{b^p} f(u^1, ..., u^{p-1}, u^p) du^p$$

are continuous functions of the parameters $u^1, ..., u^{p-1}$, and the Riemann integral of f is

$$\int\limits_U f d\mu_p = \int\limits_{U_{p-1}} f_p d\mu_{p-1}$$

where the $(p-1)-$ cube U_{p-1} = $(u^1,...,u^{p-1})|a^i \leq u^i \leq b^i, i = 1,...,p-1)$,where μ is the standard measure on a p-cube. It follows by iteration that

$$\int\limits_U f d\mu_p = \int_{a1}^{b1} (... \int\limits_{a^{p-1}}^{b^{p-1}} (\int\limits_{a^p}^{b^p} f(u^1,...,u^p)du^p)du^{p-1}...)du^1 \tag{3.2.1}$$

Now, we are armed with the necesessary tools to examine integration of forms.

Motivation. Our goal here is to prove Stoke's theorem, which is a powerful theorem , with applications in many branches of mathematics and a central role in the theory of manifolds and general relativity. Its use is indispensable . And I will dedicate a lot of time to this endeavor,[41]. At an infinitesimal level, the amount of work done to move a particle from a point $x_i \in \sum \mathbb{R}$ to a point $x_{i+1} \in \mathbb{R}$ is linearly proportional to the displacement $\triangle x_i = x_{i+1} - x_i$ and a constant of proportionality $f(x_i)$ and is approximately

$$\int_a^b f(x)dx = \sum_{i=0}^{n-1} f(x_i)\triangle x_i$$

In higher dimensions, the equivalent of a proportionality constant of a linear relationship is a linear transformation. Therefore, for each x_i , we need a linear transformation

$$\omega_{x_i} : \mathbb{R}^n \to \mathbb{R}$$

that takes in an infinitesimal displacement $\triangle x_i \in \mathbb{R}^n$and returns a scalar $\omega_{x_i}(\triangle x_i) \in \mathbb{R}$**.** Here, ω_{x_i} acts as a linear functional on the space of tangent vectors at x_i and is therefore a cotangent vector at x_i. In analogy to the 1-dimensional case above, the net work required to move from a to b along the path γ is approximately

$$\int\limits_\gamma \omega \simeq \sum_{i=0}^{n-1} \omega_{x_i}(\triangle x_i)$$

Next , a quick review on integration on Euclidean spaces. Let U be the standard measure of a rectilinear p-cube U = {(u^1,...,u^p| $a^i \leq u^i \leq b^j$)} given by μ_pU = $(b^1 - a^1)(b^2 - a^2)...(b^p - a^p)$;μ_p assigns numbers to the cubical cubes of $\mathbb{R}^p$, a measure. The Riemann integral of a real-valued function f defined on U is

$$\int_U f d\mu_p = \lim_{U_j \to 0} \sum_{f=1}^{n} f(x_j)\mu_p U_j$$

where U is broken up into N smaller p-cubes U_f and a point x_i has been chosen in each U_j. Next, integration of forms.

Integration of Forms. Let α: U $\to$ M , where U is as defined above, a measure. Let θ be a p - form , defined on a region of M , which contains the range of an oriented p - cube (α,ω). Then, $\alpha^*\theta$ is the pullback of θ to U , under α. As we will illustrate below, with an example , this is a substitution of the coordinate formula for α into the coordinate expression for θ. Hence, we get an expression $\alpha^*\theta$ = f ω , where f is a C^∞ real - valued function on U. We then define an inner product on p-forms on $\mathbb{R}^p$, such that $<\omega, \omega>_p = 1$, then

$$f = <\alpha^*\theta, \omega>_p$$

Then , the definition of the integral of θ on (α, ω) is

$$\int_{(\alpha,\omega)} \theta = \int_U <\alpha^*\theta, \omega>_p d\mu$$

The integral of a p- form on a p-chain $\sum r_i C_i$ is defined in terms of integrals on $p-$cubes :

$$\int_{\sum r_i C_i} \theta = \sum r_i \int_{C_i} \theta$$

Example. Take the circle S^1= {(x,y)| $x^2 + y^2 = 1$ in $\mathbb{R}$, with a counterclockwise orientation. Define α on [0 , 2π] by α(u)= ($cosu, sinu$). The coordinate equations for α are x = $cosu$, y = $sinu$. If $\theta = \frac{xdy - ydx}{x^2+y^2}$, then $\alpha^*\theta = \frac{cosud(sinu) - sinud(cosu)}{cos^2u + sin^2u}$ = du. Now $<du, du>_1$= 1 , so

$$\int_{(\alpha,du)} \theta = \int_{[0,2\pi]} 1 d\mu = \int_0^{2\pi} du = 2\pi$$

One more definition which is necessary for proving Stokes Theorem is the concept of Partition of Unity.

Partition of Unity**.** A smooth partition of unity on a manifold M is a collection $\{f_a{:}\alpha \in \mathrm{A}\}$ of functions $f_\alpha \in \mathfrak{J}(\mathrm{M})$ such that

(1) $0 \leq f_\alpha \leq 1$ for all $\alpha \in \mathrm{A}$

(2) $\{supp\ f_\alpha$: $\alpha \in$ A) is locally finite

(3) $\sum_\alpha f_\alpha = 1$

Property (2) can be re-stated as the partition of unity is subordinate to an open covering $\mathfrak{C}$ of M provided each set supp f_α is contained in some element of $\mathfrak{C}$.

Let f: X $\rightarrow \mathbb{R}$ be a function from a topological space X . The support of f is the closed set supp $f = [f^{-1}(\mathbb{R} \setminus 0)]^C$. Partitions of unity are useful for assembling locally defined objects into a global object or decomposing a global object into a sum of local objects.

Theorem. Stokes Theorem. Armed with the necessary tools, we can now state and prove Stoke's theorem. Let θ be a (p- 1)-form defined on the ranges of all the cubes of a p - chain C , where $p > 0$. Then

$$\int_C d\theta = \int_{\partial C} \theta \tag{3.2.2}$$

We will prove this for a single $p-$cube C = (α, ω). By linearity and partition of unity , this argument can be extended to a chain of p - cubes.

Proof. Let α : U $\rightarrow$ M , U as defined above , where θ is a p-form $\in$ M . First , we define the pull back $\alpha^*\theta = \sum_i (-1)^{i-1}\ f_i du^1 ... du^p$. By Theorem 1 : $d\alpha^*\theta = \alpha^* \mathrm{d}\theta = (\sum \partial_i f_i) du^1 ... du^p$.

$$\int_U < \alpha^* d\theta, \omega >_p d\mu_p = \sum_i \int_U \partial_i f_i d\mu_p$$

Next, we apply Fubini's theorem to the right hand side of the integral with the i*th* variable as variable of integration

$$\int_U \partial_i f_i d\mu_p$$

$$= \int_U \int_{b_i}^{b_i+c_i} \partial_i f_i(u^1,...,u^i,,,u^p)du^i d\mu_{p-1}$$

$$= \int_U [f_i(u^1,...,b^i+c^i,...,u^p) - f_i(u^1,...,b^i,...u^p)]d\mu_p$$

The second equality follows from the fundamental theorem of calculus. Therefore

$$\int_U < \alpha^* d\theta, \omega >_p d\mu_p = \int_{(\alpha,\omega)} d\theta = \int_{\partial(\alpha,\omega)} \theta$$

the first equality follows from the definition of integration of forms. By linearity;

$$\int_C d\theta = \int_{\partial C} \theta$$

Application of Stoke's Theorem . Next , I present a forme fruste version of Stoke's Theorem. I will need to use the metric of a vector space , the covariant derivative of a manifold, and the Christoffel symbol , which I will expand on in great detail in the next two chapters. Let $X \in TM$; the map

$$div : TM \to C^\infty(M)$$

is called the divergence operator . In terms of local coordinates (U,x_i), $X = X^i \frac{\partial}{\partial x^i}$, then

$$div(X) = \frac{1}{\sqrt{g.g_{ij}}} \frac{\partial}{\partial x^i}(\sqrt{g.g_{ij}} X^i), \tag{3.2.3}$$

where $g = det g_{ij}$, and $g_{ij} = (\partial_i, \partial_j)$

Proof. We know $X = X^i \partial_i$, where ∂_i is a shorter notation for $\frac{\partial}{\partial x^i}$. The covariant derivative

$\nabla_X = (\partial_i X^i + X^k \Gamma^i{}_{kj}) \mathrm{d}x^j \otimes \partial_i$

Hence, div(X) = $\partial_i X^i + X^k \Gamma^i{}_{ki}$. Using the identity $\Gamma^i{}_{ki} = \frac{1}{2} g^{ij}{}_k g_{ij} = \frac{1}{2g.g_{ij}} \partial_k g.g_{ij} = \frac{1}{\sqrt{g.g_{ij}}} \partial_k \sqrt{g.g_{ij}}$.

By substituting into ∇_X ; we obtain the identity div(X) = $\frac{1}{\sqrt{g.g_{ij}}} \partial_i (\sqrt{g.g_{ij}} X^i)$.

Interior Product. The interior product is defined to be the contraction of a differential form with a vector field. Thus , if X is a vector field on the manifold M , then

$$\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$$

is the map which sends a p-form ω to the $(p-1)$-form ι_X defined by

$$\omega(X, X_1, ..., X_{p-1}) = (\iota_X \omega)(X_1, ..., X_{p-1})$$

for any vector fields $X_1, ..., X_{p-1}$.The interior product on a 1-form α is

$$\iota_X \alpha = \alpha(X) = < \alpha, X >$$

where $<,>$ is the duality pairing between α and the vector X. If β is a $p-$form and γ is a $q-$ form, then

$$\iota_X(\beta \wedge \gamma) = (\iota_X \beta) \wedge \gamma + (-1)^p \beta \wedge (\iota_X \gamma)$$

That is , the interior product obeys Leibniz's rule. An operation satisfying linearity and the Leibnix rule is called a derivative. Note : A dual pair is a 3-tuple ($X, Y, <,>$) consisting of two vector spaces X and Y over the same field F and a bilinear map

$$<,>: X \times Y \to F$$

such that $\forall\, x \in X \setminus \{0\}, \exists\, y \in Y: (x,y) \neq 0$ and $\forall\, y \in Y \setminus \{0\}$, $\exists\, x \in X: (x,y) \neq 0$. If the vector spaces are finite dimensional, this means that the bilinear form is non-degenerate .

Theorem. A vector space V together with its dual V^* and the bilinear map

$$< x, f >:= f(x)$$

where $x \in V$ and $f \in V^*$forms a dual pair. In a local coordinate (U, x_i), let vol be defined as

$$vol = \sqrt{g} dx^i \wedge ... \wedge dx^m \tag{3.2.4}$$

where $g$is det (g_{ij}). Vol is a global m-form, called the volume form of M. Fix a smooth tangent vector field X. The interior product $\imath(X)$ is defined for every tangent vector field $X_1, ... X_{m-1}$

$$(\imath(X)vol)(X_1, ..., X_{m-1}) = vol(X_1, ..., X_{m-1})$$

Then, for every smooth tangent vector field X ,

$$d(\imath(X)vol) = div(X).vol$$

Proof. Since vol = $\sqrt{g}$ $.dx^1 ... dx^m$

$$div(X)vol = \partial_i(\sqrt{g}X^i)dx^1, ..., dx^m = d((-1)^{i+1}\sqrt{g}X^i) \wedge dx^1 \wedge ... \wedge dx^m)$$

= ω, by setting bracketed right hand expression as ω . ω is independent of choice of coordinates, so ω is a globally defined $(m-1)$ form

$$div(X)vol = d\omega = \imath(X)vol$$

second equality follows from definition. Applying Stokes Theorem , we arrive at our final result

$$\int_M div(X)vol = \int_{\partial M} \omega \tag{3.2.5}$$

This identity is very useful , and will be applied later when we derive Einstein's field equation. This is an area we will revisit later when we look at determinants of the metric tensor and its variation.

3.3 Hodge Star Operator

There are three operators that act on p-forms: wedge product , d-operator and the Hodge star operator,[13],[18]. To complete the picture, we will take a brief survey of the Hodge star operator. Let V be a finite n-dimensional vector space with inner product g , where

$$g = g_{ij}e^i \otimes e^j$$

The Hodge star operator , $*$, is a linear operator mapping p-forms on V to $(n-p)$-forms

$$*: \Omega(V)^p \rightarrow \Omega^{n-p}(V)$$

Let V have a basis $(e^1, ..., e^n)$ and let V^* have the dual basis $(e_1, ..., e_n)$. The Hodge star operator is defined as the linear operator that maps the basis elements of Ω^p(V) as

$$*(e^{i_1} \wedge ... \wedge e^{i_p}) = \frac{\sqrt{|g|}}{(n-p)!} g^{i_1 l1} ... g^{i_p l_p} \varepsilon_{l_1} ... \varepsilon_{l_p} l_p l_{p+1} ... l_n e^{l_{p+1}} \wedge ... \wedge e^{l_n},$$

$|g| = det g_{ij}$ and ε is the Levi-Civita permutation symbol. The Hodge star operator can also be defined in a coordinate free manner by the following expression

$$u \wedge *v = g(u, v) Vol(g)$$

where $g(u, v)$ is the inner product on p-forms $g(u, v) = g_{i_1 j_1} ... g_{i_p} g_{j_p} u^{i_1 ... i_p} v^{j_1 ... j^p}$ and $Vol(g)$ is the unit volume form associated with the metric, V$ol(g)$ = $\sqrt{det(g)}$e^1...e^n. Note : $**$ = (-1)$^{p(n-p)}$id , where id is the identity operator in Ω^p(V). In three dimensions . $**$ = id for p = 0,1,2,3 . In $\mathbb{R}^3$, the metric tensor $g = dx \otimes dx + dy \otimes dy + dz \otimes dz$ and the Hodge star operator is $*dx = dy \wedge dz$, $*dy = dx \wedge dz$ and $*dz = dx \wedge dy$

3.4 Correspondence Between Vectors and p-forms

In this next section, we will flesh out core concepts at a basic level in order to better our understanding,[32] . The idea here is to form a bridge between vector calculus and p-forms. Let us examine the case for 3 dimensions. One needs to note that there is a correspondence between vectors and 1-forms:

$$(a,b,c) \longleftrightarrow adx+bdy+cdz$$

Vectors also correspond to constant 2-forms:

$$(a,b,c) \longleftrightarrow adydz+bdzdx+cdxdy$$

The cross product is the product of 1 forms :

$$(adx+bdy+cdz)(rds+sdy+tdz) = (bt-cs)dydz+(cr-at)dzdx+(as-br)dxdy$$

The dot product is the product of a 1-form and a 2 -form

$$(adx+bdy+cdz)(rdydz+sdzdx+tdxdy) = (ar+bs+ct)dzdydx$$

Note that in 3 dimenssions , a scalar can be either a 0-form or 3-form. In summary;

$$\underset{f}{0-form} \overset{d}{\to} \underset{dx,grad}{1-form} \to \underset{dxdy,curl}{2-form} \overset{d}{\to} \underset{dxdydz,scalar}{3-form}$$

We have alluded to this before , but it is worth restating in words; If ω is a constant form , such as dx or dxdy , then its differential is 0.

$$d\omega = 0$$

However, consider the differential of the following 1 - from

$$d(ydx + zdy + ydz) = dydx + dzdy + dydz$$

The differential of a 1-form is a 2 -form. In general, the differential of a k-form is a $k+1$ form. If ω is a k-form for which there exists a $k-1$ form , say ν, such that

$$\omega = d\nu$$

then ω is exact.

Claim. If ω is any differential form whose coefficients have continous second partial derivatives, then

$$d(d\omega) = 0$$

The proof is available in most standard textbooksof advanced calculus. A differential form ω is closed if

$$d\omega = 0$$

If ω is exact , that is ω = dν, and has continuous partial derivatives, then ω must be closed

$$d\omega = d(d\nu) = 0$$

However, a closed differential form does not have to be exact. Consider the boundary operator ∂. If C is a closed curve, then it has no boundary

$$\partial C = \emptyset$$

Since the boundary of a simply connected surface is always a closed curve

$$\partial(\partial S) = \emptyset$$

The boundary of a boundary of a simply conneced manifold is always the empty set.

Lemma. Poincare Lemma If ω is a differential form that is closed and differentiable in a simply connected region, R , then ω is exact in R.

Note duality between differential forms and geometric objects.

$$d(d\omega) = 0 \longleftrightarrow \partial(\partial M) = \emptyset$$

This duality can be seen with Stokes Theorem

$$\int_{\partial M} \omega = \int_{M} d\omega$$

Finally we re-state some and complete the rest of the basic operators

Gradient $\nabla f \longleftrightarrow$ differential of a 0-form

Curl $\nabla\times$ F $\longleftrightarrow$ differential of a 1-form

Divergence $\nabla . f$ $\longleftrightarrow$ differential of a 2-form

Laplacian $\nabla^2 f$ $\longleftrightarrow$ re-write df as a 2-form and then take its differential

$\nabla \times (\nabla f) = 0 \longleftrightarrow d(d\omega) = 0$, ω is a 0-form

$\nabla(\nabla\times f) = 0 \longleftrightarrow d(d\omega) = 0$, ω is a 1-form

CHAPTER IV

METRICS

4.1 Riemannian Metric

Riemannian metric on a smooth manifold M is a 2-tensor field $g \in \mathscr{T}^2$(M) that is

(a) symmetric $g(X,Y) = g(Y,X)$

(b) positive definite $g(X,X) > 0$, $X \neq 0$

A Riemannian metric determines an inner product on each tangent space T_pM;

$$(X,Y) := g(X,Y) for X,Y \in T_pM$$

Hence, a Riemannian manifold is a manifold with a Riemannian metric (M, g),[29],[33]. The length or norm of any tangent vector $X \in T_pM$ is

$$|X| := < X,X >^{1/2}$$

Vectors $E_1,...,E_k$ are orthonormal if of unit length and pairwise orthogonal;

$$E_iE_j = \delta_{ij}$$

Isometry of M **.** Let φ**:** ($M^{'}$, $g^{'}$) $\rightarrow$ (M, g) , where ($M^{'}$, $g^{'}$),(M, g) are Riemannian manifolds.

A diffeomorphism φ from $M^{'}$ to M is an isometry if

$$\varphi^{*}g = g'$$

It can be shown that an isometry is an equivalence relation. Furthermore, a composition of isometries is an isometry and the inverse of an isometry is an isometry. It turns out that the set of isometries of M is a group, the isometry group of M, which is also a Lie group acting smoothly on M. A discussion of Lie groups will follow later. Let $(E_1,...,E_n)$ be a basis for TM and (φ^1, ..., φ^n) its dual basis. The Riemann metric can be expressed as

$$g = g_{ij}\varphi^i \otimes \varphi^j$$

where g_{ij}=$<E_i,E_j>$ is the coefficient matrix and is symmetric in i snd j. In a co-ordinate frame

$$g = g_{ij}dx^i \otimes dx^j$$

Because of the symmetry of g_{ij}

$$g = g_{ij}dx^i dx^j$$

For a Euclidean metric

$$g = \delta_{ij}dx^i dx^j$$

Raising and Lowering Indices. We have seen this before , but in a different format. As before , this allows us to convert vectors to covectors and vice versa. Given a metric g on M , define a map from $TM \to T^*M$, that sends a vector X to the covector $X^\flat$ by

$$X^\flat(Y) = g(X,Y)$$

In coordinate form,

$$X^\flat = g(X^i\partial_i,.) = g_{ij}X^i dx$$

where $X_j = g_{ij}X^i; X^\flat$ is obtained from X by lowering an index - hence musical notation of flat,$\flat$. SInce matrix g_{ij} is invertible , so is the flat operator. It is designated as sharp,$\sharp$. $\omega \longmapsto \omega^\sharp$. In coordinates , components of $\omega^\sharp$ has components

$$\omega^i = g^{ij}\omega_j$$

Here , we are raising the index. In juxtaposition to the vector based approach above , I will state briefly the metric tensor transformation in the index based approach format. Metric tensor tranformation between different coordinate systems, using tensor algebra rules, is as follows

$$g_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x^\beta} g_{\lambda\delta}$$

Given a co-ordinate transformation, we can compute the components of the metric tensor. Or , if we have a basis { e_i} and its dual basis $\{e^i\}$. The metric tensor g_{ij} can be computed by $g_{ij} = e_i e_j$, for the basis vector, $g^{ij} = e^i e^j$, for the dual basis and $g_j{}^i = e^i e_j$, for the contravariant and covariant basis. Later on , we will address the distance element of a metric space. However, we can also compute , literally read off the components of the metric , by using the distance element. I will illustrate with a simple example in a Euclidean 2 - dimensional space. We have $ds^2 = g_{ij}dx^i dx^j$, which reduces to $ds^2 = dx^2 + dy^2$. The metric tensor is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is simply the coefficients of the above equation in 2 $\times$2 matrix form. Next , we address the algebra of the metric tensor and try and collect all the identities that will prove very useful later. We will adopt the tensor index approach. In summary, the metric has at least three different roles , the last two properties we will expand on in later chapters.

(1) The metric allows us to compute the dot product and norm of a vector

$$A.B = g_{\mu\nu}A^\mu B_\nu$$

(2) The metric , as we will see later , will give us a connection on a manifold, through the

Christoffel symbol

(3) The metric, through a coordinate basis , will allow us to calculate length

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

4.2 Variation of Metric and Metric Determinant

Varying the metric inverse and the metric determinant, we know that $g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\lambda$, [7]. Taking the variation ;

$$\delta g^{\mu\nu} g_{\mu\lambda} + g^{\mu\nu} \delta g_{\nu\lambda} = 0$$

Multiply by $g^{\lambda\rho}$;

$$\delta g^{\mu\nu} \delta^\rho_\lambda = -g^{\lambda\rho} g^{\mu\nu} \delta g_{\nu\lambda}$$

We arrive at our first identity ;

$$\delta g^{\mu\rho} = -g^{\lambda\rho} g^{\mu\nu} \delta g_{\mu\lambda} \tag{4.2.1}$$

Next, we prove a very powerful identity , the differential of the determinant of the metric tensor.

Theorem. If g is the determinant of a metric tensor g_{ab} , then $\partial_c g = g.g^{ab} \partial_c g_{ab}$

Proof. Let $A = (a_{ij})$ be a square matrix , $a = det$A , A^{ij} the cofactor of (a$_{ij}$). Then, the inverse of A, $(b^{ij}) = \frac{1}{a}(A^{ij})^T$, T being the transpose. Fix i, and expand the determinant (a_{ij}) by the $i-th$ row. So $a = a_{ij} A^{ij}$, we are using Einstein summation convention. Differentiating both sides with respect to a$_{ij}$, we obtain

$$\frac{\partial a}{\partial a_{ij}} = A^{ij}$$

Note a_{ij} does not appear in any of the cofactors A^{ij}. Assume a$'_{ij}s$ are all functions of the coordinates x^k, the the determinant is a function of a_{ij} , which is a function of x^k. Therefore,

$$a = a(a_{ij}(x^k))$$

Differentiating with respect to x^k and using $\frac{\partial a}{\partial a_{ij}} = A^{ij}$, by the chain rule ,

$$\frac{\partial a}{dx^k} = \frac{\partial a}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial x^k}$$

$$= A^{ij} \frac{\partial a_{ij}}{\partial x^k}$$

$$= ab^{ji} \frac{\partial a_{ij}}{\partial x^k}$$

by applying definition of inverse. Noting that $a = detA = g = det g_{ij}$ and symmetry of metric tensor g^{ij}, we get

$$\partial_c g = g g^{ij} \partial_c g_{ij}$$

By differentiating $\sqrt{-g}$ and by symmetry , we obtain

$$\partial \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{ij} \partial g_{ij} \tag{4.2.2}$$

4.3 Pseudo-Riemannian Metric

Also known as semi-Reimannian metric, defined on a smooth manifold M, is a symmetric 2-tensor field g that is

(a) non-degenerate at each point $p \in M$, i.e

$$g(X,Y) = 0, \forall Y \in T_pM \longleftrightarrow X = 0$$

(b) if $g = g_{ij}\varphi^i\varphi^j$, in terms of a dual co-frame , non-degeneracy means that the metric g_{ij} is invertible , g^{ij}. By applying the Gram - Schmidt algorithm, one can construct a basis $(E_1, ...E_n)$ for T_pM such that

$$g = -(\varphi^1)^2 - ... - (\varphi^{r)^2} + (\varphi^{r+1})^2 + ... + (\varphi^n)^2$$

for $0 \leq r \leq n$, r is the index of g and it is the maximum dimension of any subspace of T_pM on which g is negative definite,[9],[13] . By Sylvester's Law of Inertia, the index is independent in the choice of basis. The most important pseudo-Reimannian metric is the Lorentz metric of index 1. An example of the Lorentz metric is the Minkowski metric on $\mathbb{R}^{n+1}$. With co-ordinates $(\zeta^1,...,\zeta^n,\tau)$, the Minkowski metric is

$$m = -(d\tau)^2 + (d\zeta^1)^2 + ... + (d\zeta^n)^2,$$

where ζ is the space direction and τ is the time direction. The ζ^i s and τ belong to subspaces on which g is positive definite and negative definite, respectively. The Minkowki metric is the invariant of Einstein's special theory of relativity, meaning the laws of physics have the same form in any coordinate system , in the absence of gravity. Before we move onto volume forms, I will outline the generalization of the gradient and the divergence operators that we encounter in vector calculus. Let $\mathfrak{J}(M)$ be the set of all smooth real-valued functions on M. Let $\mathfrak{X}(M)$ be the module over $\mathfrak{J}(M)$

Gradient. Then the gradient, grad f , of a function $f \in \mathfrak{J}$(M) is the vector field that is metrically equivalent to the differential $df \in \mathfrak{J}^*$(M) ; hence

$$< gradf, X >= df(X) = Xf$$

$X \in \mathfrak{X}(M)$. In coordinate notation,

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Hence

$$gradf = g^{ij} \frac{\partial f}{\partial x^i} \partial_j$$

Before, we define the divergence of a vector field , we will define the covariant differential of a tensor.

Covariant Differential. The covariant differential of an (r,s) tensor A on M is the $(r,s+1)$ tensor DA such that

$$DA(\theta^1,...,\theta^r,X_1,...,X_s,V) = (D_V A)(\theta^1,...,\theta^r,X_1,...,X_s)$$

for all $V,X_i \in \mathfrak{X}$(M) and $\theta^j \in \mathfrak{X}^*(M)$. When $r = s = 0$, the covariant differential is the usual differential;

$$(Df)(V) = D_V f = Vf = dfV \tag{4.3.1}$$

Divergence. The divergence of a tensor A, $divA$, is the contraction of a covariant slot in its covariant differential DA. This requires us to use frame fields, which is an orthonomal basis for a tangent space. Let D be the covariant derivative (a whole subsection will be devoted to this later). Let C be a contraction and $\varepsilon_i = < E_i, E_i >$. If V is a vector field , then

$$divV = \varepsilon_i < D_{E_i}, E_i >$$

In the coordinate system ,

$$divV = \frac{\partial V^i}{\partial x^j} + \Gamma^i_{ij} V^i \tag{4.3.2}$$

Γ^i_{ij} is the Christoffel symbol. Finally, we can now discuss volume elemnts and forms which are fundamental to our pursuit of the Einstein Field Equation.

4.4 Volume Forms

A volume element on an n-dimensional vector space V is a function ω that assigns to n vectors $v_1,...,v_n \in V$ the volume of the parallelpiped with these vectors as sides. Stated more explicitly, the determinant of a set of basis vectors is the signed hypervolume of the parallelpiped spanned by the vectors,[32]. If $\omega(v_1,...,v_n) = 0$, then the vectors are linearly dependent and the parallelpiped collapses. If ω is multilinear and if ($e_1,...,e_2$) is an orthonormal basis for V, then

$$\omega(e_1,...,e_n) = \pm 1$$

A volume element on an n-dimensional pseudo- Riemannian manifold M is a smooth n-form ω such that $\omega(e_1,...,e_n) = \pm 1$. The next two lemmas are important and we will prove both.

Lemma. On the domain U of a coordinate system ζ , there is a volume element ω such that

$$\omega(\partial_1,...,\partial_n) = \sqrt{(det} g_{ij})$$

Proof. Let $V_1,...,V_n$ be vectors on U, then $V_j = V^i{}_j \partial_i$. Define $\omega(V_1,...,V_n) = det(V^i_j)\sqrt{(det} g_{ij})$. If $(V_1,...,V_n)$ is a basis, then

$$\delta_{ij}\varepsilon_j = < V_i, V_j >= (V^r_i \partial_r, V^s_j \partial_s) = V^r g_{rs} V^s_j,$$

where $\varepsilon_j = < V_j, V_j >$. Taking determinants gives

$$(-1)^n = (detV^i_j)^2 det(g_{ij})$$

Hence

$$\omega(V_1,...,V_n) = det(V^i_j)\sqrt{(det} g_{ij}) = \pm 1 \tag{4.4.1}$$

In differential forms, the notation is

$$\omega = \sqrt{(det} g_{ij})dx^1...dx^n \tag{4.4.2}$$

Later on , we will examine the Lie derivative L_X. However, another important lemma is ;

Lemma. If ω is a local volume element on M , then

$$L_X(\omega) = (divX)\omega \tag{4.4.3}$$

Proof. Let $(E_1,...,E_n)$ be a basis such that $\omega(E_1,...,E_n) = 1$. SinceL_X is a tensor derivative , then $L_X 1 = X.1 = 0$

$$(L_X\omega)(E_1,...,E_n) = -\omega(E_1,...,L_X E_i,...E_n)$$

Let us write the Lie derivative as

$$L_X E_i = [X,E_i] = f_{ij} E_j,$$

where [ ] is the commutator. Since ω is skew symmetric , the only term that survives is $f_{ii}E_i$. Since *divX*= $\varepsilon_i < D_{E_i}, X, E_i >$, and applying simple commutator algebra ;

$$divX = \varepsilon_i < D_X E_i, E_i >= -\varepsilon_i < [X,E]_i, E_i >$$

Since $< E_i, E_i >$ is constant , the first term vanishes. Since $[X,E_i] = f_{ij}E_j$, we obtain our final result

$$divX = -f_{ii} = L_X$$

CHAPTER V

OVERVIEW OF CONNECTIONS

5.1 Differentiating Vector Forms

Let γ: $(a,b) \to M$ be a curve in a Manifold M. We need to think of a geodesic as the Riemannian equivalent of a straight line in Euclidean space,[27],[30]. First , we orthogonally project $\ddot{\gamma}$(t) onto the tangent space $\mathrm{T}_{\dot{\gamma}}(M)$. We would like to define a geodesic as a curve in M whose tangential acceleration, $\ddot{\gamma}$ is zero. Note , the velocity vector, $\dot{\gamma}$(t) , is co-ordinate invariant for each $t \in M$. Essentially, this is a ' vector field along a curve '. However, the acceleration vector, $\ddot{\gamma}$(t), is not co-ordinate invariant. If we move from t_1 to t_2 , in order to differentiate $\dot{\gamma}$(t), , we need to compute the difference quotient involving velocity vectors $\dot{\gamma}(\mathrm{t}_1)$ and $\dot{\gamma}(\mathrm{t}_2)$. However, these vectors live in different tangent spaces, $\mathrm{T}_{\gamma_{t_1}}M$ and $\mathrm{T}_{\gamma_{t_2}}$, and their subtraction is meaningless. Therefore, to obtain an acceleration of a curve in a manifold , in a co-ordinate invariant manner, the trick is to compare values of the velocity vector at different points on the manifold. In other words, to connect nearby tangent spaces.

5.2 Connections

As above, we want to compare two tangent vectors and produce a third . We do this in a tangent bundle and define a linear connection in TM, the tangent space,[27]. Hence, we define a map

$$\nabla : T(M) \times T(M) \to T(M)$$

∇ has to satisfy the following three properties ; for $X,Y \in$ T(M)

(1) $\nabla_X Y$ is linear over $C^\infty(M)$ in X

$$\nabla_{fX_1+gX_2} = f\nabla_{X_1}Y + g\nabla_{X_2}Y; f,g \in C^\infty(M)$$

(2) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y

$$\nabla_X(aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2; a,b \in R$$

(3) ∇ satisfies the product rule (Leibniz)

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y; f \in C^\infty$$

Noting, f is a scalar function , hence ∇ does not apply to it, here we have an ordinary derivative. We say $\nabla_X Y$ is the covariant derivative of Y in the direction of X. Let $(E_1,...,E_n)$ be n smooth vector fields defined on an open set U such that $(E_1,...,E_n)_{|p}$ form a basis for T_pM at every point p in U, where U is a neighbourhood of p in M. This is known as a local frame. Paraphrasing , note that the mapping is defined on a tangent bundle of a manifold. And we are mapping a velocity vector from one tangent space to another via a linear connection. Next, we examine the covariant derivative of the basis vectors $\{E\}$, remembering that we cannot subtract vectors on a non-Euclidean manifold, wthout a special operation, known as a connection. The action of ∇ is determined completely by the Christophel symbol Γ^k_{ij}. Since ∇ is a $\binom{2}{1}$ tensor operation , we expect Γ^i_{jk} to be a $\binom{1}{2}$ indexed operator. It is not linear over $C^\infty(M)$ in Y as in (1) above. Next, we express the connection in a coordinate framework. Let ∇ be our linear connection and $X,Y \in$ T(M). In a local frame, $X = X^iE_i = X^i\partial_i$, $Y = Y^jE_j = Y^i\partial_j$. Then

$$\nabla_X Y = \nabla_X(Y^jE_j)$$

$$= (XY^j)E_j + Y^j\nabla_{X^iE_i}E_j$$

$$= (XY^j)E_j + X^iY^j\nabla_{E_i}E_j$$

$$= XY^jE_j + X^iY^j\Gamma^k_{ij}E_k$$

$$\nabla_X Y = XY^jE_j + X^iY^j\Gamma^k_{ij}E_k$$

First equality follows by the product rule, in the second equality, note that X_i' s are components, third equality expresses covariant derivative of E_j and we apply dummy index to first term of last equality.

Covariant Differentiation of Tensor Fields. For every input , v, a tangent vector $\in T_pM$ at a point p, if X is a vector field, there is a unique operation $\nabla_v X$, such that $\forall\ p \in M$, all u, v $\in T_pM$, and all $a \in \mathbb{R}$ **,** the following operations hold;

$$(1)\nabla_{(u+v)}X = \nabla_u X + \nabla_v X$$

$$(2)\nabla_{av}X = a(\nabla_v X)$$

$$(3)\nabla_v(X+Y) = \nabla_v X + \nabla_v(Y), X, Y vectorfields$$

$$(4)\nabla_v(X.Y) = (\nabla_v X)Y + X(\nabla_v Y)$$

Note XY is a tensor product, $X \otimes Y$.

Covariant Derivative of a Covariant Vector Field. Let $F = F_i dx^i$ be a covector field , a 1-

form. Let $X = X^i \partial_i$ be any vector field. Then $F.X = F_i X^i$. Applyng (4);

$$\nabla_j(F.X) = \nabla_j(F^i X_i) = (\nabla_j F)_i X^i + F_i(\nabla_j X)^i$$

For derivatives of scalar functions;

$$\partial_j(F_i X^i) = (\partial_j F_i)X^i + F_i(\nabla_j X)^i$$

Equating the last two expressions and re-arranging ;

$$[(\nabla_j F)_i - \partial_j F_i]X_i = F_i[-(\nabla_j X)^i + \partial_j X^i]$$

We know that $\nabla_j X^k = \partial_j X^k + X^i \Gamma^k_{ij}$. Substituting above and re-arranging, we obtain

$$(\nabla_j F)_i = \partial_j F_i - F_k \Gamma^k_{ij} = F_{i,j}$$

5.3 Lie Brackets and Lie Derivatives

Above , we discussed how to compare vectors on a section of a tangent bundle using a connection,[27],[28]. This is a coordinate invariant method . Next, we discuss another method of taking derivatives along a section of a vector bundle ; the Lie Derivative , also a coordinate invariant method. The objects in the tangent space T(M) are first order linear operators acting on C$^\infty$ functions of the manifold. Consider such an operator $X_p(\ )$ acting on two functions f and g. At each point $p \in M$, we have, by linearity and the product rule, respectively

$$X_p(f+g) = X_p(f) + X_p(g)$$

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$

$X_p(f)$ is the directional derivative of f in the direction of X at p. Thus, X inputs a function at p and outputs a real number. Remembering that $T_p(M)$ is an n- dimensional vector space with a basis in local coordinates $(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n})$, hence

$$X_p = X^i \frac{\partial}{\partial x_i}|_p; X(f) = X^i \frac{\partial}{\partial x_i}(f)|_p$$

Let us examine XY and YX .

$$XY = X^j \partial_j (Y^i \partial_i) = X^j \partial_j Y^i \partial_i + X^j Y^i \partial^2_{ji}$$

This is a second order derivative and not a tangent vector.

$$YX = Y^j \partial_j X^i \partial_i + Y^j X^i \partial^2_{ji}$$

Subtracting;

$$XY - YX = (X^j \partial_j Y^i - Y^j \partial_j X^i)\partial_i$$

Therefore,

$$[X,Y]^i = X^j \partial_j Y^i - Y^j \partial_j X^i \tag{5.3.1}$$

, is a vector field. The Lie bracket is the Lie derivative of two smooth vector fields

$$L_X Y = [X,Y] \tag{5.3.2}$$

The bracket of two coordinate vector fields from the same coordinate system is 0, because the second partial derivatives commute on C^∞.

$$\partial_i \partial_j f - \partial_j \partial_i f = [\partial_i, \partial_j] f = 0.$$

However, this is not generally true. Properties of Lie brackets are easy to prove , and they are

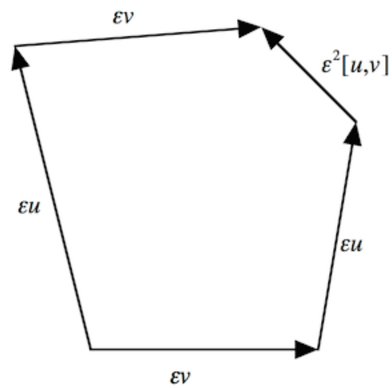

Figure 5.1: Geometric representation of a Lie bracket

(a) Linearity

$$[X+Y,Z] = [X,Z]+[Y,Z]$$

$$[X,Y+Z] = [X,Y]+[X,Z]$$

(b) Skew- symmetry

$$[X,Y] = -[Y,X]$$

(c) $[\text{f}X,gY]$ is not linear , unless f,g are constants

(d) The Jacobi Identity

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] = 0$$

We will examine the Jacobi identity in more detail later.

Geometrical interpretation of the Lie bracket . Take two vector fields, $\mathfrak{u}$ and v, on a manifold M. See Figure 5.1 for an illustration. First , we flow with $\mathfrak{u}$ to $\mathfrak{v}$. We compare this to a flow with v to $\mathfrak{u}$.

We start at point x^i and make infinitesimal shifts and Taylor expand. Moving with $\mathfrak{u}$ to $\mathfrak{v}$:

route 1

$$x^i \xrightarrow{u} x^i + \varepsilon \mathfrak{u}^i(x) \xrightarrow{v} x^i + \varepsilon \mathfrak{u}^{\mathrm{i}}(x) + \varepsilon \mathfrak{v}^i(x + \varepsilon \mathfrak{u}) = x^i + \varepsilon u^i(x) + \varepsilon \mathfrak{v}^i(x) + \varepsilon^2 \mathfrak{v}^j \mathfrak{u}^i_{,j}$$

Moving with $\mathfrak{v}$ to $\mathfrak{u}$: route 2

$$x^i \xrightarrow{v} x^i + \varepsilon \mathfrak{v}^i(x) \xrightarrow{u} x^i + \varepsilon \mathfrak{v}^i(x) + \varepsilon \mathfrak{u}^i(x + \varepsilon \mathfrak{v}) = x^i + \varepsilon v^i(x) + \varepsilon u^i(x) + \varepsilon^2 u^j v\prime_j{}^i$$

Subtracting; Route 1 - Route 2 ;

$$[\mathfrak{u}, \mathfrak{v}] = \varepsilon^2 (\mathfrak{u}^j \mathfrak{v}^i_{,j} - \mathfrak{v}^j \mathfrak{u}^i_{,j})$$

Hence the bracket is a measure of how much such parallelograms fail to close. The parallelogram collapses when [$\mathfrak{u}$,$\mathfrak{v}$] = 0 .

5.4 Lie Algebras and Lie Groups

I will give definitions and then furnish examples.

Lie Algebra . A Lie algebra over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ with a bilinear bracket operation

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

, such that $\forall\, X, Y, Z \in \mathfrak{g}$

$$(1)[X,Y] = -[Y,X], skew - asymmetry$$

$$(2)[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0, Jacobi - identity$$

Lie Group . A Lie group G is a smooth manifold that is also a group with smooth group operations that are smooth maps

$$(1)\mu : G\times G \to G, sending-(a,b)-to-ab$$

$$(2)\eta : G \to G, sending-a-to-a^{-1}$$

Example. $GL(n,\mathbb{R})$. The set $\mathfrak{gl}(n,\mathbb{R})$ of all $n\times n$ real matrices is a real vector space, hence a manifold. The map $\mathfrak{gl}$(n, $\mathbb{R}$) $\to \mathbb{R}^{n^2}$ is a diffeomorphism. $\mathfrak{gl}$(n,$\mathbb{R}$) can be made a Lie algebra by defining $[x,y] = xy-yx$, where $x,y \in\mathfrak{gl}$(n,$\mathbb{R}$) and xy is matrix multiplication.

Lie Derivatives**.** If $f \in$ C$^\infty$ and $X \in$ T(M) ;

$$L_X(f) = Xf$$

For $X,Y \in$ T(M) ;

$$L_XY = [X,Y]$$

Useful properties of Lie derivatives : For $X,Y,Z \in$ T(M)

$$(1)L_XY = -L_YX$$

$$(2)L_X[Y,Z] = [L_XY,Z]+[Y,L_XZ]$$

$$(3)L_{[X,Y]}Z = L_XL_YZ-L_YL_XZ$$

$$(4)L_X(fY) = (Xf)Y+fL_XY$$

To provide a flavor of the algebra of Lie derivatives, I will prove (3).

$L_{[X,Y]}Z = [[X,Y],Z] = [XY-YX,Z] = [XY,Z] - [YX,Z] = L_{XY}Z - L_{YX}Z = L_X L_Y Z - L_Y L_X Z$

5.5 Tetrads and Vierbein Formalism

The Verbein formalism will allow us to arrive at Einstein's field equation via a different route than varying the metric. Instead, we vary the basis at each point in the manifold,[29]. A vector basis is said to be orthonormal at a point in a manifold i

$$e_\mu . e_\nu = \eta_{\mu\nu}$$

, where $\eta_{\mu\nu}$ is the Minkowski metric. The smoothness of the manifold will allow us to do so. There is also a corresponding basis of orthonormal dual one-forms, with the usual duality condition

$$< e^{*^\mu}, e_\nu >= \delta^\mu{}_\nu$$

An orthonormal basis, unless it has a coordinate basis, does not have enough information to provide the line element for the connection. To allow this to happen, we must find a linear transformation from the orthonormal basis to the coordinate basis. In general relativity, a tetrad is a set of basis vectors $\{\gamma_m\}$, m = 0,1,2,3 attached to each point of spacetime x^μ. As above, an orthonormal tetrad , which forms a locally inertial frame at each point of the manifold, whereby the scalar product of the basis vectors form the Minkowski metric

$$\gamma_m \gamma_n = \eta_{mn}$$

Associated with the tetrad frame at each point, we have a local set of coordinates $\{\xi^m\}$, m = 0,1,2,3. Note that the local coordinates ξ^m do not extend beyond the local frame at each point .That is , we have a moving frame from point to point. A coordinate interval is

$$dx = \gamma_m d\xi^m$$

and the distance element is

$$ds^2 = dx.dx = \gamma_{mn} d\xi^m d\xi^n$$

The vierbein, German for four legs, e^{μ}_{m}, is defined to be a 4 by 4 matrix that transforms between the tetrad frame and the coordinate frame. The tetrad index, m, comes first, then the coordinate index μ.

$$\gamma_m = e^{\mu}_{m} g_{\mu}$$

The inverse vierbein matrix is e^{m}_{μ}, hence

$$e^{m}_{\mu} e^{\nu}_{m} = \delta^{\mu}_{\nu}$$

And hence

$$g_{\mu} = e^{m}_{\mu} \gamma_m$$

The metric encodes the vierbein

$$g_{\mu\nu} = \gamma_{mn} e^{m}_{\mu} e^{n}_{\nu}$$

5.6 Geodesics and Parallel Transport

Here, we will analyze curves on manifolds, address the question of covariant derivatives along curves and then define a geodesic and address their existence and uniqueness,[27],[29]. Let γ: I $\rightarrow$ M be a curve, where I $\subset \mathbb{R}$, is an interval. At any time $t \in I$, the velocity $\dot{\gamma}(t)$ is defined as

$$\dot{\gamma}(t)f = \frac{d}{dt}(f \circ \gamma)(t),$$

In coordinate representation

$$\dot{\gamma}(t) = \dot{\gamma}^{i}(t)\partial_i$$

A vector field along a curve γ: $I \to M$ is a smooth map $V\colon I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for every $t \in$ I, for example $\dot{\gamma}(t) \in T_{\gamma(t)}(M)$. Extendibility is an important idea . A vector field V along γ is said to be extendible if there exists a vecor field V' on a neighbourhood of the image of γ that is related V , direction - wise. For example, if $\gamma(t_1) = \gamma(t_2)$, but $\dot{\gamma}(t_1) \neq \dot{\gamma}(t_2)$, then γ is not extendible, as the velocity vectors at the same point , point very different directions, Figure 5. Let $\mathfrak{X}(\gamma)$ be the space of vector fields along γ. Let ∇ be a linear connection on M. Then, for every curve γ: I $\to$ M, ∇ determines a unique operator D_t: $\mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$, a connection, the covariant derivative of X along γ, that is linear over $\mathbb{R}$, satisfies Leibniz's rule and the covariant derivative is extensible. I will state this without proof;

(1) Linearity :

$$D_t(aV + bW) = aD_tV + bD_tW, a, b \in \mathbb{R}$$

(2) Product Rule :

$$D_t(fV) = \dot{f}V + fD_tV, f \in C^{\infty}(I)$$

(3) If V is extensible, then for any extension V' of V ,

$$D_tV(t) = \nabla_{\dot{\gamma}(t)}V'$$

Geodesics Let M be a manifold with a linear connection ∇. Let γ be a curve in M. Then the vector field $D_t\dot{\gamma}$ is the acceleration of γ. A curve γ is a called a geodesic if its accelerattion is zero;

$$D_t \dot{\gamma}(t) = 0 \tag{5.6.1}$$

For coordinates $\{x\}$, on some neighbourhood of a point on the manifold the geodesic equation can be shown to satify

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \tag{5.6.2}$$

(1) This is a second order differential equation . It can be converted to a first order linear differential equation by the method of change of variables. By specifying initial conditions and by the Existence and Uniqueness Theorem of first order linear differential equations, a unique solution does exist.

(2) It follows from the uniqueness theorem, for any point p and vector V, there is a unique maximal geodesic with initial point p and initial velocity V.

A physical approach to the geodesic equation follows next.

The Lagrangian of a free particle in a gravitational field is given by

$$L = \frac{1}{2} M g_{\mu\nu} \dot{x^\mu} \dot{x}^\nu \tag{5.6.3}$$

, where M is the mass. Parameterization of $x^\mu \to x^\mu(s)$, and applying the Euler Lagrange equation, which we will discuss later, under the principle of least action,

$$\frac{d}{ds}(\frac{\partial L}{\partial \dot{x^\mu}}) - \frac{\partial L}{\partial x^\mu} = 0 \tag{5.6.4}$$

, the following observations are noteworthy.

(a) For M = $\mathbb{R}^n$, the Christoffel symbols Γ^k_{ij} = 0 , then , $\ddot{x}^k$ = 0 $\Rightarrow x^k = a^i + b^i t$, a straight line; a^i, b^i are constants

(b) If $\dot{\gamma}$(t) $\neq$ 0 , a geodesic cannot slow down and stop

(c) For $x^\mu = x^0, x^1, x^2, x^3$. Let $\dot{x}^0 \simeq 1$, $\dot{x}^1$ = $\dot{x}^2$ = $\dot{x}^3$ $\ll 1$, $\ddot{x}^0$ = - Γ^1_{00} and by Newton's Law , $\ddot{x}$ = -$\nabla\phi$. Therefore ϕ= potential $\cong$ metric . This observation will become apparent later.

Theorem. Let M be a manifold with a connection ∇. Let p be a point in M and $v \in T_pM$, a vector at p. then there is a unique geodesic $p(t)$ defined on some interval around $t = 0$, so tha t $p(0) = p$, and $\dot{p}(0) = v$. This is is a re-statement of what we outlined above

Fix $p_0 \in M$, a coordinate system x^i around p_0 is said to be a local inertial frame , if all Γ^i_{jk} disappear at p_0.

The covariant derivative ∇ is said to be symmetric if

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

for all coordinate systems x^i. As stated above, based upon the uniqueness theorem, for any $p \in M$ and $V \in \mathrm{T}_p(M)$, there is a unique maximal geodesic γ: I $\rightarrow M$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$, on an open interval I. The maximal geodesic is the geodesic with initial point p and initial velocity V, written as γV. If we define $\pi : TM \rightarrow M$, one can see that $p = \pi(V)$. The next question is ; how do geodesics behave ? Do they vary with the initial point , vector or both ? Can we extract any global information from local information. It turns out that the exponential map from the tangent bundle to the manifold, by sending the vector V along the maximal geodesic γV for a time $t = 1$, gives us the information we need.

Exponential Map. Define the exponential map exp : $\mathfrak{E} \rightarrow M$ by

$$exp(V) = \gamma V(1),$$

for each $p \in M$, where $\mathfrak{E}$ is a subset of TM , $\mathfrak{E}$ = { $V \in TM$, γV is defined on an interval [0,1].

A brief introduction into exp maps (without proofs); the exponential map exp carries lines through the origin of $T_p(M)$ to geodesics of M through p. for each point $p \in M$, $\exists$ a neighbourhood U' of p in T_pM on which the exponential map $\exp_p$ is a diffeomorphism onto a neighbourhood U of p in M.. A subset S of vector space is starshaped about 0 if $v \in$ S implies $tv \in$ S, $\forall\ 0 \leq t \leq 1$. If U and $U^{'}$ are as defined above, if $U^{'}$ is star shaped about 0, then U is a normal neighbourhood of p. A broken geodesic is a piecewise smooth curve segment whose smooth subsegments are geodesics. A semi-Riemannian manifold M is connected if and only if any two points of M can be joined by a broken geodesic.

Properties of Normal Coordinates Let $\{U, (x^i)\}$ be a normal coordinate chart centered at p. Then

(a) For any $V = V^i \partial_i \in T_pM$, the geodesic γV starting at p with initial velocity vector V is represented in normal coordinates by the radial line segment $\gamma V(t) = (tV^1, \ldots, tV^n)$, as long as γV stays within U

(b) The coordinates of p are $(0, \ldots, 0)$

(c) The components of the metric at p are $g_{ij} = \delta_{ij}$

(d) Any Euclidean ball contauned in U is a geodesic ball in M

CHAPTER VI

CURVATURE

6.1 Metric Compatibility

Let g be a Riemannian or Semi-Riemannian metric on a manifold M. A linear connection ∇ is said to be compatible with g if for all vector fields X,Y,Z , the following product rule is satisfied

$$\nabla_X < Y,Z > = < \nabla_X Y,Z > + < Y,\nabla_X,Z >$$

The following conditions are equivalent

(1) ∇ is compatible with g

(2) $\nabla g = 0$

(3) If V,W are parallel vector fields along a curve γ, then < V,W > is constant

(4) If V, W are vector fields along any curve γ

$$\frac{d}{dt} < V,W > = < D_t V,W > + < V,D_t W >$$

(5) Parallel translation $P_{t_0 t_1}$: $T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ is an isometry for each t_0, t_1.

Compatibility of the metric is not enough for uniqueness of the connecttion. We need another property that involves the torsion tensor of the connection.The torsion tensor is a (2,1) tensor τ, defined by

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{6.1.1}$$

A linear connection is said to be symmetric if the torsion tensor vanishes

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{6.1.2}$$

Lie derivatives satisfy the identity

$$L_{[X,Y]} = [L_X, L_Y] = L_X L_Y - L_Y L_X. \tag{6.1.3}$$

Hence , if $[X,Y] = 0$, then L_X and L_Y commute. This result usually fails for the covariant derivative D_X,[27],[30].

6.2 Riemann Curvature

Parallel Transport,[27],[28],[30],. Let M be a manifold with a linear connection ∇. A vector field V along a curve γ is said to be parallel to γ if

$$D_t V = 0 \tag{6.2.1}$$

Armed with all these tools , now we can tackle the concept of curvature.

Motivation for Riemann Curvature Tensor

When we parallel translate a vector Z in a Euclidean plane, Riemann 2- manifold , along two cordinate bases, $X = \partial_1$ and $Y = \partial_2$, we showed geometrically, in the section on Lie derivatives that

$$\nabla_{\partial_1}\nabla_{\partial_2} Z - \nabla_{\partial_2}\nabla_{\partial_1} Z = 0$$

However, in a curved surface , we have non - commutativity of these second order covariant derivatives. Let us examine this more closely, in $\mathbb{R}^n$ with the Euclidean metric

$$\nabla_X \nabla_Y Z = \nabla_X (Y Z^k \partial_k) = X Y Z^k \partial_k$$

$$\nabla_Y \nabla_X Z = YXZ^k \partial_k$$

The difference between these two expressions

$$(XYZ^k - YXZ^k)\partial_k = \nabla_{[X,Y]} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \qquad (6.2.2)$$

This is the flatness criterion. It holds for $X, Y, Z \in \mathbb{R}^n$. If M is a Riemannian manifold, the Riemann Curvature Endomorphism is the map

$$R : \mathfrak{T}(M) \times \mathfrak{T}(M) \times \mathfrak{T}(M) \to \mathfrak{T}(M)$$

defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad (6.2.3)$$

The Riemann Curvature Tensor - basic observations and properties

(1) It is a $\binom{3}{1}$ tensor . It can be shown to be multilinear over $\mathbb{C}^\infty$ and $\mathbb{R}$

(2) It can be wriiten in any local frame with one upper and three lower indices. For example ; R = $\mathrm{R}_{ijk}{}^l dx^i \otimes dx^j \otimes dx^k \partial_l$

(3) Skew-symmetry $R(X,Y)Z = -R(Y,X)Z$

(4) It can also be defined as the covariant 4-tensor, obtained from the $\binom{3}{1}$ tensor by lowering the last index

$$Rm(X,Y,Z,W) =< R(XY)Z, W >$$

In coordinates;

$$Rm = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

Note $R_{ijkl} = g_{lm} R^m_{ijk}$

(5) A Riemann manifold is flat $\Longleftrightarrow$ its curvature tensor is zero.

Symmetries of the Curvature Tensor . I will express these in 3 ways; Rm, $<>$, and in coordinate form. For vector fields W, X, Y, Z

Symmetry 1 - follows from the definition of the curvature endomorphism

$$Rm(W,X,Y,Z) = -Rm(X,W,Y,Z)$$

$$< R(W,X)Y,Z >= - < R(X,W)Y,Z >$$

$$R_{ijkl} = -R_{jikl}$$

Symmetry 2 - follows from the compatibility of the Riemann connection with the metric

$$Rm(W,X,Y,Z) = -Rm(W,X,Z,Y)$$

$$< R(W,X)Y,Z) >= - < R(W,X)Z,Y)$$

$$R_{ijkl} = -R_{ijlk}$$

Symmetry 3 - follows from the symmetries 1, 2 and 3

$$Rm(W,X,Y,Z) = Rm(Y,Z,W,X)$$

$$< R(W,X)Y,Z >=< R(Y,Z)W,X >$$

$$R_{ijkl} = R_{klij}$$

Symmetry 4 - follows from the symmetry of the connection also known as first Bianchi identity

$$R(W,X,Y,Z)+R(X,Y,W,Z)+R(Y,W,X,Z)=0$$

$$R_{ijkl}+R_{jkil}+R_{kijl}=0 \tag{6.2.4}$$

6.3 Fundamentals of Riemannian Geometry

I left this at the end of the chapter and not at the beginning because we needed the mathematics of earlier chapters. Furthermore , it is a good entry point into General Relativity. Let (M,g) be a Riemannian or PseudoRiemannian manifold. Then, there is a unique linear connection on M that is compatible with g and symmetric. It is known as the Levi-Civita Connection,[27],[28],[29]. We will be expressing this in terms of the metric. Let X,Y,Z be vector fields in $\mathfrak{T}(M)$. Applying metric compatibility;

$$X<Y,Z>=<\nabla_X Y,Z>+<Y,\nabla_X Z>$$

$$Y<Z,X>=<\nabla_Y Z,X>+<Z,\nabla_Y X>$$

$$Z<X,Y>=<\nabla_Z X,Y>+<X,\nabla_Z,Y>$$

Noting that $<Y,[X,Z]>$ = $<Y,\nabla_X Z-\nabla_Z X>$ = $<Y,\nabla_X Z>$ - $<Y,\nabla_Z X>$, we obtain

$$<Y,\nabla_X Z>=<Y,\nabla_Z X>+<Y,[X,Z]>$$

Doing the same for the second term of the above equations, we obtain

$$X<Y,Z>=<\nabla_X Y,Z>+<Y,\nabla_Z X>+<Y,[X,Z]>$$

$$Y<Z,X>=<\nabla_Y Z,X>+<Z,\nabla_X Y>+<Z,[Y,X]>$$

$$Z < X,Y >=< \nabla_Z X,Y > + < X,\nabla_Y Z > + < X,[Z,Y] >$$

Adding the first two equations, subtracting the third and solving for

$$< \nabla_X Y,Z >= \frac{1}{2} X < Y,Z > +Y < Z,X > -Z < X,Y > - < Y,[X,Z] > - < Z,[Y,X] > + < X,[Z,Y] >$$

Since the last three terms of right hand side of this equation do not depend on the connection, the Lie brackets vanish. Using a local coordinate chart ,

$$< \nabla_{\partial_i} \partial_j, \partial_l >= \frac{1}{2} \partial_i < \partial_j, \partial_l > +\partial_j < \partial_l, \partial_i > -\partial_l < \partial_i, \partial_j >$$

Noting $< \partial_i,\partial_j > = g_{ij}$ and $\nabla_{\partial_i} \partial_j = \Gamma^m_{ij} \partial_m$. Inserting these above, we obtain

$$\Gamma^m_{ij} g_{ml} = \frac{1}{2} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{6.3.1}$$

Multiplying by g^{lk} and noting $g_{ml} g^{lk} = \delta^k_m$. Therefore,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} + \partial_l g_{ij}) \tag{6.3.2}$$

Remark;

(a) $\Gamma^i_{jk} = \Gamma^i_{kj}$, symmetry of the connection is evident

(b) The Christoffel symbol defines the connection in the chart

(c) Compatibility ∇g= 0 can be established by further computation using covariant derivative equation

The Christoffel symbol is so important, it is worth spending the time deriving it through another shorter method; Noting that $g_{\lambda\mu} = e_\lambda e_\mu$. Differentiating :

$$\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} = e_{\lambda}\frac{\partial e_{\mu}}{\partial x^{\nu}} + \frac{\partial e_{\lambda}}{\partial x^{\nu}}e_{\mu}$$

$$= e_{\lambda}e_{k}\Gamma^{k}_{\mu\nu} + e_{\mu}e_{k}\Gamma^{k}_{\lambda\nu}$$

$$= g_{\lambda k}\Gamma^{k}_{\mu\nu} + g_{\mu k}\Gamma^{k}_{\lambda\nu}$$

$$= \Gamma_{\lambda\mu\nu} + \Gamma_{\mu\lambda\nu}$$

By symmetry,

$$\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g\mu\nu}{\partial x^{\lambda}} = \Gamma_{\lambda\mu\nu} + \Gamma_{\mu\lambda\nu} + \Gamma_{\lambda\mu\nu} + \Gamma_{\nu\lambda\mu} - \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda} = 2\Gamma_{\lambda\mu\nu}$$

terms vanish by applying free torsion condition.. Therefore

$$\Gamma_{\lambda\mu\nu} = \frac{1}{2}(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{k\nu}}{\partial x^{\mu}} - \frac{\partial g\mu\nu}{\partial x^{\lambda}})$$

Application of Riemann Curvature Tensor : Geodesic Deviation Equation,[7],[14],[38].The geodesic equation relates the acceleration of the separation vector χ between two nearby geodesics to the Riemann curvature tensor. First, we define a separation 4-vector $\chi(\tau)$ that connects a point $x^{\alpha}(\tau)$ on one geodesic (fiducial) to point $x^{\alpha}(\tau)$ + $\chi^{\alpha}(\tau)$ on a nearby geodesic at the same proper time. Let u= 4-velocity vector of fiducial geodesic. Let v = separation velocity vector = $\nabla_{u}\chi$ .Then, the separation acceleration is defined as w

$$w = \nabla_{u}\nabla_{u}\chi = \nabla_{u}v$$

Noting that for any function $f \in \mathbb{R}^{\infty}$

$$\frac{df}{d\tau} = \frac{df(x^\alpha(\tau)}{d\tau} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{d\tau} = u^\alpha \frac{\partial f}{\partial x^\alpha}$$

We next calculate the coordinate basis components of v and w using the geodesic equation and applying the expression above;

$$v^\alpha = (\nabla_u \chi)^\alpha = u^\beta \nabla_\beta \chi^\alpha = \frac{d\chi^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta \chi^\gamma \tag{6.3.3}$$

$$w^\alpha = (\nabla_u v)^\alpha = u^\beta \nabla_\beta v^\alpha = \frac{dv^\alpha}{d\tau} + \Gamma^\alpha_{\delta\varepsilon} u^\delta v^\varepsilon \tag{6.3.4}$$

Next, we substitute v^α from (6.3.3) into (6.3.4);

$$w^\alpha = \frac{d^2\chi^\alpha}{d\tau^2} + \frac{d}{d\tau}(\Gamma^\alpha_{\beta\gamma} u^\beta \chi^\chi) + \Gamma^\alpha_{\delta\varepsilon} u^\delta (\frac{d\chi^\varepsilon}{d\tau} + \Gamma^\varepsilon_{\beta\gamma} u^\beta \chi^\gamma)$$

$$= \frac{d^2\chi}{d\tau^2}^\alpha + 2\Gamma^\alpha_{\beta\gamma} u^\beta \frac{d\chi^\gamma}{d\tau} + \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} u^\delta u^\beta \chi^\gamma + \Gamma^\alpha_{\beta\gamma} \frac{du^\beta}{d\tau} \chi^\gamma + \Gamma^\alpha_{\delta\varepsilon} \Gamma^\varepsilon_{\beta\gamma} u^\beta u^\delta \chi^\gamma \tag{6.3.5}$$

with re-labelling of dummy indices. Noting that the geodesic equation applies to $x^\alpha(\tau) + \chi^\alpha(\tau)$, we get

$$\frac{d^2(x^\alpha + \chi^\alpha)}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}(x^\delta + \chi^\delta)\frac{d(x^\beta + \chi\beta}{d\tau}\frac{dx^\gamma + \chi^\gamma}{d\tau} = 0$$

Taylor exapnding, since χ is small, we ignore second order terms, noting $u^\alpha = \frac{dx^\alpha}{d\tau}$ and symmetry of connection $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$, and relabelling indices, we obtain

$$\frac{d^2\chi^\alpha}{d\tau^2} + 2\Gamma^\alpha_{\beta\gamma} u^\beta \frac{dx^\gamma}{d\tau} + \frac{\partial \Gamma^\alpha_{\beta\gamma}}{dx^\delta} u^\beta u^\gamma \chi^\delta = 0$$

Substituting $\frac{d^2\chi^\alpha}{d\tau^2}$ of above expression into (6.3.5) above, we get

$$w^\alpha = -(\frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\gamma\varepsilon}\Gamma^\varepsilon_{\beta\delta} - \Gamma^\alpha_{\delta\varepsilon}\Gamma^\varepsilon_{\beta\gamma}) u^\beta \chi^\gamma u^\delta$$

$$\omega^\alpha = -R^\alpha_{\beta\gamma\delta} u^\beta \chi^\gamma u^\delta \tag{6.3.6}$$

As one can see, the separation acceleration of geodesics is directly proportional to the Riemann curvature tensor. In the weak field limit, the metric is approximatley Minkowski and the velocities of test particles are much less than c, the speed of light.

6.4 Introduction to General Relativity

Ricci Tensor. The 4 indexed Riemann curvature tensor is very complicated with $4^4 = 256$ entries. Instead, we define an index 2 covariant Ricci tensoe , Ric or Rc . The Ricci tensor is defined by contracting the first and last indices of the Riemann curvature tensor,:

$$R_{ij} = g^{km} R_{kijm} = R^k_{kij}$$

Scalar Curvature. The scalar curvature S is the trace of the Ricci tensor

$$S = R^i_i = g^{ij} R_{ij}$$

The Ricci tensor and scalar curvature are linked through the following identity, [7] ,[14] ,[38].

Lemma. The covariant derivatives of the Ricci and scalar curvatures satisfy the following identity

$$divRc = \frac{1}{2} \nabla S \tag{6.4.1}$$

$$R^j_{ij;} = \frac{1}{2} S_{;j}$$

Proof. For this proof, we need the Differential Bianchi Identity of the curvature tensor:

$$R_{abmn;l} + R_{ablm;n} + R_{abnl;m} = 0$$

Contract this equation with the tensor product $g^{bn} g^{am}$;

$$g^{bn} g^{am} (R_{abmn;l} + R_{ablm;n} + R_{abnl;m}) = 0$$

$$g^{bn}(R^m_{bmn;l} - R^m_{bml;n} + R^m_{bnl;m}) = 0$$

noting that first index contracts, the second index slides through

$$g^{bn}(R_{bn;l} - R_{bl;n} - R_b{}^m{}_{nl;m}) = 0$$

$$R^n_{n;l} - R^n_{l;n} - R^{nm}_{nl;m} = 0$$

The first term contracts to a Ricci scalar, the second term contracts to $\binom{1}{1}$ Ricci tensor.Therefore, we obtain

$$R_{;l} - R^n_{l;n} - R^m_{l;m} = 0$$

Noting n and m are dummy indices

$$R_{;l} = 2R^m_{l;m}$$

Therefore,

$$\nabla Rc = \frac{1}{2}\nabla R$$

Definition. A Riemann metric is said to be an Einstein metric if Rc = λg , where λ is a function, g is metric . Taking the trace of each side; tr Rc = S and tr g = $g_{ij}g^{ji}$= δ^i_i = dim M, M Manifold, therefore

$$S = \lambda dimM \Longrightarrow Rc = \frac{S}{dimM}g$$

Let dim M = n and taking the covariant derivative of this last expression, we obtain

$$R_{ij;k} = \frac{1}{n} S_{;k} g_{ij}$$

Taking the trace of this equation, we obtain

$$R_{ij;j} = \frac{1}{n} S_{;i}$$

Comparing with

$$R^{j}_{ij} = \frac{1}{2} S_{;i}$$

We obtain

$$\frac{1}{n} S_{;i} = \frac{1}{2} S_{;i}$$

We can conclude that for $n > 2$, $S_{;i} = 0 \Rightarrow$ S, Ricci Scalar is a constant, noting M is a connected space. Therefore S, the Ricci scalar is a constant for $n \geq 3$. The idea is to find a Riemannian metric on a given manifold that provides constant curvature. This is done by studying the critical points of the total scalar curvature functional. Given a metric $g \in$ M , the total scalar curvature functional S : M $\rightarrow \mathbb{R}$ is defined as

$$S(g) = \int_{M} S dV$$

dV = volume form of metric. The Einstein metric can be thought of as the higher dimensional analogue of the Gaussian curvature of a 2-manifold. The Gaussian curvature is intrinsic to the 2-manifold. The central them in General Relativity is that space time is modelled by a 4-manifold that satisfies the field equation

$$Rc - \frac{1}{2} R g = T \tag{6.4.2}$$

where g is the Lorentz metric and T is the energy-momentum tensor. The Hilbert action is the variational equation of a functional. The functional is defined on the space of all metrics in a 4-manifold. The theory of General Relativity states that the scalar curvature S is a crtitical point of this function. Let us examine the field equation .

Case 1: $T = 0$, Then Rc = $\frac{1}{2}$Rg. Take the trace ;

$$S = 2S \Longrightarrow S = 0 \Longrightarrow Rc = 0$$

Therefore, g is an Einstein metric

Case 2. Einstein considered adding λg to the LHS of the field equation.

$$\lambda g + Rc - \frac{1}{2}Rg = T$$

If $T = 0$, and we take the trace of both sides of the field equation, we obtain

$$4\lambda - S = 0 \Rightarrow S = 4\lambda \tag{6.4.3}$$

This means that empty space has curvature. Einstein dropped this factor. In its entire form, the field equation can be expressed as

$$Rc - \frac{1}{2}Rg + \lambda g = \kappa T \tag{6.4.4}$$

where κ is a constant. In (6.4.4) , if $T \neq 0$, we have

$$4\lambda - S = \kappa T$$

Multiply by $\frac{1}{2}g$

$$2\lambda g - \frac{1}{2}gR = \frac{1}{2}g\kappa T \tag{6.4.5}$$

Subtracting (6.4.4) - (6.4.5)

$$Rc = \kappa(T - \frac{1}{2}gT) + \lambda g \tag{6.4.6}$$

We will need this alternate form of the field equation when we deduce the Newtonian weak field limit below. Special relativity can be thought of as the flat, simply connected, Lorentz manifold of dimension 4, the Minkowski metric of index 1. Hence, we now explore the Newtonian limit of the field equation.

The Newtonian Limit. Recall the geodesic equation

$$w^{\alpha} = -R^{\alpha}_{\mu\nu\sigma} u^{\sigma} u^{\mu} \chi^{\nu}$$

The equivalent Newtonian tidal deviation equation is

$$w^{\alpha} = -\eta^{ij} (\partial_k \partial_j \Phi) \chi^k$$

, where η^{ij} is the Minkowski metric and Φ is the gravitational potential.

Taking the Newtonian limit, we make 3 assumptions:

(1) spacetime is almost flat - weak gravitational field- flat space metric η_{ij}

(2) objects move with speeds << c = 1. This means the 4-velocity can be expressed as $(1,0,0,0)$

(3) mass energy density ρ = T^u >> $T^{\mu\nu}$ components of the field source (more on this later)

Therefore, in the Newtonian limit ; $u^{\mu} = u^{\nu} = 1$, when $\nu = t$

$$w^{\alpha} \simeq -R^{i}_{i\nu i} \chi^{\nu} = -R^{i}_{tkt} \chi^k$$

Comparing this to the Newtonian tidal deviation equation, we note that

$$R^{i}_{tkt} = \eta^{ij} (\partial_j \partial_k \Phi) \simeq R^{k}_{tkt} = R_{tt} = R^{tt}$$

Using the alternate form of the field equation outlined above, we obtain

$$\nabla^2\phi \simeq R^{tt} = \kappa(T - \frac{1}{2}gT) + \lambda g$$

If ρ = T^{u} >> other $T^{\mu\nu}$, then $T = g_{\mu\nu}T^{\mu\nu} \simeq \eta_{\mu\nu}\mathrm{T}^{\mu\nu} \simeq \eta_{tt}\rho$ = -ρ, where ρ is the mass density. Therefore, $\nabla^2\phi \simeq \mathrm{R}^{tt} \simeq \kappa[\rho$ - $\frac{1}{2}$(-1)(-ρ)] - λ = $\frac{1}{2}\kappa\rho$ - λ

Remarks.

(1) set $\kappa = 8\pi G$ and assume $\lambda << 4\pi\rho G$, we obtain $\nabla^2\phi = 4\pi G\rho$ (2) If ρ = 0, in the Newtonian limit, $\nabla^2\phi$ = -λ, if $\lambda > 0$, this creates a repulsive force. Einstein reasoned that this may cause the universe to expand, and hence dropped the term. In 1929, Edwin Hubble, showed by observing the red shift of receding galaxies, that the universe was indeed expanding. After which, Einstein remarked that this was his biggest blunder

(2) Energy Momentum Tensor. Now, we briefly introduce the energy momentum tensor . A whole sub-section will be dedicated to this. Matter is a carrier of energy-momentum, mathematically it is expressed as an energy momentum tensor, T. Einstein had initially suggested that $G = kT$, where G is a function of the Ricci tensor, and k is a constant. We showed earlier, that Div Rc = $\frac{1}{2}\nabla$R, and so Div T = 0 $\Rightarrow$ div G = 0. So subtracting $\frac{1}{2}$R from Rc makes sense. Indeed, the Einstein gravitational tensor of spacetime, G, is

$$G = Rc - \frac{1}{2}Rg \tag{6.4.7}$$

If M is a spacetime containing matter with stress - energy tensor T, then

$$G = Rc - \frac{1}{2}Rg = 8\pi T$$

The Einstein equation implies that the stress-energy tensor is a symmetric $(0,2)$ tensor with zero divergence,[35].

$$divG = div(Rc - \frac{1}{2}Rg) = \frac{1}{2}\nabla R - \frac{1}{2}\nabla R = 0$$

$G = 8\pi T$ tells us how matter determines Ricci curvature , $divT$ = 0 tells us that Ricci curvature moves this matter. When T= 0, M is Ricci flat, then M is empty, a vacuum.

A perfect fluid on a spacetime M is a triple (U,ρ, p) where

(1) $U$is a timelike future pointing unit vector field on M called the flow vector field

(2) ρ is the energy density function in $T^*(M)$ and p is the pressure function in $T^*(M)$

(3) The stress-energy tensor is

$$T = (\rho + p)U^{\bullet} \otimes U^* + pg \tag{6.4.8}$$

where U^* is the one-form metrically equivalent to U. If $X,Y \perp U$, then $T(U,U) = \rho$, $T(X,U) = T(U,X) = 0, T(X,Y) = p < X,Y >$. Noting that $divT$ = 0, in coordinates;

$$T^{ij} = (\rho + p)U^i U^j + pg^{ij}$$

The divergence is

$$\sum_J T^{ij}_{;j} = \sum_J (\rho + p)_{;j} U^i U^j + (\rho + p) U^i_{;j} U^j + (\rho + p) U^i U^j_{;j} + p_{;j} g^{ij}$$

Therefore,

$$divT = U grad(\rho + p)U + (\rho + p)D_U U + (\rho + p)(divU)U + grad p$$

Proposition. The Energy Equation. $< DivT, U >$ = 0 and noting U is a unit vecor and $D_U U \perp U$

$$0 = U grad(\rho + p) + (\rho + p) divU + U grad p$$

$$U grad \rho = -(\rho + p) divU$$

This gives us the rate of change of energy density as measured by U.

Proposition. The Force Equation Substituting $U grad\rho = -(\rho + p)divU$ into $divT$, we get

$$(\rho + p)D_U U = -grad p.$$

This is equivalent to Newton's law $F = ma$, where force is equivalent to the pressure gradient and mass is replaced by $\rho + p$, while $D_U U \perp U$ is the spatial acceleration of the molecules of flow . We will later need the Maxwell Energy Momentum Tensor. in order to derive the Reissner-Nordsrtom metric of a charged non- rotating blackhole;

$$T_{ab} = \frac{1}{4\pi}(-g^{cd}F_{ac}F_{bd} + \frac{1}{4}g_{ab}F_{cd}F^{cd}),$$

where F^{ab} is the electromagnetic energy momentum tensor.

CHAPTER VII

VARIATIONAL CALCULUS AND HILBERT ACTION

7.1 Euler - Lagrange Equations - Variational Calculus and Functionals

For the functional S[y} = $\int_a^b dxF(x,y,y')$, $y(a) = A, y(b) = B$, where $F(x,u,v)$ is a real function of three real variables, a necessary and sufficient condition for the twice differentiable function $y(x)$ to be a stationary path is that it satisfies the equation

$$\frac{d}{dx}(\frac{\partial F}{\partial y'}) - \frac{\partial F}{\partial y} = 0 \tag{7.1.1}$$

, and the boundary conditions $y(a) = A, y(b) = B$. This is known as the Euler Lagrange Equation. On a stationary path, the functional may achieve a maximum or minimum value, the path being named an extremal. The nature of the stationary path is determined by the term O(ε^2) in the expansion $S[y+\ \varepsilon h]$. The limit $\triangle S[y,h]$ = $\frac{d}{d\varepsilon}S$[y + εh]$|_{\varepsilon=0}$ is linear in h and is known as the Gateaux differential.The Gateaux differential identifies stationary points of the functional,[12],[38].

Lemma. Fundamental Lemma of Calculus of Variation. If $z(x)$ is a continuous function of x for $a \leq x \leq b$, and if

$$\int_a^b dxz(x)h(x) = 0$$

for all functions $h(x)$ that are continuous for $a \leq x \leq b$ and are zero at $x = a$ and $x = b$, then $z(x) = 0$ for $a \leq x \leq b$. By applying the chain rule, integration by parts and the fundamental lemma of the calculus of variation, we can easily derive the Euler Lagrange equation. The Euler Lagrange equation identifies stationary extremal paths. The same idea applies to Lagrangian Mechanics: Define an action $S[x(t)]$ = $\int_{t1}^{t2} dtL(x,\dot{x})$, where $L(x,\dot{x}) = T(\dot{x}) - V(x)$, where $T(\dot{x})$ is the kinetic energy, $V(x)$ is the potential energy. The extremal path δS is set to 0 and defined as

$$\delta S = \delta \int_{t1}^{t2} dt L(x, \dot{x}) = 0$$

Hence, we arrive at the Principle of Least Action.

7.2 Hibert Action

The variation with respect to the metric or inverse metric is set to zero, that is

$$\delta S = 0$$

Define F : space of all metrics $\rightarrow \mathbb{R}$, then

$$F(g) = \int_M R\sqrt{(detg)}d^n x \tag{7.2.1}$$

, where g= metric, R = Ricci Scalar, $\sqrt{(detg)}d^n x$ is the volume form. We want F(g + δg) = 0 for all variations δg, i.e. we want $\delta F(g)$ = 0. In space-time, we have a 4-manifold, with a metric signature

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, (+, -, -, -)$$

or ($-, +, +, +$) , dependent on convention, and the volume element, $d^4 V$ is

$$d^4 V = \sqrt{(-g)} d^4 x$$

where $g = detg_{\mu\nu}$. We showed in subsection 3.3 that the variation of the metric, $\delta\sqrt{(-g)}$, is $\delta\sqrt{(-g)}$ = -$\frac{1}{2}\sqrt{(-g)}g_{\mu\nu}\delta g^{\mu\nu}$. As a quick reminder, the Ricci tensor is a contracted Riemann curvature tensor

$$R_{ij} = R^k_{ikj}$$

and the Ricci curvature scalar S,

$$R = g^{kl} R_{kl} \tag{7.2.2}$$

Also, a manifold for which] Rc = kg, k a constant is an Einstein metric. We need Stokes theorem, which we proved in subsection 2.3, and states; for a differential form ω, on a manifold M with a boundary ∂M

$$\int_M d\omega = \int_{\partial M} \omega$$

This is useful when we compute the second term of the Hilbert action. The metric compatibility is also useful. The covariant derivative of a metric is zero. By combining

$$\nabla_i g_{kl} = \partial_j g_{kl} - \Gamma^m_{jk} g_{ml} - \Gamma^m_{jl} g_{km}$$

and

$$\Gamma^m_{ij} = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji})$$

and noting $g^{mn} g_{ml}$= δ^n_l, we get

$$\nabla_j g_{kl} = 0$$

Now we can construct the Hilbert action. Define the action as S,

$$S = \int R \sqrt{(-g)} d^4 x \tag{7.2.3}$$

R, the Ricci scalar curvature, is a good intuitive guess for our functional because it is integrable and it is gauge invariant(see section). Note catena of events on our 4-manifold

$$metric, g \overset{derivative}{\longrightarrow} \Gamma^i_{jk} \overset{derivative}{\longrightarrow} R_{jkl}{}^i \overset{contraction}{\longrightarrow} S$$

where Γ^i_{jk} is the Levi-Civita Connection, $R_{jkl}{}^i$ is the Riemann Curvature Tensor and S is the Scalar Curvature. Applying variation of our action;

$$\delta S = \delta \int d^4x \sqrt{(-g)}R = 0$$

$$\delta S = \delta \int d^4x (\sqrt{(-g)}\delta R + \delta\sqrt{(-g)}R) = 0$$

Substituting for R and $\delta(\sqrt{(-g)}$;

$$\delta S = \int d^4x \sqrt{(-g)}\delta(g^{km}R_{km}) - \frac{1}{2}\sqrt{(-g)}g_{km}\delta g^{km}R = 0$$

$$\delta S = \int d^4x \sqrt{(-g)}R_{km}\delta g^{km} + g^{km}\sqrt{(-g)}\delta R_{km} - \frac{1}{2}g_{km}(\sqrt{-g})R\delta g^{km} = 0$$

Next, we must show that the second term $\int$ $d^4x\sqrt{(-g)}g^{km}\delta R_{km} = 0$. This is so because since R = $g^{km}R_{km}$,

$$\delta R = \delta g^{km}R_{km} + g^{km}\delta R_{km} = \delta g^{km}R_{km} + g^{km}\nabla_l \delta\Gamma^l_{mk} - \nabla_m \delta\Gamma^l_{lk}$$

The first term disappears as it is the derivative of a metric. The second term is the covariant derivative of a vector field. When multiplied by $\sqrt{(-g)}$ it becomes a total derivative, which makes it equivalent to the divergence of a vector field. Hence, when integrated, as above, we can apply Stokes Theorem. By Stokes theorem, when this term is integrated over a volume element d^4x, the entire term only contributes at the boundary of the manifold. For spacetime, the universe, the boundary is infintely far away. That is ,

$$\int_M d\omega = \int_{\partial M} \omega = 0$$

Therefore,

$$\delta S = \int d^4x (R_{km}\delta g^{km} - \frac{1}{2} g_{km} R \delta g^{km}) = 0$$

$$\delta S = \int d^4x \delta g^{km} (R_{km} - \frac{1}{2} g_{km} R) = 0$$

By the fundamental lemma of the calculus of variation

$$R_{km} - \frac{1}{2} g_{km} R = 0$$

CHAPTER VIII

TRADITIONAL APPLICATIONS OF GENERAL RELATIVITY

8.1 Newtonian Limit

Without gravitation, spacetime posseses the Minkowski metric $\eta_{\mu\nu}$,[37]. Weak gravitational fields cause small curvature in spacetime. The metric takes the form of

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, |h_{\mu\nu}| << 1$$

The metric is also stationary, that is, $\partial_0 g_{\mu\nu} = 0$. The geodesic expression for a free falling particle is

$$\frac{d^2x}{d\tau^2} + \Gamma^{\mu}_{\nu\sigma}\frac{dx^{\nu}}{d\tau}\frac{dx^{\sigma}}{d\tau} = 0 \tag{8.1.1}$$

We set $\frac{dx^i}{dt} << c$, for $(i = 1,2,3)$, $\frac{dx^i}{d\tau} << \frac{dx^0}{d\tau}$ and $x^0 = ct$. Computing the Levi-Civita connection

$$\begin{aligned}\Gamma^{\mu}_{00} &= \frac{1}{2}g^{\kappa\mu}(\partial_0 g_{0\kappa} + \partial_0 g_{0\kappa} - \partial_\kappa g_{00}) \\ &= -\frac{1}{2}g^{\kappa\mu}\partial_\kappa g_{00} = -\frac{1}{2}\eta^{\kappa\mu}\partial_\kappa h_{00}\end{aligned} \tag{8.1.2}$$

Noting that $\frac{dt}{d\tau} = 1$, substituting into equation

$$\frac{d^2\overline{x}}{dt^2} = -\frac{1}{2}c^2\nabla h_{00}$$

Compare with Newton's equation $\frac{d^2\vec{x}}{dt^2}$ = - $\frac{\partial\phi}{\partial x}$, we get

$$h_{00} = \frac{2\phi}{c^2}$$

Therefore,

$$g_{00} = 1 + h_{00} = (1 + \frac{2\phi}{c^2}) \tag{8.1.3}$$

This equation tells us that spacetime curvature causes the time coordinate t to be different from the proper time. Consider a clock at rest, so that $\frac{dx^i}{dt} = 0$. The proper time $d\tau$ between two ticks of the clock is given by

$$c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} dx^0 dx^0 = g_{00} c^2 dt^2$$

Therefore,

$$d\tau = (1 + 2\frac{\phi}{c^2})^{\frac{1}{2}} dt$$

This is the interval in proper time $d\tau$ that corresponds to an interval dt for a stationary observer in the vicinity of a massive object with gravitational potential ϕ. Since ϕ is negative, the proper time interval is shorter than the corresponding interval for a stationary observer at a large distance to the object, where $\phi \longrightarrow 0$ and and where $dt = d\tau$. The space time interval is given by

$$ds^2 = -(1 + \frac{2\phi}{c^2})(cdt)^2 + dx^2 + dy^2 + dz^2$$

This equation tells us that Newtonian gravitation corresponds solely to a curvature of time.

8.2 Gravitational Red-shift

If a stationary light emitter emits photons at a frequency ν_1, what is the observed frequency? Since this is a light signal $ds^2 = c^2 d\tau^2 = 0 = dx^{i2}$, $i = 0,1,2,3$. That is, the world line corresponding to the propagation of light would be a null geodesic,[28]. If we have a stationary light emitter at x_1 and a stationary observer at x_2, then since the coordinate time for two emission cycles is the same

$$c^2 d\tau_1^2 = g_{00}(x_1) dt^2 = c^2 d\tau_2^2 = g_{00}(x_2) dt^2$$

Therefore, setting $c = 1$

$$dt = \frac{d\tau_1}{\sqrt{g_{00}(x_1)}} = \frac{d\tau_2}{\sqrt{g_{00}(x_2)}}$$

Hence, the ratio between the observed and emitted frequencies is

$$\frac{\nu_2}{\nu_1} = \frac{d\tau_1}{d\tau_2} = \sqrt{\frac{g_{00}(x_1)}{g_{00}(x_2)}}$$

In a weak field, substituting $g_{00} = 1 + 2\frac{\phi}{c^2}$

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{1 + \frac{2\phi_1}{c^2}}{1 + \frac{2\phi_2}{c^2}}}$$

Therefore,

$$\frac{\nu_2}{\nu_1} \simeq 1 + \frac{\phi_1}{c^2} - \frac{\phi_2}{c^2}$$

$$z_{grav} = \frac{\Delta\nu}{\nu} = \frac{\nu_1 - \nu_2}{\nu_1} = 1 - \frac{\nu_2}{\nu_1} = \frac{\phi_2 - \phi_1}{c^2}$$

8.3 Advance of Perihelion of Mercury

Astronomers needed an additional mass nearer to the sun than mercury to explain the strange advance of the perihelion of mercury. No such mass was found,. Newton's theory of gravitation was found to be imperfect. This is the one body problem of general relativity. We assume that a centrally massive body produces a spherically symmetric gravitational field. The general relativity solution is the Schwarzschild solution which provides a solution. Hence, we begin with the Schwarzschild spherically symmetric vacuum solution(the Schwarzschild solution will be reviewed in the next section). The test particle moves in a time-like geodesic.

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{1 - \frac{2M}{r}}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{8.3.1}$$

Since the test mass, Mercury, moves along a geodesic, the Lagrangian, L, is the kinetic energy $g_{\alpha\beta}\,\dot{x}^{\alpha}\dot{x}^{\beta} = 1$. Hence, setting $\dot{x} = \frac{dx}{d\tau}$, where τ is the proper time;

$$L = \frac{m}{2}[(1-\frac{2m}{r})\dot{t}^2 - \frac{1}{1-\frac{2m}{r}}\dot{r}^2 - r^2\dot{\theta}^2 - r^2 sin^2\theta\dot{\phi}^2]$$

where $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$. Applying the Euler Lagrange Equation

$$\frac{\partial L}{\partial x^{\alpha}} - \frac{d}{d\tau}(\frac{\partial L}{\partial \dot{x}^{\alpha}}) = 0$$

When a= 0 , we get

$$\frac{d}{d\tau}[(1-\frac{2m}{r})\dot{t}] = 0 \tag{8.3.2}$$

When a = 2,

$$\frac{d}{d\tau}(r^2\dot{\phi}) - r^2 sin\theta cos\theta\dot{\phi}^2 = 0 \tag{8.3.3}$$

When a= 3,

$$\frac{d}{d\tau}(r^2 sin^2\theta\dot{\phi}) = 0$$

The case a= 1, that is r, is messy to differentiate, and will be left out. At $\tau = 0$, assume $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$, then (28.3) gives

$$\frac{d}{d\tau}(r^2\dot{\phi}) = 0$$

$$\Longrightarrow r^2\dot{\phi} = h = constant$$

Integrating (8.3.2);

$$(1-\frac{2m}{r})\dot{t} = k, constant \tag{8.3.4}$$

Substituting (8.3.2) and $\theta = \frac{\pi}{2}$ into (8.3.3) we get

$$\frac{k^2}{1-\frac{2m}{r}} - \frac{\dot{r}^2}{1-\frac{2m}{r}} - r^2\dot{\phi}^2 = 1 \tag{8.3.5}$$

Let $u = \frac{1}{r}$, then $\dot{r} = \frac{dr}{d\tau} = \frac{d}{d\tau}(\frac{1}{u}) = -\frac{1}{u^2}(\frac{du}{d\phi})(\frac{d\phi}{d\tau}) = -\frac{1}{u^2}(\frac{du}{d\phi})hu^2 = -h\frac{du}{d\phi}$. Substituting $\dot{r} = -h\frac{du}{d\phi}$, $r = 1/u$ and $r^2\dot{\phi} = h$ into (28.5)

$$\frac{k^2}{1-2mu} - \frac{h^2(\frac{du}{d\phi})^2}{1-2mu} - h^2u^2 = 1$$

Multiply through by $\frac{1-2mu}{h^2}$;

$$\frac{k^2}{h^2} - (\frac{du}{d\phi})^2 - u^2(1-2mu) = \frac{1-2mu}{h^2}$$

Re-arranging;

$$(\frac{du}{d\phi})^2 + u^2 = \frac{k^2-1}{h^2} + \frac{2m}{h^2}u + 2mu^3$$

Differentiating with respect to u;

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2 \tag{8.3.6}$$

Re-writing in simpler form and let $\varepsilon = \frac{2m^2}{h^2}$;

$$u'' + u' = \frac{m}{h^2} + \varepsilon(\frac{h^2u^3}{m}) \tag{8.3.7}$$

Apply a perturbation method, where $u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$ and substitute into Eq.

$$u_0'' + u_0 - \frac{m}{h^2} + \varepsilon(u_1'' + u_1 - \frac{h^2u_0^2}{m}) + O(\varepsilon^2) = 0$$

Equating ε^0 coefficients

$$u_0 = \frac{m}{h^2}(1 + ecos\phi)$$

Equating coefficients of $\varepsilon$and substituting for u_0

$$u_1{}'' + u_1 = \frac{h^2 u_0^2}{m} = \frac{m}{h^2}(1 + ecos\phi)^2 = \frac{m}{h^2}(1 + 2ecos\phi + \varepsilon^2 cos^2\phi)$$

$$u_1'' + u_1 = \frac{m}{h^2}(1 + \frac{1}{2}e^2) + \frac{2me}{h^2}cos\phi + \frac{me^2}{2h^2}cos2\phi \tag{8.3.8}$$

Let us try a general solution $u_1 = A + B\phi sin\phi + Ccos2\phi$

$$u_1' = Bsin\phi + B\phi cos\phi - 2Csin2\phi$$

$$u_1'' = 2Bcos\phi - B\phi sin\phi - 4Ccos2\phi$$

Therefore,

$$u_1{}'' + u_1 = A + 2Bcos\phi - 3Ccos2\phi$$

Comparing with (8.3.8), we get $A = \frac{m}{h^2}(1 + \frac{e^2}{2})$, B = $\frac{me}{h^2}$ and $C = \frac{-me^2}{6h^2}$. Hence,

$$u_1 = \frac{m}{h^2}(1 + \frac{e^2}{2}) + \frac{me}{h^2}\phi sin\phi - \frac{me^2}{6h^2}cos2\phi$$

Finally,

$$u \simeq u_0 + \varepsilon u_1 = \frac{m}{h^2}(1 + ecos\phi) + \varepsilon\frac{m}{h^2}(1 + \frac{e^2}{2}) + \varepsilon\frac{me}{h^2}\phi sin\phi - \frac{\varepsilon me^2}{6h^2}cos2\phi$$

$$u \simeq \frac{m}{h^2}(1 + ecos\phi + \varepsilon\phi sin\phi)$$

$$u \simeq \frac{m}{h^2}[1+cos[\phi(1-\varepsilon)] \tag{8.3.9}$$

One can see from (8.3.9) that Mercury's orbit is no longer an ellipse, but is still periodic with period 1 - ε = 1 - $\frac{2m^2}{h^2}$. Its period is T;

$$T = \frac{2\pi}{1-\varepsilon} \simeq 2\pi(1+\varepsilon)$$

Therefore, precession is

$$precession \simeq 2\pi\varepsilon = 6\frac{\pi m^2}{h^2}$$

That is, a planet will travel in an ellipse, but the axis of the ellipse will move by an amount equal to $2\pi\varepsilon$ between two points of closest approach. This is the time honored and famous precession of the perihelion,[38].

8.4 The Bending of Light

As above, we begin with the Schwarzschild solution, express the Lagrangian, compute the Euler Lagrange equation and solve the differential equation by perturbation methods,[28],[38]. However, we have one crucial difference, we are dealing with a null geodesic and ds = 0. We obtain a modified equation to that of (8.3.6)

$$\frac{d^2u}{d\phi^2} + u = 3mu^2$$

The homogenous solution is

$$\frac{d^2u}{d\phi^2} + u = 0$$

which has a solution u_0 = $\frac{1}{D}sin(\phi$ - $\phi_0)$, where D is a constant. Then by a perturbation method, set $u = u_0 + mu_1$, by setting ϕ_0 = 0, and neglecting terms of order $(mu)^2$, we obtain

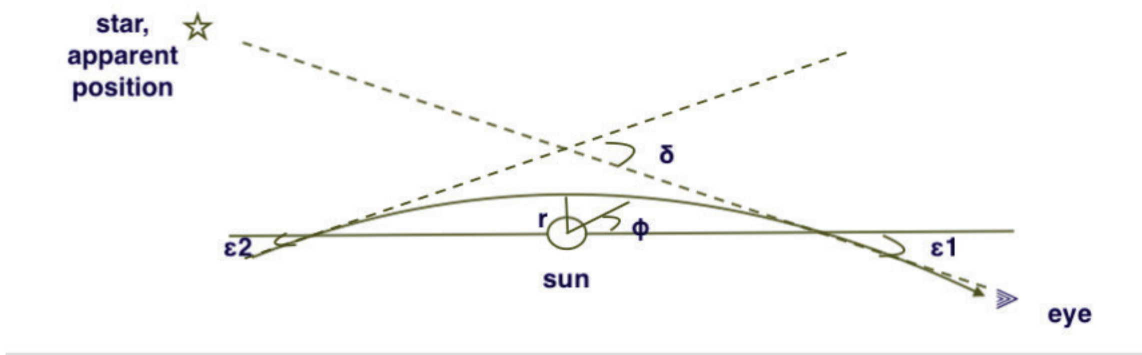

Figure 8.1: Bending of light

$$u_1'' + u_1 = u_0^2 = \frac{sin^2\phi}{D^2}$$

This has a solution $u_1 = (1 + Ccos\phi + cos^2\phi)/3D^2$, where C is a constant of integration. The general solution is

$$u \simeq \frac{sin\phi}{D} + \frac{m(1 + cos\phi + cos^2\phi)}{D^2}$$

Since $u= \frac{1}{r}$, as $r \longrightarrow \infty$, $u \longrightarrow 0$ and the right hand side vanishes. Let the angle of the asymptotes be -ε_1 and π + ε_2, see Figure 8.1 for an illustration.

By applying small angle formula, we get

$$-\frac{\varepsilon_1}{D} + \frac{m}{D^2}(2 + C) = 0$$

and

$$-\frac{\varepsilon_2}{D} + \frac{m}{D^2}(2 - C) = 0$$

The angle of deflection of a light ray

$$\delta = \varepsilon_1 + \varepsilon_2 = \frac{4m}{D} = \frac{4GM}{c^2 D}$$

The deflection of light which grazes the sun has been shown experimentally to be 1.75 seconds of arc. This agrees with the theoretical calculation.This was confirmed by Sir Arthur Eddington in 1919 during a total eclipse, by measuring the apparent position of stars.. If one considers a family of light rays arriving in parallel from a distant source, then the presence of a massive object will cause the light rays to bend and produce a caustic line on the axis. This is the phenomenon of gravitational lensing.

CHAPTER IX

PRINCIPLES OF COSMOLOGY

9.1 Derivation of Basic Friedmann Equations for a Matter and Radiation Dominated Universe

We assume that the universe is expanding adiabatically and that the cosmos is isotropic and homogeneous. By isotropy we mean the universe has no preferrred direction, that is , axially symmetric. By homogeneity, the temperature and density is the same everywhere. In practical terms, homogeneity means a relatively uniform cosmic microwave background. By adiabatic expansion, we mean, if in relative terms, the universe is expanding slowly, say doubling its size every 10 billion years, then the number of nodes of any wave on the boundary of the expansion does not change. Or, the relative distances between galaxies does not change. Under these conditions, we imagine a grid covering the universe. We introduce $a(t)$, the scaling factor, of the grid. We take a galaxy of unit mass $m = 1$, situated a unit distance from the center of an imaginary sphere. Then, by Newton's theorem, the force experienced by this unit mass galaxy is only accounted for by the total mass of galaxies within the sphere,[14],[38]. We have, the distance, D, of the unit galaxy, from the center

$$D = a(t)x$$

Here, $x = 1$. Then

$$D = a(t)$$

Since the kinetic energy, with $m = 1$, is

$$\frac{1}{2}a\dot{(t)}^2$$

and the potential energy, with $m = 1$, is

$$-\frac{GM}{a(t)}$$

and the volume of a sphere is, with mass density ρ, is

$$\frac{4}{3}\pi a(t)^3$$

In terms of energy, kinetic and potential, we define a factor - k, such that

$$\frac{1}{2}a\dot{(t)}^2 + \frac{4}{3}\pi\frac{a(t)^3\rho G}{a} = -k$$

We will drop the t for convenience. Re-arranging and dividing by a^2

$$(\frac{\dot{a}}{a})^2\rho - \frac{k}{a^2} = \frac{8\pi}{3}G\rho \tag{9.1.1}$$

Compare this with the Einstein field equation for $x^0 = t$;

$$R^{00} - \frac{1}{2}g^{00}R = \frac{8\pi}{3}T^{00}$$

The left hand side in both cases refers to the geometry of spacetime and the right hand side to matter and energy density respectively. Noting that ρ is the unit mass density

$$\rho = \frac{M}{a^3}$$

We substitute into (9.1.1) and get the matter dominated form of Friedmann's equations

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\frac{M}{a^3} - \frac{k}{a^2}$$

and assuming ρ is very small

$$(\frac{\dot{a}}{a})^2 \cong \frac{1}{a^2}$$

Substituting $a = bt^p$, where b and p are constants, and assuming ρ is very small

$$(\frac{\dot{a}}{a})^2 = \frac{p^2}{t^2} = \frac{1}{b^3 t^{3p}} \Longrightarrow p = \frac{2}{3}$$

Therefore,

$$a = bt^{\frac{2}{3}}$$

Note that Hubble's constant, H(t), is

$$\frac{\dot{a}}{a} = H(t)$$

As the universe expands in a matter dominant universe, the expansion is proportional to $t^{\frac{2}{3}}$.

In an energy dominated universe; again, think of a unit volume,V , where

$$V = a^3$$

By Planck's law , the energy, E, of a photon is

$$E = \hbar\nu = \frac{\hbar c}{\lambda}$$

where $\hbar$ is Planck's constant, $\nu$is the frequency and λ is the wavelength. If the photon occupies the unit volume, then λ = a. Setting c = 1 and $\hbar = 1$ we get

$$E = \nu = \frac{1}{a}$$

We know that ρ = $\frac{M}{a^3}$. Substituting into (9.1.1), we obtain

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\frac{1}{a^4} - \frac{k}{a^2}$$

k is so small, $\frac{k}{a^2}$ can be ignored; so

$$(\frac{\dot{a}}{a})^2 \simeq \frac{1}{a^4}$$

Hence,

$$\frac{\dot{a}}{a} \simeq \frac{1}{a^2} \Longrightarrow \dot{a} = \frac{1}{a}$$

Integrating, we get for an radiation dominated universe

$$a^2 \propto t \Longrightarrow a \propto t^{\frac{1}{2}}$$

Combining the equations for both a matter and radiation dominated universe, we get

$$(\frac{\dot{a}}{a})^2 = \frac{c_1}{a^3} + \frac{c_2}{a^4}$$

where c_1 and c_2 are constants.

Remark. For small a, $\frac{1}{a^4}$ dominates $\frac{1}{a^3}$, and in the early universe , radiation dominates. For large a, beyond 2 billion years, $\frac{1}{a^3}$ dominates, and matter domnates. By 10,000 years after the big bang, the universe transitioned from a radiation to a matter dominated universe. The photons had smaller wavelengths, higher energies and hence the temperature of the universe was higher. The temperature was high enough that the hydrogen and helium atoms were ionized, and the universe was opaque. The temperature was around 3000 K. Today it is 3 K. Since E $\propto$ T $\propto \frac{1}{a}$, the universe was 1000 times smaller. When the universe cooled down, if we look at the ratio

$$\frac{t_{today}^{\frac{2}{3}}}{t_{ionized}^{\frac{2}{3}}} = 1000$$

Since the time today is roughly 10 billion years, it is actually 13.6 billion years, substituting into the above equation, we obtain $t^{\frac{2}{3}}{}_{ionized} = \frac{10^{10}}{3\times 10^{4}} \Longrightarrow t_{ionized}$ = 300,000 years, the universe was completely ionized, hot and opaque, a plasma of ions. Light cannot penetrate and observations are not possible. Beyond this, there is no ionization, and light can penetrate. This boundary is the surface of last scattering, and is approximately 300,000 years after the big bang. Since neutrinos have no charge, they can penetrate deeper, to a certain extent. Gravitons even deeper. Due to very large red shift, the surface of last scattering cannot be observed.

9.2 Friedmann- Robertson-Walker Metric (FRW Metric)

The FRW metric describes an expanding homogeneous universe, [8],[38]. On a large scale, the universe is both isotropic (no preferred direction) and homogeneous (the same everywhere). Homogeneity means that at any given time the temperature and density are the same everywhere. Noting that nothing bypasses you faster than the speed of light, in spatial terms

$$c^2dt^2 = ds^2 + g_{ij}dx^i dx^j \tag{9.2.1}$$

, where $dl^2 = g_{ij}dx^i dx^j$. For dl^2, we use spherical coordinates (ρ,θ,ϕ) , ρ is the radius and θ, ϕ are the azimuthal and altitudinal angles, respectively.

$$dl^2 = d\rho^2 + \rho^2 d\Omega^2 + d\omega^2$$

, where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ and

$$x^2 + y^2 + z^2 + \omega^2 = \rho^2 + \omega^2 = R^2 \tag{9.2.2}$$

Differentiating

$$2\rho d\rho + 2\omega d\omega = 0$$

Therefore

$$d\omega^2 = \frac{\rho^2 d\rho^2}{\omega^2} = \frac{\rho^2 d\rho^2}{R^2 - \rho^2}$$

Thus

$$dl^2 = d\rho^2 + \frac{\rho^2 d\rho^2}{R^2 - \rho^2} + \rho^2 d\Omega^2$$

Hence

$$dl^2 = \frac{d\rho^2}{1 - (\frac{\rho}{R})^2} + \rho^2 d\Omega^2$$

Let ρ = Rr , dl^2 = $R^2(\frac{dr^2}{1-kr^2}$ + r^2dΩ^2), where k = -1, 0 or +1. R is a function of t , $R(t)$, not position, to preserve homogeneity. Thus, we arrive at FRW,

$$ds^2 = c^2 dt^2 - R(t)^2 (\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2) \quad (9.2.3)$$

Geometry of the Universe. There are 3 possibilities:

(1) k = 1, positive curvature, closed universe ,

(2) k = 0, zero curvature, flat universe,

(3) k = -1 , negative curvature, open universe.

Let k = 1, r = sinχ, then

$$ds^2 = c^2 dt^2 - R(t)^2 (d\chi^2 + sin^2\chi d\Omega^2)$$

The circumference , C, of a circle of radius r= $\int R$dχ = $R\chi$, is

$$C = 2\pi R sin\chi = 2\pi R sin(\frac{r}{R})$$

The area of a sphere of same radius is

$$A = 4\pi(Rsin\chi)^2 = 4\pi R^2 sin^2(\frac{r}{R})$$

And the volume is

$$V = \int_0^{\chi}(4\pi R^2 sin^2\chi)Rd\chi = 2\pi R^3[\frac{r}{R} - \frac{1}{2}sin(\frac{2r}{R})]$$

As $r \to \pi R$, C and A $\to$ 0 and $V \to 2\pi^2 R^3$is finite, hence a closed universe. The FRW metric can be written as

$$ds^2 = c^2dt^2 - R^2(t)(d\chi^2 + S_k^2(\chi)d\Omega^2) \tag{9.2.4}$$

where sin χ, for k=1; $S_k(\chi) = \chi$, for k = 0; and sinh χ, for k = -1.

Next, we will will flesh out the above analysis in more detail. Imagine a set of observers, each at rest relative to the motion of a nearby mass. Each observer is equipped with a clock. In the metric description of spacetime, physical distances and time are described by the line element. Neighboring events along the world line of an observer are separated by dx^{α}= 0, α = $1,2,3$; spatial coordinates. Because the observer has fixed spatial coordinates, and by dx^0 = dt, which is the proper time interval read from a clock, we get the invariant interval connecting the events is

$$ds^2 = g_{ij}dx^idx^j = g_{00}dx^0dx^0 = g_{00}dt^2 = dt^2$$

Therefore,

$$g_{00} = 1$$

Now, consider two intervals $dx^i = (0, dx^{\alpha})$ and $dy^j = (dt, 0)$, running from an event in spacetime. The first dx^i connects comoving observers at the same time frame, t. The second connects two events along the path of one of the comoving observers. We want to synchronize the clocks to satisfy $g_{0\alpha}$ = 0 for α = 1, 2, 3 because in a homogeneous and isotropic universe there is no preferred direction for $g_{0\alpha}$ to point. Therefore,

$$dx.dy = g_{0\alpha} dt.dx^{\alpha} = 0$$

$dx.dy$ vanishes in locally Minkowski space. This means that every observer sees that neighbouring observer 's clocks are synchronized to his. We arrive at the Cosmological Principle, which states; isotropy allows synchronization of neighboring clocks and homogeneity allows synchronization through space. Hence, we arrive at the line element

$$ds^2 = dt^2 + g_{\alpha\beta} dx^{\alpha} dx^{\beta} = dt^2 - dl^2,$$

dl^2being the proper spatial separation between events at the same world time. Poisson's equation for the Newtonian gravitational acceleration g in a small region generalizes to

$$\nabla.g = -4\pi G(\rho + 3p),$$

where $\rho(x)$ = mass density and $p(x)$ = pressure. This equation says that gravitational mass density acts as the source of gravitational acceleration. Homogeneity and isotropy require that

(1) mean mass density and pressure are functions of time,

(2) the spatial part of the mertic tensor can evolve through a univeral function of time $a(t)^2$,

(3) each galaxy has a fixed spatial coordinate, x^{α},

(4) the proper physical distance dl between a pair of comoving galaxies scales with time as l(t) $\propto a(t)$ where

$$ds^2 = dt^2 - dl^2 = dt^2 - a(t)^2 g_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

Consider an expanding sphere with mean flow of matter so that there is no net flux of material through any part of the sphere. A fixed point on the surface has fixed comoving spatial coordinates χ, θ and ϕ in the line element, so that the proper physical radius of the sphere varies as

$l(t) = l_0 a(t)$, with l_0 a constant . Birkhoff's theorem states that material outside the sphere cannot have any gravitational effect on the material that is inside. Since $M = \rho V = \frac{4}{3}\pi(\rho + 3p)l^3$, l= radius. The gravitational acceleration

$$\ddot{l} = -\frac{GM}{L^2} = -\frac{4}{3}\pi G(\rho + 3p)l$$

This is the equation for the evolution of a homogenous mass distribution. Therefore, substituting $\ddot{l} = l_0\ddot{a}$ into above equation, we get

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p)$$

When ρ(t) and $p(t)$ are written as the sum of mean values ρ_0(t) and p_0(t) and the cosmological constant Λ, we obtain the standard relativistic form for the acceleration of the cosmological expansion

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho_0 + 3p_0) + \frac{\Lambda}{3}$$

Since ρ is mass per unit volume and mass is equivalent to energy, the net energy U within a sphere is U = ρV. Differentiating;

$$dU = \rho dV + V d\rho$$

But, the change in energy is dU, where, by Boyle's Law

$$dU = -pdV$$

Therefore, re-arranging; we obtain the energy conservation equation

$$\dot{\rho} = -(\rho + p)\frac{\dot{V}}{V} = -(\rho + p)\frac{\dot{l}}{l}$$

and since V $\propto \dot{l}$, eliminating p , we get

$$\ddot{l} = \frac{8}{3}\pi G\rho l + \frac{4}{3}\pi G\dot{\rho}\frac{l^2}{\dot{l}}$$

which is a perfect differential

$$\dot{l}^2 = \frac{8}{3}\pi G\rho l^2 + K$$

, where K is constant of integration . Combining, we obtain

$$\dot{\rho}_0 = -3(\rho_0 + p_0)\frac{\dot{a}}{a}$$

As we did above, we eliminate $\dot{\rho}$, and derive the second cosmological equation

$$(\frac{\dot{a}}{a})^2 = \frac{8}{3}\pi G\rho_0 + \frac{K}{a^2} + \frac{\Lambda}{3}$$

The cosmological equation in the standard form is

$$H^2 = (\frac{\dot{a}}{a})^2 = \frac{8}{3}\pi G\rho_0 \pm \frac{1}{a^2 R^2} + \frac{\Lambda}{3}$$

The sign in front of the curvature term is negative in the closed line element and positive in the open line element. H is an expression for Hubble's parameter.

Remarks. (1) Returning to the energy conservation equation $\dot{\rho}$ = -3(ρ + p)$\frac{\dot{a}}{a}$. At zero pressure, $\dot{\rho}_0$ = -3$\rho_0\frac{\dot{a}}{a}$, with a solution $\rho_0 \propto 1/a(t)^3$. This only says that mass per unit volume varies inversely as the volume.

(2) When the mass density dominates, the expansion equation reduces to $(\frac{\dot{a}}{a})^2 = \frac{8}{3}\pi\rho_0$

(3)) Solving , we get $a \propto t^{\frac{2}{3}}$ and $t = \frac{2}{3H} = \frac{1}{\sqrt{(6\pi G\rho_0)}}$

(4) The zero of world time is the singularity at $a \to 0$, where $\rho \to \infty$

(5) When the curvature term is dominant $\propto \frac{1}{a^2}$ and is positive, the geometry is open.

(6) If the cosmological constant Λ dominates, the solutions are hyperbolic sines and

cosines, as in the De Sitter solution, to be discussed below.

(7) The mean mass density, ρ_0 , varies as the inverse of the cube of $a(t)$, at a higher power than the curvature and cosmological constant. This means if $a \to 0$, and we go far back in time, a very large density comes into play at the singularity.

In summary, the expansion rate, H(t), the mean mass density, ρ_0 , the radius of curvature of space, $a(t)R$, and the cosmological constant, Λ , describe spacetime geometry.

9.3 Dark Matter and Dark Energy

Vera Rubin's observations on spiral galaxies led to the conclusion that spiral galaxies have flat rotation curves, instead of a decrement in star rotation velocities the further one travels away from the galactic center. As above, one would expect from Newtonian gravity and from $F = ma$ that

$$\frac{GM}{r^2} = \frac{v^2}{r}$$

where r is the radius of the star from the galactic center, $v$is the rotation velocity and M is the mass within a sphere of radius r. Hence.

$$v = \sqrt{\frac{MG}{r}}$$

and therefore, one would expect that the star rotation velocity is inversely proportional to $\sqrt{r}$. However, observations suggest otherwise, a flat rotaion curve, as in Figure 9.1 illustrated below.

This can be explained by a halo of non-luminous dark matter of very little friction, non-collapsing, surrounding the galaxy. Its mass is estimated to be 10 times larger than the galaxy. As explained above, matter energy density scales with inverse of a^3 and radiation scales with inverse of a^4. Dark matter scales with a^3. Dark energy scales with a^0; it does not change.

If we think of a photon traveling in one dimension bouncing between two walls, separated by a distance, l, the force , F, exerted on the wall is given by

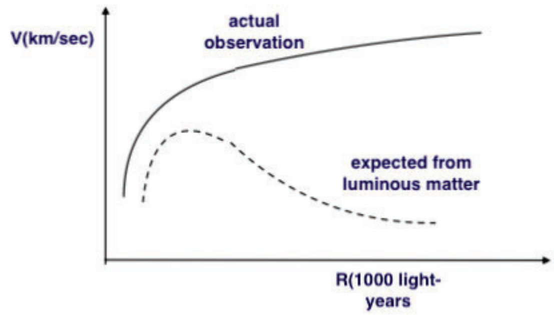

Figure 9.1: Flat rotation curve of spiral galaxies

$$F = \frac{dp}{dt} = \frac{2p}{2l/c} = \frac{pc}{l}$$

where, c , is the speed of light, p is the momentum , and $2p = p$ - $(-p)$ is due to the conservation of momentum. Since the energy, E, is equal to pc; we get the energy density, ρ;

$$F = \frac{E}{l} = \rho$$

In three dimensions, if we think of a rectanguloid of end area, A, and length, l, then

$$F = \frac{E}{lA}$$

where $lA = V$, the volume of the rectanguloid, and the pressure, P, is the energy density, ρ. In each direction,

$$P = \frac{1}{3}\rho$$

For matter,

$$\rho_M = \frac{c}{a^3}, a \sim t^{\frac{2}{3}}$$

Here, the pressure, $P = 0$, as the particles are relatively fixed and not bouncing off the walls.

For radiation,

$$\rho_R = \frac{c}{a^4}, t \sim a^{\frac{1}{2}}$$

and as shown above $P = \frac{1}{3}\rho$.

Question is, what is ρ for empty space ? We will next try to answer this question. In fact, it turns out that ρ is constant for emoty space. As empty space expands, the energy density remains the same; an amazing finding, It is as if energy is created to keep ρ a constant. The general expression for P is

$$P = \omega\rho$$

where ω is a constant. Let us examine a cube of space , filled with particles, with sides of area, A, and pressure, P, at each end. The work done, E , by the particles in pushing the sides by a displacement dx is

$$E = PdxA = PV$$

where V is the volume displaced. Then

$$dE = -PdV$$

Since

$$E = \rho V$$

Differentiating

$$dE = \rho dV + V d\rho$$

Hence

$$\rho dV + V d\rho = -\rho dV$$

Re-arranging

$$V d\rho = -(P+\rho)dV$$

Substituting $P = \omega\rho$

$$V d\rho = -(\omega+1)\rho dV$$

Dividing by ρ and integrating

$$\int \frac{d\rho}{\rho} = -(\omega+1)\int \frac{dV}{V}$$

$$ln\rho = -(\omega+1)lnV = ln\frac{1}{V^{\omega+1}}$$

$$\rho = \frac{1}{V^{\omega+1}}$$

Since $V = a^3$

$$\rho = \frac{1}{a^{3(\omega+1)}}$$

We recover our previous expressions for matter and energy dominated universe, with a unique circumstance, when ω = -1. In summary;

when $\omega = 0$, $\rho = \frac{1}{a^3}$, $P = 0$ and $a = t^{3^{\frac{2}{}}}$

when $\omega = \frac{1}{3}$, $\rho = \frac{1}{a^4}$, $P = \frac{\rho}{3}$ and $a = t^{\frac{1}{2}}$

when ω = -1 ρ = a constant, P = - ρ and $a = e^{\sqrt{\Lambda}t}$

when ω = -1, ρ = ρ_0, a constant;

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_0$$

$\frac{8\pi G\rho_0}{3}$ = Λ is known as the cosmological constant; we get

$$(\frac{\dot{a}}{a}) = \sqrt{\Lambda}$$

$$\dot{a} = a\sqrt{\Lambda}$$

Integrating,

$$a = Ce^{\sqrt{(\Lambda)}t}$$

where C is a constant. So in a vacuum; ω =-1 , P= -ρ, and a increases exponentially with time. However, this is not what happens. The universe doubles in size every 10 billion years. ρ_0 is balance by luminous matter and dark matter. ρ_0 accounts for vacuum energy. It is constant with expansion of space, being constantly created. The current hypothesis is that the universe was initially radiation dominated and expanding as a power of $t^{\frac{1}{2}}$. At around 10, 000 years , it became matter dominated and expanded as a power of $t^{\frac{2}{3}}$. We are now in a cosmological constant-dominated universe that is exponentially expanding as $e^{\sqrt{\Lambda}t}$. The latter is supported by Perlmutter's observation of Type 1A supernovae, for which he received the Nobel prize. In summary, the energy density scales differently over time. Today, the cosmological constant accounts for 0.7 of all energy density, matter is 0.3, radiation is extremely small. The energy density of matter is accounted by a fraction of 0.25 for dark matter and 0.05 fraction for luminous matter. The distance to the horizon of the universe can be calculated simply. Taking ρ_0 to be the energy density of a cosmological dominated universe and H, the Hubble constanr

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G\rho_0}{3}} = H$$

Integrating,

$$a \propto e^{Ht}$$

Since at the horizon $v = c$, and since $v = HD$, where D is the distance to the horizon; setting $c = 1$;

$$1 = D\sqrt{\frac{8\pi G\rho_0}{3}}$$

So

$$D = \frac{1}{\sqrt{\frac{8\pi G\rho_0}{3}}}$$

D turns out to be 12 billion light years. The horizon is the distance at which galaxies are moving away at the speed of light. It is a fixed distance. It is to be noted that the cosmological constant, Λ, is the vacuum energy density, ρ_0, which does not dilute as the universe expands.

9.4 De Sitter Solution

The De Sitter metric is a solution for the Einstein field equation with a positive cosmological constant, leading to an expanding universe. We begin with an empty universe, that is, we set our energy momentum tensor to zero; we go from

$$Rc - \frac{1}{2}Rg + \Lambda g = -T$$

, where symbols, as before, with Λ being the cosmological constant, to

$$Rc = \Lambda g$$

We proceed along similar lines as we did for the Schwarzschild metric. As before, we begin with a static metric with spherical symmetry

$$ds^2 = f(r)dt^2 - g(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Notice, this time I use a different metric signature(+,-,-,-). It will not change our final result, but we must specify which convention we deploy.Here $f(r)$ and $g(r)$ are both ≥ 0, hence we get our familiar metric

$$ds^2 = e^{A(r)}dt^2 - e^{B(r)}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Yet again, we run through the usual cascade of reading out our metric components, calculating the Christoffel symbols (connections), and computing the Ricci tensors, and finally substituting into our contracted field equation. The components of the Ricci tensors are

$$R_{tt} = -e^{A-B}(\frac{1}{2}A^{''} - \frac{1}{4}A^{'}B^{'} + \frac{1}{4}(A^{'})^2 + \frac{A^{'}}{r})$$

$$R_{rr} = \frac{1}{2}A^{''} - \frac{1}{4}A^{'}B^{'} + \frac{1}{4}(A^{'})^2 - \frac{B^{'}}{r}$$

$$R_{\theta\theta} = e^{-B}(1 + \frac{1}{2}r(A^{'} - B^{'})) - 1$$

$$R_{\phi\phi} = R_{\theta\theta}sin^2\theta$$

where the prime, $'$, is the derivative with respect to r. Setting these components equal to Λg = 0, for the R$_{tt}$ and R$_{rr}$ components, and with a little algebra, we obtain, as we showed before $A^{'} = -B^{'}$ gives A = -B, by setting constant of integration = 0. For the R$_{tt}$ equation

$$e^{A}(1+rA^{'}) = 1-\Lambda r^2$$

Let α = e^{Ar}, we get the differential equation,

$$\alpha + r\alpha^{'} = 1-\Lambda r^2$$

which can be written as

$$\frac{d}{dr}(r\alpha) = \frac{d}{dr}(r - \frac{\Lambda}{r}r^3)$$

which gives us

$$r\alpha = r - \frac{\Lambda}{3}r^3 + M$$

, where M is the integration constant. Substituting this into our line element equation, we get the De Sitter metric

$$ds^2 = (1-\frac{M}{r}-\frac{1}{3}\Lambda r^2)dt^2 - \frac{1}{1-\frac{M}{r}-\frac{1}{3}\Lambda r^2}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{9.4.1}$$

where M is the spherical mass at the origin.

Remark. (1)For Λ = 0, we obtain the Schwarzschild solution, which models the curvature of spacetime with M at the origin; (2) For M = 0, we obtain the De Sitter metric

$$ds^2 = (1-\frac{1}{3}\Lambda r^2)dt^2 - \frac{1}{1-\frac{1}{3}\Lambda r^2}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Historical perspective. De Sitter in 1917 pointed out that one can find another solution to Einstein's field equation for a universe that is homogeneous, isotropic and static. De Sitter sets ρ_0 =p_0 = 0 and $\frac{1}{R^2}$ = 0. In this limit, by simply solving our second cosmological equation we get,

$$a(t) = e^{H_\Lambda t}$$

, where $H_\Lambda t$= $(\frac{\Lambda}{3})^{\frac{1}{2}}$. Our line element reduces to

$$ds^2 = dt^2 - e^{2H_\Lambda t}(dr^2 + r^2 d\Omega)$$

In De Sitter's original paper, he changed the time and radial space variables from $t \rightarrow \hat{t}$ and $r \rightarrow \hat{r}$, with

$$e^{H_\Lambda t} = cos\hat{r}e^{H_\Lambda \hat{t}}$$

$$H_\Lambda r = tan\hat{r}e^{-H_\Lambda \hat{t}}$$

His line element looked like this

$$ds^2 = cos^2\hat{r}d\hat{t}^2 - H_\Lambda^{-2}[d\hat{r}^2 + sin^2\hat{r}d\Omega]$$

Remarks.

(1) This solution is time invariant, as spacetime is determined by one parameter only, Λ

(2) The source term in the field equation would be T = $(\Lambda g/8)\pi$G This is proportional to the metric tensor g, which is invariant under a Lorentz velocity transformation . That is, there is nothing to define a preferred velocity

(3) The term $H_\Lambda\hat{r}$ = $tan\hat{r}$e$^{-H_\Lambda t}$ indicates that a freely moving particle is accelerated in the direction of increasing $\hat{r}$ and that acceleration increases with distance. That is, freely moving particles, scatter and accelerate away from each other. At this time, Slipher noted that the nebulae (galaxies) had spectra that were red-shifted.This was interpreted as the De Sitter scattering effect. However, the De Sitter solution does not define a preferred velocity. If initial conditions are assigned such that galaxies move on geodesics with fixed spatial conditions, then the pattern

of relative velocities of galaxies is independent of the galaxy to which the distance and velocities are referred. The redshift of a galaxy is proportional to its distance. The situation is different in a matter filled universe, here the streaming motion of matter has to yield to the energy momentum tensor. Hubble finally published his 1929 data for a linear redshift-distance relation. The observation that the universe is filled with galaxies would argue for Einstein's matter filled static universe, but galaxy redshifts would favor De Sitter's solution. In 1930, this conundrum was disambiguated by Lemaitre's resolution, an expanding matter filled solution. In this situation, the universe is expanding because of a repulsive cosmological constant.

9.5 Cosmic Inflation

We begin with a scalar field $V(\phi)$, with relatively high vacuum energy(zero point energy), in a rapidly expanding inflaton,[34],[38]. If the cosmic scaling factor is $a(t)$; then the Lagrangian, $\mathscr{L}$, is

$$\mathscr{L} = a^3(t)[\dot{\phi}^2 - V(\phi)$$

Next, we find the equation of motion by computing the Euler Lagrange equation;

$$\frac{d}{dt}a^3(t)\dot{\phi} = -a^3(t)\frac{\partial V}{\partial \phi}$$

$$a^3(t)\ddot{\phi} + 3a^2(t)\dot{a}(t)\dot{\phi} = -a^3(t)\frac{\partial V}{\partial \phi}$$

$$\ddot{\phi} + 3(\frac{\dot{a}}{a})\dot{\phi} = -\frac{\partial V}{\partial \phi}$$

$$\ddot{\phi} + 3H\phi = -\frac{\partial V}{\partial \phi}$$

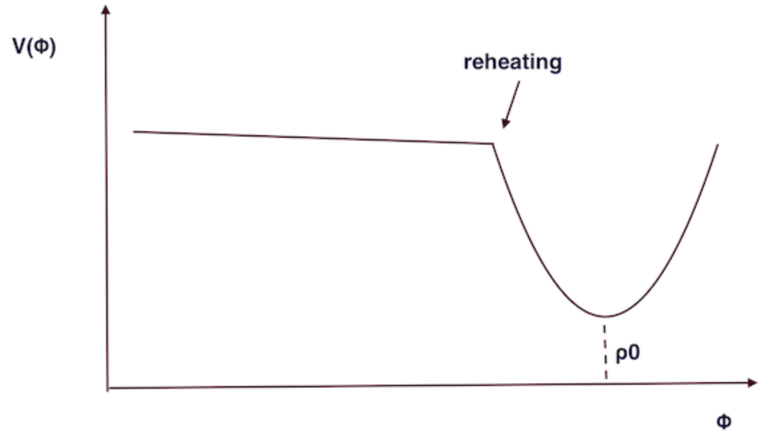

Figure 9.2: Scalar field of inflation

This is the equation of viscosity. The first term is the acceleration, the second term the viscous drag force (Hubble friction) and the third, the force. An imaginary particle in this scalar field would get to a terminal velocity. In this inflaton, the vacuum energy is so large that the universe is e-folding, expanding veru rapidly e^{60}. However, at around 300,000 years, there is a sudden drop in vacuum energy, see Figure 9.2 for an illustration.

As shown in section 32, the expansion is exceptionally high, the flat section of the curve is known as e-folding, where,

$$a \propto e^{Ht}$$

where H is very high, a red-shift close to 1000.

At this point, we introduce the ideas of quantum fluctuation and vacuum energy (zero point energy). To do so, we will need Heisenberg's uncertainty principle and the dynamics of harmonic oscillators. The Heisenberg uncertainty principle simply states that given a momentum, p, and position ,x, both these variables cannot be instantly determined together. This principle has far reaching consequences, as one determines from the relativistic energy-momentum relation of particles

$$E^2 = p^2c^2 + (m_0^2c^2)^2$$

Since p and x can never be zero at the same time, E can never be zero. At this point, we introduce harmonic oscillators

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$$

The quantized Planck energy of particles is

$$E = (n+1)\hbar\omega$$

The zero point energy, when $n = 0$, is

$$E = \frac{1}{2}\hbar\omega$$

When $\dot{x} = 0$;

$$\frac{1}{2}\omega^2 x^2 = \frac{1}{2}\hbar\omega \Longrightarrow x = \sqrt{\frac{\hbar}{\omega}}$$

For planar waves,

$$x = \sqrt{\frac{\hbar}{\omega}e^{i\omega t}}$$

For a damped oscillator;

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = 0$$

where the second term is the damping factor(friction), and $\gamma$is the drag coefficient. Solving, by substituting $x = e^{\alpha t}$;

$$\alpha^2 + \gamma\alpha + \omega^2 = 0$$

$$\alpha = -\frac{\gamma}{2} \pm i\frac{\sqrt{4\omega^2 - \gamma^2}}{2}$$

Case 1. Viscosity is small $\gamma^2 < 4\omega^2$; underdamped oscillator

$$\alpha = -\frac{\gamma}{2} \pm i\frac{\sqrt{4\omega^2 - \gamma^2}}{2}$$

$$x = ce^{-\frac{\gamma}{2}t} e^{\pm\frac{i}{2}\sqrt{4\omega^2 - \gamma^2}t}$$

Multiplicative factor means oscillation with dissipation

Case 2. $\gamma^2 > 4\omega^2$; overdamped oscillator

$$\alpha = -\frac{\gamma}{2} \pm \frac{\sqrt{\gamma^2 - 4\omega^2}}{2}$$

Both roots are negative.

$$x = ce^{-|\alpha|t}$$

Exponential drop-off

Case 3. Critical case; $\gamma^2 = 4\omega^2 \implies \gamma = 2\omega$; crossover from one state to another.

Case 4. When restoring force is zero, $\omega = 0$, and oscillator is completely dominated by friction, γ.

$$\alpha^2 + \gamma\alpha + = 0$$

$$\alpha(\alpha + \gamma) = 0$$

For α = 0, α = -γ

$$x = c_0 + c_1 e^{-\gamma t}$$

This asymptotically goes into a constant ,c_0 , exponentiallly. Comes to rest quickly. Note; ω varies very slowly, begins large, in the oscillating regime, γ sets in, frequency decreases, then ω = 2γ, transition phase, then stops at some point. Next, we need the equations of motion of the inflaton. This is a scalar field with a potential $V(\phi)$. It varies as in figure 8.2. The value of the field varies as shown in diagram. The field equation for a scalar field, where $V^{'}(\phi)$ = 0, the linear part of diagram. This is wave equation for a scalar field of flat potential; if not flat, we equate with -$\frac{\partial V}{\partial \phi}$

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0$$

The universe is expanding. We use the equation we derived above

$$\ddot{\phi} + 3H\phi = 0$$

If ϕ varies, we need to reproduce wave equation; and noting that $D = a(t)dx$;

$$\frac{\partial^{\ddot{2}} \phi}{\partial t^2} + 3H\frac{\partial \phi}{\partial t} - \frac{1}{a(t)^2}\frac{\partial^2 \phi}{\partial x^2} = 0$$

The solution has a waveform

$$\phi_k(t)e^{ik.x}$$

Substituting;

$$\ddot{\phi}_k(t) + 3H\dot{\phi}_k(t) + \frac{k^2}{a(t)^2}\phi_k = 0$$

This is the damped harmonic oscillator, where

$$3H = \gamma$$

$$\omega^2 = \frac{k^2}{a(t)^2} \Longrightarrow \omega = \frac{k}{a(t)}$$

Note, ω depends on time, as the universe expands, a(t) increases with time, and the restoring force decreases with time. The waves will stretch, they oscillate for a while, they then come to rest for a non-zero value for the field, for each k. This is what happens with inflation. Now quantum mechanics kicks in, with zero point oscillations. There is an oscillating field for each k. These oscillations are initially very fast, they slow down, then freeze. From underdamped, they transition to overdamped, then freezing. When does this critical damping happen ? This happens at γ= 2ω, where $\gamma = 3H$. and ω= $\frac{k}{a(t)}$. Therefore,

$$3H = 2\frac{k}{a} \Longrightarrow a(t) = \frac{2k}{3H}$$

Large k means small wavelength, high frequency, and transition happens at large a(t). Let us replace k by wavelength. Since

$$\lambda = \frac{2\pi a(t)}{k} \Longrightarrow k = \frac{2\pi a(t)}{\lambda}$$

Hence,

$$a(t) = \frac{2}{3H}(\frac{2\pi a(t)}{\lambda}) \Longrightarrow \lambda = \frac{4\pi}{3H}$$

This is exactly the event horizon- the boundary of last scattering; since by Hubble's law; $v = HD$. setting c= 1

$$1 = HD \Longrightarrow D = \frac{1}{H}$$

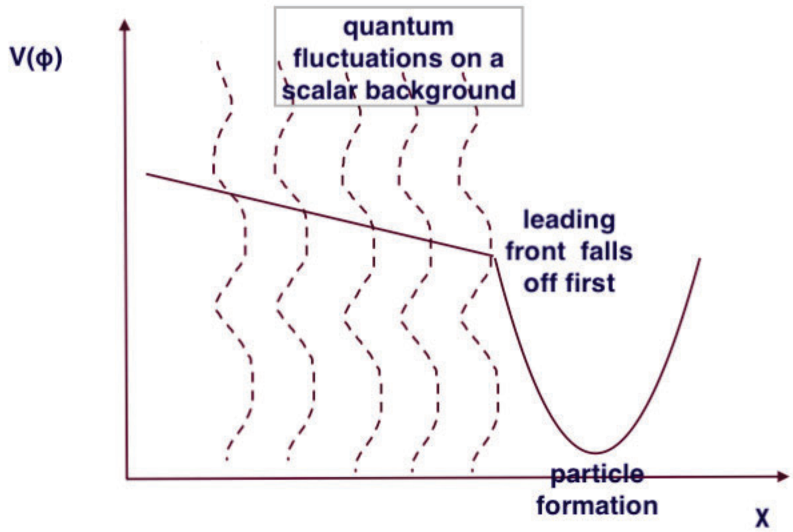

Figure 8.2 Quantum noise superimposed on a scalar background

So the distance to the horizon is λ. The wave is expanding with the universe; the Hubble constant was very large. The wave freezes and stops oscillating when D = $\lambda = \frac{1}{H}$. Since all the waves will eventually feeeze, there is a source of a new population of waves. They come from the vacuum energy, quantum fluctuations.While the scalar field slowly decreases, the quantum fluctuations are buzzing in the background. We then slide down and go over the edge, as shown in Figure 8.2. The potential energy turns into particles. There is less vacuum energy, the expansion is no longer exponential and the energy density dilutes. However, from point to point, the field varies due to quantum noise. Over the edge, in the vicinity of the $'$cliff$'$, the energy density is diluting, the universe is no longer exponentially expanding, the potential energy is converted into particles. The trough of the wave is lagging behind, so we obtain a fractal of variation of energy over the edge. The variation is 1 in 100,000 as determined theoretically and by WMAP measurements of the cosmic microwave background. The density fluctuations are the seeds of structure. This is the origin of matter in the universe. This process is depicted graphically in Figure 9.5 below.

CHAPTER X

SOLUTIONS TO THE FIELD EQUATIONS

10.1 Schwarzschild Metric

Metrics with spherical symmetry,[8],[38]. When a coordinate transformation from x^i to y^i is made, the metric tensor g_{ij} will change to $g^{'}_{ij}$ by tensor transdormation rules, as outlined in subsection 1.2, that is , they are not form invariant. However, as outlined below, there are instances when they are form invariant. A metric g_{ij}, that is form invariant under a group of orthogonal transformations $\bar{x} = Ax$ and AA^T = I, is said to be spherically symmetric, about the origin, where x^i, y^i are spatial coordinates , $i = 1,2,3$ and $x^0 = y^0 = t$, is unaltered. Invariants of this group of coordinate transformtions are

$$x^2+y^2+z^2, xdx+ydy+zdz, dx^2+dy^2+dz^2$$

In spherical polar coordinates, the invariants are

$$r^2, rdr, dr^2+r^2d\theta^2+r^2sin^2\theta d\phi^2$$

It follows that $r, dr, d\theta^2+sin^2\theta d\phi^2$ are invariants. Hence, the most general metric with spherical symmetry looks like

$$ds^2 = A(r,t)dr^2+B(r,t)(d\theta^2+sin^2\theta d\phi^2)+C(r,t)drdt+D(r,t)dt^2$$

Next, we replace r by a new coordinate $r^{'}$, according to the transformation $r^{'2} = B(r,t)$, so that

$$ds^2 = E(r',t)dr'^2 + r'^2(d\theta^2 + sin^2\theta d\phi^2) + F(r',t)dr^{'}dt + G(r',t)dt^2$$

Compare the above equation with the space time metric in spherical polar coordinates

$$ds^2 = dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) - c^2dt^2$$

For convenience, we will drop the primes and obtain $E(r,t)$ = 1, $F(r,t)$ = 0 and $G(r,t)$ = -c^2. Let us examine the special case when the gravitational field is static, that is , the functions E, F, G are independent of t. We then obtain

$$ds^2 = adr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) - bc^2dt^2$$

, where a, b are functions of r. So for our metric, $x^1 = r, x^2 = \theta, x^3 = \phi$, $x^4 = t$; g_{11} = a, $g_{22} = r^2$, $g_{33} = r^2sin^2\theta$ and $g_{44} = -bc^2$. Thus g^{11} = $\frac{1}{a}$, g^{22} = $\frac{1}{r^2}$, g^{33} = $\frac{1}{r^2sin^2\theta}$, g^{44} = -$\frac{1}{bc^2}$. Putting $a = e^{\alpha}$, b= e^{β}, we calculate the Christoffel symbols, and denoting prime ($^{'}$) , differentiation with respect to r;

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$

and noting the symmetry Γ^k_{ij} = Γ^k_{ji}

$$\Gamma^1_{11} = \frac{1}{2}g^{11}(\partial_1 g_{11}) = \frac{1}{2}e^{-\alpha}\frac{\partial}{\partial r}(e^{\alpha}) = \frac{1}{2}\alpha'$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) = \frac{1}{2}\frac{1}{r^2}\partial_1 r^2 = \frac{1}{r},$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\partial_1 g_{33} = \frac{1}{2}\frac{1}{r^2sin^2\theta}\frac{\partial}{\partial r}(r^2sin^2\theta) = \frac{1}{r}$$

$$\Gamma^4_{14} = \Gamma^4_{41} = \frac{1}{2}g^{44}\partial_i g_{44} = \frac{1}{2}(-\frac{1}{c^2}e^{-\beta})\frac{\partial}{\partial r}(-c^2e^{\beta}) = \frac{1}{2}\beta'$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\partial_i g_{22} = \frac{1}{2}e^{-\alpha}\frac{\partial}{\partial r}(r^2) = re^{-\alpha}$$

$$\Gamma^3_{32} = \Gamma^3_{23} = \frac{1}{2}g^{33}\partial_2 g_{33} = \frac{1}{2}(\frac{1}{r^2sin^2\theta})\frac{\partial}{\partial\theta}(r^2sin^2\theta) = cot\theta$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\partial_1(-g_{33}) = \frac{1}{2}e^{-\alpha}\frac{\partial}{\partial r}(-r^2sin^2\theta) = -re^{-\alpha}sin^2\theta$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\partial_2 g_{33} = \frac{1}{2}\frac{1}{r^2}\frac{\partial}{\partial\theta}(-r^2sin^2\theta) = -sin\theta cos\theta$$

$$\Gamma^1_{44} = \frac{1}{2}g^{11}\partial_1 g_{44} = \frac{1}{2}e^{-\alpha}\frac{\partial}{\partial r}(c^2e^{\beta}) = \frac{1}{2}c^2\beta'e^{\beta-\alpha}$$

All other Γ's are zero. Now we can calculate the non-zero components of the Ricci tensor

$$R_{ij} = \partial_l\Gamma^l_{ij} - \partial_j\Gamma^l_{il} + \Gamma^m_{ij}\Gamma^l_{lm} - \Gamma^m_{il}\Gamma^l_{jn}$$

We obtain the following non zero components of the Ricci tensor.

$$R_{11} = \frac{1}{2}\beta'' + \frac{1}{4}\beta'^2 - \frac{1}{4}\alpha'\beta' - \frac{1}{r}\alpha'$$

$$R_{22} = e^{-\alpha}(\frac{1}{2}r\beta' - \frac{1}{2}r\alpha' + 1) - 1$$

$$R_{33} = R_{22}sin^2\theta$$

$$R_{44} = c^2(\beta - \alpha)(-\frac{1}{2}\beta'' - \frac{1}{4}\beta'^2 + \frac{1}{4}\alpha'\beta' - \frac{1}{r}\beta')$$

Schwarzschild solution. When all space outside our spherically symmetric object is empty, we get $T_{ij} = 0$ and $R = 0$; therefore, $Rc - \frac{1}{2}Rg = 0 \Longrightarrow Rc = 0$. Subtracting R_{44} - R_{11} ; $\alpha + \beta$ = a constant. At infinity , in the absence of a gravitational fields

$$\alpha + \beta = 0$$

Going to equation $R_{22} = 0$ and eliminating β, we obtain

$$r\alpha' = 1 - e^{\alpha}$$

This is a separable differential equation, easily solved

$$a = e^{\alpha}(1 - 2m/r)^{-1}$$

where m= constant of integration, therefore

$$b = e^{\beta} = 1 - 2m/r$$

Finally, we arrive at the metric for a spherically symmetric body in empty space

$$ds^2 = (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) - c^2(1 - \frac{2m}{r})dt^2 \tag{10.1.1}$$

Remarks.

(1) The metric is not valid for $r = 2m$. We will show next that $r = 2GM/c^2$. This is the Schwarzschid radius. For Earth, this radius is 9 mm

(2) We know from above calculation that $g_{44} = -bc^2 = -(1 - \frac{2m}{r})c^2$. We know the potential U at a distance r from a spherical body of mass M is $U = -GM/r$, where $g_{44} = 1 + 2U/c^2 = 1 -$

$2GM/c^2r$. Comparing, we get $m = GM/c^2$

(3) Note again that all components of the metric are time independent

(4) When $m = 0$, the metric reduces to flat Minkowski space.

Theorem. Birkhoff's Theorem. For a spherically symmetric distribution of matter, Einstein's field equations have a unique solution.

If space is empty $T_{ij} = 0$, in some region that includes the point of symmetry, the solution in this empty space is the flat spacetime of special relativity, with a line element that can be written as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Newton's Theorem says the Newtonian gravitational acceleration inside a hollow spherical mass vanishes.The relativistic generalization states that spacetime is flat in a hole centered inside a spherically symmetric distribution of matter. A consequence of Birkhoff's theorem is that a spherically symmetric pulsating star cannot emit gravitational waves.

At this point, we will introduce the concept of a Killing vector field. A Killing vector field on a semi-Riemannian manifold is a vector field X for which the Lie derivative of the metric tensor vanishes.

$$L_X g = 0$$

Thus, under the flow of X, the metric does not change. We need the concept of a local isometry. A local isometry from one pseudoReimannian manifold to another is a map which pulls back the metric tensor on the second manifold to the metric tensor on the first.When such a map is a diffeomorphism, it is also known as an isometry. In simple terms, the flow generates a symmetry. Flows generated by Killing fields are continuous isometries of the manifold. Geometrically speaking, moving each point on an object in the direction of the Killing vector does not distort distances on the object. As stated above, a vector field X is a Killing field if the Lie derivative with respect to X of the metric vanishes; $L_X g$= 0. In terms of the Levi-Civita connection

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

In local coordinates, we get the Killing equation

$$\nabla_\mu X_\nu + \nabla_\nu X \mu = 0 \tag{10.1.2}$$

Example. (1) The vector field rotating clockwise on a circle , with same length at each point, is a Killing vector field,

(2) In general relativity, the gravitational field distorts the 4-manifold. In the absence of a gravitational field, in which nothing changes with time, the time vector will be a Killing vector.

The Lie derivative of a metric g along the vector V is

$$(L_V g)_{\mu\nu} = V^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + \frac{\partial V^\rho}{\partial x^\mu} g_{\mu\nu} + \frac{\partial V^\rho}{\partial x^\nu} g_{\mu\rho}$$

$L_V g$= 0, means that the metric g is invariant under the flow of V. Paraphrasing, if we travel along the flow of a Killing vector field, the metric, and hence spacetime in our 4-manifold, is unchanged. It is a way of expressing translational invariance of the metric g, in a coordinate invariant way. I will illustrate the machinery of Killing vectors by explicitly working out the metric for the 2-sphere

$$ds^2 = d\theta^2 + sin^2\theta d\phi^2$$

The metric components a

$$g_{\vartheta\theta} = g^{\theta\theta} = 1, g_{\phi\phi} = sin^2\theta = \frac{1}{g_{\phi\phi}}, g_{\phi\theta} = g_{\theta\phi} = 0$$

Next, we calculate the Christoffel symbol

$$\Gamma^i_{kl} = \frac{1}{2} g^{im}(g_{mk,l} + g_{ml,k} - g_{kl,m})$$

We obtain

$$\Gamma^\theta_{\theta\theta} = \Gamma^\phi_{\phi\phi} = \Gamma^\phi_{\theta\theta} = \Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = 0$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta}(\frac{\partial}{\partial\varphi} g_{\theta\theta} + \frac{\partial}{\partial\phi} g_{\theta\phi} - \frac{\partial}{\partial\theta}(g_{\phi\phi}) = -sin\theta cos\theta$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = cot\theta$$

The Killing equations are then

$$\nabla_\theta K_\theta = \partial_\theta K_\theta - \Gamma^\alpha_{\theta\theta} K_\alpha = \partial_\theta K_\theta = 0$$

$$\nabla_\phi K_\phi = \partial_\phi K_\phi - \Gamma^\alpha_{\phi\phi} K_\alpha = \partial_\phi K_\phi + sin\theta cos\theta K_\theta = 0$$

$$\nabla_\theta K_\phi + \nabla_\phi K_\theta = \partial_\theta K_\phi - \Gamma^\alpha_{\theta\phi} K_\alpha + \partial_\phi K_\theta - \Gamma^\alpha_{\phi\theta} K_\alpha = \partial_\theta K_\phi + \partial_\phi K_\theta - 2cot\theta K_\phi$$

K_θ and K_α components of the Killing vector are functions $K_\theta(\theta,\phi)$ and $K_\phi(\theta,\alpha)$. Note K_θ does not depend on θ; therefore, $\partial_\theta K_\theta = 0 \Longrightarrow K_\theta = K_\vartheta(\phi)$. Take the derivative of the third equality of the above equation with respect to ϕ;

$$\partial_\theta(\partial_\phi K_\phi) + \partial^2_\phi K_\theta - 2cot\theta(\partial_\phi K_\phi) = 0$$

Substituting;

$$\partial_\theta[-sin\theta cos\theta K_\theta] + \partial_\phi^2 K_\theta - 2cot\theta[-sin\theta cos\theta K_\theta] = 0$$

, from which we derive

$$\partial_\phi^2 K_\theta + K_\theta = 0$$

, which is the harmonic oscillator equation, with solution

$$K_\theta = Asin\phi + Bcos\phi$$

, where A and B are constants. Next, we need to find K_ϕ; by substituting

$$\partial_\phi K_\phi = -sin\theta cos\theta[Asin\phi + Bcos\phi]$$

The solution of this differential equation is;

$$K_\theta = Asin\theta cos\theta cos\phi - Bsin\theta cos\theta sin\phi + F(\theta)$$

, where F(θ) is the solution of the homogeneous equation $\partial_\phi K_\phi = 0$. Substitute our solutions , we arrive at

$$\partial_\theta F - 2cot\theta F = 0$$

Dividing by $\sin^2\theta$, we arrive at

$$\partial_\theta[\frac{F}{sin^{2\theta}}] = 0$$

Hence

$$F(\theta) = Csin^2\theta,$$

where C is a constant. Therefore, we arrive at the following solutions for the Killing vectors;

$$K_\theta = Asin\phi + Bcos\phi$$

$$K_\phi = Asin\theta cos\theta cos\phi - Bsin\theta cos\theta sin\phi + Csin^2\theta$$

Noting that the constants A, B and C are independent. Setting A= 0, B =1 and C = 0, we get

$$K_{1\theta} = cos\phi and K_{1\phi} = -sin\theta cos\theta sin\phi$$

Also set A = -1, $B = C$= 0 and $A = B = 0$, C = 1. But we will not show the results here (easy exercise).

A solution is stationary if it is time independent. A metric is stationary if there is a coordinate system in which the metric is time independent

$$\frac{\partial x_{ab}}{\partial x_0} = 0$$

, where x^0 is a timelike coordinate.

10.2 Reissner-Nordstrum Metric

If Q is the total charge and J the angular momentum , then blackholes can be classified as follows,[35] ;

A non- rotating, uncharged blackhole is a Schwarzschild blackhole.

A rotating uncharged charged blackhole is a Kerr blackhole.

A charged non-rotating blackhole is a Reissner-Nordstrum blackhole.

A rotating charged blackhole is a Kerr-Newman blackhole.

Next, we explore the charged, non rotating solution of the field equation.

The Field of a Charged Mass Point. Our goal is to find a static, asymptotically flat, spherically symmetric solution of the Einstein-Maxwell field equation;

$$G_{ab} = 8\pi T_{ab}$$

, where T_{ab} is the Maxwell energy-momentum tensor. This tensor, it turns out, is trace free .Therefore, taking the trace ;

$$R - \frac{1}{2}R(4) = 0$$

Therefore, R = 0, and the scalar curvature vanishes. Hence, $R_{ab} = 8\pi T_{ab}$. We need to make several observations. Firstly, Maxwell's equation in a source-free region states that the divergence of the energy momentum tensor, $T_{ab} = F_{ab}$ is zero.

$$\nabla F_{ab} = 0$$

Secondly, we assume spherical symmetry and the line element reduces to

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2),$$

where ν and λ are functions of t and r. Thirdly, we dictate that ν and λ are functions of r only, that is, a static field only. I will now take a little survey of the electromagnetic energy tensor. As we noted earlier, in the language of differential forms, the divergence of a vector is the exterior derivative of a 2-form on $\mathbb{R}^3$. The curl of a vector is the exterior derivative of a 1-form on $\mathbb{R}^3$. So, we treat the magnetic field, not as a vector, but as a 2-form and the electric field as a 1-form, so

$$B = B_1 dx^2 \wedge dx^0 + B_2 dx^3 \wedge dx^0 + B_3 dx^1 \wedge dx^2$$

$$E = E_1 dx^1 + E_2 dx^2 + E_3 dx^3$$

The electric and magnetic fields are considered inhabitants of spacetime and as before, the manifold M is assumed to be a semi-Riemannian manifold equipped with the Minkowski metric, that is a Lorentzian 4 -manifold, spacetime. We assume spacetime can be split into a 3-manifold S, space with a Riemann metric and another space R for time. Then,

$$M = R \times S$$

Let $x^i (i = 1,2,3)$ denote local coordinates on an open interval U $\subseteq$ S, and let x^0 denote the coordinate on $\mathbb{R}$. We can then combine the electric and magnetic fields into a unified electromagnetic field F, which is a 2-form on $\mathbb{R} \times$ U $\subseteq$ M, defined by

$$F = B + E \wedge dx^0$$

In components

$$F = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$$

Explicitly, we have

$$F = E_1 dx^1 \wedge dx^0 + E_2 dx^2 \wedge dx^0 + E_3 dx^3 \wedge dx^0 + B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^3$$

In the traditional way of formulating Maxwell's equation, the homogeneous and inhomogeneous versions are related by reversing the roles of E and B. In the language of differential forms, this reveral is done by using the Hodge star operator, by treating E as a 2-form and B as a 1-form;

$$*F = -B_1 dx^1 \wedge dx^0 - B_2 dx^2 \wedge dx^0 - B_3 dx^3 \wedge dx^0 + E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^3$$

$$(*F)_{\alpha\beta} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}$$

In our radially symmetric electrostatic field of charged point particle, the Maxwell tensor reduces to

$$F_{ab} = E(r) \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

, since it only depends on r. Therefore, applying $\nabla F_{ab} = 0$, plugging into it F_{ab} and the metric tensor g_{ab} which is diag $(e^{\nu}, e^{-\lambda}, -r^2, -r^2 sin^2\theta)$; we first compute the Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{ji} + \partial_j g_{il} - \partial_l g_{ij})$. Note coordinates are $(r, \theta, \phi, t) = (0, 1, 2, 3)$. We get

$$(e^{-\frac{1}{2}(\nu+\lambda)} r^2 E)' = 0$$

Integrating,

$$E = e^{\frac{1}{2}(\nu+\lambda)} \varepsilon / r^2$$

, ε being a constant of integration. As $\nu, \lambda \to 0$, $E = \varepsilon/r^2$, the inverse square law. As $r \to \infty$, we have an asymptotically flat solution. Next, we calculate the Ricci tensor

$$R_{ij} = \partial_l \Gamma^l_{ij} - \partial_j \Gamma^l_{il} + \Gamma^m_{ij} \Gamma^l_{lm} - \Gamma^m_{il} \Gamma^l_{jm}$$

And we apply $R_{ab} = 8\pi T_{ab}$, by plugging in $T_{ab} = \frac{1}{4\pi}(-g^{cd} F_{ac} F_{bd} + \frac{1}{4} g_{ab} F_{cd} F^{cd})$; the T_{00} and T_{11} equation lead to

$$\nu' + \lambda' = 0$$

as $\nu, \lambda \to 0$ as $r \to \infty$, ν = -λ. The T_{22} equation leads to

$$(re^{\nu})' = 1 - (\varepsilon^2/r^2)$$

Integrating

$$e^{\nu} = 1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}$$

, m is a constant of integration. Therefore,

$$ds^2 = (1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2})dt^2 - (1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2})^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)$$

When ε = 0, this reduces to the Schwarzchild line element. Before we analyze the Reissner-Nordstrom equation, let us introduce the concept of a nullcone. In Minkowski space time, the norm of a vector is

$$X^2 = g_{ab}X^aX^b$$

The vecor is said to be timelike if $X^2 > 0$, spacelike if $X^2 < 0$ and null or lightlike if $X = 0$. Two vectors X^a and X^b are orthogonal if their inner product is zero;

$$g_{ab}X^aY^b = 0$$

Hence, a null vector is orthogonal to itself. The set of all null vectors at a point P in a Minkowski manifold forms a double cone called the null cone or light cone (see figure 4 below). In Minkowski coordinates, the null vectors at P satisfy

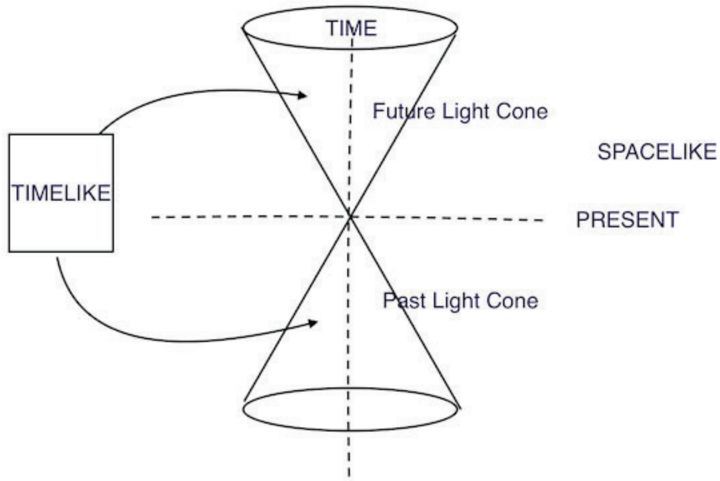

Figure 10.1: Light cone

$$\eta_{ab}X^aX^b = 0,$$

That is,

$$-(X^0)^2 + (X^1)^2 + (X^3)^3 + (X^4) = 0$$

This is the equation of a double cone. This null cone lies in the tangent space T_pM at P. If we define the timelike vector as T^a in Minkowski coordinates by $T^a = (1,0,0,0)$ then a timelike or null vector is said to be future-pointing if $\eta_{ab}X^aT^b > 0$ and past-pointing if $\eta_{ab}X^aT^b < 0$. See Figure 10.1 below.

10.3 Kerr Solution

The Kerr metric tensor is in terms of the coordinates t, r, θ, ϕ on $R^4 = \mathbb{R}^3 \times \mathbb{R}^1$, the time coordinate t on $\mathbb{R}^1$. Kerr spacetime depends on two paramaters M > 0, its mass, and $a \neq 0$, its angular momentum per unit mass. By setting a= 0, the Kerr spacetime reverts to Schwarzchild spacetime. By setting M = 0, empty space, Minkowski spacetime remains,[35]. The two most commonly encountered functions in Kerr spacetime are

$$\rho^2 = r^2 + a^2 cos^2\theta$$

and

$$\Delta = r^2 - 2Mr + a^2$$

Therefore,

$$\frac{\Delta}{r^2} = 1 - 2\frac{M}{r} + \frac{a^2}{r^2}$$

When a = 0, the Kerr metric reduces to the Schwarzschild metric. Thus, $\rho^2 \rightarrow r^2$ and

$$\frac{\Delta}{r^2} = h(r) = 1 - \frac{2M}{r}$$

The Boyer Lindquist coordinates fail when (a) sinθ = 0, z-axis; (b) ρ^2 = 0 and (c) Δ = 0; this gives the horizon of Kerr spacetime.

The Kerr metric in Boyer-Lindquist coordinates is as follows

$$ds^2 = -dt^2 + \rho^2(\frac{dr^2}{\Delta} + d\theta^2) + (r^2 + a^2 sin^2\theta)d\phi^2 + \frac{2M}{\rho^2}(a.sin^2\theta.d\phi - dt)^2 \qquad (10.3.1)$$

where $\Delta(r) = r^2 - 2Mr + a^2$ and $\rho^2(r,\theta) = r^2 + a^2 cos^2\theta$. The coordinate (t, r, θ, ϕ)are known as Boyer-Lindquist coordinates. M = mass of blackhole, a = its angular momentum per unit mass.

Remarks.(1) The metric is stationary, meaning none of the coefficient terms depend on time, t

(2) The metric is axisymmetric, none of the coefficient are dependent on ϕ

(3) Not static ; not invariant under time reversal $t \rightarrow -t$

(4) Invariant for simultaneous inversion of t and ϕ. The double sign change $t \rightarrow -t$ and $\phi \rightarrow -\phi$ gives an isometry. Running time backwards reverses the rotation

(5) The Kerr metric is asymptotically flat ; as $r \rightarrow \infty$, the Kerr metric reduces to the

Minkowki metric. This means that far from the blackhole, the gravitational field is weak

(6) If a, angular velocity, $\rightarrow 0$, with M $\neq 0$; $\rho^2 \rightarrow r(r-2M)$, $\rho^2 \rightarrow r^2$, the metric reduces to the Schwarzschild metric; ds^2 = -(1 -2$\frac{M}{r}$)dt^2 + (1 - 2$\frac{M}{r}$)$^{-1}$dr^2 + r^2(dθ^2 + sin$^2\theta$dϕ^2)

(7) Let x = $\sqrt{(r^2+a^2)}$. sinθcosϕ y= $\sqrt{(r^2+a^2)}$.sinθsinϕ and $z = rcos\theta$. Then $dx^2 + dy^2 + dz^2$ = $\frac{r^2+a^2cos^2\theta}{r^2+a^2}dr^2$ + $(r^2+a^2cos^2\theta)d\theta^2$ + $(r^2+a^2)sin^2\theta d\phi^2$. As M $\rightarrow 0$, with $a \neq 0$,$ds^2 = -dt^2 + dx^2$ + dy^2 $+ dz^2$

(8) As noted above, the metric is singular for Δ = $r^2 - 2Mr + a^2$ = 0 and ρ^2 = $r^2 + a^2cos^2\theta$ = 0 , r = 0, θ = $\frac{\pi}{2}$

(9) Since the line element is independent of both t and ϕ we conclude that there are two killing vector fields, ∂_t and ∂_ϕ. These isometries express time invariance and axial symmetry espectively.

The Boyer-Linquist form of the line element is

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 + 2g_{\phi t}d\phi dt$$

where g_{tt} = -(1 + 2$\frac{M}{\rho^2}$), g_{rr} = $\frac{\rho^2}{\Delta}$, $g_{\theta\theta}$ = ρ^2, $g_{\phi\phi} = r^2 + a^2sin^2\theta + \frac{2M}{\rho^2}asin^2\theta$ and $g_{\phi t} = g_{t\phi} = -2Mra\frac{sin^2\theta}{\rho^2}$. It is easy to show that for the Boyer-Lindquist coordinates; (1) $g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -\nabla sin^2\theta$ and (2) $det(g_{ij}) = -\rho^4 sin^2\vartheta$

10.4 Kerr-Newman Metric

There is a modification of the Kerr metric for a source that carries an electric charge, e,[35]. Replacing the previous definition ∇ by

$$\Delta = r^2 - 2Mr + a^2 + e^2$$

leads to the Kerr-Newman metric.

Theorem.If (M,g) is an asymptotically flat, stationary axisymmetric vacuum spacetime, non-singular on and outside a connected horizon, then (M,g) is a member of the two-parameter Kerr family of solutions. The parameters are mass M and angular momentum J, where $J = aM$, the total angular momentum.

(1) The final state of gravitational collapse is stationary. The initial state is complicated, with many independent multipole moments of the gravitational

field. All information about the initial state is radiated away during collapse except M and J.

(2) The Einstein-Maxwell generalization is; (M,g) belongs to the Kerr-Newman family of solutions with four parameters $(M.J,Q,P)$.

In Boyer-Lindquist coordinates, the Kerr Newman solution is

$$ds^2 = -\frac{\Delta - a^2 sin^2\theta}{\rho^2}dt^2 - 2asin^2\theta\frac{r^2+a^2-\Delta}{\rho^2}dtd\phi + \frac{(r^2+a^2)^2 - \Delta a^2 sin^2\theta}{\rho^2}sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where $\rho^2 = r^2 + a^2cos^2\theta$, Δ =$r^2 - 2Mr + a^2 + e^2$, and $e = \sqrt{(P^2+Q^2)}$

Remark. (1) The metric is asymptotically flat as $r \to \infty$

(2) If a = 0, the metric reduces to the Reissner Nordstrom solution

(3) The transformation $\phi \to$ -ϕ has the same effect as $a \to -a$

(4) The metric has a discrete isometry $t \to -t$ and $\phi \to$ -ϕ

CHAPTER XI

GRAVITATIONAL WAVES

11.1 Linearizing the Field Equation

Einstein's field equation is a non-linear partial differential equation. In 1916, Einstein linearized his field equation

$$Rc - \frac{1}{2}Rg = \kappa T$$

by assuming that the metric $g_{\mu\nu}$ is a perturbed Minkowski metric $\eta_{\mu\nu}$, [24].

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu} \tag{11.1.1}$$

He linearized his field eqaution to O(ε) and obtained

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_{\alpha\beta}\eta^{\alpha\beta}$$

$$\Box\tilde{h}_{\mu\nu} = 2\kappa T_{\mu\nu}$$

and

$$\Box = \eta_{\mu\nu}\partial^{\mu}\partial^{\nu}$$

When $T_{\mu\nu} = 0$;

$$\frac{1}{2}\eta_{\mu\nu}\partial^{\mu}\partial^{\nu}h_{\mu\nu} = \Box h_{\mu\nu} = 0 \tag{11.1.2}$$

This allowed Einstein to conclude that solutions to this equation are plane waves traveling with the speed of light. He called these waves gravitational waves. He was haunted by the fact that these waves may be fictitious, an artefact of the linearization process. The gravitational plane wave is a spacetime that

(a) satisfies the vacuum field equation, $R_{\mu\nu} = 0$,

(b) has a 5-dimensional group of isometries as plane electromagnetic waves do,

(c) has to carry energy.

The Metric and the Gravitational Wave Equation. This involves three topics.

(1) Linearization of the field equation

(2) Demonstration of gauge transformations in the linearized regime

(3) Writing of a wave equation for small deviations from the background spacetime.

As opposed to Newton gravity, action at a distance is not allowed in Special and General Relativity. Instead variation in gravitational attraction is transmitted via gravitational waves. The traditional approach assumes that the waves are described by a small perturbation to flat space

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = (\eta_{\mu\nu} + h_{\mu\nu})dx^{\mu}dx^{\nu}$$

where $\eta_{\mu\nu}$ is the Minkowski metric for flat spacetime and $h_{\mu\nu}$ is the small perturbation known as the wave metric. If one makes the linear approximation above, then the Einstein wave equation can be reduced to a vacuum wave equation for the metric perturbation $h_{\mu\nu}$

$$\square h^{\mu\nu} = (-\frac{\partial^2}{\partial t^2} + \Delta^2)h^{\mu\nu} = 0 \longrightarrow \eta^{\alpha\beta}h^{\mu\nu}_{,\alpha\beta} = 0 \tag{11.1.3}$$

$\square$ is the wave operator and ∇ is the Laplacian. This is a wave equation, hence assume the plane wave solution

$$h^{\mu\nu} = A^{\mu\nu}exp(ik_{\alpha}x^{\alpha})$$

where $A^{\mu\nu}$ is a tensor with constant components and k_{α} is a 1-form with constant components.

Taking a first derivative

$$h^{\mu\nu}_{,\alpha} = k_\alpha h^{\mu\nu}$$

Taking a second derivative, we get back the wave equation

$$\eta^{\alpha\beta} h^{\mu\nu}_{,\alpha\beta} = \eta^{\alpha\beta} k_\alpha k_\beta h^{\mu\nu} = 0$$

This is true if k_α is a null vector

$$\eta^{\alpha\beta} k_\alpha k_\beta = k_\alpha k^\alpha = 0$$

k^α is the k-vector and has components $k^\alpha = \{\omega, \vec{k}\}$. Null normalization gives

$$k_\alpha k^\alpha = 0 \longrightarrow \omega^2 = k^2$$

The wave equation above has a gauge condition known as de Donger gauge or Lorentz gauge or Lorentz gauge or Hilbert gauge.

$$h^{\mu\nu}_{,\nu} = 0.$$

Next, we will take a brief detour and discuss gauge transformations in detail after which we will return to gravitational waves.

Gauge Invariance . The gauge principle was first recognized in electromagnetism. We first define the electromagnetic 4-current density, $\mathrm{j}^\mu = \{\rho, \mathrm{j}\}$, where ρ is the electric charge density, and j is the 3 dimensional electric current .The electromagnetic 4-potential $A_\mu(x) = \{\phi(x), A(x)\}$ is a 1-form, where ϕ is the electric potential and A is the magnetic potential. The electromagnetic field tensor is a 2-form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Using the above definitions, Maxwell's equations can be expressed in tensor form

$$\partial_{[\alpha} F_{\mu\nu]} = 0$$

, where [ ] denote anti-symmetrization

$$\partial_\mu F^{\mu\nu} = j^\nu$$

If we take the transformation

$$A_\mu(x) \longrightarrow A'_\mu(x) + \partial_\mu \alpha(x)$$

, for any differentiable function $\alpha(x)$. $F_{\mu\nu}$ remains unchanged due to equality of mixed partials. Thus, Maxwell's equations are unaltered by adding a gradient. Such a transformation is known as a gauge transformation. It turns out that these transformations form an Abelian group for single continuous parameter $\alpha(x)$.

Gauge Invariance in General Relativity. The gauge transformations are rigid motions in spacetime. We begin with our connection and line element

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g \mu\nu)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

When a covector is parallel transported a distance dx^ν, its components vary as

$$dv^\lambda = -v^\mu \Gamma^\lambda_{\mu\nu} dx^\nu$$

or

$$dv_\mu = v_\lambda \Gamma^\lambda_{\mu\nu} dx^\nu$$

Hence,

$$dv^2 = d(v_\nu v^\nu) = dv_\nu v^\nu + v_\nu dv^\nu$$

$$= v_\lambda \Gamma^\lambda_{\mu\nu} dx^\mu v^\nu - v_\nu v^\mu \Gamma^\nu_{\mu\lambda} dx^\lambda = 0$$

So , the length is invariant under parallel transport. Next, how can derivatives remain gauge invariant under a transformation ? With change of coordinates from unprimed to primed

$$\partial_{\mu'} v^{\nu'} = (\partial_{\mu'} x^\mu \partial_\nu x^{\nu'} \partial_\mu + \partial_{\mu'} x^\mu \partial_{\mu\nu} v^{\nu'}) v^\nu$$

The first term is tensorial, the second is not. As before, we remedy this by defining our covariant derivative through a connection

$$(\nabla_\mu)^\lambda_\nu = \delta^\lambda_\nu \partial_\mu + \Gamma^\lambda_{\mu\nu}$$

so that the covariant derivative transforms tensorially. Through covariant derivatives, global invariance is preserved locally. The Riemann curvature tensor is expressed in terms of a connection, or a covariant derivative, as before

$$R^\lambda_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\sigma\nu} - \partial_\nu \Gamma^\lambda_{\sigma\mu} + \Gamma^\lambda_{\alpha\mu} \Gamma^\alpha_{\sigma\nu} - \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\sigma\mu}$$

$$= [\nabla_\sigma, \nabla_\mu]^\lambda_\nu$$

This second expression highlights the non-commutativity of parallel transport, which tells us about the curvature of spacetime. Gauge freedom in relativity means the freedom to choose coordinates. Returning to gravitational waves, the de Donder gauge does not use up all the gauge freedom because small changes in coordinates

$$\bar{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}$$

preserves the gauge if $\xi^{\alpha\beta}_{\ ,\beta}$ = 0. For the wave amplitude, A$^{\mu\nu}$, we state, without proof the de Donder gauge

$$A^{\mu\nu}k_{\nu} = 0$$

, which implies A$^{\mu\nu}$ is orthogonal to k^{α}. The transverse traceless gauge are the 3 conditions:

(1) A$^{\mu\nu}k_{\alpha}$ = 0

(2) A$^{\alpha}_{\alpha}$ = 0 and

(3) A$_{\mu\nu}u^{\nu}$ = 0, where u^{ν} is the 4-velocity. By restricting the gauge freedom in the wave equation, we are removing waving of the coordinates, and what is left is the waving of the curvature of spacetime.The Transverse Traceless Gauge has 2 independent components of A$_{\mu\nu}$

$$A^{TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To observe the effects of the gravitational waves, we imagine two test particles. Both begin at rest x^{α}_{1} = $(0,0,0,0)$ and x^{α}_{2}= $(0,\varepsilon,0,0)$. The proper distance between them is

$$l = \int \sqrt{ds^2} = \int |g_{\alpha\beta}dx^{\alpha}dx^{\beta}|^{\frac{1}{2}}$$

$$l = \int_0^{\varepsilon} |g_{xx}|^{\frac{1}{2}} \simeq |g_{xx}(x=0)|^{\frac{1}{2}}\varepsilon \simeq [1 + \frac{1}{2}h^{TT}_{xx}(x=0)]\varepsilon$$

Our imposed solution for $h^{TT}_{\mu\nu}$ is a travelling plane wave which depends on time, which is a geodesic deviation. The gravitational wave is distorting spacetime by geodesic deviation, which we calculate by computing the components of the Riemann curvature tensor $\mathrm{R}^{\alpha}_{\beta\gamma\delta}$ in the TT gauge in the presence of $h^{TT}_{\alpha\beta}$.

$$R^{TT}_{j0k0} = -\frac{1}{2}\partial_t^2 h^{TT}_{jk}$$

, $j,k = 1,2,3$. When $\mathrm{A}_{xx} \neq 0$ and $\mathrm{A}_{xy} = 0 \longrightarrow +$ polarization state ; compression of geodesics in one direction and stretching in orthogonal direction(during half-cycle) and vice vera during second half cycle. When $\mathrm{A}_{xx} = 0$ and $\mathrm{A}_{xy} \neq 0 \longrightarrow \times$, cross polarization. This area of mathematics is a very fertile field of research in the study of detection of gravitational waves, and further treatment of this matter will be deferred for the interested reader to pursue.

11.2 Multipolar Moments

In studying fields, such as electromagnetism and gravitation, a very important concept is the idea of multipole moments,[16] ,[40]. This idea will be initially illustrated for an electric charge source. Let the charge density be $\rho(x)$. We want to know what the potential is at a point x from the source x_0. The general solution for the potential is

$$V(x) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(x)}{\mathbf{x}-\mathbf{x_0}}dx^3 \tag{11.2.1}$$

, where x and x_0 are vectors, x= r$\hat{r}$ and x_0= r$_0\hat{r_0}$. Let the angle between the vectors x and x_0 be θ, then by cosine similarity

$$x - x_0 = \sqrt{(x^2 + x_0^2 - 2xx_0cos\theta)} = r\sqrt{(1 + \frac{r_0^2 - 2r_0rcos\theta}{r^2}})$$

The reciprocal of this difference is nothing but a Legendre polynomial and it can be expressed as

$$\frac{1}{|x-x_0|} = \sum_{l=0}^{\infty}\frac{r_0^l}{r^{l+1}}P_l cos\theta \tag{11.2.2}$$

A quick note on Legendre Polynomials.

(1) The series converges when $r > r_0$;

(2) The expression gives gravitational potential at a point mass or Coulomb potential associated with a point charge

(3) They occur as solutions of Poisson's equation $\nabla^2\phi = 0$. Substituting into V(x), we get

$$V(x) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_0^l \rho(x_0)P_l(cos\theta)d^3x \tag{11.2.3}$$

We let $Q_l = \frac{1}{4\pi\varepsilon_0}\int r_0^l\rho(x)P_l(\cos\theta)d^3x$, be the moments. We separate the Q_l's and obtain our potential

$$V(x) = \sum_{l=0}^{\infty}\frac{Q_l}{r^{l+1}} \tag{11.2.4}$$

The Zeroth Moment Q_0

$$Q_0 = \frac{1}{4\pi\varepsilon_0}\int_0^0 \rho(x)P_0(cos\alpha)d^3x = \frac{1}{4\pi\varepsilon_0}\int \rho(x)d^3x = \frac{1}{4\pi\varepsilon_0}Q_{total}$$

Remarks.

(1) Q_{total} is independent of observation point x. It characterizes only the source

(2) The potential of a monopole is given by Coulomb's Law for a single point charge $V(x) = \frac{Q}{4\pi\varepsilon_0 r_0}$

(3) For any source , at a large distance $r >> r_0$, the dominant term is the Coulomb potential of the total charge $V(x) \cong \frac{Q_{total}}{4\pi\varepsilon_0 r}$. Higher multipole moments give corrections to this.

Dipole Moment Q_1

$$\begin{aligned}Q_1 &= \frac{1}{4\pi\varepsilon_0}\int \rho(x)P_1(cos\theta)d^3x \\ &= \frac{1}{4\pi\varepsilon_0}\int \rho(x)cos\theta d^3x \\ &= \frac{1}{4\pi\varepsilon_0}\int \rho(x)d^3x\end{aligned}$$

The dipole moment is defined as $p = \int x\rho(x)d^3x$

Quadrupole Moment. For the quadrupole moment, we need to compute

$$Q_2 = \frac{1}{4\pi\varepsilon_0}\int d^3xr^2\rho(x)P_2(cos\theta)$$

$$P_2(cos\theta) = 3x^2 - 1$$

, is the Legendre polynomial, where

$$r^2(3x^2 - 1) = 3(\hat{r_0}.\hat{r})^2 - r.r$$

In tensor notation,

$$Q_{ij} = \int d^3x\rho(x)(3x_ix_j - r^2\delta_{ij})$$

The trace Q_{ij} is

$$tr(Q_{ij}) = \sum_{i=1}^{3} Q_{ii} = \int_{i=1}^{3} d^3x\rho(x)\sum(3x_ix_i - r^2\delta_{ii})$$

$$= \int d^3x\rho(x)(3r^2 - 3r^2) = 0$$

Therefore, the quadrupole moment tensor is a rank 2 tensor and is traceless.

Gravitational Quadrupole. The mass quadrupole is analagous to the electric charge quadrupole. The mass monopole represents the total mass energy in a system, which is conserved, hence it gives off no energy. The mass dipole corresponds to the center of mass of a system, and its first derivative is the momentum, which is also a conserved quantity, so the mass dipole also emits no radiation. The mass quadrupole, however, can change in time, and is the lowest order contribution to gravitational radiation. The simplest example of a radiating system is a pair of of mass points, as in binary blackholes. Since the dipole moment is constant, we can place the coordinate origin between the two points. So the dipole moment is zero. If we scale the coordinates such as the points are a unit distance from the center, the quadupole moment will be

$$Q_{ij} = M(3x_ix_j - \delta_{ij}) \tag{11.2.5}$$

Now that we have laid out the bare skeletons of gravitational waves, we will dive into the trenches of tensor calculus and demonstrate the linearized Einstein tensor in its full glory. As before, we consider a metric that differs from the flat Minkowoski metric, η_{ab}, by a small perturbation, h_{ab}. Let $|\varepsilon| << 1$. Then

$$g_{ab} = \eta_{ab} + \varepsilon h_{ab}$$

, working to O(ε^2). Next, we compute the Christoffel symbol $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(g_{bc,d} + g_{cd,b} - g_{db,c})$. Let us begin the computation;

$$g_{bc,d} = \eta_{bc,d} + \varepsilon h_{bc,d} = \varepsilon h_{bc,d}$$

$$g_{cd,b} = \varepsilon h_{cd,b}$$

and

$$g_{db,c} = \varepsilon h_{db,c}$$

Next, we need to calculate g^{ab}, noting that $g_{ab}g^{bc} = \delta^c_a$. Then

$$(\eta_{ab} + \varepsilon h_{ab})(\eta^{bc} + k\varepsilon h^{bc}) = \delta^c_a$$

, where k is a constant to be determined. So

$$\eta_{ab}\eta^{bc} + \varepsilon h_{ab}\eta^{bc} + k\varepsilon\eta_{ab}h^{bc} + O(\varepsilon^2) = \delta^c_a$$

Hence

$$\delta_a^c + \varepsilon(h_{ab}\eta^{bc} + k\eta_{ab}h^{bc} = \delta_a^c$$

Therefore,

$$h_{ab}\eta^{bc} = -k\eta_{ab}h^{bc} \tag{11.2.6}$$

Note $k\eta_{ab}$h^{bc} = $k\eta_{ab}\eta^{be}\eta^{cf}h_{ef}$ = $k\delta_a^e\eta^{cf}h_{ef}$ = $k\eta^{cf}h_{af}$ = $k\eta^{cb}h_{ab}$ = $k\eta^{bc}h_{ab}$ = -$h_{ab}\eta^{bc}$(from above). Therefore k = - 1 and g^{ab} = η^{ab} - εh^{ab}. Returning to our Christoffel symbol

$$\Gamma^a_{bc} = \frac{\varepsilon}{2}(\eta^{ad} - \varepsilon h^{ad})(h_{bc,d} + h_{cd,b} - h_{db,c})$$

Computing to O(ε^2);

$$\Gamma^a_{bc} = \frac{\varepsilon}{2}\eta^{ad}(h_{bc,d} + h_{cd,b} - h_{db,c}) \tag{11.2.7}$$

Next, we calculate the Riemann tensor

$$R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$$

The last two terms contain ε^2, so we can ignore. Hence, our Riemann tensor looks like

$$R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc}$$

Plugging in our Christoffel symbols;

$$R^a_{bcd} = \partial_c[\frac{1}{2}\varepsilon\eta^{ae}(h_{bd,e} + h_{de,b} - h_{eb,d})] - \partial_d[\frac{1}{2}\varepsilon\eta^{af}(h_{bc,f} + h_{cf,b} - h_{fb,c})]$$

$$R^a_{bcd} = \frac{1}{2}\varepsilon(\eta^{ae}h_{bd,ce} + \eta^{ae}h_{de,cb} - \eta^{ae}h_{eb,cd}) - \frac{1}{2}\varepsilon(\eta^{af}h_{bc,df} + \eta^{af}h_{cf,db} - \eta^{af}h_{fb,dc})$$

Re-indexing $e \longrightarrow f$ in the first expression

$$R^a_{bcd} = \frac{1}{2}\varepsilon(\eta^{af}h_{bd,cf} + \eta^{af}h_{df,cb} - \eta^{af}h_{fb,cd}) - \frac{1}{2}\varepsilon(\eta^{af}h_{bc,df} + \eta^{af}h_{cf,db} - \eta^{af}h_{fb,dc})$$

$$R^a_{bcd} = \frac{1}{2}\varepsilon\eta^{af}(h_{bd,cf} + h_{df,cb} - h_{bc,df} - h_{cf,db})$$

Now

$$R_{ab} = R^c_{acb}, R^a_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} \Rightarrow R^a_b = \partial_c\Gamma^c_{bd} - \partial_d\Gamma^c_{bc}$$

Define the d'Alembertian operator, $\Box$

$$\Box = -\frac{\partial^2}{\partial t} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \eta^{ab}\partial_a\partial_b$$

Noting $h = \eta^{cd}h_{cd}$ and summing over a and c ;

$$R^a_{bad} = R_{bd} = \frac{1}{2}\varepsilon\eta^{af}(h_{bd,af} + h_{df,ab} - h_{ba,df} - h_{af,db})$$

Now we have ordinary derivatives, so they commute, and relabeling indices; $b \longrightarrow a$, $d \longrightarrow b$

$$R_{ab} = \frac{1}{2}\varepsilon\eta^{af}(h_{ab,af} + h_{bf,aa} - h_{aa,bf} - h_{af,ab})$$

$$R_{ab} = \frac{1}{2}\varepsilon(h^f_{b,af} + h^a_{b,aa} - h^f_{a,bf} - h_{,ab})$$

f, being a dummy index;

$$R_{ab} = \frac{1}{2}\varepsilon(h^c_{a,bc} + h^c_{b,ac} - \Box_{ab} - h_{,ab})$$

The Ricci tensor becomes

$$R_{ab} = \frac{1}{2}\varepsilon(h^c_{a,bc} + h^c_{b,ac} - \Box_{ab} - h_{ab})$$

And the Ricci scalar is

$$R = \varepsilon(h^{cd}_{,cd} - \Box)$$

Plugging into the Einstein tensor, we get

$$G_{ab} = \frac{1}{2}\varepsilon(h^c_{a,bc} + h^c_{b,ac} - \Box h_{ab} - h_{.ab} - \eta_{ab}h^{cd}_{,cd} + \eta_{ab}\Box h)$$

The Trace Reverse. Let ψ_{ab} = h_{ab} - $\frac{1}{2}\eta_{ab}$h and noting ψ_{ab} = $\eta_{ac}\psi^c_b$, h_{ab} = $\eta_{ac}h^c_b$ and $\eta_{ab}h$ = $\eta_{ac}h^c{}_b$ .. Substituting into ψ_{ab};

$$\eta_{ac}\psi^c_b = \eta_{ac}h^c_b - \frac{1}{2}\eta_{ac}\delta^c_b h$$

$$\Longrightarrow \psi^c_b = h^c_b - \frac{1}{2}\delta^c_b h$$

Setting $b = c$ and summing;

$$\psi^c_c = h^c_c - \frac{1}{2}\delta^c_c h$$

Setting ψ = ψ^c_c and h = h^c_c, noting δ^c_c = 4

$$\psi = h - \frac{1}{2}(4)h = -h$$

This is why ψ_{ab} is known as the trace reverse of h_{ab}. Substituting ψ and ψ_{ab} into G$_{ab}$ and using the trace reverse, we get

$$G_{ab} = \frac{1}{2}\varepsilon(\psi^c_{a,bc} + \psi^c_{b,ac} - \Box\psi_{ab} - \eta_{ab}\psi^{cd}_{,cd})$$

To obtain the wave equation, we perform a gauge transformation. This is a transfromation that leaves $R^a{}_{bcd}$ and R_{ab} and R unchanged. Define the coordinate transformation $x^{a'} = x^a + \varepsilon\phi^a$, where $\phi(a)$ is a function of position and $|\phi^a{}_{,b}| << 1$. It can be seen that this coordinate transfomation changes h_{ab} as

$$h'_{ab} = h_{ab} - \phi_{a,b} - \phi_{b,a}$$

And, the derivative ψ_{ab} changes as

$$\psi'^a_{b,a} = \psi^a_{b,a} - \Box\phi$$

We are free to choose ϕ as long as the Riemann tensor retains the same form. We demand that $\Box\phi = \Box\psi^b{}_{a,b} \Longrightarrow \psi'^a_{b,a} = 0$. Substitute this into Einstein tensor G_{ab} and into the full field equation, we get

$$\frac{1}{2}\varepsilon\,\Box\,\psi_{ab} = -\kappa T_{ab} \tag{11.2.8}$$

In a vacuum , $\Box\psi_{ab} = 0$. The d'Alembertian operator is nothing more than the wave equation for waves traveling at the speed of light. The choice of gauge is known as de Donder or Einstein gauge. Hence,

$$\Box\psi_{ab} = \Box(h_{ab} - \frac{1}{2}\eta_{ab}h) = \Box h_{ab} - \frac{1}{2}\eta_{ab}\,\Box\,h = 0$$

Recalling ψ = -$h$and multiplying above by η^{ab}

$$\eta^{ab}\,\Box\,\psi_{ab} = \Box(\eta^{ab}\psi_{ab}) = \Box(\psi^b_b) = \Box\psi = -\,\Box\,h = 0$$

Therefore,

$$\Box\,h_{ab} = 0 \tag{11.2.9}$$

Therefore the study of gravitational waves reduces to the study of $\Box h_{ab} = 0$. Two Nobel prizes were awarded for observations on gravitational waves. The first for indirect observations of gravitational waves of Taylor-Hulse neutron star binary system. Einstein predicted that the orbiting period should decrease, the stars speed up due to loss of gravitational energy. This was confirmed by measuring the beam of light emitted by the binary system. In 2017, this year, the Nobel prize was awarded for direct observation of gravitational waves, the LIGO experiment, by measuring the deflection of a laser beam. For this reason, I will spend extra time summarizing gravitational waves, as presented above.

Important definitions and concepts.

Hypersurface. A hypersurface Σ is an n-dimensional subspace (submanifold) of a $n+1$ dimensional manifold M, $\Sigma \subset$ M. There are two ways of describing hypersurfaces; by

(1) Embedding $\Phi : \Sigma \hookrightarrow$ M ; if Σ has coordinates y^a and M coordinates x^a, then Φ: $x^\alpha = x^\alpha(y^\alpha)$,

(2) Subspace of M ; $\Sigma \subset$ M, such that $\Sigma = \{x \in$ M; S(x) = 0$\}$, for real valued function S on M.

Example. Standard 2-sphere S^2 of radius r_0 in $\mathbb{R}^3$. In the first prescription, $x^\alpha = (x^1, x^2, x^3)$ $\in \mathbb{R}^3$ and $y^\alpha = (\theta,\phi) \in S^2$. Then $x^\alpha(y^\alpha)$: $x^1(\theta,\phi) = r_0 \sin\theta \cos\phi$, $x^1(\theta,\phi) = r_0 \sin\theta \sin\phi$ and $x^3(\theta,\phi)$ $= r_0 \cos\theta$. In the second prescription $S(x^\alpha) = (x^1)^2 + (x^2)^2 + (x^3)^2 - (r_0)^2 = 0$

Retarded Time - Minkowski Space Time- A photon emitted at time $t = t_0$ reaches an observer located at a distance $r \geq 0$, from the source at time $t = t_1$. Then

$$t = t_0 + \frac{r}{c} = t_0 + r$$

, $c = 1$. The time $t_0 = t - r$ is defined as the retarded time. Setting $u = t - r = k$, a constant is a hypersurface, the future directed null cone with vertex $r = 0$, t = k.

11.3 Spherical Harmonics and Blackhole Perturbation Theory

The mathematical techniques of spherical harmonics allow us to study blackhole perturbation theory in spherically symmetric spacetime,[16], [36]. A blackhole's horizon is sphere-like and well adapted to spherical coordinates. As these are isolated systems in general relativity, the exact boundary conditions are imposed at infinity, which requires a compactification of space, which is achieved with the compactification of the radial coordinate only. Spherical coordinates can simplify the Poisson-like equations. However, there is the issue of coordinate singularities. The transformation from spherical (r, θ, ϕ) to Cartesian coordinates (x, y, z) is obtained by x = $r$sinθcosϕ, y = $r$sinθsinϕ and z = $r$cosθ. Note r = 0 if and only if $x = y = z$ =0 is singular in spherical coordinates because neither θ or ϕ can be uniquely defined. The same happens for the z-axis, where θ = 0 or π, and ϕ cannot be defined. Let us analyze the Laplace operator in spherical coordinates

$$\nabla = \frac{\partial^2}{\partial^2 r} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(\frac{\partial^2}{\partial \theta^2} + \frac{1}{tan\theta}\frac{\partial}{\partial \theta} + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial \phi^2})$$

r = sinθ = 0 gives singularities on the z-axis. This is not the case in Cartesian coordinates. An analytic function in Cartesian coordinates looks like

$$f(x,y,z) = \sum_{n,p,q} a_{npq} x^n y^p z^q$$

Substituting for spherical coordinates, and re-arranging in terms of ϕ, we get

$$f(r,\theta,\phi) = \sum_{m,p,q} b_{mpq} r^{|m|+2p+q} sin^{|m|+2p}\theta cos^q \theta e^{im\phi}$$

Setting $l = |m| + 2p + q$ and with transformations of trigonometric functions in θ, we can express the angular parts in terms of the spherical harmonics $\mathrm{Y}_l^m(\theta,\phi)$

$$Y_l^m(\theta,\phi) = \sqrt{(\frac{2l+1(l-m)!}{4\pi(l+m)!}} P_l^m(cos\theta)e^{im\phi} \tag{11.3.1}$$

, where $l \geq 0$, $|m| \leq 1$. $P_l^m(\cos\theta)$ are the associated Legendre functions defined by

$$P_l^m(x) = \frac{(l+m)!}{(l-m)!}\frac{1}{2^l l!}\frac{1}{\sqrt{(1-x^2)^m}}\frac{d^{l-m}}{dx^{l-m}}(1-x^2)^l$$

, $m \geq 0$, and

$$P_l^{-m}(x) = \frac{(l-m)!}{(l+m)1} P_l^m(x), m < 0$$

For a given couple (l, m), we obtain the following regularity conditions for $l \geq 2$ and $m \geq 2$. By Taylor expansion;

(1) near $\theta = 0$, $f(\theta) \sim \sin^{|m|}\theta$ and

(2) near r= 0, $f(r) \sim r^l$. This family of functions have three very important properties;

(a) They represent an orthogonal set of regular functions defined on the sphere. Any regular scalar field $f(\theta,\phi)$ defined on the sphere can be decomposed into spherical harmonics;

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-1}^{m=1} f_{lm} Y_l^m(\theta,\phi)$$

(b) Since the harmonics are regular, they take care of the coordinate singularity on the z-axis and

(c) They are eigenfunctions of the angular part of the Laplace operator

$$\forall (l,m) \nabla_{\phi\theta} Y_l^m(\theta,\phi) = -l(l+1) Y_l^m(\theta,\phi),$$

the associated eigenvalues being -$l(l+1)$.

Tensorial Components. For vector or tensor fields, a vector basis, triad, must be specified to express the components. The choice of basis is independent of choice of coordinates. We have 2 options:

(1) The Cartesian triad, (∂_x,∂_y,∂_z) and

(2) Orthonormal spherical basis ($\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{rsin\theta}\frac{\partial}{\partial\phi}$). Let us analyze the pros and cons of these options with an example.

Consider the gradient V^i = V$^i\phi$ of the scalar field $\phi = x$. The gradient in Cartesian coordinates is a regular vector field

$$V^x = 1, V^y = 0$$

and

$$V^z = 0$$

The spherical components of V are

$$V_r = sin\theta cos\phi$$

$$V^{\theta} = cos\theta cos\phi$$

$$V^{\phi} = -sin\phi$$

We have several problems with spherical coordinates, which are not good for a scalar field.;

(a) all components are multi-defined at the origin

(b) V$^{\theta}$ and V$^{\phi}$ are multi-defined on the z-axis and

(c) If V$^{\theta}$ is set to zero; the square of the norm is multi-defined . We begin by considering a small perturbation, h_{ab}, of the Schwarzschild geometry. Then;

$$g_{ab} = g^S_{ab} + h_{ab}$$

where

$$g^S_{ab} dx^a dx^b = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2)$$

Plugging $g_{ab} = g^S_{ab} + h_{ab}$ into the field equation with T$_{ab} = 0$ and applying Schwarzschild metric and R$_{ab}$ - $\frac{1}{2} g_{ab}$R = 0 and keeping only terms linear in h_{ab} , we obtain,

$$\mathscr{E}_{ab} = -\frac{1}{2}\nabla^c\nabla_c h_{ab} - \frac{1}{2}\nabla_a\nabla_b h^c_c + \nabla^c\nabla_{(a}h_{b)c} + \frac{1}{2}g_{ab}(\nabla^c\nabla_c h^d_d - \nabla^c\nabla^d h_{cd}) = 0$$

where ∇_a is the derivative operator compatible with the background geometry. The indices are raised and lowered with the background metric. Next, we need to fix our gauge, our equation is invariant under the transformation

$$h_{ab} \longrightarrow h_{ab} + L_{\xi} g_{ab} = h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a$$

where ξ_a is an arbitrary vector and L_{ξ} is our Lie derivative. Next, we decompose the components of the metric perturbation into s, v, t; scalar, vector and tensor harmonics respectively;

$$h_{ab} = \begin{pmatrix} s_1 & s_2 & v_1 & v_1 \\ s_2 & s_3 & v_2 & v_2 \\ v_1 & v_2 & t + s_4 & t \\ v_1 & v_2 & t & t - s_4 \end{pmatrix}$$

Consider the metric of the 2-sphere

$$\gamma_{ab} dx^A dx^B = d\theta^2 + sin^2\theta d\phi^2$$

Since the scalar harmonics, Y_{lm}, define a complete set of functions on the 2-sphere, we can use them to construct two types of vectors;

(1) Even parity, $\nabla_A \mathrm{Y}_{lm}$, where ∇_{lm} is compatible with γ_{AB} and

(2) Odd parity, $\gamma_{BC} \varepsilon^{AB} \nabla_A \mathrm{Y}_{lm}$, where ε^{AB} is the Levi-Civita symbol. To define a tensor harmonic, we take the derivatives of (1) and (2)

(3) Even parity tensors, $\nabla_A \nabla_B \mathrm{Y}_{lm}$

(4) Odd parity tensors, $\gamma_{AC} \varepsilon^{CD} \nabla_D \nabla_B \mathrm{Y}_{lm}$. Under a parity transformation $\theta \longrightarrow \pi$ - θ and $\phi \longrightarrow \pi$ + ϕ , even parity picks up a minus sign according to (-1)^{l}and odd parity picks up a minus sign according to (-1)$^{l+1}$. The two sectors of the metric perturbation are

$$h_{ab}^{odd} = \begin{pmatrix} 0 & 0 & v_1 & v_1 \\ 0 & 0 & v_2 & v_2 \\ v_1 & v_2 & t & t \\ v_1 & v_2 & t & t \end{pmatrix}$$

, all terms odd.

$$h_{ab}^{even} = \begin{pmatrix} s_1 & s_2 & v_1 & v_1 \\ s_2 & s_3 & v_2 & v_2 \\ v_1 & v_2 & t+s_4 & t \\ v_1 & v_2 & t & t-s_4 \end{pmatrix}$$

, all terms even, except s_1 , s_2 , s_3 and s_4. Taking the odd-parity sector , the Regge-Wheeler gauge vector is

$$\xi^a = (0, 0, \Lambda \varepsilon^{AB} \nabla_B Y_{lm})$$

Λ is chosen such that the odd parity part of the metric perturbation is

$$h_{ab}^{odd} = \begin{pmatrix} 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & 0 \\ v_1 & v_2 & 0 & 0 \end{pmatrix}$$

, all terms odd. The Regge Wheeler Zerrili equation is

$$-\frac{\partial^2 \phi_{l.m}^{o,e}}{\partial t^2} + \frac{\partial^2 \phi_{lm}^{o,e}}{\partial t^{*2}} - V_l^{o,e}(r)\phi_{l,m}^{o,e} = 0$$

, where $r^* = r + ln(\frac{r}{2M} - 1)$, pushes the horizon to infinity.

Regge-Wheeler Approach to the Stability of the Schwarzschild Metric. Our usual Schwarzschild Metric

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) = g_{\mu\nu}dx^\mu dx^\nu$$

If the metric is perturbed, we have two possible states;

(1) Stability - undamped oscillations about the equilibrium state represented by the

Schwarzschild background

(2) Instability - oscillations grow exponentially with time.

For perturbations; Regge and Wheeler have given the normal modes into which any arbitrary perturbation on a spherically symmetric background can be decomposed. These modes can be expressed in the form of products of four factors each of which is a function of t, r, θ and ϕ. This is achieved by the use of generalized tensor spherical coordinates. These modes are associated with angular momentum ,l, and its projection on the z-axis of the mass M. For any given value of l, there are two independent classes of perturbation characterized by their parities; (-1)^l - even parity and (-1)$^{l+1}$ - odd parity. Then a perturbation matrix is generated by suitable gauge transformations. The time dependence of the perturbation is given by e^{-ikt}. Why ? We will illustrate with a hydrodynamic analogy - an oscillating charged liquid drop. Assume we have a spherical liquid drop of incompressible fluid of radius R, mass M, mass density ρ, charge density ρ_e and total charge q. Let us analyze the surface vibrations, ignoring gravity. The variables are

(1) restoring force - surface tension,

(2) Disruptive - Coulomb repulsion, ϕ_e.

In the equilibrium state, the Coulomb potential ϕ_e is related to the charge density ρ_e by Poisson's equation

$$\nabla^2\phi_e = -4\pi\rho_e, \tag{11.3.2}$$

which has solution $\phi_e = \frac{q}{r}$, $r \geq R$ and $\phi_e = 2\pi\rho_e(R^2 - \frac{1}{3}r^2)$, $r \leq R$. Note this equation is independent of time. We will now perturb the drop from the equilibrium position, so that the surface of the drop varies slightly from the sphere

$$r_s = R + \varepsilon\xi(\theta, \phi, t)$$

This is the process of linearizing, ε is a very small constant, to order O(ε). The perturbed Poisson Equation is,

$$\nabla^2\phi_e^{'} = -4\pi\rho_e^{'}$$

, $\phi_e^{'}$ and $\rho_e^{'}$ are the perturbed Coulomb potential and charge density. Now, we also have a velocity potential ϕ_v at every point on the surface. As the variables describing the initial equilibrium state are independent of time, the time dependence of the perturbations can be expressed as

$$exp(-i\omega t)$$

, where ω is the angular frequency of a particular mode of oscillation. Because of spherical symmetry, we can assume that

$$\xi = e^{-i\omega t}Y_{lm}(\theta\phi)$$

, where $Y_{ml}(\theta\phi)$ is the spherical harmonic of angular momentum l, with projection on z-axis given by m. The equation governing the velocity potential ϕ_v is

$$\nabla^2\phi_v = 0$$

, whose solution is

$$\phi_v = Ce^{-i\omega t}r^lY_{lm}$$

where C is determined by equating the normal gradient of ϕ_v to the time derivative of the displacement ξ at $r = R$. Next we return to the Schwarzschild metric.

Perturbing the Field Equation. Let the background Schwarzschild metric be $g_{\mu\nu}$ and the superimposed perturbation by $h_{\mu\nu}$. The Einstein field equation for the Schwarzschild exterior metric is given by

$$R_{\mu\nu}(g) = 0$$

, $g = g_{\mu\nu}$ is the background Schwarzschild metric. For the perturbed spacetime

$$R_{\mu\nu}(g+h) = 0$$

, the total metric is $g_{\mu\nu} + h_{\mu\nu}$. We are assuming the perturbed spacetime is still empty. Expanding to first order of h

$$R_{\mu\nu}(g) + \delta h_{\mu\nu}(g) = 0$$

Since $\mathrm{R}_{\mu\nu}(h) = 0$, $\delta \mathrm{R}_{\mu\nu}(h) = 0$, where

$$\delta R_{\mu\nu} = -\delta\Gamma^{\beta}_{\mu\nu;\beta} + \delta\Gamma^{\beta}_{\mu\beta;\nu}$$

Boundary Conditions. The two boundaries are spatial infinity and $r = 2m$. We know that the Schwarzschild metric is asymptotically flat. Since the blackhole metric contains a singularity at $r = 2m$, the behaviorof the perturbation is liable to be unphysical.We must then employ a coordinate system which is singularity free at $r = 2m$. Hence, we employ Krukal coordinates.

Perturbation Analysis in Kruskal Coordinates. Let us use coordinates u and v instead of r and t. The metric takes the form

$$ds^2 = -(1 - 2\frac{m}{r})dudv + r^2 d\Omega^2$$

, where $r = r(u,v)$, where u and v are the Eddington-Fingelstein coordinates (v,r,θ,ϕ) or (u,r,θ,ϕ), $u = t - r^*$ is the retarded time coordinate ,$v = t + r^*$ is the advanced time coordinate, where $r^* = r$ + $2mlog|\frac{r}{2m-1}|$is a solution of $\frac{dr^*}{dr}$ = $f(r)^{-1}$. Since the metric does not explicitly depend on t, dt can be substituted using

$$dt = dv - dr^* = dv - \frac{dr}{f(r)}$$

, like wise for u. With these coordinates, the horizon is still infintely far away, noting $2r^* = v - u$, so we introduce new coordinates

$$U = -e^{-u/4m} \tag{11.3.3}$$

and

$$V = e^{v/4m} \tag{11.3.4}$$

so the horizon is at U = 0 or V = 0.

A Brief Historical Tour of Spherical Harmonics. Laplace in 1782 determined that the gravitational potential at a point x associated to a set of point masses m_i located at x_i is given by

$$V(x) = \sum_i \frac{m_i}{|x - x_i|}$$

Legendre then showed the expansion of the Newtonian potential in powers of $r = |x|$ and$r_1 = |x_1|$. He showed for r$\leq r_l$

$$\frac{1}{|x - x_1|} = \sum_{l=0}^{\infty} P_l(cos\gamma)\frac{r^l}{r_1^{l+1}}$$

, where γ is the angle between the vectors x and x_1. The functions P_l are the Legendre polynomials, a special case of spherical harmonics. Lord Kelvin in 1867 introduced solid spherical harmonics which are homogeneous polynomial solutions of Laplace's equation

$$\nabla^2 u = 0$$

In simple terms, spherical harmonics represent the fundamental modes of vibrations of a sphere. Compare this to a Fourier series,which represents the fundamental modes of vibration of a string. Also compare to the eigenfunctions of the square of the orbital angular momentum operator -$\imath\hbar$r$\times\nabla$, which represent the different quantized configuration of atomic orbits. Spherical harmonics are defined as the eigenfunctions of the angular part of the Laplacian in three dimensions, the Laplacian on the sphere. In spherical coordinates, the Laplacian is

$$\nabla^2 = \frac{1}{r^2 sin^{2\theta}}(\frac{\partial}{\partial r} r^2 sin\theta \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} sin\theta \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi} csc\theta \frac{\partial}{\partial \phi})$$

The Laplacian $\nabla^2 f = 0$ can be solved via separation of variables. We make the ansatz below to separate the radial and angular parts of the solution

$$f(r,\theta,\phi) = R(r)Y(\theta,\phi)$$

We obtain two eigenvalue functions

$$l(l+1)R(r) = \frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r} R(r))$$

and

$$-l(l+1)Y(\theta,\phi) = \frac{1}{sin\theta}\frac{\partial}{\partial \theta}(sin\theta \frac{\partial Y(\theta,\phi)}{\partial \theta} + \frac{1}{sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial \phi^2}$$

, where $l(l+1)$ is the separation constant. Multiplying the first equation on both sides by Y(θ,ϕ) and the second equation by R(r) on both sides, and adding them up, yields the original solution to Laplace's equation in spherical coordinates. The separation constant reflects the two eigenvalue functions of different signs. The angular equation can also be solved by separation of variables. We make the ansatz

$$Y(\theta,\phi) = \Theta(\theta) exp(im\phi)$$

, based upon analogy to two dimensional angular Laplacian, where m is a separation constant, which can take on negative values. We arrive at

$$sin\theta \frac{\partial}{\partial \theta}(sin\theta \frac{\partial \Theta(\theta)}{\partial \theta}) = m^2 \Theta(\theta) - l(l+1) sin^2\theta \Theta(\theta)$$

The solution for $\Theta(\theta)$ can be found by putting the equation into a canonical form, the solutions of which are given in terms of the Legendre polynomial

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\frac{d^{l+m}}{dx^{l=M}}(x^2-1)^l$$

, $l \geq 0$, |m| $\leq$ l. The general solution for each linearly independent Y(θ,ϕ) are the spherical harmonics,with

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}}P_l^m(cos\theta)exp(im\phi)$$

Every spherical harmonic is labelled by the integer l and m , the order and degree of a solution. The following low lying spherical harmonics; Y_0^0(θ,ϕ) is spherically symmetric, monopole moment of a function on the sphere and Y_1^m(θ,ϕ) are axially symmetric, represent the dipole moment. Armed with all the preliminaries, we will now complete our task of perturbing Schwarzschild metric.

Perturbation of Schwarzschild Metric. We begin with our metric $g^S{}_{\mu\nu}$

$$ds^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

, where f= 1 - $\frac{2M}{r}$. Our perturbations is

$$g_{\mu\nu} = g^S_{\mu\nu} + h_{\mu\nu}, h_{\mu\nu} << g^S_{\mu\nu}$$

Which is effected by a perturbation of the energy-momentum tensor of the blackhole,

$$T_{\mu\nu} = T^S_{\mu\nu} + \delta T_{\mu\nu}$$

Then, we have

$$\delta G_{\mu\sigma} = \delta(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = -8\pi G\delta T_{\mu\nu}$$

We are looking at a solution outside the blackhole, where $T^S_{\mu\nu} = 0$. The question we ask; what is $h_{\mu\nu}$ for $\delta T_{\mu\nu}$? Next, we deploy multipole expansion and execute the following algorithm. Decompose $h_{\mu\nu}$ by tensorial spherical harmonics into even and odd parity modes. The even parity mode gives us the Zerrili-Moncrief function $\psi_{ZM}(r,t)$ and the odd parity mode gives us the Regge-Wheeler function $\psi_{RW}(r,t)$. Both solutions give us a total solution $\psi(r,t)$. Because of spherical symmetry, we drop the odd parity solution. The two dimensional spherical space S^2 is denoted by $\theta^A = (\theta,\phi)$. Orthogonal to this, we have the two dimensional Lorentzian space

$$x^a = (x^0, x^1) = (t, r)$$

Hence, the line element of the Schwarzschild metric is

$$ds^2 = g_{ab}dx^a dx^b + r^2\Omega_{AB}d\theta^A d\theta^B,$$

where $\Omega_{AB}d\theta^A d\theta^B = d\theta^2 + \sin^2\theta d\phi^2$. The covariant derivative, symbolized as $l$is calculated from Ω_{AB} and not g_{ab}. It can be shown that $\Omega_{AB|C} = 0$. Using these properties and symmetry of the perturbation tensor, $h_{\mu\nu}$ can be split as follows

$$h_{\mu\nu} = \begin{pmatrix} h_{ab} & h_{aA} \\ h_{aA} & h_{AB} \end{pmatrix}$$

Remarks

(1) h_{ab} behaves as a scalar - no angular dependence of indices

(2) h_{aA} behaves as a vector - angular dependence of one of the indices and

(3) h_{AB}behaves as a tensor - angular dependence of both indices

Next, we expand all three parts in spherical harmonics. Let us first look at the scalar spherical harmonic function Y^{lm}, a standard function defined as

$$Y^{lm}_{\theta\phi} = Ne^{im\phi}P^{lm}(cos(\theta))$$

l is called the degree and, m, the order of the function, both integers that satisfy $l \geq 1$ and $|m| \leq$ l. N is the constant normalization factor, a function of l and m. P^{lm} is associated with Legendre functions. It is known that $l = 0$, and $l = 1$ modes are non-radiative.Why ? Energy conservation makes monopole radiation impossible, while conservation of momentum makes dipole radiation impossible. A changing dipole moment corresponds to the motion of the center of density in the case of gravitation. An isolated system will never change its center of density because of conservationof momentum.In electromagnetism, dipole radiation is possible because because negative and positive charges exist, wheres in gravitation, we only have positive mass. Because of axial symmetry, in the problem of radial infall, only m = 0 modes are excited. Quadrupole radiation is the lowest possible radiation mode for gravitational waves. The quadrupole moment measures the shape of the system, and there is no conservation law that forbids a change in shape of an isolated system, for instance, a particle falling into a blackhole.

The Quadrupole Formula. A solution to the wave equation can be found by integrating over the source. In electromagnetism, the vector potential A^{μ}, can be expressed as an integral over the source, the current, J^{μ}. Similarly, in general relativity, the wave tensor $h_{\mu\nu}$ may be expressed as an integral over the stress-energy tensor $T_{\mu\nu}$

$$h_{\mu\nu}(t,\vec{x}) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(\vec{x'},t-|\vec{x}-\vec{x'}|/c)}{|\vec{x}-\vec{x'}|}d^3x' \tag{11.3.5}$$

Let $I^{jk} = \int d^3x\rho(t,\vec{x})x^j x^k$ and the reduced quadrupole moment tensor be $\Im^{jk} = I^{jk} - \frac{1}{3}\delta^{jk}\delta_{lm}I^{lm}$. Not all sources nned to be treated relativistically. If they are slow motion, then the expressioncan be treated in the weak field limit and reduces to

$$h^{TT}_{jk} = \frac{2G}{c^4}\frac{1}{r}\Im^{TT}_{jk}(t-\frac{r}{c})$$

Einstein solved the quadrupole formula for gravitational radiation by solving the linearized field equations with a source term

$$\Box h_{\mu\nu}(t,\vec{x}) = -\kappa T_{\mu\nu}(t,\vec{x})$$

Illustration - The Two-Body Problem. The quadrupole formula can be used for any system as long as we can compute I_{jk}. Let us consider the circular binary system - two stars , point masses m_1 and m_2 on the $x-y$ plane:

$$x_1^i = r(\theta)\frac{\mu}{m_1}.(cos\theta, sin\theta, 0)$$

$$x_2^i = r(\theta)\frac{\mu}{m_2}.(-cos\theta, -sin\theta, 0)$$

where, θ = anomaly, angular position of star with orbit, which changes with time, μ is the reduced mass $\frac{m_1 m_2}{m_1+m_2}$, $r(\theta)$ is the radius of the orbit defined in terms of the semi-major axis, a, and eccentricity e ; $r(\theta) = \frac{a(1-e^2)}{1+ecos\theta}$, a = semi-major axis. For circular orbits; $\theta = \omega$t = 2πft = $2\pi\frac{t}{T}$ and $e = 0$, so $r(\theta) = a$ = a constant. The mass density is expressed in terms of the delta function

$$\rho = \delta(z)[m_1\delta(x-x_1)\delta(y-y_1) + m_2\delta(x-x_2)\delta(y-y_2)]$$

We now calculate the components of the quadrupole tensor

$$I^{xx} = \int d^3x\rho(x^2) = m_1x_1^2 + m_2x_2^2$$

$$= (\frac{\mu^2a^2}{m_1^2}m_1 + \frac{\mu^2a^2}{m_1^2}m_2)cos^2(\omega t)$$

$$= \mu^2a^2(\frac{1}{m_1} + \frac{1}{m_2})cos^2(\omega t)$$

$$= \mu a^2 cos^2(\omega t)$$

$$= \frac{1}{2}\mu a^2(1 + cos(2\omega t)$$

Note from cos(2ωt) - the gravitational wave frequency in a circular binary is twice the orbital frequency. That is, for each cycle made by the binary motion, the gravitational wave signal goes through two full cycles. There are two maxima and two minima per orbit. For this reason, gravitational waves are called quadrupolar waves. The other components of the quadrupole tensor are

$$I^{yy} = \mu a^2 sin^2 \omega t = \frac{1}{2}\mu a^2(1 - cos(2\omega t)$$

$$I^{xy} = I^{yx} = \mu a^2 cos(\omega t) sin(\omega t) = \frac{1}{2}\mu a^2 sin(2\omega t)$$

Next, the trace subtraction

$$\frac{1}{3}\delta^{ij}\delta_{lm}I^{lm} = \frac{1}{3}\delta^{ij}\mu a^2[\frac{1}{2}(1 + cos(2\omega t) + \frac{1}{2}(1 - cos(2\omega t)]$$

$$= \frac{1}{3}\delta^{ij}\mu a^2$$

Hence, we can write down the components of $\mathfrak{T}^{ij}$

$$\mathfrak{T}^{ij} = \frac{1}{2}\mu a^2 \begin{pmatrix} cos(2\omega t) + \frac{1}{3} & sin(2\omega t) & 0 \\ sin(2\omega t) & -cos(2\omega t) + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

$\ddot{\mathfrak{T}}$ can easily be calculated.

CHAPTER XII

ADM FORMALISM AND NUMERICAL RELATIVITY

12.1 Introduction to ADM Formalism

First, we apply the $3+1$ (space and time) decomposition of space time, [8],[11]. Realize that the Einstein field equation consists of 10 partial differential equations (elliptic and hyperbolic) for 10 metric components and they are not easy to solve. Direct numerical integration of the Einstein equation is the most robust way of finding solutions. The formulation we discuss is the Arnowitt, Deser and Misner, or ADM , system. This formulation divides the Einstein equations into (1) Constraint Equations, and (2) Evolution Equations. Before embarking on ADM formalism, we will take a brief tour of partial differential equations, PDEs.

The Cauchy problem. For a PDE defined on $\mathbb{R}^n$ and a smooth manifold S $\subset \mathbb{R}^n$ of dimension $n-1$, the Cauchy problem consists of finding a solution u of the differential equation of order m that satisfies

$$u(x) = f_0(x)$$

, for all $x \in$ S

$$\frac{\partial^k u(x)}{\partial x^k} = f_k(x)$$

, for $k = 1, 2, ..., m$ and al l $x \in$ S, where f_k are given functions defined on the surface S, known as Cauchy data. S is known as a Cauchy surface and n is a normal vector to S.

Theorem. Cauchy-Kowalevski Theorem . Cauchy problems have unique solutions under certain conditions; (a) Cauchy data are real analytic functions and (b) The coefficients of the

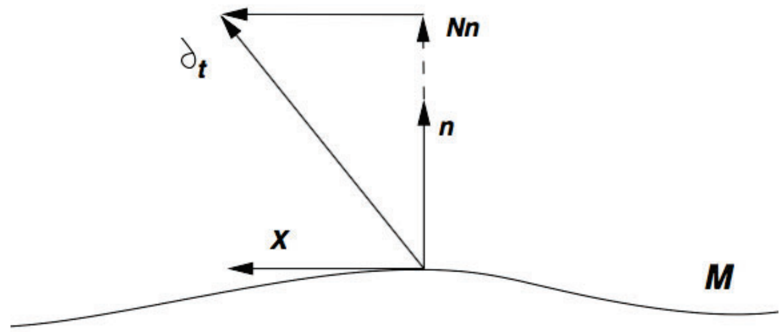

Figure 12.1: Time evolution of space-time; α or N is the lapse function or β^i or $N^t$is shift vector

PDEs are real analytic functions. However, the existence and uniqueness problem is not settled. Pathological behavior can arise.

Pathology. An example of pathology is the sequence of Cauchy problems for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

, with boundary conditions $u(x,0) = 0$ and $\frac{\partial u}{\partial y}(x,0) = \frac{sin(nx)}{n}$, n an integer. The derivative of u with respect to y approaches 0 uniformly in x as n increases, but the solution is

$$u(x,y) = \frac{sinh(ny)sin(nx)}{n^2}$$

The solution approaches infinity if nx is not an integer multiple of π. The Cauchy problem for the Laplacian equation is said to be not well posed. The ADM formulation gives the fundamental idea of time evolution of space and time; such as foliations of 3-dimensional hypersurface, Σ. See Figure 12.1 below.

The metric is expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

, where $\alpha \equiv 1/\sqrt{-g^{00}}$, is the lapse function and $\beta_j \equiv g_{0j}$, is the shift vector. The projection operator or intrinsic 3-metric g_{ij} is defined as

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

, where $n_\mu = (-\alpha, 0, 0, 0)$, $n^\mu = g^{\mu\nu} n_\nu$ = (1/α,-β^i/α) is the unit normal vector of the hypersurface Σ.

Hamiltonian Field Theory. Next, we will introduce Hamiltonian field theory. The Hamiltonian density is the continuous analogue for fields; it is a function of the fields, the conjugate momentum fields, and the space and time coordinates. For a one scalar field, $\phi(x,t)$, the Hamiltonian density is defined from the Lagrangian density by

$$\mathscr{H}(\phi, \pi, x, t) = \dot{\phi}\pi - \mathscr{L}(\phi, \nabla\phi, \partial\phi/\partial t, x, t)$$

,where x is the position vector, $\phi(x,t)$ has a conjugate momentum field , $\pi(x,t) = \frac{\partial \mathscr{L}}{\partial \dot{\phi}}$, $\dot{\phi} = \frac{\partial \phi}{\partial t}$. See appendix A.

The Action. The time integral of the Lagrangian is called the action denoted by S. In field theory, a distinction is made between

(1) The Lagrangian, L, of which the time integral is the action

$$S = \int L dt$$

(2) The Lagrangian density, which one integrates over all spacetime to get the action

$$S = \int \mathscr{L}(\phi, \nabla\phi, \partial\phi/\partial t, x, t) d^3x dt$$

The spatial volume integral of the Lagrangian density is the Lagrangian, in 3d

$$L = \int \mathscr{L} d^3x$$

12.2 Extrinsic Geometry and Codazzi - Gauss Equations

The intrinsic geometry of space is described by the metric and its derivatives, connections and Riemann curvature tensor,[8],[13] ,[31]. However, the hypersurface Σ, is embedded in the ambient space M. How Σ bends inside M can only be captured by the extrinsic geometry of Σ. We define the extrinsic curvature of Σ in M by

$$K_{\alpha\beta} = \frac{1}{2} h_{\alpha}^{\gamma} h_{\beta}^{\delta} L_N g_{\gamma\delta} = h_{\alpha}^{\gamma} h_{\beta}^{\delta} \nabla_{\gamma} N_{\delta} \tag{12.2.1}$$

where $L_N g_{\alpha\beta}$ is the Lie derivative of the metric$g_{\alpha\beta}$ along the normal direction N. See appendix D

$$L_N g_{\alpha\beta} = \nabla_{\alpha} N_{\beta} + \nabla_{\beta} N_{\alpha}$$

and $h^{\alpha}{}_{\beta}$ = $\delta^{\alpha}{}_{\beta}$ - $\varepsilon N^{\alpha} N_{\beta}$;$N^{\alpha}$ is the normalized normal vector, $\varepsilon = \pm 1$. With Σ being the spacelike or timelike hypersurface Σ, we can construct the induced metric from the metric $g_{\alpha\beta}$ on the ambient space M. With $N^{\alpha} N_{\alpha} = \varepsilon = \pm 1$, we construct the tensor h$_{\alpha\beta}$ defined on or in a neighbourhood of Σ by

$$h_{\alpha\beta} = g_{\alpha\beta} - \varepsilon N_{\alpha} N_{\beta}$$

The tensor has the following properties:

(1) It is orthogonal to N^{α}; $N^{\alpha} h_{\alpha\beta} = 0$,$h_{\alpha\beta} N^{\beta} = 0$, very easy to show,

(2) For vectors V^{α} orthogonal to N^{α}, i.e. tangential to Σ, the scalar product with respect to $h_{\alpha\beta}$ is identical to that with respect to $g_{\alpha\beta}$

$$V^{\beta} N_{\beta} = 0 \Rightarrow h_{\alpha\beta} V^{\beta} = g_{\alpha\beta} V^{\beta}$$

Properties (1) and (2) imply that h$_{\alpha\beta}$ restricted to Σ is the metric induced on Σ by $g_{\alpha\beta}$. The induced metric, $h_{\alpha\beta}$, is known as the First Fundamental Form of Σ. The extrinsic curvature tensor, $K_{\alpha\beta}$, is known as the second fundamental form of Σ.

Gauss and Codazzi Equations for a Hypersurface. Given vectors X and Y on $\mathbb{R}^n$, the Euclidean covariant derivative is the same as the directional derivative

$$D_X Y = X(Y)$$

Euclidean space has zero Riemann curvature tensor

$$Rm(X,Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z = 0$$

This boils down to

$$[X,Y](Z^i) = X(Y(Z^i) - Y(X(Z^i)$$

A subset $\mathrm{M}^k \subset \mathbb{R}^n$ is an embedded submanifold if for each $p \in \mathrm{M}^k$, there exists a neighbourhood U of p in $\mathbb{R}^n$ and a diffeomorphism ϕ: U $\longrightarrow \phi$(U) onto an open subset of $\mathbb{R}^n$ such that ϕ(M $\cap$ U) is the intersection of a k-dimensional plane with ϕ(U). If$k = n-1$, then M^{n-1} is an embedded hypersurface. Let $\mathrm{M}^n \subset \mathbb{R}^{n+1}$ be an embedded hypersurface.

First Fundamental Form. The induced Riemannian metric I on M, also called the first fundamental form is defined as

$$I(U,V) \doteqdot < U,V >_{R^{n+1}}$$

for $U,V \in \mathrm{T}_p\mathrm{M}$, where p $\in$ M. This is the restriction of the Euclidean inner product to the tangent spaces T_pM. We identify T_pM with an n-dimensional subspace of $\mathbb{R}^{n+1}$. The first fundamental form gives us information about the intrinsic curvature of the surface.

At each $p \in$ M, there are exactly two unit vectors perpendicular to T_pM, called unit normals. If M is orientable, then there are exactly two choices of C^∞ unit normal vector fields defined at all point of M. Let N : M $\longrightarrow \mathbb{R}^{n+1}$ be such a choice. Given U $\in \mathrm{T}_p$M, we have the directional derivative $D_U(N) = U(N)$, which represents the change in the normal.

The Second Fundamental Form. Given $p \in M$, the second fundamental form at p,

$$\mathrm{II} : T_pM \times T_pM \longrightarrow R$$

is defined by,

$$\mathrm{II}(U,V) \doteqdot < D_UN, V >$$

, for $U,V \in T_pM$. The second fundamental form gives us information about the extrinsic curvature. It is the interior product of the directional derivative of N along U with V.

Remarks. (1) $< D_UN, N > = U|N|^2 = 0$, $|N| \equiv 1$, so $D_UN \in T_pM$, (2) Given $U,V \in T_pM$, extend U and V to vector fields $\bar{U}$ and $\bar{V}$ in a neighborhood of p in M. Then

$$\mathrm{II}(U,V) = < D_UN, V >$$

$$= U < N, \bar{V} > - < N, D_U\bar{V} >$$

$$= - < N, D_U\bar{V} > = < N, \bar{V} > = 0$$

Hence, it can be shown that $\mathrm{II}(U,V) = \mathrm{II}(V,U)$. The normal component of $V \in T_p\mathbb{R}^{n+1}$ is $V^N \doteqdot < V, N > N$. The tangential component of V is $V^T \doteqdot V - < V, N > N$. The induced affine connection on M is $\nabla(X,Y) = \nabla_XY$, where

$$\nabla_XY \doteqdot (\nabla_XY)^T$$

$$= D_XY - < D_XY, N > N$$

$$= D_X Y - \mathrm{II}(X,Y)N$$

Next, we need the Weingarten map, L

$$L : T_pM \longrightarrow T_pM$$

, defined by $L(U) = D_U N$. By definition, II(U,V) = $< D_U N, V >$ = $< L(U), V >$. Since II is symmetric, the linear map L is self-adjoint, i.e. $< L(U), V > = < U, L(V) >$. We now derive the Gauss and Codazzi equations, which are a pair of fundamental equations for hypersurfaces. We begin with

$$D_X Y = \nabla_X Y - \mathrm{II}(X,Y)N \tag{12.2.2}$$

If $II(X,Y)N = 0$, then $D_X Y = \nabla_X Y$. Take directional derivative of both sides of (12.2.2)

$$D_X(D_Y Z) = D_X(\nabla_Y Z - \mathrm{II}(Y,Z)N)$$

$$= \nabla_X(\nabla_Y Z) - \mathrm{II}(X, \nabla_Y Z)N - X(\mathrm{II}(Y,Z))N - \mathrm{II}(Y,Z)D_X N$$

Anti-symmetrizing X and Y and inserting (12.2.2) into Riemann curvature tensor equation (12.2.3)

$$D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z = Rm(X,Y)Z = 0 \tag{12.2.3}$$

$$\nabla_X(\nabla_Y Z) - \mathrm{II}(X, \nabla_Y Z)N - X(\mathrm{II}(\mathrm{Y},\mathrm{Z}))\mathrm{N} - X(\mathrm{II}(Y,Z)N - \mathrm{II}(Y,Z)L(X)$$

$$-\nabla_Y(\nabla_X Z) + \mathrm{II}(Y, \nabla_X Z)N + Y(\mathrm{II}(X,Z))N + \mathrm{II}(X,Z)L(Y)$$

$$-\nabla_{[X,Y]}Z + \mathrm{II}([X,Y],Z)N = 0$$

We collect like terms;

$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z - \mathrm{II}(Y,Z)L(X) + \mathrm{II}(X,Z)L(Y)$$

$$+(-X(\mathrm{II}(Y,Z) + \mathrm{II}(Y,\nabla_X Z) + \mathrm{II}(\nabla_X Y,Z))N$$

$$+(Y(\mathrm{II}(X,Z)) - \mathrm{II}(X,\nabla_Y Z) - \mathrm{II}(\nabla_Y X,Z))N = 0$$

, where L is the Weigarten map. By applying the definition of the Riemann curvature tensor and the covariant derivative of a 2-tensor,

$$(\nabla_X T)(Y,Z) = X(T(Y,Z) - T(\nabla_X Y,Z) - T(Y,\nabla_X Z)$$

, we get

$$Rm(X,Y)Z - \mathrm{II}(Y,Z)L(X) + \mathrm{II}(X,Z)L(Y) + (-(\nabla_X \mathrm{II})(Y,Z) + (\nabla_Y \mathrm{II})(X,Z)))N = 0$$

Taking the tangential and normal components of this equation, we get the Gauss's Equation :

$$Rm(X,Y)Z = \mathrm{II}(Y,Z)L(X) - \mathrm{II}(X,Z)L(Y) \tag{12.2.4}$$

and the Codazzi Equation :

$$(\nabla_X \mathrm{II})(Y,Z) = (\nabla_Y \mathrm{II})(X,Z) \tag{12.2.5}$$

Before Einstein's equations can be solved numerically, they need to be cast into a suitable initial value form. We will explore the original ADM formulation, which we introduced above. There are more robust models, which we will not introduce here. The gravitational fields are described in terms of the spatial metric and the extrinsic curvature, a method symbolised as $\dot{g}$ -$\dot{K}$. These equations satisfy some initial constraints and they can be integrated forward in time. As before, we describe our metric as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j)dt$$

where α is the lapse function , β^i is the shift vector and γ_{ij} is the spatial metric .The two evolution equations are: the extrinsic curvature, K_{ij} , which can be defined by the equation

$$\frac{d}{dt}\gamma_{ij} = -2\alpha K_{ij}$$

, where $\frac{d}{dt}$ = $\frac{\partial}{\partial t}$ - $\mathfrak{L}_\beta$, where $\mathfrak{L}_\beta$ is the Lie derivative with respect to $\beta_i$and, the evolution equation for the extrinsic curvature K_{ij}

$$\frac{d}{dt}K_{ij} = -D_i D_j \alpha + \alpha(R_{ij} - 2K_{il}K^l_j + KK_{ij} - M_{ij})$$

where D_i is the covariant derivative associated with γ_{ij}, R_{ij} is the three dimensional Ricci tensor ,

$$R_{ij} = \frac{1}{2}\gamma^{kl}(\gamma_{kj,il} + \gamma_{il.kj} - \gamma_{kl,ij} - \gamma_{ij,kl}) + \gamma^{kl}(\Gamma^m_{il}\Gamma_{mkj} - \Gamma^m_{ij}\Gamma_{mkl})$$

$$R = \gamma^{ij}R_{ij}$$

ρ, S_i and S_{ij} are matter sources,which are projections of the stress energy tensor with respect to the unit normal vector n$_\alpha$

$$\rho = n_\alpha n_\beta T^{\alpha\beta}$$

$$S_i = -\gamma_{i\alpha} n_\beta T^{\alpha\beta}$$

$$S_{ij} = \gamma_{i\alpha}\gamma_{j\beta} T^{\alpha\beta}$$

$$M_{ij} = S_{ij} + \frac{1}{2}\gamma^{ij}(\rho - S)$$

, where$S = \gamma^{ij} S_{ij}$

12.3 Hamiltonian Formulation and General Relativity

Einsteins equation can also be split into the Hamiltonian and momentum constraints; see appendix A . The Hamiltonian constraint is

$$R - K_{ij}K^{ij} + K^2 = 2\rho \tag{12.3.1}$$

The momentum constraint is

$$D_j K_i^j - D_i K = S_i \tag{12.3.2}$$

Ingredients of ADM formalism

(1) The formalism assumes time is foliated into a family of spacelike surfaces Σ_t, with coordinates on each slice given by x^i

(2) The dynamic variables are the metric tensor of three dimensional spatial slices, $\gamma_{ij}(t,x^k)$ and their conjugate momenta $\pi^{ij}(t,x^k)$. These provide 12 variables

(3) Conjugate momenta; the canonical coordinates, q_i and p$_i$ in phase space are used in the Hamiltonian formalism. The canonical coordinates satisfy the Poissson bracket relations

$\{q_i, q_j\} = 0$, $\{p_i, p\} = 0$ and $q_i p_j = \delta_{ij}$. See appendix B.

(4) There are four Lagrange multipliers; the lapse function α and the shift vector β^i.

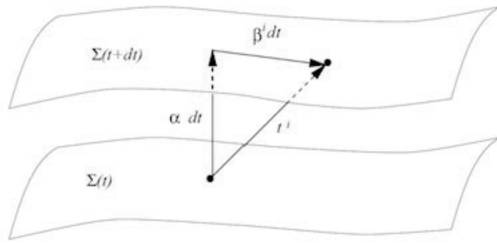

Figure 12.2: Foliation of spacetime

These describe how the leaves Σ_t of the foliation of spacetime are welded together

(5) Greek indices are spacetime indices ; $(0,1,2,3)$, Latin infices are spatial indices; $(1,2,3)$. A quantity suspended with a superscript (4) has a 3 and 4 dimensional version

such as the metric tensor for 3d slices g_{ij} and and the metric tensor for for the 4d spacetime ${}^{(4)}g_{\mu\nu}$

(6) As usual, the trace is $\pi = g^{ij}\pi_{ij}$

(7) The starting point for the ADM formalism is the Lagrangian

$$\mathscr{L} =^{(4)} R\sqrt{^{(4)}g}$$

where g is the determinant of the four dimensional metric tensor for the full spacetime

(8) The desired outcome is to define an embedding of the three dimensional spatial slices in the four dimensional spacetime. The metric of the three dimensional slices, $g_{ij} =^{(4)} g_{ij}$ will be the generalized coordinates for a Hamiltonian formulation.

(9) Foliation of space time is the breaking of the spacetime manifold into a one-parameter family of three dimensional spacelike hypersurfaces parametrized by a time function t. The hypersurfaces have timelike normal vectors and spacelike tangent vectors. Let n^a be a unit normal vector field to the hypersurface Σ_t and t^a be a vector field on the spacetime manifold. See Figure 12.2 below.

t^acan be seen as the flow of time through spacetime, and h$_{ab}$ is the induced spatial metric on every surface Σ_t.The spatial metric is related to the spacetime metric by

$$h_{ab} = g_{ab} + n_a n_b$$

$t^a$is decomposed into a normal and tangential components with respect to the surfaces Σ_t,where

$$t^a \nabla_a = 1$$

The lapse function, α, measures the rate of flow of proper time τ with respect to coordinate time t as one moves normally to Σ_t along n^a

$$\alpha = -g_{\alpha\beta} t^a n^b$$

The shift vector, β^a, measures how much the local spatial coordinate system shifts tangential to Σ_t when moving from Σ_1 to Σ_2 along n^a From Figure 10.2, we see that

$$\alpha n^a + \beta^a = t^a$$

$$\Rightarrow n^a = \frac{t^a - \beta^a}{\alpha}$$

So

$$g^{ab} = h^{ab} - n^a n^b = h^{ab} - \alpha^{-2}(t^a - \beta^a)(t^b - \beta^b)$$

(10) As we move from one hypersurface to the next along the time flow of t^a,the components of h_{ab} change on each successive hypersurface in accordance with Einstein's field equation. In the ADM formalism, we need h_{ab} and its time derivative $\dot{h}_{ab}$ as initial data.

(11) We obtain the Hamiltonian density, $\mathscr{H}$, from the Lagrangian density, $\mathscr{L}$ = R$\sqrt{-g}$ where $g = det g_{ab}$

(12) We require our Lagrangian, $\mathscr{L}$, in terms of variables that describe Σ_t we have $\sqrt{-g}$ =

αh , $h = deth_{ab}$

(13) Multiply Einstein tensor G_{ab} by $n^a n^b$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$$

$$2G_{ab} n^a n^b = 2R_{ab} n^a n^b - R g_{ab} n^a n^b$$

$$2G_{ab} n^a n^b = 2R_{ab} n^a n^b - R$$

$$R = 2(R_{ab} n^a n^b - G_{ab} n^a n^b)$$

(14) From the Gauss-Codacci equation, we get the following constraint relation

$$G_{ab} n^a n^b = \frac{1}{2}[{}^{(3)}R - K_{ab} K^{ab} + K^2]$$

, where K$_{ab}$ is the extrinsic curvature of Σ_t and K it's trace

(15) From the definition of the Ricci tensor R$_{ab}$

$$R_{ab} = R^c_{acb}$$

$$R_{ab} n^a n^b = R^c_{acb} n^b n^a$$

$$= -(\nabla_a \nabla_c - \nabla_c \nabla_a) n^c n^a$$

$$= -n^a (\nabla_a \nabla_c - \nabla_c \nabla_a) n^a$$

$$= -(\nabla_a n^a)(\nabla_c n^c) - \nabla_a(n^a \nabla_c n^c) + (\nabla_c n^a)(\nabla_a n^c) + \nabla_c(n^a \nabla_a n^c)$$

$$= -K^2 + K_{ac}K^{ac}$$

, the second and fourth term are divergences and can be neglected

(16) The Lagrangian density in terms of the variables of the hypersurface becomes

$$\mathscr{L} = \sqrt{(-g)}R$$

$$= \alpha\sqrt{h}R$$

$$= 2\alpha\sqrt{h}(G_{ab}n^a n^b - R_{ab}n^a n^b)$$

$$= 2\alpha\sqrt{h}(\frac{1}{2}[{}^{(3)}R - K_{ab}K^{ab} + K^2] - K^2 + K_{ab}K^{ab})$$

$$\mathscr{L} = \alpha\sqrt{h}({}^{(3)}R + K_{ab}K^{ab} - K^2)$$

(17) Hamiltonian Dynamics**.** See appendix B for the basics.

The Lagrangian density, $\mathscr{L}$, is the kinetic energy minus the potential energy of a system. $\mathscr{L}$ is expressed in terms of generalized coordinates q_i and $\dot{q}_i$. The moments, p_i, can be defined by

$$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i}$$

The resulting systems of equations can be solved to obtain $\dot{q}_i$ as a function of p_i.

The Hamiltonian density, $\mathscr{H}$, is then defined as

$$\mathscr{H} = \sum p_i \dot{q}_i - \mathscr{L}(q_i, \dot{q}_i)$$

Hamilton's equations are

$$\dot{q}_i = \frac{\partial \mathscr{H}}{\partial p_i}$$

and

$$\dot{p}_i = -\frac{\partial \mathscr{H}}{\partial q_i}$$

For general relativity, q_i is replaced by h_{ab} and p_i is replaced with p^{ab}. Therefore,

$$\mathscr{H} = p^{ab}\dot{h}_{ab} - \mathscr{L}(q_i, \dot{q}_i)$$

What we need is a variation of $\mathscr{H}$ with respect to α and β_a. By the definition of momentum

$$\begin{aligned} p^{ab} = \frac{\partial \mathscr{L}}{\partial \dot{h}_{ab}} &= \sqrt{h}\alpha[\frac{\partial^{(3)}R}{\partial \dot{h}_{ab}} + \frac{\partial (K_{ab}K^{ab})}{\partial \dot{h}_{ab}} - \frac{\partial K^2}{\partial \dot{h}_{ab}}] \\ &= \sqrt{h}\alpha(K^{ab} - h^{ab}K), \end{aligned}$$

where $\frac{\partial K_{ab}}{\partial \dot{h}_{ab}} = \frac{1}{2\alpha}$, $\frac{\partial^{(3)}R}{\partial \dot{h}_{ab}} = 0$, $\frac{\partial K^2}{\partial \dot{h}_{ab}} = \frac{h^{ab}K}{\alpha}$. The extrinsic curvature of a surface Σ is defined as

$$K_{ab} = \nabla_a n_b,$$

where n_b is a field orthogonal to Σ and tangent to timelike geodesics that do not intersect. Our goal is to link K_{ab} in terms of α, β_a and h_{ab}. Noting $g_{ab} = h_{ab}$ - $n_a n_b$, the Lie derivative of g_{ab} can be expressed as

$$L g_{ab} = 2\nabla_a n_b$$

and

$$\alpha = -g_{ab}t^a n^b$$

and

$$\beta = h^a_b t^b$$

Therefore,

$$K_{ab} = \frac{1}{2} L g_{ab}$$

$$= \frac{1}{2} L(h_{ab} - n_a n_b)$$

$$= \frac{1}{2} L_n h_{ab}$$

$$= \frac{1}{2}[n^c \nabla_c h_{ab} + h_{cb} \nabla_a v^c + h_{ac} \nabla_b v^c]$$

$$= \frac{1}{2\alpha}[\alpha n^c \nabla_c h_{ab} + h_{cb} \nabla_a \alpha v^c + h_{ac} \nabla_b \alpha v^c]$$

$$= \frac{1}{2\alpha} h^c_a h^d_b [L_t h_{cd} - L_\beta h_{cd}]$$

$$= \frac{1}{2\alpha} h^c_a h^d_b [\dot{h}_{ab} - D_a \beta_b - D_b \beta_a]$$

(18) The Hamiltonian, $\mathscr{H}$.

$$\mathscr{H} = p^{ab} \dot{h}_{ab} - \mathscr{L}(q_i, \dot{q}_i)$$

Substituting $\mathscr{L} = \alpha \sqrt{h}({}^{(3)}R + K_{ab} K^{ab} - K^2)$ and K_{ab} above, we get

$$\mathscr{H} = -\sqrt{h}\alpha^{(3)}R + \frac{\alpha}{\sqrt{h}}[p^{ab} p_{ab} - \frac{1}{2} p^2] + 2p^{ab} D_a \beta_b$$

, where p is the trace of p

$$=\sqrt{h}\alpha[-^{(3)}R+h^{-1}p^{ab}p_{ab}-\frac{1}{2}h^{-1}p^{2}]-2\beta_{b}[D_{a}(h^{-1}p^{ab})]+2D_{a}(h^{-\frac{1}{2}}\beta_{b}p^{ab})$$

We neglect the boundary term in the last line as we assume a sufficiently large spatial surface.

(19) Constraint Equations**.** We determine the Hamiltonian H by integrating $\mathscr{H}$ over the hypersurface Σ_t using the fixed spatial element, $^{(3)}e$ **,** where $^{(3)}e\sqrt{h}$ **=** ε**,** ε is the natural volume element associated with the metric h_{ab}

$$H=\int_{\Sigma}\mathscr{H}^{(3)}e$$

We then perform a variation of H with respect to α and β. and then obtain two constraint equations

$$-^{(3)}R+\frac{p^{ab}p_{ab}}{h}-\frac{p^{2}}{2h}=0$$

, hence the Hamiltonian is constrained, and

$$D_{a}(\frac{p^{ab}}{\sqrt{h}})=0,p^{ab}$$

cannot change with respect to a

(20) Evolution Equations. We expand the Hamiltonian equations, as is done in Classical mechanics

$$\mathscr{L}_{t}h^{ab}=\dot{h}_{ab}=\frac{\delta H}{\delta p^{ab}}=\frac{2\alpha}{\sqrt{h}}(p_{ab}-\frac{h_{ab}p}{2})+D_{a}\beta_{b}+D_{b}\beta_{a}$$

$$\mathscr{L}_{t}p^{ab}=\dot{p}^{ab}=-\frac{\delta H}{\delta h_{ab}}$$

$$=-\alpha\sqrt{h}(^{(3)}R^{ab}-\frac{^{(3)}Rh^{ab}}{2})+\frac{\alpha h^{ab}}{2\sqrt{h}}(p_{cd}p^{cd}-\frac{p^{2}}{2})-2\frac{\alpha}{\sqrt{h}}(p^{ac}p_{c}^{b}-p\frac{p^{ab}}{2})+\sqrt{h}(D^{a}D^{b}\alpha-h^{ab}D^{c}D_{c}\alpha$$

$$+\sqrt{h}D_c(\frac{\beta^c p^{ab}}{\sqrt{h}}) - 2p^{ca}D_c\beta^b$$

(21) The two contraint and two evolution equations constitute a Hamiltonian formulation of general relativity. Given initial conditions that satisfy the constraint equations, the universe at any point in spacetime, can be modelled by an evolving system using the evolution equations.

CHAPTER XIII

HOYLE NARLIKAR THEORY AND CONCEPT OF MASS IN GENERAL RELATIVITY

13.1 Hoyle - Narlikar Theory

In 1964 Fred Hoyle and Jayant Narlikar proposed a gravitational theory that incorporated the idea of Mach's theory that the inertia of a particle is due to the rest of the particles in the universe,[22],[23]. Let $a, b, c...$ be the particles of the universe. Let g_{ik} be the metric tensor of the Riemannian spacetime manifold. Let da be an element of the proper time of the world line of particle a, and da^i be the coordinate differentials along the worldline of a. Then

$$da^2 = g_{ik} da^i da^k$$

Let A, B, C,... be world points on the world lines of $a, b, c,...$ and let X be a typical point in the spacetime manifold. Hoyle and Narlikar derive their theory from the action S,

$$S = \sum \sum_{a \neq b} \int \int G(a, b) da db$$

where the integration is over the worldlines of particles $a, b, c,...$; G is a Green's function that satisfies the wave equation

$$\Box + \frac{1}{6} R = \frac{\delta^4}{\sqrt{-g}}$$

where R is the scalar curvature of the Riemann spacettime manifold, g is the determinant of g_{ij}, δ is the Dirac delta function and $\Box f = g^{ij} f_{;ij}$ for any function f

$$\delta(X,X^{'}) = \delta(X_1 - X_1^{'})\delta(X_2 - X_2^{'})\delta(X_3 - X_3^{'})\delta(X_4 - X_4^{'})$$

where X_1, X_2, X_3, X_4 are the coordinates of the point X. Since the double sum in the action S is symmetrical between all pairs of particles a, b, only that part of G(a,b) that is symmetrical between a and b will contribute to the action of S. Hence, the action can be written as

$$S = \sum\sum_{a \neq b} \iint G^{*}(a,b)dadb$$

where $G^{*}(a,b) = \frac{1}{2}G(a,b) + \frac{1}{2}G(b,a)$. Thus G^{*} must be the time symmetric Green function, and can be written as $G^{*} = \frac{1}{2}G_{ret} + \frac{1}{2}G_{adv}$, where G_{ret} and G_{adv} are the retarded and advanced Green functions. The Hoyle-Narlikar theory makes the following assumptions; for the mass at position x_a, $m(x_a)$, to become a direct particle field, it must arise from all the other mass in the universe. Since mass is scalar, it can be expressed through a scalar Green function. Furthermore, the action must be symmetric between any pair of particles. Let each particle b give rise to a mass field at a point x, be given by $m^{(b)}(x)$. Let x_a be the path of particle a. The contribution of particle b to the mass of particle a at position x_a and summing for all particles b

$$m(x_a) = \sum_b m^{(b)}(x_a) = \sum_b = \int G^{*}(x_a x_b)dx_b$$

This gives the mass at point x_a due to all particles including those at position x_a . The full mass field is the sum of half retarded and half advanced fields, where

$$G^{*}(x_a,x_b) = \frac{1}{2}(G(x_a,x_b + \frac{1}{2}G(x_b,x_a))$$

is a time symmetric Green's function.

In 1965, Stephen Hawking's argued in his PhD thesis that the Hoyle - Narlikar theory was incompatible with an expanding universe, since the advanced mass field would be infinite. However, with emerging observational data of an accelerating expanding universe with a cosmic event horizon, we will show that Hawking's argument is not entirely true. We begin with the FRW metric

$$ds^2 = dt^2 - R^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)]$$

The geometric factor k = +1 for a closed universe, 0 for flat and -1 for an open universe. R(t) is the scaling factor. By computing the Ricci scalar, Ricci tensor using Christoffel symbols and substituting into Einstein's field equation(see Appendix K), we obtain Friedmann's equations;

$$R\ddot{R} + 2\dot{R}^2 + 2k = 4\pi G(\rho - p)R^2$$

$$3\ddot{R} = -4\pi G(\rho + 3p)R$$

Eliminating $\ddot{R}$ and seting H = $\frac{\dot{R}}{R}$;

$$H^2 = (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \tag{13.1.1}$$

For Einstein de Sitter flat space, k = 0, where $\eta_{\mu\nu}$ is the flat space metric. We define a new conformal function given by Ω(t) = R(t). We define a new time coordinate τsuch that

$$ds^2 = \Omega^2[d\tau^2 - dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2] = \Omega^2\eta_{\mu\nu}dx^\mu dx^\nu$$

Mass ,m^*, in the new Minkowski frame is Rm. Let n be the particle density in the original frame, then the corresponding density in the new frame is $n^* is R^3 n = L^{-3}$. Writing ρ = nm and using $8\pi G$=$\frac{6}{m^2}$,from Hoyle-Narlikar Theory, substituting into (45.1);

$$\frac{\dot{R}^2}{R^2} = \frac{2\rho}{m^2} = \frac{2n}{m}$$

$$\dot{R}^2 = \frac{2nR^3}{mR} = \frac{2n^*}{mR} = (\frac{2L^{-3}}{m})\frac{1}{R}$$

The term in brackets is a constant. Integrating, using the boundary condition R = 0 at t = 0,

$$\int_0^t R^{\frac{1}{2}} dR = \int_0^t (\frac{2L^{-3}}{m})^{\frac{1}{2}} dt$$

$$R(t) = (\frac{2L^{-3}}{m})^{\frac{2}{3}} (\frac{3}{2})^{\frac{2}{3}} t^{\frac{2}{3}} \tag{13.1.2}$$

Now τ is defined as

$$\tau = \int_0^t \frac{dt}{R(t)} = 3(\frac{2}{3})^{\frac{2}{3}} (\frac{m}{2L^{-3}})^{\frac{1}{3}} t^{\frac{1}{3}}$$

$$\frac{1}{2}\tau = (\frac{m}{2L^{-3}})^{\frac{1}{3}} (\frac{3t}{2})^{\frac{1}{3}}$$

Substituting into (45.2);

$$R(\tau) = \frac{1}{4}\tau^2 (\frac{2L^{-3}}{m})$$

Since $m^* = m\mathrm{R}(t)$; we obtain

$$m^* = \frac{1}{2}\tau^2 L^{-3} = mR(t)$$

$$R(t) = \frac{1}{2m}\tau^2 L^{-3}$$

Let $\mathrm{T}^2 = (\frac{2m}{L^{-3}})$, we get

$$R(\tau) = \frac{\tau^2}{T^2}$$

As shown above, the scale factor $\mathrm{R}(t) \propto \mathrm{t}^{\frac{2}{3}}$ and $\Omega = \mathrm{R}(\tau) = (\frac{\tau}{T})^2$. As mentioned above, Green's function obeys the wave equation

$$G^*(x_a,x_b)+\frac{1}{6}RG^*(x_a,x_b)=\frac{\delta^4(x_a,x_b)}{\sqrt{-g}}$$

The mass field , m, is given by

$$m(\tau_1)=\frac{1}{2}(m_{ret}+m_{adv})=\int G^*N\sqrt{-g}\Omega(\tau_2)4\pi r^2 dr d\tau_2$$

, where N = nR^{-3}. For the retarded mass field,

$$m_{ret}(\tau_1)=\frac{1}{\Omega(\tau_1)}\int\frac{N\delta(r+\tau_2-\tau_1)\Omega^3(\tau_2)}{4\pi r}4\pi r^2 dr d\tau_2$$

The r integral is performed using the delta function to give

$$m_{ret}(\tau_1)=\frac{T^2}{\tau_1{}^2}\int_0^{\tau_1}n(\tau_1-\tau_2)d\tau_2=\frac{1}{2}nT^2=\frac{n}{L^{-3}}m$$

where τ_1 is the current age of the universe and the integral is over the past light cone. For the advanced wave, we have

$$m_{adv}(\tau_1)=\frac{1}{\Omega(\tau_1)}\int\frac{N\delta(r-\tau_2+\tau_1)\Omega^3(\tau_2)}{4\pi r}4\pi r^2 dr d\tau_2$$

We then integrate over the future light cone

$$m_{adv}(\tau_1)=(\frac{T}{\tau_1})^2\int_{\tau_1}^{\infty}n(\tau_2-\tau_1)d\tau_2\longrightarrow\infty$$

where τ_1 is the present time (age of the universe) and τ_2 is a future time, which can go to infinity. This is the divergence problem described by Hawkings.

Solution for a non-divergent advanced mass. The particle horizon, H$_p$, is the distance beyond which an observer cannot see at the current time. Let t$_0$ be the current age of the universe,

$$H_p = R(t_0)\int_0^{t_0} \frac{cdt}{R(t)}$$

The event horizon is due to an accelerating universe and is the distance beyond which the observer will never see.

$$H_e = R(t_0)\int_{t_0}^{\infty} \frac{cdt}{R(t)}$$

Consider a scale factor which allows for acceleration

$$R(t) = (\frac{t}{T})^{\frac{3}{2}}$$

The proper time in comoving coordinates becomes

$$\tau = \int_0^{t} \frac{dt}{R(t)} = 2T^{\frac{3}{2}}t^{\frac{-1}{2}}$$

and so

$$t^{\frac{3}{2}} = 8\frac{T^{\frac{9}{2}}}{\tau^3}$$

Hence

$$R(\tau) = 8\frac{T^3}{\tau^3}$$

The event horizon is given by

$$H_e = R(\tau_1)\int_{\tau_1}^{\infty} \frac{cdt}{R(t} = T^{\frac{3}{2}}(\frac{\tau_1}{T})^{\frac{3}{2}}\int_{\tau_1}^{\infty} ct^{\frac{-3}{2}}dt = 2c\tau_1$$

Hence $H_e/c = 2\tau_1$ is the upper limit of for the m_{adv} integral. The advanced mass field integral over the future light cone can be written with limits from τ_1 to $H_e/c = 2\tau_1$

$$m_{adv}(\tau_1) = \frac{1}{R(\tau_1)} \int \frac{N\delta(r - \tau_2 + \tau_1)\Omega^3(\tau_2)}{4\pi r} 4\pi r^2 dr d\tau_2$$

$$= \frac{n\tau_1^3}{8T^3} \int_{\tau_1}^{\tau_2} (\tau_2 - \tau_1) n d\tau_2$$

$$\frac{n\tau_1^3}{8T^3} [\frac{\tau_2^2}{2} - \tau_2\tau_1]_{\tau_1}^{2\tau_2}$$

$$\frac{n\tau_1^3}{8T^3} (\frac{\tau_1^2}{2}) = \frac{n}{L^{-3}} m$$

where n = L^{-3} is the density of particles in the universe. Hence, the advanced mass component is not infinite. Noteworthy, is that Hoyle Narlikar theory deploys the action at a distance principle.

13.2 The concept of mass in general relativity

Newton's second law states that , the force, F , is mass times acceleration

$$F = m\frac{d^2x}{dt^2}$$

It states how an external foce interferes wth the movement of an object. It does not tell us about the origin of forces. Newton's law of Gravitation does, two masses m_1 and m_2 separated by a distance r, are mutually attracted to each other by a force F, with gravitation constant G, where

$$F = G\frac{m_1 m_2}{r^2}$$

However, this theory does not tell us how the gravitational forces are transmitted. In 1895, Maxwell showed that electromagnetic waves always travel at constant speed c in a vacuum

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

According to Newton, the speed of light measured by an inertial frame travelling at constant velocity v should be $c \pm v$. According to Maxwell, the speed of light should be c in any referential frame. In 1905, Einstein solved this conundrum by ntroducing the spacetime manifold, where the line element is the Minkowski metric

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)$$

For light, a null geodesic ds^2= 0, then

$$c^2dt^2 = dx^2 + dy^2 + dz^2$$

Then,

$$c^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

This equation tells us that time ticks at the speed of light. Since c is a constant, the faster we travel, that is, the more distance we cover , the slower time ticks.This still does not tell us how gravitational forces are transmitted. However, Einstein's general theory of relativity does explain it. The universe is modelled by an N^4 (= $M^3 \times \mathbb{R}$) spacetime manifold, where 3 dimensions correspond to space and 1 dimension corresponds to time. The metric is g_{ij} and the field equation relates the geometry of space, curvature, to the energy- momentum tensor; as before

$$Ric - \frac{1}{2}Rg_{ij} = T_{ij}$$

In 1916, K. Schwarzchild provided the first solution; it the metric

$$g_{ij} = (1 + \frac{m}{2r})^4 \eta_{ij}$$

, where M^3 = $R\mathbb{R}^3$ - B$_{m/2}$(0), $m \geq 0$ and η_{ij} is the standard Euclidean metric on $\mathbb{R}^3$. This is the blackhole solution with a mass m placed at the origin 0. The 2-dimensional sphere ∂B$_{m/2}$(0) of radius $m/2$ is an event horizon. The parameter $m \geq 0$ is identified with the mass of a blackhole; the deviation of a geodesic, curvature, is proportional to m. This raises the question; can we define a notion of mass of arbitrary space-time ?

13.3 ADM Mass in General Relativity

The definition of mass in general relativity becomes a delicate issue, that is, a notion which becomes invariant under change of coordinates. It is easier to conceptualize the mass of a point or the mass of the entire universe, but the issue becomes trickier for a local mass between these two trivial extremes. In order to define a notion of this local mass, we need to make a few assumptions:

(1) whatever definition we propose has to coincide with the notion of the mass of a Schwrazchild blackhole, the parameter m,

(2) the mass of the whole universe or the mass of very large objects are confined to a bounded, compact region, so that the gravitational effects tend to decay at infinity. that is, the metric g_{ij} is asymptotically flat. That is , there is a comapct set $K \subset M^3$ such that $M^3 \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and g_{ij} tends to the Euclidean metric at infinity. Under these assumptions, in 1960, Arnowitt, Deser and Misner,[1], introduced the following definition of mass, the ADM mass

$$m(g_{ij}) = \frac{1}{16\pi} \int_{S_\infty} (g_{ij,j} - g_{jj,i}) d\nu$$

where S_∞ is the sphere at infinity, ν is the area element of S_∞ . Bartnik showed that the ADM mass is independent of choice of coordinates and $m(g_{ij}) = m$, in the case of a Schwarzchild blackhole.

Theorem. Positive Mass Theorem. Let (M^3, g_{ij}) be an asymptotically flat Riemannian manifold of scalar curvature $R(g_{ij}) \geq 0$, at every point. Then $m(g_{ij}) \geq 0$. Furthermore, $m(g_{ij})$ = 0 if and only if $M^3 = \mathbb{R}^3$ and $g_{ij} = \eta_{ij}$; that is, the ADM mass is zero exactly for the vacuum spacetime.

Remark. The term $g_{ij,j} - g_{jj,i}$ has a negative sign when the potential energy surpasses the kinetic energy, so that the positivity of ADM mass is not obvious. However, the positive mass theorem states that m(g_{ij}) ≥ 0, when the local density of energy(as measured by the scalar curvature, R) is non-negative everywhere R(g_{ij})≥ 0.

Re-iterating; general relativity models the world by a four-dimensional spacetime manifold with a Lorentz metric g_{ij}. The metric represents the gravitational field and it features in two major ways;

(1) the metric determines the dynamics; the trajectories or worldlines of free falling point particles are geodesics

(2) the metric tensor satisfies the field equation Ric - $\frac{1}{2}$Rg_{ij} = T$_{ij}$, where g_{ij} is analagous to the gravitational potential and the energy-momentum tensor T$_{\mu\nu}$ is analagous to the mass density. The Schwarzchild metric is the famous solution and represents the gravitational field of a static point particle - a blackhole - of mass m. It is a singular Lorentz metric on $\mathbb{R}^4$, when restricted to any constant time three plane , is asymptotically flat of order 1 and has the form

$$g_{ij}(z) = (1 + m\rho^{-1})\delta_{ij} + O(\rho^{-2})$$

in suitable coordinates, ρ = $2r$ and δ_{ij} is the Euclidean metric. More realistic solutions of the Einstein field equation model isolated gravitational systems like a binary star in an otherwise empty universe. When such a system is observed at a great distance, its gravitational field should resemble that of a point mass. Thus the spacetime modelling the system should be asymptotically Schwarzchild, and should admit spacelike hypersurfaces which are asymptotically flat Riemannian 3-manifolds. In this contect, the study of solutions of the field equations on asymptotically flat manifolds began. One way to do this is via the Hilbert action integral;

$$A(g_{ij}) = \int_X S dV$$

where X is our manifold, S is the scalar curvature and dV is the volume form of $g_{ij}$and A(g_{ij}) is the first variation.

A quick note on geometric notations. If g is a Riemannian metric, dV_g denotesits Riemannian density, which is defined wether or not the manifold M is oriented. In local cordinates dV_g = $(detg)^{\frac{1}{2}}|dx|$, where dx is the Euclidean volume form $dx^1 \wedge dx^2 ... \wedge dx^n$ on $\mathbb{R}^n$. The divergence operator is the formal adjoint ∇^* of ∇, given on 1-forms by ∇^* = - $\omega_i{}^i$. On a compact manifold with boundary, it satifies the divergence theorem;

$$\int_M \nabla^* \omega dV_g = -\int_{\partial M} \omega(N) dV_{\tilde{g}}$$

where $\tilde{g}$ is the induced metric on ∂M and N is the outward unit normal.

Lemma. Let (X, g) be a Riemannian or pseudo-Riemannian manifold. Let h be a smooth symmetric 2-tensor. Let g_t be a one-parameter family of metrics, with h = $\frac{dg_t}{dt}|_{t=0}$. Let S$_t$ and dV_t be the scalar curvature and volume form of g_t , then

$$\frac{d}{dt}(S_t dV_t)_{|t=0} = -h(h^{jk} G_{jk} + \nabla^* \xi) dV_g \qquad (13.3.1)$$

where G is the Einstein tensor G$_{jk}$ = R$_{jk}$ - $\frac{1}{2}$Sg_{jk} and ξ is the 1-form ξ = - (∇^* h + ∇(tr$_g$h)) = $(h_{jk,}{}^k$ - $h^k{}_{k,j})dx^j$.

Proof. The variation of the volume form dV = $\sqrt{detg_t}dx$ is

$$\frac{d}{dt} dV_{t|t=0} = \frac{1}{2}\sqrt{detg_t} g^{jk} h_{jk} dx = \frac{1}{2} g^{jk} h_{jk} dV_g \qquad (13.3.2)$$

The scalar curvture S = $g_t{}^{jk}$(R$_{jk}$)$_t$. And, since R$_{jk}$ = $\partial_i \Gamma^i_{kj}$ - $\partial_k \Gamma^i_{ij}$ + $\Gamma^i_{il}\Gamma^l_{jk}$ - $\Gamma^i_{kl}\Gamma^l_{ij}$ and $\partial_l \Gamma^k_{ij}$ = $\frac{1}{2} g^{km} \partial_l$ ($\partial_j g_{jm}$ + $\partial_j g_{im}$ - $\partial_m g_{ij}$). In g-normal coordinates $\partial_i(g_{jk})$ = 0, we have

$$\partial_l \Gamma^k_{ij} = \frac{1}{2} g^{km}(\partial_j g_{im} - \partial_m g_{ij})$$

and

$$R_{jk} = -\partial_k \Gamma^i_{ij} = -\partial_k[\frac{1}{2} g^{il}(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji})] = -\partial_k[\frac{1}{2} g^{il}(\partial_j g_{il} - \partial_l g_{ji})]$$

Therefore,

$$\frac{d}{dt} S_{t|t=o} = \frac{d}{dt}[g_t^{jk}(R_{jk})_t]_{t=o} = g_t^{jk} \frac{d}{dt}(R_{jk})_{|t=0} + \frac{d}{dt}(g_t^{jk}) R_{jk|t=0}$$

$$\frac{d}{dt}(R_{jk})_{|t=0} = -\frac{1}{2}\frac{d}{dt}[\partial_k g^{il} \partial_j g_{il} + g^{il} \partial_k \partial_j g_{il} - \partial_k g^{il} \partial_l g_{ji} - g^{il} \partial_k \partial_l g_{ji}]_{|t=0}$$

By using property of normal coordinates;

$$\frac{d}{dt}(R_{jk})_{|t=0} = -\frac{1}{2}\frac{d}{dt}[g^{il} \partial_k(\partial_j g_{il} - \partial_l g_{ji})] = -\frac{1}{2} g^{il} \partial_k(\partial_j h_{il} - \partial_l h_{ji})$$

Therefore,

$$\frac{d}{dt} S_{t|t=o} = h_t^{ik} R_{jk_{|t=0}} - \frac{1}{2} g^{jk} g^{il} \partial_k(\partial_j h_{il} - \partial_l h_{ij})$$

Since $\xi = (h^k_{jk,} - h^k_{k,j}) dx^j$ and since $h_{jk,l} = \partial_l h_{jk} - \Gamma^m_{kl} h_{jm} - \Gamma^m_{jl} h_{mk}$, then at the origin of normal coordinates, we have

$$-\nabla^* \xi = h^k_{k,jj} - h^k_{jk,j} = \partial_j \partial_k h_{jk} - \partial_j \partial_j h_{kk} + (\partial_k \Gamma^m_{jk} - \partial_j \Gamma^m_{kk}) h_{jm}$$

$$= \partial_j \partial_k h_{jk} - \partial_j \partial_j h_{kk} - R_{jm} h_{jm} \qquad (13.3.3)$$

Combining ; we get the result.

Applying lemma above and the divergence theorem; at $t = 0$, we have

$$\frac{d}{dt} A(g_t) = \int_X h^{jk} G_{jk} dV_g$$

The Lorecntz metric is a critical point of A if it satisfies the vacuum equation $G_{ij} = 0$. For compactly supported variations, the second divergence term disappears. However, for a an asymptotically flat Riemannian manifold (N,g), we can then look for metrics that are critica l for A(g) under all variations that maintain the asymptotically flat structure, not just the compactly supported ones. The second divergence term then comes into play. If we integrate Eq 11.3, over a large sphere S_R and take the limit as R$\longrightarrow\infty$

$$\frac{d}{dt}A(g_t)_{|t=0} = \int_N h^{jk} G_{jk} dV_g - lim_{R\to\infty} \int_{S_R} \xi(N) dV_g$$

where ξ_j = ($\partial_i h_{ij}$ - $\partial_j h_{ii}$)(1 + O(ρ^{-1})).This boundary term , second term) is the variation of a geometric invariant called the mass.

Definition. Given an aymptotically flat Riemannian manifold (N,g) with asymptotic coordinates $\{z^j\}$, we define the mass as

$$m(g) = lim_{R\to\infty}\omega^{-1}\int_{S_R}\mu dz$$

where μ is the mass density vector field defined on N_∞;

$$\mu = (\partial_i g_{ij} - \partial_j g_{ii})\partial_j$$

From the previous discussion, the family g_t of asymptotically flat metrics

$$\frac{d}{dt}(A(g_t) + m(g_t))_{|t=0} = \int_N h^{jk} G_{jk} dV_g$$

13.4 Energy in General Relativity

An isolated system aasume that the sources are confined to a finite region and assume that the fields are weak far from the sources. This kind of system is assumed to have a finite total energy. One way of defining isolated systems is by setting initial conditions for Einstein's equations. The notion of total energy has been discovered and formulated using a Hamiltonian formulation, which involves the study of initial conditions,[8],[11].

Initial Data Set. This is given by S, a connected 3-dimensional manifold ; h$_{ij}$, a positive definite Riemannian metric; K_{ij}, a symmetric tensor field, the second fundamental form; μ, a scalar field; and j^i, a vector field on S. D is the Levi-Civita Connection and R is the scalar curvature. These satisfy the following constraint equations on S ;

$$D_j K^{ij} - D^j K = -8\pi j^i$$

$$R - K_{ij}K^{ij} + K^2 = 16\pi\mu$$

where $K = K_{ij}h^{ij}$; i and j are 3-dimensional indices that are raised and lowered with the metric h_{ij} and its inverse h^{ij}. The matter fields are assumed to satisfy the dominant energy condition

$$\mu \geq \sqrt{j^i j_i}$$

Energy Conditions. Weak energy condition:

$$T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$$

where $T_{\mu\nu}$ is the energy-momentum tensor and ξ^μ are timelike vectors. This is the energy density measured by the observed ξ^μ.

Dominant energy condition. For all future directed timelike vectors ξ^μ, the vector $T^\mu_\nu\xi^\nu$, should be future directed timelike or null vector. This is the energy momentum current density. It implies that the speed of the energy flow is less than or equal the speed of light. An equivalent formulation is;

$$T^{\mu\nu}\xi^\mu k^\nu$$

, where k^ν is the null vector.The initial data model an isolated system if the fields are weak far away from the sources. Let B_R be a ball of finite radius R in $\mathbb{R}^3$. The exterior region $U = \mathbb{R}^3\ B_R$ is called an end. On U we consider Cartesian coordinates $x^i$with Euclidean radius $r = (\sum_{i=1}^{i=3}(x^i)^2)^{\frac{1}{2}}$ and let δ_{ij} be the Euclidean metric.

A 3-dimensional manifold S is called Euclidean at infinity, if there exists a compact subset $\mathscr{K}$ of S such that S$\backslash\mathscr{K}$ is the disjoint union of a finite number of ends U$_k$. The initial data set ($S, h_{ij}, K_{ij}, \mu, j^i$) is called asymptotically flat if S is Euclidean at infinity and at every end the metric h_{ij} and the tensor K_{ij} satisfy the following fall off conditions

$$h_{ij} = \delta_{ij} + \gamma_{ij}$$

and

$$K_{ij}$$

where γ_{ij} = O (r^{-1}) , $\partial_k\gamma_{ij}$ = O(r^{-2}) , $\partial_l\partial_k\gamma_{ij}$ = O(r^{-3}) and $\partial_k K_{ij}$= O(r^{-3}). The most important case is when S = $\mathbb{R}^3$, we only have one end U = $\mathbb{R}^3$ /B$_R$. Initial data for standard configurations of matter like stars, galaxies and gravitational collapse are modelled with S = $\mathbb{R}^3$. However, initial conditions with multiple ends and non-trivial interior $\mathscr{K}$ also appear in blackhole initial data. The total energy known as ADM energy is defined as an integral over 2 -spheres at infinity at every end by the following formula

$$E = \frac{1}{16\pi} lim_{r\to\infty} \oint_{S_r} (\partial_j h_{ij} - \partial_i h_{jj}) s^j ds_0$$

and

$$P_i = \frac{1}{8\pi} lim_{r\to\infty} \oint_{S_r} (K_{ik} - K h_{ik}) s^k ds_0$$

where s^i is its exterior unit normal and ds_0 is the surface element of the 2-sphere . For every end U$_k$, we have a corresponding energy E$_k$, and linear momentum $P^i{}_k$, which can have different values. The total mass of the spacetime is defined by

$$M = \sqrt{E^2 - P_i P_j \delta_{ij}}$$

Theorem. Positive Energy Theorem. Let ($S, h_{ij}, K_{ij}, \mu, j^i$) be an asymptotically flat , with possible many asymptotic ends, complete, initial data set. such that the dominant energy condition holds. Then the energy and linear momentum (E, P_i) satisfies

$$E \geq \sqrt{P_i P_j \delta_{ij}} \geq 0$$

at every end. Moreover, E = 0 at any end if and only if the initial data correspond to the Minkowski spacetime.

Remarks. A complete Riemannian manifold (S, h_{ij}) means no singularities are present on the initial conditions. The theorem is non-trivial where S = $\mathbb{R}^3$; when no matter field is present , that is μ= j^i = 0, this corresponds to the positivity of the energy of the pure vacuum gravitational waves. A remarkable feature of the asymptotic conditions is that they imply that the total energy can be expressed exclusively in terms of the Riemannian metric h_{ij} of the initial data and the linear momentum in terms of h$_{ij}$ and the second fundamental form K_{ij}. In the literature, E is known as the total mass and is denoted as m or M. The definition of the total energy has three main ingredients; the end U, the coordinate system x^i and the Riemannian metric h_{ij}. Critical to the definition of E is the fall off conditions of the metric. We need to elaborate further on this point. Given an end U with coordinates x^i, and an arbitrary real number α, we say that the metric h_{ij} on U is asymptotically flat of degree αif as $r \longrightarrow \infty$

$$h_{ij} = \delta_{ij} + \gamma_{ij}$$

falls off in U as γ_{ij} = O(r$^{-\alpha}$), $\partial_k \gamma_{ij}$ = O($r^{-\alpha-1}$). The trick is to determine the appropriate α decay. It turns out the energy can only be coordinate independent if we impose that $\alpha > \frac{1}{2}$.

Theorem. Let U be an end with a Riemannian metric h$_{ij}$such that it satisfies the fall off condition with $\alpha > \frac{1}{2}$. Assume that the scalar curvature R is integrable in U , that is

$$\int_U |R| dv < \infty$$

Then the energy is unique and it is finite.

This means if we calculate the energy in any coordinate system for which the metric satisfies the decay condition with $\alpha > \frac{1}{2}$, we obtain the same result. This theorem ensures that the energy is a geometric invariant of the Riemannain metric in the end U. Historically, this theorem was proven after the positive energy theorem.

Positivity. The model example is given by the initial data for the Schwarzchild blackhole, with metric on U given by

$$h_{ij} = \psi^4 \delta_{ij}$$

where $\psi = (1 + \frac{C}{2r})$, C is a constant. To ensure positivity of the energy, two conditions need to be imposed;

(1) positivity of the local energy given by the dominant energy condition

(2) the manifold should be complete or should have black hole boundaries

Time symmetric initial data fulfill the following criteria:

(1) $K_{ij} = 0$

(2) Characterized only by a Riemannian metric h_{ij}

(3) Satisfy the dominant energy condition, and

(4) $R \geq 0$

Corollary. Riemannian positive mass theorem. Let (S, h_{ij}) be a complete, asymptotically flat, Riemannian manifold. Assume that the scalar curvature is non-negative. Then the energy is non-negative at every end and it is zero at one end if and only if the metric is flat. The interesting aspect of this corollary is that it does not appeal to the constraint equations, the second fundamental form or to the matter fields. This theorem appeals only to Riemannian geometry.

Multiple Ends. Take out a point in $\mathbb{R}^3$, then the manifold S = $\mathbb{R}^3\setminus 0$ is asymptotic Euclidean at two ends, U_0 and U_1. Let $B_2$and B_1 be two balls centered at the origin; $B_1\subset B_2$. Then, $U_1 = B_1\setminus \{0\}$ and $U_0= \mathbb{R}^3\setminus B_2$. In the same way, $\mathbb{R}^3$ minus a finite number N of points i_k is a Euclidean manifold with N + 1 ends. For each i_k, take a small ball B_k of radius r_k centered at i_k, where the B_k's are disjoint. Take B_R with large R, such that B_R contains all points i_k. Let the compact set be $\mathscr{K} = B_R\setminus \sum_{k=1}^{N} B_k$, and the open sets $U_k = B_k\setminus i_k$, for $1\leq k \leq N$, , then U_0 is given by $\mathbb{R}^3\setminus B_R$.

Example. Consider the manifold S = $\mathbb{R}^3\setminus \{0\}$ and the metric given by

$$h_{ij} = \psi^4\delta_{ij}$$

and

$$\psi = 1 + \frac{C}{2r}$$

The function is smooth for anu value of C except at r= -$\frac{C}{2}$. That is, h_{ij} is smooth on S only when $C \geq 0$. S has two asymptotic ends. On $U_0 = \mathbb{R}^3\setminus B^2$, the metric is aymptotically flat in the coordinates x^i. But not so at the end $U_1 = B_1\setminus \{0\}$ in the neighbourhood of r = 0. The components are singular at r = 0. We apply the following coordinate transformations

$$y^i = (\frac{C}{2})^2 \frac{1}{r^2} x^i$$

and

$$\rho = (\frac{C}{2})^2 \frac{1}{r}$$

In terms of these coordinates, the metric has the form

$$h'_{ij} = (1 + \frac{C}{2\rho})^4 \delta_{ij}$$

We have two energies, one for each end.The two are equal and given by the constant C. The positivity of the mass is guaranteed by the completeness of the metric. When $C < 0$, the metric is defined on a manifold with boundary $S \setminus B_{-\frac{2}{C}}$ and the metric vanishes at the boundary $\partial B_{-\frac{2}{C}}$ .Proof of the positive mass theorem is complicated. The Schoen-Yau proof employs the properties of minimal surfaces in Riemannian manifolds. Consider a 2-surface on a 3-dimensional Riemannian manifold (S, h_{ij}), with unit normal vector n^i. The surface is minimal if its mean curvature $H = D_i n^i$ vanishes. This is equivalent to saying that the area of the surface is an extremum under variation.Examples of a minimum surface are a helicoid $(x_1, x_2, x_3) = (tcoss, tsins, s)$ and a catenoid, revolve acosh($\frac{z}{a}$) around the z-axis. A minimal surface Σ is stable if its area is a local minimum under variations. This is equivalent to the following condition

$$\int_S (R - K + \frac{1}{2} \parallel A \parallel^2) f^2 \leq \int_S \parallel \nabla f \parallel^2$$

where all functions f have compact support in the surface Σ, R is the scalar curvature of the Riemann manifold (S, h_{ij}) , K is the Gaussian curvature of the surface Σ, and A is the second fundamental from of Σ.

CHAPTER XIV

QUANTUM GRAVITY

14.1 Background

General relativity is a classical macroscopic theory. For a microscopic description, we need to combine general relativity with quantum mechanics. This is quantum gravity. Quantum gravity is a union of general relativity with quantum mechanics, with the Planck scale as its natural scale. The intersection of these two fields occurs at the event horizon of a blackhole.This was described in 1975 by Stephen Hawking in the context of a quantum description of particle fields in the background Schwarzchild geometry,[3], [19],[20],[21]. where he considered the quantum fluctuation of creating a particle and anti-particle pair from a vacuum. Energy momentum conservation requires that

$$p^{\mu}+\tilde{p}^{\mu}=0$$

, where p is the angular momentum. We contract with the Killing vector -$g_{\mu\nu}\mathrm{K}^{\mu}_{(t)}$, we obtain the constants of motion along their respective geodesics after creation,

$$\kappa+\bar{\kappa}=0$$

or

$$E(\infty)+\tilde{E}(\infty)=0$$

If both particles could reach $r=\infty$, then E(∞) and $\tilde{E}(\infty)$ would be the energies measured by local observers at infinity. Such energies must be positive and the equality is not satisfied. However, if this quantum fluctuation takes place close to the event horizon of a blackhole, one particle, say p , travels to $r = \infty$ and E(∞) is positive and the other , say $\tilde{p}$, falls into the singularity and if $\tilde{E}(\infty)$ is negative, then the energy relation is satisfied at the macroscopic time-scale. This latter energy can be negative, because within the event horizon g_{tt} is positive, so that the t coordinate is space-like. The result is that to a distant observer, the blackhole emits a particle with positive energy. This is known as the Hawking radiation. Hawking reasoned that inside the static limit (distance from singularity at which particles can no longer escape via radially directed motion), the Killing vector field K^{μ} that encodes time-translation symmetries is space-like,$|K^{\mu}| > 0$. So inside the static limit, particles can have negative energies,

$$E = -K^{\mu} p_{\mu}$$

His intuition was that a blackhole should have a blackbody spectrum and emit radiation according to Planck's law,

$$B_{\nu}(\nu, T) = (\frac{2\hbar\nu^3}{c^2})\frac{1}{e^{\frac{\hbar\nu}{\kappa T}} - 1}$$

where B_{ν} is the spectral radiance, measured in power per unit area; ν is frequency at absolute temperature T, $\hbar$ is Planck's constant , c is the speed of light and κ is the Boltzmann constant.

14.2 Schwarzschild Metric, Kruskal Coordinates, Rindler Coordinates and Event Horizon

The Schwarzchild metric describes the spacetime outside a gravitationally collapsed non-spinning star with zero charge. It is a spherically symmetric static solution of the vacuum Einstein field equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0$$

The line element is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

where t is the time, r is the radius and Ω is the solid angle on the 2-sphere. At r = 2GM, we have a singularity. Both g_{00} and $g_{rr} \longrightarrow \infty$. The surface $r - 2GM$ is known as the event horizon. Let ξ = $r - 2GM$. The metric then looks like

$$ds^2 = -\frac{\xi}{2GM}dt^2 + \frac{2GM}{\xi}d\xi^2 + (2GM)^2 d\Omega^2$$

Introducing a new coordinate ρ , where ρ^2 = ($8GM$) ξ, and hence dξ $^2 \frac{2GM}{\xi}$ = dρ^2. The metric takes the form

$$ds^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2 + (2GM)^2 d\Omega^2$$

From the metric, one can see that ρ measures the geodesic radial distance. The geometry factorizes. One factor is a 2-sphere of radius 2GM and the other is the (ρ, t) space

$$ds_2^2 = -\frac{\rho^2}{16G^2M^2}dt^2 + d\rho^2 \tag{14.2.1}$$

This 1 + 1 dimensional spacetime is a flat Minkowski space written in Rindler coordinates.

Rindler Coordinates. We begin with the 1 + 1 Minkowski space with a flat metric

$$ds^2 = -dT^2 + dX^2$$

In light cone coordinates,

$$U = T + X and V = T - X$$

The line element takes the form

$$ds^2 = -dUdV$$

We next change the coordinates to Rindler coordinates (u, v).

$$U = \frac{1}{\kappa}e^{\kappa u}, V = -\frac{1}{\kappa}e^{-\kappa v}$$

In these coordinates, the line element looks like

$$ds^2 = -dUdV = -e^{\kappa(u-v)}dudv$$

We make further coordinate changes

$$u = t + x, v = t - x, \rho = \frac{1}{\kappa}e^{\kappa x}$$

The line element can be written as

$$ds^2 = e^{2\kappa x}(-dt^2 + dx^2) = -\rho^2\kappa^2 dt^2 + d\rho^2 \tag{14.2.2}$$

Comparing to (14.2.1); we obtain

$$\kappa = \frac{1}{4GM}$$

κ is known as the surface gravity of the blackhole. This analysis demonstrates that Schwarzchild radius near $r = 2GM$ is not singular. The coordinates u, v of the Rindler metric do not cover all of Minkowski space; when u and v vary over the full range

$$-\infty \leq u \leq \infty, -\infty \leq v \leq \infty$$

the Minkowski coordinates vary only over the quadrant

$$0 \leq U \leq \infty, -\infty \leq V \leq 0$$

Kruskal coordinates cover the entire Schwarzchild spacetime. We introduce the 'tortoise coordinate '

$$r^* = r + 2GMlog(\frac{r-2GM}{2GM})$$

In the (r^*, t) coordinate system , the metric is conformally flat

$$ds^2 = (1 - \frac{2GM}{r})(-dt^2 + dr^{*2})$$

Near the horizon the coordinate r^* is similar to x in (50.2). Hence $u = t + r^*$ and $v = t - r^*$ are like the Rindler (u, v) coordinates. In U, V coordinates, the metric takes the form

$$ds^2 = -e^{-(u-v)\kappa} dUdV = -\frac{2GM}{r} e^{-\frac{r}{2GM}} dUdV$$

Now the metric is perfectly regular at $r = 2GM$ which is the surface $UV = 0$ and there is no singularity there.The singularity a t$r = 0$ cannpot be removed by a coordinate change because tidal forces become infinite.

Event Horizon. In the metric of (14.2.1) ,

$$ds^2 = e^{2\kappa x}(-dt^2 + dx^2) = -\rho^2\kappa^2 dt^2 + d\rho^2$$

near the horizon, the constant radius surfaces are determined by

$$\rho^2 = \frac{1}{\kappa^2} e^{2\kappa x} = \frac{1}{\kappa^2} e^{\kappa u} e^{-\kappa v} = -UV = constant$$

These surfaces are hyperbolas. We have three kinds of observers :

(1) For $r >> 2\ GM$, an observer is inertial, free falling in flat space. The trajectory is

timelike and is a straightline going upwards on a spacetimediagram. For $r > 2GM$, to stay at a fixed disatnce from a blackhole, the observer must boost the rockets to overcome gravity. Far away, the required acceleration is negligible, and the observer is almost free falling

(2) Near $r = 2GM$, the constant r lines are hyperbolas which are the trajectories of observers in uniform acceleration. The observer is definitely not free- falling. The acceleration is substantial.

(3) At $r = 2GM$, these trajectories are lightlike. This means that an observer who wishes to stay at $r = 2GM$ has to move at the speed of light.

In summary, the surface defined by r = constant is

$$timelike; r > 2GM$$

$$spacelike; r < 2GM$$

$$null(lightlike); r = 2GM$$

An event horizon is a stationary, null surface;

(1) Stationary because it is a hypersurface $r = 2GM$ that does not change with time. The Killing vector $\frac{\partial}{\partial t}$ leaves it invariant

(2) Null because g^{rr}vanishes at $r = 2GM$

Blackhole Parameters. A blackhole is an asymptotically flat spacetime that contains a region which is not in the backward lightcone of future timelike infinity for $r >> 2GM$. The boundary is a stationary null surface known as the event horizon. The fixed t slice of the event horizon is a two sphere. For a blackholes the properties are specified mass mass, charge and angular momentum. For a Schwarzchild blackhole;

(1) The radius of the event horizon is the radius of the 2-sphere. For a Schwarzchild blackhole, the radius $r = 2GM$.

(2) The area of the event horizon is $4\pi r^2$. For a Schwarzchild blackhole, $A = 16G^2M^2$

(3) The surface gravity $\kappa = 1/4GM$

14.3 Blackhole Entropy

The laws of blackhole entropy are similar to the laws of thermodynamics,[5], [6]:

(1) Zeroth Law. The surface gravity, κ, is constant on the event horizon. This is true for stationary, spherically symmetric horizons and for non-spherical horizons of spinning blackholes.

(2) First Law. Energy is conserved

$$dE = TdS + \mu dQ + \Omega dJ$$

where, E = energy, E = chrage with chemical energy = μ, J is the spin with chemical potential Ω.

For blackholes,

$$dM = \frac{\kappa}{8\pi G} dA + \mu dQ + \Omega dJ$$

For a Schwarzchild blackhole, $\mu = \Omega = 0$, because there is no charge or spin.

(3) Second Law. The total entropy never decreases $\Delta S \geq 0$. Thus for a blackhole, the net area in any process never decreases. For example, for two Schwarzchild blackholes, of masses M_1 and M_2, that coalesce, we have

$$(M_1 + M_2)^2 \geq M_1^2 + M_2^2$$

Similarities between laws of thermodynamics and blackhole physics;

Laws of Thermodynamics...

1) Temperature is constant throughout a body at equilibrium

2) Energy is conserved: dE = TdS + μdQ + ΩdJ

3) Entropy never decreases; $\Delta S \geq 0$

Laws of Blackhole Mechanics...

1) Surface gravity is constant on the event horizon. κ= constant

2) Energy is conserved: dM = $\frac{\kappa}{8\pi}$dA + μdQ + ΩdJ

3) Area never decreases; ΔA ≥ 0

A = area of horizon, M = mass of blackhole, κ is surface gravity

By the first law of thermodynamics, if a blackhole has energy E and entropy S, then it must also have a temperature T given by

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

For a Schwarzchild blackhole, the area and entropy scale as

$$S \sim M^2$$

Therefore,

$$\frac{1}{T} = \frac{\partial S}{\partial M} \sim \frac{\partial M^2}{\partial M} \sim M$$

Hawking reasoned that if a blackhole has temperature, then it must radiate. He calculated that

$$T = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi GM}$$

Bekenstein argued heuristically that for the first law of blackhole mechanics , one might say that

$$T = \varepsilon\kappa$$

where ε is a constant. And, entropy is a finite multiple η of area of event horizon

$$S = \eta A$$

with $8\pi\eta\varepsilon$ = 1. Bekenstein proposed that η is finite and and equal to $\frac{ln2}{8\pi}$. Then ε = $\frac{1}{ln2}$, and so

$$T = \frac{\kappa}{ln2}$$

Classic Thermodynamic Derivation of Blackhole Entropy**.** Blackbody radiation obeys the Stefan- Boltzmann Law

$$\frac{E}{V} = \sigma T^4$$

and

$$\frac{S}{V} = \frac{4}{3}\sigma T^3$$

where E = energy, T = temperature , V = volume, S = entropy, σ = $\frac{\pi^2}{15\hbar^3}$, an integration constant derived quantum mechanically.

For blackholes, the corresponding Bekenstein-Hawking formula is

$$T = \frac{\hbar\kappa}{2\pi}$$

and

$$S = \frac{A}{4\hbar}$$

These were first derived quantum mechanically. Here A = $4\pi(r_+^2 + a^2)$ = area; κ = $\frac{2\pi(r_+ - r_-)}{A}$ = surface gravity, $r_\pm$ = M $\pm$ $\sqrt{M^2 - a^2 - Q^2}$, $a = \frac{J}{M}$ and M, J and Q are the mass, angular momentum and charge of the blackhole.

Isolated blackholes evolve asymptotically toward a limit which is deecribed by 3 parameters, the mass, angular momentum and charge

$$S = S(M, J, Q)$$

Two blackholes of the same area have the same entropy. The entropy of a blackhole is onlya function of its area

$$S = f(A)$$

Differentiating and substituting parameters,

$$dS = f^{'}(A)dA = \frac{8\pi}{\kappa}f^{'}(A)(dM - \Omega dJ - \Phi dQ)$$

where $\Omega = \frac{4\pi}{A}$ and $\Phi = \frac{4\pi r_+ Q}{A}$. Setting $\tau = \kappa = \frac{\kappa}{8\pi f^{'}(A)}$ and re-arranging, we get

$$dM = \tau dS + \Omega dJ + \Phi dQ$$

For the Schwarzchild blackhole, J = Q = 0. Hence,

$$T = \tau = \frac{\kappa}{8\pi f^{'}(A)}$$

Geroch suggested that if one adiabatically lowers a perfectly reflecting box filled with electromagnetic radiation at a temperature T >> T_{BH} to a Schwarzchild radius r close to the event horizon. One then exchanges radiation with the hole, then adiabatically raises the box, then the local temperature at which the exchange takes place must be

$$T_\chi = \frac{T}{\sqrt{1 - \frac{2M}{r}}}$$

where the factor $\chi = \sqrt{1 - \frac{2M}{r}}$ is the red-shift factor. The gravitational pull (geodesic deviation) κ_χ felt by a local stationary observer is

$$\kappa_\chi = -\frac{d\chi}{dr} \simeq \frac{\kappa}{\chi}$$

Combining the last three equations,

$$T_\chi = \frac{1}{8\pi f'(A)}\kappa_\chi$$

This says that the local temperature depends on the local pull felt by the observer and not by the presence of the local blackhole. Therefore f'(A) is a universal constant.That is,

$$f'(A) = \xi$$

where ξ is a constant. Integrating,

$$f(A) = \xi A + C$$

, where C is a constant of integration. Since for M = 0, S = 0 $\Rightarrow$ C = 0. Therefore,

$$S = \chi A$$

and

$$T = \frac{\kappa}{8\pi\xi}$$

One cannot evaluate ξ thermodynamically without quantum mechanics

14.4 Classic Field Theory

We describe the dynamics of classical fields, how they change with time, using the Lagrangian and the principal of least action. The evolution of a system progresses along the path of least action, where the action S is defined in terms of the Lagrangian. Euler-Lagrange Equations and the Principle of Least Action. The action is

$$S = \int \mathscr{L}(\phi, \partial_\mu \phi) d^4x$$

where $\mathscr{L}$ is the Lagrangian and the integration is over spacetime. The field will adopt the configuration which reduces the action to a minimum. To find this configuration, we look for a field configuration such that infinitesimally small variations of the field leave the action unchanged

$$\phi(x) \longrightarrow \phi(x) + \delta\phi(x) \Rightarrow S \longrightarrow S + \delta S, \delta S = 0$$

$$S = \int \mathscr{L}(\phi, \delta_\mu \phi) d^4x$$

$$\delta S = \delta\phi \frac{\partial}{\partial\phi} \int \mathscr{L}(\phi, \partial_\mu \phi) d^4x + \delta(\partial_\mu \phi) \frac{\partial}{\partial(\partial_\mu \phi)} \int \mathscr{L}(\phi, \partial_\nu \phi) d^4x$$

Since $\delta(\partial_\nu \phi) = \partial_\mu(\delta\phi)$;

$$\partial_\mu(\delta\phi \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}) = \partial_\mu(\delta\phi) \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} + \delta\phi \partial_\mu(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)})$$

So,

$$\delta S = \int [\delta\phi \frac{\partial \mathscr{L}}{\partial\phi} + \partial_\mu(\delta\phi \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}) - \delta\phi \partial_\mu(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)})] d^4x$$

The second term of the right hand side is a total derivative and it vanishes at the boundary. Hence, the Euler Lagrange equation is

$$\frac{\partial \mathscr{L}}{\partial\phi} - \partial_\mu(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}) = 0$$

Consider the Lagrangian $\mathscr{L} = \partial_\mu \phi \partial^\mu \phi$; then

$$\frac{\partial \mathscr{L}}{\partial\phi} = 0, \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \Longrightarrow \partial_\mu(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}) = \partial_\mu \partial^\mu \phi$$

So the Euler Lagrange equation turns into the wave equation

$$\partial_\mu \partial^\mu \phi = (\frac{\partial^2}{\partial t^2} - \nabla^2)\phi = 0$$

Noether's Theorem. States that if the action is unchanged under a transformation, then there exits a conserved current associated with the symmetry.

Consider infinitesimal transformations of the coordinates and the fields, parametrised by the infinitesimal parameter ω^ν;

$$x^\mu \longrightarrow x'^\mu = x^\mu + \delta x^\mu = x^\mu + X^\mu_\nu \omega^\nu$$

$$\phi \longrightarrow \phi' = \phi + \delta\phi = \phi + \Phi_\nu \omega^\nu$$

The change in the Lagrangian is

$$\delta\mathscr{L} = \partial_\mu \mathscr{L} \delta x^\mu + \frac{\partial \mathscr{L}}{\partial \phi} \delta_\phi \phi + \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} \delta_\phi(\partial_\nu \phi)$$

$$\delta\mathscr{L} = \partial_\mu \mathscr{L} \delta x^\mu + \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)}\right) \delta_\phi \phi + \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} \partial_\nu(\partial_\phi \phi)$$

$$\delta\mathscr{L} = \partial_\mu \mathscr{L} \delta x^\mu + \partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)}\right) \delta_\phi \phi$$

The change in action is

$$\delta S = \delta\left(\int d^4 x \mathscr{L}\right)$$

The integration measure also changes due to tranformation via the Jacobian;

$$d^4 x' = \left|\frac{\partial x'}{\partial x}\right| d^4 x = (1 + \partial_\mu \delta x^\mu) dx^4$$

The change in the action is

$$\delta S = \int^4 x(\delta\mathscr{L} + \mathscr{L} \partial_\mu \delta x^\mu)$$

$$\delta S=\int d^4x(\delta_\mu\mathscr{L}\delta x^\mu+\partial_\nu(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi)}\delta_\phi\phi)+\mathscr{L}\partial_\mu\delta\mathrm{x}^\mu)$$

$$\delta S=\int d^4x\partial_\mu(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi)}\delta_\phi\phi)+\mathscr{L}\delta\mathrm{x}^\mu)$$

Noting that δx^μ = $\mathrm{X}^\mu{}_\nu\omega^\nu$, $\delta\phi$ = $\Phi_\nu\omega^\nu$ and $\delta\phi$ = $\delta_\phi\phi$ + $\partial_\mu\phi\delta\mathrm{x}^\mu$;

$$\delta_\phi\phi=\delta\phi-\partial_\mu\phi\delta x^\mu=(\Phi_\nu-\partial_\mu\phi X^\mu_\nu)\omega^\nu$$

We can write the change in the action in terms of the divergence of a current

$$\delta S=-\int dx^4\partial_\mu j^\mu_\nu\omega^\nu$$

where

$$j^\mu_\nu=(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi-g^\mu_\rho\mathscr{L})X^\rho_\nu-\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\Phi_\nu \tag{14.4.1}$$

To ensure that the action is invariant under this transformation, this must be a conserved current;

$$\partial_\mu j^\mu_\nu=0$$

This is Noether's theorem.

Energy-Momentum Tensor**.** The most common application of Noether's theorem is its application to space-time translation.

$$x^\mu\longrightarrow\acute{x}^\mu=x^\mu+X^\mu_\nu\omega^\nu\Longrightarrow X^\mu_\nu=g^\mu_\nu$$

$$\phi\longrightarrow\phi^{'}=\phi+\delta\phi=\phi+\Phi_\nu\omega^\nu\Longrightarrow\Phi_\nu=0$$

Substituting into (52.1)

$$T^{\mu\nu} = (\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - g^\mu_\rho \mathscr{L} \tag{14.4.2}$$

This is the energy-momentum tensor for the field ϕ.

Green's Function Revisited. Imagine a field satisfies a differential equation of the form

$$D\phi(x) = \rho(x)$$

where D is a differential operator and ρ is a source. A great example is Poisson's equation

$$\nabla^2 \phi = \rho(x)$$

Let G(x,y) be a solution, but with a point source at $x = y$, so that

$$DG(x,y) = \delta(x-y)$$

Then

$$D\int G(x,y)\rho(y)dy = \int \delta(x-y)\rho(y)dy = \rho(x)$$

In other words,

$$\phi(x) = \int G(x,y)\rho(y)dy$$

is a solution to the original equation. The function G(x,y) is a Green's function. Green's functions let us convert the problem of solving a differential equation into the problem of doing an interval.

Quantum Mechanics Synopsis,[10],[36]. A quantum mechanical state can be completely described by a state vector in an infintely dimensional complex vector space known as a Hilbert space. We use Dirac's bra and ket notation. All information about the state is contained in the vector. A vector is written as

$$|\psi>$$

Its complex conjugate is written as

$$<\psi| \equiv |\psi>^*$$

Observables. Every observaable A has a corresponding linear Hermitian operator $\hat{A}$ acting on the Hilbert space

$$\hat{A} = \hat{A}^\dagger$$

for which there is complete set of orthonormal eigenvectors $| a >$ with eigenvalue a

$$\hat{A} = |a>= a|a>$$

Since these eigenvectors span the space, they are complete, we can write

$$\int |a><a|da = 1$$

Any state vector can be written as

$$|\psi>= \int \psi_A(a)|a> da$$

The function $\psi_A(a)$ is the wavefunction in the eigenspace of $\hat{A}$ and can be obtained via

$$<a|\psi>= \int \psi_A(b) <a|b> db = \int \psi_A(b)\delta(a-b)db = \psi_A(b)$$

, where we have used the orthonormality relation $<a|b> = \delta\ (\ a-b\ )$. A measurement of the observable A will return a result with probability

$$| < a|\psi > |^2 = |\psi_A(a)|^2$$

After the measurement, the state will no longer be $|\ \psi >$ but will have collapsed onto the corresponding eigenvector $|a >$.

The Position and Momentum Eigenbases. The position and momentum bases correpond to the position $\hat{x}$ and momentum $\hat{p}$ operators. The position space wave function

$$< x|\psi >= \psi_x(x)$$

and the probability of finding the particle at position x is

$$| < x|\psi > |^2 = |\psi_x(x)|^2$$

Since $|\ x>$ and $|\ p>$ are not aligned bases, the state cannot be an eigenvector of position and momentum simultaneously. Also, since the measurements change the state of the system, the order of measurement is important

$$\hat{x}\hat{p}|\psi >\neq \hat{p}\hat{x}|\psi >$$

Commutator . We postulate a commutator relation

$$[\hat{x},\hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$$

Note these operators are observables, hence Hermitian. If we operate on the vector < x, y >

$$[\hat{x},\hat{p}] < x,y >=< x|[\hat{x},\hat{p}],y >=< x|\hat{x}\hat{p}|y > - < x|\hat{p}\hat{x}|y >= (x-y) < x|\hat{p}|y >$$

$$i\hbar\delta(x-y) = (x-y) < x|\hat{p}|y >$$

$$< x|\hat{p}|y >= i\hbar\frac{\delta(x-y)}{x-y} = i\hbar\frac{\partial}{\partial y}\delta(x-y)$$

The last term follows from the property of the Dirac delta functions. However,

$$< x|\hat{p}|\psi >= \int < x|\hat{p}|y >< y|\psi > dy = \int (i\hbar\frac{\partial}{\partial y}\delta(x-y)) < y|\psi > dy = -i\hbar\frac{\partial}{\partial x}\psi_x(x)$$

Therefore,

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

is a suitable representation for the one-dimensional momentum operator.Thus, the position and momentum operator are related in this way

$$p < x|p >= -i\hbar\frac{\partial}{\partial x} < x|p >$$

Solving this differential equation gives

$$< x|p >= \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}$$

Taking the adjoint

$$< p|x >= \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{-ipx}{\hbar}}$$

The bases are related by Fourier transformation

$$|p >= \frac{1}{\sqrt{2\pi\hbar}}\int e^{\frac{ipx}{\hbar}}|x > dx$$

and

$$|x> = \frac{1}{\sqrt{2\pi\hbar}} \int e^{\frac{-ipx}{\hbar}} |p> dp$$

This embodies the Heisenberg Uncertainty principle.

The Schrodinger Equation. This is simply a statement about conservation of energy; Total Energy, $\hat{H}$= kinetic energy + potential energy

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$

Multiply Schrodinger's equation and its adjoint with ψ^* and ψ respectively;

$$\psi^* i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \psi \nabla^{*2} \psi + V\psi^* \psi$$

$$\psi i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \psi \nabla^2 \psi^* + V\psi\psi^*$$

Subtracting;

$$i\hbar \frac{\partial [\psi^* \psi]}{\partial t} = \frac{\hbar^2}{2m} [-\psi^* \nabla^2 \psi + \psi \nabla^2 \psi^*] = \frac{\hbar^2}{2m} \nabla . [-\psi^* \nabla \psi + \psi \nabla \psi^*]$$

$$\frac{\partial [\psi^* \psi]}{\partial t} + \frac{\hbar}{2im} \nabla . [\psi \nabla \psi - \psi \nabla \psi^*] = 0$$

Compare with the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla . J = 0$$

where ρ = $|\psi|^2$ and J = $\frac{\hbar}{2im}[\psi^* \nabla \psi$ -$\psi \nabla \psi]$; ρ the probability is conserved.

The Harmonic Oscillator and Ladder Operators. The quantum harmonic oscillator is defined by the Hamiltonian, $\hat{H}$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

Define the ladder oprators

$$\hat{a} = \frac{1}{\sqrt{2}}(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{\hbar}{m\omega}}\hat{p})$$

$$\hat{a^{\dagger}} = \frac{1}{\sqrt{2}}(\sqrt{\frac{m\omega}{\hbar}}\hat{x} - i\sqrt{\frac{\hbar}{m\omega}}\hat{p})$$

Then,

$$[\hat{a}, \hat{a}^{\dagger}] = 1$$

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}$$

and

$$[\hat{H}, \hat{a}^{\dagger}] = \hbar\omega\hat{a}^{\dagger}$$

The Hamiltonian can be written as

$$\hat{H} = \frac{1}{2}\hbar\omega(\hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a}) = \hbar\omega(\hat{a}\hat{a}^{\dagger} + \frac{1}{2})$$

Consider an eigenstate $| n >$ with an eigenvalue E_n, so that

$$\hat{H}|n> = E_n|n>$$

Then

$$\hat{H}\hat{a}^{\dagger}|n>=([\hat{H,}\hat{a}^{\dagger}]+a^{\dagger}\hat{H})|n>=(\hbar\omega\hat{a}^{\dagger}+\hat{a}^{\dagger}\hat{H})|n>=(\hbar\omega\hat{a}^{\dagger}+\hat{a}^{\dagger}E_n)|n>$$

$$=(E_n+\hbar\omega)\hat{a}^{\dagger}|n>$$

Hence $\hat{a}^{\dagger}$ | n > is an eigenstate but with an eigenvalue E$_{n+1}$ = E$_n$ + $\hbar\omega$. Similarly $\hat{H}\hat{a}$ | n > = (E$_n$ - $\hbar\omega$)$\hat{a}$ | n >.$\hat{a}^{\dagger}$ is a creation operator - it creates one quantum of energy $\hbar\omega$ - while $\hat{a}$ is an annihilation operator. By definition, the ground state | 0 > has the lowest energy , so we must have $\hat{a}|0>=0$. The ground state energy is

$$\hat{H}|0>=\hbar\omega(\hat{a}\hat{a}^{\dagger}+\frac{1}{2})|0>=\frac{1}{2}\hbar\omega|0>$$

Therefore, the remarkable finding that the ground state has non-zero energy.

14.5 Scalar Field Theory

The simplest theory is that of a free real scalar field with Lagrangian

$$\mathscr{L}=\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi-\frac{1}{2}m^2\phi^2$$

The Euler Lagrange Equation gives us;

$$\frac{\partial\mathscr{L}}{\partial\phi}=-m^2$$

$$\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi)}=\partial^{\mu}\phi\Longrightarrow\partial_{\mu}(\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi)})=\partial_{\mu}\partial^{\mu}\phi$$

Hence, we obtain the Klein Gordon Equation

$$(\partial^2+m^2)\phi=0$$

where $\partial^2 = \partial^\mu \partial_\mu = \Box$,[10], [36].

The Klein - Gordon Equation. The Schrodinger equation for the quantum wave function is based on the nonrelativistic expression for the energy of a particle. The Klein-Gordon equation is the first step to more complex relativistic equations. For now, we will restrict ourselves to a spinless particle in empty space. If there is no potential energy, classical physics says that the energy E is just the kinetic energy of the particle

$$\frac{p^2}{2m}$$

, p is the linear momentum, m is the mass. Quantum mechanics replaces the energy E by the operator

$$i\hbar\frac{\partial}{\partial t}$$

and the momentum p by

$$-i\hbar\nabla$$

Then it applies the resulting operators on the wave function Ψ. This then produces the Schrodinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi$$

Solutions with a definite value E for the energy take the form

$$\Psi = ce^{-\frac{iEt}{\hbar}}\psi$$

Substituting into the Schrodinger equation produces the Hamiltonian eigenvalue problem

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi$$

ψ is called the energy eigenfunction. The energy E of a particle in empty space is the kinetic energy plus the rest mass energy mc^2. According to special relativity, the mass in motion m_v is related to the mass at rest m by

$$m_v = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The true knietic energy T is

$$T = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} c^2 - mc^2$$

Quantum mechanics does not use the speed of a particle, but its momentum, p = m_vv. The total kinetic energy, kinetic plus rest mass energy , after substituting for p ,then m_v gives

$$E = T + mc^2 = \sqrt{(mc^2)^2 + p^2c^2}$$

Square both sides and substituting into the Schrodinger equation yields the Klein Gordon equation

$$-\frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} + \nabla^2 \Psi = (\frac{mc^2}{\hbar c})^2 \Psi$$

In empty space,

$$\nabla^2 \psi = -k^2 \psi$$

where for The Klein-Gordon equation

$$k = \frac{\sqrt{E^2 - (mc^2)^2}}{\hbar c}$$

and for the Schrodinger equation

$$k = \frac{\sqrt{2mE}}{\hbar}$$

k is the wave number. Schrodinger's equation, non-relativistic, does not include the rest mass. It is an approximation of the Klein-Gordon equation. Relativistic or not, the magnitude of linear momentum is given by the de Broglie relation

$$p = \hbar k$$

Note, the momentum operator is

$$\hat{p} = -i\hbar \Longrightarrow \hat{p}^2 = -\hbar^2 \nabla^2$$

Relativistic or not, the energy E is associated with the operator

$$i\hbar \frac{\partial}{\partial t}$$

This means that the the time-dependent factor in states of definite energy is

$$e^{-\frac{iEt}{\hbar}}$$

Hence the energy can be associated with an angular frequency ω by writing the exponential as

$$e^{-i\omega t}$$

The relationship between energy and frequency is then given by the Planck-Einstein relation

$$E = \hbar \omega$$

The wave number k is the quantum number of linear momentum and ω is the angular frequency of energy. The Schrodinger equation is square integrable

$$\int |\Psi|^2 d^3 r = 1$$

The integral represents the probability of finding the paricle. It is 1 wherever you look. It must be somewhere. Because the integral stays at 1, whenever you look, the particle must be somewhere. This ensures that the particle cannot disappear and no second particle can show up out of nowhere. The Klein Gordon equation is not square integrable. Therefore, the number of particles is not necessatily preserved. In relativity, the mass-energy equivalence allows particles to be created or destroyed.The integral that the Klein Gordon Equation preserves is

$$\int |\frac{1}{c}\frac{\partial\Psi}{\partial t}|^2 + |\nabla\Psi|^2 + |\frac{mc^2}{\hbar c}\Psi|^2 d^3 r$$

Even though the square energy E^2 is positive, the energy E is positive or negative. The negative energy solutions represent anti-particles. Ant-particles have positive energy and move backwards in time. They cannot have negative energy because there is no ground state, no lower limit to the energy. Consider two hypotehtical wave functions of the form

$$e^{-\frac{iEt}{\hbar}}\psi_1$$

and

$$e^{\frac{iEt}{\hbar}}\psi_2$$

The first wave function is a particle of positive energy E . The second wave function is an anti-particle of positive energy that moves back in time. We will next explore these ideas with the quantum mechanics formalism. The vector r represnts (x, y, z). $d^3 r$ is omitted for brevity. We wll first show that

$$\int |\Psi|^2 = 1$$

is a constant. In index notation, the Schrodinger equation in free space is

$$i\hbar\frac{\partial\Psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\Psi_{ii}=0$$

Note that any solution of the Schrodinger equation takes the form

$$\Psi = c_n e^{-iE_nt}\psi_n(r)$$

, the Einstein summation convention applies.Since ψ_n is an energy eigenfunction, we can apply the orthonormality of eigenfunctions. Then

$$\int|\Psi|^2=\int\Psi^*\Psi=c_n^*c_n$$

This does not depend on time, and the normalization requirement makes it 1. For the Klein=Gordon equation, the solution is

$$\Psi = c_n e^{-iE_nt}\psi_n(r)+d_n e^{iE_nt}\psi_n(r)$$

The first sum is the particles states and the second sum is the anti-particle states.

$$\int|\Psi|^2=c_n^*c_n+d_n^*d+c_n^*d_n e^{2iE_nt}(r)+d_n^*c_n e^{-2iE_nt}$$

The final two terms in the sum oscillate, so the integral is no longer constant. If d$_n$ = 0, we only have particle states, if c$_n$= 0 , we only have anti-particle states ; only then is the integral constant.The Klein Gordon equation preserves the sum of integral

$$\int|\frac{1}{c}\frac{\partial\Psi}{\partial t}|^2+|\nabla\Psi|^2+|\frac{mc^2}{\hbar c}\Psi|^2$$

As before, multiply the Klein Gordon equation by $\frac{\partial\Psi^*}{\partial t}$ and add to the complex conjuagte of the obtained equation; let us analyze the three terms;

$$\int \frac{1}{c^2}(\frac{\partial\Psi^*}{\partial t}\frac{\partial^2\Psi}{\partial t^2}+\frac{\partial\Psi}{\partial t}\frac{\partial^2\Psi^*}{\partial t^2})=\frac{1}{c^2}\frac{d}{dt}\int\frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial t}=0$$

The second and third terms also reduce to zero.

Heisenberg Uncertainty Principle. For non-commuting Hermitian operators, we establish a bound on the uncertainty in their expectation values. Given a state $|\psi>$, the mean square uncertainty is defined as

$$(\Delta A)^2 =< \psi|\hat{U}^2\psi >$$

and

$$(\Delta B)^2 =< \psi|\hat{V}^2\psi >$$

where the operators

$$\hat{U} = \hat{A}- < \psi|\hat{A}\psi >$$

and

$$\hat{V} = \hat{B}- < \psi|\hat{B}\psi >$$

Next, we take the scalar product of $\hat{U} \mid \psi > + \mathrm{i}\lambda\ \hat{V} \mid \psi >$

$$[\hat{U}|\psi > +i\lambda\hat{V}|\psi >][\hat{U} < \psi| - i\lambda\hat{V} < \psi|]$$

Since these are Hermitian operators,

$$< \psi|\hat{U}^2\psi > +\lambda^2 < \psi|\hat{V}^2\psi > +i\lambda < \hat{U}\psi|\hat{V}\psi > -i\lambda < \hat{V}\psi|\hat{U}\psi > \geq 0$$

Hence,

$$(\Delta A)^2 + \lambda^2 (\Delta B)^2 + i\lambda < \psi|[\hat{U},\hat{V}]|\psi > \geq 0$$

Minimising expression with respect to λ

$$2\lambda(\Delta B)^2 + i < \psi|[\hat{U},\hat{V}]|\psi >= 0$$

Therefore,

$$\lambda = \frac{-i}{2}\frac{< \psi|[\hat{U},\hat{V}]|\psi >}{\Delta B^2}$$

Substituting λ back into the inequality;

$$(\Delta A)^2(\Delta B)^2 \geq -\frac{1}{4} < \psi|[\hat{U},\hat{V}]|\psi >^2$$

Therefore, for non-commuting operators , the uncertainities obey the following inequality

$$\Delta A \Delta B \geq \frac{i}{2} < [\hat{U},\hat{V}] >= \frac{1}{2} < [\hat{A},\hat{B}] >$$

since $< \hat{A} >$ and $< \hat{B} >$ are just constants. If the commutator is a constant, as in the case of the conjugate operators

$$[\hat{p},x] = -i\hbar$$

the expectation values can be dropped and we obtain the relation

$$(\Delta A)(\Delta B) \geq \frac{i}{2}[\hat{A},\hat{B}]$$

For momentum and poistion, this result recovers the Heisenberg Uncertainty Principle

$$\Delta p \Delta x \geq \frac{i}{2} < [\hat{p},x] >= \frac{\hbar}{2}$$

For time and energy , [$\hat{E}$, t] = i$\hbar$;

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Density Matrices**.** These are density operators - they encode all the accessible information about a quantum mechanical system. The stste vectors | ψ > on a Hilbert space describe ' pure ' states which are idealized descriptions that cannot characterize incoherent mixtures which commonly occur in nature.

General properties of density matrices: Consider n observable A in the pure state | ψ > with the expectation value given by

$$< A >_{\psi} = < \psi | A | \psi >$$

Definition. The desnity matrix ρ for the pure state is given by

$$\rho := | \psi >< \psi |$$

The density matrix has the following properties

(1) projector $\rho^2 = \rho$

(2) hermicity $\rho^{\dagger} = \rho$

(3) normalization Tr ρ = 1

(4) positivity $\rho \geq 0$

The trace of an operator D is given by

$$TrD := \sum_{n} < n | D | n >$$

where { | n> } is a basis .

Example.Take the operator

$$D = |\psi >< \phi|$$

and let us calculate the trace

$$TrD = \sum_n < n|\psi >< \phi|n >= \sum_n < \phi|n >< n|\psi >=< \phi|\psi >$$

Theorem. The expectation value of an observable A in a ttate, represented by a density matrix ρ, is given by $< A >_\rho$ = Tr (ρA}.

Proof. Tr (ρA) = Tr (| ψ > < ψ | >) = $\sum_n$ (< n | ψ > < ψ | > n > = $\sum_n$ (< ψ | > n > < n | ψ > = < ψ | A | ψ > = < A >

14.6 Complex Structure

Definition. Suppose we have a real vector space V. A linear operator J : V $\longrightarrow$ V is called a complex structure if J^2 = -1 . Note that V has to be even or∞-dimensional for there to exist a complex structure onV.

Choosing a J amounts to choosing a decomposition into positive and negative frequencies. Suppose we have a decomposition

$$V_C = V \oplus iV = V^{(+)} + V^{(-)}$$

and projection operators

$$P^{\pm} : V_C \longrightarrow V^{(\pm)}$$

such that

$$P^+ + P^- = I$$

and

$$P^{+} = (P^{-})^{*}$$

It easily follows that

$$P^{+}P^{-} = P^{-}P^{+} = 0$$

Given such a decomposition, we can define

$$J\varphi = iP^{+}\varphi - iP^{-}\varphi$$

We must show that J is a linear map V $\longrightarrow$ V. We embed V into V_C by

$$v \longmapsto (v, 0)$$

where

$$J(\varphi, 0) = iP^{+}(\varphi, 0) - iP^{-}(\varphi, 0)$$

It can be easily shown that J^2 = - I. Conversely, if we have a J, we can define projectors P^{+}, P^{-}. Suppose we have J : V $\longrightarrow$ V with J^2 = - I. Extend J to V_C as above. Define

$$P^{+}\varphi = \frac{1}{2i}[i\varphi + J\varphi]$$

and

$$P^{-}\varphi = \frac{1}{2i}[i\varphi - J\varphi]$$

It is easy to check that thesehave the required properties. Thus, we have a decomposition of the vector space of complex solutions into two components; equivalent inro positive and negative frequency components. This way, we gave a complex structure on the real vector space of classical solutions. We begin with the classic theory. Our phase space V consists of real solutions to the Klein-Gordon equations

$$(\Box - m^2)\varphi = 0$$

Suppose we have an n-dimensional manifold M, and a smooth $(n-1)$ dimensional submanifold S. Given φ,

$$\dot{\varphi} = \overrightarrow{n}.\nabla\varphi$$

on S, and a partial differential equation. The Cauchy problem consists of finding solutions ψ to the partial differential equation on M, which agree with φ on S. The symplectic structure on our manifold is a hypersurface integral

$$\Omega(\varphi_1, \varphi_2) = \int_{\Sigma} (\varphi_1 \nabla \varphi_2 - \varphi_2 \nabla \varphi_1).d\sigma$$

where Σ is the Cauchy surface. Using this we can write a solution of the Cauchy problem as

$$\varphi(x) = \int_{\Sigma} d\sigma(y)[\varphi(y)\nabla G(x,y) - G(x,y)\nabla\varphi(y)]$$

where G is the Green's function . So we have a real vector space V and a symplectic 2-form Ω on V. The traditional approach was to use Fourier transforms to decompose the real solutions into positive and negative frequency components. Instead, here we complexify the complex space of solutions

$$V_c = V \oplus iV$$

Scalar multiplication on this space is defined by

$$(a+ib)(u,v) = (au-bv)(av+bu)$$

with the Klein-Gordon inner product

$$< \varphi, \varphi >= i \int_{\Sigma} (\psi^* \nabla \varphi - \varphi \nabla \psi^*) d\sigma$$

14.7 Symplectic Geometry

Motivation. Firstly, as will be seen later, symplectic geometry is an even dimensional geometry . It lives on even dimensional spaces and measures the size of even dimensional objects. It is naturally associated with the field of complex numbers. The concept arose in the study of classical mechanical systems. The trajectory of a system is determined by its poistion q and its momnetum p. This pair of real numbers (p , q) give a point in the plane $\mathbb{R}^2$. In this case the symplectic structure ω is an area form in the plane

$$dp \wedge dq$$

It measures the area of each open region S in the plane. We think of this region as oriented. We choose a direction in which we traverse the boundary ∂S. Hence the area is signed. By Stokes Theorem

$$\int_S dp \wedge dq = \int_{\partial S} pdq$$

We think of pdq as the action around the boundary. It has been shown that this area is preserved under time evolution . That is, for a set of particles with position and momentum in the region S_1 at time t_1, the position and momentum of the particles at time t_2 will form a region S_2 of the same area, see Figure 14.1 below.

In quantum mechanics, area has another meaning. Heisenberg's uncertainty principle states that position and momentum can be known simultaneously with a certain degree of accuracy. We can think of a particle as lying not at asingle point in a plane, but rather lying in a region of the plane. The quantization principle says that the area of this region is quantized. One

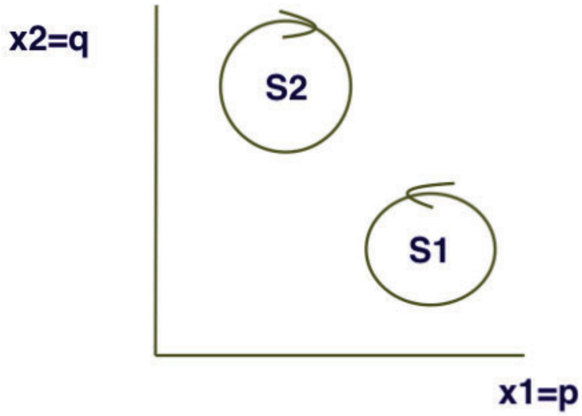

Figure 14.1: Signed symplectic areas; S_1has positive sign , S_2 negative sign

can think of the symplectic area as a measure of the entaglement of position and momentum. Suppose we are given a particle of mass m in $\mathbb{R}^n$ acted upon by a conservative force F(x)

$$F(x) = -\nabla U(x)$$

where U(x) is the potential energy.Then the Lagrangian of the system is

$$L = \frac{1}{2}m\dot{x}^2 - U(x)$$

Applying the Euler - Lagrange equation;

$$\frac{d}{dt}(m\dot{x}) + \nabla U = m\ddot{x} - F(x) = 0$$

We define the Hamiltonian system as

$$H(t,x,p) = \sum p^j \dot{x}^j - L$$

where p is the momentum. As shown in appendix C , Hamilton's equations become

$$\frac{\partial H}{\partial p} = \dot{x}, \frac{\partial H}{\partial x} = -\dot{p}$$

If we take the coordinates $z = (x^i, p_i) \in \mathbb{R}^{2n}$, then the Hamilton equations become

$$\frac{dz}{dt} = \begin{pmatrix} p_i \frac{dx^i}{dt} \\ x^i \frac{dp_i}{dt} \end{pmatrix} = \begin{pmatrix} p_i \frac{\partial H}{\partial p_i} \\ -x^i \frac{\partial H}{\partial x^i} \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial \\ \partial \end{pmatrix} H(t,z) = -\omega_0 \nabla H(t,z)$$

where ω_0 = $\begin{pmatrix} 0 & -id_{R^n} \\ id_{R^n} & 0 \end{pmatrix}$. The time evolution of the system is given as the flow along the Hamiltonian vector field - $\omega_0 \nabla$H(t,z). It is to be noted that our geometry is always even dimensional, as each positional coordinate is accompanied by a momentum coordinate. Poistion and momentum are intertwined by a skew-symmetric non-degenrate bilinear from ω_0.

An object in the plane has two position coordinates q_1 and $q_2$and two velocity coordinates $p_1$and p_2. So it is described by a point

$$(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$$

The symplectic form ω measures the signed area of 2-dimensional surfaces S in $\mathbb{R}^4$by adding the areas of the projections of S to the (x_1, x_2) and the (x_3, x_4) plane; see Figure 14.2 below.

$$\omega = area(proj_{12}S_1) + area(proj_{34}S_2)$$

ω is a differential 2-form written as

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

The area is evaluated by integrating the form over the surface S

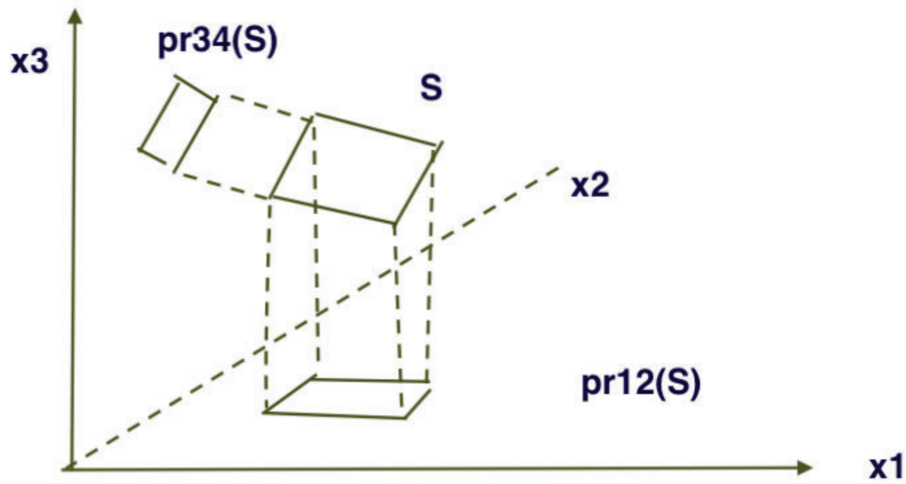

Figure 14.2: Symplectic area projection; ω(S) is the sum of proj_{12} and proj_{34}

$$\omega(S) = \int_S \omega$$

For particles moving in n dimensions, the symplectic area form is the sum of contributions from each of the n pairs of directions

$$\omega_0 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + ... + dx_{2n-1} \wedge dx_{2n}$$

Definition. A symplectic form ω on any even dimensional smooth manifold M is a closed non-degenerate 2 -form, where the non-degeneracy condition is that for each non-zero tangent direction v, there is another direction w such that the area ω(v,w) of the infinitesimal parallelolgram spanned by these vectors is non-zero.

Formalism. Symplectic geometry is the geometry of a closed skew-symmetric form. Its concepts are expressed in differential forms. It is a two dimensional geometry that meaures the area of complex curves instead of the length of real curves. Let M^{2n} be a smooth closed manifold, that is , a compact smooth manifold without a boundary. A symplectic structure ω on M is closed

$$d\omega = 0$$

and a non-degenerate smooth 2-form

$$\omega^n = \omega \wedge ...... \wedge \omega \neq 0$$

Thus, the intrinsic measurements one can make on a symplectic manifold are 2-dimensional. If S is a little piece of 2-dimensional surface then one can meaure

$$\int_S \omega = area_\omega S$$

By Stokes' theorem, the closedness of ω is equivalent to saying that this integral does not change when one deforms S keeping its boundary fixed.

$$\int_S \omega = \int_{\partial S} d\omega = constant$$

The non- degeneracy condition is equivalent to the fact that ω induces an isomorphism

$$T_x M \longrightarrow T_x^* M$$

$$vectorfields \longrightarrow 1 - forms$$

$$X \longmapsto \iota_X \omega = \omega(X, .)$$

Example. The linear form ω_0 = dx_1 $\wedge$ dy_1 .... $dx_n \wedge dy_n$ on Euclidean space $\mathbb{R}^{2n}$. The isomorphism is given by

$$X = \frac{\partial}{\partial x_j} \longmapsto \iota_X \omega_0 = dy_j$$

$$Y = \frac{\partial}{\partial y_j} \longmapsto \iota_Y \omega_0 = -dx_j$$

In Riemann geometry, one identifies the tangent space $T_x\mathbb{R}^{2n}$ of vectors and the cotangent space $T_x^*\mathbb{R}^{2n}$ of covectors (1-forms) by making the following identification

$$\frac{\partial}{\partial x_j} = dx_j$$

$$\frac{\partial}{\partial y_j} = dy_j$$

The isomorphism given by a symplectic form differs from this by a rotation through a quarter of a turn. If $z_j = x_i + y_j$, then this quarter turn correspnds to multiplication bi i.

Theorem. Darboux's Theorem. Every symplectic form is locally diffeomorphic to ω_0 = $dx_1 \wedge dx_2$ .... $dx_{2n-1} \wedge dx_{2n}$. Paraphrasing; given a symplectic form ωon a manifold M and any point on M, one can always find coordinates ($x_1, x_2, ..., x_{2n}$) defined in an open neighbouhood U of this point such that in this coordinate system ω is given on the whole open set U by ω_0 =$dx_1 \wedge dx_2$..........$dx_{2n-1} \wedge$ dx_{2n}

Darboux's theorem says that all symplectic structures are locally indistinguishable. Thus, locally all symplectic forms are the same. In other words , all symplectic invariants are global in nature. This local uniqueness of symplectic structures gives them a rich group of automorphisms. A symplectic form ω has an important invariant ω, called its homology class [ω]. This class is determined by the areas ω(S) of all closed surfaces S in M. For compact M, the class [ω] is determined by a finite number of these areas ω(S_i) and so contains only a finite number of information. An important theory of Moser says that one cannot change the symplectic form in any important way by deforming it, provided that the homology class is unchanged. If ω_t , where $t \in [0,1]$, is a smooth path of symplectic forms such that [ω_0] = [ω_t] for all t , then these forms are the same. The idea is that we cannot find new structures by deforming the old ones, provided that we fix the the integrals of our forms over all closed surfaces. This Moser's stabilty theorem.

Symplectomorphism . There are many ways to move the points of the underlying space M without changing the symplectic structure ω. Such a movement is a symplectomorphism ;

(1) ϕ is a diffeomorphism ; a bijective and smooth map ϕ : M $\longrightarrow$ M , giving rise to the movement $x \mapsto \phi$ (x) of the points x of the space M

(2) it preserves symplectic area, that is , ω(S) = ω(ϕ(S)) for all little pieces of surface S.

(3) since $\omega \wedge \omega$ is a volume form, symplectomorphisms preserve volume.

Example. In 2 - dimensions, a symplectomorphism ϕ is an area preserving transformation. For example ψ (x_1, x_2) = ($2x_1, \frac{1}{2}x_2$) . In 4 -dimensions, the situation is different. Let B be round ball of radius 1 in $\mathbb{R}^4$;

$$B = (x_1, x_2, x_3, x_4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq 1$$

Note that ϕ has to preserve pairs (x_1, x_2) and $(x_3, x_4$) and is made by combining area preserving transformations in each of these two planes. There are symplectomorphisms that mix pairs. If B is our unit ball above , then let Z(r) be a cylinder

$$Z(r) = D^2(r) \times R^2 = (x_1, x_2, x_3, x_4 : x_1^2 + x_2^2 \leq r^2 \subset R^4$$

Gromov showed that one cannot squeeze a ball into a cylinder .

Theorem. Gromov's Non-squeezing Theorem. If $r < 1$, there is no symplectomorphism ϕ such that ϕ (B) $\subset$ Z (r). In order to sketch a proof of this theorem, we need to go further.

Almost Complex Structures and J Holomorphic Curves**.** There is connection between symplectic forms and complex numbers. As a quck recap , we first we survey manifolds as we did in Chapters 1 and 2. A differentiable manifold is a space in which one can do calculus. Locally, it looks Euclidean and globally, it has interesting properties. As in calculus, one approximates curves or surfaces near a point $x \in$ M by the closest linear objects, tangent lines or planes. The collection of all possible tangent directions at a point x is called the tangent space T_x M to M at x. It is a vector space of the same dimension as M. As the point x varies over M, the collection

$U_{x\in M}T_xM$ of all these planes forms a tangent bundleof M. If M = $\mathbb{R}^{2n}$, a Euclidean space, then one can identify each of its tangent spaces $T_x\mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$. However , most manifolds, such as a sphere, curve around and do not contain their tangent spaces.

Almost complex structure**.** An almost complex structure at a point x of a manifold M is a linear transformation J_x of th tangent space T_xM at x whose square is -1.Geometrically J_x rotates by a quarter turn . Thus the tangent space T_xM becomes a complex vector space. We can think of J_x as playing the role of multiplication by i. An almost complete structure J on M is a collection J_x of such transformations , one for each point x. J_x varies smootly as a function of x. One can choose J to be compatible with the symplectic form ω, so that at all points $x \in$M

$$\omega(J_xv, J_xw) = \omega(v, w)$$

This tells us that rotation by J_x preserves symplectic area. And,

$$\omega(v, J_xv) > 0$$

This tells us that every complex line has a positive symplectic area. We can think of ω (v, w) as the symplectic area of a small infinitesimal parallelogram spanned by the vectors v and w. Associated to each J there is a Riemannian metric, a symmetric inner product g_J on the tangent space T_xM , for any $v, w \in T_xM$

$$g_J(v, w) = \omega(v, Jw)$$

This gives a way of measring lengths nd angles and depends on the choice of J, not just ω. Note that one dimensional measurements vanish since $\omega(v, v)$ = - $\omega(v, v)$., by skew symmetry. Hence symplectic geometry is a 2-dimesnional geometry that measures the area of a complex curve instead of the length of a real curve.

Formalism. In Riemann geometry, we have a family of complex structures J . J is an automorphism that turns TM into a complex vector bundle

$$J : TM \longrightarrow TM$$

and

$$J^2 = -I$$

We also have the compatibility condition

$$\omega(x,y) = \omega(Jx,Jy)$$

and

$$\omega(x,Jx) > 0, x \neq 0$$

This implies that the bilinear form

$$g_J : g_J(x,y) = \omega(x,Jy)$$

is a Riemann metric. A symplectic manifold has the form ω and then there is a family of J imposed at the tangent space level. The only intinsic measurements that one can make on a symplectic manifold are two-dimensional. If S is a little piece of two-dimensional surface , then one can measure

$$\int_S \omega = area_\omega S$$

J-Holomorphic curves. A real curve in a manifold M is a path in M ; it is the image of a map f

$$f : U \longrightarrow M$$

where U is a subinterval of the real line $\mathbb{R}$. A $J-$ holomorphic curve in an almost complex manifold (M , J) is the complex analogue of this It is the image of a complex map f

$$f : \Sigma \longrightarrow M$$

for some complex curve $\Sigma \in$ (M , J). The domain Σ is either a 2-dimensional disc D or the 2-sphere $S^2 = \mathbb{C} \cup \{ \infty \}$, which is the complex plane completes by adding a point at ∞. Given any symplectic submanifold Q of (M , ω) , one can find and ω- compatible J on M that restricts to an almost complex structure on Q, that is

$$J(TQ) = TQ$$

If Q = $\mathbb{C}$, is 2 -dimensional, it will be a complex submanifold with respect to any J such that

$$J(TC) = TC$$

Such a $\mathbb{C}$ is known as a J - holomorphic curve. Gromov realized that J-holomorphic curves are equivalent to geodesic curves in Riemannian geometry. The simplest holomorphic map f, from $\mathbb{C}$ to $\mathbb{C}$ is a holomorphic function. The condition of holomorphicity is charcaterized by the Cauchy Riemann equation

$$\frac{\partial f}{\partial z} = 0$$

This tells us that the derivative of f is a complex linear map fron $\mathbb{C}$ to $\mathbb{C}$. There always is a Riemann surface (Σ, j) that maps onto $\mathbb{C}$ by a map u

$$u : \Sigma \longrightarrow M$$

that satisfies the generalized Cauchy Riemann equation

$$du \circ j = J \circ du$$

j is the complex structure on the Riemann surface. This tells us that, if we complexify du, then du takes up the complex structure of the Riemann surface.

Example. A simple example. Suppose you have a single particle. Then the phase space has coordinates (p,q). The symplectic form ω is

$$\omega = dp \wedge dq$$

which means that in terms of this basis, the symplectic form is given by the skew symmetric matrix, ω

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Note

$$\omega^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

When we apply to two vectors in the phase space $v_1 = (p_1 q_1)$ and $v_2 = (p_2 q_2)$, we get

$$\omega(v_1, v_2) = \begin{pmatrix} p_1 & q_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \begin{pmatrix} -q_1 & p_1 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = p_1 q_2 - q_1 p_2$$

Compatibility condition $\omega\ (v_1, v_2\ ) = \omega(Jv_1, Jv_2\ )$

$$\begin{pmatrix} -q_1 & p_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -q_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} -p_1 & -q_1 \end{pmatrix} \begin{pmatrix} -q_2 \\ p_2 \end{pmatrix} = p_1 q_2 - q_1 p_2$$

and non degeneracy $\omega\ (v_1, Jv_1) > 0$

$$\begin{pmatrix} p_1 & q_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} -q_1 & p_1 \end{pmatrix} \begin{pmatrix} -q_1 \\ p_1 \end{pmatrix} = p_1^2 + q_1^2 > 0$$

14.8 Hamiltonian Mechanics and Symplectic Geometry.

For the case of a single particle of mass m moving in a potential V , where the poistion q_i and momenta p_i , i - $1,2,3$; the time evolution of the system is dtermined by the Hamiltonian, H

$$H = \frac{1}{2m} p_i^2 + V(q_i)$$

The Hamiltonian equations describe the time evolution of the state of the system

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

A quick note on interior product**.** Usually written as $\iota_X \omega$ or X$\lrcorner\omega$, the interior product is defined to be the contraction of a differential form with a vector field. If X is a vector field on a manifold M, then

$$\iota_X : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$$

is the map which sends a $p-$form ω to the $(p-1)$ form $\iota_X(\omega)$ defined by the property that

$$(\iota_X \omega)(X_1, ..., X_{p-1}) = \omega(X, X_1, ..., X_{p-1})$$

for any vector fields X$_1$,......, X$_p$. The interior product on 1 - forms α

$$\iota_X \alpha = \alpha(X) =< \alpha, X >$$

where $<\,,\,>$ is the duality pairing between α and the vector. If β is a p -form and γ is a q-form then

$$\iota_X(\beta \wedge \gamma) = (\iota_X \beta \wedge \gamma) + (-1)^p \beta \wedge (\iota_X \gamma)$$

By anti-symmetry of forms

$$\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega$$

and so

$$\iota_x^2 = 0$$

The interior product relates the exterior derivative and Lie derivative of differential forms by Cartan's identity

$$\mathscr{L}_X \omega = d(\iota_X \omega) + \iota_X d\omega$$

If we choose a function f , we use a symmetric non- degenerate 2 -form, the inner product $<\,,\,>$, to produce a map from functions to vector fields

$$f \longrightarrow \nabla_f ; < \nabla_f, . >= -df$$

Starting with the symplectic form ω

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

the Hamiltonian function H produces a vector field X_H , symplectic gradient of H , with the symmetric 2-form coming from the inner product replaced by the anti-symmetric 2-form ω. Starting with a Hamiltonian function H, one produces a vector field X_H as follows

$$H \longrightarrow X_H : \omega(X_H,) = \imath_{X_H}\omega = -dH$$

Whereas the flow along a gradient vector field of f changes the value of f as fast as possible, flow along X_H keeps the value of H constant since

$$dH = -\omega(X_H,)$$

$$dH(X_H) = -\omega(X_H, X_H) = 0$$

since ω is anti - symmetric . Now dH is

$$-dH = -\frac{\partial H}{\partial q_i}dq_i - \frac{\partial H}{\partial p_i}dp_i$$

and

$$dH = \imath_{X_H}(dp_i \wedge dq_i)$$

Therefore

$$-\frac{\partial H}{\partial q_i}dq_i - \frac{\partial H}{\partial p_i}dp_i = \imath_{X_H}(dp_i \wedge dq_i) = (dp_i \wedge dq_i)(X_H) = -(dq_i \wedge dp_i)(X_H)$$

Which implies that

$$X_H = -\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}$$

Using Cartan's formula

$$\mathscr{L}_{X_H}\omega = d(\imath_{X_H}\omega) + \imath_{X_H}d\omega$$

Since dω= 0 and $d^2(f)$ = 0,

$$\mathscr{L}_{X_H}\omega = -d(dH) = 0$$

Definition. Hamiltonian Vector Field. A vector field X that satisfies

$$\mathscr{L}_X\omega = 0$$

is called a Hamiltonian vector field and the space of such vector fields on $\mathbb{R}^{2n}$will be denote by Vect ($\mathbb{R}^{2n}$, ω). The flow of X is a family of maps

$$\phi_t^X : M \longrightarrow M$$

where t $\in\mathbb{R}$, such that the path $\phi_t^X(p)$ is everywhere tangent to the vector field X . A vector field X is said to be symplectic if its flow ϕ_t^X: M $\longrightarrow$ M consists of symplectomorphisms

$$(\phi_t^X)^*\omega = \omega, \forall t$$

Note that ω is closed and $d\omega$ = 0, closed 1 -forms . Every function H on M gives rise to a vector field X$_H$ via the correspondence

$$dH = \iota_{X_H}\omega$$

and hence to a flow ϕ_t^M on M called the Hamiltonian flow of H. Note that

$$dH(X_H) = \omega(X_H, X_H) = 0$$

the orbits $\phi_t^H(p)$ of this flow lie entirely in the level sets H = constant of the Hamiltonian. H is the energy of the system and it is conserved.

Penrose Diagram

These conformal diagrams gave us an idea of the causal structure of the spacetime, wether the past or future light cones of two specified points intersect, as in Figure 14.3 below. In Minkowski space, this is always true for any two points. Not so in curved spacetime. Let us analyze the case for the Schwatzchild solution which describes the spherically symmetric vacuum spacetime. Schwarschild metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

As M$\longrightarrow$ 0, we recover the Minkowski space. As r$\longrightarrow \infty$, the metric becomes progressively Minkowskian ; asymptotic flatness. Next, we explore the causal structure as defined by light-cones. Consider radial null cones, where θ and ϕ are constant and ds^2 = 0

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 = 0$$

from which we get

$$\frac{dt}{dr} = \pm(1-\frac{2GM}{r})^{-1}$$

This measures the slope of the light cones on a spacetime diagram of the $t-r$ plane. For large r, the slope is ± 1, as expected in flat space. As r approaches $2GM$, we obtain $\frac{dt}{dr} \longrightarrow \pm\infty$ and the light cones ' close up' and do not allow causality. What we do here is switch to Eddington-Finkelstein coordinates which are adapted to radial null geodesics or a Schwarzchild geometry. Distant spacetimes are crunched into the diagram, so that they converge at points in the corners of the diagram and staright lines become hyperbolas. Two lines drawn at 45 degrees sholud intersect in the diagram only if the corresponding two light rays intersect in the actual spacetime. The diagonal boundaries orrespond to the infinity or to the singularities where light rays must end. An infinite static Minkowski universe, coordinates (x,t) is related to penrose coordinates $(u,v\ )$ by

$$tan(u \pm v) = x \pm t$$

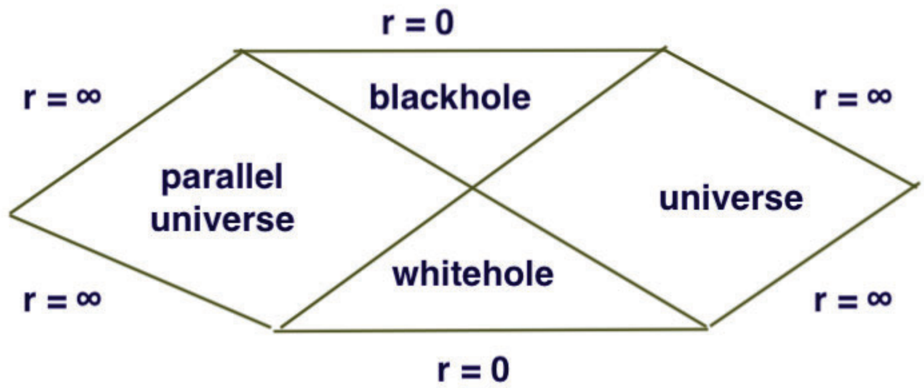

Figure 14.3: Penrose diagram for Schwarzchild blackhole

14.9 Statistical Mechanics

Consider an observable x that takes with probability $p(x_i)$ the value x_i. In total there are N such possible values , $i = 1, 2, ..., N$. We have

$$\sum_{i=1}^{i=N} p(x_i) = 1$$

The mean value of x is given as

$$< x >= \sum_{i=1}^{i=N} p(x_i) x_i$$

For an arbitrary function $f(x)$

$$< f(x) >= \sum_{i=1}^{i=N} p(x_i) f(x_i)$$

If $f(x) = x^n$, the $n - th$ moment of the distribution function is

$$< x^n >= \sum_{i=1}^{i=N} p(x_i) x_i^n$$

The variance of the distribution is the mean square deviation

$$< x^2 > - < x >^2$$

Partition Function. Consider a system T embedded into a heat bath B. Let E be the total energy of both systems. Let $p_i$be the probability of microstate of energy E$_i$.Assuming that all microstates are equally probable Then p_i is proportional to the number of microstates of the heat bath with energy $E-E_i$. Then

$$p_i = \frac{\Omega_b(E-E_i)}{\Omega_B(E)}$$

Then

$$klnp_1 = kln\Omega_B(E-E_i) - kln\Omega_B(E)$$

Assuming $E >> E_i$ and Taylor expanding Ω_B to first order in E_i

$$klnp_i \simeq -\frac{\partial}{\partial E}[kln\Omega_B(E-E_i)]E_i$$

Noting tha $S = kln\Omega_B(E)$ and using the thermodynamic identity $\frac{\partial S}{\partial E} = \frac{1}{T}$

$$klnp_i \simeq -\frac{\partial S}{\partial E}E_i = -\frac{E_i}{T}$$

$$\implies p_i \propto e^{\frac{-E_i}{kT}} = e^{-\beta E_i}$$

where S = entropy , k= Boltzamnn constant , β = 1 / kT.Note that the sum of all p_i = 1. The partition function is defined as the normalization constant Z

$$Z = \sum_i e^{-\beta E_i}$$

In quantum mechanics , where H = Hamiltonian

$$Z = trexp(-\beta H) = \sum_n < n|exp(-\beta H)|n >= \sum_n < n|n > exp(-E_n) = \sum_n exp(-\beta E_n)$$

Hence we define the density operator ρ

$$\rho = \frac{1}{Z} exp(-\beta H)$$

where Z = tr exp(-βH)

The Unruh Effect

In section 12.6 , we chose a complex structure to define the positive and negative energy frequency solutions of our theory. Next, we analyze how two different observers view this. Assume we have two complete and orthonormal solutions to the Klein Gordon equation , { f_i} and { g_j};

$$< f_i, f_j >= - < f_i^*, f_j^* >= \delta_{ij}$$

$$< g_i, g_j >= - < g_i^* g_j^* >= \delta_{ij}$$

If $\hat{\phi}(x)$ is the field operator, then the creation and annihilation operators are defined by

$$\hat{\phi}(x) = \sum_i (\hat{a} f_i + \hat{a}^\dagger f_i^*)$$

The vacuum | 0 > is defined as the unique state that is killed by all the annihilation operators

$$\hat{a}_i|0 >= 0$$

Since we are dealing with complete stes, we can write f_i and g_i in terms of each other;

$$g_j = \sum_i (\alpha_{ij} f_i + \beta_{ij} f_i^*; g_j^* = \sum_i (\alpha_{ij}^* f^*{}_i + \beta_{ij}^* f_i)$$

$$f_i = \sum_j (\alpha_{ji} g_j + \beta_{ji} g_j^*); f_i^* = \sum_j (\alpha_{ji}^* g_j^* + \beta_{ji}^* g_j)$$

where $\alpha_{ij} = < g_j, f_i >$. We could equally expand the field operator in the g-basis. Then,

$$\hat{\phi}(x) = \sum_j (\hat{b_j} f_i + \hat{b_j}^\dagger f_j^*)$$

for some for some other creation and annihilators $\hat{b}_j$ and $\hat{b}_j^\dagger$. We also have another vacuum defined by $\hat{b}_j | 0 >' = 0$.

Bogoliubov Transformations. The next step is to see how one vacuum looks like in another basis. Expressing an annihilator in terms of another basis, we obtain the Bogoliubov transformations;

$$\hat{a}_i = \sum_j [\alpha_{ji} \hat{b}_j + \beta_{ji} \hat{b}_j^\dagger]; \hat{a}_i^\dagger = \sum_j [\alpha_{ji}^* \hat{b}_j^\dagger + \beta_{ji}^* \hat{b}_j]$$

$$\hat{b}_j = \sum_i [\alpha_{ij} \hat{b}_i + \beta_{ij} \hat{b_i}^\dagger; \hat{b}^\dagger = \sum_i [\alpha_{ij}^* \hat{b_i}^\dagger + \beta_{ij}^* \hat{b}_i$$

The number operators are

$$\hat{N}_i = \hat{a_i}^\dagger \hat{a}_i$$

and

$$\hat{N_j}' = \hat{b_j}^\dagger \hat{b_j}$$

We then take one vacuum and calculate the expectation value of the other number operator

$$< 0|^{\prime}\hat{N}_i|0 >^{\prime} = < \hat{N}_i > = < \hat{a}_i{}^{\dagger}\hat{a}_i > \neq 0 = \sum |\beta_{ijj}|^2$$

Under the Bogoliubov transformation, the annihilation operator picks up a creation part, and the expectation value of the number operator is non-zero. It is this mixing of the creation and annihilation operators that is responsible for particle creation.

WKB Ansatz. The wave function for a particle of energy E moving in a potential V is

$$\psi = Ae^{\frac{i}{\hbar}pq}$$

where A is the amplitude, λ is the wavelength = $\frac{2\pi}{k}$, where $k = \frac{p}{\hbar}$, and the momentum

$$p = \pm\sqrt{2m(E-V)}$$

Here the potential varies slowly over many wavelengths - this is the semi-classical WKB approximate solution of the Schrodinger equation. It fails at turning points , where the particle momentum , p = 0. Consider the time indepedent Schrodinger equation in 1 dimension

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(q)}{\partial q^2} + (V-E)\psi(q) = 0$$

The potential V(q) grows fast as $q \longrightarrow \pm\infty$. For any E, the classical particle motion is confined . See Figure 14.4 below.

The variable wave number form of the Schrodinger equation is

$$\frac{\partial^2\psi}{\partial q^2} + k^2\psi = 0$$

An ansatz solution is

$$\psi = Ae^{\frac{i}{\hbar}S}$$

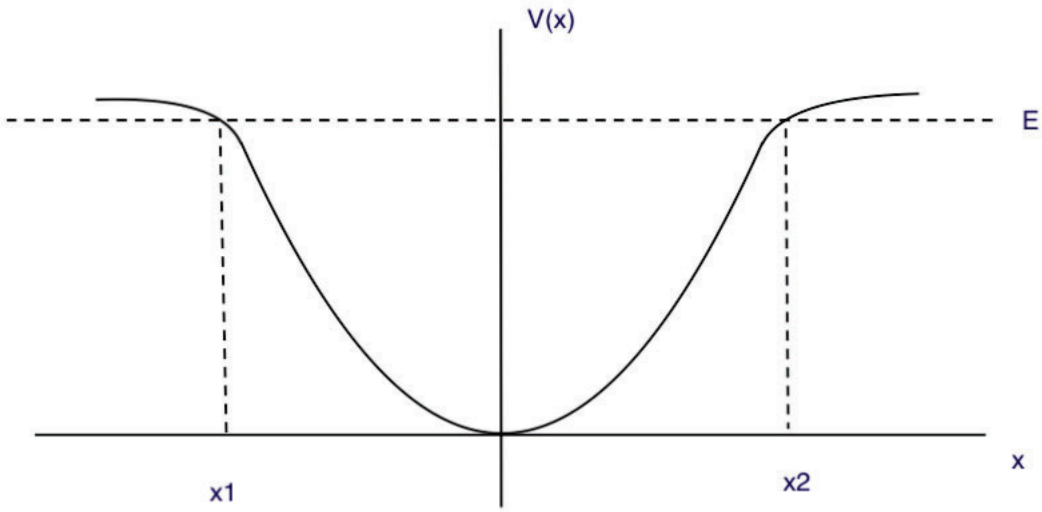

Figure 14.4: 1- dimensional potential ; location of two turning points at fixed energy E

where A and S are real functions of q, the position. Substituting yields two solutions; one real and one imaginary. We will use prime ’ as derivative with respect to q;

$$(S')^2 = p^2 + \hbar^2 \frac{A''}{A} \tag{14.9.1}$$

and

$$S''A + 2S'A' = \frac{d}{dq}(S'A^2) = 0 \tag{14.9.2}$$

The WKB - Wentzel - Kramers - Brillouin - semiclassical approximation consists of dropping the $\hbar^2$ term. Since $p = \hbar k$, this amounts to assuming that

$$k^2 = \frac{p^2}{\hbar^2} >> \frac{A''}{A}$$

This implies that the phase of the wavefunction is changing much faster than the amplitude. Setting $\hbar = 0$ and integrating (14.9.1) yields

$$S = \int_{q_1}^{q2} pdq$$

This is the action of a particle trajectory at constant energy. Integrating (14.9.2) yields

$$A = \frac{C}{\sqrt{p}}, C = \sqrt{p(q_1)}\psi(q_1)$$

The integration constant C is fixed by the value of the wave function at the initial value q_1. The WKB or semi-classical ansatz wave function is given by

$$\psi = \frac{C}{\sqrt{p}} e^{\frac{iS}{\hbar}}$$

The WKB ansatz generalizes the free motion wave function

$$\psi = Ae^{\frac{i}{\hbar}S}$$

with the probablity density | A |^2 for finding a particle at q , now inversely proportional to the velocity at that point. The phase

$$\frac{1}{\hbar} qp$$

is replaced by the action

$$\frac{1}{\hbar} \int pdq$$

This is all fine except at tuening points where all the energy is potential. So the assumption that k >> $\frac{A''}{A}$ does not hold.

14.10 Hawking Blackbody Spectrum - Hawking Radiation - Quantum Tunnelling

There are many approaches to show that blackholes radiate. We will demonstrate the tunneling mechanism approach,[2],[4]. The idea is that a particle-antiparticle pair forms close to the event horizon. The ingoing mode is trapped inside the horizon. The outgoing mode can quantum mechanically tunnel through the event horizon. This is observed at infinity as a Hawking flux. We will demonstrate the blackbody spectrum of a spherically symmetric geometry that is asymptotically flat.

Klein Gordon Solution**.** We will first briefly review the Klein Gordon field. This is the simplest relativistic classic field. It is a scalar field, φ, simply a function on spacetime , $x^{\alpha} = (t,x,y,z)$

$$\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}$$

With $\Box$ as the d' Alembertian operator, the Klein Gordon equation is given by

$$\Box\varphi - m^2\varphi = 0$$

m is the mass of the Klein Gordon field. When m= 0 , we simply get the wave equation. Here we set $c = \hbar = 1$. This field equation is characterized by particles with rest mass m and no other structure such as spin or charge. One way to solve this is by Fourier expansion

$$\varphi(x) = (\frac{1}{2\pi})^{\frac{3}{2}} \int_{R^3} d^3k\hat{\varphi}_k(t)e^{ik.r}$$

where k = ($k_x, k_y, k_z) \in \mathbb{R}^3$ and since φ is a real function

$$\varphi_{-k} = \varphi_k^*$$

The Klein Gordon equation implies the following equation for the Fourier transform $\hat{\varphi}_k(t)$ of φ

$$\ddot{\hat{\varphi}}_k + (k^2 + m^2)\hat{\varphi}_k = 0$$

The solution to this equation is

$$\hat{\varphi}_k(t) = a_k e^{-i\omega_k t} + b_k e^{i\omega_k t}$$

where $\omega_k = \sqrt{k^2+m^2}$ and a_k and b_k are complex constants for each k . The reality condition on φ implies that

$$b_k = a^*_{-k}$$

Hence φ solves the Klein Gordon equation if and only if it takes the form

$$\varphi(x) = (\frac{1}{2\pi})^{\frac{3}{2}} \int_{R^3} d^3k(a_k e^{ikr-i\omega_k t} + a_k^* e^{-ikr+i\omega_k t})$$

The Gordon Klein equation is just an infinite collection of uncoupled harmonic oscillator equations for the real and imaginary parts of $\hat{\varphi}_k$(t) with natural frequency ω_k. The Klein Gordon field is akin to a dynamical system with an infinite number of degrees of freedom. We first note that

$$\varphi = \varphi^+ + \varphi^-$$

where

$$\varphi^+ = (\frac{1}{2\pi})^{\frac{3}{2}} \int_{R^3} d^3k a_k e^{ikr-i\omega_k t}$$

and

$$\varphi^- = (\frac{1}{2\pi})^{\frac{3}{2}} \int_{R^3} d^3k a_k^* e^{-ikr+i\omega_k t}$$

These are the positive frequency and negative frequency solutions to the Klein Gordon equations.

The Meaning Behind Positive and Negative Frequencies**.** Let us begin with a classic electromagnetic plane wave travelling along the z-axis. The electric field $\overrightarrow{E}$ is expressed as

$$\overrightarrow{E} = E_0\hat{x}(coskz - \omega t)$$

We break up this equation into positive and negative frequency parts.

$$\overrightarrow{E} = E_0\hat{x}(e^{-i\omega t+ikx)} + e^{i\omega t-ikx)}$$

By convention, $e^{-i\omega t}$ is the positive frequency component, ω>0. e$^{i\omega t}$ is the negative frequency component. The important point is that in physics, everything is real, so the positive and negative components have to exist in equal measure. Next, we turn to massive particles . These show wave phenomena, such as interference, diffraction etc., whose wavelengths are determined by De Broglie

$$\lambda = \frac{\hbar}{p}$$

or

$$k = \frac{p}{\hbar}$$

A single material particle travelling along can only have positive energy. The Einstein - Planck formula states

$$E = \hbar\omega$$

$$E > 0 \Longrightarrow \omega > 0$$

The conclusion is that a wave function for a single material particle can only have a positive frequency part. So Ψ, whatever it represents, must vary as follows

$$\Psi \sim e^{-i\omega t+ikx}$$

The fact that e^{ikz} goes with $e^{-i\omega t}$ tells us that the particle is travelling along the + z-axis. If it was travelling in the opposite -z-axis, we would then have e^{-ikz}. Regardless of the direction of travel, the time dependence must be of the form

$$e^{-i\omega t}$$

with

$$\omega = \frac{E}{\hbar} > 0$$

Hence complex numbers allow wave-like phenomena and positive frequencies only for free particles, that can only be allowed to have positive energies.

Formulation of Problem. One bit of information is associated with the knowledge of existence of one particle. 1 bit of information correspnds to $ln2$ of entropy. This can be derived from the Shannon formula for entropy H and p_k which denotes the probability of the system being in the k-th state.

$$H = -\sum_k p_k ln p_k$$

For the particle entropy, $ln\, 2$ arises if the chances of the particle existing or not are both equal to 1/2. We will later analyze the explicit forms of quantum mechanical modes of the states inside and outside the blackhole horizon. This yields the probabilities of individual ingoing modes being trapped inside the blackhole horizon, which is unity, or tunnelling out of the blackhole horizon and escaping to infinity to be perceived as Hawking radiation. The effective field theories become two dimensional near the event horizon of a blackhole. Upon transforming to the tortoise coordinates and performing the partial wave decomposition, it can be shown that the effective radial potentials for partial wave modes of the scalar field vanish ecponentially fast near the horizon.Thus physics near the horizon can be described using an infinite collection of (1 + 1) - dimensional manifolds This allows us to use the metric below in the ($t-r$) sector. We first consider, as before, a blackhole that is spherically symmetric, with static spacetime and asymptotically flat metric of the form

$$ds^2 = -F(r)dt^2 - \frac{dr^2}{F(r)} - r^2 d\Omega^2$$

where the event horizon is defined by $F(r) = 0$. We next consider the massless Klein Gordon equation

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \phi = 0$$

In the $(r-t)$ sector, this reduces to

$$\frac{1}{F(r)}\partial_t^2\phi + F^{'}(r)\partial_r\phi + F(r)\partial_r^2\phi = 0 \tag{14.10.1}$$

Taking the standard WKB ansatz

$$\phi(r,t) = e^{\frac{-i}{\hbar}S(r,t)}$$

Substituting the expansion for $S(r,t)$ into (58.1) and let $\hbar \longrightarrow 0$;

$$S(r,t) = S_0(r,t) + \sum_{i=1}^{\infty}\hbar S_i(r,t)$$

we obtain

$$\partial_t S_0(r,t) = \pm F(r)\partial_r S_0(r,t) \tag{14.10.2}$$

Since the metric is stationary, it has a timelike Killing vector. We choose an ansatz for $S_0(r,t)$

$$S_0(r,t) = \omega t + \bar{S_0}(r) \tag{14.10.3}$$

where ω is the conserved quantity corresponding to the timelike Killing vector. Substituting this into (14.8 .2) and integrating

$$\tilde{S}_0(r) = \pm\omega \int \frac{dr}{\sqrt{F(r)}} \tag{14.10.4}$$

The limits of integration are chosen so that the particle just goes through the horizon r= r_H. The (+) sign in front of the integral indicates that the particle is ingoing (L) and the (-) sign indicates the particle is outgoing (R) . Therefore,

$$S_0(r,t) = \omega t \pm \omega \int \frac{dr}{\sqrt{F(r)}}$$

The key idea behind quantum particle production in curved spacetime is that the definition of a particle is vacuum dependent. It depends on the choice of the reference frame. Since the theory is generally covariant, any time coordinate defined locally within a patch, is a legitimate choice with which to define positive and negative frequency modes. Hawking considered a massless quantum scalar field moving in the background of a collapsing star. If the quantum field was initially in the quantum state, no particle state, defined in the asymptotic past, then at late times, it will appear as if particles are present in that state. Hawking showed, by calculating the Bogoliubov coefficients between the two sets of vacuum states defined at asymptotic past and future respectively, that the spectrum of emitted particles is that of a blackbody. The idea is that pair production occurs inside the event horizon of a blackhole. One member of the pair corresponds to the ingoing mode and the other member corresponds to the outgoing mode. The outgoing mode is allowed to follow classically forbiden trajectories, by starting just behind the horizon. This mode travels back in time, since the horizon is locally to the future of the external region.The classical one particle action becomes complex and so the tunnelling amplitude is governed by the imaginary part of this action for the outgoing mode. However, the action for the ingoing mode must be real, since classically a particle can fall behind a horizon. Since this is a near horizon theory and the tunnelling occurs radally, the phenomenon is dominated by the two dimensional $(t-r)$ metric.This follows from the fact that near the horizon all the angular part can be neglected and the solution of the field equation corresponds angular quantum momentum number l= 0, which is known as the s-wave. The essence of tunnelling based calculations is the computation of the imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related to the Boltzamnn factor for the emission at the Hawking temperature. Therefore the ingoing, left mode , $\phi^{(L)}$, and outgoing , $\phi^{(R)}$, right mode ,solutions of the Klein Gordon equations are

$$\phi^{(L)} = e^{\frac{-i\omega}{\hbar}(t+\int \frac{dr}{\sqrt{F(r)}})}$$

$$\phi^{(R)} = e^{\frac{-i\omega}{\hbar}(t-\int \frac{dr}{\sqrt{F(r)}})}$$

A mode will be called ingoing if its radial momentum eigenvalue is negative and outgoing if its radial momentum eigenvalue is positive. For a wavefunction ϕ, the momentum eigenvalue equation is

$$\hat{p}_r\phi = p_r\phi$$

where $\hat{p}_r$ = -$i\hbar\frac{\partial}{\partial r}$. Applying momentum operator to $\phi^{(L)}$, we get $p_r^{(L)}$ = - $\frac{\omega}{\sqrt{(F(r)}}$, which is negative and p$_r^{(L)}$ = - $\frac{\omega}{\sqrt{(F(r)}}$ which is positive. For the tunnelling of a particle across the horizon, the nature of the coordinates change. The timelike coordinate t outside the horizon changes to a spacelike coordinate inside the horizon. This indicates that the t coordinate may have an imaginary part upon crossing the horizon. There will be a temporal contribution to the probabilities for the ingoing and outgoing particles. The ingoing and outgoing probabilities of the particle are

$$P^{(L)} = |\phi^{(L)}|^2 = e^{\frac{2\omega}{\hbar}(Imt+Im\int \frac{dr}{\sqrt{(F)r)}})}$$

$$P^{(R)} = |\phi^{(R)}|^2 = e^{\frac{2\omega}{\hbar}(Imt - Im\int \frac{dr}{\sqrt{(F)r)}})}$$

The ingoing probability P$^{(L)}$ has to be unity in the classic limit, $\hbar \longrightarrow 0$, when there is no reflection and everything is absorbed. Thus P$^{(L)}$ = 1 leads to

$$Imt = -Im\int \frac{dr}{\sqrt{F(r)}}$$

To find Im t for the outgoing particle we deploy Kruskal coordinates which are well behaved throughout spacetime. The Kruskal time, T, and space, X, coordinates inside and outside the horizon are defined in terms of Schwarzchild coordinates.

$$T_{in} = e^{\kappa(r^*_{in})} cosh(\kappa t_{in}); X_{in} = e^{\kappa(r^*_{in})} sinh(\kappa t_{in})$$

$$T_{out} = e^{\kappa(r^*_{out})} sinh(\kappa t_{out}); X_{out} = e^{\kappa(r^*_{out})} cosh(\kappa t_{out})$$

where κ= $\frac{F^{'}(r_H)}{2}$ is the surface gravity of the event horizon and r* = $\int \frac{dr}{\sqrt{F(r0}}$. Next, we introduce the null tortoise coordinates which are defined as

$$u = t - r^*$$

$$v = t + r^*$$

Substituting S$_0$ with r^* expressed in terms of u and v into $\phi(r,t\ )$, we obtain the right and left modes of both sectors

$$\phi^{(R)}_{in} = e^{\frac{-i}{\hbar}\omega u_{in}}; \phi^{(L)}_{in} = e^{\frac{-i}{\hbar}\omega v_{in}}$$

$$\phi_{out}^{(R)} = e^{\frac{-i}{\hbar}\omega u_{out}}; \phi_{out}^{(L)} = e^{\frac{-i}{\hbar}\omega v_{out}}$$

Next, we express

$$S_0(r,t) = \omega t \pm \omega \int \frac{dr}{\sqrt{F(r)}}$$

in null tortoise coordinates

$$(S_0)_{in}^{(R)} = \omega(t_{in} - r_{in}^*) = \omega u_{in}$$

$$S_0)_{in}^{(L)} = \omega(t_{in} + r_{in}^*) = \omega v_{in}$$

$$(S_0)_{out}^{(R)} = \omega(t_{out} - r_{out}^*) = \omega u_{out}$$

$$S_0)_{out}^{(L)} = \omega(t_{out} + r_{out}^*) = \omega v_{out}$$

Substituting these equations into

$$\phi(r,t) = e^{\frac{-i}{\hbar}S(r,t)}$$

yields the right and left modes for both sectors

$$\phi_{in}^{(R)} = e^{-(\frac{i}{\hbar})\omega u_{in}}; \phi_{in}^{(L)} = e^{-(\frac{i}{\hbar})\omega v_{in}}$$

$$\phi_{out}^{(R)} = e^{-(\frac{i}{\hbar})\omega u_{out}}; \phi_{out}^{(L)} = e^{-(\frac{i}{\hbar})\omega v_{out}}$$

In the tunnelling formalism, a virtual pair of particles is produced in thr blackhole. One of this pair can quantum mechanically tunnel through the horizon. This particle is observed at infinity while the other goes to the center of the blackhole. While crossing the horizon the nature of the coordinate changes in the following way. The Kruskal time, T, and space , X, coordinates inside and outside the horizon are defined by

$$T_{in} = e^{\kappa(r^*_{in})} cosh(\kappa t_{in}); X_{in} = e^{\kappa(r^*_{in})} sinh(\kappa t_{in})$$

$$T_{out} = e^{\kappa(r^*_{out})} sinh(\kappa t_{out}); X_{out} = e^{\kappa(r^*_{out})} cosh(\kappa t_{out})$$

These two sets of coordinated are connected by the following relations

$$t_{in} = t_{out} - i\frac{\pi}{2\kappa} \tag{14.10.5}$$

$$r^*_{in} = r^*_{out} + i\frac{\pi}{2\kappa} \tag{14.10.6}$$

This indicates that when a particle travels from inside to outside the horizon, the t coordinate picks up an imaginary term $\frac{-\pi}{2\kappa}$. The Kruskal coordinates get identified as

$$T_{in} = T_{out}$$

$$X_{in} = X_{out}$$

For the Schwatzchild metric, the surface gravity, κ , is

$$\kappa = \frac{1}{4M}$$

and thus the extra term cnnecting t_{in} and t_{out} is given by

$$-2\pi iM$$

Deploying $u = t - r^*$ and $v = t + r^*$, and (14.10.5) and (14.10.6) ; we get

$$u_{in} = t_{in} - r^*_{in} = t_{out} - i\frac{\pi}{2\kappa} - r^*_{in} = t_{out} - i\frac{\pi}{2\kappa} - r^*_{out} - i\frac{\pi}{2\kappa} = u_{out} - i\frac{\pi}{\kappa}$$

and by similar reasoning

$$v_{in} = v_{out}$$

Under these transformations the modes which are travelling in the in sector and out sectors of the blackhole horizon are connected through the expressions

$$\phi^{(R)}_{in} = e^{-\frac{\pi\omega}{\hbar\omega}} \phi^{(R)}_{out}$$

$$\phi^{(L)}_{in} = \phi^{(L)}_{out}$$

Since the left moving mode travels towards the center of the blackhole, its probability to go inside, as measured by an external observer is unity;

$$P^{\{L)} = |\phi^{(L)}_{in}|^2 = |\phi^{(L)}_{out}|^2 = 1$$

and the tunnelling probability as seen by an external observer through the event horizon is

$$P^{(R)} = |\phi^{(R)}_{in}|^2 = |e^{-\frac{\pi\omega}{\hbar\kappa}} \phi^{(R)}_{out}|^2 = e^{-\frac{\pi\omega}{\hbar\kappa}} \tag{14.10.7}$$

Next we apply the method of detailed balance to derive the Hawking temperature.

Method of Detailed Balance. Einstein coefficients measure the probability of absorption or emission of light by an atom or molecule. The A coefficient is related to the rate of spontaneous emission of light and the B coefficient to the absorption and stimulated emission of light.

Let n_i be the number of particles in state i; then in thermal equilibrium

$$\frac{dn_i}{dt} = -emission + absorption = -B_{12}n_1\rho(\nu) + B_{21}n_2\rho(\nu) + A_{21}n_2 = 0 \tag{14.10.8}$$

where $\rho(\nu)$ is the spectral energy density at the frequency of transition. From the Boltzmann distribution, we have for the excited number of atomic species

$$\frac{n_i}{n} = \frac{g_i e^{-\frac{E_I}{kT}}}{Z}$$

where n is the total number density of atomic species , k is Boltzmann constant, T is temperature, g_i is degeneracy of state i, Z is the partition coefficient. From Planck's law of black body radiation at temperature T, we have the spectral energy density at frequency ν

$$\rho(\nu) = \frac{8\pi\hbar\nu^3}{c^3}\frac{1}{e^{\frac{\hbar\nu}{kT}} - 1}$$

By substituting $\rho(\nu)$ into equation above, $\frac{dn_1}{dt}$ and re-arranging, we obatin

$$\frac{A_{21}}{B_{21}} = \frac{8\pi\hbar\nu^3}{c^3}; \frac{B_{21}}{B_{12}} = \frac{g_1}{g_2}$$

By the method of detailed balance

$$P^{(R)} = e^{\frac{-\omega}{T_H}} P^{(L)} = e^{\frac{-\omega}{T_H}}$$

Comparing this to P^R above, we arrive at the Hawking temperature

$$T_H = \frac{\hbar\kappa}{2\pi}$$

Note that

$$\frac{P_{out}}{P_{in}} = e^{\frac{-\omega}{T}}$$

is the outgoing/ingoing probability where ω is the observed energy at that point , also

$$P_{out/in} = |wavefunction|^2$$

Note if $\hbar \longrightarrow 0$, the tunnelling probability goes to 0, which is expected , since classically a black-hole cannot radiate.

Blackbody Spectrum from Tunneling Mechanism. To find the blackbody spectrum and Hawking flux, we consider an n number of non-interacting virtual particles that are created inside the blackhole. Each of these pairs is represented by the modes derived above and re-listed below.

$$\phi_{in}^{(R)} = e^{-\frac{\pi\omega}{\hbar\kappa}} \phi_{out}^{(R)}$$

$$\phi_{in}^{(L)} = \phi_{out}^{(L)}$$

Then the physical state of the system observed from outside is given by

$$|\Psi >= N \sum_n |n_{in}^{(L)} > \otimes |n_{in}^{(R)} >= N \sum_n e^{\frac{-\pi n\omega}{\hbar\kappa}} |n_{out}^{(L)} > \otimes |n_{out}^{(R)} >$$

N is the normalization constant, determined by the normalization condition

$$< \Psi|\Psi >= 1$$

Hence

$$N = \frac{1}{(\sum_n e^{\frac{-2\pi n\omega}{\hbar\kappa}})^{\frac{1}{2}}}$$

For bosons; n runs from $0 \longrightarrow \infty$

$$|\Psi >= \sum_{n=0}^{n=\infty} N |n_{in}^{(L)} > \otimes |n_{n}^{(R)} >= N \sum_{n=0}^{n=\infty} e^{\frac{-\pi n\omega}{\hbar\kappa}} |n_{out}^{(L)} > \otimes |n_{out}^{(R)} >$$

$$|\Psi|^2 = N^2 (\sum_{n=0}^{n=\infty} e^{\frac{-2\pi n\omega}{\kappa\hbar}}) = 1$$

$$\Longrightarrow N = (1 - e^{\frac{-2\pi\omega}{\kappa\hbar}})^{\frac{1}{2}}$$

By the Pauli exclusion principle , for fermions , n can only be 0 or 1

$$N = (1 - e^{\frac{-2\pi\omega}{\kappa\hbar}})^{-\frac{1}{2}}$$

We will next do the analysis for bosons only, the analysis is similar for fermions.Therefore, the normalized physical states of the bosons is

$$|\Psi >= (1 - e^{-\frac{2\pi\omega}{\hbar\kappa}})^{\frac{1}{2}} \sum_n e^{\frac{-\pi n\omega}{\hbar\kappa}} |n_{out}^{(L)} > \otimes |n_{out}^{(R)} >$$

For bosons, the density matrix operator for the system is

$$\hat{\rho} = (1 - e^{-\frac{2\pi\omega}{\hbar\kappa}})^{\frac{1}{2}} \sum_{n,m} e^{\frac{-\pi n\omega}{\hbar\kappa}} e^{\frac{-\pi m\omega}{\hbar\kappa}} ||n_{out}^{(L)} > \otimes |n_{out}^{(R)} > |m_{out}^{(R)} > \otimes |m_{out}^{(L)} >$$

The ingoing modes are completely trapped, they do not contribute to the emission spectrum from the blackhole event horizon. Hence we trace out the ingoing L modes, and we obtain the density matrix for the outgoing R modes

$$\hat{\rho} = (1 - e^{-\frac{2\pi\omega}{\hbar\kappa}})^{\frac{1}{2}} \sum_{n,} e^{\frac{-2\pi n\omega}{\hbar\kappa}} ||n_{out}^{(R)} > \otimes |n_{out}^{(R)} >$$

The average number of particles detected at asymptotic infinity is given by

$$< n >= trace(\hat{n}\hat{\rho}^{(R)}) = (1 - e^{-\frac{2\pi\omega}{\hbar\kappa}})^{\frac{1}{2}} \sum_{n,} e^{\frac{-2\pi n\omega}{\hbar\kappa}} = (1 - e^{-\frac{2\pi\omega}{\hbar\kappa}})^{\frac{1}{2}} (\frac{-\hbar\kappa}{2\pi}) \frac{\partial}{\partial\omega} (\sum_{n,} e^{\frac{-2\pi n\omega}{\hbar\kappa}})$$

$$= (1 - e^{-\frac{2\pi\omega}{\hbar\kappa}})^{\frac{1}{2}}(-\frac{\hbar\kappa}{2\pi})\frac{\partial}{\partial\omega}(\frac{1}{1 - e^{\frac{-2\pi n\omega}{\hbar\kappa}}})$$

$$= \frac{1}{e^{\frac{2\pi n\omega}{\hbar\kappa}} - 1}$$

This is the Bose distribution. A similar analysis or fermions leads to the Fermi distribution

$$< n >= \frac{1}{e^{\frac{2\pi n\omega}{\hbar\kappa}} + 1}$$

Both these distributions correspond to a black body spectrum. Hence, from the density matrix constructed from the modes, we were able to reproduce the black body spectrum.

14.11 An Introduction to String Theory

Single Particle Dirac Equation. Turning the relativistic equation

$$E^2 = p^2 + m^2$$

into a partial differential equation by the usual substitution

$$p = -i\nabla, E = i\frac{\partial}{\partial t}$$

results in the Klein-Gordon Equation

$$(\Box + m^2)\psi(x) = 0$$

The problem is that this wave equation has a negative energy solution. In order to overcome this problem, Dirac tried the ansatz

$$(i\beta^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi(x) = 0 \tag{14.11.1}$$

To overcome this problem, Dirac set

$$\gamma^{\mu} = \beta^{\mu}$$

and

$$\gamma^{\mu}\partial_{\mu}\gamma^{\nu}\partial_{\nu} = \partial^{\mu}\partial_{\mu}$$

which implies the following identities

$$(\gamma^{0})^{2} = 1$$

$$(\gamma^{i})^{2} = -1$$

and the anticommutation relation for $\mu \neq \nu$

$$\gamma^{\mu}, \gamma^{\nu} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g_{\mu\nu} = 0$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

At least one of the factors in the Dirac equation is equal to zero. By convention, the second term is set to zero. We obtain the Dirac spinor and $\psi(x)$ is called the Dirac spinor;

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0$$

The originial Dirac representation for γ matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where σ^i are the Pauli matrices ;

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Properties of the Pauli matrices

$$\sigma^{i\dagger} = \sigma^i$$

$$\sigma^{i*} = (i\sigma^2)(\sigma^i)(i\sigma^2)$$

$$[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k$$

$$\sigma^i, \sigma^j = 2\delta^{ij}$$

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$$

where ε^{ijk} is the anti-symmetric Levi-Civita tensor, ε^{123} = ε^{231} = ε^{312} = 1 and ε^{213} = ε^{321} = ε^{132} = -1, all other components are zero.

Euler B Functions and the Origins of String Theory

The definition of the gamma function proposed by Euler is

$$\Gamma(x) = \int_o^1 [-ln\sigma]^{x-1} d\sigma$$

where Re(x) > 0. Setting $\sigma = e^{-t}$, we get

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

where R(x) > 0 . The notation Γ(z) is due to Lagrange. If Re (z) > 0, then

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$$

Γ (z) converges absolutely. Integrating by parts, we get

$$\Gamma(z+1) = z\Gamma(z)$$

This is known as the Euler function of the second kind. The gamma function has many interesting properties with many applications in many branches of mathematics. It will take a whole chapter to summarize these properties. For string theory, we will need the beta function, which is defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

where Re x and Re $y > 0$. The beta function is related to the gamma function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

This is known as the Euler function of the first kind. It can be proven easily by substitution and change of variable.

In the 1960s, analysis of experimental data showed the surprising finding that the spins of elementary particles are proprtional to the sqaures of their masses. See Figure 14.5 below. This plot for meson resonances (fermion) also holds for baryon resonances and they can be adjusted on linear trajectories, with a universal slope, whose value can be fixed around 1 GeV^{-2}. These trajectories are known as Regge trajectories.

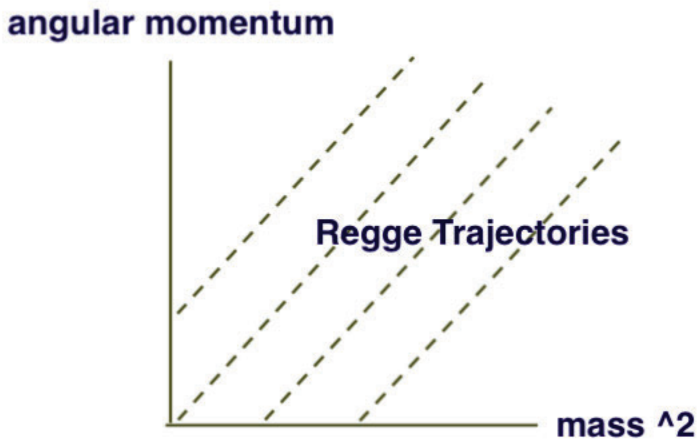

Figure 14.5: Regge Trajectory; a plot of spin vs mass^2; a linear relation

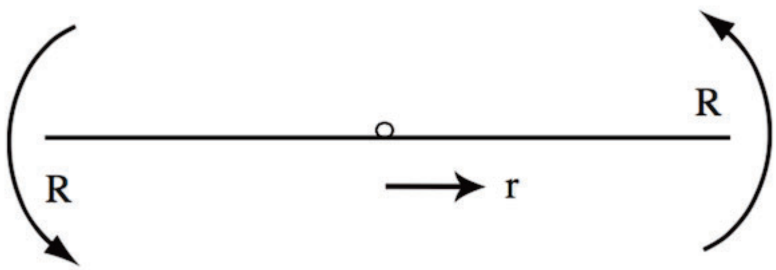

Figure 14.6: Meson resonance composed of a rotating string,; where v(r) = rotating velocity and $\frac{v(r)}{c} = \frac{r}{R}$

Limiting the discussion to the case of the meson, it is asssumed that the meson is composed of two quarks linked by a string, see Figure 14.6 below. The string is characterized by a linear energy density k

The total energy of the rotating string is

$$E = mc^2 = 2\int_0^R \frac{kdr}{\sqrt{1-(\frac{v}{c})^2}} = 2\int_0^R \frac{kdr}{\sqrt{1-(\frac{r}{R})^2}} = \pi kR$$

Therefore,

$$R = \frac{mc^2}{\pi k}$$

The total angular momentum is

$$J = 2\int_0^R \frac{kdr}{\sqrt{1-(\frac{v}{c})^2}} rvdr = \frac{1}{2}\pi k R^2$$

Substituting R into J, we get

$$J = (\frac{c^4}{2\pi k})m^2$$

Therefore

$$J \propto m^2$$

Hence, the string-like potential is adequate to explain the phenomenon underlying the Regge trajectory. If we set $\alpha^* = \frac{c^4}{2\pi k}$, the force binding the quark is independent of the radius

$$F = \partial_R E = \pi k = \frac{c^4}{2\alpha}$$

14.12 Elastic Scattering.

We examine the case of two ingoing mesons yielding two outgoing mesons and no other particle is produced, hence the term elastic. See Figure 14.7 below.

Mandelstam Variables. These are numerical quantities that encode the energy, momentum, and angles of particles in a scattering process in a Lorentz-invariant fasshion. They are used for scattering processes of two particles to two particles. If the Minkowski metric is $(1,-1,-1,-1)$, the Mandelstram variables s,t,u are defined as

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$

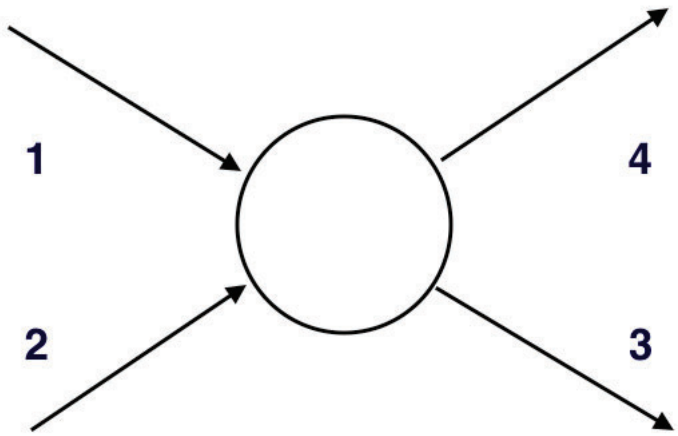

Figure 14.7: Elastic scattering of two mesons. Two incoming particles (1, 2) form two outgoing particles (3, 4)

$$t = (p_1 - p_3)^2 = (p_4 - p_3)^2$$

$$u = (p_1 - p_4)^2 = (p_3 - p_2)^2$$

where p_1, p_2, p_3 and p_4 are the four-momenta. s is known as the square of the center of mass energy (invariant mass) and t, u are the aquare of the four momenta transfer. The letters s, t, u are also used in the terms -channel (space channel), t - channel (time channel) and u- channel. These channels represent different Feynman diagrams or different possible scatttering events where the interaction involves the exchange of an intermediate particle whose squared four- momenta equals s, t and u respectively. See Figure 14.8 below.

The $s-$ channel corresponds to the particles 1,2 joining into an intermediate particle that eventually splits into 3,4; the s-channel is the only way that resonances and new unstable particles may be discovered provided their lifetimes are long enough that they are directly detectable. The t - channel represents the process in which particle 1 emits the intermediate particle and becomes

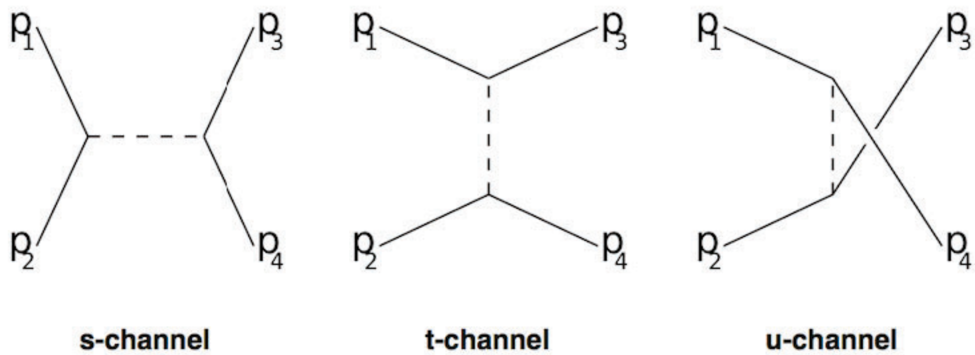

Figure 14.8: Feynman diagrams of possible particle interaction

the final particle 3, while particle 2 absorbs the intermediate particle and becomes particle 4. The *u*- channel reverses the role of particles 3 and 4 in the *t*-channel.In the relativistic limit, the momentum is large, the energy E becomes essentially the momentum norm

$$E^2 = p.p + m_0^2$$

$$E^2 \cong p.p$$

Setting c = 1 ,

$$p_i^2 = m_i^2$$

$$s = (p_1^2 + p_2^2)^2 = p_1^2 + p_2^2 + 2p_1p_2 \approx 2p_1p_2 \approx 2p_3p_4$$

Similarly,

$$t \approx -2p_1p_3 \approx -2p_2p_4$$

$$u \approx -2p_1p_4 \approx -2p_2p_3$$

It can also be shown that

$$s+t+u=\sum_{i=1}^{4} m_i^2$$

The scattering amplitude depends on the trajectories α, which can be defined in the various channels. The poles in this model play the role of resonances. The constraint is that the poles can occur in one or the other chaneel , say s or t , but not simultaneously. The amplitude is constructed using the beta function

$$A(s,t)=\frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

This way, the Euler B function reproduces the scattering amplitude in terms of the Regge trajectories.

Regge Field Theory

Regge theory is concerned with the classification of elementary particles and resonances and with the collisions of elementary particles at high energies. It forms a basis for the study of strong interactions of elementary particles. To understand this theory we will take a brief survey of the lingo of high energy physics, introduce some of the experimental results, and the basic theoretical ideas, hence culminating in the idea of duality and the Veneziano representation. A Regge pole is a singularity of the form

$$\frac{1}{J-\alpha}$$

where J is the angular momentum and α is a function of the energy of colliding particles, in a suitably chosen scattering amplitude. Regge models are phenomenonological in nature, they are developed nd modified under the stimulus of experimental results. Regge theory is based on the use of complex angular momentum combined with relativistic collision theory.

Basic Definitions. Certain quantities are conserved absolutely in interactions of elementary particles. Total energy, momentum and charge are conserved. Total strangeness is conserved in strong but not in weak interactions. Pions are represented as (π^+, π^0 , π^-) , kaons (K^- , K^0 , $\overline{K}^0$, K^+) , nucleons (n, p) , hyperons (Λ^0, Σ^-, Σ^0, Σ^+, Ξ^0, Ξ^-, Ω^-)

Spin J.

(1) A fermion (eg, proton or neutron) has half-odd- integer spin , in units $\hbar = \frac{h}{2\pi}$. These particles obey the Pauli Exclusion Principle and the Fermi-Dirac statistics.

(2) A boson has integer or zero spin, e.g. pions have J = 0, the ρ meson have J = 1 . They obey Bose- Einstein statistics.

Strangeness S. A strangeness quantum number is associated with each particle

. π mesons and nucleons have S = 0

. Λ and Σ baryons have S = -1

. K^+ mesons have S = + 1, K^- mesons have S = -1.With the rules of strangeness conservation, the following reaction is not allowed

$$\pi^- + p \longrightarrow \Sigma^+ + K^-$$

S = 0 on the left hand side, and S = -2 on the right hand side.Let us consider the pion-nucleon elastic scattering

$$\pi_1 + N_1 \longrightarrow \pi_2 + N_2$$

The corresponding four vectors will be written q_1 and q_2 for the incoming and outgoing pions and p_1, p_2 for the ingoing and ougoing nucleons. See Figure 14.9 below. Energy and momentum must be conserved

$$q_1 + p_1 = q_2 + p_2$$

In the πN system, conservation of energy gives

$$k^2 = k'^2$$

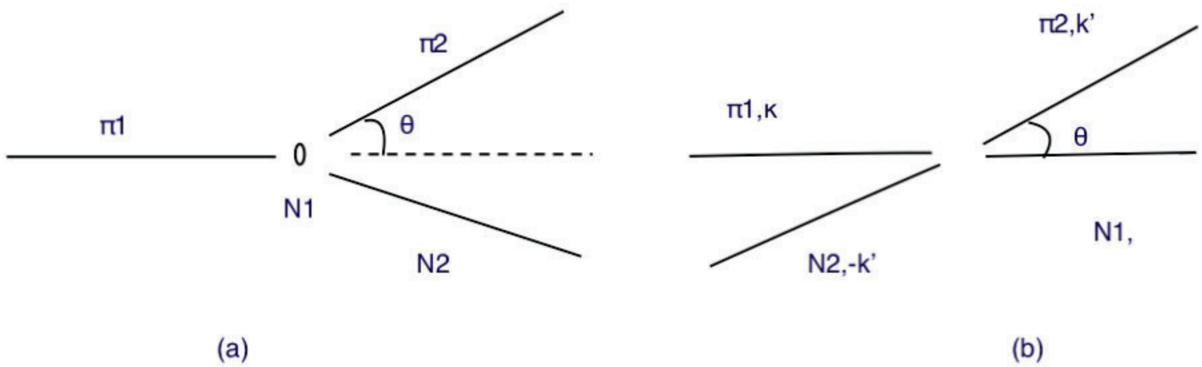

Figure 14.9: Pion - nucleon interaction ; (a) N_1 at rest , (b) Momenta for CM = center of mass system

where k , $k^{'}$ denote the initial and final monenta in the CM system and the total energy W is

$$W = \sqrt{s} = \sqrt{\mu^2 + k^2} + \sqrt{M^2 + k^2}$$

where μ is the mass of the pion . s is the square of the total energy in the CM system. The scattering angle θ in the CM system is related to the three momenta k and $k^{'}$ by

$$k.k^{'} = |k||k^{'}|cos\theta$$

The momentum transfer from nucleon to pion in the CM collision is

$$k^{'} - k$$

t is defined to be minus the momentum transfer squared

$$t = -(k^{'} - k)^2 = -2k^2(1 - cos\theta)$$

As in section 12.13, the relativistic invariants s and t are given in terms of the four -vector equations

$$s = (q_1 + p_1)^2 = (q_2 + p_2)^2$$

$$t = (q_1 - q_2)^2 = (p_1 - p_2)^2$$

Scattering Amplitude. This is the probability anplitude of the outgoing spherical wave relative to the incoming plane wave in a scattering process. The latter is described by the wave function

$$\psi(r) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

where $r \equiv (x, y, z)$ is the position vector, e^{ikz} is is the incoming plane wave with the wave number k along the z-axis: $\frac{e^{ikr}}{r}$ is the outgoing spherical wave , θ is the scattering angle and $f(\theta)$ is the scattering amplitude. The dimension of the scattering amplitude is length. The scattering amplitude is a probability amplitude and the differential cross-section as a function of scattering angle is given as its modulus squared

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

Differential Cross Section. Consider scattering process, as in Figure 14.10 below;

The impact parameter b and the scattering angle ,θ , have a one-to-one functional dependence on each other. The differential size of the cross section is the area element of the plane of the impact parameter

$$d\sigma = bdbd\varphi$$

The solid angle dΩ is

$$d\Omega = sin\theta d\theta d\sigma$$

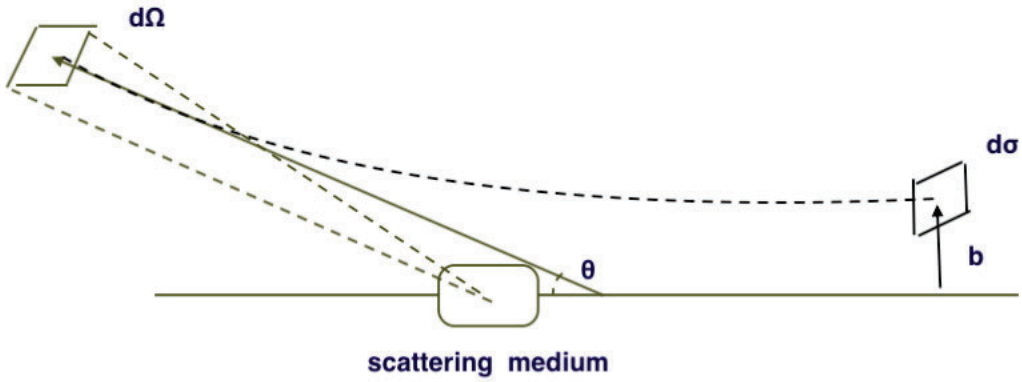

Figure 14.10: Scattering process; single particle scattered by a stationary target. Cylindrical coordinates azimuthal angle φ, the target at the origin, incident beam aligned with z-axis, θ is the scattering angle, measured between incident and scattered beam and the impact parameter b is the perpendicular offset of the trajectory of incoming particle

The differential cross section is the quotient of these quantities $\frac{d\sigma}{d\Omega}$,

$$\frac{d\sigma}{d(cos\theta)} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\sigma}{d\Omega}d\varphi$$

It is a function of the scattering angle θ and the impact parameter b, plus observables such as the momentum of the incoming particle. In cylidrically symmetric situations, about the beam axis, the azimuthal angle, φ, is not changed by the scattering process and the differential cross section is as above.

Quantum Scattering. In the time- dependent formalism of quantum scattering, the initial wave function, before scattering is taken to be a plane wave ith definite momentum k;

$$\phi_{-}(r) \overset{r\to\infty}{=\longrightarrow} e^{ikz}$$

where z and r are the relative coordinates between the projectile and the target. The arrow indicates that this only describes the asymptotic behavior f the wave function when the projectile and the target are too far apart for the interaction to have any effect. After the scattering takes place, the wave function takes on the following asymptotic form

$$\phi_+(r) \overset{r\to\infty}{=\longrightarrow} f(\theta,\phi)\frac{e^{ikr}}{r}$$

where f is the scattering aomplitude, a function of the angular coordinates. This is valid for a short range energy-conserving interaction. The full wave function of the system behaves asymptotically as the sum

$$\phi(r) = \phi_-(r) + \phi_+(r)$$

The differential cross section is related to the scattering amplitude

$$\frac{d\sigma}{d\Omega}(\theta,\phi) = |f(\theta,\phi)|^2$$

This can be interpreted as the probability density for finding the scattered projectile at a given angle.

The Scattering Matrix. The transition of a closed system of particles from an inital state of | k> to a final state |f > is described in quantum theory by the S - matrix

$$|f> = S|k>$$

The matrix elements of the S - matrix

$$S_{fk} = < f|S|k>$$

where we have Hermitian conjugacy

$$|f>^* = < f|$$

S$_{fk}$ can be represented in the form

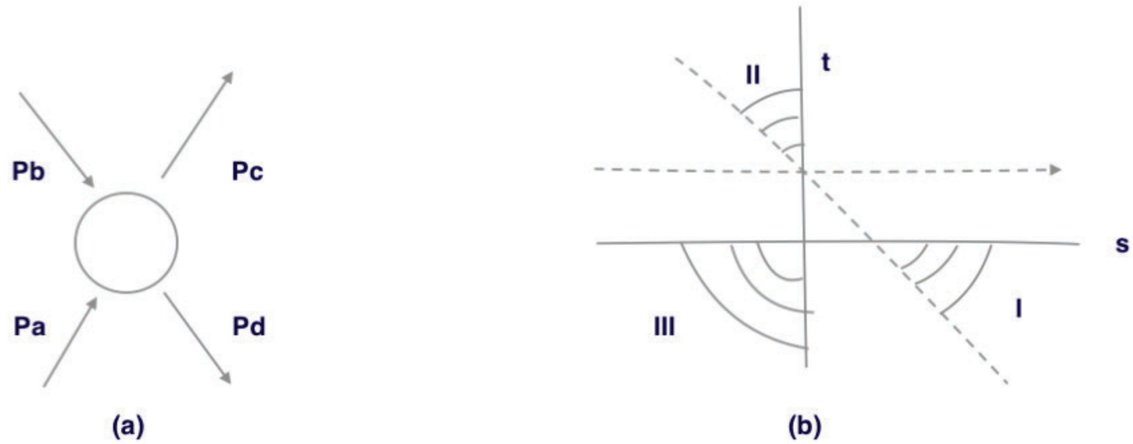

Figure 14.11: Two particle elastic collision ; (a) energy-momentum four vectors for reaction ab$\longrightarrow cd$, (b) s, t plot for two body collision, shaded areas are allowed physical regions, - 1 $\leq$ cos $\theta \leq 1$, s $\geq$ (M + $\mu)^2$, - $4k^2 \leq t \leq 0$

$$S_{fk} = \delta_{fk} + i(2\pi)^4\delta^{(4)}(P_i - P_k)T_{fk}$$

Remarks.

(1) δ_{fk} = 1 if the state does not change , that is , $|f>$ = $|\, k>$; no interaction

(2) $\delta^{(4)}$(P_i - P_k) represents the conservation of energy and momentum

(3) T_{fk} is the transition scattering amplitude from the state $|\, k>$ to the state $|\, f>$.

Relativistic Kinematics of Two- Body Collisions. If we neglect the spin of the nucleon, the whole result of the scattering process can be expressed in terms of s and t. The scattering amplitude F is a function of s and t only, F (s,t). We next introduce the concept of relativistic crossing symmetry. we re-label the ingoing and outgoing momenta as in Figure14.11 below

For process I; The outgoing momenta are $-p_c$, - p_d and their energies are $-p_c{}^2$, -p_d^2. Conservation of total energy and momentum requires that

$$p_a + p_b + p_c + p_d = 0$$

As outlned in section 12,13

$$s+t+u=4m^2$$

assuming all particles have equal mass. In the CM system for particles a and b, we write k^2 for the square momentum and θ for the scattering angle. Then

$$s=4(k^2+m^2)$$

$$t=-2k^2(1-cos\theta)$$

$$u=-2k^2(1+cos\theta)$$

The physical values of s and t for process I are shown in the shaded region I. The shaded region II and III correspond to the values s and t (or u) for which the following processes are physically allowed

$$II.b+\bar{d}\longrightarrow\bar{a}+c$$

$$III.b+\bar{c}\longrightarrow\bar{a}+d$$

where $\bar{a}$ amd $\bar{d}$ are the anti-particles of a and d respectively. t is the energy sqaured of particles b and $\bar{d}$ and u is the energy squared of b and $\bar{c}$. The principle of crossing symmtery asserts that the same scattering amplitudes F(s,t) describes all three processes I, II , and III, provided suitable values of s and t are chosen in each case. It applies when the particles have zero spin. The Mandelstam variables have a simple physical meaning. In the center- of - mass system , cms , $a+b\longrightarrow c+d$, the s -channel, s is the square of the total energy of the colliding particles and $t=-(p_a-p_c)^2$ is the square of the momentum transfer from a to c . In the cms reaction II, the roles of t and s are reversed. The variables u and t play similar roles in the t and u channel. Then , we can represent the physical region of any reaction on the Mnadelstam plane. For instance, reaction II, coresponds to

$$t \geq 4m^2, s \leq 0, u \leq 0$$

Threshold Singularities. Let us consider the singualrities of the amplitude. For an illustration, we can consider elastic scattering of neutral pions;

$$\pi^0 + \pi^0 \longrightarrow \pi^0 + \pi^0$$

It is assumed, in accordance with experiments, that pions are the lightest stable hadrons and that there is no bound state of two neutral pions. Then the amplitude has no singularities at $s < 4\mu^2$. The first threshold lies at $s = (2\mu)^2$. It corresponds to the two-particle intermediate state. The next three particle threshold could have appeared at $s = (3\mu)^2$. However, the second threshold in the pion scattering amplitude is situated a t $s = (4\mu)^2$ - the four particle state, since the transition of two pions into three is forbidden by G-parity conservation.

Collision Amplitude. Differential cross sections are measured in the lab. The counter records the number of pions elastically scattered into a solid angle dΩ per unit time. The elastic differential cross section for scattering of pions on protons is defined by

$$\frac{d\sigma}{d\Omega} = \frac{N_{out}}{N_{in}}$$

where the numerator enotes the number of elastically scattered pions per unit time crossing the area r^2dΩ. The denominator denotes the number of pions in the incident beam per unit area per unit time divided by the number of protons per unit area in the hydrogen target. See Figure 14.12 as an illustration.

If ϕ is the azimuthal angle, then

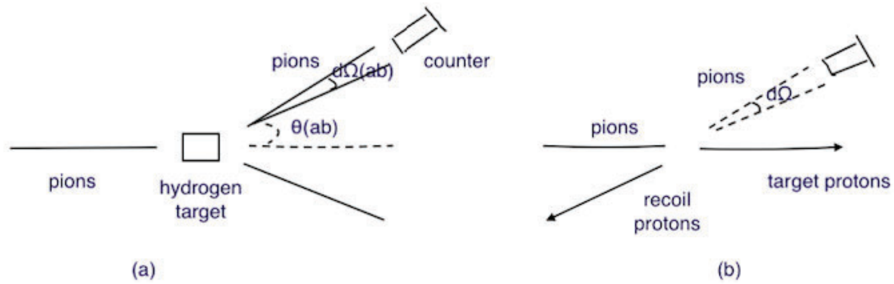

Figure 14.12: Solid angle of elastic scattering process; (a) differential cross section (b) scattering into solid angle dΩ in the CM system

$$d\Omega = d(cos\theta)d\phi$$

Ignoring proton spin, there is no dependence of the differential section on ϕ. Hence

$$\int \phi \frac{d\sigma}{d\Omega} = 2\pi \frac{d\sigma}{d\Omega} = \frac{d\sigma}{d(cos\theta)}$$

From $t = -2k^2(1 - cos\theta)$

$$dt = 2k^2 d(cos\theta)$$

Hence

$$\frac{d\sigma}{dt} = (\frac{\pi}{k^2})\frac{d\sigma}{d\Omega}$$

If we neglect nucleon spin, this invariant differential cross section may be expressed in terms of a single invariant scattering amplitude F(s,t)

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s k^2}|F(s,t)|^2$$

Total cross sections are equal to the sum of all allowed elastic and reaction cross sections including those involving many- body productios

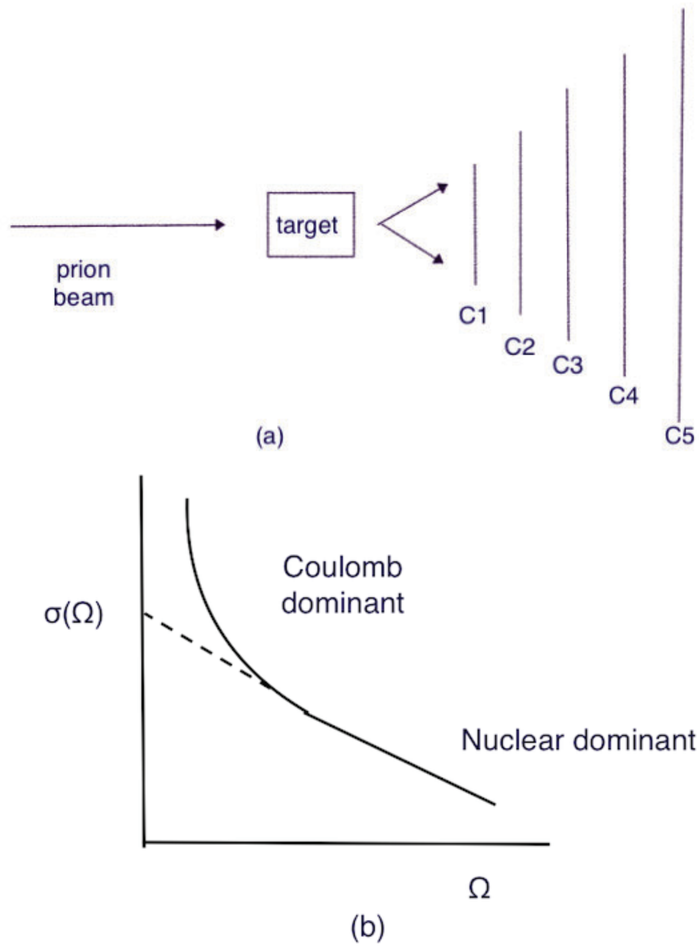

Figure 14.13: Measuring particle scattering angle; (a) Counters C 1 to C5 for measuring total cross sections , (b) Extrapolating to zero solid angle Ω to give a total cross section σ_T

$$\sigma_T(\pi^+p) \equiv \sigma_T(\pi^+p, total) = \sum \sigma(\pi^+p \longrightarrow anything)$$

In practice, σ_T (π^+p) is obtained experimentally by measuring the number of particles removed from the incident prion beam. This is achieved by extrapolating to zero angle as illustrated in Figure 14.13 below.

The counters C_i subtend an angle Ω_i at the target. They detect the passage of charged particles. From the flux through these counters, and from the flux through the incident beam, one can obtain the number of particles removed from the incident beam by collision with the target.

For each solid angle Ω_i one obtains, per unit time

$$\sigma(\Omega_i) = \frac{\pi^+(not\Omega_i)}{\pi^+(target)}$$

where the numerator is the number of π^+ not going through angle Ω_i and the denominator is the number of π^+ in incident beam per proton per target. Extrapolating to $\Omega_i = 0$, one obtains the total cross sectional area as in Fig (b);

$$\sigma_T(\pi^+ p, s) = lim_{\Omega\to 0}\sigma(\Omega)$$

Total cross sections are are related to elastic scattering amplitudes by the optical theorem

$$\sigma_T(\pi^+ p) = \frac{ImF(s,0)}{2k\sqrt{s}}$$

where ImF(s,0) denotes the imaginary part of the π^+pforward elastic scattering amplitude.

Analyticity and Crossing Symmetry. Even for the simplest case of identical spinless particles, it is necessary to regard F(s,t), a function of complex variables s and t. As an illustration, we will consider the case of forward scattering , $t = 0$, for process I. One finds that the forward amplitude F($s,0)$ is an analytic function of the complex variable s throughout the complex s plane, exccept for branch cuts along the real axis. The complex s plane for F $(s,0)$ is illustrated in Figure 14.14 below. There is a branch cut at the real value,

$$s = (2m)^2 = 4m^2$$

which is a the threshold at which process I is a physical process allowed by the kinematic conditions, along the line $t = 0$. There are other branch points in the s plane, for example at the thresholds for production of new particles, $s = (4m)^2$, $(6m)^2$, ..., in the case of pion-pion scattering. These all lie along the real axis.

In addition, there are branch points at the corresponding values of u for process III, of

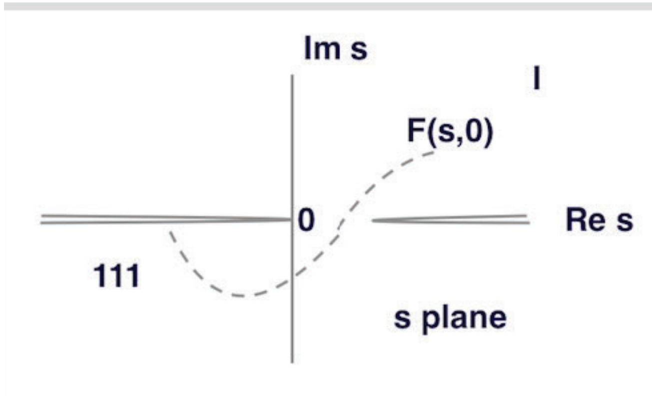

Figure 14.14: The complex s plane for t = 0; branch cut along the real axis, and the path of analytic continuation from I to III

which the leading branch point is at $u = 4m^2$. In this case, with $t = 0$, the threshold $u = 4m^2$ corresponds to $s = 0$, and the attached branch cut is drawn along the left hand real axis. Between the branch points $s = 0$ and $s = 4m^2$, the amplitude F $(s,0)$ is real. This means that the amplitude is Hermitian , hence

$$F(s^*,0) = F^*(s,0)$$

where the star denotes complex conjugation. The physical amplitude for process I is obtained by taking the limit on top of the right hand branch cut. When s is real and greater than $4m^2$, we have

$$F(s,0) = lim_{\varepsilon\to 0}F(s+i\varepsilon,0)$$

The physical amplitude for process III.where u is the energy, is obtained in the s plane by taking the limit on to the real axis below the left hand branch cut. The analytic properties of F$(s,0)$ permit us to derive dispersion relation provided that F ($s,0) \longrightarrow 0$ as $|s| \longrightarrow \infty$ in any direction in the s complex plane. Since F $(s,0)$ is regular inside the closed contour C, we can apply Cauchy's theorem

$$F(s,0) = \frac{1}{2\pi i}\int_C \frac{F(s',0)}{s'-s}ds'$$

Scattering Amplitudes and The S Matrix. The state of a particle at time $|\psi(t)>$ is

$$|\psi(t)>= U(t)|\psi>$$

where $|\psi>$ is the state at time zero and U(t) is the evolution operator;

$$U(t) = e^{\frac{-iHt}{\hbar}}$$

where

$$H = H_0 + V$$

$$H_0 = \sqrt{p^2c^2 + m^2c^4}$$

In non-relativistic mechanics,

$$H_0 = \frac{p^2}{2m}$$

where $p^2 = p.p$ and $V = (x^i, p^i)$, $i = 1,2,3$; x^i are position cocordinates and p^i the momenta. The potential V specifies the interaction of the particle with the fixed target. The physical system is a spinless particle of rest mass m interacting with a target fixed at the origin of the coordinate system. The Cartesian coordinate system and momenta satisfy the fundamental quantum conditions

$$[x^i, x^k] = 0$$

$$[p^i, p^k] = 0$$

$$[x^j, p^k] = i\hbar\delta_{jk}$$

where j,k= 1,2,3 .

In- and Out- Asymptotes. The essential feature of scattering is that the particle behaves as a free particle well before and well after collision with the target. A scatterung state satisfies

$$U(t)|\psi> \longrightarrow U_0(t)|\psi_{in}>; t \longrightarrow -\infty$$

$$U(t)|\psi> = U_0(t)|\psi_{out}>; t \longrightarrow +\infty$$

for some | ψ_{in} > and | ψ_{out} >, where $U_0(t)$ is the free particle evolution operator. The Moller Operators are define by

$$\Omega_{\pm} = lim_{t \to \mp} U^{\dagger} U_0(t)$$

where $U^{\dagger}$ is the Hermitian conjugate. The Moller operators relate the actual state of the system with the free particle particle in- and out-states ;

$$|\psi> = \Omega_{+}|\psi_{in}>$$

$$|\psi> = \Omega_{-}|\psi_{out}>$$

Scattering Operator/Matrix S. The scattering operator S relates the out-state with the in-state without direct reference to the actual state. It follows that

$$|\psi_{out}> = S|\psi_{in}>$$

where

$$S = \Omega_{-}^{\dagger} \Omega_{+}$$

The main goal of scattering theory is to express the out-asymptote $U_0(t)|\ \psi_{out}>$ in terms of the in-asymptote $U_0(t)|\ \psi_{in}>$ without further direct reference to the experimentally indeterminate details of the scattering state U(t) | ψ >. The main goal of scattering theory is to determine the scattering operator S.

Unitarity.

$$S^{\dagger}S = SS^{\dagger} = I$$

This can be easily derived by using the Moller operators. Unitarity has a simple physical meaning; the sum of probabilities of all processes which are possible at a given energy is equal to unity. If S = I + A,t hen

$$i(A - A^{\dagger}) = -AA^{\dagger}$$

Representing the amplitude A as the sum of real and imaginary parts , A = ReA + iImA, the unitarity condition can be expressed as

$$2ImA = AA^{\dagger}$$

Conservation of Energy. It can also be shown that

$$U_0(t)SU_0^{\dagger}(t) = S$$

and therefore, the energy is conserved in the scattering process

$$[S, H_0] = 0$$

H_0 corresponds to the fact that the particle is asymptotically free. The conservation of energy in the scattering process may be expressed as

$$< \psi_{in}|H_0|\psi_{in} >=< \psi_{out}|H_0|\psi_{out} >$$

Scattering Amplitude. The momentum representation of the scattering operator, < q | S | p >, the S matrix, which corresponds to the scattering of a particle with initial momentum p to final momentum q has the form

$$< q|S|p >= \delta(p-q) - 2\pi i\delta(\varepsilon_p - \varepsilon_q)t(p,q)$$

Not all states of the system are scattering states. If the potential is attractive and sufficiently strong, there may also be bound states

$$|1 >, |2 >, ..., |n_b >$$

which satisfy

$$H|b >= \varepsilon_b|b >; b = 1, 2, ..., n_b$$

No in- or out-asymptotes exist for the states

$$|\psi_b(t) >= U(t)|b >= e^{-\frac{i\varepsilon_b t}{\hbar}}|b >; b = 1, 2, , ..., n_b$$

Spectral Decomposition of the Free Particle Hamiltonian. The free particle Hamiltonian is a function of momentum , where

$$H_0 = \int d^3p|p > \varepsilon_p < p|$$

where

$$\varepsilon_p = \sqrt{p^2c^2 + m^2c^4}$$

Lorentz Factor γ.

$$\gamma = \frac{\varepsilon_p}{mc^2} = \frac{\omega}{c\mu} = \sqrt{1 + \frac{k^2}{\mu^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where $\varepsilon_p = \hbar\omega$ and $\mu = \frac{mc}{\hbar}$, the inverse of the Compton wavelemgth.

Green's Operators. These are defined as

$$G_0(z) = \frac{1}{z - H_0}$$

$$G(z) = \frac{1}{z - H}$$

where H_0 is the free-particle Hamiltonian, H is the Hamiltonian including interaction, V is the inetraction potential and z is a complex number, which has the dimensions of energy, for which the inverses exist. It follows fromd H_0 and ε_p above that

$$G_0(z) = \int d^3p|p> \frac{1}{z - \varepsilon_p} < p|$$

The function $< \phi \mid G_0(z) \mid \psi >$ has branch points at mc^2 and ∞.

Preliminary Scattering Theory - Partial Wave Analysis**.** This is a technique for solving scattering problems by decomposing each wave into its constituent angular momentum components and solving using boundary conditions. The sccenario is as follows; a steady beam of particles scatters off a spherically symmetric potential V(r) . This is short ranged, meaning as $r \longrightarrow \infty$, the particles behave like free particles. The particle is described as a plane wave travelling along the z-axis

$$e^{ikz}$$

It is assumed that the beam is switched on for times longer than the times for particle interaction with the scattering potential. . Hence, we solve the wave function $\Psi(r)$ for the particle beam

$$[-\frac{\hbar^2}{2m}\nabla^2 + V(r)]\Psi(r) = E\Psi(r)$$

We make the following ansatz

$$\Psi(r) = \Psi_0(r) + \Psi_s(r)$$

where Ψ_0(r) $\propto$ e^{ikz} is the incomng plane wave and Ψ_s(r) is the scattered part. It is the asymptotic form of the latter that is of interest because detection of particles occurs far away from the origin. At these distances, particles are free and Ψ_s(r) is a solution of the free Schrodinger equation. We therefore investigate the plane wave expansion

$$e^{ikz} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l cos\theta$$

The spherical Bessel function j$_l$(kr) asymptotically behaves like

$$j_l(kr) \longrightarrow \frac{1}{2ikr}(e^{i(kr-\frac{l\pi}{2})} - e^{-i(kr-\frac{l\pi}{2})}$$

This corresponds to an outgoing and and an incoming spherical wave. For the scattered wave function, only outgoing parts are expected. We therefore expect at large distance

$$\Psi_s(r) \propto \frac{e^{ikr}}{r}$$

and set the asymptotic form of the scattered wave to

$$\Psi_s(r) \longrightarrow f(\theta, k)\frac{e^{ikr}}{r}$$

where $f(\theta, k)$ is the scattering amplitude, which is dependent on the elevation angle θ, and the energy. For the entire wave function

$$\Psi(r) \longrightarrow \Psi^{+}(r) = e^{ikz} + f(\theta,k)\frac{e^{ikr}}{r}$$

Partial Wave Expansion. In case of spherically symmetric potential V(r), the scattering wave function may be expanded in spherical harmonics which reduce to Legendre polynomials because of azimuthal symmetry, no dependence on ϕ,

$$\Psi(r) = \sum_{l=0}^{\infty} \frac{u_l(r)}{r} P_l(cos\theta)$$

In the standard scattering problem, the incoming beam is assumed to take the form of a plane wave of wave number k, which can be decomposed into partial waves using the plane wave expansions in terms of spherical Bessel functions and Legendre polynomials. Consider equal mass spinless particles having a single scattering amplitude F(s, t) . The partial wave expansion tahes the form

$$F(s,t) = \frac{8\pi W}{k} \sum_{l=0}^{\infty} (2l+1) f_l(s) P_l cos\theta$$

where $W^2 = s = 4\ (m^2 + k^2)$ and $\cos\ \theta = z$

$$f_l(s) = \frac{1}{2}\int_{-1}^{1} d(cos\theta)\frac{kF(s,t)}{8\pi W} p_l cos\theta$$

The unitarity of the S- matrix corresponds to conservation of probability. As outlined in the section on unitarity

$$i(f_l^* - f_l) = 2|f_l|^2$$

Hence,

$$f_l(s) = \frac{e^{(2i\delta_l)-1}}{2i}$$

where the phase shift δ_l is real in the elastic collisions. It is convenient to analyze the amplitude as a partial wave series. This is how resonances are derived from experimental differential cross - sections. The resonance would occur as a pole in the partial wave amplitude f_l corresponding to angular momentum l. Near s = a - ib, we have a resonance pole

$$f_l(s) \simeq \frac{g}{s-a+ib}$$

a and b are both dependent on l and s. Writing

$$s_0(l,s) = a - ib$$

we get

$$f_l(s) \simeq \frac{g}{s-s_0(l,s)}$$

Yukawa Potential. This is a potential of the form

$$V(r) = -g^2\frac{e^{-kmr}}{r}$$

where g is a magnitude scaling constant, m is the mass of the particle,r is the radial distance of the particle and k is another scaling constant. The potential is monotone increasing in r and is negative, implying the force is atractive. The Coulomb potential of electromagnetism is an exapmle of a Yukawa potential with e^{-kmr} =1 everywhere, interpreted as the photon mass equal to zero. In interactions between a meson field and a fermion field, the constant g is equal to the gauge coupling constant between those fields. Its Fourier transform is

$$V(r) = -\frac{g^2}{(2\pi)^3}\int e^{ikr}\frac{4\pi}{k^2+m^2}d^3k$$

The fraction $\frac{4\pi}{k^2+m^2}$ is the propagator or Green's function of the Klein Gordon equation.A comparison of the long range potential strength for Yukawa and Coulomb potential is shown in the

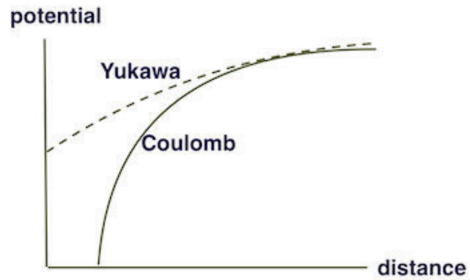

Figure 14.15: Comparison of Yukawa and Coulomb Potential

Figure 14.15 below.

It can be seen that the Coulomb potential has effect over a great distance whereas the Yukawa potential approaches zero rather quickly.

The Propagator Term. Consider the electrostatic potential about a charged point particle. This is given by

$$\nabla^2 \phi = 0$$

which has the solution

$$\phi = \frac{e}{4\pi\varepsilon_0 r}$$

This describes the potential for a force mediated by massless particles, the photons. For a particle with mass, the relativistic equation

$$E^2 = p^2c^2 + m^2c^4$$

can be converted to a wave equation by the subtitutions

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}; p_x \longrightarrow -i\hbar\frac{\partial}{\partial x}$$

Hence,

$$-\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = (m^2 c^4 - \hbar^2 c^2 \nabla^2)\phi$$

Or, in the static, time-independent case, this leads to

$$(\nabla^2 - \frac{m^2 c^2}{\hbar^2})\phi = 0$$

which gives $\nabla^2 \phi = 0$, for the massless case. For a point source with spherical symmetry,

$$\nabla^2 \phi \longrightarrow \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d\phi}{dr}) = \frac{1}{r^2}(2r\frac{d}{dr} + r^2 \frac{d^2}{dr^2})\phi = (\frac{2}{r}\frac{d}{dr} + \frac{d^2}{dr^2})\phi$$

Note

$$\frac{d}{dr}(r\phi) = r\frac{d\phi}{dr} + \phi$$

and

$$\frac{d^2}{dr^2}(r\phi) = \frac{d\phi}{dr} + r\frac{d^2\phi}{dr^2} + \frac{d\phi}{dr} = r\frac{d^2\phi}{dr^2} + 2\frac{d\phi}{dr}$$

$$\frac{1}{r}\frac{d^2}{dr^2}(r\phi) = \frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = \nabla^2\phi = \frac{m^2c^2}{\hbar^2}\phi$$

Therefore

$$\frac{d^2}{dr^2}(r\phi) = \frac{m^2c^2}{\hbar^2} r\phi$$

A solution of this differential equation is

$$\phi = g^2 \frac{e^{-\frac{mcr}{\hbar}}}{r}$$

where g is a constant, the coupling strength. This is the Yukawa form of the potential. It was originally introduced to describe the nuclear interaction between protons and neutrons due to pion exchange. For non-relativistic scattering on a Yukawa potential, Regge showed that a unique analytic extension $f(l,s)$ could be defined such that at integer values it is equal to the partial wave amplitude

$$f(l,s) = f_l(s); l = 1,2,3,...$$

Using Regge's analytic extension, the location $s_0(l,s)$ of the resonance pole becomes in $f_l(s)$ becomes an analytic function. One can solve the equation

$$s = s_0(l,s)$$

to give

$$l = \alpha_0(s)$$

We can re-write $f\ (l,\ s)$ as

$$f(l,s) = \frac{r}{l - \alpha_0(s)}$$

The pole in the partial wave amplitude is located in the complex l plane ans its position is a function of s. This is called a Regge pole, and the path as it follows as s moves through real values is called a Regge trajectory. The Regge trajectory correlate sequences of particles and resonances. The Regge pole model states that there exist complex Regge trajectory functions $\alpha_n(s)$, depending on $s = W^2$ (the square of the center of mass energy), that correlate certain sequences of particles or resonances. In relativistic theory the particles associated with a given function $\alpha_n(s)$ have the same internal quantum numbers - baryon number, isospin, parity, strangeness etc. - but they have spins that differ by units of two. In nonrelativistic potential scattering a given function α_n(s) correlates sequences of bound states or resonances, and, in the absence of exchange forces, the spins (angular momenta) will differ by only one unit.

The Veneziano Representation. Veneziano considered the reaction

$$\pi\pi \longrightarrow \pi\omega$$

We take all the particles to have the same mass. Then

$$s+t+u=4m^2$$

The three channels s,t,u denote the energy squared;

$$s-channel;\pi^+\pi^0 \longrightarrow \pi^-\omega$$

$$t-channel;\pi^-\pi^- \longrightarrow \pi^0\omega$$

$$u-channel;\pi^+\omega \longrightarrow \pi^+\pi^0$$

In each of the three channels, the ρ meson represents an intermediate stage or resonance. In the s-channel, the amplitude should be expressible in terms of a sum over successive resonances

$$A(s,t)=\sum_n \frac{r}{s-s_n}$$

where $s_n = m_n^2$ and m_n is the mass of the nth resonance on the ρ Regge trajectory. n = 0 gives the ρ meson and

$$\alpha(m_n^2)=2n+1$$

Next, we make a linear approximation for the trajectory and take $\varepsilon(s)$ to be small

$$\alpha(s) \simeq \alpha_0+s+i\varepsilon(s)$$

At high energies, the amplitude will be dominated by the Regge pole exchange in the t-channel, which is also the ρ-meson, thus as $s \longrightarrow \infty$

$$A(s,t) \sim \beta(t) s^{\alpha(t)-1}$$

The idea of duality asserts that A(s,t) above is the asymptotic form of the sum over resonances

$$\sum_n \frac{r}{s-s_n} \longrightarrow \beta(t) s^{\alpha(t)-1}$$

Veneziano made the important observation that the Euler beta function has the desired property of giving both forms of A(s,t) that can be written symmetrically in s and t. He, therefore took as a trial, a scattering amplitude

$$A(s,t) = B(1-\alpha(s), 1-\alpha(t))$$

$$= \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))}$$

Noting that the complex valued gamma function is undefined for non positive integers, but they can be defined in the Riemann sphere as ∞. The reciprocal gamma function is well defined and analytic at these values. Therefore, This function has poles at fixed s and at fixed t,

$$1-\alpha(s) = -n$$

$$1-\alpha(t) = -m$$

for m,n = 0, 1, 2, ... It also has lines of zeros, due to the poles of the gamma function in the denominator, when

$$2-\alpha(s)-\alpha(t)=-l;l=0,1,2...$$

CHAPTER XV

STRING THEORY

15.1 Primer on String Theory

Infinite Momentum Boost (Light Cone Frame),[10]. We begin with a set of particles in a spacetime coordinate system (x,y,z,t). We boost the system in the z-axis. Any axis will do. We boost it to a great velocity, near the speed of light. We obtain a huge momentum in the z-axis. The particles are Lorentz contracted and time dilated. The proper time is τ. The energy, E, of the system is

$$E = \sqrt{p_x^2 + p_y^2 + p_z^2 +} m^2$$

where p is the momentum and m is the rest mass (internal binding energy). p_z is very large. Nothing happens to p_x, p_y and m. We Taylor expand for large p_z

$$E = p_z \sqrt{1 + \frac{p_x^2 + p_y^2 + m^2}{p_z^2}}$$

$$E \simeq p_z \left(1 + \frac{p_x^2 + p_y^2 + m^2}{2p_z^2}\right) \tag{15.1.1}$$

For a system of particles ; $\sum p_z$ is constant, by the Law of Conservation of Momentum . It can be dropped off, as we are only interested in energy differences. Therefore,

$$E = \sum \left(\frac{p_x^2 + p_y^2}{2p_z} + \frac{m^2}{2p_z}\right)$$

For simplicity of calculations , we will ignore p_y. Note, for large p_z, the energy is very small in the $x-y$ plane. Since

$$H = E = i\hbar \frac{\partial}{\partial t}$$

where H is the Hamiltonian, the time evolution operator. The system changes very slowly, when the energy is very small. This is due to time dilation. In the plane perpendicular to the boost, non-relativistic quantum mechanics applies. Let us next analyze a simple string, the energy of a simple string can be expressed according to Hooke's law as

$$E = \frac{1}{2}kx^2$$

where x is the length of the string and k is the stiffness constant. The important point to note here is that the energy of the string is proprtional the the mass squared

$$E \propto m^2$$

If the string is moving, we can express the energy as

$$E = \frac{p^2}{2m} + B$$

where B is the binding energy, the internal energy of the system. Thinking of the string as a collection of particles, the energy is the sum total of kinetic and potential energy.

$$E = m\sum_i \frac{\dot{x}_i^2}{2} + k\frac{(x_i - x_{i+1})^2}{2}$$

If we parametrize the string with a parameter , σ, whose length ranges from 0 to π, the Hamiltonian can be expressed as

$$H = \int_0^\pi d\sigma [\frac{\dot{x}(\sigma)^2}{2} + \frac{1}{2}(\frac{\partial x}{\partial \sigma})^2]$$

Noting that $\dot{x}(\sigma) = \frac{\partial x}{\partial t}$ and $\frac{\partial x}{\partial \sigma} = \frac{L}{\pi}$, we see that

$$L^2 \propto m^2 \Longrightarrow L \propto m$$

Since we are dealing with a vibrating string, we apply Hooke's Law , where the binding energy is the potential energy. Here, the energy is proportional to the mass and not the mass squared. At the quantum level , the energy is quantized and the Regge trajectory has no reason to end. Furthermore, as we explained above, perpendicular to the boost, the physics is non-relativistic with one less dimension. The binding energy does not depend on p. When $p = 0$, $B = mc^2$. Rearranging Eq. 13.1;

$$H = i\hbar \frac{\partial}{\partial t} = (E - p_z)p_z = \frac{p_x^2 + p_y^2}{2} + \frac{m^2}{2}$$

In the limit as $p_z \longrightarrow \infty$, we have non - relativistic physics in the $x - y$ plane. We see that the kinematics are independent of the state of motion. The binding energy, second term, is proportional to m^2. The left hand side is known as the light cone energy.

Strings in Two Dimensions. We think of a string as a collection of mass points, N mass points. Parametrize with $\sigma(\tau)$ from 0 to π;

$$E = \sum_i (\mu \frac{\dot{x}_i{}^2}{2} + k \frac{\Delta x^{2_i}}{2})$$

where $\mu$is the non - relativistic, or analogue mass and $\mu = \frac{1}{N}$, k is the spring constant. We set $k = \frac{N}{\pi^2}$ and $\Delta\sigma = \frac{\pi}{N}$. Let us first analyze the kinetic energy, K.E.

$$K.E. = \sum_i \frac{1}{2N} \dot{x}^2 = \frac{1}{2\pi} \sum_i \Delta\sigma \dot{x}_i{}^2$$

In the limit, we get

$$K.E = \frac{1}{2\pi} \int_0^{2\pi} (\frac{\partial x}{\partial \tau})^2 d\sigma$$

where τ is re-scaled time, proper time , moving with system. The potential energy P.E. is

$$P.E = \frac{1}{2\pi}\int_0^{\pi}(\frac{\partial x}{\partial \sigma})^2 d\sigma$$

Therefore,

$$H = \frac{1}{2\pi}\int_0^{\pi}(\frac{\partial x}{\partial \tau})^2 + (\frac{\partial x}{\partial \sigma})^2 d\sigma$$

This is similar to the wave eqiuation, the energy of a wave field. Hence, we arrive at wave physics. What about the boundary conditions ? This is determined by Newton's law applied to mass points. By Hooke's law

$$F \propto displacement$$

$$F \propto k\Delta x = \frac{N}{\pi}\Delta x$$

In the limit,

$$F = N\frac{\partial x}{\partial \sigma}\partial \sigma = N\frac{\partial x}{\partial \sigma}\frac{1}{N} = \frac{\partial x}{\partial \sigma}$$

By Newton's second law,

$$F \propto \frac{\partial x}{\partial \sigma} = \mu \ddot{x}^2 = \frac{1}{N}\ddot{x}$$

Therefore

$$\ddot{x} \propto N\frac{\partial x}{\partial \sigma}$$

Note $\ddot{x} \longrightarrow \infty$ as N $\longrightarrow \infty$, which is not tenable. Therefore, we set the Neumann boundary condition

$$\frac{\partial x}{\partial \sigma} = 0$$

Next, we Fourier analyze x_i and substitute into the Lagrangian. For Neumann boundary conditions,

$$x(\sigma) = \sum_{n=0}^{\infty} x_n(\tau) cosn\sigma$$

Substitute into Lagrangian, L

$$L = \frac{1}{2\pi} \int_0^{\pi} (\frac{\partial x}{\partial \tau})^2 - (\frac{\partial x}{\partial \sigma})^2 d\sigma$$

$$\frac{\partial X}{\partial \tau} = \dot{x}(\sigma) = \sum_{n=0}^{\infty} \dot{x}_n cosn\sigma$$

$$(\frac{\partial x}{\partial \tau})^2 = \sum_{n=0,m=0} \dot{x}_m \dot{x}_n cosn\sigma cosm\sigma$$

Using the trigonometric identity ; $\int_0^{\pi}$cosnσcosmσdσ = $\delta_{mn}\frac{\pi}{2}$

$$(\frac{\partial x}{\partial \tau})^2 = \frac{\dot{x}_0^2}{2} + \sum_n + \frac{1}{4}\dot{x}_n^2$$

where x_0 is the average position, the center of mass and the second term is the internal relative motion.

$$\frac{\partial x}{\partial \sigma} = -\sum_{n=0}^{\infty} n x_n sinn\sigma$$

$$(\frac{\partial x}{\partial \sigma})^2 = \frac{1}{2\pi} \sum_{n,m} x_n x_m \int_0^{2\pi} sinn\sigma sinm\sigma$$

$$= \frac{1}{2\pi}\sum_{n,m} n^2 x_n^2 . \frac{\pi}{2} = \frac{1}{4} n^2 {x_n}^2$$

Therefore,

$$L = \frac{\dot{x}_0^2}{2} + \sum_n \frac{1}{4}\dot{x}_n^2 - \frac{1}{4}\sum_n n^2 x_n^2$$

If we drop out the first term, which is a constant, we obtain the Lagrangian of a harmonic oscillator with frequency n. We have an infinite collection of harmonic oscillators which are not coupled to each other. The frequency of the nth oscilltor is n. $\omega_n = n$. The center of mass has no oscillation, $\omega_0 = 0$. x_0 has no restoring force. The third term is the restoring force. The second and third terms are the internal energy which is proportional to m^2. The frequencies are n = 1, 2, 3, ...

Constructing the Hamiltonian. Replacing $\dot{x}$ by the momentum p, let $p = \frac{\dot{x}}{2}$, then the Hamiltonian H

$$H = p^2 + \frac{n^2 x^2}{4}$$

In order to find the operators in the space of states, we use the identity

$$a^2 + b^2 = (a + ib)(a - ib)$$

Hence

$$H = (\frac{nx}{2} + ip)(\frac{nx}{2} - ip)$$

The commutator relations for the annihilation, a^-, and creation a^+ , operators are

$$[a^+, a^-] = 1$$

and

$$[p,x] = i$$

Noting [$\frac{nx}{2}$,$\frac{nx}{2}$] = 0 and [ip, ip] = 0, only cross terms survive. Therefore,

$$(\frac{nx}{2} + ip)(\frac{nx}{2} - ip) = \frac{nxp}{2}i - \frac{npx}{2}i = 2\frac{n}{2}i[x,p] = n$$

Divide by n

$$\frac{1}{\sqrt{n}}(\frac{nx}{2} + ip)\frac{1}{\sqrt{n}}(\frac{nx}{2} - ip) = 1$$

Then, we have

$$a^- = \frac{\sqrt{n}}{2}x + \frac{1}{\sqrt{n}}p$$

and

$$a^+ = \frac{\sqrt{n}}{2}x - \frac{1}{\sqrt{n}}p$$

Each oscillation of a string has creation and annihilation operators with frequency n. Solving for x_n

$$x_n = \frac{a^= + a^-}{\sqrt{n}}$$

The spin number of a massless particle, such as a photon, is 2 because we cannot bring it to rest and rotate out and boost in another direction. Polarization of light is linear or circular. The spin is perpendicular to granslation. Linear polarization is a quantum superposition of left and right , with equal probability, on average no spin. When meaured, it collapses into left or right handed. Linear polarization state

$$|x>$$

For circular polarization

$$|r>=|x>+i|y>$$

and

$$|l>=|x>-i|y>$$

Energy of ground state

$$E \longrightarrow {m_0}^2$$

Unexcited oscillator

$$|0>$$

Excite by integer amount of energy

$$a_1^+|0>=m_0^2+1$$

Closed Strings. For closed strings, σ ranges from 0 to 2π. The direction of wave propagation is with the direction of increasing or decreasing σ. For a closed string

$$x(0)=x(2\pi)$$

Because, we have neither Dirichlet nor Neumann boundary conditions, we are not restricted to only cosnσ with Fourier decomposition, hence

$$X(\sigma) = \sum_n x_n e^{in\sigma}$$

Since we have left and right propagating waves in the string

$$x(\sigma) = \sum_{n>0} x_n e^{in\sigma} + \sum_{n>0} x_{-n} n e^{-in\sigma} + x_0$$

As before, we calculate the Lagrangian, and obtain

$$L = \int_0^{2\pi} (\frac{\partial x}{\partial \tau})^2 - (\frac{\partial x}{\partial \sigma})^2 d\sigma$$

Note, integral limits are from 0 to 2. By a clever manipulation, we get the forward and backward moving waves, which are expressed as

$$L = \int_0^{2\pi} \frac{1}{2}(\frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial \sigma})^2 + \frac{1}{2}(\frac{\partial x}{\partial \tau} - \frac{\partial x}{\partial \sigma})^2 d\sigma$$

15.2 Nambu-Goto Action

A particle sweeps out a worldline in Minkowski space. A closed string sweeps out a worldsheet. Let us parametrize the world sheet by the timelike coordinate τ and spacelike coordinate σ. Let σ be periodic with range $\sigma \in [\ 0,\ 2\pi\ )$. The two worldsheet coordinates are packaged as σ^α= (τ, σ), α = 0, 1. See Figure 15.1 below.

The string sweeps a surface in spacetime which defines a map from the world sheet to Minkowski space, X$^\mu$(σ, τ) with $\mu = 0, 1, 2, ..., D-1$.

$$\sigma^\alpha \longrightarrow X^\mu(\sigma, \tau)$$

The worldsheet is a curved surface embedded in spacetime. For closed strings, X$^\mu$(σ, τ) = X$^\mu$ (σ+ 2π, τ). The spacetime is referred to as the target space. Next, we need an action that describes the dynamics of the string. The action of the string should be proportional to the area A

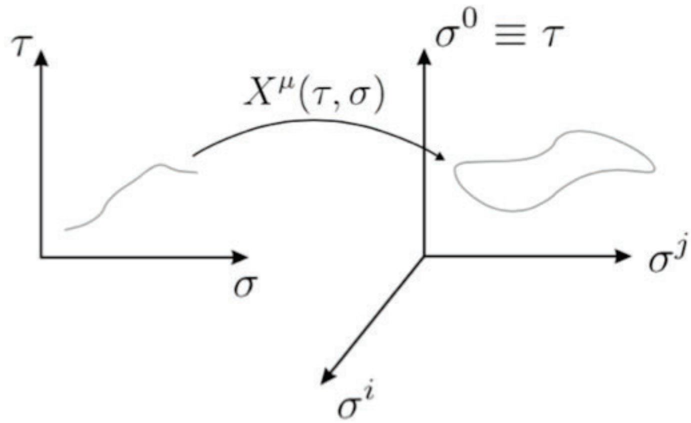

Figure 15.1: Worldsheet coordinates $\sigma^\alpha = (\sigma, \tau)$; embedding coordinate $X^\mu(\sigma, \tau)$

of the worldsheet, just as the action of the point particle is proportional to the length of the world-line. The induced metric, $\gamma_{\alpha\beta}$, on this surface is the pull-back of the flat metric on Minkowski space

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\eta_{\mu\nu}$$

Then, the action, which is proportional, to the area of the worldsheet isgiven by

$$S = -T\int d^2\sigma\sqrt{-det(\gamma)}$$

where T is the constant of proportionality, the tension of the string, the mass per unit length. The pullback of the metric is given by

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\acute{X} \\ \acute{X}\dot{X} & \acute{X}^2 \end{pmatrix}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$ and $\acute{X}^\mu = \frac{\partial X^\mu}{\partial \sigma}$. The action then takes the form

$$S = -T\int d\sigma^2\sqrt{-(\dot{X})^2(\acute{X})^2 + (\dot{X}.\acute{X})^2}$$

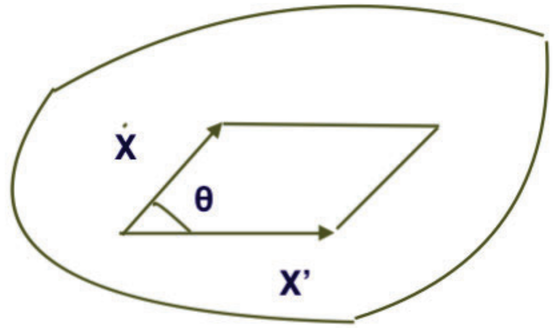

Figure 15.2: Area swept out by a worldsheet

This is the Nambu-Goto action for a relativistic string.The action is the area swept out by the worldsheet, as illustrated below in Figure 15.2

$$dA = d\tau d\sigma |\dot{X}||X^{'}|sin\theta|$$

$$= d\tau d\sigma \sqrt{X'^2 \dot{X}^2 sin^2\theta}$$

$$= d\tau d\sigma \sqrt{X'^2 \dot{X^2}(1 - cos^2\theta)}$$

$$= d\tau d\sigma \sqrt{X'^2 \dot{X^2} - (X'.\dot{X)}^2}$$

$$= d\xi^2 \sqrt{-det}\gamma$$

Writing the Minkowski coordinates as X$^\mu$ = (t, $\vec{x}$). Let $X^0 \equiv t = R\tau$, where R is a constant. Let the kinetic energy, $\frac{d\vec{x}}{d\tau}$ = 0. We have dt = Rdτ. Then, the action is

$$S = -T \int d\tau d\sigma \sqrt{(\frac{d\vec{x}}{d\sigma})^2 + (\frac{d\vec{x}}{d\sigma})^2} = -T \int d\tau d\sigma \sqrt{(\frac{d\vec{x}}{d\sigma})^2} = -T \int dt$$

where the right hand equality is the spatial length of the string. Hence, when the kinetic energy is zero, the action is proprtional to the time integral of the potential energy; potential energy = T $\times$ (spatial length of string). So, T is the energy per unit length. The string's energy increases with length, quite unlike that of an elastic band whose energy increases quadratically with length(Hooke's law). To minimize energy, the string will want to shrink to zero. The tension , T, per unit length , is expressed in terms of $\alpha^{'}$, the universal Regge slope

$$T = \frac{1}{2\pi\alpha^{'}}$$

Polyakov Action

This is a quantization friendly action for the string. It takes the form

$$S_p = \frac{-T}{2}\int d^2\sigma\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

where $g \equiv det g_{\alpha\beta}$. Here $g_{\alpha\beta}$ is a new field.

Equations of Motion. Starting with the Nombu-Goto action of a closed string

$$S = -T\int d^2\sigma\sqrt{-det(\gamma)}$$

Noting that

$$\delta\sqrt{-det(g)} = \frac{1}{2}\sqrt{-det(g)}g^{\alpha\beta}\delta g_{\alpha\beta}$$

Applying the Euler Lagrange equation to the Nambu-Goto equation, that is extremising ;

$$\partial_\alpha(\frac{\partial\sqrt{-det\gamma}}{\partial(\partial_\alpha X^\mu)}) = \partial_\alpha(\sqrt{-det(g)}g^{\alpha\beta}\partial_\beta X^\mu) = 0$$

Re-writing the Polyakov action as

$$S[h,X] = -\frac{1}{4\pi\alpha^{'}}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

where σ^{α} = (τ , σ) are coordinates on the string worldsheet, $h_{\alpha\beta}$ (τ,σ) is an independent dynamical variable (the worldsheet metric) and $X^{\mu\sigma}$ are scalar fields that describe the embedding of the worldsheet in a Minkowski target space with metric $\eta_{\mu\nu}$.. The string tension T , the Regge slope $\alpha^{'}$. and the string length l_s are related by

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s}$$

We will show that the Polyakov action is equivalent to the Nambu-Goto action. We vary the action S[h, X] with respect to h_{ab}

$$\frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_{\gamma}X^{\mu}\partial_{\delta}X^{\nu}\eta_{\mu\nu} = \partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu} \tag{15.2.1}$$

Hence, the worldsheet metric is proportional $h_{\alpha\beta}$ is proportional to the induced metric $\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu}$. Then we can set as

$$h_{\alpha\beta} = e^{2\phi}\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu} \tag{15.2.2}$$

We plug this ansatz back into the equation of motion

$$\frac{1}{2}e^{2\phi}\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu}h^{\gamma\delta}e^{-2\phi}h_{\gamma\delta} = \partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu}$$

Noting that

$$det(h_{\alpha\beta}) = e^{4\phi}det(\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu})$$

From Polyakov action, we get

$$S[X] = -\frac{1}{2\pi\alpha'}\int d\sigma\sqrt{-det(\partial_{\alpha}X^{\mu}\partial_{\beta}X^{\nu}\eta_{\mu\nu}}$$

which is the Nambu-Goto action. The fact that ϕ dropped out of the Polyakov action reflects an underlying symmetry.

Symmetries of the Polyakov Action.

Sidelights on Lorentz and Poincare groups. . Lorentz Invariance. Special relativity is a theory of transformation rules. The allowed set of transformations are those that leave the interval ds^2 invariant. The interval is the same in all frames of refernce.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

Under the general transformation

$$dx^\mu \longrightarrow \Lambda^\mu_\nu dx^\nu$$

the allowed $\Lambda^{'}$s satisfy the following conditions

$$\eta = \Lambda^T \eta \Lambda$$

They are separated into two classes - rotations and boosts, where det $(\Lambda\ ) = 1$. Most often, we think of a Lorentz transformation as a boost, special Lorentz transformation, along an axis, say, x-axis. the time and position coordinates between inertial frames S and $S^{'}$ are related as

$$x^{'} = \gamma(x - vt)$$

$$t^{'} = \gamma(t - \frac{vx}{c^2})$$

where

$$\gamma = \sqrt{1 - \frac{v^2}{c^2}}$$

The Rotation Group. A 3 - dimensional rotation may be expressed as

$$x_i = x_i^{'} = R_{ij} x_j \tag{15.2.3}$$

The rotation group may be considered to be the set of all 3 $\times$ 3 matrices R where the inverse R^{-1} is the same as the transpose R^{T}

$$O(3) = R : R_{kl} \in R; R^T = 1$$

Rotations , ϕ, around the $x-$, $y-$, and $z-$axis , are represented as

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\phi & sin\phi \\ 0 & -sin\phi & cos\phi \end{pmatrix}$$

$$R_y = \begin{pmatrix} cos\phi & 0 & -sin\phi \\ 0 & 1 & 0 \\ sin\phi & 0 & cos\phi \end{pmatrix}$$

$$R_z = \begin{pmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotations around an axis $\overrightarrow{n}$ constitute a subgroup

$$R_{\overrightarrow{n}}(\phi_1) R_{\overrightarrow{n}}(\phi_2) = R_{\overrightarrow{n}}(\phi_1 + \phi_2)$$

which follows from

$$cos\phi_1 cos\phi_2 - sin\phi_1 sin\phi_2 = cos(\phi_1 + \phi_2)$$

$$cos\phi_1 sin\phi_2 + sin\phi_1 cos\phi_2 = sin(\phi_1 + \phi_2)$$

Such rotations about the same axis commute with each other and they constitute an Abelian subgroup. In general, rotations around different axes will not commute.

Lorentz Transformation**.** A Lorentz transformation of a 4-vector x^{μ} = (x^0, $\overrightarrow{x}$) is in analogy with Eq. 13.3 given by

$$x^{\mu} \longrightarrow x_{\mu}{}' = \Lambda^{\mu}_{\nu} x \nu$$

where

$$(x^{'})^2 = x^{'\mu} x^{'}_{\mu} = x^{\mu} x_{\mu} = x^2$$

implies

$$\Lambda^{\mu}_{\alpha} x^{\nu} \Lambda^{\alpha}_{\mu} x^{\mu} x_{\mu} = g^{\mu}_{\nu} x^{\mu} x_{\mu}$$

Therefore,

$$\Lambda^{\mu}_{\alpha} \Lambda^{\alpha}_{\mu} = g^{\mu}_{\nu}$$

Implying

$$(\Lambda^{-1})^{\mu}_{\nu} = \Lambda^{\mu}_{\nu}$$

The Lorentz group may be interpreted as the set of all 4 $\times$ 4 matrices where the inverse is equal to the transpose.

$$\mathscr{L} = \{O(1,3) = \Lambda; \Lambda^{\mu}_{\nu} \in R; \Lambda^{T} = \Lambda^{-1}\}$$

Poincare Group. This is a group of Minkowski spacetime isometries. It is a ten - generator non-Abelian Lie group. As noted above, a Minkowski spacetime isometry has the property that the interval between events is left invariant. In Minkowski space, ignoring the effects of gravity, there are 10 degrees of freedom of the isometries, which are

Poincare Symmetry. This is the full symmetry of special relativity. It includes

(1) Translations or displacements in time and space , P, form an Abelian Lie group of translations in spacetime

(2) Rotations in space , from a non-Abelian Lie group of 3-dimensional rotations J

(3) Boosts, transfromations connectiong two uniformly moving bodies, K

The symmetries J and K form the Lorentz group. The semi-product of the translation group J and the Lorentz group form the Poincare group. The Poincare group is the group of Minkowski spacetime isometries. It is a 10-dimenssional non-compact Lie group. The Abelian group of translations is a normal subgroup, while the Lorentz group is also a sub-group, the stabilizer of the origin. The Poincare group is the semi-product of the translations and Lorentz group

$$R^{1,3} \rtimes O(1,3)$$

with group multiplication

$$(\alpha, f).(\beta g) = (\alpha + f.\beta, f.g)$$

CHAPTER XVI

ADVANCED QUANTUM MECHANICS

16.1 Feynman Propagators

The propagator gives the probability amplitude for a particle to travel from one place, (x,t) to another $(x^{'},t^{'})$ in a given time. In non-relativistic quantum mechanics, it is the Green's function for the Schrodinger equation,[10], [36]. If a system has a Hamiltonian, H, the propagator function is expressed as

$$G(x,t;x^{'},t^{'}) = \frac{1}{i\hbar}\Theta(t-t^{'})K(x,t;x^{'}t^{'})$$

where $\Theta(t-t^{'})$ is the Heaviside step function and $K(x,t,x^{'},t^{'})$ is the kernel of the differential operator, which is the propagator, This function satisfies

$$(i\hbar\frac{\partial}{\partial t} - H)G(x,t;x^{'}t^{'}) = \delta(x-x^{'})(t-t^{'})$$

where $\delta(x-x^{'})(t-t^{'})$ is the Dirac delta function. This is equation is simply the definition of our Green function for our operator. Note that

$$i\hbar\frac{\partial}{\partial t} = H$$

is the time-dependent Schrodinger equation. $K(x,t;x^{'},t^{'})$ can also be written as

$$K(x,t;x^{'},t^{'}) = < x|\hat{U}(t,t^{'})|x^{'} >$$

where $\hat{U}(t,t^{'})$ is the unitary time evolution operator from a state at time t to a state at time $t^{'}$

$$\hat{U}(t,t^{'}) = e^{\frac{i}{\hbar}iH(t-t^{'})}$$

We can also apply the Lagrangian and obtain a path integral.

For the Lagrangian, L, and a parametrized distance function $D[q(t)]$, with boundary conditions $q(t) = x$ and $q(t^{'}) = x^{'}$, the quantum mechanical propagator is

$$K(x,t;x^{'},t^{'}) = \int exp\{\frac{i}{\hbar}\int_{t}^{t^{'}} L(\dot{q},q,t)dt\}D[q(t)]$$

The propagator lets us find the state of a system, given an initial state and a time interval. The new state is given by

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x^{'},t^{'})K(x,t;x^{'},t^{'})dx^{'}$$

If $K(x,t;x^{'},t^{'})$ depends only on the difference $x - x^{'}$, then we have a convolution of the initial state and propagator.

Next, we will compute the simplest case; the propagator for a one dimensional particle, with Hamiltonian, H

$$H = \frac{\hat{p}^2}{2m}$$

Since the system is time-independent, we will set $t^{'} = 0$, and therefore

$$\hat{U(t)} = \hat{U}(t,0) = e^{-\frac{itH}{\hbar}}$$

Hence,

$$K(x,x^{'},t) = < x|e^{-\frac{it\hat{p}^2}{2m\hbar}}|x^{'} >$$

Inserting a momentum identity operator, we obtain

$$K(x,x^{'},t) = \int dp < x|e^{-\frac{it\hat{p}^2}{2m\hbar}}|p >< p|x^{'} >$$

By applying eigenfunction-eigenvalue property, we get

$$K(x,x^{'},t) = \int dpe^{-\frac{itp^2}{2m\hbar}} < x|p >< p|x^{'} >$$

Since $<x|p >< p|x^{'} >= p^*(x^{'})p(x) = e^{\frac{ip(x-x^{'})}{\hbar}}$, since these are eigenfunctions.

$$K(x,x^{'},t) = \int dpe^{-\frac{itp^2}{2m\hbar}} \frac{1}{2\pi\hbar} e^{\frac{i(p(x-x^{'})}{\hbar}}$$

Therefore

$$K(x,x^{'},t) = \int \frac{dp}{2\pi\hbar} e^{\frac{i}{\hbar}\{p(x-x^{'})-\frac{p^2t}{2m})}$$

Integrating, we get

$$K(x,x^{'},t) = \sqrt{(\frac{m}{2\pi\hbar t})} e^{\frac{i}{\hbar}\frac{m(x-x^{'})^2}{2t}}$$

Next, we compute the path integral for a particle moving in one dimension. The Hamiltonian H now has two components

$$H = \frac{p^2}{2m} + V(x)$$

As before, the Hamiltonian is time independent and we get the unitary operator and propagator as

$$U(t,0) = e^{-\frac{iHt}{\hbar}}$$

$$K(x,x^{'},t) =< x|U(t)|x^{'} >$$

We next break up the time interval t into N number of smaller intervals, of duration ε such that

$$\varepsilon = \frac{t}{N}$$

The unitary operator becomes

$$U(t) = U(\varepsilon)^N$$

and the time evolution operator for time ε becomes

$$U(\varepsilon) = e^{-i\varepsilon\frac{T+V}{\hbar}}$$

Taylor expanding

$$U(\varepsilon) = 1 - \frac{i\varepsilon}{\hbar}(T+V) + O(\varepsilon^2) = e^{-\frac{-i\varepsilon T}{\hbar}} e^{-\frac{i\varepsilon V}{\hbar}} + O(\varepsilon^2)$$

Raising both sides to the power of N

$$U(t) = (e^{-\frac{i\varepsilon T}{\hbar}} e^{-\frac{i\varepsilon V}{\hbar}})^N + O(\frac{1}{N})$$

Therefore

$$K(x,x^{'},t) = lim_{N\to\infty} < x|(e^{-\frac{i\varepsilon T}{\hbar}} e^{-\frac{i\varepsilon V}{\hbar}})^N|x^{'} >$$

We have $N-1$ resolutions of identity. Let us compute one of them in the momentum space,

$$< x_{j+1}|e^{-\frac{i\varepsilon T}{\hbar}} e^{-\frac{i\varepsilon V}{\hbar}}|x_j >= \int dp < x_{j+1}|e^{-\frac{i\varepsilon \hat{p}^2}{2m\hbar}}|p >< p|e^{-\frac{i\varepsilon V(\hat{x})}{\hbar}}|x_j >$$

The operators $\hat{x}$ and $\hat{p}$ act on their own eigenstates as before for the free particle computation, we get

$$\int \frac{dp}{2\pi\hbar} e^{\{\frac{i}{\hbar}(-\frac{\varepsilon p^2}{2m}+p(x_{j+1}-x_j)-\varepsilon V(x_j)\}} = \sqrt{\frac{m}{2\pi i\varepsilon\hbar}} e^{\{\frac{i}{\hbar}[m\frac{(x_{j+1}-x_j)^2}{2\varepsilon}-\varepsilon V(x_j)]\}}$$

Hence we get a product of exponentials

$$K(x,x^{'},t) = lim_{N\to\infty}(\frac{m}{2\pi i\hbar\varepsilon})^{\frac{N}{2}} \int dx_1......dx_{N-1} e^{\{\frac{i\varepsilon}{\hbar}\sum_{j=0}^{N-1}[m\frac{(x_{j+1}-x_j)^2}{2\varepsilon^2}-V(x_j)]\}}$$

16.2 Second Quantization

First quantization is the quantization of particles due to the commutation relation between position and momentum, that is

$$[\hat{x},\hat{p}] = i\hbar$$

First quantization refers to using eigenvalues and eigenfucntions to solve the Schrodinger equation. Second quantization refers to the situation where the number of particles vary, such as an electromagnetic field, where the number of photons vary. Here, the quantized field $\Psi(r)$ is expressed via a different commutation relation

$$[\Psi(r),\Psi(r^{'})] = \delta(r-r^{'})$$

Here, we introduce a formalism that deals with many interacting particles. No new theory is introduced, just a new formalism.

(1) We deploy the symmetry and anti-symmetry properties for bosons and fermions

$$\Psi(r_1,r_2,,......,r_n) = \nu\Psi(r_2,r_1,......,r_n)$$

ν = 1 for bosons and -1 for fermions. Ψ is thescalar field function, not to be confused with ψ, the wave function

(2) If we imagine filling up the states $\alpha_1, ..., \alpha_n$, we obtain

$$\Psi_{\alpha_1,......,\alpha_n}(r_1, ..., r_n) = \frac{1}{\sqrt{n!}} \sum_n \nu^n \phi_{\alpha_1}(r_1)......\phi_{\alpha_n}(r_n)$$

where the sum is over the number of permutations. N ia positive integer.

(3) The equation of state is given by

$$\Psi_{\alpha_1,......,\alpha_n}(r_1, ..., r_n) = \zeta|\alpha_1, ..., \alpha_n >$$

One needs to think of the α_i's as fill -up states.

(3) Computations are done by deploying the creation and annihilation operators as in Appendix O. We construct a Hilbert space $\mathscr{H}_\nu^{\otimes \mathscr{N}}$ of N identical particles, where ν = 1 for bosons and -1 for fermions. The vacuum state $\mathscr{H}^0$ is

$$\mathscr{H}^0 = \lambda|\phi(0) >; \lambda \in \mathbb{C}$$

where $|\phi(0) >= |0 >$ is a unit vector. The elements $|\phi >$ in the space $\mathscr{F}$ take the form

$$\phi >= |\phi(0), ..., \phi(n) >, ...\} = \{|\phi(n) >\}_n$$

The Fock space $\mathscr{F}$ where these state vectors exist is defined as

$$F_\nu(\mathscr{H}) = \{|\phi >\in \mathscr{F}; < \phi|\phi >< \infty\}$$

Properties of the Foch space

(1) The Foch space is generated by $|0 >$

(2) Which is acted on by the creation operator, $a^*|\phi >$

$$a^*(\phi)|\phi_1, ..., \phi_n >= \sqrt{n+1}|\phi_1, ..., \phi_n >$$

and the annihilation operator $a|\phi>$ defined by

$$a|\phi>=(a^*|\phi>)^*$$

the asterisk * is the Hermitian conjugate.

(3) Occupation number representation Given a basis $\{|\phi_i>\}$ of one-particle space $\mathscr{H}$, the basis in the Foch space is

$$|n_1,.....n_r>$$

$n_r=0,1,2.....$ for bosons, $n_r=0,1$ for fermions

(4) Scalar multiplication and addition

$$|\phi>+|\psi>=\{|\phi>+|\psi>\}_n$$

(5) Inner product on the space $\mathscr{F}$

$$<\phi|\psi>=\sum_{n=0}^{\infty}<\phi|\psi>$$

where $<\phi(n)|\psi(n)>$ is an inner product in $\mathscr{H}^{\otimes}$.

If $A(n)$ is an observable for n particles, it acts on the Fock space as

$$A|\phi>=\sum_{n=0}^{\infty}A(n)|\phi(n)>$$

The particle number operator N is defined as

$$N|\phi>=\sum_{N=0}^{\infty}n|\phi(n)>$$

Formation of States from the Vacuum. All states of n independent particles can be obtained by successive applications of the creation operators on the vacuum.

$$\frac{1}{\sqrt{n}} a^*(\phi_1)......a^*\phi_n)|0>=|\phi_1......\phi_n>$$

Such states are typical of a scattering process, where have a determined number of particles.

For superposition of n particles

$$|\phi>=e^{-\frac{1}{2}<\phi|\phi>}\sum_{n=0}^{\infty}\frac{[a^*(\phi)]^n}{n!}|0>$$

In terms of the creation and annihilation operators, the particle number operator N is

$$N=\sum_r a*(\psi_r)a(\psi_r)$$

If the particle spin is σ and the particle density in X is

$$n(X)=\sum_\sigma n(X,\sigma)$$

then the density of particles with spin σ is

$$n(X,\sigma)=a^*(X,\sigma)a(X,\sigma)$$

If E_k is the kinetic energy of free particles, the Hamiltonian H is

$$\sum_{k\sigma} E_k a^*_{k\sigma} a_{k\sigma}$$

k represents the configuration space. The momentum P of free particles

$$P=\sum_{k\sigma}\hbar k a^*_{k\sigma} a_{k\sigma}$$

Consider an operator D acting on a two particle space $\mathscr{H}\otimes\mathscr{H}$

$$D|\phi_1,......,\phi_n>=\sum_{i<j}^{n} D_{ij}|\phi_1,......,,\phi_n>$$

We express D in terms of the creation and annihilation operators

$$D = \frac{1}{2} \sum_{r_1 r_2 s_1 s_2} < r_1 r_2 | D | s_1 s_2 > a^*_{r_1} a^*_{r_2} a_{s_1} a_{s_2}$$

Consider a two-body potential $V(x_1 - x_2)$ which is invariant under translations and independent of spin

$$< x_1 \sigma_1, x_2 \sigma_2 | V | x_1^{'} \sigma_1, x_2^{'} \sigma_2 >= V(x_1 - x_2) \delta(x_1 - x_1^{'}) \delta(x_2 - x_2^{'}) \delta_{\sigma_1} \delta_{\sigma_2}$$

Since in the basis $|X\sigma>$, for a one particle

$$V = \int dX V(X) n(X)$$

and in the basis of plane waves $|k\sigma>$

$$\tilde{V}(k) = \int dX e^{-ikX} V(X)$$

$$V = \frac{1}{2} \{ \int dx \int dx_2 \sum_{\sigma_1 \sigma_2} V(x_1 - x_2) \} a^*(x_1, \sigma_1) a^*(x_2, \sigma_2) a(x_1, \sigma_1) a(x_2, \sigma_2)$$

When expressed in the plane wave basis $|k\sigma>$, the momentum representation is best understood via Feynman diagrams as in Figure 16.1 below;

We obtain

$$< k\sigma_1, k\sigma_2 | V 1 k_1 \sigma_1{}', k_{\sigma_2} >{}' = \delta_{\sigma_1, \sigma_2'} \delta_{\sigma_2 \sigma_2'} \frac{1}{L^3} \sum_k \tilde{V}_L(k) \delta_{k_1, k_1' + k} \delta_{k_2, k_2'}$$

where

$$\tilde{V}_L = \int dX e^{-ikX} V(X)$$

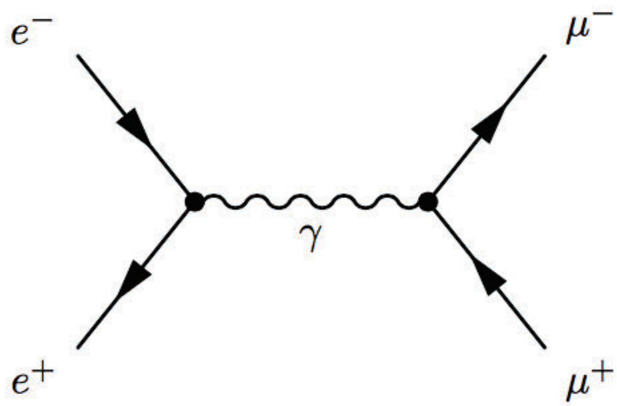

Figure 16.1: Feynman diagram, two particle interaction

Therefore

$$V = \frac{1}{2L^3} \sum_{k_1 k_2 k} \sum_{\sigma_1 \sigma_2} \tilde{V}_L(k) a^*_{(k_1+k)\sigma_1} a^*_{(k_2-k)\sigma_2} a_{k_2\sigma_2} a_{k_1\sigma_1}$$

16.3 Lie Groups in Quantum Mechanics

Lie groups properties

(1) structure - algebraic - the Lie group, with the usual identity, inverse and associativity

(2) analyticity - smoothness- differentiability - manifold - this means that the components of a matrix in the group can vary smoothly over a field, either $\mathbb{C}$ or $\mathbb{R}$

(3) accompanied by a Lie algebra

A Lie algebra is a vector space over a field endowed with a Lie bracket operation, $F \times F \longrightarrow F$. Let X and Y be smooth vector fields on a manifold, M. The Lie bracket $[X,Y]$ of X and Y is the vector field which acts on a function $f \in F$ to give

$$[X,Y]f = [XY - YX]f$$

Lie brackets satisfy;

(a) Bilinearity

$$[aX+bY,Z]=a[X,Z]+b[Y,Z]$$

$$[Z,aX+bY]=a[Z,X]+b[Z,Y]$$

for $a,b \in F$.

(b)The Jacobi Identity

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$$

and

(c) Skew-symmetry

$$[X,Y]=-[Y,X]$$

The cross product is a Lie bracket for the real space $\mathbb{R}^3$. The Lie algebra is usually written in lower case Fraktur.

Example 1.

General Linear Group of rank n, $GL_n(\mathbb{R})$. Matrix multiplication is the group operation, matrix inversions gives the inverse elements, and the identity matrix gives the neutral element.

Example 2. Special Orthogonal Group, SO_2. These are rotations in the two dimensional space, $\mathbb{R}^2$. These are orthogonal matrices with determinant 1;

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where

$$a^2+b^2=1$$

Example 3. Unitary matrices, U(n), over $\mathbb{C}$. These are defined by

$$A * A^{\dagger} = A^{\dagger} * A = I$$

where $A^{\dagger}$ is the Hermitian(conjugate transpose).

Let G be a Lie group. Let $x(t)$ be a smooth curve in G passing through the unit element 1 of G such that

$$x(0) = 1$$

Let $T(G)$ be the tangent space of G at 1, that is, all matrices of the form

$$x^{'}(0) = 1$$

Then $T(G)$ is the Lie algebra of G and a vector space over $\mathbb{R}$. Let $y(t) = x(kt), k \in \mathbb{R}$. Then

$$y^{'}(0) = kx^{'}(0) = k \in \mathbb{R}$$

So $T(G)$ is closed under multiplication. Let $z(t) = x(t)y(t)$; then

$$z^{'}(0) = x(0)y^{'}(0) + x^{'}(0)y(0) = k_1.1 + 1.k_2 = l; k_1, k_2, l \in \mathbb{R}$$

Hence $T(G)$ is closed under addition. Lie algebras arose from considering elements of G close to 1. Suppose ε is very small, then

$$x(\varepsilon) \approx 1 + \varepsilon x^{'}(0)$$

$x(\varepsilon)$ is known as the infinitesimal generator of G. Let n be a large integer and $t \in \mathbb{R}$. Settting ε= $\frac{t}{n}$, we obtain

$$x(\varepsilon)^n \approx (1+\frac{tx^{'}(0)}{n})^n \approx e^{tx^{'}(0)}$$

Let $x^{'}(0) = v$ be a finite element of $T(G)$. Then we have a map $\mathbb{R}\times T(G) \longmapsto G$

$$(t,v) \longmapsto e^{tv}$$

Therefore, for any Lie group G with Lie algebra T(G), we have a mapping

$$exp : T(G) \longrightarrow G$$

such that $exp(0) = 1$ and

$$exp(t_1+t_2)v = exp(t_1 v)exp(t_2 v)$$

for $t_1, t_2 \in \mathbb{R}$ and $v \in T(G)$.

Returning to SO(2), Differentiating the condition $A^t A = 1$ and substituting $A(0) = 1$.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$$

So the Lie algebra of SO(2) is $\mathfrak{so}(2)$, of the form

$$\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$$

We set up a one-to-one correspondence between

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \Longleftrightarrow a+ib$$

where $a^2+b^2 = 1$. For $\mathfrak{so}(2)$, , we have a correspondence with ic. We get a map from $\mathbb{R} \times \mathfrak{so}(2)$,

$\longmapsto$ ($a+ib; a = cos\theta + isin\theta$), unit circle.

$$(t, ib) \longrightarrow e^{ibt}$$

Parametrizing the SO(2) group, with φ as the rotation angle of the transformation

$$R(\varphi) = \begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix}$$

By using trigonometric identities for the sum of two angles,

$$R(\varphi_1 + \varphi_2) = R(\varphi_1)R(\varphi_{2)}$$

The idea behind the infinitesimal generator is that rather than considering the group as a whole, we consider an infinitesimal transformation around the identity. Then, any finite transfromation can be constructed by repeated application or integration of this infinitesimal transformation. By Taylor expanding $R(\varphi)$, we obtain

$$R(\varphi) = R(0) + \frac{dR}{d\varphi}_{|\varphi=0} + \frac{1}{2}\frac{d^2R}{d\varphi}_{|\varphi=0} + ...$$

By using the matrix $R(\varphi)$ and by setting , X = $\frac{dR(\varphi_1)}{d\varphi_1}_{|\varphi_1=0}$ = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we obtain

$$\frac{dR(\varphi)}{d\varphi}_{|\varphi=0} = X$$

and

$$\frac{dR(\varphi)}{d\varphi} = XR(\varphi)$$

Differentiating n times

$$\frac{d^nR(\varphi)}{d\varphi^n}_{|\varphi=0} = X^n$$

we obtain

$$R(\varphi) = 1 + X\varphi + \frac{1}{2}X^2\varphi^2 + ... = e^{\varphi X}$$

where $X^0 = I$ and $X^2 = I$. Therefore,

$$e^{\varphi X} = Icos\varphi + Xsin\varphi = \begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix}$$

See section below on matrix rotation.

Operator Form of Generators.

To derive the operator associated with infinitesimal rotations, we expand $R(\varphi)$ to first order

$$x' = xcos\varphi - ysin\varphi = x - yd\varphi$$

$$y' = xxin\varphi + ycos\varphi = xd\varphi + y$$

An arbitrary differential function $F(x,y)$ then transforms as

$$F(x',y') = F(x - yd\varphi, xd\varphi + y)$$

Retaining terms of order $d\varphi$

$$F(x'y') = F(x,y) + (-y\frac{dF}{dx} + x\frac{\partial F}{\partial y})d\varphi$$

Since F is an arbitrary function, we can associate infinitesimal rotations with the operator

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

Turning to $SO(3)$; consider rotation about z-axis by angle φ_3 , rotation about x-axis by angle φ_1 and rotation about y-axis by angle φ_2;

$$R_3(\varphi_3) = \begin{pmatrix} cos\varphi_3 & -sin\varphi_3 & 0 \\ sin\varphi_3 & cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_1(\varphi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\varphi_1 & -sin\varphi_1 \\ 0 & sin\varphi_1 & cos\varphi_1 \end{pmatrix}, R_2(\varphi_2) = \begin{pmatrix} cos\varphi_2 & 0 & sin\varphi_2 \\ 0 & 1 & 0 \\ -sin\varphi_2 & 0 & cos\varphi_2 \end{pmatrix}$$

$$X_3 = \frac{dR_3}{d\varphi_3}_{|\varphi_3=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_1 = \frac{dR_1}{d\varphi_1}_{|\varphi_1=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, X_2 = \frac{dR_2}{d\varphi_2}_{|\varphi_2=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

It is easy to compute the commutator relations

$$[X_1, X_2] = X_3, [X_{2,}, X_3] = X_1, [X_{3,} X_1] = X_2$$

More generally, using the Levi-Civita symbol

$$[X_i, X_j] = \varepsilon_{ijk} X_k$$

Applying the same method for infinitesimal rotations, in $SO(3)$, we obtain the transformation, expanding to first order

$$\begin{pmatrix} x' \\ y' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 1 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

As before, substituting into a differentiable function F(x,y,z) and expanding to first order in φ_i, we obtain the differential operators

$$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

Since the angular momentum L is

$$L = r \times p = r \times (-i\hbar\nabla)$$

we obtain the components of L

$$L_1 = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$$

$$L_2 = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$$

$$L_3 = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

for the x, y, z components of L. Hence the angular momentum relations are

$$L_i = -i\hbar X_i; i = 1, 2, 3$$

and

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$$

Therefore, the vector components of the angular momentum operator are associated with the generators of infinitesimal rotations about the corresponding axis. Also, we have shown that the exponential map is the bridge between a Lie group and a Lie algebra.

Next, we will review the mathematical background of rotation matices. We begin with square matrices A and B, then

$$exp(A+B) = exp(A)expB)$$

This relation only holds if A commutes with B , that is, $AB = BA$. Since

$$I = exp(0) = exp(A+(-A)) = exp(A)exp(-A)$$

where I is the identity matrix, it follows that

$$exp(A)^{-1} = exp(-A)$$

It can also be easily shown that

$$exp(A)^T = exp(A^T)$$

The Cayley-Hamilton Theory

This theorem states that we can subsitute a square matrix M into its characteristic equation. If I is the identity and the characteristic equation is

$$det(M-I) = p_0 + p_1 t + ... + p_n t^n = 0$$

Then this condition holds

$$det(M-I) = p_0 + p_1 M + ... + p_n M^n = 0$$

Next, consider a skew-symmetric matrix S,

$$S^T = -S$$

Define $R = exp(S)$; then

$$R^T = exp(S)^T = exp(S^T) = exp(-S) = exp(S)^{-1} = R^{-1}$$

Hence R must be an orthogonal matrix. In fact, they are rotational matrices with $detR = 1$. The set of all orthogonal matrices of size n with determinant +1 form $SO(n)$, the special orthogonal group. The set of all orthogonal matrices with $det + 1$ or -1 form $O(n)$, the general orthogonal group; $det + 1$, the rotation matrices, $det - 1$, the reflection matrices. Each component, the rotation and reflection matrices, are path connected. The curve of orthogonal matrices

$$exp(tS); t \in [0,1]$$

is a path connecting $I(t = 0)$ and $R = exp(S)$ for $t = 1$. So R and I must have the same determinant, which is 1. Therefore, R is a rotation matrix.

Rotation Matrix in Two Dimensions. In this case, the skew-symmetric matrix S is

$$S = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where θ is a real number. The rotation matrix R is

$$R = exp(S)$$

The characteristic equation for S is

$$det(S - tI) = t^2 + \theta^2 = 0 \Longrightarrow S^2 = -\theta^2 I$$

Substituting these into the power expansion of $exp(S)$

$$R = exp(S)$$

$$= I + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \frac{S^4}{4!} + \frac{S^5}{5!} + ...$$

$$= I + S - \frac{\theta^2}{2!}I - \frac{\theta^2}{3!}S + \frac{\theta^4}{4!}I + \frac{\theta^5}{5!} -$$

$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ......) + (1 - \frac{\theta^3}{3!} + \frac{\theta^4}{5!} - ......)S$$

By defining $\hat{S} = \frac{S}{\theta}$

$$= I cos\theta + \hat{S}\frac{sin\theta}{\theta}$$

$$= \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

By a similar method, we obtain the rotation matrix in three dimensions

$$R = I + \hat{S} sin\theta - \hat{S}^2(1 - cos\theta)$$

16.4 Gauge Transformations

This is a field theory in which the equations of motion remain unchanged after a transformation,[10],[36]. The ability to move from point to point in the coordinate system without a change in the equation of motion is known as gauge invariance. A local gauge is a coordinate system that can change from point to point. A simple example is adding a constant C to the potential. For example, transform ϕ by adding C

$$\phi^{'} = \phi + C$$

to a system, with potential ϕ

$$F = -\nabla\phi$$

Then

$$F = -\nabla\phi^{'} = -\nabla(\phi + C) = -\nabla C$$

We can also add the gradient of a scalar function, χ, to the vector potential, A. Let $A^{'} = A + \nabla\chi$. Then, in the case of a magnetic field, $B = \nabla \times A$, we get

$$B = \nabla \times A^{'} = \nabla \times (A + \nabla_{\chi}) = \nabla \times A$$

Therefore, the gauge transformation leaves the magnetic field invariant. However, this gauge invariance may not be the case for an electric field, E, where

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

Substituting $A^{'}$,

$$E = -\nabla\phi - \frac{\partial(A + \nabla_{\chi})}{\partial t} = E - \frac{\partial\nabla_{\chi}}{\partial t}$$

This does not result in gauge invariance. To remedy the situation, we apply a gauge transformation to the scalar potential ϕ; we apply the transformation

$$\phi^{'} = \phi + \frac{\partial\chi}{\partial t}$$

E can be written as

$$E = -\nabla(\phi + \frac{\partial \chi}{\partial t}) - \frac{\partial A}{\partial t} - \nabla\frac{\partial \chi}{\partial t} = -\nabla\phi - \frac{\partial A}{\partial t} = E$$

This way, the electric field is invariant. Taking the divergence of E,

$$\nabla.E = \nabla^2\phi + \frac{\partial \nabla.A}{\partial t}$$

Substituting into Gauss's Law; $\nabla.E = -\mu_0\rho$

$$\nabla.E = \nabla^2\phi + \frac{\partial \nabla.A}{\partial t} = -\mu_0\rho$$

where μ_0 is the permeability of free space and ρ is the total charge density. Substituting B and A into Ampere's Law; $\nabla \times B = \mu_0 J + \mu_0\varepsilon_0\frac{\partial E}{\partial t}$, where ε_0 is the permittivity of free space and J is the total current density

$$\nabla \times (\nabla \times A) = \mu_0 J + \frac{1}{c^2}(\frac{\partial}{\partial t}(\nabla\phi + \frac{\partial}{\partial t}\frac{\partial A}{\partial t}))$$

Since $\nabla \times (\nabla \times A) = \nabla(\nabla.A) - \nabla.(\nabla A)$; by re-arranging, we obtain

$$\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \nabla(\nabla.A + \frac{1}{c^2}\frac{\partial \phi}{\partial t}) = -\mu_0 J$$

By setting,

$$\nabla.A + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

we arrive at the Lorenz Gauge Condition. Note that what is left

$$\nabla^2 A + \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$

This is just the wave equation describing a wave travelling at the speed of light.

Yang-Mills Theory

Let us first recall some basic differential geometry. We attach to each point x on a manifold a vector space V_x, which is the fiber over x. The manifold plus plus fibers at each point is the vector bundle, E We define a section s to be a function which takes each point on the manifold to a vector $s(x)$ in the fiber V_x. Put another way, to each point x on the manifold we define the fiber V_x to be the tangent space T_xM. The resulting vector bundle is the tangent bundle, TM. If the fiber is T_x^*M, then the resulting bundle is the cotangemt bundle, T^*M. The $p^{'}th$ exterior product of a vector space V is the vectoe space $\Lambda^p(V)$ spanned by vectors of the form $v_1 \wedge ...... \wedge v_p$. By convention $\Lambda^0(V) = \mathbb{R}$, a one-dimensional vector space.

Yang-Mills theory is a gauge theory. We fix a manifold M and consider a vector bundle E defined over M, known as G-bundle. We choose a fixed group G and a representation V of that group. Then each fiber V_x is a copy of the representation. The resulting structure is a G-bundle. A section of a G-bundle assigns to each point x on a manifold a group element $g(x)$ acting on V_x. The function $g(x)$ is the gauge transformation or gauge symmetry. The group G is the gauge group. Yang-Mill theories are invariant under such transformations. A simple example is to set up a Yang-Mills theory for Maxwell's equations. The steps are

(1) Pick a manifold; in this case; Minkowski space

(2) pick a gauge group; $U(1)$; unit circle in the complex plane; $z = e^{i\theta}$, where $|z| = 1$

(3) pick a representation of that group to act as a fiber; choose V_x=$\mathbb{C}$

So a gauge transformation is just a function $e^{i\theta(x)}$ which acts by multiplication on a section $s(x)$. We have defined a G-bundle. Rather than choosing a certain section on the bundle (a complex function in Minkowski space), we define a connection D_w and introduce the curvature. The Yang-mills equations contrain the curvature. Given v and w are vector fields, s is a section and D is a connection; we define the curvature $F(v,w)$ as

$$F(v,w)s = D_v D_w s - D_w D_v s - D_{[v,w]} s$$

where $[v,w]$ is the commutator of the vestor fields. The curvature $F[v,w]$ inputs a section and outputs another section. In local coordinate basis

$$v = v^j \partial_j . w = w^k \partial_k$$

$$F(v,w) = v^j w^k F_{jk}$$

$$[\partial_j, \partial_k] = 0$$

$$D_0 s = 0$$

We obtain,

$$F_{jk}(s) = (D_j D_k - D_k D_j)(s)$$

where $D_j = D_{\partial_j}$. Since $D_w s = w^j \partial_j s + A_j s$, where A_j is the connection matrix($A^{\alpha}_{\beta j}$), we obtain

$$F_{jk} s = (\partial_j A_k - \partial_k A_j + [A_j, A_k] s$$

In electromagnetism, the matrices A_j are 1×1 matrices, complex numbers, nd the commutator vanishes, giving

$$F_{jk} = \partial_j A_k - \partial_k A_j$$

This is the Faraday tensor for electromagnetism. In the general Yang-Mills theory, the curvature F plays a role analogous to the Faraday tensor in electromagnetism.

Next, we define a new vector bundle $End(E)$, whose fibers consist of linear maps of the fibers V_x onto themselves. A section $v(x)$ of $End(E)$ assigns to each point x a linear map $v(x)$ from V_x into itself. A section of $End(E)$ is a gauge symmetry. Curvature is an $End(E)$ -valued 2-form

$$F = F_{jk} \otimes dx \wedge dx^k$$

Substituting $v = v^j \partial_j$ and $w = w^k \partial_k$ and since $F_{jk} = -F_{kj}$

$$F_{jk} \otimes dx \wedge dx^k(v \otimes w) = \frac{F_{jk}(v^j w^k - v^k w^j)}{2} = F(v, w)$$

By Leibniz's rule, if s is a section of E and T is a section of $End(E)$, then

$$D_v(Ts) = (D_v T)(s) + T(D_v s)$$

Re-arranging

$$(D_v T)(s) = D_v(Ts) - T(D_v s) = [D_v, T]$$

In coordinate notation,

$$(DjT)(s) = D_j(Ts) - TD_j s$$

$$= \partial_j(Ts) + A_j Ts - T\partial_j s - TA_j s$$

$$= (\partial_j T + [A_j, T])(s)$$

Now, we can state the first of the Yang-Mills equations

$$d_D F = 0$$

We have a connection D defined on sections of a G-bundle E. We use the connection to define

(1) curvature F , which is an $End(E)$- valued-2-form

(2) a connectionn $End(E)$

(3) the latter connection is then used to define an exterior covariant derivative d_D, which is applied to F, giving rise to an End(E) valued 3-from, which is required to vanish.

The second Yang-Mills equation is

$$*d_D * F = J$$

$*$ is a Hodge star operator. J is the current, an $End(E)$- valued-one-form, where

$$J = J_k \otimes dx^k$$

If v and w are one-forms, then the following relations hold

$$*(v \wedge w) = v \times w$$

$$*(dv) = curlv$$

$$*d * v = divv$$

The Yang-Mills equations satisfy gauge symmetry. We will not prove this here.

16.5 The Standard Model of Particle Physics

In order to dive into this area, we need the mathematical background of the Faraday tensor, the electromagnetic field tensor,[40]. This is a mathematical construct that describes the electromagnetic field in spacetime. The electromagnetic tensor, F, is the exterior derivative of the electromagnetic four potential, A, a differential 1-form. Hence, it is a differential 2-form, an anti-symmetric rank 2 tensor field on Minkowski space

$$F = dA$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$$

The relationsship between the electric, E, magnetic field, B, and the Faraday tensor is

$$E_i = cF_{0i}$$

$$B_i = -\frac{1}{2}\varepsilon_{ijk}F^{jk}$$

where ε_{ijk} is the Levi-Civita ymbol. The Faraday tensor Hodge dual is

$$G^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$$

Maxwell's equations reduce to

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

$$\partial_{[\alpha}F_{\beta\gamma]} = 0$$

where J^β is the 4-current. The first identity leads to the continuity equation, conservation of charge;

$$\partial_\alpha J^\alpha = 0$$

The second identity is the Bianchi identity;

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0$$

The inner product of the Faraday tensor is Lorentz invariant

$$F_{\mu\nu}F^{\mu\nu} = 2(B^2 - \frac{E^2}{c^2})$$

The equations of electromagnetism can be derived from the action S and applying the Euler Lagrange equations

$$S = \int (-\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - J^{\mu}A_{\mu})dx^4$$

The Lagrangian density is

$$L = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - J^{\mu}A_{\mu}$$

$$= -\frac{1}{4\mu_0}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\nu})(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) - J^{\mu}A_{\nu})$$

$$= -\frac{1}{4\mu_0}(\partial_{\mu}A_{\nu}\partial^{\mu}A^{\nu} - \partial_{\nu}A_{\mu}\partial^{\mu}A^{\nu} - \partial_{\mu}A_{\nu}\partial^{\nu}A^{\mu} + \partial_{\nu}A_{\mu}\partial^{\nu}A^{\mu}) - J^{\mu}A_{\mu}$$

$$= -\frac{1}{2\mu_0}(\partial_{\mu}A_{\nu}\partial^{\mu}A^{\nu} - \partial_{\nu}A_{\mu}\partial^{\mu}A^{\nu}) - J^{\mu}A_{\mu}$$

Substituting into the Euler Lagrangian Equation

$$\partial_{\mu}(\frac{\partial L}{\partial(\partial_{\mu}A_{\nu})}) - \frac{\partial L}{\partial A_{\nu}} = 0$$

This becomes

$$-\partial_{\mu}\frac{1}{\mu_0}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) + J^{\nu} = 0$$

Hence, we arrive at

$$\partial_{\mu}F^{\mu\nu} = \mu_o J^{\nu}$$

Substituting Gauss's Law

$$\nabla . E = \frac{\rho}{\varepsilon_0}$$

we arrive at

$$\frac{1}{c} E^i = -F^{0i}$$

Substituting Ampere"s Law

$$\nabla \times B - \frac{1}{c^2} \frac{\partial E}{\partial t} = \mu_0 J$$

we arrive at

$$\varepsilon^{ijk} B_k = -F^{ij}$$

The Lagrangian of quantum electrodynamics incorporates the creation and annihilation of photons and electrons, where the first term containing the Dirac spinor, ψ, is the Dirac field

$$L = \bar{\psi}(i\hbar c \gamma^{\alpha} D_{\alpha} - mc^2)\psi - \frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta}$$

where γ^{α} are the Dirac matrices.

A brief summary of quantum field theory ensues. A field is a function which depends on spacetime coordinates.It could be a scalar function, a vector-based or tensor valued function of any rank. Spinors are a special case of tensors. They are characterized by the way they respond to rotation of coordinates. Spinor fields describe fields of half-integer spin particles. Spin is useful for classifying particles. Integer spin particles are associated with bosons(Einstein-Bose statistics) and half-integer spin particles are associated with fermions(Fermi-Dirac statistics). In classical fields, the interaction between particles is done via forces. In quantum field theory, the interaction is executed by force carrying virtual particles. The number of virtual particles exchanged is proportional to the product of charges in the interaction. Gauge theory is the foundation of the Standard Model.

A gauge field theory has two main ingredients

(1) gauge invariance - the Lagrangian is invariant

(2) the invariance of the Lagrangian is under a Lie group of local transformations

Quantum Electrodynamics (QED) and U(1)

QED is an Abelian gauge theory with the symmetry group U(1), the circle group, which has one generator. There is one gauge boson. We begin with the action of the Dirac field

$$S = \int dx^4 < \psi | i\hbar rc\gamma^{\mu}\partial_{\mu} - mc^2 | \psi >$$

Then consider the transformation of U(1)

$$|\psi > \longrightarrow e^{i\Lambda(x)} | \psi >$$

with the covariant derivative

$$D_{\mu} = \partial_{\mu} + ieA_{\mu}$$

Applying the gauge transformation of the vector potential

$$A_{\mu} \longrightarrow A_{\mu} - \frac{1}{e}\partial_{\mu}\Lambda$$

The interaction Lagrangian becomes

$$L = J^{\mu}A_{\mu} = \frac{e}{\hbar} < \psi | \gamma^{\mu} | \psi > A_{\mu}$$

$$\Longrightarrow L = < \psi | i\hbar c\gamma^{\mu}D_{\mu} - mc | \psi >$$

Using the gauge principle and classical electrodynamics, we can write the classic Lagrangian and the QED Lagrangian

$$L_{classical} == -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - J^{\mu}A_{\mu}$$

$$L_{QED} =< \psi|i\hbar c\gamma^{\mu}D_{\mu} - mc|\psi> -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

, where $F_{\mu\nu}$=$\partial_{\mu}A_{\nu}$- $\partial_{\nu}A_{\mu}$, is the Faraday tensor. QED describes interactions between charged particles and ones with dipole moments. However, quantum chromodynamics, QCD, strong interactions and electroweak interactions are associated with non-Abelian symmetry groups.

SU(3) and the Quark Model; Quantum Chromodynamics, QCD

Hadrons are bound states of quarks that interact strongly via gauge boson exchanges. Quarks are confined to hadrons and mesons. One cannot isolate a single quark, as energy flux is formed between two quarks, as they are pulled apart, forming quark anti-quark production. QCD is a gauge theory of the SU(3) gauge group obtained by taking the color charge to define a local symmetry, a symmetry that acts independently at each point in space. This requires the introduction of gauge bosons, called gluons. Since SU(3) has 8 generators, we get 8 gluons and their fields. QCD is described by the Lagrangian

$$L = \psi_i^{\dagger}(i(\gamma^{\mu}D_{\mu} - m\delta_{ij})\psi_j - \frac{1}{4}G^{\alpha}_{\mu\nu}G^{\mu\nu}_{\alpha}$$

The gluon field strength tensor is given by

$$G^{\alpha}_{\mu\nu} = \partial_{\mu}A^{\alpha}_{\nu} - \partial_{\nu}A^{\alpha}_{\mu} + gf^{abc}A^{b}_{\mu}A^{c}_{\nu}$$

Weak Interaction

It is the only interaction capable of changing the flavor of quarks, changing one type of quark to another. It si propagated by gauge bosons, that have significant masses. There are two types of weak interactions

(1) charged current interactions, such as, beta decay

(2) neutral current interaction, such as Z-boson decay

Electroweak Model and Higgs Mechanism

At energies of order of magnitude of 100GeV, electromagnetic and weak interactions become indistinguishable. This interaction is described by the electroweak model. Mathematically,

(a) unification by SU(2)×U(1) group

(b) weak hypercharge is acquired from U(1)

(c) isospin from SU(2)

At lower energy, Higgs, Engler et all, postulated a 0-spin scalar field that forms a condensate in all space, the Higgs field. Spontaneous symmetry breaking occurs when E < 100GeV as follows

$$\begin{pmatrix} \gamma \\ Z^0 \end{pmatrix} = \begin{pmatrix} cos\theta_W & sin\theta_W \\ -sin\theta_W & cos\theta_W \end{pmatrix} \begin{pmatrix} B^0 \\ W^0 \end{pmatrix}$$

where θ_W is the weak mixing angle, where $\sin\theta = \frac{m_W}{m_Z}$. An analogy is refraction of ordinary light by a prism into different color wavelengths.

In summary:

(1) There are two main types of particles - fermions and bosons

(2) The group

$$U(1) \times SU(2) \times SU(3)$$

has 12 generators - under local symmetry, we get 12 vector gauge fields and their quanta, gauge bosons; force mediators.

(3) There is one scalar boson, the Higgs boson

(4) Only left handed particles interact weakly. Hence they are massless. They acquire mass by coupling to the Higgs field. W± bosons and Z-bosons.

(5) Fermions are matter particles. There are two types; quarks(strongly interacting) and leptons(bind to the strong force). Another copy of fermions is anti-matter.

(6) Gravity is not included in this model.

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APPENDIX A

APPENDIX A

EULER LAGRANGE EQUATION

Euler Lagrange Equation - A heuristic approach. Much of physical theory can be formulated in versions of this principle as this provides a general formulation. Newton's law is stated in Cartesian coordinates. The formulation of Hamilton's principle can be stated in other coordinate systems, such as cylindrical or spherical coordinate systems. Here, as will be seen later, we can incorporate forces of constraint, which we do not know in advance. This is not possible with the Newtonian formulation.

Let us consider a simple example; the one-dimensional motion of a single particle with mass m, whose position is $x(t)$ at time t. Let $F(x)$ be a force acting on the particle, where $V(x)$ is the scalar potential and

$$F = -\frac{dV}{dx} \tag{1.0.1}$$

Newton's law in one dimension states

$$m\ddot{x} - F(x,t) = 0 \tag{1.0.2}$$

Substitute (1.0.1) into (1.0.2); and noting that the particle's momentum is $\dot{p}$ = m$\dot{x}$;

$$\dot{p} + \frac{\partial V}{\partial x} = 0 \tag{1.0.3}$$

Noting that the kinetic energy T = $\frac{1}{2}m\dot{x}^2$, we can express $\dot{p}$ as

$$\dot{p} = \frac{d}{dt}(m\dot{x}) = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}(\frac{1}{2}m\dot{x}^2) = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} \tag{1.0.4}$$

Combining (1.0.3) and (1.0.4);

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{x}}) + \frac{\partial V}{\partial x} = 0 \tag{1.0.5}$$

Note that the kinetic energy T is a function of $\dot{x}$ and not x, whereas V is a function of x and not $\dot{x}$. We define the Lagrangian, L

$$L(\dot{x}, x) = T(\dot{x}) - V(x)$$

with the property that $\frac{\partial L}{\partial \dot{x}}$ = $\frac{\partial T}{\partial \dot{x}}$ and $\frac{\partial L}{\partial x}$ = - $\frac{\partial V}{\partial x}$. Substituting the above identities into (66.5), we obtain the Euler Lagrange Equation,

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial \dot{x}} = 0$$

Notice how it all fits together; the first term is $m\ddot{x}$ and the second term is F.

APPENDIX B

APPENDIX B

HAMILTON'S PRINCIPLE

Hamilton's Principle. The path $x(t)$ that satisfies the Lagrange equation ia also the one that makes the function *S*, called the action, a minimum.The action is

$$S[x(t)] = \int_{t_1}^{t_2} L(x,\dot{x})dt$$

Hamilton's principle states

$$\delta S = \delta \int_{t_1}^{t_2} L(x,\dot{x})dt = 0$$

. where *L* = T - V for the path $x(t)$. This idea can be extended to multiple particles in three dimensions. For *N* particles with poistions $r_1,...,r_N$ and masses $m_1,...,m_N$, the kinetic energy of the system is the sum of the individual kinetic energies;

$$T = \sum_{k=1}^{N} \frac{m_k \dot{r}_k^2}{2}$$

The potential energy ,V , depends on the position of all the particles, V$(r_1,...,r_N)$. The Lagrangian is

$$L(\dot{r_1},...,\dot{r_N},r_1,...,r_N) = T(\dot{r_1},...,\dot{r_N}) - V(r_1,...,r_N)$$

The System Euler Lagrange Equation is

$$\frac{d}{dt}\frac{\partial L(x_k,\dot{x_k})}{\partial \dot{x_k}} - \frac{\partial L(x_k,\dot{x_k}}{\partial x_k} = 0$$

The system Hamilton Principle is

$$\delta S = \delta \int_{t_1}^{t_2} L(\dot{r}_1, ..., \dot{r}_N, r_1, ..., r_N)dt = 0$$

APPENDIX C

APPENDIX C

HAMILTON'S EQUATIONS

Hamilton's Equations **.** In Cartesian coordinates, the momentum p is

$$p = m\ddot{x} = \frac{\partial}{\partial \dot{x}}(\frac{1}{2}m\dot{x}^2) = \frac{\partial L(x,\dot{x})}{\partial \dot{x}}$$

For generalized coordinates q_k, it is defined as

$$p = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}$$

This momentum p is known as the canonical conjugate of the corresponding position q. Hence,

$$\dot{p} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})$$

Therefore, the Euler Lagrange equation

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = \frac{\partial L}{\partial q}$$

can be written as

$$\dot{p} = \frac{\partial L(q,\dot{q})}{\partial q}$$

The problem here is that L is a function of q and $\dot{q}$ and not p. So, then we associate the function $H(q,p,t)$ with $L(q,\dot{q},t)$ by the Hamiltonian relation, which in the one-dimensional case is

$$H(q,p,t) = \dot{q}p - L(q,\dot{q},t) \tag{3.0.1}$$

If we sum over all (q_i, p_i),

$$H(q_i, p_i, t) = \sum_j \dot{q}_j p_j - L(q_i \dot{q}_i, t)$$

Differentiating (3.0.1);

$$dH = \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

Since $p = \frac{\partial L}{\partial \dot{q}}$,

$$dH = \dot{q}dp + pd\dot{q} - pd\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt = \dot{q}dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt \tag{3.0.2}$$

However,

$$dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial t}dt \tag{3.0.3}$$

Therefore, equating the coefficients of equations (3.0.2) and (3.0.3), we obtain Hamilton's one dimensional equations

$$\frac{\partial H}{\partial p} = \dot{q} \tag{3.0.4}$$

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p} \tag{3.0.5}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{3.0.6}$$

Returning to equation (3.0.3),

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial t} \tag{3.0.7}$$

Substituting equation (3.0.4) and (3.0.5) into (3.0.7), we get

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Remark. If $\frac{dH}{dt} = 0$, then H is time independent. Therefore, H is a constant of motion.

Example. In Cartesian coordinates, $p\dot{x} = m\dot{x}^2 = 2T$, where T is the kinetic energy. For a conservative system. $H= p\dot{x}$ - $L= 2T - L = 2T - (T - V) = T + V$, the total energy of a conservative system.

Poisson Brackets**.** These are important in the time evolution of functions of the variables p and q.

Example. A one - dimensional Poisson bracket is defined for two functions A and B that depend on canonically conjugate variables p and q as

$$\{A,B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

For a multi-dimensional Poisson Bracket

$$\{A,B\} = \sum_j (\frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j})$$

Example. Let us look at the time derivative of a function A (q,p,t);

$$\frac{dA}{dt} = \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial t}$$

Applying Hamilton's one dimensional equations;

$$\frac{dA}{dt} = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial t}$$

Hence, we get the Evolution Equation

$$\frac{dA}{dt} = \{A,H\} + \frac{\partial A}{\partial t} \tag{3.0.8}$$

Remark. If H is independent of time, let A= q, then $\dot{q}$ = $\{q, H\}$ and if $A = p$, then $\dot{p}$ = - $\{p,H\ \}$
The Poisson Brackets can be applied to p and q;

$$\{q,p\} = \frac{\partial q}{\partial q}\frac{\partial p}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial p}{\partial q} = 1$$

and $\{q,q\} = 0$. When a pair of variables satisfy the relation $\{r,s\} = 1$, r and s are said to be canonically conjugate. For N variables, $\{q_i,q\} = 0$, $\{p_i,p\} = 0$ and $\{q_i,p\} = \delta_{ij}$

APPENDIX D

APPENDIX D

DIFFERENTIAL GEOMETRY OF CURVED SURFACES

Differential Geometry of Curved Surfaces. We want to analyze the connection between ds^2 and the curvature of a given manifold. We wll study a two-dimensional curved surface embedded in a three dimensional Euclidean space. Equation of the surface is

$$z = F(x, y) \tag{4.0.1}$$

where F is smooth. Let us fix a point P at the origin such that F(0, 0) = 0. Let the tangent plane at P be the (x, y)-plane. Then $(\frac{\partial F}{\partial y})_{|x=y=0} = (\frac{\partial F}{\partial x})_{x=y=0} = 0$. Expanding Eq. (69.1) as a Taylor series with P being the point of expansion

$$z = \frac{1}{2} f_{ij} x^i x^j + ...$$

where $x^1 = x, y^1 = y$ and $f_{ij} = f_{ji} = (\frac{\partial^2 F}{\partial x^i \partial x^j})_{|P}$ .The derivatives f_{ij} determine the degree of curvature of the surface at the point P. The principal curvatures , κ_1 and κ_2, of the surface are determined by the eigenvalues of the matrix f_{ij}.The eigenvalue equation is

$$|f_{ij} - \Lambda \delta_{ij}| = 0$$

where

$$\Lambda^2 - \Lambda(f_{11} + f_{22}) + (f_{11} f_{22} - f_{12} f_{21}) = 0$$

Therefore,

$$\kappa_1 + \kappa_2 = f_{11} + f_{22}$$

and

$$\kappa_1 \kappa_2 = f_{11} f_{22} - f_{12} f_{21} = |f_{ij}| = det(f_{ij})$$

If x^1 and x^2 are measured in the direction of the principal axes, then

$$f_{ij} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

then

$$z = \frac{1}{2}\kappa_1 (x^1)^2 + \frac{1}{2}\kappa_2 (x^2)^2 + ... \tag{4.0.2}$$

The Gaussian Curvature, K, of the surface at a point P is defined as

$$K = \kappa_1 \kappa_2 = det(f_{ij})$$

K represents an intrinsic property of the surface. For a cylindrical surface, the Gaussian curvature is 0. κ_1 = 1/R and κ_2 is 0. Indeed a cylindrical surface can be unrolled onto a flat surface without gaps or wrinkles. This is not possible for a sphere, which has a Gaussian curvature of 1/R^2.

Relation of Gaussian curvature to the Metric. For our surface

$$ds^2 = dx^i dx^i + dz^2; i = 1,2 \tag{4.0.3}$$

Differentiating (4.0.2)

$$dz = \kappa_1 x^1 dx^1 + \kappa_2 x^2 dx^2 + ... = f_{ij} x^i dx^j + ...$$

Substituting above equation into (4.0.3)

$$ds^2 \simeq dx^i dx^i + f_{ij}x^i dx^j f_{ij}x^i dx^j + ... = (1 + f_{li}f_{kj}x^l x^k)dx^i dx^j + ... = (\delta_{ij} + f_{li}f_{kj}x^l x^k)dx^i dx^k$$

What we need next are geodesics on the surface,

$$\delta \int_A^B ds = 0$$

where $ds = [(\delta_{ij} + f_{li}f_{kl}x^l x^k)dx^i dx^k]^{\frac{1}{2}}$.

Geometrical Interpretation of Curvature. Consider a geodesic polygon ABC...., which is formed by geodesics AB, BC,... Let α,β,γ, denote the interior angles of the polygon, then the exterior angles are denoted by $\alpha^{'} = \pi - \alpha$, $\beta^{'} = \pi - \beta$, etc. Let ξ^i be a vector at the point A, parallel to the geodesic AB. Let us parallel transport the vector along the geodesic AB, the inclination of the vector with the geodesic BC will be -$\beta^{'}$. Continue by parallel transport along the geodesics BC, CD,..., the vector will finally arrive at the point A, where its inclination with the geodesic AB will be - (....+$\gamma^{'}$+ $\beta^{'}$ + $\alpha^{'}$) or 2π - ($\alpha^{'}$ + $\beta^{'}$ + $\gamma^{'}$+...). This is the inclination of the transported vector with the original vector at the point A. When the surface is flat, parallel transporting the vector along a closed contour brings the vector back to its original inclination, that is

$$\alpha^{'} + \beta^{'} + \gamma^{'} + ... = 2\pi$$

Therefore,

$$\alpha + \beta + \gamma + ... = (n-2)\pi$$

For a curved surface, the transported vector is inclined to the original vector at an angle $\Delta\theta$, expressed as

$$\Delta\theta = \int K dS$$

, dS is the area of the surface. For a curved surface

$$(\alpha^{'} + \beta^{'} + \gamma^{'} + ...) + \int K dS = 2\pi$$

Hence,

$$(\alpha + \beta + \gamma + ...) = (n-2)\pi + \int K dS$$

For a geodesic triangle, n= 3,

$$\alpha + \beta + \gamma = \pi + \int K dS$$

For a sphere, $K = 1/R^2$, therefore

$$\alpha + \beta + \gamma = \pi + (S/R^2)$$

The interesting thing here is that one does not need to leave the surface in order to determine R, an intrinsic feature.

APPENDIX E

APPENDIX E

FIRST AND SECOND FUNDAMENTAL FORMS

First and Second Fundamental Forms

First Fundamental Form. Denoted by the Roman numeral I. Let $X(u,v)$ be a parametric surface. Then the inner product of two tangent vectors is

$$I(aX_u+bX_v,cX_u+dX_v)=ac<X_u,X_u>+(ad+bc)<X_u,X_v>+bd<X_v,X_v>=Eac+F(ad+bc)+Gbd$$

where E, F and G are the coefficients of the first fundamental form. When the first fundamental form is written with only one argument, it denotes the inner product of the vector with itself.

$$I(v)=<v,v>=|v|^2$$

In tensor notation,

$$g_{ij}=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}=\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

The components of this tensor are calculated as the scalar product of tangent vectors X_1 and X_2.The first fundamental form completely describes the metric properties of a surface. The line element may be expressed as

$$ds^2=Edu^2+2Fdudv+Gdv^2$$

The area element is

$$dA = |X_u \times X_v| dudv = \sqrt{EG - F^2} dudv$$

Second Fundamental Form. We begin with a parametric surface S in $\mathbb{R}^3$. Let $z = f(x, y)$ be a smooth surface . Let the plane $z = 0$ be tangent to the surface at the origin. Then f and its partial derivatives with respect to x and y vanish at the origin (0,0). The two variable Taylor expansion of f at (0, 0) is

$$z = L\frac{x^2}{2} + Mxy + N\frac{y^2}{2}$$

where the second fundamental form is written as

$$II = Ldx^2 + 2Mdxdy + Ndy^2$$

Let $r = r(u, v)$ be a smooth parametrization of a surface in $\mathbb{R}^3$. The unit normal vector n to the surface is

$$n = \frac{r_u \times r_v}{|r_u \times r_v|}$$

The second fundamental from is as above, where $L = r_{uu}.n$, $M = r_{uv}.n$ and $N = r_{vv}.n$ In tensor notation,

$$II = b_{\alpha\beta} du^{\alpha} du^{\beta}$$

where $b_{\alpha\beta} = r^{\gamma}_{\alpha\beta} n_{\gamma}$; where the coefficients of $b_{\alpha\beta}$ are the projections of the partial derivatives of r at that point onto the normal line to S. The Gaussian curvature of a surface is given by

$$K = \frac{detII}{detI} = \frac{LN - M^2}{EG - F^2}$$

Theorema Egregium of Gauss states that the Gaussian curvature of a surface can be expressed only in terms of the first fundamental form and its derivatives, so that K is an intrinsic invariant of surface. Alternative definitions.

$$K = \frac{< (\nabla_2 \nabla_1 - \nabla_1 \nabla_2) e_1, e_2 >}{det g}$$

where ∇_i is the covarient derivative and g_{ij} is the metric tensor. Also, at a point p in $\mathbb{R}^3$, the the Gaussian curvature is given by

$$K(p) = det(S(p))$$

where S is the shape operator.

Gauss Map. This maps a surface in $\mathbb{R}^3$ to the unit sphere $\mathbb{R}^2$. Given a surface X lying in $\mathbb{R}^3$, the Gauss map is a continuous map N: X$\longrightarrow$S^2, such that N(p) is a unit vector orthogonal at p, the normal vector to X at p. The Gauss map can be defined globally if and only if the surface is orientable. The Jacobian determinant of the Gauss map is the Gaussian curvature, and the differential of the Gauss map is called the shape operator.

$$\iint_R |N_u \times N_v| dudv = \iint_R K |X_u \times X_v| dudv = \iint_R K dA$$

Shape Operator. Also known as Weingarten map. This is a type of extrinsic curvature. We take the differential df of a Gauss map f. For each point x on the surface S, we look at two tangent vectors v and w and take the inner product,

$$< df(v), w >$$

noting that both $df(v)$ and w lie in E^3. $df(v)$ is written as S$_x$. Note the tangent space at each point is an inner product space. Properties:

(1) The shape operator is self-adjoint

$$< df(v), w > = < v, df(w) >$$

(2) The eigenvalues of S_x are the principal curvatures κ_1 and κ_2 at x, the eigenvectors are the corresponding principal directions

(3) The determinant of the shape operator at a point is the Gaussian curvature.

(4) The mean curvature is half the trace of the shape operator. In tensor notation, the Weingarten map can be expressed as

$$\partial_a n = K_a^b r_b$$

, where K_{ab} are components of the surface's curvature tensor. Paraphrasing, the rate of change of the unit normal vector depends on the curvature of the surface for a given point.

Geodesic Curvature. The geodesic curvature k_g at a point of a curve $c(t)$, parametrized by arc length, on an oriented surface, is defined as

$$k_g = \ddot{c}(t).n(t)$$

where $n(t)$ is the principal unit normal. The geodesic curvature at a point is an intrinsic variant dependent only on the metric near that point. A unit speed curve on a surface is geodesic if and only if

(1) the geodesic curvature vanishes on all points on the curve

(2) the acceleration vector, $\ddot{c}\ (t)$ is normal to the surface

Frenet - Serret Frame. We look at a curve in $\mathbb{R}^3$. Let T be the unit tangent vector to the curve, N the unit normal vector (the derivative of T with respect to arc length) and B, the binormal unit vector (the cross product of T and N). Let κ be the curvature and τ the torsion of the curve. The formulas are

$$\frac{dT}{ds} = \kappa N$$

$$\frac{dN}{ds} = -\kappa T + \tau B$$

$$\frac{dB}{ds} = -\tau N$$

(1) Torsion can be defined as the rate of change T or B per unit arc length with respect to N; by convention minus sign with respect to B, (2) The rate of change of N with respect to arc length depends on the rate of change of T and B. In matrix form;

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Intrinsic and Extrinsic Curvature - This a fundamental idea in differential geometry. As the name implies, intrinsic curvature can be computed using parameters within a manifold. For extrinsic curvature, we need to go outside our manifold.

Intrinsic curvature - depends on the metric - examples; Gaussian curvature and Riemann curvature tensor. When we parallel transport a vector tangent to the path of translation around a loop, if it fails to return to the same orientation, this is meaured by the intrinsic curvature.

Extrinsic curvature - depends on embedding in a higher dimensional manifold ; curvature and torsion as in Frenet formulas. Here, we parallel transport a vector normal to the surface , the difference in the two normal vectors δn , after making a loop , defines the extrinsic curvature δn = Kδ, for a unit of translation δe around the curve.

Hessian. The Gaussian curvature of a surface at a point p is the determinant of the Hessian matrix of f, where f is the function of two variables. Let S be the surface given by $z = f(x,y)$, where f : $\mathbb{R}^2 \rightarrow \mathbb{R}$ is smooth. The first fundamental form of S is:

$$g(X,Y) = (1+(\partial_X f)^2)X^2 + 2(\partial_X f)(\partial_Y f)XY + (1+(\partial_Y f)^2)Y^2$$

The second fundamental form is:

$$h(X,Y) = \frac{1}{\sqrt{1+(\partial_X f)^2+(\partial_Y f)^2}}((\partial_{XX} f)X^2 + 2(\partial_X f)(\partial_Y f)XY + (\partial_{YY} f)Y^2)$$

The Gaussian curvature of S is, noting that at critical points $\partial_x f = \partial_Y f = 0$

$$K = \frac{det(h)}{det(g)} = \frac{(\partial_X f)(\partial_Y f) - (\partial_{XY} f)^2}{(1+(\partial_X f)^2+(\partial_Y f)^2}$$

$$K = (\partial_X f)(\partial_Y f) - (\partial_{XY} f)^2 = det(Hess f)$$

APPENDIX F

APPENDIX F

BASICS OF SPHERICAL HARMONICS

Basics of Spherical Harmonics. Spherical harmonics are a complete set of orthogonal functions on the sphere. They are organized by spatial angular frequency.They are also a basis functions for SO(3), the group of rotations in three dimensions. They take their simplest form in Cartesian coordinates, where they can be defined as homogeneous polynomials of degree l in (x,y,z) and obey Laplace's equation. In any spherically symmetric system, energy eigenstates can be given by wave functions of the form

$$\psi(\overrightarrow{x}) = R(r)Y_l^m(\theta,\phi)$$

where $\overrightarrow{x}$ is the position vector. It is best to understand spherical harmonics as deployed in orbital angular momentum in quantum mechanics. The orbital angular momentum operator is defined as

$$\overrightarrow{L} = \overrightarrow{x} \times \overrightarrow{p}$$

where $\overrightarrow{p}$ is the momentum operator . The commutation relation among the components of the angular momentum is

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$$

We now use the quantum formalism to find the spherical harmonics

$$Y_m^l(\theta,\varphi) = < \theta,\varphi|l,m >$$

Here we are deploying the bra - ket notation. In quantum mechanics,

$$\hat{H}Y_l^m(\theta,\phi) = E_l Y_i^m(\theta,\phi)$$

where $E_l = \frac{\hbar^2}{2I}l(l+1)$, l and m are quantum numbers where l= 0,1,2,3,4,... and $m = -j,...,+j$; where $m = 2l+1$

Anatomy of spherical harmonic: $Y_l^m(\theta,\phi) = s_m N_l^{|m|} P_l^{|m|}(\cos\theta) e^{im\phi}$

(1) By convention; s_m = -1 for positive odd m , +1 otherwise

(2) Normalization constant $\int_0^{2\pi}\int_0^{\pi}(Y_l^m)^* Y_l^m \sin\theta d\theta d\phi = 1$

(3) $P_l^{|m|}$ = Legendre polynomial

Let us look at the first few simple harmonics. These can be looked up in tables.

$Y_0^0 = \sqrt{\frac{1}{4\pi}}$ - this is a constant , no nodes, equally likely to see particle in any direction

$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$, nodes at $\theta = \pm\frac{\pi}{2}$, we get a dumbell shaped harmonic

$Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta . e^{\pm i\phi}$, when $m \neq 0$, ϕ appears and it gets more complicated, here we get nodes at the north and south poles, and we get a bilobed harmonic

$Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$, $Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{\pm i\phi}$, ... ; note double trigonometric terms, making functions more complicated with multiple nodes

APPENDIX G

APPENDIX G

3 + 1 FOLIATION OF SPACETIME

3 + 1 Foliations of Spacetime.

Let V^4 be a smooth 4-dimensional manifold (Lorentzian spacetime). Let the metric g have signature $(-1,1,1,1)$. A hypersurface in V^4 is an embedded submanifold if i is the mapping

$$i; M^3 \hookrightarrow V^4$$

M is spacelike if the induced bilinear form γ_{ij} = i^*g is a Riemannian metric on M

M is spacelike if at each point $x \in$ M , there is a timelike future unit normal vector n. See Figure G.1 for an illustration.

If X and Y are vector fields tangent to M, we can consider them as vectors in V and decompose the directional derivative D_XY into tangential and normal components

$$D_X Y = \nabla_X Y + II(X,Y)n$$

where ∇ is the Levi Civita Connection of the induced Riemmanian metric on M, II is the second fundamental form (bilinear form, rank 2 tensor, extrinsic curvature). We know from the definition of the second fundamental form that

$$II(X,Y) = g(D_X,Y)n$$

We also know that $[X,Y]$, the Lie bracket of X and Y is tangent to M, $[X,Y] \in$ TM and that II is a symmetric from II(X,Y) = II(Y,X). $X = X_i \partial_i$ is the shift vector. The lapse function is usually

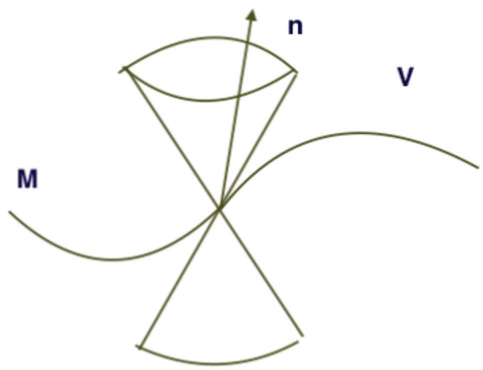

Figure G.1: Light cone, with normal vector n

donated as N or α and the time evolution vector field ∂_t can be expressed as in Figure G.2 below;

$$\partial_t = \alpha n + X$$

The metric can be expressed in terms of the lapse function and shift vector as

$$g_{ij} = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + X^i dt)(dx^i + X^j dt)$$

The second fundamental form II is given by

$$II_{ij} = II(\partial_i \partial_j) = \frac{1}{2}\alpha^{-1}(\partial_t \gamma_{ij} - \mathfrak{L}_X \gamma_{ij})$$

where $\mathscr{L}_X \gamma_{ij} = \nabla_i X_j + \nabla_j X_i$ is the Lie derivative in M of the spatial metric γin the direction $X \in$ TM. Thus, we obtain the evolution of the spatial metric

$$\partial_t \gamma_{ij} = 2\alpha II_{ij} + \mathfrak{L}_{\mathfrak{X}} \gamma_{ij}$$

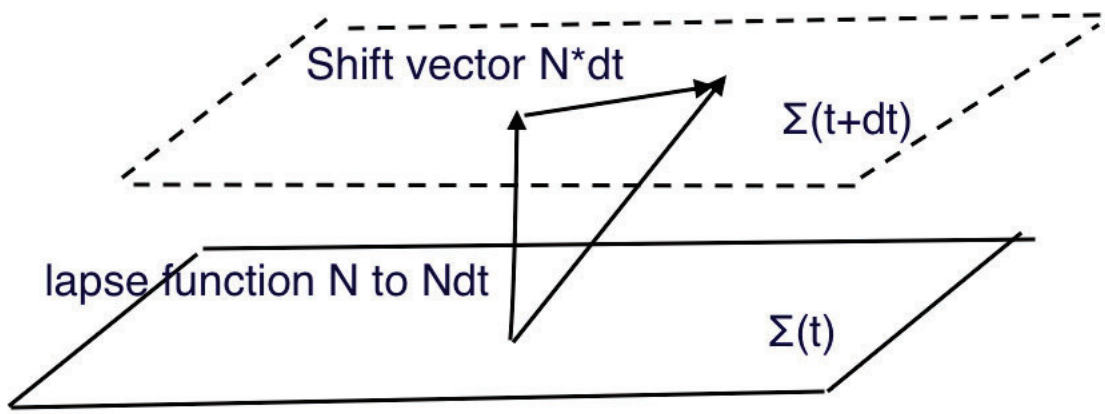

Figure G.2: Hypersurface; relation between time evolution ∂_y, normal vector, shift vector and lapse function. $\Sigma(t)$ is hypersurface at time t.

APPENDIX H

APPENDIX H

GEODESICS

Geodesics. Consider a timelike curve C with a parametric equation $x^a = x^a(u)$. Then, the inetrval s, between two points, P_1 and P_2 on C is given by

$$s = \int_{P_1}^{P_2} (g_{ab}\frac{dx^a}{du}\frac{dx^b}{du})^{\frac{1}{2}} du \tag{8.0.1}$$

The timelike metric geodesic is stationary under small variations of s, that vanish at endpoints. By applying the Euler Lagrange equation, we get

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = (\ddot{s}/\dot{s})\dot{x}^a \tag{8.0.2}$$

where dot differentiation is with respect to u. If we choose u such that

$$u = \alpha s + \beta$$

where α and β are constants, then the right hand side of (8.0.2) vanishes. This is an affine connection. When $u = s$, we get from (73.1) that

$$g_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 1$$

Similar results apply to spacelike geodesics. Geodesics for which the distance between any two points is zero are called null geodesics. Here, the right hand side of (8.0.2) is also zero but

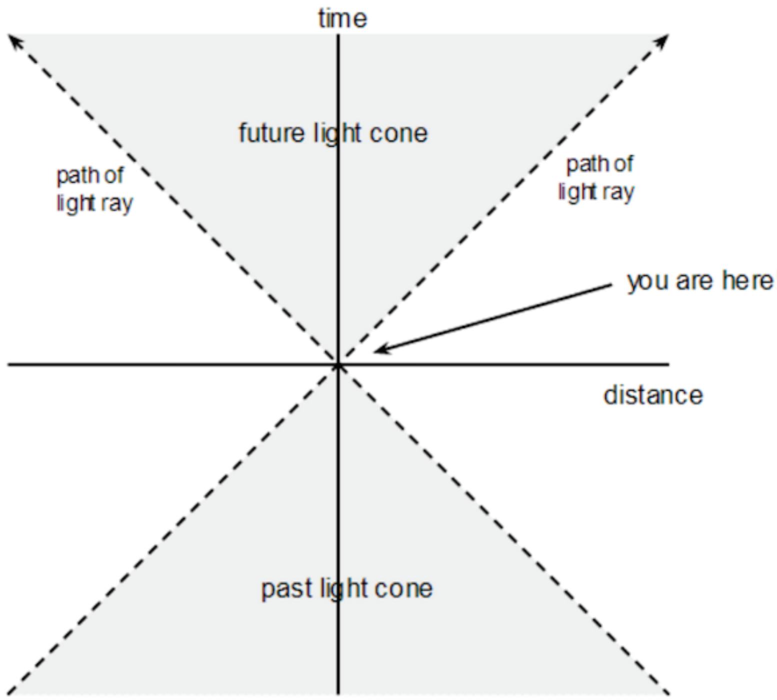

Figure H.1: Lightcone

$$g_{ab}\frac{dx^a}{du}\frac{dx^b}{du} = 0$$

For spacelike coordinates,

$$g_{ab}\frac{dx^a}{du}\frac{dx^b}{du} = -1$$

A null geodesic is the path that a massless particle, such as a photon, follows. It's interval in 4-dimensional spacetime is equal to zero. It has no proper time attached to it. They are the edges of a light cone on a lightcone diagram, projecting at 45 degrees to the horizontal. See Figure as an illustration. It is also known as the light-like geodesic.

A null geodesic is a geodesic, whose tangent vector is a light-like vector everywhere on the geodesic, that is, $x(s)$ is a geodesic and

$$g_{\mu\nu}\cdot\frac{dx^\mu}{ds}\frac{dx^\nu}{dx} = 0$$

for all s, where s is an affine parameter along the curve.

Timelike interval. For two events separated by a timelike interval, sufficient time passes between them that there could be a cause-effect relationship. This holds true for particles travelling at less than the speed of light.

Spacelike interval. When two intervals are separated by a spacelike interval, there is no time between the two occurrences, and hence no causal relationship. The events are not located in each other's future or past cone.

APPENDIX I

APPENDIX I

REVIEW OF NEWTONIAN THEORY

Review of Newtonian Theory. A free falling particle has the equation of motion in an inertial coordinate system,

$$m_i \frac{d^2x}{dt^2} = m_p g(x,t)$$

where g is the gravitational field constant, m_i, the inertial mass and m_p, the passive gravitational mass. Due to equality of inertial and gravitational mass, principle of equivalence, $m_i = m_p$.and hence

$$\frac{d^2x}{dt^2} = g(x,t)$$

This means that all particles fall with the same accelertaion in a gravitational field. Next, we move to a non-inertial coordinate system

$$\tilde{x} = x + b(t)$$

where

$$\frac{d^2\tilde{x}}{dt^2} = \tilde{g}(x,t) = g(x,t) - \ddot{b}(t)$$

Here, we need to think of Einstein's Gendaken - or thought experiments with " lifts". Next, we analyze the motion of two neighbouring particles with position x and $\tilde{x} = x + N$, then

$$\frac{d^2x}{dt^2} = g(x,t)$$

and,

$$\frac{d^2\tilde{x}}{dt^2} = g(x+N,t)$$

So, subtracting and Taylor expanding,

$$\frac{d^2N}{dt^2} = (N.grad)g + O(N^2)$$

In tensor notation,

$$\frac{d^2N}{dt^2} = (\partial_j g_i)N_j$$

The Tidal Tensor is

$$E_{ij} = -\partial_j g_i$$

Hence, we obtain the Geodesic Deviation Equation is

$$\frac{d^2N}{dt^2} + E_{ij}N_J = 0$$

Since the gravitational field is consevative , $\nabla\times$ g = 0, therefore, we can introduce the gravitational potential U,

$$g = -\nabla U$$

or

$$g_i = -\partial_i U$$

Therefore, the tidal tensor can be thought of as the Hessian of the gravitational potential

$$E_{ij} = E_{ji} = \partial_i \partial_j U$$

Poisson's Equation states

$$E_{ii} = 4\pi G\rho$$

This is the field equation of Newtonian gravity . Note that since E$_{ij}$= -∂_{ji}, we have

$$\partial_k E_{ij} = \partial_i E_{kj}$$

or

$$E_{i[j,k]} = 0$$

This is the Bianchi identity.

Weak Equivalence Principle. All freely falling bodies, with same initial velocities, follow the same path, if there are negligible mass - mass interactions.

Strong Equivalence Principle. Inertial mass is the same as gravitational mass.

APPENDIX J

APPENDIX J

RETARDED AND ADVANCED POTENTIALS

Retarded and Advanced Potentials**.**

We begin with a light signal emitted from position $r^{'}$. This would reach position r at time t. We define the retarded time as

$$t_r = t - \frac{r - r^{'}}{c}$$

The advanced time is defined as

$$t_a = t + \frac{r - r^{'}}{c}$$

The time-dependent Maxwell's equations are

$$\Box^2 \phi = -\frac{\rho}{\varepsilon_0}$$

and

$$\Box^2 A = -\mu_0 j$$

where $\Box$ is the d'Alembertian opertor

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

, ϕ is scalar potential, A is the general plane wave vector field

$$A = A_0 e^{i(kr-\omega t)}$$

Green's Function. Let $\mathscr{D}(x)$ be a linear differential operator. Let Ω be a subset of $\mathbb{R}^n$. A Green's function at G = G(x,s) at the point $s \in \Omega$ is any solution of the form

$$\mathscr{D}G(s,x) = \delta(s-x)$$

where δ is the Dirac delta function.

Motivation. Multiplying the above identity by a function $f(s)$ and integrating with respect to s and since $\mathscr{D}$ acts only on x, we get

$$\int \mathscr{D}G(x,s)f(s)ds = \mathscr{D}\int G(x,s)f(s)ds = \int \delta(s-x)f(s)ds = f(x)$$

Set $\int G(x,s)f(s)ds = u(x)$, then

$$\mathscr{D}u(x) = f(x)$$

Let us analyze Poisson's equation

$$\nabla^2 u = v$$

where $v = v(r)$ is the source function. The potential $u(r)$ satsifies the boundary condition $u(r) \longrightarrow 0$ as $|r| \longrightarrow \infty$. This is a linear partial differential equation. The Green's function $G(r,r^{'})$ is a potential which satisfies these boundary conditions, generated by a unit amplitude point source located at $r^{'}$. Therefore,

$$\nabla^2 G(r,r^{'}) = \delta(r-r^{'})$$

The source function $v(r)$ can be expressed as a weighted sum of point sources

$$v(r) = \int \delta(r - r^{'})v(r^{'})d^3r^{'}$$

The generated potential $u(r)$ is the weighted sum of point source driven potentials i.e. Green's functions

$$u(r) = \int G(r, r^{'})v(r, r^{'})d^3(r^{'})$$

Since the Green's function for Poisson's equation is

$$G(r, r^{'}) = -\frac{1}{4\pi}\frac{1}{|r - r|^{'}}$$

This the point source driven potential. It is spherically symmetric about the source, and decreases smoothly with increasing distance from the source. Therefore, the general solution for Poisson's equation is

$$u(r) = -\frac{1}{4\pi}\int \frac{v(r^{'})}{|r - r|^{'}}d^3r^{'}$$

Next, we need to solve the wave equation

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2})u = v$$

where v(r, t) is a time-varying source function. The potential $u(r, t)$ satsifies the boundary condition $u(r) \longrightarrow 0$, as $|r| \longrightarrow \infty$ and $|t| \rightarrow \infty$. This equation is linear, so a Green's function method of solution is again appropriate. The Green's function $G(r, r^{'}; t, t^{'})$ is the potential generated by a point impulse located at a position $r^{'}$ and applied at time $t^{'}$. Therefore,

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2})G(r, r^{'}; t, t^{'}) = \delta(r - r^{'})\delta(t - t^{'})$$

Again, a general source $v(r, t)$ can be built from a weighted sum of point impulses

$$v(r,t) = \iint \delta(r-r^{'})\delta(t-t^{'})v(r^{'},t^{'})d^3r^{'}dt^{'}$$

The generated potential can be written as the weighted sum of point impulse driven potentials

$$u(r,t) = \iint G(r,r^{'},t,t^{'})v(r^{'},t^{'})d^3r^{'}dt^{'}$$

For Green's function, it can be shown that

$$G(r,r^{'};t,t^{'}) = -\frac{1}{4\pi}\frac{\delta(t-t^{'}-\frac{|r-r^{'}|}{c}}{|r-r^{'}|}$$

Returning to the time- dependent Maxwell equation $\Box^2\phi$ = - $\frac{\rho}{\varepsilon_0}$, where

$$\phi(r,t) = \frac{1}{4\pi\varepsilon_0}\iint \frac{\rho(r^{'},t-\frac{r-r^{'}}{c}\}}{|r-r^{'}|}d^3r^{'}$$

Every charge in the universe is continuously performing this integral. The information that the charge receives from the rest of the universe is carried by spherical waves at the speed of light. So the further the charge, the more out OF date the information, so our charge uses the retarded charge density, as in our equation. However, we can also look at advanced time

$$t_a = t + \frac{|r-r^{'}|}{c}$$

Here, Green's function corresponding to the advanced potential solution is

$$\phi(r,t) = \frac{\rho(r^{'},t^{'})}{4\pi\varepsilon_0}\frac{\delta(t-t^{'}+\frac{|r-r|^{'}}{c})}{|r-r^{'}|}$$

This says that a charge density at position $r^{'}$ and $t^{'}$ emits a spherical wave in the scalar potential which propagates back in time. The d'Alembertain equation $\Box^2\phi = -\frac{\rho}{\varepsilon_0}$ is invariant under transformation t$\longrightarrow$ -t and $x\longrightarrow$ -x. However, Green's function is asymmetric in time. So, then we use the completely symmetric Green's function

$$\phi(r,t) = \frac{\rho(r^{'},t^{'})}{4\pi\varepsilon_0}\frac{1}{2}(\frac{\delta(t-t^{'}-\frac{|r-r^{'}|}{c}}{|r-r^{'}|}+\frac{\delta(t-t^{'}+\frac{|r-r^{'}|}{c}}{|r-r^{'}|})$$

In other words, a charge emits half of its waves running forward in time (retarded waves) and the other half running backwards in time (advanced waves). Consider a charge interacting with the rest of the universe. Assume that the rest of the universe is a perfect reflector of advanced waves and a perfect absorber of retarded waves. The waves emitted by the charge can be written as

$$F = \frac{1}{2}(retarded) + \frac{1}{2}(advanced)$$

and the response of the rest of the universe is written as

$$R = \frac{1}{2}(retarded) - \frac{1}{2}(advanced)$$

If we add $F + R =$ retarded, which fits with everyday experience.

APPENDIX K

APPENDIX K

FRIEDMANN EQUATIONS FOR FLAT SPACETIME

Deriving Friedmann Equations for Flat Spacetime. To simplify the derivation, it is assumed that space is flat (not spacetime), a zero cosmological constant. and a perfect fluid. We also make the following assumptions;

Cosmological constant, Λ. Astronomical observations tell us that this constant is close to 0. To simplify Einstein's equation, we will set this to 0.

Isotropy. The cosmic microwave background, CMB looks almost the same in every direction.

Homogeneity. The stress-energy tensor $T_{\mu\nu}$ is the same at every point in space. This assumption holds only at large scales.

Spatial flatness. This means that locally, space is Euclidean. This is useful only if the time coordinates of two points are equal. If space is expanding or contracting, the spatial distance function will depend upon time.. We assume that space is completely fat at every point of time, but we will allow the distance function to change over time. For every point of spacetime, $g_{\mu\nu}$ is the Minkowski metric, and varies smoothly within a neighbourhood of a point. The Minkowski metric has the sign $(-1,1,1,1)$;

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

However, since space is expanding or contracting, to allow for this, we introduce a scale factor $a(t)$;

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

Hence g_{00} = -1 , $g_{11} = g_{22} = g_{33} = a^2(t)$

Stress-Energy Tensor, $T_{\mu\nu}$. The modern universe is so empty that we can model it as a perfect fluid

$$T^{\mu\nu} = pg^{\mu\nu} + (p+\rho)u^{\mu}u^{\nu}$$

where p is the pressure, ρ is the mass energy density and u^{μ}= (1,0,0,0).

Hence, $T^{00} = \rho$ and $T^{11} = T^{22} = T^{33} = \frac{p}{a^2}$

We need the components of the covariant form; $T_{00} = g_{0k}g_{0\lambda}T^{k\lambda} = g_{00}g_{00}T^{00} = \rho$ and $T_{11} = T_{22} = T_{33} = g_{3k}g_{3\lambda}\ T^{k\lambda} = g_{33}g_{33}T^{33} = a^2 p$

The Ricci scalar is defined by contraction of the mixed Ricci tensor; $R = R_{\mu}{}^{\mu} = R_0{}^0 + R_1{}^1 + R_2{}^2 + R_3{}^3$

Only 3 of the 64 partial derivatives of the metric tensor are non-zero; $\frac{\partial g_{11}}{\partial x^0} = \frac{\partial g_{22}}{\partial x^0} = \frac{\partial g_{33}}{\partial x^0} =$ 2a(t)$\dot{a}$(t)

Only 9 of the 64 Christoffel symbols are non-zero; $\Gamma^0{}_{11} = \Gamma^0{}_{22} = \Gamma^0{}_{33}$ = a(t)$\dot{a}$(t) and $\Gamma^1_{01} = \Gamma^1_{01} = \Gamma^2_{20} = \Gamma^2_{02} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a(t)}}{a(t)}$

For the Ricci tensor; $\frac{\partial \Gamma^0_{ii}}{\partial t}$ = a$\ddot{a}$(t) + $\dot{a}^2$(t) and $\frac{\partial \Gamma^i_{0i}}{\partial t} = \frac{\partial \Gamma^i_{i0}}{\partial t} = \frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2$, $R_{00} = -3\frac{\ddot{a}}{a}$ and $R_{ii} = a\ddot{a} + 2\dot{a}^2$, i = 1, 2, 3

The mixed Ricci scalar $R_0{}^0 = g^{0k}R_{ok} = 3\frac{\ddot{a}}{a}$ and $R_1{}^1 = R_2{}^2 = R_3{}^3 = g^{1k}R_{1k} = \frac{1}{a^2}(a\ddot{a} + 2\dot{a}^2) = \frac{\ddot{a}}{a} + 2(\frac{\dot{a}}{a})^2$, so the Ricci scalar R is $R = R_k{}^k = 6(\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2)$

Friedmann Equations. Substitute the above results into Einstein's field equations

$$R_{00} - \frac{1}{2}Rg_{00} = 8\pi T_{00}$$

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}6(\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2)(-1) = 8\pi\rho$$

$$3(\frac{\dot{a}}{a})^2 = 8\pi\rho \tag{11.0.1}$$

This is the first Friedmann equation.

$$R_{11} - \frac{1}{2}Rg_{11} = 8\pi T_{11}$$

$$a\ddot{a} + 2\dot{a}^2 - \frac{1}{2}6(\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2)a^2 = 8\pi a^2 p$$

$$-2(\frac{\ddot{a}}{a}) - (\frac{\dot{a}}{a})^2 = 8\pi p \tag{11.0.2}$$

Multiply (76.2) by 3 and add to (76.1), we get

$$3\frac{\ddot{a}}{a} = -4\pi(\rho + 3p)$$

This is the second Friedmann equation.

APPENDIX L

APPENDIX L

INTRODUCTION TO QUANTUM MECHANICS

Introduction to Quantum Mechanics.

Bra-Ket Notations. The rule to turn inner products into bracket pairs is

$$< u, v > = < u|v >$$

Also the bra - ket pairs are conjugate to each other

$$< u, v > = < u, v >^*$$

Two vectors for which

$$< u|v > = 0$$

are orthogonal. For the norm

$$|v|^2 = < v|v >$$

The Schwarz inequality for any pair of vectors u, v reads

$$| < u, v > | \leq |u||v|$$

Le t$a = (a_i)$ and $b = (b_i)$ be two vectors in a complex dimesnional space, then

$$< a|b >= a_i^* b_i$$

Consider the complex vector space of functions $f(x)$ and $g(x) \in \mathfrak{C}$ with $x \in [0,L]$

$$< f,g >= \int_0^L f^*(x)g(x)dx$$

A set of of basis vectors $\{e_i\}$ satisfies

$$< e_i|e_j >= \delta_{ij}$$

and is orthonormal. An arbirary vector can be written as a linear superposition of basis states

$$v = \alpha_i e_i$$

The coefficients are determined by the inner product

$$< e_k|v >=< e_k|\alpha_i e_i >= \alpha_k < e_k|e_i >= \alpha_k$$

Therefore,

$$v = e_i < e_i|v >$$

Bras are different objects. They belong to the space V^* dual to V. Elements of V^* are linear maps from V to $\mathfrak{C}$. If $v \in V$ and a linear function $\phi \in V^*$, such that $\phi(v)$ is the action of the function of the vector v, which is a number, then in braket notation, we have the following replacements

$$v \longrightarrow |v >$$

$$\phi \longrightarrow < u|$$

$$\phi_u v \longrightarrow < u|v >$$

If kets are viewed as column vectors, then the bras are row vectors. If $a = (a_1, ..., a_i)$and b = $\begin{pmatrix} b_1 \\ . \\ . \\ . \\ b_i \end{pmatrix}$. Then,

$$< a|b >= a_i^* b_i$$

Viewed another way;

$$< u| : |v > \longrightarrow < u|v >$$

Any linear map from V to $\mathfrak{C}$ defines a bra and the corresponding underlying vector . Let v = $\begin{pmatrix} v_1 \\ . \\ . \\ . \\ v_n \end{pmatrix}$. A linear map $f(v)$ acting on a vector v gives a number

$$f(v) = \alpha_1^* v_1 + ... + \alpha_n^* v_n$$

The bra vector is the row vector

$$< \alpha| = (\alpha_1^*, ..., \alpha_n^*)$$

and the associated ket vector is

$$\begin{pmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_n \end{pmatrix}$$

By construction;

$$f(v) = < \alpha|v >$$

The rule to pass from kets to bras is

$$|v >= \alpha_1|a_1 > +\alpha_2|a_2 > \longleftrightarrow < v| = \alpha_1^* < a_1| + \alpha_2^* < a_2|$$

The ket $|e_i >$ is simply called | i > and the orthonormal condition reads

$$< i, j >= \delta_{ij}$$

The expansion of a vector now reads

$$|v >= |i > \alpha_i$$

The coefficeints are

$$\alpha_k = < k|v >$$

so that the ket vector is

$$|v >= |i >< i|v >$$

Introduction to Operators. Let Ω be an operator in a vector space V . Then

$$\Omega : V \longrightarrow V$$

Then, if $|a> \in V$, then $\Omega |a> \in V$. The operator Ω is linear , then

$$\Omega(|a> + |b>) = \Omega|a> + \Omega|b>$$

and

$$\Omega(\alpha|a>) = \alpha\Omega|a>$$

A linear operator on V is also a linear operator on V^*

$$\Omega : V^* \longrightarrow V$$

This is written as

$$<a| \longrightarrow <a|\Omega \in V^*$$

The object $<a|$ is the bra acting on the ket $|b>$ to give the number

$$<a|\Omega|b>$$

The object

$$\Omega = |a><b|$$

is a linear operator on V and V^*.

$$\Omega|v> \equiv |a><b|v> \sim |a>$$

since $<b|v>$ is a number.

$$< w|\Omega \equiv < w|a >< b| \sim < b|$$

since $< w|a >$ is a number.

Operators as Matrices. Consider two vectors expanded in an orthonormal basis | n>

$$|a >= |n > a_n$$

and

$$|b >= |n > b_n$$

and let | b > be obtained from | a> by the action Ω

$$\Omega|a >= |b > \Longrightarrow \Omega|n > a_n = |n > b_n$$

Acting with bra < m | on both sides

$$< m|\Omega|n > a_n = < m|n > b_n = b_m$$

We define the matrix elements

$$\Omega_{mn} = < m|\Omega|n >$$

Therefore,

$$\Omega_{mn} a_n = b_n$$

This is the matrix version of

$$\Omega|a >= |b >$$

We also claim that the operator Ω can be reclaimed from Ω_{mn}

$$\Omega = |m> \Omega_{mn} < n|$$

We can see this if we compute the following matrix

$$< m^{'}|\Omega|n^{'} >= \Omega_{mn} < m^{'}|m >< n|n^{'} >= \Omega_{mn}\delta_{m^{'}m}\delta_{n^{'}n} = \Omega_{m^{'}n^{'}}$$

Identity operator, I. | b> - orthonormal set of basis vectors;

$$\sum_{i} |b_i >< b_k| = \delta_{ij}$$

, where $<b_i|b_k> = 0$, $i \neq k$ and 1 , when $i= k$.

Ket vector;

$$|V >= V_i|b_i >= V_1|b_1 > +V_2|b_2 > +... + V_n|b_n >$$

, V_i is a complex number. For example,

$$< b_3|V >=< b_3|[V_1|b_1 + ... + V_n|b_n] >=< b_3|V_3|b_3 >= V_3$$

That is,

$$< b_i|V >= V_i$$

and

$$V = V_i|b_i >= |b_i >< b_i|V >$$

where the nose to nose > < is 1 or I, the identity.

Matrix representation of an operator with respect to orthonormal basis.

$$T_{ij} =< v_i|\hat{T}|v_j >$$

where $< v_j|$ is the bra , $\hat{T}$ is the operator and $| v_j>$ is the ket. Let us take two orthonormal basis vectors $| v_i>$ and $| v_2>$, where

$$\begin{pmatrix} < v_1|\hat{T}|v_1 > & < v_1|\hat{T}|v_2 > \\ < v_2|\hat{T}|v_1 > & < v_2|\hat{T}|v_2 > \end{pmatrix}$$

where the linear operator $\hat{T}$ is given by $\hat{T}| v_1> = 2 | v_1>$ and $\hat{T}| v_2> = 3|v_1> - i|v_2>$. Then,

$$\begin{pmatrix} < v_1|2v_1|v_1 > & < v_1|(3|v_1 > -i|v_2 >) \\ < v_2|2|v_1 > & < v_2|(3|v_1 > -i|v_2 >) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & -i \end{pmatrix}$$

Operator adjoint - adjoint of a matrix.

$$\hat{L}|A >= |C >$$

$$< A|\hat{L}^T =< C|$$

$$< S|\hat{L} =< S|\hat{L}|n >< n| =< S|D >< n >= \alpha < n|$$

where $< S | D >$ is an inner product, a number α.

Bra vector is complex conjugate of ket vector.

$$< B|C >=< C|B >^*$$

$$< B|\hat{L}|A >=< A|\hat{L}^T|B >^*$$

where $\hat{L} = (\hat{L}^T)^*$

Bra-kets in action. $| \alpha>$ represents a quantum state. This is a ket vector. We say that a physical system is in quantum state α, where α is a physical quantity such as spin or momentum. If we have two distinct quantum states $| \alpha_1 >$ and $| \alpha_2 >$, then the following is a possible state of the system

$$|\psi >= c_1|\alpha_1 > +c_2|\alpha_2 >$$

where c_1 and c_2 are complex numbers. The number of linearly independent kets required to express any other ket is the dimension of a vector space known as the Hilbert space. The bra vector $< \beta |$ belongs to a different vector space. It is the dual of the ket vector $| \beta >$. We define the anti linear relation

$$c_1|\alpha_1 > +c_2|\alpha_2 >= c_1^* < \alpha_1| + c_2^* < \alpha_2|$$

The bra is the conjugate of the ket. The bra's and ket's can line up back to back $< \alpha | \beta > =$ a complex number equal to the value of the inner product of ket $| \alpha>$ and ket $| \beta >$. Hence,

$$< \alpha|\beta >=< \beta|\alpha >^*$$

Let X , the outer product be

$$|\alpha >< \beta|$$

Then the arbitrary ket

$$X|\psi >= |\alpha >< \beta||\psi >=< \beta|\psi > |\alpha >$$

The outer product X is an operator in Hilbert space. It acts on ket $| \psi >$ and turns it into another ket. If we take an operator A and operate on a ket, then

$$A|\alpha>=<\alpha|A^{\dagger}$$

$A^{\dagger}$ is the Hermitian conjugate of the operator A. When A = $A^{\dagger}$, then A is called a Hermitian operator.

Hermitian Operator.

$$<B|\hat{L}|A>=<A|\hat{L}^{T}|B>^{*}$$

Compare with

$$L_{mn}=\hat{L}_{nm}^{T\;*}$$

If $\hat{L} = \hat{L}^{T}$, then

$$<A|\hat{L}|A>=<A|\hat{L}|A>^{*}$$

Then all diagonals are real; off diagonals are complex. Compare with $L^{*}_{mn} = L^{T}_{nm}$

$$\hat{C}|A>=\lambda_{a}|A>$$

, where $\hat{C}$ is eigenfunction and λ_a is eigenvalue. If $\hat{H}| A > = \lambda_a \mid A >$, then eigenvalues are always real,

$$<a|\hat{H}|a>=<a|\lambda_{a}|a>=\lambda_{a}<a|a>$$

λ_a's are the observables - spin, angular momentum, position, energy etc. Hermitian eigenvalues are the observables. If $\hat{H}| a> = \lambda_a| a>$ and $\hat{H} \mid b > = \lambda_b \mid b >$, and λ_a and λ_b are orthogonal , then

$$<b|\hat{H}|a>=\lambda_{a}<b|a>$$

and

$$< a|\hat{H}|b >= \lambda_b < a|b >$$

Then since $< b \mid \hat{H} \mid a >$ and $< a \mid \hat{H} \mid b>$ are complex conjugates, then

$$< b|\hat{H}|a >=< a|\hat{H}|b >^*$$

Hermitian eigenvalues are linearly independent and orthogonal;

$$\lambda_b < a|b >= \lambda_a^* < b|a >^* = \lambda_a < a|b >$$

Therefore,

$$(\lambda_b - \lambda_a) < a|b >= 0$$

Thus,

$$\lambda_a = \lambda_b or < a|b >= 0$$

If $\lambda_a \neq \lambda_b$, then $\mid a >$ and $\mid b >$ are orthonormal. In summary, if H $\mid a>$ = $\lambda_a \mid a>$, H $\mid b >$ = $\lambda_b \mid b >$, and $\lambda_a \neq \lambda_b$, then $\mid a >$ and $\mid b >$ are orthonormal.

Position Operator - Physical state of a system - Hilbert Space - infinite dimensional space - inner product space.

Inner product of two functions

$$< A|B >= \int_c^d A^* B dx$$

Quantum mechanical particle $\longleftrightarrow \mid B >$, state vector , is a ket vector

Wave function of a particle ,$\psi \longleftrightarrow \mid B >$

Probabilty of finding a particle at a particular location

$$\psi_x \psi_x^*$$

where x is the position. How do we pull out of state vector ?

$$< x|B >= \psi_x$$

Properties of position operator : Hermitian

$$< \psi|\hat{x}|\psi >= \int \psi^* \psi dx$$

$$x\psi_x = \lambda \psi_x$$

$$(x-\lambda)\psi_x = 0$$

$$x = \lambda; \psi_x \neq 0$$

Everywhere else , ψ_x is zero.

Dirac Delta Function - eigenfunction for position operator

$$\delta(x-\lambda) = \{ {}^{\infty, x=\lambda}_{0, x\neq\lambda}$$

Eigenvalue is any value of λ on x- axis, whole continuous spectrum. If $\lambda_i \neq \lambda_j$, at λ_1, we have $\delta($ x - $\lambda_1)$; at λ_2,we have δ (x- λ_2) ; zero everywhere, that is, orthogonal eigenvectors.

$$\int_{-\infty}^{\infty} \delta(x-\lambda)dx = 1$$

$\int_{-\infty}^{\infty} \psi_i^* \psi_j$ d$x < \infty$, that is a square integrable function. $\int_{-\infty}^{\infty}$ [δ(x - λ) $]^2$ not a square integrable function , that is , does not lie in Hilbert space.

The dirac delta function is

(1) the eigenvalue of position operator

(2) eigenvalues are continuous functions

(3) eigenfunctions not part of Hilbert space.

Momentum Operator.

$$\widehat{p}_x = -i\hbar\frac{\partial}{\partial x}$$

$\psi^*\psi$ is probabibilty of finding particle at position x. When $\hat{p}_x$ operates on a wave function ψ, it behaves as a Hermitian operaor and $< \psi \mid \hat{H} \mid \psi >$ must be real.

$$< \psi|\hat{p}|\psi >= \int_{-\infty}^{\infty} \psi^*(-i\hbar\frac{\partial \psi}{\partial x})dx = -i\hbar\int_{-\infty}^{\infty} \psi^* d\psi$$

Integrating by parts

$$-i\hbar\int_{-\infty}^{\infty} \psi^* d\psi = -i\hbar[\psi^*\psi]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \psi d\psi^*$$

The first term on right hand side vanishes, symmetric function. We are left with an entity that is its own conjugate. Therefore, it is a real entity. Hence the linear momentum operator is Hermitian with real eigenvalues and eigenfunctions; orthogonal and complete.

Eigenfunctions of momentum operator. $\hat{p}_x$ = -i$\hbar\frac{\partial}{\partial x}$ is a Hermitian operator. Therefore,

$$-i\hbar\frac{\partial \psi}{\partial x} = p\psi$$

where p is a constant. Integrating

$$\int \frac{d\psi}{\psi} = \frac{i}{\hbar}\int p dx$$

Therefore,

$$\psi = Ae^{\frac{ip}{\hbar}x}$$

Hence,

$$i\hbar\frac{\partial\psi}{\partial x} = p\psi$$

So

$$\int_{-\infty}^{\infty}\psi^*\psi dx = \int_{-\infty}^{\infty} e^{\frac{-ip'}{\hbar}x}.e^{\frac{ip}{\hbar}x}dx = \int_{-\infty}^{\infty} e^{\frac{i(p-p')}{\hbar}x}dx$$

We need Fourier transforms and the Dirac delta function to solve this integral.

Operator Methods in Quantum Mechanics. As a quick summary of the above, in the Dirac notation, a state vector or wave function, ψ, is represented as a ' ket '. Just as we can express any three-dimensional vector in terms of the basis vectors, $r = xe_1 + ye_2 + ze_3$, so we can expand any wavefunction as a superposition of basis state vectors

$$|\psi| = \lambda_1|\psi_1| + \lambda_2|\psi_2| + \ldots$$

Alongside the ket, we can define the bra , $< \psi$ |. Together, the bra snd ket, define the scalar product

$$< \phi|\psi >= \int_{-\infty}^{+\infty} dx\phi^*(x)\psi(x)$$

from which follows the identity

$$< \phi|\psi >^* = < \psi|\phi >$$

In this formulation, the real space representation of the wavefunction is recovered from the inner product

$$\psi(x) = <x|\psi>$$

The momentum space wavefunction is obtained from

$$\psi(p) = <p|\psi>$$

An operator $\hat{A}$ is a mathematical object that maps one state vector, $|\hat{A}>$, into another, $|\phi>$, that is,

$$\hat{A}|\psi> = |\phi>$$

If a is real and

$$\hat{A}|\psi> = a|\psi>$$

then $|\psi>$ is said to be an eigenstate or eigenfunction of $\hat{A}$ with eigenvalue of a. For any observable A, there is an operator $\hat{A}$ which acts on the wavefunction so that, if a system is in a state described by $|\psi>$, the expectation value of A is

$$<A> = <\psi|\hat{A}|\psi> = \int_{-\infty}^{\infty} \psi^*(x)\hat{A}\psi(x)dx$$

Every operator corresponding to an observable is linear and Hermitian. For any two wave function $|\psi>$ and $|\psi>$ and any two complex numbers α and β, linearity implies

$$\hat{A}(\alpha|\psi> + \beta|\phi>) = \alpha\hat{A}|\psi> + \beta\hat{A}|\phi>$$

For any linear operator $\hat{A}$, the Hermitian conjugate operator or adjoint is defined by

$$<\phi|\hat{A}\psi> = \int dx\phi^*(\hat{A}\psi) = \int dx\psi(\hat{A}^\dagger\phi)^* = <\hat{A}^\dagger\phi|\psi>$$

Operators are their own Hermitian conjugates

$$< \psi|\hat{H}|\psi >^{*}= [\int_{-\infty}^{\infty} dx\psi^{*}(x)\hat{H}\psi(x)]^{*} = \int_{-\infty}^{\infty} dx\psi(x)(\hat{H}\psi(x))^{*} =< \hat{H}\psi|\psi >$$

That is

$$< \hat{H}\psi|\psi >=< \hat{H}^{\dagger}\psi|\psi >\Longrightarrow \hat{H} = \hat{H}^{\dagger}$$

Operators that are their own Hermitian conjugate are called Hermitian or self-adjoint. Eigenfunctions of Hermitian operators

$$\hat{H}|i >= E_i|i >$$

form an orthonormal complete basis

$$< i|j >= \delta_{ij}$$

For a complete set of states $| i>$, we can expand a state function $| \psi >$ as

$$|\psi >= \sum_i |i >< i|\psi >$$

The expansion of vectors $| \phi > = \sum_i b_i| i >$ and $| \psi > = \sum_i c_i| i >$ allows the dot product to be taken

$$< \phi|\psi >= \sum_i b_i^{*} c_i$$

Time-Evolution Operator. We can evolve a wavefunction forward in time by applying the time-evolution operator. For a Hamiltonian, which is time-independent, we have

$$|\psi(t) >= \hat{U}|\psi(0)|$$

, where the time evolution operator $\hat{U}$ follow from integrating the time dependent Schrodinger equation

$$\hat{H}|\psi>= i\hbar\partial_t|\psi>$$

By inserting the identity I = $\sum_i$ | i > < i | , where the states | i > are the eigenstates of the Hamiltonian with eigenvalues E$_i$, we get

$$|\psi(t)| = e^{\frac{-i\hat{H}t}{\hbar}}\sum_i |i>< i|\psi(0) = \sum_i |i>< i|\psi(0)e^{\frac{-i\hat{H}t}{\hbar}}$$

The time evolution operator is an example of a Unitary opertor. They are transformations which preserve the scalar product

$$< \phi|\psi >=< \hat{U}\phi|\hat{U}\psi >=< \phi|\hat{U}^{\dagger}U|\psi >$$

That is,

$$\hat{U}^{\dagger}\hat{U} = I$$

APPENDIX M

APPENDIX M

FUNCTIONAL ANALYSIS - THE VERY BASICS

Functional Analysis - an introduction

Hilbert Space . An inner product space X is a vector space with an inner product $< x, y >$ defined on it. This is used to define a norm || - || by

$$||x|| =< x,x >^{\frac{1}{2}}$$

and orthogonality by

$$< x,y >= 0$$

A Hilbert space is a complete inner product space. An inner product on X is a mapping of $X \times X$ into the scalar field K of X, that is for every pair of vectors x and y there is associated a scalar which is written as

$$< x,y >$$

and is called the inner product of x and y, such that for all vectors x, y and z and scalars α we have

$$< x+y,z >=< x,z > + < y,z >$$

$$< \alpha x,y >= \alpha < x,y >$$

$$< x, y >= \overline{< y, x >}$$

$$< x, x > \geq = 0$$

$$< x, x >= 0 \Longleftrightarrow x = 0$$

An inner product on X defines a norm on X given by

$$||x|| =< x, x >^{\frac{1}{2}}$$

and a metric on X is given by

$$d(x, y) = ||x - y|| = \sqrt{< x - y, x - y >}$$

Hence inner product spaces are normed spaces and Hilbert spaces are Banach spaces.

Cauchy Sequence. A sequence (x_n) in a metric space X = (X, d), is said to be Cauchy if for every $\varepsilon > 0$, there is an $N = N$ (ε) such that

$$d(x_m, x_n) < \varepsilon, \forall m, n > N$$

The space is said to be complete if every Cauchy sequence in X converges, that is, it has a limit which is an element of X.

The real line and the complex plane are complete metric spaces. Omission of a point a from the real line yields an incomplerte space $\mathbb{R} \setminus \{a\}$. Omission of all irrational numbers leads to the rational line, which is incomplete. Let $X = (0, 1]$ with the metric $d(x, y)$= | $x - y$| and the sequence (x_n) , where x_n = { $\frac{1}{n}$ } ; $n = 1, 2, ...$ This is a Cauchy sequence , but it does not converge because $0 \notin$ X.

Normed Space is a vector space with a metric defined by a norm and a Banach Space is a normed space which is a complete metric space. A mapping from a normed space X into a normed space Y is called an operator. A mapping from from X into the scalar field $\mathbb{C}$ or $\mathbb{R}$ is called a functional.

Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$. These are Banach spaces with norm defined by

$$||x|| = (\sum_{j=1}^{n} |\xi_j|^2)^{\frac{1}{2}}$$

APPENDIX N

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BESSEL FUNCTIONS

Bessel Functions. These are canonical solutions of $y(x)$ of Bessel's differential equation for an arbitary complex number α, the order of the Bessel function.

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0$$

the most important cases are when α is an integer or half-integer. Bessel functions for integer α are also known as cylindrical harmonics because they appear in the solution to Laplace's equation in cylindrical coordinates. Spherical bessel equations with half-integer $\alpha$are obtained when Helmholtz equation

$$\nabla^2A+k^2A=0$$

is solved in spherical coordinates.where k is the wave number and A is the amplitude. Bessel's equation arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates. Hence, Bessel functions are important in problems of wave propagation and static potentials. In solving in cylindrical coordinates, one obtains Bessel functions of integer order $\alpha = n$; in spherical problems, one obtains half-integer order , $\alpha = n + \frac{1}{2}$. Because it is a second order differential equation, there must be two linearly independent solutions. These solutions come in different formulations, depending on circumstances.

Bessel Functions of the First Kind. $J_\alpha\ (x)$. These are solutions that are finite at the origin, $x = 0$, for integer or positive α. and diverge for negative or non-integer α. The function can be defined by series expansion around $x = 0$ by applying the method of Frobenius to Bessel's equation

$$J_{\alpha}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}(\frac{x}{2})^{2m+\alpha}$$

Method of Frobenius **.** This is a way to find an infinite series solution for a second order ordinary differential equation of the form

$$z^2u^{''} + p(z)zu^{'} + q(z)u = 0$$

where $u^{'} = \frac{du}{dz}$, $u^{''} = \frac{d^2u}{dz^2}$, in the vicinity of the regular singular point z = 0. Dividing by z^2, we obtain the differential equation

$$u^{''} + \frac{p(z)}{z}u^{'} + \frac{q(z)}{z^2}u = 0$$

which is not solvable with power series methods at $z = 0$, if $\frac{p(z)}{z}$ and $\frac{p(z)}{z^2}$ are not analytic. If $p(x)$ and $q(z)$ are analytic, we can seek a power series solution of the form

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}; A_0 \neq 0$$

By differentiating and substtuting, we get the expression for the lowest power of z

$$r(r-1) + p(0)r + q(0) = I(r)$$

$I(r)$ is the indicial polynomial. The series solution $u_r(z) = \sum_{r=0}^{\infty} A_k z^{k+r}$ satisfies

$$z^2 u_r(z)^{''} + p(z)zu_r(z)^{'} + q(z)u_r(z) = I(r)z^r$$

We choose one of the roots of the indicial quadratic equation for r in $u_r(z)$, we gain a solution to the differential equation. If the difference in the roots is not an integer, we get another linearly independent solution in the other root.

Returning to Bessel functions of the first kind, this is an entre function if α is an integer. Otherwise, it is a multi-valued function with singularity at zero. The graphs of Bessel functions look roughly like sine or cosine functions that decay proportionally to $\frac{1}{\sqrt{x}}$. For non-integer α, the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent and are two solutions of the differential equations. For integer order α, the following relationship is valid

$$J_{-n}(x) = (-1)^n J_n(x)$$

This means that the two solutions are no longer linearly independent. In this case, the second linearly independent solution is found by the Bessel function of the second kind.

Bessel Function of The Second Kind. Denoted by $Y_\alpha(x)$, they are solutions of Bessel's differential equations that have a singularity at the origin, $x = 0$, nd are multi-valued. For non-integer α, $Y_\alpha(x)$ is related to $J_\alpha(x)$ by

$$Y_\alpha(x) = \frac{J_\alpha(x)cos\alpha\pi - J_{-\alpha}(x)}{sin\alpha x}$$

In the case of integer order n , the function is defined by taking the limit as a non-integer α tends to n

$$Y_n(x) = lim_{\alpha\to n}Y_\alpha(x)$$

If n is a non-negative integer, we have a long and complicated series, which can be checked in any advanced differential calculus textbook. $Y_\alpha(x)$ is necessary as the second linearly-independent solution of the Bessel's equation when $\alpha$is an integer. It can be considered as a natural partner of $J_\alpha(x)$. Both $\mathrm{J}_\alpha(x)$ and $\mathrm{Y}_\alpha(x)$ are holomorphic functions of x on the complex plane cut along the negative real axis. When α is an integer, the Bessel functions J are entire functions of x. If x is held fixed at a non-zero value, then he Bessel functions are entire functions of α.

Hankel Functions . $H_\alpha^{(1)}$ and $H_\alpha^{(2)}$ of the First and Second Kind**.** These are defined as

$$H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(x)$$

APPENDIX O

APPENDIX O

ADVANCED QUANTUM MECHANICS

Advanced Quantum Mechanics

Preliminaries. Since $|V> = V_i|b_i>$ and $< b_i|\ V> = V_i$,

$$|V>=< b_i|b_i> |V>$$

The outer product $< b_i|b_i> =$ I , the identity operator. In integral form;

$$\int dk|k>< k|$$

The inner product of two vectors ϕ and ψ is defined as

$$<\phi|\psi>=\int dx\phi^*(x)\psi(x)$$

For positions x and y

$$< y|\psi>=\int dx\psi(x)\delta(x-y)=\psi(y)$$

where δ is the Dirac delta function.

Probability, P, of finding the particle at a position y is

$$P(y)=|< y,\psi>|^2=<\psi|y>< y|\psi>=\psi^*(y)\psi(y)$$

For the operator x, the eigenfunction is $\delta(x-y)$, the Dirac delta function, which means multiply by x.

For the operator $p =$ - $i\hbar\frac{\partial}{\partial x}$, the eigenfunction is $e^{-i\hbar x}$, where - $i\hbar\frac{\partial}{\partial x}$, = $k\psi$, ψ = e^{-ikx}, $\hbar$ = 1

Note that;

$$[x, p] = i\hbar$$

Hermitian operators are the observables - eigenvalues different, eigenfunctions orthogonal, form a basis.

Unitary Operators are evolution operators;

$$|\psi(t+T)> = U(T)|\psi(T)>$$

$$t = 0 \Longrightarrow U(0) = 1$$

Set

$$U(\varepsilon) = 1 - \frac{i\varepsilon H}{\hbar}, U^{\dagger}(\varepsilon) = 1 + \frac{i\varepsilon H}{\hbar}$$

$$U(0) = 1$$

$$U^{\dagger}U(\varepsilon) = 1$$

$$H^{\dagger} = H$$

H is Hermitian ; observable, orthonormal basis, real eigenvalues (energy)

Time - dependent Schrodinger wave equation;

From

$$|\psi(t+\varepsilon)>=(1-\frac{i\varepsilon H}{\hbar})|\psi(t+\varepsilon)$$

where $U(\ \varepsilon\ ) = (\ 1 - \frac{i\varepsilon H}{\hbar})I$, we obtain to the limit of ε

$$\frac{\partial|\psi(t)>}{\partial t}=-\frac{iH}{\hbar}|\psi(t)>$$

Substituting $H = \frac{p^2}{2m}$ and $p = -i\hbar\frac{\partial}{\partial x}$ into time dependent Schrodinger wave equation, we obtain

$$\frac{\partial|\psi>}{\partial t}=\frac{i\hbar}{2m}\frac{\partial^2|\psi>}{\partial x^2}$$

Expectation Values

K= observable/ Hermitian , λ_n = Eigenvalues , $|n>$ = eigenvectors, orthonormal basis

$$<\psi|K|\psi>=\sum_n<\psi|K|n><n|\psi>$$

$$K|n>=\lambda_n|n>$$

$$<\psi|K|\psi>=\sum_n<\psi|n><n>|\psi>=\sum_n|<\psi|n>|^2\lambda_n$$

Hamiltonian

$$H=\frac{p^2}{2m}+V(x)$$

$$\frac{\partial H}{\partial p}=\dot{x}$$

$$\frac{\partial H}{\partial x} = -\dot{p}$$

Harmonic Oscillator

$$H = \frac{p^2}{2} - + \frac{1}{2}\omega^2 x^2$$

$$x|\psi(x) \longrightarrow x\psi(x)$$

$$p|\psi(x) \longrightarrow -i\hbar\frac{\partial\psi(x)}{\partial x}$$

$$i\frac{\partial\psi}{\partial t} = H\psi$$

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}\omega^2 x^2\psi$$

Applying $H \mid \psi >= E \mid \psi >$

$$\psi(x) = e^{-\frac{\omega^2}{2}x^2}$$

Annihilation and Creation Operators

$$H = \frac{1}{2}(p^2 + \omega^2) = \frac{1}{2}(p + i\omega x)(p - i\omega x) + \frac{\omega}{2}$$

$$b^+ = \frac{1}{2}(p + i\omega x)$$

$$b^- = \frac{1}{2}(p - i\omega x)$$

$$[b^+, b^-] = 2\omega$$

$$a^+ = \frac{b^+}{\sqrt{2\omega}}, a^- = \frac{b^-}{\sqrt{2\omega}}$$

$$[a^+, a^-] = -1$$

$$H = \frac{1}{2}b^+b^- + \frac{\omega}{2}$$

Applying the creation operator at the ground state; energy is quantized as follows

$$|0> \longrightarrow \frac{1}{2}\omega$$

$$a^+|0> \longrightarrow \frac{3}{2}\omega$$

$$a^+a^+|0> \longrightarrow \frac{5}{2}\omega$$

etc.

Scalar Fields, $\Psi(x)$.

$$\Psi(x) = \sum_i a_i^- \psi_i(x)$$

$$\Psi^\dagger(x) = \sum_i a^\dagger{}_i \psi^*(x)$$

Since $< i|x> \psi\ (x)$, $< x|i> = \psi^*(\ x\ )$ and $|\ i > = a_i^+\ |\ 0 >$, we get

$$|x>=\sum_i |i><i|x>=\sum_i \Psi_i^{\dagger}(x)|i>=\sum_i \Psi_i^{\dagger}(x)a_i^{+}|0>=\Psi^{\dagger}(x)|0>$$

The Number Operator

$$\int dx = \Psi^{\dagger}(x)\Psi(x) = \int dx \sum_{i,j} a_i^{+}\psi_i^{*}a^{-}{}_j\psi_j(x) = \sum_{i,j} a_i^{+}a_j^{-}\delta_{ij} = \sum_i a_i^{+}a_i$$

APPENDIX P

APPENDIX P

FOURIER TRANSFORMS

Fourier Transforms. The Fourier transform decomposes a function into a sum of sinusoidal basis functions, each of which is a complex exponential of different frequency. The Fourier transform works for both periodic and non-periodic functions. The Fourier transform of a function $g(t)$ is defined by

$$\mathscr{F}\{g(t)\} = G(f) = \int_{-\infty}^{+\infty} g(t)e^{-2\pi i f(t)}dt$$

As a result, $G(f)$ gives how much power $g(t)$ contains at the frequency f. Meanwhile, g can be obtained from G by the inverse Fourier transform

$$\mathscr{F}^{-1}G(f) = \int_{-\infty}^{+\infty} G(f)e^{2\pi f t}dt = g(t)$$

As an example, we will compute the Fourier transform of the box function (square wave, square pulse) ; we define $g(t)$ of amplitude, a , where $|t| > \frac{T}{2} = 0$. The Fourier transform is

$$\mathscr{F}\{g(t\} = G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi i f t}dt$$

$$= \int_{\frac{-T}{2}}^{\frac{T}{2}} ae^{-2\pi i f t}dt = -\frac{a}{2\pi i f}[e^{-2\pi i f t}]_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{a}{\pi f}[\frac{e^{\pi i f T} - e^{-\pi i f T}}{2i}] = \frac{a}{\pi f}sin(\pi f T) = aT sinc(fT)$$

where sinc(ft) = $\frac{sin(\pi t)}{\pi t}$. By L' Hopital's rule sinc(0) = 1. The Fourier transforms for T = 10 and T = 1 are illustrated in Figure P.1

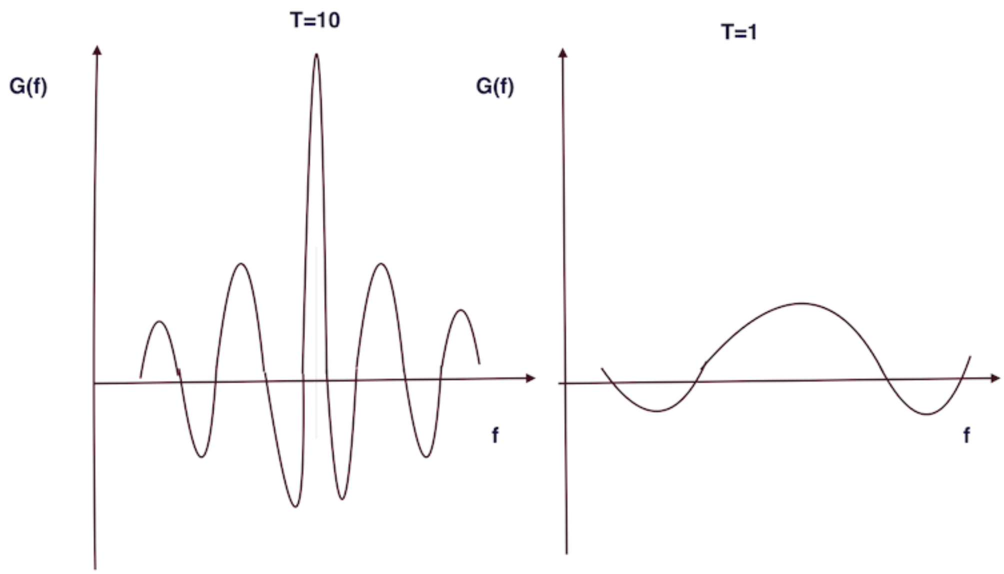

Figure P.1: Fourier transforms for box function; T = 10 and T = 1

The wider pulse, T = 10, produces a higher frequency Fourier spectrum, with more energy. The thinner pulse, T = 1, produces a wider spectrum, with less energy. We will next list properties of Fourier transforms without proof. The proofs are relatively simple and can be obtained from any standard textbook of mathematical methods.

Linearity

If c_1 and c_2 are complex or real

$$\mathscr{F}\{c_1 g(t) + c_2 h(t)\} = c_1 G(F) + c_2 H(f)$$

Shift property

For $g(t-a)$. where a is a real number

$$\mathscr{F}\{g(t-a)\} = e^{-i2\pi f a} G(f)$$

Time delay alters the phase, not the magnitude; noting $|e^{i2\pi f a}| = 1$. The same holds for position function $g(x-a)$.

Scaling property

For a constant, c

$$\mathscr{F}\{g(ct)\} = \frac{1}{|c|}G(\frac{f}{c})$$

Derivative property

$$\mathscr{F}\{dg(t)\} = i2\pi f G(f)$$

Convolution property

The convolution of two functions in time is defined as

$$g(t) * h(t) = \int_{-\infty}^{+\infty} g(\tau)h(t-\tau)d\tau$$

Then

$$\mathscr{F}\{g(t) * h(t)\} = G(f)H(f)$$

Modulation property

A function is modulated by another function if they are multiplied in time. The Fourier transform of the product is

$$\mathscr{F} g(t)h(t) = G(f) * H(f)$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

The left hand side is the energy of the function. This identity says that the energy of $g(t)$is the same as the energy contained in $G(f)$.

Duality property

If $G(f)$ is the Fourier transform of $g(t)$, then the Fourier transform of $G(t)$ is

$$\mathscr{F}G(t) = g(-f)$$

That is, its parity.

APPENDIX Q

APPENDIX Q

MISCELLANEOUS

Miscellaneous.

A. A quick and non formal way to obtain gravitational waves from a perturbation of the metric; omitting indices and ignoring O (h^2)

$$R = \partial\Gamma + \Gamma\Gamma + ...$$

$$\Gamma = \frac{1}{2}g^{-1}\partial g + ...$$

$$R \sim \partial[(\eta - h)\partial h] + O(h^2)$$

$$\partial^2 h = 0$$

B. Newtonian physics

$$ma = F = -m\nabla\phi(x)$$

Noting that $g = a$ is a gradient of a potential ϕ , we arrive at Poisson's equation

$$\nabla^2\phi = 4\pi G\rho(x)$$

and

$$\phi = -\frac{GM}{r}$$

where ϕ is a solution of $\nabla^2\phi$, we arrive at

$$F = -\frac{GMm}{r^2}$$

C. Derivation of the Continuity Equation

Let σ be the charge density and j the flow of charge (current)

$$\dot{\sigma} + \nabla . j = 0$$

$$\frac{\partial \sigma}{\partial t} + \frac{\partial j^m}{\partial x^m} = 0$$

where $t = x^0$, σ = j^0 and (σ, j^m) = ($\mathrm{j}^0, j^1, j^2, j^3$)

$$\frac{\partial j^\mu}{\partial x^m} = 0$$

And for a non flat manifiold, the covariant derivative is

$$\frac{Dj^\mu}{Dx^m} = 0$$

D. Dimensions . Reeturning to the metric $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

$g_{\mu\nu}$ is dimensionless. ds^2 has dimensions L^2, where L is length. The connection is

$$\Gamma^m_{ij} = \frac{1}{2}g^{ml}(\partial_j g_{il} + \partial_i g_{lj} + \partial_l g_{ji})$$

The units of Γ are $\frac{1}{L}$. The Riemann curvature tensor is

$$R_{ij} = \partial_l \Gamma^l_{ij} - \partial_j \Gamma^l_{il} + \Gamma^m_{ij}\Gamma^l_{lm} - \Gamma^m_{il}\Gamma^l_{jn}$$

The units are $\frac{1}{L^2}$.

E. Kruskal Coordinates - the easy way. Starting with the Schwarzchild metric

$$d\tau^2 = (1 - \frac{2MG}{r'})dt'^2 - (1 - \frac{2MG}{r'})^{-1}dr'^2 - r'^2 d\Omega^2$$

where $d\Omega^2$ is the 2-sphere d$\theta^2 + cos^2\theta d\phi$. Note that $r' = 2MG$ is the Schwarzchild radius, R_s, the radius of the event horizon. Setting $R_s = 2MG$

$$d\tau^2 = (1 - \frac{R_s}{r'})dt'^2 - (1 - \frac{R_s}{r'})^{-1}dr'^2 - r'^2 d\Omega^2$$

Setting $r = \frac{r'}{R_s}$ and t = $\frac{t'}{R_s}$

$$d\tau^2 = [(1 - \frac{1}{r})dt^2 - (1 - \frac{1}{r})^{-1}dr^2 - r^2 d\Omega^2]R_s^2$$

This is analogous to $2MG = 1$. Rescaling, and replacing r with ρ, the radial distance from the event horizon, where θ and ϕ = 0; we obtain

$$\rho = \int_1^r \sqrt{\frac{r}{r-1}}dr$$

and hence

$$d\rho^2 = (\frac{r}{r-1})dr^2$$

Re- scaling time ; ω= $\frac{t}{2}$, we obtain

$$d\tau^2 = F(\rho)\rho^2 d\omega^2 - d\rho^2 - r(\rho)^2 d\Omega.$$

When ρ is small, we obtain the metric of flat space in hyperbolic polar coordinates

$$\rho^2 d\omega^2 - d\rho^2$$

where

$$X = \rho cosh\omega$$

$$Y = \rho sinh\omega$$

$$X^2 - T^2 = \rho^2$$

F. Minkowski Metric. The metric of flat spacetime in special relativity, Cartesian , with 1 negative time eigenvalue and 3 positive space eigenvalues.

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

However $g_{\mu\nu}$ is a function of space. Consider the metric of polar coordinates in two - dimensional flat space

$$ds^2 = r^2 d\theta^2 + dr^2$$

Note that the metric is not constant in curvilinear coordinates, but is a function of space

$$\begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix}$$

Minkowski metrc;

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu$$

Time -like coordinates :

$$d\tau^2 > 0$$

Space - like coordinates

$$d\tau^2 < 0$$

Null - like coordinares

$$d\tau^2 = 0$$

G. Derivation of Geodesic Equation**.** Let the tangent vector along a parameterized curve be

$$t^m = \frac{dx^m}{ds}$$

where s is the length of the parametrized curve. The covariant derivative of a vector along the curve is

$$D_m V^m = \frac{\partial V^m}{\partial x^m} + \Gamma^n_{mr} V^r$$

Multiplying by displacement dx^m we get

$$D_m V^m dx^m = \frac{\partial V^m}{\partial x^m} dx^m + \Gamma^n_{mr} V^r dx^m = 0$$

$$dV^m + \Gamma^n_{mr} V^r dx^m = 0$$

Replacing V by t

$$dt^m + \Gamma^n_{mr} t^r dx^m = 0$$

Divding by ds

$$\frac{dt^m}{ds} + \Gamma^n_{mr} t^r \frac{dx^m}{ds} = 0$$

$$\frac{d^2x}{ds^2} + \Gamma^n_{mr} \frac{dx^r}{ds}\frac{dx^m}{ds} = 0$$

H. Tensor Algebra

Contravariant Vector

$$(V^{'})^m = \frac{\partial y^m}{\partial x^p} V^p$$

Covariant Vector , where $S(x)$ is a scalar

$$\frac{\partial S}{\partial y^m} = \frac{\partial S}{\partial x^p}\frac{\partial x^p}{\partial y^m}$$

$$W^{'}_m = \frac{\partial x^p}{\partial y^m} W_p$$

Relationship between Contravariant and Covariant Basis Vectors

$$\frac{\partial y^m}{\partial x^q}\frac{\partial x^p}{\partial y^m} = \delta^p_q$$

Tensor Addition

$$T^{m\ldots}_{\ldots p} \pm S^{m\ldots}_{\ldots p} = (T \pm S)^{m\ldots}_{\ldots p}$$

Tensor Product (Outer Product)

$$V^m \otimes W_n = T^m_n$$

$$V^m \otimes W^n = T^{mn}$$

Transformation of Rank 2 Contravariant Tensor

$$T'^{nm} = \frac{\partial y^n}{\partial x^\alpha}\frac{\partial y^m}{\partial x^\beta}T^{\alpha\beta}$$

Transformation of a $(1,1)$ Tensor

$$T'^n_m = \frac{\partial y^n}{\partial x^\alpha}\frac{\partial x^\beta}{y^m}T^\alpha_\beta$$

Tensor Contraction

$$\frac{\partial x^b}{\partial y^m}\frac{\partial y^m}{\partial x_a} = \frac{\partial x^b}{\partial x^a} = \delta^b_a$$

$$V'^m W'_n = \frac{\partial y^m}{\partial x^a}\frac{\partial x^b}{\partial y^n}V^a W_b$$

$$V'^m W'_m = \frac{\partial y^m}{\partial x^a}\frac{\partial x^b}{\partial y^m}V^a W_b = V^a W_a = scalar$$

Raising and Lowering Indices

$$T^n g_{mn} = T_m$$

$$T_n g^{mn} = T^m$$

Metric Tensor, g_{mn}

$$ds^2 = g_{mn} dx^m dx^n$$

1) varies from point to point

$$g_{mn} = e_m e_n$$

2) in spacetime 4×4 metric, 10 independent components

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}$$

3) non- zero eigenvalues - invertible ; g^{mn}

4) if metric tensor can be transformed to Minkowski metric, η_{mn}, then space is flat

$$g'_{mn} = \frac{\partial x^a}{\partial y^m} \frac{\partial x^b}{\partial y^n} \eta_{ab}$$

5) always 1 negative eigenvalue and 3 positive eigenvalues

BIOGRAPHICAL SKETCH

Hassan Kesserwani studied mathematics for one year at Imperial College, University of London. He obtained a bachelor's degree in mathematics from Troy State University, Alabama. He earned a masters degree in mathematics from the University of Texas, Rio Grande Valley, in December 2018. For correspondence, e-mail is neuro1815@yahoo.com. He has a special interest in applied mathematics, especially in the fields of general relativity and quantum mechanics.