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The First-Integral Method for Duffing-Van Der Pol-Type Oscillator System

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THE FIRST-INTEGRAL METHOD FOR

DUFFING–VAN DER POL–TYPE OSCILLATOR SYSTEM

A Thesis

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ABSTRACT

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In this thesis, we restrict our attention to nonlinear Duffing–van der Pol–type oscillator system by means of the First-integral method. This system has physical relevance as a model in certain flow-induced structural vibration problems, which includes the van der Pol oscillator and the damped Duffing oscillator etc as particular cases. Firstly we apply the Division Theorem for two variables in the complex domain, which is based on the ring theory of commutative algebra, to explore a quasi-polynomial first integral to an equivalent autonomous system. Then through a certain parametric condition, we derive a more general first integral of the Duffing–van der Pol–type oscillator system.

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CHAPTER I

INTRODUCTION

In this paper, we consider a general Duffing-van der Pol-type oscillator system of the form

$$\ddot{u} + (\delta + \beta u^n)\dot{u} - \mu u + \alpha u^{n+1} = 0, \tag{1}$$

where an over-dot represents differentiation with respect to the independent variable ξ, and all coefficients $\delta,\ \beta,\ \mu$and α are real constants with $\delta \cdot \beta \cdot \mu \cdot \alpha \neq 0$. It can also be regarded as a general combination of the van der Pol oscillator and damped Duffing equation, since the choices $\delta \neq 0,\ \beta \neq 0,\ \mu \neq 0,\ \alpha = 0$ and $n = 2$ leads equation (1) to the van der Pol oscillator [1]

$$\ddot{u} + (\delta + \beta u^2)\dot{u} - \mu u = 0, \tag{2}$$

which was proposed by the Dutch physicist Balthasar van der Pol, who pioneered the fields of radio and telecommunications [2-7]. The choices $\delta \neq 0,\ \mu \neq 0,\ \alpha \neq 0,\ \beta = 0$ and $n = 2$ leads equation (1) to the damped Duffing equation [8, 9]

$$\ddot{u} + \delta\dot{u} - \mu u + \alpha u^3 = 0, \tag{3}$$

which is a nonlinear dynamical system. When we choose $\delta \neq 0,\ \mu \neq 0,\ \alpha \neq 0,\ \beta = 0$ and $n = 1$, equation (1) becomes the damped Helmholtz oscillator [11, 12]

$$\ddot{u}+\delta\dot{u}-\mu u+\alpha u^2=0. \tag{4}$$

Furthermore, if we take $\delta\neq 0,\ \mu\neq 0,\ \alpha\neq 0,\ \beta\neq 0$ and $n=2$, equation (1) becomes the standard form of the Duffing-van der Pol oscillator, whose autonomous version (force-free) takes the form

$$\ddot{u}+(\delta+\beta u^2)\dot{u}-\mu u+\alpha u^3=0. \tag{5}$$

This non-linear differential equation (1) is used in physics, engineering, electronics, biology, neurology and many other disciplines [9, 39, 14-21]. Therefore, it is one of the most intensively studied systems in non-linear dynamics [9, 13-19]. It is well known that there are a great number of theoretical works dealing with equations (2)-(5) [9, 20, 25, 26], and applications of these four equations and related systems can be seen in quite a few scientific areas [11, 27, 28]. Much research on Duffing-van der Pol system has been done [22-24]. In 1997, Holms and Rand made a study of the local and global bifurcation of the Duffing-van der Pol system [20]. In 1998, Maccari investigated the main resonance of the Duffing-van der Pol system using asymptotic perturbation method and obtained the sufficient conditions for period-doubling motion of the system [29]. In 2006, Dong et al. investigated the local bifurcation of the Duffing-van der Pol system to multi-frequency excitations[30]. It is well known that the Duffing-van der Pol system is a system with multi-parameters.

In the present paper, we study the nonlinear Duffing-van der Pol-type oscillator system (1) to obtain its first integrals under certain parametric conditions by using the Division Theorem for two variables in the complex domain based on the ring theory of commutative algebra, which is currently called the first-integral method. The paper is organized into four chapters, with the introduction as chapter one. In the next chapter, we construct a particular first-integral for equation (1) by applying the first-integral method . In chapter III, we derive the general first integral of the Duffing-van der Pol-type oscillator system (1). In chapter IV, we present a brief conclusion.

CHAPTER II

THE FIRST-INTEGRAL METHOD

In this chapter, we consider the Duffing-van der Pol-type oscillator (1) by appling the first-integral method [31].

2.1 Preliminaries

2.1.1 Duffing-van der Pol-type oscillator

In order to make the paper well self-contained and present our results in a straightforward manner, in this section we will focus our attention in reviewing the first-integral method for solving second-order ODEs based on the ring theory of commutative algebra.

Consider the oscillator equation (1) in the following form:

$$\ddot{u} = -(\delta + \beta u^n)\dot{u} + \mu u - \alpha u^{n+1} = F\ (\xi,\ u,\ u'), \tag{6}$$

where u' denotes differentiation with respect to ξ. To investigate the integrability of this equation, the Division Theorem will be used. Let $x = u,\ y = u_\xi$, then equation (6) is equivalent to an autonomous system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(\delta + \beta x^n)y + \mu x - \alpha x^{n+1}. \end{cases} \tag{7}$$

By the qualitative theory of ordinary differential equations [40], if we can find two first-integrals to equation (7) under the same conditions, then the general solutions to equation (7) can be expressed explicitly. However, generally, it is difficult for us to realize this, even for one first-integral, because for a given plane autonomous system, there is no systematic theory that can tell us how to find its first-integrals, nor is there a logical way to tell us what these first-integrals are.

In this section, we apply the first-integral method to obtain a particular first-integral of equation (1). That is , we will apply the Hilbert-Nullstellensatz Theorem to study and obtain a first-integral to equation (7) which reduces equation (6) to a first-order integrable ordinary differential equation.

For convenience, let us first recall the Hilbert-Nullstellensatz Theorem [33-34].

2.1.2 Hilbert-Nullstellensatz Theorem

Hilbert-Nullstellensatz Theorem (Zero-locus-Theorem) is a theorem which makes a fundamental relationship between the geometric and algebraic aspects of algebraic geometry. The Hilbert-Nullstellensatz Theorem relates algebraic sets to ideals in polynomial rings over algebraically closed fields. The theorem was first proved by David Hilbert, whom it is named after.

Formulation of Hilbert-Nullstellensatz Theorem [10].

Let k be a field and L an algebraic closure of k.

(i) Every ideal γ of $k\ [X_1,\ X_2,\ \cdots,\ X_n]$ not containing 1 admits at least one zero in L^n.

(ii) Let $x = (x_1,\ x_2,\ \cdots\ x_n)$, $y = (y_1,\ y_2,\ \cdots\ y_n)$ be two elements of L^n; for the set of polynomials of $k\ [X_1,\ X_2,\ \cdots,\ X_n]$ zero at x to be identical with the set of polynomials of $k\ [X_1,\ X_2,\ \cdots,\ X_n]$ zero at y, it is necessary and sufficient that there exists a k-automorphism s of L such that $y_i = s(x_i)$ for $1 \leq i \leq n$.

(iii) For an ideal α of $k\ [X_1,\ X_2,\ \cdots,\ X_n]$ to be maximal, it is necessary and sufficient that there exists an x in L^n such that α is the set of polynomials of $k\ [X_1,\ X_2,\ \cdots,\ X_n]$ zero at x.

(iv) For a polynomial Q of $k\ [X_1,\ X_2,\ \cdots,\ X_n]$ to be zero on the set of zeros in L^n of an ideal γ of $k\ [X_1,\ X_2,\ \cdots,\ X_n]$, it is necessary and sufficient that there exist an integral $m > 0$ such that $Q^m \in \gamma$.

2.1.3 Division Theorem

Following immediately from the Hilbert-Nullstellensatz Theorem, we obtain the Division Theorem for two variables in the complex domain $\mathbb{C}$.

Formulation of Division Theorem.

Suppose that $P\ (w,\ z)$ and $Q\ (w,\ z)$ are polynomials in $\mathbb{C}\ [w,\ z]$, and $P\ (w,\ z)$ is irreducible in $\mathbb{C}\ [w,\ z]$. If $Q\ (w,\ z)$ vanishes at all zero points of $P\ (w,\ z)$, then there exists a polynomial $G\ (w,\ z)$ in $\mathbb{C}\ [w,\ z]$ such that

$$Q\ (w,\ z) = P\ (w,\ z) \cdot G\ (w,\ z). \tag{8}$$

This is of some interest to ask whether the above Division Theorem can be proven by using the complex theory. The answer is 'Yes'. Next, we would like to present a direct and simple proof [31]:

<Proof> For convenience, we give the following lemmas:

Lemma (2.1) Suppose that $U(\omega,z)$ and $V(\omega,z)$ are polynomials in $\mathbb{C}[\omega,z]$, and $U(\omega,z)$ is irreducible in $\mathbb{C}[\omega,z]$. Suppose that $R(\omega,z)$ is a non-constant polynomial and a factor of $U(\omega,z) \cdot V(\omega,z)$, and $\deg R(\omega,z) < \deg U(\omega,z)$ with respect to ω. Then $R(\omega,z)|V(\omega,z)$.

Lemma (2.2) Suppose that $P(\omega,z)$ is an irreducible polynomial in $\mathbb{C}[\omega,z]$ and that $P_\omega(\omega,z)$ is the partial derivative with respect to ω. Then there exist two polynomials $A(\omega,z), B(\omega,z)$, and a nonzero polynomial $D(z)$ in $\mathbb{C}[\omega,z]$, such that

$$A(\omega,z)\cdot P(\omega,z)+B(\omega,z)\cdot P_\omega(\omega,z)=D(z). \tag{9}$$

The proofs of Lemma (2.1) and Lemma (2.2) can be seen in [13].

Notice that a polynomial $P(\omega,z)$ in $\mathbb{C}[\omega,z]$ can be written as

$$P(\omega,z)=\sum_{k=o}^{n} p_k(z)\omega^k, \tag{10}$$

where $p_k(z) \quad (k=0,1,\cdots,n)$ are polynomials in z and $p_n(z)\not\equiv 0$. If $P(\omega,z)$ is an irreducible polynomial in $\mathbb{C}[\omega,z]$, then $p_k(z),(k=0,1,\cdots,n)$ are all relatively prime. For any fixed $z_0\in\mathbb{C}$, $P(\omega,z_0)$ is a polynomial in ω. By the Fundamental Theorem of Algebra, it has n zeros in $\mathbb{C}$.

Definition (2.3) If z_0 is a complex number such that the polynomial $P(\omega,z_0)$ does not have n distinct zeros in $\mathbb{C}$, then z_0 is called a *Special Zero Point* of the polynomial $P(\omega,z)$.

Lemma (2.4) If $P(\omega,z)$ is an irreducible polynomial in $\mathbb{C}[\omega,z]$, then $P(\omega,z)$ has at most finitely many Special Zero Points in $\mathbb{C}$.

<Proof> Write $P(\omega,z)$ in equation (10) and consider the set

$$M=\{z|z\in C, \quad p_n(z)=0, \quad or \quad D(z)=0\}.$$

By equation (9), it is easily noted that M is a finite set. Suppose that $z^*\in\mathbb{C}\setminus M$, then the polynomial $P(\omega,z^*)$ with respect to ω must have n distinct zeros. Hence, the set of *Special Zero Points* of $P(\omega,z)$ is a subset of M. Therefore, $P(\omega,z)$ has at most finitely many Special Zero Points.

Next, we prove Division Theorem using the above lemmas. For any $z\in\mathbb{C}\setminus M$, by Lemma (2.4), the polynomial $P(\omega,z)$ with respect to ω must have n distinct roots $r_i(i=1,2,\cdots,n)$. By the hypothesis, r_i $(i=1,2,\cdots,n)$ are also the roots of $Q(\omega,z)$. Hence the degree for polynomial

$Q(\omega,z)$ with respect to ω is greater than or equal to n.

Assume that

$$Q(\omega,z)=\sum_{k=0}^{m}q_k(z)\omega^k,$$

where $q_k(z)$ $(k=0,1,\cdots,m)$ are polynomials in z, $q_m(z)\not\equiv 0$, and $m\geq n$.

By the division theory for the polynomials in one variable, we have

$$Q(\omega,z)=h(\omega,z)\cdot P(\omega,z), \tag{11}$$

where

$$\begin{cases} h(\omega,z)=\sum_{k=0}^{m-n}h_k(z)\omega^k \\ h_{m-n}(z)=q_m(z)/p_n(z) \\ h_{m-n-1}(z)=\frac{1}{p_n(z)}[q_{m-1}(z)-\frac{q_m(z)p_{n-1}(z)}{p_n(z)}]=\frac{q_{m-1}*(z)}{p_n^2(z)} \\ \cdots\cdots \\ h_{m-n-i}(z)=\frac{q_{m-i}*(z)}{p_n^{i+1}(z)} \\ h_0(z)=\frac{q_n*(z)}{p_n^{m-n-1}(z)}. \end{cases} \tag{12}$$

Notice that the polynomials $q_{m-i}^*(z)$ could be obtained from $q_k(z)$ and $p_k(z)$ by applying the operations of addition, subtraction, multiplication and division. The denominators and numerators of equation (12) may have common factors.

Suppose that $u(z)$ is a polynomial with the least degree such that $u(z)\cdot h(\omega,z)$ is a polynomial in $\mathbb{C}[\omega,z]$. That is

$$u(z)h(\omega,z)=G_1(\omega,z), \tag{13}$$

where $u(z)$ and $G_1(\omega,z)$ are polynomials in $\mathbb{C}[\omega,z]$. Note that there is no nontrivial common factor between $u(z)$ and $G_1(\omega,z)$. By equations (11) and (13), we get

$$u(z)\cdot Q(\omega,z) = G_1(\omega,z)\cdot P(\omega,z). \tag{14}$$

If $u(z)$ is a nonzero constant, then we obtain the desired result. If $u(z)$ is a non-constant polynomial, then since $P(\omega,z)$ is irreducible, Lemma (2.1) implies that $u(z)$ must divide $G_1(\omega,z)$. This yields a contradiction with the above assumption that $u(z)$ and $G_1(\omega,z)$ have no nontrivial common factor in $\mathbb{C}[\omega,z]$. Therefore, $u(z)$ must be nonzero constant. Letting $G(\omega,z) = [\frac{1}{u(z)}]\cdot G_1(\omega,z)$, from equation (14) we obtain equation (8). So the proof of Division Theorem is complete.

In the next section, we are going to apply the Division Theorem to seek the first-integral to system (7).

2.2 Application of Division Theorem for Duffing-van der Pol-type oscillator

Suppose that $x = x(\xi)$ *and* $y = y(\xi)$ are the nontrivial solutions to equation (7), and $p(x,\ y) = \sum_{i=0}^{i=m} a_i(x)y^i$ is an irreducible polynomial in $\mathbb{C}\ [x,\ y]$ such that

$$p\ [x(\xi),\ y(\xi)] = \sum_{i=0}^{m} a_i(x)y^i = 0, \tag{15}$$

where $a_i(x)$ $(i = 0,\ 1,\ \cdots,\ m)$ are polynomials of x and are all relatively prime in $\mathbb{C}\ [x,\ y]$, and $a_m(x) \neq 0$. Equation (15) is also called the first-integral to equation (7). We start our study by assuming $m = 2$ in equation (15), that is, $p(x,\ y) = a_0(x) + a_1(x)y + a_2(x)y^2$.

Note that $\frac{dp}{d\xi}$ is a polynomial in x and y, and $p[x(\xi),\ y(\xi)] = 0$ implies $\frac{dp}{d\xi} = 0$. By the Division Theorem, there exists a polynomial $H\ (x,\ y) = \rho(x) + \eta(x)y$ in $\mathbb{C}\ [x,\ y]$ such that

$$\begin{aligned} \frac{dp}{d\xi} &= H\ (x,y)\cdot p(x,y) \\ &= [\rho(x)+\eta(x)y]\cdot\left[a_0(x)+a_1(x)y+a_2(x)y^2\right] \\ &= [\rho(x)\cdot a_0(x)] + [\rho(x)\cdot a_1(x)+\eta(x)\cdot a_0(x)]\,y \\ &\quad +[\rho(x)\cdot a_2(x)+\eta(x)\cdot a_1(x)]\,y^2+[\eta(x)\cdot a_2(x)]\,y^3. \end{aligned} \tag{16}$$

Moreover, it is clear that we can obtain

$$\begin{aligned} \frac{dp}{d\xi} &= \left(\frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial \xi}\right) \\ &= a_1(x)(\mu x-\alpha x^{n+1}) \\ &\quad +\left\{2a_2(x)(\mu x-\alpha x^{n+1})-a_1(x)\,[\delta+\beta x^n]+a_0'(x)\right\}y \\ &\quad +\{a_1'(x)-2a_2(x)\,[\delta+\beta x^n]\}\,y^2+a_2'(x)y^3. \end{aligned} \tag{17}$$

On equating the coefficients of y^ion both sides of above equations (16) and (17), we have

$$\begin{cases} \mathbf{a}'(\mathbf{x}) = A(x)\cdot\mathbf{a}(\mathbf{x}) \\ \\ \left[0,\ \alpha x^{n+1}-\mu x,\ \rho(x)\right]\cdot\mathbf{a}(\mathbf{x}) = 0, \end{cases} \tag{18}$$

where

$$\mathbf{a}(\mathbf{x}) = \begin{bmatrix} a_2(x) \\ a_1(x) \\ a_0(x) \end{bmatrix}, \tag{19}$$

and

$$A(x) = \begin{bmatrix} \eta(x) & 0 & 0 \\ 2\beta x^n + 2\delta + \rho(x) & \eta(x) & 0 \\ 2(\alpha x^{n+1} - \mu x) & \beta x^n + \delta + \rho(x) & \eta(x) \end{bmatrix}. \tag{20}$$

Since $a_i(x)$ are polynomials, from equation (18), we deduce that $a_2(x)$ is a constant and $\eta(x) = 0$. For simplification, we take $a_2(x) = 1$ and solve equation (18). We have

$$\mathbf{a}(\mathbf{x}) = \begin{bmatrix} 1 \\ \int [2\beta x^n + 2\delta + \rho(x)]\, dx \\ \int \left[\beta x^n a_1(x) + \delta a_1(x) + \rho(x) \cdot a_1(x) + 2(\alpha x^{n+1} - \mu x)\right] dx \end{bmatrix}. \tag{21}$$

We can determine the degree of polynomials $\rho(x)$ *and* $a_1(x)$ through equations (18) and (21).

1. If $deg\ \rho(x) = k > n > 0 \Rightarrow deg\ a_1(x) = k+1 \ \ and \ \ deg\ a_0(x) = 2k+2$. From (18) $deg\left[(\alpha x^{n+1} - \mu x)a_1(x)\right] = k+n+2$ but $deg\,[\rho(x) \cdot a_0(x)] = 3k+2 \Rightarrow k+n+2 = 3k+2 \Rightarrow k = \frac{n}{2} \Rightarrow \frac{n}{2} > n > 0$ $(contradiction)$.

2. If $deg\ \rho(x) = k \quad and \quad n > k > 0 \Rightarrow deg\ a_1(x) = n+1 \quad and \quad deg\ a_0(x) = 2n+2$. From (18) $deg\left[(\alpha x^{n+1} - \mu x)a_1(x)\right] = 2n+2$ but $deg\,[\rho(x) \cdot a_0(x)] = 2n+k+2 \quad \Rightarrow \quad k = 0$ but $k > 0$ $(contradiction)$.

3. If $deg\ \rho(x) = k = 0 \quad \Rightarrow \quad deg\ a_1(x) = n+1$.

Thus, let us assume

$$a_1(x) = B_2 x^{n+1} + B_1 x + B_0. \tag{22}$$

Through equation (21), we find

$$\begin{aligned} a_1(x) &= \int [2\beta x^n + 2\delta + \rho(x)] dx \\ &= \frac{2\beta}{n+1} x^{n+1} + (2\delta + \rho(x))x + B_0. \end{aligned} \tag{23}$$

So from equations (22) and (23) we can immediately obtain

$$\begin{cases} B_1 = 2\delta + \rho(x) \\ B_2 = \frac{2\beta}{n+1}\ . \end{cases} \tag{24}$$

From equations (21), (22) and (24), we can also deduce

$$\begin{aligned} a_0(x) &= \int [\beta x^n a_1(x) + \delta a_1(x) + \rho(x) \cdot a_1(x) + 2(\alpha x^{n+1} - \mu x)] dx \\ &= \int [\beta B_2 x^{2n+1} + (B_1\beta + B_2\delta + (B_1 - 2\delta)B_2 + 2\alpha)x^{n+1} + \beta B_0 x^n \\ &\quad + (B_1\delta + (B_1 - 2\delta)B_1 - 2\mu)x + B_0(B_1 - 2\delta) + B_0\delta] dx \\ &= \frac{\beta B_2}{2(n+1)} \cdot x^{2n+2} + \frac{(2\alpha + \beta B_1 + B_2(B_1 - \delta))}{n+2} \cdot x^{n+2} + \frac{\beta B_0}{n+1} \cdot x^{n+1} \\ &\quad + (\frac{B_1(B_1 - \delta)}{2} - \mu)x^2 + B_0(B_1 - \delta)x + D, \end{aligned} \tag{25}$$

where D is an arbitrary integration constant.

Substituting equations (22) and (24) into equation (18) that is $(\alpha x^{n+1}-\mu x)a_1(x)+\rho(x)a_0(x)=0$, where $\rho(x)=B_1-2\delta$, we have

$$\begin{aligned}(\alpha x^{n+1}-\mu x)\cdot(B_2x^{n+1}+B_1x+B_0)&\\+(B_1-2\delta)\cdot(\tfrac{\beta B_2}{2(n+1)}\cdot x^{2n+2}+\tfrac{(2\alpha+\beta B_1+B_2(B_1-\delta))}{n+2}\cdot x^{n+2}&\\+\tfrac{\beta B_0}{n+1}\cdot x^{n+1}+(\tfrac{B_1(B_1-\delta)}{2}-\mu)x^2+B_0(B_1-\delta)x+D)&=0\end{aligned}\tag{26}$$

$\Longleftrightarrow$

$$\begin{aligned}(\alpha B_2+\tfrac{\beta B_1B_2}{2(n+1)}-\tfrac{\beta\delta B_2}{n+1})\cdot x^{2n+2}+(\alpha B_1-\mu B_2+\tfrac{B_1^2B_2}{n+2}&\\-\tfrac{4\alpha\delta}{n+2}+\tfrac{2\alpha B_1}{n+2}+\tfrac{\beta B_1^2}{n+2}+\tfrac{2\delta^2B_2}{n+2}-\tfrac{2\beta\delta B_1}{n+2}-\tfrac{3\delta B_1B_2}{n+2})\cdot x^{n+2}&\\+(\alpha B_0-\tfrac{2\beta\delta B_0}{n+1}+\tfrac{\beta B_0B_1}{n+1})\cdot x^{n+1}&\\+(\tfrac{B_1^3}{2}+2\delta\mu-2\mu B_1-\tfrac{3\delta B_1^2}{2}+\delta^2B_1)\cdot x^2&\\+(2\delta^2B_0+B_1^2B_0-\mu B_0-3\delta B_1B_0)\cdot x&\\+(B_1D-2\delta D)&=0.\end{aligned}\tag{27}$$

Setting all coefficients of x^i $(i=2n+2,\ n+2,\ n+1,\ 2,\ 1,\ 0)$ to be zero, we have

$$\left\{\begin{aligned}\alpha B_2+\tfrac{\beta B_1B_2}{2(n+1)}-\tfrac{\beta\delta B_2}{n+1}&=0\cdots(I)\\\alpha B_1-\mu B_2+\tfrac{B_1^2B_2}{n+2}-\tfrac{4\alpha\delta}{n+2}+\tfrac{2\alpha B_1}{n+2}+\tfrac{\beta B_1^2}{n+2}+\tfrac{2\delta^2B_2}{n+2}-\tfrac{2\beta\delta B_1}{n+2}-\tfrac{3\delta B_1B_2}{n+2}&=0\cdots(II)\\\alpha B_0-\tfrac{2\beta\delta B_0}{n+1}+\tfrac{\beta B_0B_1}{n+1}&=0\cdots(III)\\\tfrac{B_1^3}{2}+2\delta\mu-2\mu B_1-\tfrac{3\delta B_1^2}{2}+\delta^2B_1&=0\cdots(IV)\\2\delta^2B_0+B_1^2B_0-\mu B_0-3\delta B_1B_0&=0\cdots(V)\\(B_1-2\delta)D&=0\cdots(VI).\end{aligned}\right.\tag{28}$$

Let us take the integration constant $D=0$.

Also from (I) , we can deduce $B_1=2\delta-\frac{2\alpha(n+1)}{\beta}$.

From(III), we can deduce $B_0=0$ *or* $B_1=2\delta-\frac{\alpha(n+1)}{\beta}$.

From(IV), we can deduce $B_1 = \delta$ *or* $B_1 = \delta \pm \sqrt{\delta^2 + 4\mu}$.
From (V), we can deduce $B_0 = 0$ *or* $B_1 = \frac{3\delta \pm \sqrt{\delta^2+4\mu}}{2}$.

Consider the case $B_0 = 0$.

2.2.1 Case I

Then, we assume

$$B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta} = \delta + \sqrt{\delta^2 + 4\mu}, \tag{29}$$

or

$$B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta} = \delta - \sqrt{\delta^2 + 4\mu}. \tag{30}$$

We can deduce

$$\begin{cases} 2\delta - \frac{2\alpha(n+1)}{\beta} & = & \delta + \sqrt{\delta^2 + 4\mu} & \Rightarrow & \mu = (\frac{\alpha(n+1)}{\beta})^2 - \delta(\frac{\alpha(n+1)}{\beta}) \\ \\ 2\delta - \frac{2\alpha(n+1)}{\beta} & = & \delta - \sqrt{\delta^2 + 4\mu} & \Rightarrow & \mu = (\frac{\alpha(n+1)}{\beta})^2 - \delta(\frac{\alpha(n+1)}{\beta}), \end{cases} \tag{31}$$

or

$$\beta^2\mu = \alpha^2(n+1)^2 - \alpha\beta\delta(n+1). \tag{32}$$

Plugging $B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta}$ back into equation (II) of equation system (28), we give

$$B_2 = \frac{2\alpha^2\beta(n+1)(n-2) - 2\alpha\beta^2\delta n}{-4\alpha^2(n+1)^2 + 2\alpha\beta\delta(n+1) + \beta^2\mu(n+2)}. \tag{33}$$

Using equation (32) to simplify B_2, we obtain

$$\begin{aligned} B_2 &= \frac{2\alpha^2\beta(n+1)(n-2)-2\alpha\beta^2\delta n}{\alpha^2(n+1)^2(n-2)-\alpha\beta\delta(n+1)n} \\ &= \frac{2\beta}{n+1}. \end{aligned} \tag{34}$$

Hence, under the parametric condition:

$$\beta^2\mu = \alpha^2(n+1)^2 - \alpha\beta\delta(n+1), \tag{35}$$

equation (28) has solution as

$$\begin{cases} B_1 &= 2\delta - \frac{2\alpha(n+1)}{\beta} \\ B_2 &= \frac{2\beta}{n+1} \\ B_0 &= 0 \\ D &= 0. \end{cases} \tag{36}$$

Using equation (35) to rewrite equation (22), we have

$$a_1(x) = \frac{2\beta}{n+1}\cdot x^{n+1} + (2\delta - \frac{2\alpha(n+1)}{\beta})\cdot x. \tag{37}$$

Substituting equation (36) into equation (25), and simplifying by equation (35), we get

$$\begin{aligned} a_0(x) &= \tfrac{\beta B_2}{2(n+1)} \cdot x^{2n+2} + \tfrac{(2\alpha+\beta B_1+B_2(B_1-\delta))}{n+2} \cdot x^{n+2} + \tfrac{\beta B_0}{n+1} \cdot x^{n+1} \\ &\quad +(\tfrac{B_1(B_1-\delta)}{2} - \mu)x^2 + B_0(B_1-\delta)x + D \\ &= \tfrac{\beta^2}{(n+1)^2} \cdot x^{2n+2} + (\tfrac{2\alpha+2\beta\delta-2\alpha(n+1)+\frac{2\beta}{n+1}(\delta-\frac{2\alpha(n+1)}{\beta})}{n+2}) \cdot x^{n+2} \\ &\quad +(\tfrac{(2\delta-\frac{2\alpha(n+1)}{\beta})\cdot(\delta-\frac{2\alpha(n+1)}{\beta})}{2} - \mu) \cdot x^2 \\ &= \tfrac{\beta^2}{(n+1)^2} \cdot x^{2n+2} + (\tfrac{2\beta\delta}{n+1} - 2\alpha) \cdot x^{n+2} \\ &\quad +(\delta^2 + \mu - \tfrac{\alpha\delta(n+1)}{\beta}) \cdot x^2. \end{aligned} \tag{38}$$

Substituting $a_0(x)$, $a_1(x)$ and $a_2(x) = 1$ into $p(x,y) = a_0(x) + a_1(x)y + a_2(x)y^2$ and setting $p(x,y) = 0$, we get

$$\begin{aligned} & y^2 + (\tfrac{2\beta}{n+1} \cdot x^{n+1} + (2\delta - \tfrac{2\alpha(n+1)}{\beta}) \cdot x) \cdot y \\ & \quad + \Big[\tfrac{\beta^2}{(n+1)^2} \cdot x^{2n+2} + (\tfrac{2\beta\delta}{n+1} - 2\alpha) \cdot x^{n+2} \\ & \qquad + (\delta^2 + \mu - \tfrac{\alpha\delta(n+1)}{\beta}) \cdot x^2\Big] = 0. \end{aligned} \tag{39}$$

From equation (39), y can be expressed in terms of x under the parametric condition (35), i.e.

$$y = \tfrac{1}{2} \cdot \left[\tfrac{-2\beta}{n+1} \cdot x^{n+1} - (2\delta - \tfrac{2\alpha(n+1)}{\beta}) \cdot x\right]$$

$$\pm\tfrac{1}{2}\sqrt{(\tfrac{2\beta}{n+1} \cdot x^{n+1} + (2\delta - \tfrac{2\alpha(n+1)}{\beta}) \cdot x)^2 - 4(\tfrac{\beta^2}{(n+1)^2} \cdot x^{2n+2} + (\tfrac{2\beta\delta}{n+1} - 2\alpha) \cdot x^{n+2} + (\delta^2 + \mu - \tfrac{\alpha\delta(n+1)}{\beta}) \cdot x^2)}$$

$\Longrightarrow$

$$\begin{aligned} y &= \tfrac{1}{2} \cdot \left[\tfrac{-2\beta}{n+1} \cdot x^{n+1} - (2\delta - \tfrac{2\alpha(n+1)}{\beta}) \cdot x\right] \\ &= \left(\tfrac{\alpha(n+1)}{\beta} - \delta\right) \cdot x - \tfrac{\beta}{n+1} \cdot x^{n+1}. \end{aligned} \tag{40}$$

2.2.2 Case II

Choose

$$B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta} = \delta, \tag{41}$$

the we have

$$2\delta - \frac{2\alpha(n+1)}{\beta} = \delta \;\;\Rightarrow\;\; \delta = \frac{2\alpha(n+1)}{\beta}. \tag{42}$$

Plugging $B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta}$ back into equation (II) of equation (28), we get

$$B_2 = \frac{2\alpha^2\beta(n+1)(n-2) - 2\alpha\beta^2\delta n}{-4\alpha^2(n+1)^2 + 2\alpha\beta\delta(n+1) + \beta^2\mu(n+2)}. \tag{43}$$

Then, simplifying B_2 by equation (42), we have

$$B_2 = -\frac{2\alpha^2(n+1)}{\beta\mu}. \tag{44}$$

From (24) and equation (44), we get

$$\begin{aligned} \tfrac{2\beta}{n+1} &= -\tfrac{2\alpha^2(n+1)}{\beta\mu} \\ \Rightarrow \\ \mu &= -\tfrac{\alpha^2(n+1)^2}{\beta^2}. \end{aligned} \tag{45}$$

So under the parametric condition

$$\begin{cases} \mu = -\frac{\alpha^2(n+1)^2}{\beta^2} \\ \\ \delta = \frac{2\alpha(n+1)}{\beta}, \end{cases} \tag{46}$$

equation (28) has the following solution:

$$\begin{cases} B_1 &= 2\delta - \frac{2\alpha(n+1)}{\beta} \\ B_2 &= \frac{2\beta}{n+1} \\ B_0 &= 0 \\ D &= 0. \end{cases} \tag{47}$$

Plugging equation (47) back into equation (22), we get

$$a_1(x) = \frac{2\beta}{n+1} \cdot x^{n+1} + (2\delta - \frac{2\alpha(n+1)}{\beta}) \cdot x. \tag{48}$$

Plugging equation (47) back into equation (25) and simplifying by equation (46), we can obtain

$$\begin{aligned} a_0(x) &= \tfrac{\beta B_2}{2(n+1)} \cdot x^{2n+2} + \tfrac{(2\alpha+\beta B_1+B_2(B_1-\delta))}{n+2} \cdot x^{n+2} + \tfrac{\beta B_0}{n+1} \cdot x^{n+1} \\ &\quad + (\tfrac{B_1(B_1-\delta)}{2} - \mu)x^2 + B_0(B_1-\delta)x + D \\ &= \tfrac{\beta^2}{(n+1)^2} \cdot x^{2n+2} + (\tfrac{2\alpha+2\beta\delta-2\alpha(n+1)+\frac{2\beta}{n+1}(\delta-\frac{2\alpha(n+1)}{\beta})}{n+2}) \cdot x^{n+2} \\ &\quad + (\tfrac{(2\delta-\frac{2\alpha(n+1)}{\beta})\cdot(\delta-\frac{2\alpha(n+1)}{\beta})}{2} - \mu) \cdot x^2 \\ &= \tfrac{\beta^2}{(n+1)^2} \cdot x^{2n+2} + (\tfrac{2\beta\delta}{n+1} - 2\alpha) \cdot x^{n+2} - \mu \cdot x^2. \end{aligned} \tag{49}$$

Substituting $a_0(x)$, $a_1(x)$ and $a_2(x) = 1$ into $p(x,y) = a_0(x) + a_1(x)y + a_2(x)y^2$ and setting $p(x,y) = 0$, we have

$$\begin{aligned} & y^2 + (\tfrac{2\beta}{n+1} \cdot x^{n+1} + (2\delta - \tfrac{2\alpha(n+1)}{\beta}) \cdot x) \cdot y + \\ & + \left[\tfrac{\beta^2}{(n+1)^2} \cdot x^{2n+2} + (\tfrac{2\beta\delta}{n+1} - 2\alpha) \cdot x^{n+2} - \mu \cdot x^2\right] = 0. \end{aligned} \tag{50}$$

From the above equation, y can be expressed in terms of x under parametric condition (46), i.e

$$y = \frac{1}{2}\cdot\left[\frac{-2\beta}{n+1}\cdot x^{n+1} - (2\delta - \frac{2\alpha(n+1)}{\beta})\cdot x\right]$$

$$\pm\frac{1}{2}\sqrt{(\frac{2\beta}{n+1}\cdot x^{n+1} + (2\delta - \frac{2\alpha(n+1)}{\beta})\cdot x)^2 - 4(\frac{\beta^2}{(n+1)^2}\cdot x^{2n+2} + (\frac{2\beta\delta}{n+1} - 2\alpha)\cdot x^{n+2} - \mu\cdot x^2)}$$

$\Longrightarrow$

$$\begin{aligned} y &= \tfrac{1}{2}\cdot\left[\tfrac{-2\beta}{n+1}\cdot x^{n+1} - (2\delta - \tfrac{2\alpha(n+1)}{\beta})\cdot x\right] \\ &= \left(\tfrac{\alpha(n+1)}{\beta} - \delta\right)\cdot x - \tfrac{\beta}{n+1}\cdot x^{n+1}. \end{aligned} \tag{51}$$

Let us consider the case $B_0 = 1$.

2.2.3 Case III

From equation (I), (III) and (V) of equation system (28), we can get

$$B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta} = 2\delta - \frac{\alpha(n+1)}{\beta} = \frac{3\delta \pm \sqrt{\delta^2 + 4\mu}}{2}. \tag{52}$$

Note that

$$2\delta - \frac{2\alpha(n+1)}{\beta} \neq 2\delta - \frac{\alpha(n+1)}{\beta}. \tag{53}$$

It is a contradiction and $B_0 \neq 1$.

Since $\alpha \neq 0$. Hence, equation system (28) has only one solution, which is

$$\begin{cases} B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta} \\ B_2 = \frac{2\beta}{n+1} \\ B_0 = 0 \\ D = 0. \end{cases} \tag{54}$$

2.2.4 Conclusion

From the previous works, we find

$$y = \left(\frac{\alpha(n+1)}{\beta} - \delta\right) \cdot x - \frac{\beta}{n+1} \cdot x^{n+1}, \tag{55}$$

under the parametric conditions of case I and case II. It is clear by equation (7) that equation (55) is a first-integral of the Duffing-van der Pol-type oscillator. Comparing the two parametric conditions between case I [equation (35)] and case II [equation (46)], we obtain that under certain transformations, equation (46) is equivalent to equation (35). Hence, we can conclude that the only one parametric condition is

$$\beta^2\mu = \alpha^2(n+1)^2 - \alpha\beta\delta(n+1). \tag{56}$$

So far, we have derived one particular first-integral of the Duffing-van der Pol-type oscillator under certain parametric condition (56). In the next chapter, we are going to look for a more general first-integral of the Duffing-van der Pol-type oscillator.

CHAPTER III

THE GENERAL FIRST INTEGRAL

In this chapter, we concentrate on obtaining a more general first integral of the Duffing-van der Pol-type oscillator (1).

3.1 Preparation

From the previous works, we know

$$\frac{dp}{d\xi} = H\ (x,y) \cdot p(x,y), \tag{57}$$

where $H\ (x,\ y) = \rho(x) + \eta(x)y$ is a polynomial in $\mathbb{C}\ [x,\ y]$. It is clear we can deduce

$$\eta(x) = 0, \tag{58}$$

which was mentioned on page 9, from equation (18). Furthermore, equation (24) indicates

$$\rho(x) = B_1 - 2\delta. \tag{59}$$

Through equation (54), we find

$$B_1 = 2\delta - \frac{2\alpha(n+1)}{\beta}, \tag{60}$$

which gives

$$\rho(x) = -\frac{2\alpha(n+1)}{\beta} \tag{61}$$

by equation (59).

Substituting equations (58) and (61) into $H\ (x,\ y) = \rho(x) + \eta(x)y$, we derive

$$H\ (x,\ y) = -\frac{2\alpha(n+1)}{\beta}. \tag{62}$$

Thus, equation (57) will be rewritten as

$$\frac{dp}{d\xi} = \left[-\frac{2\alpha(n+1)}{\beta}\right] \cdot p(x,y), \tag{63}$$

where $-\frac{2\alpha(n+1)}{\beta}$ is a constant. The solution is given by $p(x,y) = C \cdot e^{\left[-\frac{2\alpha(n+1)}{\beta}\right]\cdot\xi}$ with arbitrary number C.

As we know,

$$p(x,\ y) = a_0(x) + a_1(x)y + a_2(x)y^2, \tag{64}$$

which indicates

$$a_0(x) + a_1(x)y + a_2(x)y^2 = C \cdot e^{\left[-\frac{2\alpha(n+1)}{\beta}\right]\cdot\xi}. \tag{65}$$

3.2 The general first integral

Let us discuss equation (65) with two cases. Additionally, as we have mentioned on page 10, we take $a_2(x) = 1$ for simplification.

1. Under the parametric condition (35), equations (37) and (38) give

$$\begin{cases} a_1(x) &= \frac{2\beta}{n+1}\cdot x^{n+1} + (2\delta - \frac{2\alpha(n+1)}{\beta})\cdot x \\ \\ a_0(x) &= \frac{\beta^2}{(n+1)^2}\cdot x^{2n+2} + (\frac{2\beta\delta}{n+1} - 2\alpha)\cdot x^{n+2} + (\delta^2 + \mu - \frac{\alpha\delta(n+1)}{\beta})\cdot x^2. \end{cases} \tag{66}$$

Substituting $a_2(x) = 1$ and equation (66) into equation (65), we have

$$\begin{aligned} y^2 + (\tfrac{2\beta}{n+1}\cdot x^{n+1} + (2\delta - \tfrac{2\alpha(n+1)}{\beta})\cdot x)\cdot y \\ + \Big[\tfrac{\beta^2}{(n+1)^2}\cdot x^{2n+2} + (\tfrac{2\beta\delta}{n+1} - 2\alpha)\cdot x^{n+2} \\ + (\delta^2 + \mu - \tfrac{\alpha\delta(n+1)}{\beta})\cdot x^2\Big] &= C\cdot e^{\left[-\frac{2\alpha(n+1)}{\beta}\right]\cdot\xi}. \end{aligned} \tag{67}$$

After applying the parametric condition (35), the L.H.S. of equation (67) can be factored into a perfect square, then we derive

$$\left(y + \left(\delta - \frac{\alpha(n+1)}{\beta}\right)x + \frac{\beta}{n+1}x^{n+1}\right)^2 = C\cdot e^{\left[-\frac{2\alpha(n+1)}{\beta}\right]\cdot\xi}, \tag{68}$$

which implies

$$\left(y + \left(\delta - \frac{\alpha(n+1)}{\beta}\right)x + \frac{\beta}{n+1}x^{n+1}\right)\cdot e^{\left[\frac{\alpha(n+1)}{\beta}\right]\cdot\xi} = I, \tag{69}$$

where $I = C^{\frac{1}{2}} \in \mathbb{C}$.

2. Under the parametric condition (46), equations (48) and (49) give

$$\begin{cases} a_1(x) &= \frac{2\beta}{n+1}\cdot x^{n+1} + (2\delta - \frac{2\alpha(n+1)}{\beta})\cdot x \\ \\ a_0(x) &= \frac{\beta^2}{(n+1)^2}\cdot x^{2n+2} + (\frac{2\beta\delta}{n+1} - 2\alpha)\cdot x^{n+2} - \mu\cdot x^2. \end{cases} \tag{70}$$

Substituting $a_2(x)=1$ and equation (70) into equation (65), we have

$$\begin{aligned} y^2+(\tfrac{2\beta}{n+1}\cdot x^{n+1}+(2\delta-\tfrac{2\alpha(n+1)}{\beta})\cdot x)\cdot y+ & \\ +\left[\tfrac{\beta^2}{(n+1)^2}\cdot x^{2n+2}+(\tfrac{2\beta\delta}{n+1}-2\alpha)\cdot x^{n+2}-\mu\cdot x^2\right] &= C\cdot e^{\left[-\frac{2\alpha(n+1)}{\beta}\right]\cdot\xi}. \end{aligned} \tag{71}$$

After applying the parametric condition (46), the L.H.S. of equation (71) can be factored into a perfect square, then we obtain

$$\left(y+\left(\delta-\frac{\alpha\,(n+1)}{\beta}\right)x+\frac{\beta}{n+1}x^{n+1}\right)^2=C\cdot e^{\left[-\frac{2\alpha(n+1)}{\beta}\right]\cdot\xi}, \tag{72}$$

which implies

$$\left(y+\left(\delta-\frac{\alpha\,(n+1)}{\beta}\right)x+\frac{\beta}{n+1}x^{n+1}\right)\cdot e^{\left[\frac{\alpha(n+1)}{\beta}\right]\cdot\xi}=I, \tag{73}$$

where $I=C^{\frac{1}{2}}\in\mathbb{C}$.

Since the parametric condition (46) is a particular case of parametric condition (35). Hence, we can conclude that under the parametric condition

$$\beta^2\mu=\alpha^2(n+1)^2-\alpha\beta\delta(n+1),$$

the nonlinear Duffing-van der Pol-type oscillator system (1) has only one general first integral, which is

$$\left(y+\left(\delta-\frac{\alpha\,(n+1)}{\beta}\right)x+\frac{\beta}{n+1}x^{n+1}\right)\cdot e^{\left[\frac{\alpha(n+1)}{\beta}\right]\cdot\xi}=I,$$

where $I=C^{\frac{1}{2}}\in\mathbb{C}$.

CHAPTER IV

CONCLUSION

The first-integral method used in this paper was first proposed by Dr. Feng [31] in solving Burgers-Kvd equation which is based on the ring theory of commutative algebra. It is a very useful method for nonlinear dynamical equations[31, 35]. Recently, this method has been widely used by many mathematicians, such as in [36-38] and by the references therein. The method described herein is not only efficient, but also has the merit of being widely applicable. We can apply this technique to many nonlinear equations, such as the nonlinear Schr$\ddot{o}$dinger equation, the generalized Klein-Gordon equation, and the higher order KdV-like equation. We believe that this method will be advantageous for a rather diverse group of scientists.

In this paper, the first-integral method was applied successfully for solving the nonlinear Duffing-van der Pol-type oscillator system. The first-integrals of the Duffing-van der Pol-type oscillator system with four parameters have been studied and established in the previous chapters. During chapter II, we applied the first-integral method to obtain one particular first integral of the Duffing-van der Pol-type oscillator (1) under certain parametric conditions. We obtained the parametric conditions by solving equation system (28). It has been discussed for three cases, which are parametric conditions (35), (46) and (52). However, after further examining the parametric condition (46) we found that under certain transformations, the parametric condition (46) can be transferred into parametric condition (35). Moreover, the parametric condition (52) of case III is in contradiction to equation (53). Hence, we can conclude that there is only one parametric condition of equation system (28), which is the parametric condition (56). Afterward, we applied the solution of equation system (28) to obtain a particular first integral of the Duffing-van der Pol-type oscillator by setting $p(x,y) = 0$. Finally, in order to complete this paper, we achieved the general first integral of the Duffing-van der Pol-type oscillator (1) under the only parametric condition (56) in chapter III.

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