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Coupled Telegraph and SIR Model of Information and Diseases

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COUPLED TELEGRAPH AND SIR MODEL

OF INFORMATION AND DISEASES

A Thesis

by

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COUPLED TELEGRAPH AND SIR MODEL

OF INFORMATION AND DISEASES

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ABSTRACT

Galarza, Jose de Jesus, <u>Coupled Telegraph and SIR Model of Information and Diseases</u>. Master of Science (MS), May, 2017, 73 pp., 6 tables, 22 figures, 52 references, 11 titles.

In this work, the effect of information propagation on disease spread and vaccination uptake through networks is studied. In this model the information reaches different people at different distances from the center of information containing the health data. We use a pair of Telegraph equations to depict the vaccine and disease information propagation on a network embedded into a straight line. The Telegraph equation is coupled with an SIR (Susceptible-Infected-Recovered) model to examine the anticipated mutual influence. Numerical simulations and stability analysis were made to study the model. We show how the propagation of information about the disease impacts the probability of vaccination and as a consequence the vaccination rate.

DEDICATION

To my family. Without them I would not be here.

ACKNOWLEDGMENTS

Special acknowledgment goes to Dr. Erwin Suazo and Dr. Tamer Oraby. They helped me all the way through this thesis.

To the committee members Dr. Zhijun Qiao and Dr. Josef Sifuentes for the valuable feedback to make this thesis possible.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In the area of epidemiology there is a need to research the spread of diseases. In order to study this we start by researching the behavior of the population. As Brauer and Castillo-Chavez explain in [2, p.xviii], we can address questions raised by a complex demographic processes through the use of mathematical models. Mathematical models that study the spread of diseases are typically refereed as epidemic models. There is an effort to produce models that can predict the spread of diseases; we have several descriptions of these epidemic models, and one in particular upon which we will focus our attention is the SIR (Susceptible-Infected-Recovered) model.

1.1 SIR Model

The SIR model is one in which according to Brauer, Castillo-Chavez [2, p.350] and Murray [6, p.320] we have three sections in the population, which are the susceptible population S, the infected population I and the recovered population R.

The interaction between them can be seen in the next diagram:
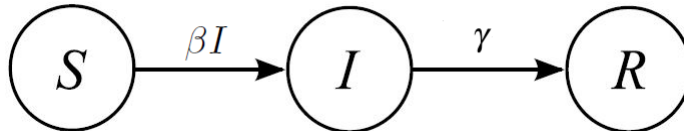
Figure 1.1: SIR Model Diagram by Brauer and Castillo-Chavez [2, p.351]

The interaction we can clearly see in the Figure by Brauer and Castillo-Chavez [2, p.351] is that the susceptible people can only become infected, and the infected people can become recovered. As we can see the recovered section of the people remain there.

Usually the only way that one can become infected is when there is interaction between the susceptible people and the infected ones, and as such a fraction of population will become infected; we will call this effect the rate of infection β. The rate at which people become recovered or cured in the infected section is called the recovery rate γ. We always regard β and γ as positive quantities. The model as described by Brauer and Castillo-Chavez [2, p.351] can be seen as:

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta SI \\
\frac{dI}{dt} &= \beta SI - \gamma I \\
\frac{dR}{dt} &= \gamma I.
\end{aligned}
\tag{1.1}
$$

Of course we see that births and deaths are not considered in this simple model. The variables of interest can be considered as proportions such that $S + I + R = 1$. Brauer and Castillo-Chavez [2, p.352] considered them as actual population sizes such that $S + I + R = N$. Also it is important to notice that one variable is redundant; in most of the cases we ignore the R variable, which means we only need to analyze the differential equations for S and for I.

We are going to describe the analysis made in the book of Brauer and Castillo-Chavez [2, p.353], which they start by taking the difference between the derivatives of S and I, yielding the following ODE:

$$
\frac{d(S+I)}{dt} = -\gamma I.
\tag{1.2}
$$

The following initial conditions described by Brauer and Castillo-Chavez [2, p.353] are:

$$
S(0) = S_0 \quad I(0) = I_0 \quad S_0 + I_0 = N.
\tag{1.3}
$$

As Brauer and Castillo-Chavez [2, p.354] explain, the functions S and I need to be non-negative and can be regarded as smooth functions. As such $S + I$ is a smooth non-negative decreasing function, which will tend to a limit as t goes to infinity. Recall from Analysis that the derivative of smooth non-negative decreasing function will tend to zero as t goes to infinity. We see then that the function $S + I$ has a limit as t approaches infinity. Let's call this limit S_∞. As such we have the following:

$$\lim_{t \to \infty} \frac{d(S+I)}{dt} = \lim_{t \to \infty} -\gamma I = 0. \tag{1.4}$$

But this also implies that:

$$\lim_{t \to \infty} I = 0. \tag{1.5}$$

Now an important aspect of this is to find the so-called final size relation. As explained by Brauer and Castillo-Chavez [2, p.354] this can be obtained by integrating the differential equations from zero to infinity. Thus we obtain the following:

$$\int_0^\infty \frac{d(S+I)}{dt} dt = S_\infty - (S_0 + I_0) = S_\infty - N. \tag{1.6}$$

But then we have that

$$-\gamma \int_0^\infty I(t)dt = S_\infty - N$$
$$\int_0^\infty I(t)dt = \frac{1}{\gamma}(N - S_\infty). \tag{1.7}$$

3

We can then immediately solve the first differential equation as follows:

$$
\begin{aligned}
\frac{dS}{dt} &= -\beta SI \\
\frac{dS}{S} &= -\beta I\, dt \\
\int_{S(0)}^{S(\infty)} \frac{dS}{S} &= -\beta \int_0^\infty I(t)\, dt \\
\ln(S_\infty) - \ln(S_0) &= -\frac{\beta}{\gamma}(N - S_\infty) \\
\ln(S_0) - \ln(S_\infty) &= \frac{\beta}{\gamma}(N - S_\infty) \\
\ln\frac{S_0}{S_\infty} &= \frac{\beta}{\gamma}(N - S_\infty).
\end{aligned}
\tag{1.8}
$$

Now notice that this result is the final size relation mentioned by Brauer and Castillo-Chavez [2, p.354]. Another way to analyze the model is to understand the behavior of the derivatives as explained by Murray in [6, p.321]. This can be seen in the differential equations we have, and we notice that I will increase as long as $S > \gamma/\beta$. Notice that even though this is true, the differential equation of S tells us that it is decreasing in time and I will ultimately stop increasing. This does not mean it is not epidemic, since in order to make it real we know that the model must stop if either any of the quantities reaches N or 1 if the variables are defined as proportions. Murray [6, p.321] shows us this behavior in the phase plane of S and I respectively in Figure 1.2.

1.2 Description of Diffusion of Information

In this section we will talk about the diffusion models which describe how information propagates as noted by Wang [11, p.9]. It is important to point out that these models which describe this process are either ordinary differential equations or partial differential equations. The diffusion models are used most of the time for information propagation, which is derived from a random walk process which is the way we model how information spreads over users. A detailed discussion on applications of diffusion can be referred in Murray [7, p.75].

Most of these models come from a general PDE model as described in Wang [11, p.9]
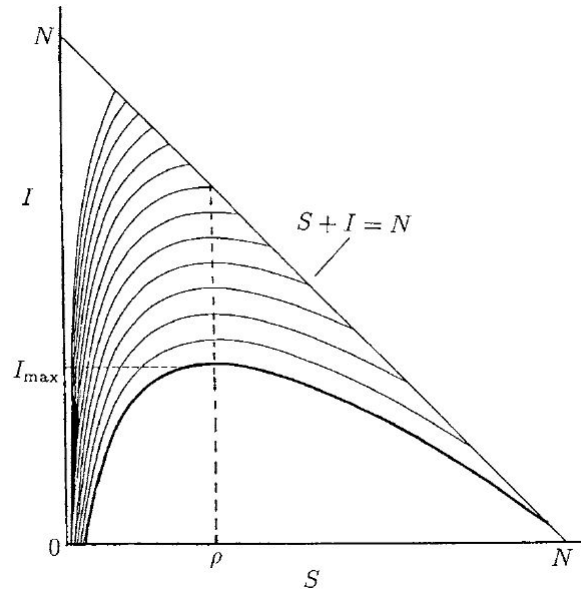
Figure 1.2: SIR Phase Plane Example by Murray [6, p.321]

which describes information flow:

$$\frac{\partial M}{\partial t} \;=\; d\frac{\partial^2 M}{\partial x^2} + f(M,x,t).$$ (1.9)

Symbol	Description
$d\frac{\partial^2 M}{\partial x^2}$	Diffusion process (random walk)
$f(M,x,t)$	Local growth (birth and death)

Table 1.1: General Information Flow Model Variables Table by Wang [11, p.9]

With this in mind let us talk of the Diffusive Logistic Model which is the simplest one we can describe. The following equations are given by Wang [11, p.12] as

$$\begin{aligned}
\frac{\partial M}{\partial t} &= d\frac{\partial^2 M}{\partial x^2} + rM\left(1 - \frac{M}{K}\right) \\
M(x,1) &= \phi(x), \ell \leq x \leq L \\
\frac{\partial M}{\partial x}(\ell,t) &= \frac{\partial M}{\partial x}(L,t) = 0, \ t \geq 1.
\end{aligned} \tag{1.10}$$

The description of each variable are in Table 1.2. Here $\phi(x)$ as explained by Wang [11, p.12] is the density of influenced users at distance x at the initial time a news story is submitted. The choice to start at $t = 1$ is due to the data they have in their research as explained in [11, p.11].

Symbol	Description
M	Density of influenced users with a distance of x at a time t
d	Popularity of information
r	Intrinsic growth rate
K	Maximum possible density of influenced users at a given distance
L	Upper bound of the distance from the source of information
ℓ	Lower bound of the distance from the source of information
$\phi(x)$	Initial density function

Table 1.2: Diffusive Logistic Model Variables Table by Wang [11, p.12]

The next model was proposed by Wu and Xia et al. [10, p.312], named the Linear diffusive model. The difference here as Wang pointed out [11, p.14], is that it takes into account effects of heterogeneity at a distance x denoted by $h(x)$ and the news decay with respect to time which we will denote as $r(t)$. Thus we have our next model as

$$\begin{aligned}
\frac{\partial M}{\partial t} &= d\frac{\partial^2 M}{\partial x^2} + r(t)h(x)M \\
M(x,1) &= \phi(x),\ \ell \leq x \leq L \\
\frac{\partial M}{\partial x}(\ell,t) &= \frac{\partial M}{\partial x}(L,t) = 0,\ t > 1.
\end{aligned} \tag{1.11}$$

Symbol	Description
M	Density of influenced users with a distance of x at a time t
d	Popularity of information
$r(t)$	News decay with respect to time
$h(x)$	Heterogeneity at a distance x
K	Maximum possible density of influenced users at a given distance
L	Upper bound of the distance from the source of information
ℓ	Lower bound of the distance from the source of information
$\phi(x)$	Initial density function

Table 1.3: Linear Diffusive Model Variables Table by Wu and Xia [10, p.312]

The next model proposed by Wang [11, p.16] has the popularity of information as a function of the distance. Thus we have that

$$d(x) = d_0 e^{-bx}. \tag{1.12}$$

Also it takes into consideration the heterogeneity $h(x)$ and the news decay with respect to time $r(t)$. Thus the model named logistic model with variable content-based diffusion is the following:

Symbol	Description
M	Density of influenced users with a distance of x at a time t
d	Popularity of information
$r(t)$	News decay with respect to time
$h(x)$	Heterogeneity at a distance x
K	Maximum possible density of influenced users at a given distance
L	Upper bound of the distance from the source of information
ℓ	Lower bound of the distance from the source of information
$\phi(x)$	Initial density function

Table 1.4: Logistic Model With Variable Content-Based Diffusion Variables Table by Wang [11, p.12,p.16]

$$
\begin{aligned}
\frac{\partial M}{\partial t} &= \frac{\partial \left(d_0 e^{-bx} M_x \right)}{\partial x} + r(t) M \left(h(x) - \frac{M}{K} \right) \\
M(x,1) &= \phi(x), \ \ell \leq x \leq L \\
\frac{\partial M}{\partial x}(\ell,t) &= \frac{\partial M}{\partial x}(L,t) = 0, \ t > 1.
\end{aligned}
\tag{1.13}
$$

A general model for spreading information on a social network was studied by Dai et al. [3, p.1614] is

$$
\begin{aligned}
\frac{\partial M(t,x)}{\partial t} - \frac{\partial}{\partial x}\left(a(x) \frac{\partial M(t,x)}{\partial x} \right) &= \lambda r_\infty u \left(h(x) - \frac{u}{K} \right), \ l < x < L \\
M(1,x) &= \phi(x), \ \ell \leq x \leq L \\
\frac{\partial M(t,\ell)}{\partial x} &= 0 \quad t > 1 \\
\frac{\partial M(t,L)}{\partial x} + \beta M(t,L) &= 0 \quad t > 1.
\end{aligned}
\tag{1.14}
$$

Taking $u(x) = \lim_{t \to +\infty} M(t,x)$ and $r_\infty = \lim_{t \to +\infty} r(t)$ and taking the time derivatives to zero, we see that the following is the steady state equations of 1.14 which are

$$-(a(x)u')' = \lambda r_\infty u\left(h(x) - \frac{u}{K}\right), \ \ell < x < L$$

$$u'(\ell) = 0 \tag{1.15}$$

$$u'(L) + \beta u(L) = 0.$$

Having the principal eigenvalue λ_1^+ from the following problem, the principal result from Dai [3, p.1622] is that the Logistic model with variable content-based diffusion has $\lambda < \lambda_1^+/r_\infty$ then the information will vanish with finite time. If $\lambda > \lambda_1^+/r_\infty$ then information diffusion last forever.

1.3 Telegraph From Random Walk

In this section we will discuss how the telegraph equation is derived from a random walk, an important aspect in our study used later on. As such let us comment on the motivation of the idea and derivation described by Keller [5, p.1120].

The motivation or the paradox, another way of describing the situation, is how is it possible that the diffusion equation with infinite speed is derived from a random walk with steps with finite speed. We can resolve this paradox as explained by Keller [5, p.1120] by showing that the diffusion approximation only holds for a limited range of distance. As such let us start to describing how this is derived. Murray also has a derivation of the diffusion equation from a simple random walk, of which can be seen in [6, p.395].

Consider a particle which is moving on the x axis, and at position $x = k$ in which each step has a probability p to move to the left and probability q to move to the right. This can be seen in Figure 1.3, with time interval equal to 1 between steps and each step at time $t = i$ is $h_i = \pm 1$.

For our case in particular we have that $p = q = 1/2$ and we have that the steps are assumed to be independent. So denoting the position at a time t as x_t, we can calculate it as $x_t = \sum_{i=1}^{t} h_i$. Now for large time t, the central limit theorem as explained in the book by Ross [9, p.210], states that the probability density of the sum x_t is asymptotic to a normal density with zero mean and variance of t. In other words we have that
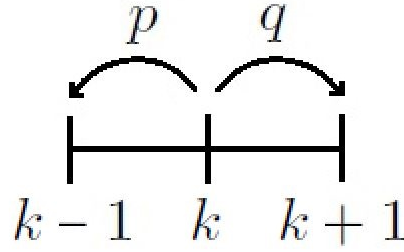
9

Figure 1.3: Simple Random Walk Step Diagram

$$p(x,t) \sim \frac{e^{-x^2/2t}}{(2\pi t)^{1/2}}, \qquad |x| = \mathcal{O}(\sqrt{t}). \tag{1.16}$$

Notice that the term in the right hand side, as Keller points out in [5, p.1120], is the solution for the following diffusion equation with the coefficient as $D = 1/2$ with initial condition as follows:

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}, \qquad p(x,0) = \delta(x). \tag{1.17}$$

Special care is needed when using the previous equation since it states there is a positive probability no matter how large x_t. But this is not true since we know that the bound for $|x_t| \leq t$, so we need another equation to state this fact which is the following:

$$p(x,t) = 0, \qquad |x| > t. \tag{1.18}$$

Equation 1.17 is only valid for were $x = \mathcal{O}(\sqrt{t})$. But then we can obtain probability densities for values of $|x|$ greater than $\mathcal{O}(\sqrt{t})$. To do this, as Keller [5, p.1120] describes, consider the steps j to the right of the particle up to time t which will yield the following equation (notice in

10

the parentheses we are subtracting the step on the left):

$$x_{ij} = j - (t - j) = 2j - t. \tag{1.19}$$

The probability of x_{ij} by the binomial distribution is as follows:

$$p(x_{ij}, t) = \frac{t!}{j!(t-j)!2}, \quad j = 0, 1, \ldots, t \tag{1.20}$$

$$= \quad 0 \quad\quad j < 0 \text{ or } j > t.$$

A simplification of the previous equation which later will be used for the validation of the telegraph equation is obtained for large t. The procedure given according to Keller [5, p.1120] is by dividing by 2 and using the Stirling formula $(t - j)!$ and making $j = (x + t)/2$, as Keller [5, p.1120] points out, we obtain the following:

$$p(x_{ij}, t) \sim \frac{1}{\sqrt{2\pi t}} \left(1 + \frac{x}{t}\right)^{-\frac{1}{2}(t+x+1)} \left(1 - \frac{x}{t}\right)^{-\frac{1}{2}(t-x+1)} \quad \left|\frac{x}{t}\right| < 1,\ t \gg 1, \tag{1.21}$$

$$= \quad\quad\quad\quad 0 \quad\quad\quad\quad\quad \left|\frac{x}{t}\right| > 1.$$

The next step is to derive a better approximation than the diffusion equation. In order to do this we use the procedure denoted by Keller [5, p.1121]. We denote $x_j = j$, where j is an integer and t is a positive integer starting from zero. We then proceed to find the probability $p(x_j, t + 1)$ as follows:

$$p(x_j, t + 1) = \frac{1}{2}\left(p(x_{j-1}, t) + p(x_{j+1}, t)\right). \tag{1.22}$$

11

Subtracting by $p(x_j,t)$ we get

$$p(x_j,t+1) - p(x_j,t) \ = \ \frac{1}{2}\Big(p(x_{j-1},t) - 2p(x_j,t) + p(x_{j+1},t)\Big). \qquad (1.23)$$

As explained by Keller [5, p.1121] if $p(x,t)$ is smooth function on x and t then the derivatives of an order higher than two can be disregarded. This will yield:

$$\frac{\partial p(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial t^2} \ = \ \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}. \qquad (1.24)$$

This is the telegraph equation. This equation propagates p with speed less than 1 which is the same speed as the random walk steps. This is the result with finite speed which was required from the random walk. Keller [5, p.1121] illustrates the accuracy of this equation when he substitutes the large deviation approximation in the the telegraph left hand side and subtracts the right hand side substitution; the error is plotted in the Figure 1.4, and we see that the error is really low to the order of 10^{-7}. (There is a typo in Keller's [5, p.1121] plot in the magnitude, but the caption has the correct magnitude.)
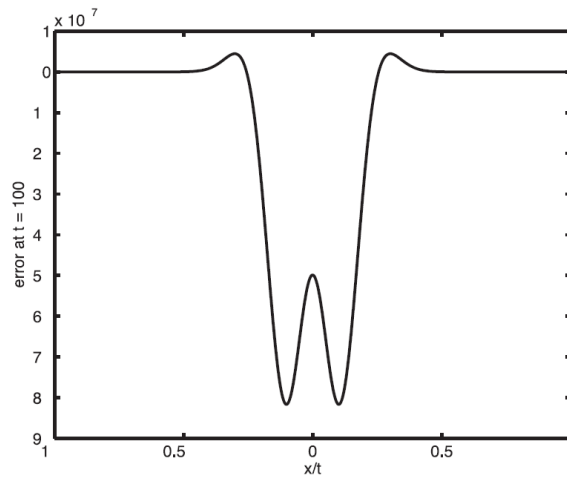
Figure 1.4: Telegraph Equation Error Graph by Keller [5, p.1121]

12

An important thing to notice is this is a natural way to get the telegraph equation from a random walk. The telegraph equation occurs in various physical phenomena, and that is the important use of this equations. Since it describes more accurately the random walk where it takes into account large deviation it stands to reason that is better suited to model the information system.

CHAPTER II

MODEL DESCRIPTION AND METHODS

In this chapter we will explain the details of the model, the numerical solution and algorithm.

2.1 Description of the Model

The following model was considered to solve numerically:

$$\frac{\partial S(x,t)}{\partial t} \;=\; -\beta\,(1 - p(x,t))S(x,t)I(t) - p(x,t)S(x,t) \tag{2.1}$$

$$\frac{dI(t)}{dt} \;=\; \beta I(t) \int_0^\infty (1 - p(x,t))S(x,t)dx - \gamma I(t) \tag{2.2}$$

$$\frac{dR(t)}{dt} \;=\; \int_0^\infty p(x,t)S(x,t)dx + \gamma I(t). \tag{2.3}$$

Here $S(x,t)$ is the density of the susceptible individuals who are at distance x from the center of information 0. $I(t)$ and $R(t)$ are the proportion of infected and recovered individuals in the population. $p(x,t)$ is the probability that an individual at distance x is going to accept to vaccinate at time t. The model diagram can be seen in Figure 2.1 inspired by the SIR model Figure 1.1 by Brauer and Castillo-Chavez [2, p.351]. The rate of change of the accumulated vaccinated individuals at a time t is given by $\frac{dV(t)}{dt} = \int_0^\infty p(x,t)S(x,t)dx.$
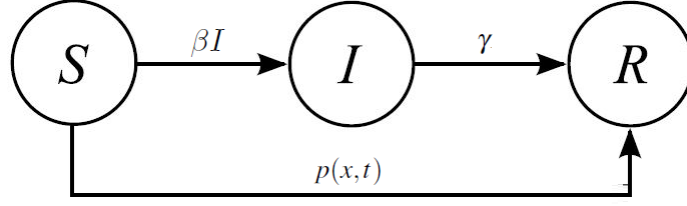
Figure 2.1: Model Diagram

The information about the disease and vaccine at time t and distance x are $M_I(x,t)$ and $M_V(x,t)$, respectively. Now the network of information will be modeled the following way:

$$\tau_I \frac{\partial^2 M_I}{\partial t^2} + \alpha_I \frac{\partial M_I}{\partial t} = D_I \frac{\partial^2 M_I}{\partial x^2} + \gamma_I (1 - M_I - M_V) M_I \qquad (2.4)$$

$$\tau_V \frac{\partial^2 M_v}{\partial t^2} + \alpha_V \frac{\partial M_V}{\partial t} = D_V \frac{\partial^2 M_v}{\partial x^2} + \gamma_V (1 - M_I - M_V) M_V. \qquad (2.5)$$

With $\tau_I \neq 0$ and $\tau_V \neq 0$, and the rest of the constants are arbitrary. From that type of information about the disease and vaccination, people make the decision to vaccinate with probability:

$$p(x,t) = \frac{1}{1 + e^{-c(M_I(x,t) - M_V(x,t))}}.$$

The probability being based in a logistic equation is a common practice when you having a competition of information as we can see in Oraby's paper in [8, p.4]. Here in the exponential we see the difference for $M_I(x,t)$ and $M_V(x,t)$. If the information about the disease is more prevalent, meaning that $M_I(x,t) > M_V(x,t)$ will make the decision to vaccinate more probable. If the secondary effects of the disease is more prevalent, meaning that $M_I(x,t) < M_V(x,t)$ will make the decision to vaccinate less probable. If information have the same impact meaning $M_I(x,t) = M_v(x,t)$ the probability is 0.5 as expected.

2.2 Numerical Method

The discretization that we are going to use is finite differentiation in space, so we can change this partial differential equation into a coupled Ordinary differential equation, then reduce it to a first order system and apply the fourth order Runge Kutta. This procedure is in fact called the method of lines for solving the partial differential equation. For some integrals in the main model we will use numerical integration, in particular the trapezoid rule which is already integrated in MATLAB to integrate a grid of lines.

2.2.1 Numerical Solving Methods for Partial Differential Equations

There are various methods for solving partial differential equations, but in our research we are going to use the 'Method of Lines' by which we discretize the space domain with finite difference which makes our partial differential equation into a Coupled ODE's. Afterwards, we use MATLAB ode45 into the the system of coupled ODEs time domain which uses explicit Runge Kutta of fourth order in the time domain.

The methods we usually compare are Crank-Nicholson which is a second order method in time, and is also implicit in time and the use of Linear State Space method, but since we usually encounter PDEs with nonlinear coefficients we discard this one. In summary we just need to compare to Crank-Nicholson since it is also widely popular.

It is common to use a known result, so we use the wave equation with initial value $U(x,0) = \sin(x)$. And as such we compare the two results: First we see our exact solution is given by Figure 2.2. Then we see the numerical results from the Method of Lines with a fourth order difference in time on the left and Crank-Nicholson on the right with their respective errors in Figure 2.3.
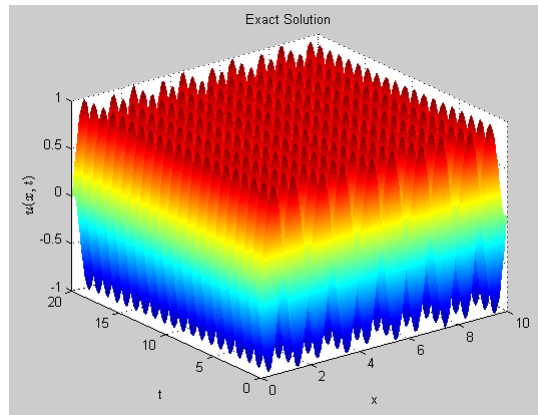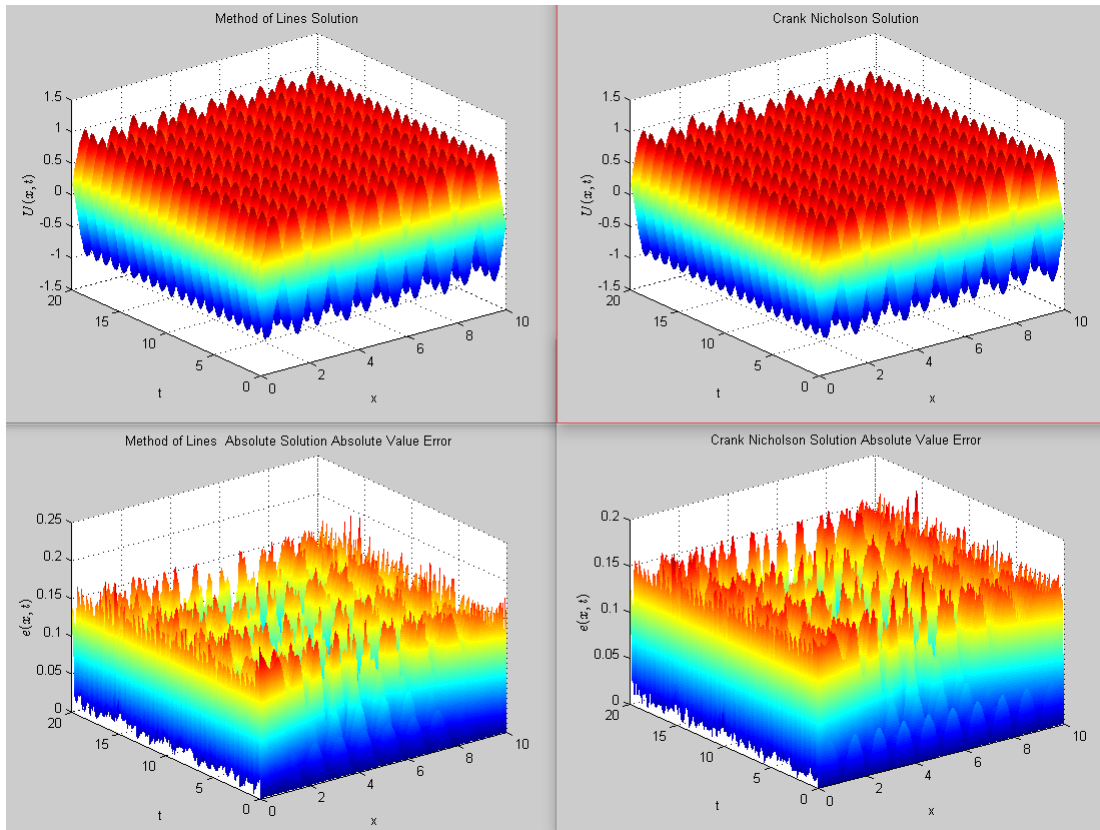
Figure 2.2: Wave Equation Results 1

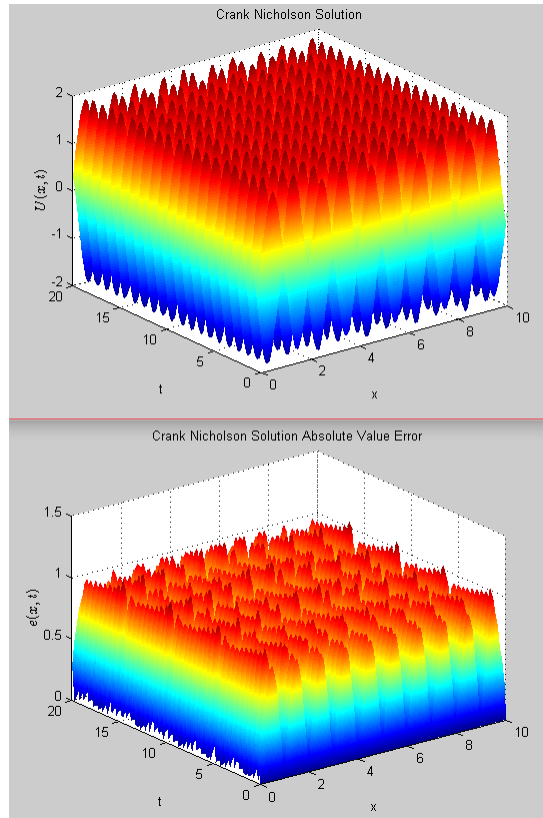Figure 2.3: Wave Equation Results 2

17

Figure 2.4: Wave Equation Results 3

We see that the errors are really similar, but as will be the case for stability sometimes it is required to increase the order of the discretization, in particular in the space dimension. For example, trying to make a higher order approximation for Crank-Nicholson immediately gives bad results (stable none the less).

With this in mind we see that although Crank-Nicholson is a good alternative we might want to stay in with the Method of Lines because at the very least we have more freedom modifying the numerical method to our requirements.

Regarding implementation speed, we compare this aspect when we implemented the Method of Lines and Crank-Nicholson to a Coupled Telegraph Fisher. We see that Crank-Nicholson is slower to implement, because there is a need to derive the matrices required for the calculations before starting coding, which takes extra time.

2.2.2 Space Discretization

In this subsection we will talk about how we do the discrertization. As we previously discussed, we are going to approximate the PDE of the telegraph equation into a coupled ODEs. Now from Fernberg [4, p.702,p.704] we have the following tables for central and forward differences (backward difference coefficients have the same value but for odd derivatives change sign):

Derivative	Accuracy	0	1	2	3	4	5	6	7
1	1	-1	1						
	2	-3/2	2	-1/2					
	3	-11/6	3	-3/2	1/3				
	4	-25/12	4	-3	4/3	-1/4			
	5	-137/60	5	-5	10/3	-5/4	1/5		
	6	-49/20	6	-15/2	20/3	-15/4	6/5	-1/6	
2	1	1	-2	1					
	2	2	-5	4	-1				
	3	35/12	-26/3	19/2	-14/3	11/12			
	4	15/4	-77/64	107/6	-13	61/12	-5/6		
	5	203/45	-87/5	117/4	-254/9	33/2	-27/5	137/180	
	6	469/90	-223/10	879/20	-949/18	-41	201/10	1019/180	-7/10

Table 2.1: Forward Difference Coefficients Table by Fornberg [4, p.704]

Derivative	Accuracy	-4	-3	-2	-1	0	1	2	3	4
1	2				-1/2	0	1/2			
	4			1/12	-2/3	0	2/3	-1/12		
	6		-1/60	3/20	-3/4	0	3/4	-3/20	1/60	
	8	1/280	-4/105	1/5	-4/5	0	4/5	-1/5	4/105	-1/280
2	2				1	-2	1			
	4			-1/12	4/3	-5/2	4/3	-1/12		
	6		1/90	-3/20	3/2	-49/18	3/2	-3/20	1/90	
	8	-1/560	8/315	-1/5	8/5	-205/72	8/5	-1/5	4/105	-1/560

Table 2.2: Central Difference Coefficients Table by Fornberg [4, p.702]

Having these coefficients we are going to set the coupled approximation of the PDE of the telegraph as a coupled first order ODE with finite difference, depending on what point you are in the boundary. Notice that the values of M_I and M_v come from previous time steps, as well as the time derivatives used in the calculation. The space coordinates will be denoted as x_k for clarity.

If we are at $k = 1$ a fourth order accuracy forward differences was chosen, and thus we will have our telegraph coupled ODE as follows:

$$
\begin{aligned}
\frac{d^2 M_{I,k}}{dt^2} = & \frac{D_I}{\tau_I} \left(M_{I,k} - 2M_{I,k+1} \right. \\
& \left. + M_{I,k+2} \right) / \Delta x^2 \\
& + \frac{\gamma_I}{\tau_I} \left(1 - M_{I,k} - M_{V,k} \right) M_{I,k} \\
& - \frac{\alpha_I}{\tau_I} \frac{dM_{I,k}}{dt}
\end{aligned}
\tag{2.6}
$$

$$\frac{d^2 M_{V,k}}{dt^2} = \frac{D_V}{\tau_V} \left(M_{V,k} - 2M_{V,k+1} \right.$$
$$+ M_{V,k+2} \left. \right) / \Delta x^2 \quad (2.7)$$
$$+ \frac{\gamma_V}{\tau_V} \left(1 - M_{I,k} - M_{V,k} \right) M_{V,k}$$
$$- \frac{\alpha_V}{\tau_V} \frac{dM_{V,k}}{dt}.$$

If we are at $k = 2$ or $k = n - 1$ a second order central differences accuracy was chosen, and thus we will have our telegraph coupled ODE as follows:

$$\frac{d^2 M_{I,k}}{dt^2} = \frac{D_I}{\tau_I} \left(M_{I,k-1} - 2M_{I,k} \right.$$
$$+ M_{I,k+1} \left. \right) / \Delta x^2 \quad (2.8)$$
$$+ \frac{\gamma_I}{\tau_I} \left(1 - M_{I,k} - M_{V,k} \right) M_{I,k}$$
$$- \frac{\alpha_I}{\tau_I} \frac{dM_{I,k}}{dt}$$

$$\frac{d^2 M_{V,k}}{dt^2} = \frac{D_V}{\tau_V} \left(M_{V,k-1} - 2M_{V,k} \right.$$
$$+ M_{V,k+1} \left. \right) / \Delta x^2 \quad (2.9)$$
$$+ \frac{\gamma_V}{\tau_V} \left(1 - M_{I,k} - M_{V,k} \right) M_{V,k}$$
$$- \frac{\alpha_V}{\tau_V} \frac{dM_{V,k}}{dt}.$$

If we are at $k = 3$, $k = 4$, $k = n - 3$ or $k = n - 2$ a fourth order central differences accuracy was chosen, and thus we will have our telegraph coupled ODE as follows:

$$\frac{d^2 M_{I,k}}{dt^2} = \frac{D_I}{\tau_I}\left(-\frac{1}{12}M_{I,k-2} + \frac{4}{3}M_{I,k-1}\right.$$
$$-\frac{5}{2}M_{I,k} + \frac{4}{3}M_{I,k+1}$$
$$\left.-\frac{1}{12}M_{I,k+2}\right)/\Delta x^2 \qquad (2.10)$$
$$+\frac{\gamma_I}{\tau_I}\left(1 - M_{I,k} - M_{V,k}\right)M_{I,k}$$
$$-\frac{\alpha_I}{\tau_I}\frac{dM_{I,k}}{dt}$$

$$\frac{d^2 M_{V,k}}{dt^2} = \frac{D_V}{\tau_V}\left(-\frac{1}{12}M_{V,k-2} + \frac{4}{3}M_{V,k-1}\right.$$
$$-\frac{5}{2}M_{V,k} + \frac{4}{3}M_{V,k+1}$$
$$\left.-\frac{1}{12}M_{V,k+2}\right)/\Delta x^2 \qquad (2.11)$$
$$+\frac{\gamma_V}{\tau_V}\left(1 - M_{I,k} - M_{V,k}\right)M_{V,k}$$
$$-\frac{\alpha_V}{\tau_V}\frac{dM_{V,k}}{dt}.$$

If we are at $k = n$ a fourth order accuracy backward differences was chosen, and thus we will have our telegraph coupled ODE as follows:

$$\frac{d^2 M_{I,k}}{dt^2} = \frac{D_I}{\tau_I}\left(M_{I,k} - 2M_{I,k-1}\right.$$
$$\left.+M_{I,k-2}\right)/\Delta x^2 \qquad (2.12)$$
$$+\frac{\gamma_I}{\tau_I}\left(1 - M_{I,k} - M_{V,k}\right)M_{I,k}$$
$$-\frac{\alpha_I}{\tau_I}\frac{dM_{I,k}}{dt}$$

$$\frac{d^2 M_{V,k}}{dt^2} = \frac{D_V}{\tau_V} \left(M_{V,k} - 2M_{V,k-1} \right.$$
$$+ M_{V,k-2} \left. \right) / \Delta x^2 \qquad (2.13)$$
$$+ \frac{\gamma_V}{\tau_V} \left(1 - M_{I,k} - M_{V,k} \right) M_{V,k}$$
$$- \frac{\alpha_V}{\tau_V} \frac{dM_{V,k}}{dt}.$$

For every other point we will use eight order central differences as follows, and we will have our telegraph coupled ODE as follows:

$$\frac{d^2 M_{I,k}}{dt^2} = \frac{D_I}{\tau_I} \left(-\frac{1}{560} M_{I,k-4} + \frac{8}{315} M_{I,k-3} \right.$$
$$-\frac{1}{5} M_{I,k-2} + \frac{8}{5} M_{I,k-1}$$
$$-\frac{205}{72} M_{I,k} + \frac{8}{5} M_{I,k+1}$$
$$-\frac{1}{5} M_{I,k+2} + \frac{8}{315} M_{I,k+3} \qquad (2.14)$$
$$-\frac{1}{560} M_{I,k+4} \left. \right) / \Delta x^2$$
$$+ \frac{\gamma_I}{\tau_I} \left(1 - M_{I,k} - M_{V,k} \right) M_{I,k}$$
$$- \frac{\alpha_I}{\tau_I} \frac{dM_{I,k}}{dt}$$

23

$$
\begin{aligned}
\frac{d^2 M_{V,k}}{dt^2} \;=\; & \frac{D_V}{\tau_V}\Bigg(-\frac{1}{560}M_{V,k-4}+\frac{8}{315}M_{V,k-3} \\
& -\frac{1}{5}M_{V,k-2}+\frac{8}{5}M_{V,k-1} \\
& -\frac{205}{72}M_{V,k}+\frac{8}{5}M_{V,k+1} \\
& -\frac{1}{5}M_{V,k+2}+\frac{8}{315}M_{V,k+3} \\
& -\frac{1}{560}M_{V,k+4}\Bigg)/\Delta x^2 \\
& +\frac{\gamma_V}{\tau_V}\left(1-M_{I,k}-M_{V,k}\right)M_{V,k} \\
& -\frac{\alpha_V}{\tau_V}\frac{dM_{V,k}}{dt}.
\end{aligned}
\tag{2.15}
$$

Finally after describing these equations we can calculate the probability as described at the start of the section. The next step is to calculate the values of the derivatives of S, I and R at the time step. Note that we use the values of the previous step to make the calculation with the probability of the actual step. Let us denote the function as in MATLAB notation as $\text{trapz}(X)$ which is the numerical integration of the discrete values of the vector X. Products in MATLAB are point wise, so if we input a product of two vectors XY in $\text{trapz}(XY)$ it will be valid. Thus we have that:

$$
\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= -\beta(1-p)SI-pS \\
\frac{dI(t)}{dt} &= \beta I(t)\text{trapz}\left((1-p)S\right)-\gamma I \\
\frac{dR(t)}{dt} &= \text{trapz}\left(pS\right)+\gamma I(t).
\end{aligned}
\tag{2.16}
$$

This completes the Runge Kutta step to calculate the time derivatives.

2.3 Algorithm

We are going to explain the general algorithm in MATLAB, in which case there are two main ones:

Pseudocode 1 Main program

begin

Initialize the constant variables

Discretize the space and time dimension

Create vectors T and X of the discretization

Create initial conditions for M_I and M_v

Create initial conditions for $S(x,t)$, $I(t)$ and $R(t)$

Save all initial conditions in a vector y_0 for the solver

Use ode45 with the discretize model with initial conditions

Reassign variables of the solution y which the solver gives

Graph results

end

We will include an initial condition for a second test defined as:

$$\frac{\partial M_I}{\partial t} = I(t) \tag{2.17}$$

$$\frac{\partial M_V}{\partial t} = V(t). \tag{2.18}$$

Now the next algorithm is applied in each of the time step t, in which case we approximately calculate the derivatives along the x axis with our space discretization values at that time step. This is used for the solver ode45 from MATLAB. The algorithm is as follows:

Algorithm 1 Function algorithm for ode45 solver

Input t, y, X, dx

Output $\frac{dy}{dt}$

begin

Reassignation of the variables from y

for $x \in X$

Calculate the derivatives M_I and M_v at the time step

The description of the calculation is explained in the numerical method subsection

end

Calculate the probability P

If $I < 0$ from previous time step then $I = 0$

Calculate the derivatives of S, I, R

Reassign all results to the output $\frac{dy}{dt}$

end

CHAPTER III

RESULTS AND DISCUSSION

In the previous section we discussed a numerical approach to solving the model. In this section we will talk about exact solutions for the telegrapher equation, a solution for a coupled model with some special coefficients and some simulations with different scenarios and discuss the results obtained.

3.1 Explicit Solutions-Traveling Waves

Theorem 1 *The following is an arbitrary nonlinear telegraph equation:*

$$\tau\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} \;=\; D\frac{\partial^2 u}{\partial x^2} + \gamma(1 - u^p)u \tag{3.1}$$

Looking for a solution for $u(x,t) = f(kx - \omega t) = f(z)$ with boundary conditions $f(-\infty) = 1$ and $f(\infty) = 0$. If $Dk^2 - \tau\omega^2 = 1$ then it has the Ablowitz-Zappatella type solution

$$f(z) \;=\; \left(1 + me^{\pm p\sqrt{\frac{\gamma}{4+2p}}z}\right)^{-\frac{2}{p}}$$

with

$$z \;=\; kx - \omega t$$

$$\omega_{1,2} \;=\; \pm\frac{4+p}{\alpha}\sqrt{\frac{\gamma}{4+2p}}$$

$$k \;=\; \sqrt{\frac{\tau\omega^2 + 1}{D}}.$$

Proof: Making $u(x,t) = f(kx - \omega t) = f(z)$, thus

$$\frac{\partial u}{\partial t} = \frac{df}{dz}\frac{dz}{dt} = \frac{df}{dz}(-\omega) = -\omega\frac{df}{dz} \tag{3.2}$$

$$\frac{\partial^2 u}{\partial t^2} = \omega^2\frac{d^2 f}{dz^2} \tag{3.3}$$

$$\frac{\partial u}{\partial x} = k\frac{df}{dz} \tag{3.4}$$

$$\frac{\partial^2 u}{\partial x^2} = k^2\frac{d^2 f}{dz^2} \tag{3.5}$$

$$u^{p+1} = (f(z))^{p+1}, \tag{3.6}$$

but making

$$Dk^2 - \tau\omega^2 = 1 \tag{3.7}$$

$$k = \sqrt{\frac{\tau\omega^2 + 1}{D}}, \tag{3.8}$$

thus we have

$$\tau\omega^2\frac{d^2 f}{dz^2} - \alpha\omega\frac{df}{dz} = Dk^2\frac{d^2 f}{dz^2} + \gamma f - \gamma f^{p+1} \tag{3.9}$$

$$(\tau\omega^2 - Dk^2)\frac{d^2 f}{dz^2} - \alpha\omega\frac{df}{dz} - \gamma f + \gamma f^{p+1} = 0 \tag{3.10}$$

$$(Dk^2 - \tau\omega^2)\frac{d^2 f}{dz^2} + \alpha\omega\frac{df}{dz} + \gamma f - \gamma f^{p+1} = 0 \tag{3.11}$$

$$\frac{d^2 f}{dz^2} + \frac{\alpha\omega}{Dk^2 - \tau\omega^2}\frac{df}{dz} + \frac{\gamma}{Dk^2 - \tau\omega^2}f - \frac{\gamma}{Dk^2 - \tau\omega^2}f^{p+1} = 0. \tag{3.12}$$

Since $Dk^2 - \tau\omega^2 = 1$

$$\frac{d^2 f}{dz^2} + \alpha\omega\frac{df}{dz} + \gamma f - \gamma f^{p+1} = 0. \tag{3.13}$$

Following Ablowitz-Zappatella [1, p.383] we start with:

28

$$\tau\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \gamma(1 - u^p)u \quad\Longrightarrow\quad \frac{d^2 f}{dz^2} + \alpha\omega\frac{df}{dz} + \gamma f - \gamma f^{p+1} = 0. \qquad (3.14)$$

We look for a solution of the form:

$$f(z) \;=\; \left(1 + me^{nz}\right)^{-r} \qquad m, n > 0 \qquad r > 0. \qquad (3.15)$$

For (3.13) we have $z = kx - \omega t$, so we obtain

$$\frac{df}{dz} \;=\; -r\left(1 + me^{nz}\right)^{-r-1}\left(nme^{nz}\right) \qquad (3.16)$$

$$\frac{d^2 f}{dz^2} \;=\; -r(-r-1)\left(1 + me^{nz}\right)^{-r-2}\left(nme^{nz}\right)\left(nme^{nz}\right) \qquad (3.17)$$

$$-r\left(1 + me^{nz}\right)^{-r-1}\left(n^2 me^{nz}\right)$$

$$=\; r(r+1)\left(1 + me^{nz}\right)^{-r-2}n^2 m^2 e^{2nz} \qquad (3.18)$$

$$-r\left(1 + me^{nz}\right)^{-r-1}\left(n^2 me^{nz}\right)$$

$$f^{p+1} \;=\; \left(1 + me^{nz}\right)^{-r(p+1)}. \qquad (3.19)$$

So (3.13) becomes

$$\frac{d^2 f}{dz^2} + \alpha\omega\frac{df}{dz} + \gamma f - \gamma f^{p+1} \;=\; r(r+1)\left(1 + me^{nz}\right)^{-r-2}n^2 m^2 e^{2nz}$$

$$-r\left(1 + me^{nz}\right)^{-r-1}\left(n^2 me^{nz}\right)$$

$$-r\alpha\omega\left(1 + me^{nz}\right)^{-r-1}\left(nme^{nz}\right) \qquad (3.20)$$

$$+\gamma\left(1 + me^{nz}\right)^{-r}$$

$$-\gamma\left(1 + me^{nz}\right)^{-r(p+1)},$$

29

but then if we factor $(1+me^{nz})^{-r-2}$ we will get

$$
\begin{aligned}
\frac{d^2 f}{dz^2} + \alpha\omega\frac{df}{dz} + \gamma f - \gamma f^{p+1} \; &= \; (1+me^{nz})^{-r-2}\Big[r(r+1)n^2 m^2 e^{2nz} \\
&\quad -r(1+me^{nz})(n^2 me^{nz}) \\
&\quad -r\alpha\omega(1+me^{nz})(nme^{nz}) \\
&\quad +\gamma(1+me^{nz})^2 \\
&\quad -\gamma(1+me^{nz})^{-rp+2}\Big] \\[4pt]
&= \; (1+me^{nz})^{-r-2}\Big[r(r+1)n^2 m^2 e^{2nz} \\
&\quad -rn^2 me^{nz} - rn^2 m^2 e^{2nz} \\
&\quad -r\alpha\omega nme^{nz} - r\alpha\omega nm^2 e^{2nz} \\
&\quad +\gamma + 2\gamma me^{nz} + \gamma m^2 e^{2nz} \\
&\quad -\gamma(1+me^{nz})^{-rp+2}\Big] \\[4pt]
&= \; (1+me^{nz})^{-r-2}\Big[m^2 e^{2nz}\big(r(r+1)n^2 - rn^2 - r\alpha\omega n + \gamma\big) \\
&\quad me^{nz}\big(-rn^2 - r\alpha\omega n + 2\gamma\big) \\
&\quad +\gamma - \gamma(1+me^{nz})^{-rp+2}\Big].
\end{aligned}
$$

(3.21)

Choosing to solve $\frac{d^2 f}{dz^2} + \alpha\omega\frac{df}{dz} + \gamma f - \gamma f^{p+1} = 0$ we have

$$
r(r+1)n^2 - rn^2 - r\alpha\omega n + \gamma \;=\; 0 \tag{3.22}
$$

$$
-rn^2 - r\alpha\omega n + \gamma \;=\; -\gamma. \tag{3.23}
$$

Subtracting them we get

$$r(r+1)n^2 - rn^2 - r\alpha\omega n + \gamma \;=\; 0 \tag{3.24}$$

$$+rn^2 + r\alpha\omega n - \gamma \;=\; \gamma. \tag{3.25}$$

which will yield

$$r(r+1)n^2 - \gamma \;=\; 0 \tag{3.26}$$

$$r(r+1)n^2 \;=\; \gamma. \tag{3.27}$$

With

$$-rp + 2 \;=\; 0 \tag{3.28}$$

it will yield

$$r \;=\; \frac{2}{p}, \tag{3.29}$$

we will have

$$\frac{2}{p}\left(\frac{2}{p}+1\right)n^2 \;=\; \gamma \tag{3.30}$$

$$\frac{2}{p}\left(\frac{2+p}{p}\right)n^2 \;=\; \gamma \tag{3.31}$$

$$\left(\frac{4+2p}{p^2}\right)n^2 \;=\; \gamma. \tag{3.32}$$

Thus we have

$$n \;=\; \pm\sqrt{\frac{p^2\gamma}{4+2p}} \tag{3.33}$$

$$\;=\; \pm p\sqrt{\frac{\gamma}{4+2p}}. \tag{3.34}$$

31

Notice that for ω we have

$$
\begin{aligned}
r(r+1)n^2 - rn^2 - r\alpha\omega n + \gamma &= 0 \\
rn^2 + r\alpha\omega n &= 2\gamma \\
\frac{2}{p}\frac{p^2\gamma}{4+2p} + \frac{2}{p}\alpha\omega\left(\pm p\sqrt{\frac{\gamma}{4+2p}}\right) &= 2\gamma \\
\frac{p\gamma}{4+2p} + \alpha\omega \pm \sqrt{\frac{\gamma}{4+2p}} &= \gamma \\
\alpha\omega \pm \sqrt{\frac{\gamma}{4+2p}} &= \gamma - \frac{p\gamma}{4+2p} \\
\alpha\omega \pm \sqrt{\frac{\gamma}{4+2p}} &= \frac{4\gamma + 2p\gamma - p\gamma}{4+2p} \\
\alpha\omega \pm \sqrt{\frac{\gamma}{4+2p}} &= \gamma\frac{4+p}{4+2p} \\
\omega &= \pm\frac{4+p}{\alpha}\sqrt{\frac{\gamma}{4+2p}}.
\end{aligned}
\tag{3.35}
$$

We see that there are two speeds (forward direction and a backward direction):

$$
\omega_{1,2} = \pm\frac{4+p}{\alpha}\sqrt{\frac{\gamma}{4+2p}}.
\tag{3.36}
$$

But then

$$
\begin{aligned}
k &= \sqrt{\frac{\tau\omega^2 + 1}{D}} \\
&= \sqrt{\frac{\tau\left(\pm\frac{4+p}{\alpha}\sqrt{\frac{\gamma}{4+2p}}\right)^2 + 1}{D}} \\
&= \sqrt{\frac{\frac{(4+p)^2}{\alpha^2(4+2p)}\tau\gamma + 1}{D}}.
\end{aligned}
\tag{3.37}
$$

Therefore a solution is given by

$$
f(z) = \left(1 + me^{\pm p\sqrt{\frac{\gamma}{4+2p}}z}\right)^{-\frac{2}{p}}.
\tag{3.38}
$$

32

Notice that for $p = 1$ we will have

$$f(z) = \left(1 + me^{\pm\sqrt{\frac{\gamma}{6}}z}\right)^{-2},$$

(3.39)

result used in the next section.

3.1.1 A Coupled Model

We can make a solution of the difference of the coupled telegraph for some special coefficients.

Theorem 2 *If we have the following coupled telegraph equation*

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (1 - u + v)u$$

(3.40)

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (1 - u + v)v$$

(3.41)

with boundary conditions $u(-\infty) = 1 + k\ u(\infty) = L, v(-\infty) = k, v(\infty) = L$, then we have that $u - v = \left(1 + me^{\pm\sqrt{\frac{1}{6}}z}\right)^{-2}$ with $z = kx - \omega t$.

Proof: Taking the difference we will get

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 v}{\partial t^2} + \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 v}{\partial x^2}$$
$$+ (1 - u + v)u - (1 - u + v)v$$
$$\frac{\partial^2 (u - v)}{\partial t^2} + \frac{\partial (u - v)}{\partial t} = \frac{\partial^2 (u - v)}{\partial x^2} + (1 - u + v)(u - v)$$
$$\frac{\partial^2 (u - v)}{\partial t^2} + \frac{\partial (u - v)}{\partial t} = \frac{\partial^2 (u - v)}{\partial x^2} + (1 - (u - v))(u - v).$$

(3.42)

Making $f = u - v$ we will have

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + (1 - f)f,$$

(3.43)

for which we already have an explicit solution explained in Theorem 1. This solution can

be used to obtain the values of the probability p.

3.2 Numerical Solution

Limitations of explicit solution are serious; for example if we change the coefficients in our model we will stop having explicit solutions. In this section we will show the numerical solution, which avoids these kind of problems in exchange accuracy as discussed in the previous sections.

Since we already have our model, numerical method, and our algorithms we can now start programming our solution into MATLAB. It is important to create two files for the main program and for the ode45 solver which are named SIRTel1 and odefcn respectively. It is important to note that this program will solve the telegraph and the SIR model at the same time.

It is important to have these two files in the same folder as in Figure 3.1 so it can be run in MATLAB main program.
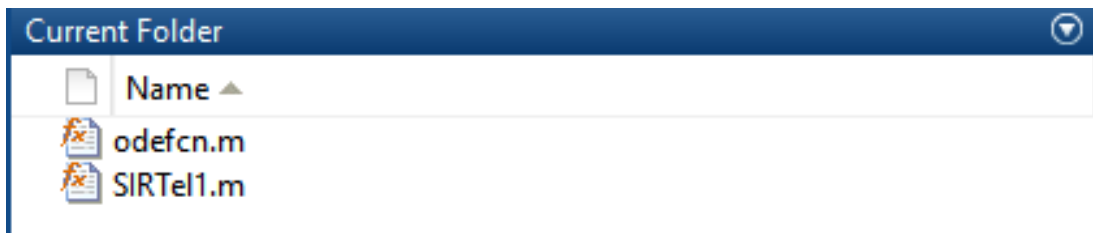
Figure 3.1: MATLAB Folder With Files

The parameters used in the telegraph equation are the following:

$$\begin{aligned} \tau_I &= 1 & \tau_V &= 1 \\ \alpha_I &= 1 & \alpha_V &= 1 \\ D_I &= 1 & D_V &= 1 \\ \gamma_I &= 1 & \gamma_V &= 1. \end{aligned} \tag{3.44}$$

The initial distribution used in all simulations for $S(x,0)$ is a scaled Beta(1,2). The parameter used in the probability is $c = 1$ and the parameters used for the SIR model are $\beta = 1$ and $\gamma = 1$.

34

Now for the time and space discretization we have these parameters:

$$
\begin{aligned}
n &= 200 \\
m &= 100 \\
x_0 &= 0 \\
x_n &= 10 \\
t_0 &= 0 \\
t_m &= 1.5. \\
S(0) &= 0.6 \\
I(0) &= 0.3
\end{aligned}
\tag{3.45}
$$

Now we see the following results from the MATLAB code:
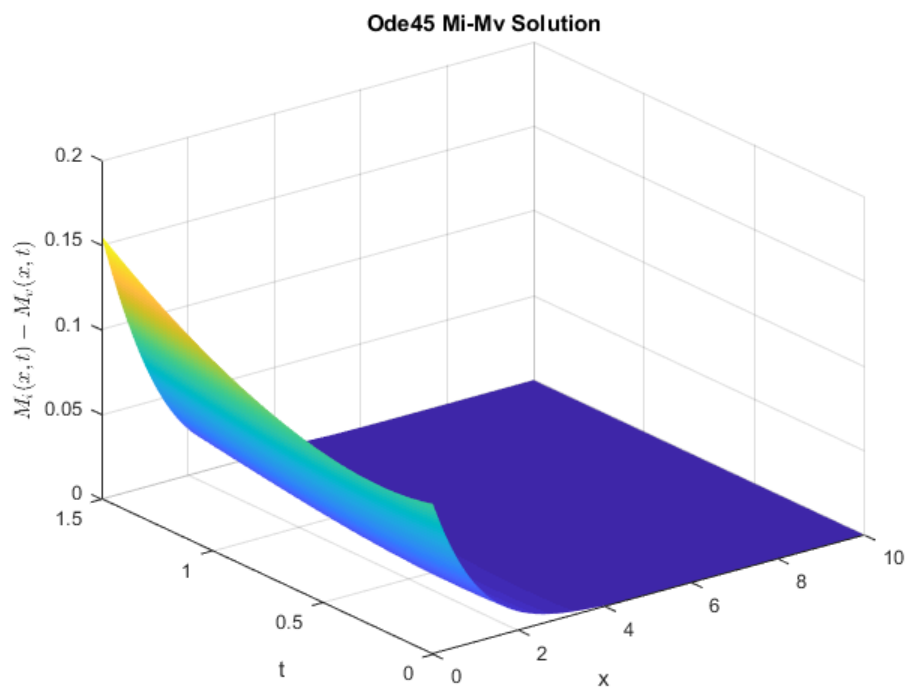

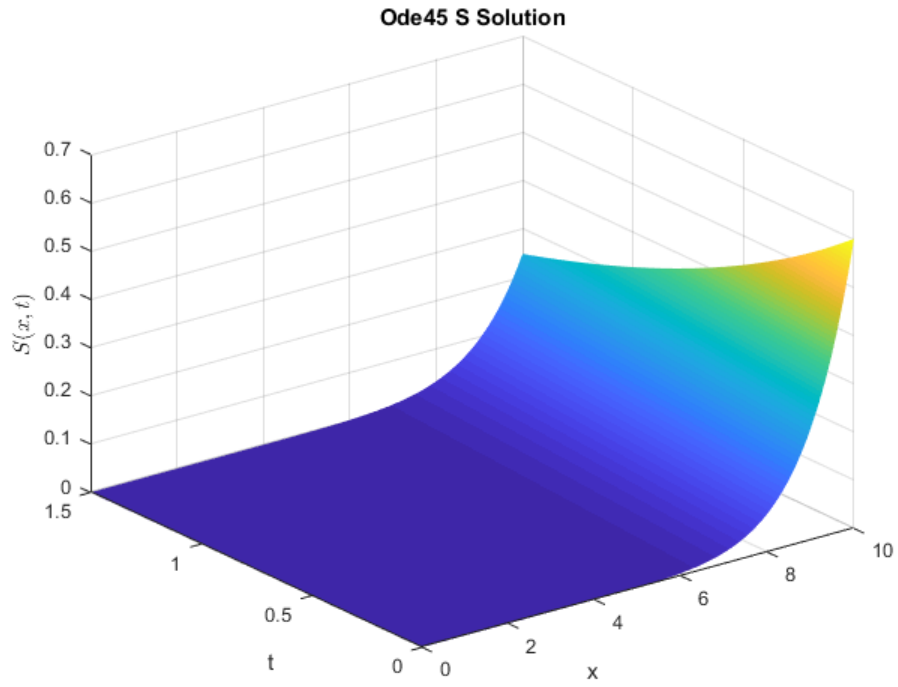Figure 3.2: $M_I(x,t) - M_v(x,t)$ Solution 1
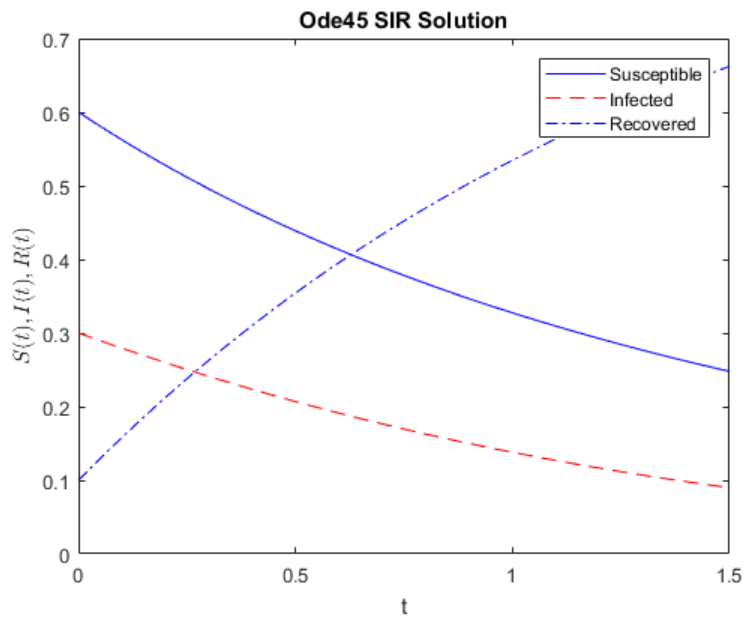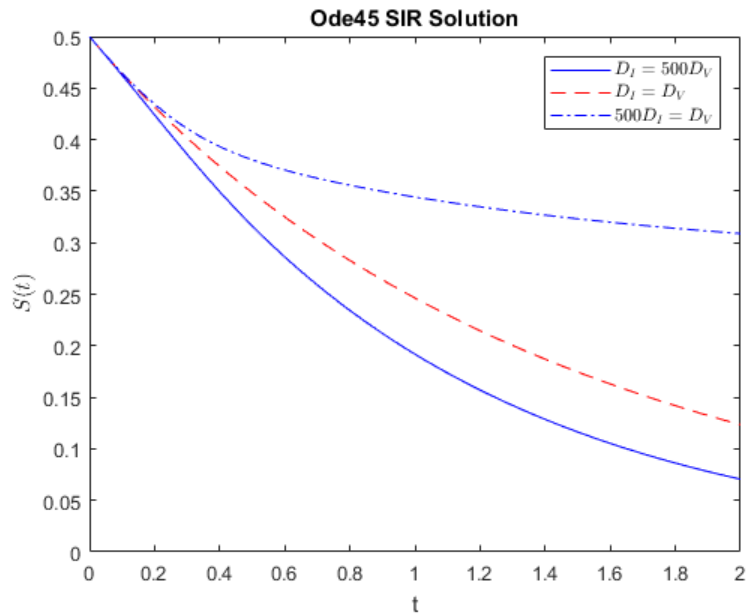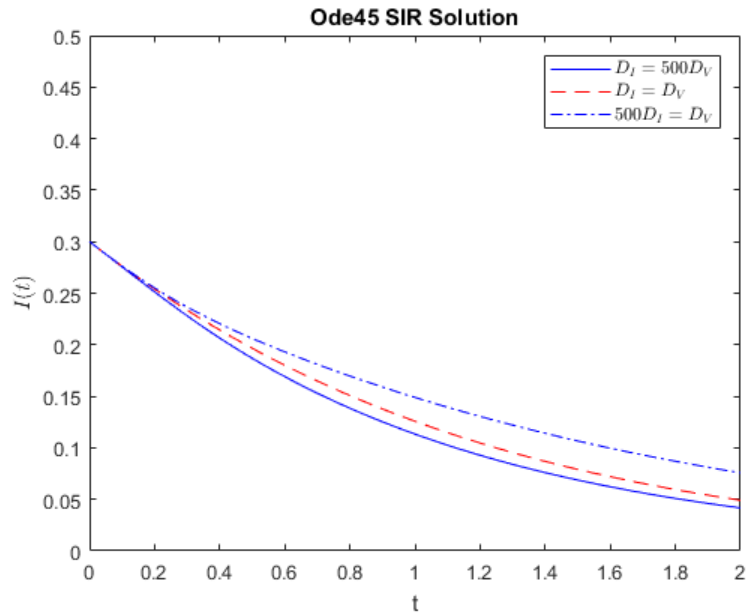
Figure 3.3: $S(x,t)$ Solution 1

Figure 3.4: *SIR* Solution 1

3.3 Simulation Analysis

In order to see if the telegraph indeed makes an impact to the model we are going to plot several solutions wherein all the parameters in the telegraph equations are one, except we vary the diffusion coefficient as $D_I = 500D_v$, $D_I = D_V$ and $500D_I = D_V$. The time and space discretization $n = 200$, $m = 50$, $x_0 = 0$, $x_n = 10$, $t_0 = 0$ and $t_m = 2$.. Here the time units are in days. The parameter used for the probability is $c = 50$ and the parameters used for the SIR model are $\beta = 1$, $\gamma = 1$.

Figure 3.5: $S(t)$ Solution 1

Figure 3.6: $I(t)$ Solution 1

In the next simulation the parameter used for the probability will remain the same and the parameters used for the SIR model are $\beta = 10$, $\gamma = 1$.
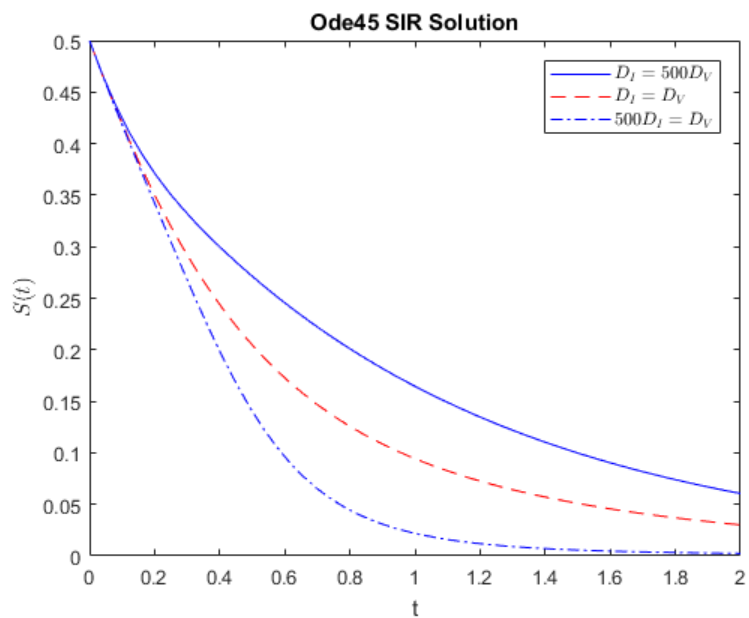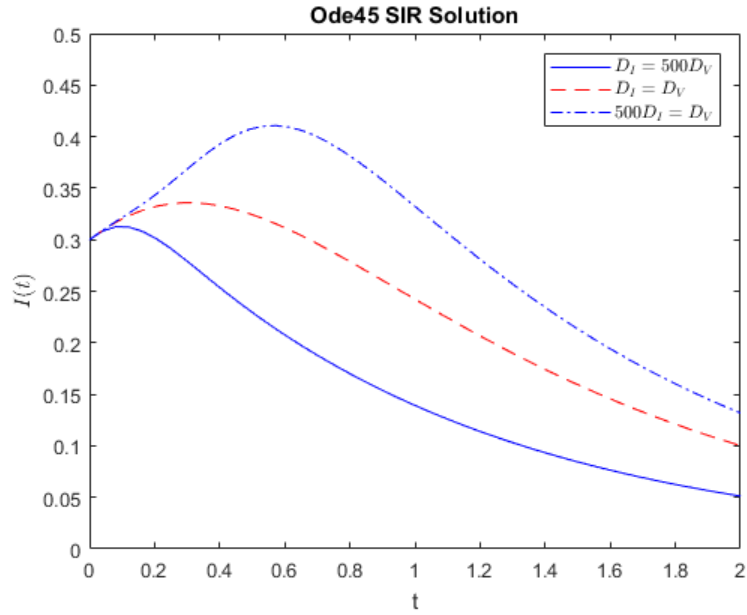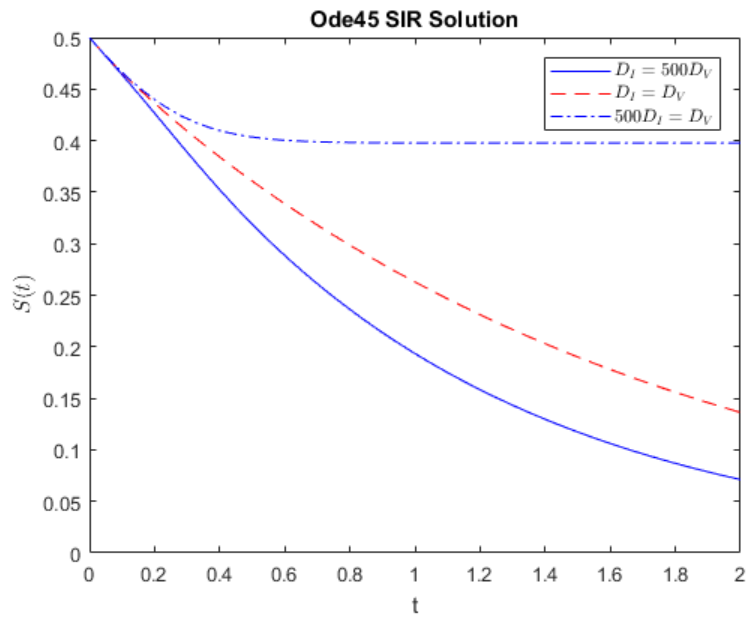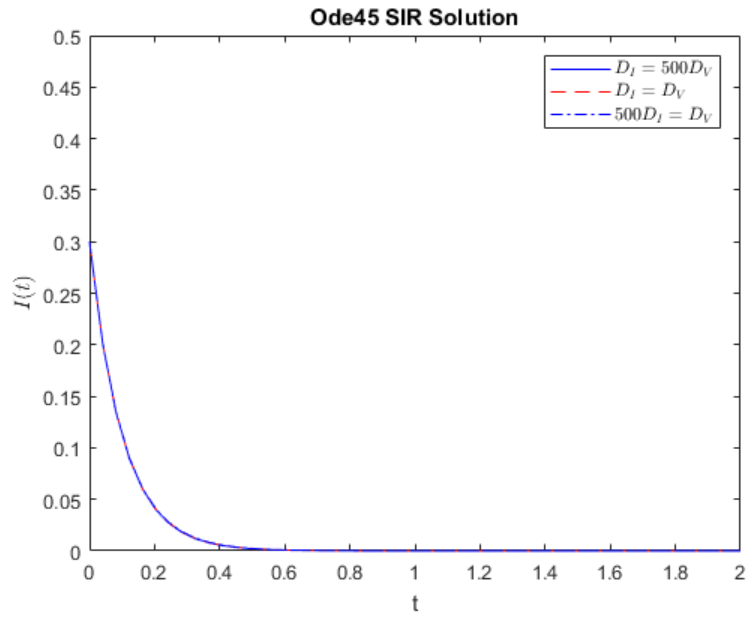

Figure 3.7: $S(t)$ Solution 2

Figure 3.8: $I(t)$ Solution 2

In the next simulation the parameter used for the probability will remain the same and the parameters used for the SIR model are β = 1, γ = 10.

Figure 3.9: $S(t)$ Solution 3

Figure 3.10: $I(t)$ Solution 3

Now will run a long term simulation with more realistic values of β and γ for the time and space discretization we have these parameters:

$$
\begin{aligned}
n &= 200 \\
m &= 50 \\
x_0 &= -50 \\
x_n &= 50 \\
t_0 &= 0 \\
t_m &= 90 \\
S(0) &= 0.5 \\
I(0) &= 0.3.
\end{aligned}
$$

The parameter used for the probability is $c = 50$ and the parameters used for the SIR model are $\beta = 1/20$, $\gamma = 1/20$.
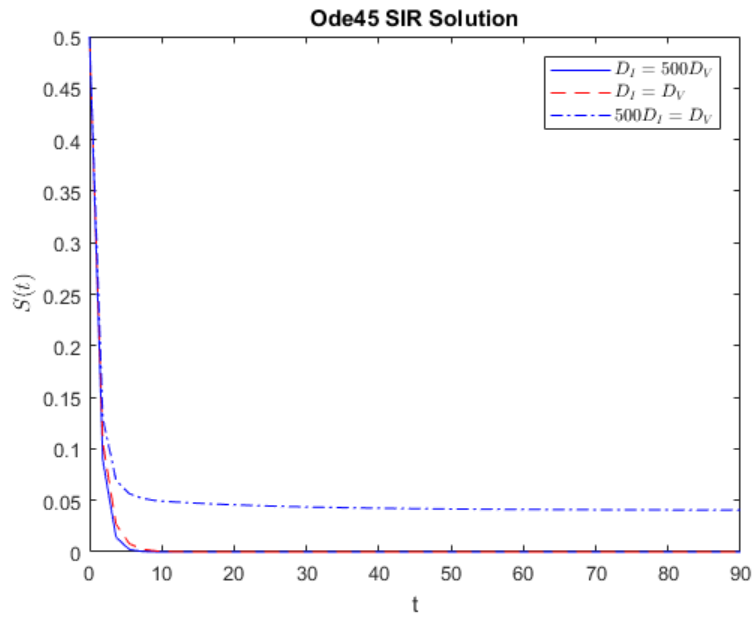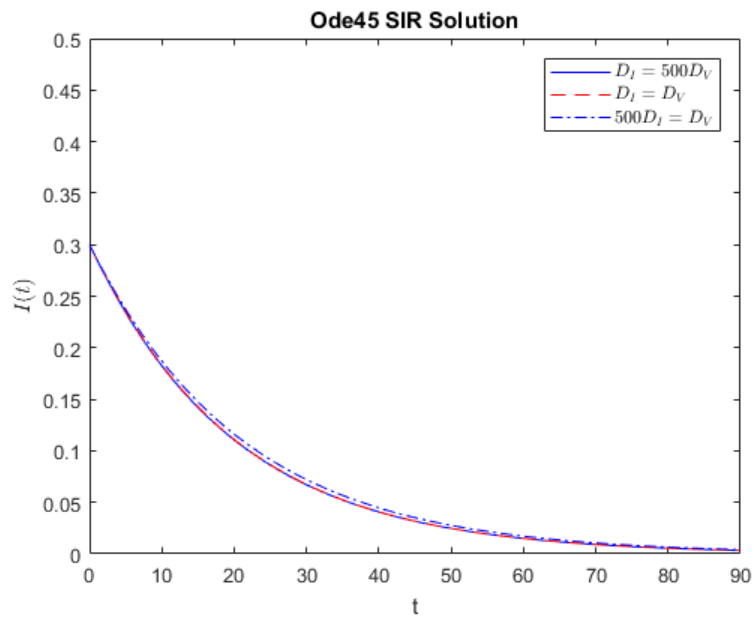

Figure 3.11: $S(t)$ Solution 4

Figure 3.12: $I(t)$ Solution 4

In the next simulation the parameter used for the probability will remain the same and the parameters used for the SIR model are $\beta = .8$, $\gamma = 1/20$.
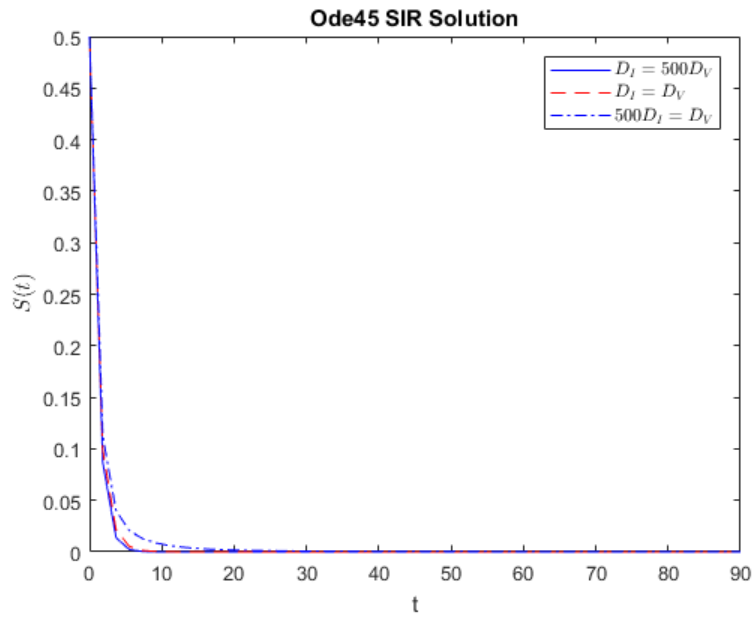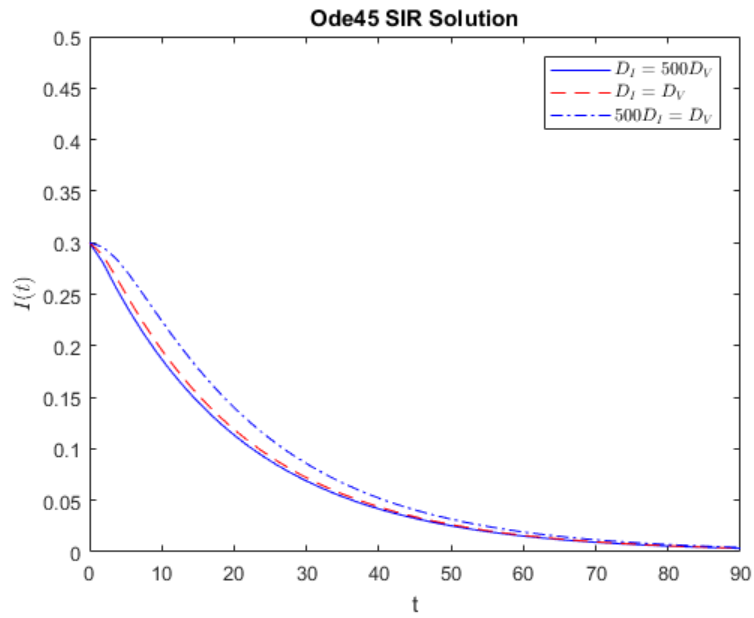

Figure 3.13: $S(t)$ Solution 5

Figure 3.14: $I(t)$ Solution 5

We see the behavior as expected in this extreme example. The telegraph equation does affect the distribution of the probability of vaccination and does have a direct impact to the *SIR* model as a whole.

3.4 Asymptotic Behavior

In this section we are going to analyze the stable points in the SIR model.

Theorem 3 *The following SIR model*

$$\frac{dS(t)}{dt} = -\beta I(t) \int_0^\infty (1-p(x,t))S(x,t)dx - \int_0^\infty p(x,t)S(x,t)dx \tag{3.46}$$

$$\frac{dI(t)}{dt} = \beta I(t) \int_0^\infty (1-p(x,t))S(x,t)dx - \gamma I(t) \tag{3.47}$$

$$\frac{dR(t)}{dt} = \int_0^\infty p(x,t)S(x,t)dx + \gamma I(t) \tag{3.48}$$

with $I(t),S(t)$ and $R(t)$ are non-negative and with positive transmission parameter $\beta > 0$ and recovery rate $\gamma > 0$. If the initial conditions $I(0) > 0$ and $S(0) > 0$, then the following limits are true:

$$\lim_{t \to \infty} S(t) = S_\infty \tag{3.49}$$

$$\lim_{t \to \infty} I(t) = 0 \tag{3.50}$$

$$\lim_{t \to \infty} R(t) = R_\infty. \tag{3.51}$$

Proof: If we add the derivatives of $S(t)$ and $I(t)$, we will have the following:

$$\frac{d}{dt}(S(t)+I(t)) = -\gamma I(t) - \int_0^\infty p(x,t)S(x,t)dx. \tag{3.52}$$

Brauer and Castillo-Chavez points out in [2, p.354] that the derivative of a non-negative smooth decreasing function must tend to zero as $t \to \infty$, and as such we have that

$$\lim_{t \to \infty} \left(\frac{d}{dt}(S(t)+I(t)) \right) = 0$$

$$\lim_{t \to \infty} \left(-\gamma I(t) - \int_0^\infty p(x,t)S(x,t)dx \right) = 0.$$

43

Since both terms are always positive then both must converge to 0, so

$$\lim_{t\to\infty} \gamma I(t) \;=\; 0 \tag{3.53}$$

$$\lim_{t\to\infty} \int_0^\infty p(x,t)S(x,t)dx \;=\; 0. \tag{3.54}$$

Therefore we have that.

$$\lim_{t\to\infty} I(t) \;=\; 0. \tag{3.55}$$

$$\lim_{t\to\infty} \left(S(t)+I(t)\right)) \;=\; S_\infty. $$

$$\tag{3.56}$$

And thus

$$\lim_{t\to\infty} \left(S(t)+I(t)+R(t)\right) \;=\; 1$$

$$\lim_{t\to\infty} S(t) + \lim_{t\to\infty} I(t) + \lim_{t\to\infty} R(t) \;=\; 1$$

$$S_\infty + R_\infty \;=\; 1$$

$$R_\infty \;=\; 1 - S_\infty. \tag{3.57}$$

As desired.

Theorem 4 *The following SIR model without recovery rate*

$$\frac{dS(t)}{dt} \;=\; -\beta I(t) \int_0^\infty (1-p(x,t))S(x,t)dx - \int_0^\infty p(x,t)S(x,t)dx \tag{3.58}$$

$$\frac{dI(t)}{dt} \;=\; \beta I(t) \int_0^\infty (1-p(x,t))S(x,t)dx \tag{3.59}$$

$$\frac{dR(t)}{dt} \;=\; \int_0^\infty p(x,t)S(x,t)dx \tag{3.60}$$

44

with $I(t), S(t)$ and $R(t)$ are non-negative and with positive transmission parameter $\beta > 0$. If the initial conditions $I(0) > 0$ and $S(0) > 0$, then the following limits are true:

$$\lim_{t \to \infty} R(t) \ = \ R_\infty \tag{3.61}$$

where $0 \leq R_\infty \leq 1$.

Proof: If we add the derivatives of $S(t)$ and $I(t)$, we will have the following:

$$\frac{d}{dt}(S(t) + I(t)) \ = \ -\int_0^\infty p(x,t)S(x,t)\,dx. \tag{3.62}$$

Brauer and Castillo-Chavez points out in [2, p.354] that the derivative of a non-negative smooth decreasing function must tend to zero as $t \to \infty$ and as such we have that

$$\lim_{t \to \infty} \left(\frac{d}{dt}(S(t) + I(t)) \right) \ = \ 0$$
$$\lim_{t \to \infty} \int_0^\infty p(x,t)S(x,t)\,dx \ = \ 0.$$

This implies the following

$$\lim_{t \to \infty} \left(\frac{dR(t)}{dt} \right) \ = \ 0$$

.

And thus

$$\lim_{t \to \infty} R(t) \ = \ R_\infty, \tag{3.63}$$

as desired.

45

CHAPTER IV

CONCLUSIONS AND FUTURE WORK

The research on how to model networks have become increasingly important since the arrival of social networks in which information spreads at a faster rate since previous decades. This information shapes the behavior in which of the population makes decisions regarding vaccination and as consequence shape the way the disease will spread.

As we see in The Coupled Telegraph and SIR Model, the information about the disease and how it propagates impacts the probability of vaccination and as a consequence the vaccination rate. We can see that this vaccination rates shape how long the disease will last in the population and with numerical simulation we confirm this hypothesis.

The model that we study here has limitations. Complexity can be added to the SIR model such as death and birth rates, incubation periods and reinfection. Also it is also important to notice that infection rates changes with age of the individuals, as such in this model is not considered. Further study can be done with other types of information, such as an anti-vaccination information, speed of propagation and changing the probability definition for the decision of vaccination.

This model is intended as a first approach to understand how if we integrate social networks into classical models the behavior will change. This is intended to be a first step and further work needs to be done making more realistic models were social networks play an important role in the population.

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APPENDIX A

APPENDIX A

MATLAB CODE

Main program MATLAB file:

```matlab
close all
%Constants initializations
tau=3;
alpha=2;
D=2;
gamma=.5;
mc=2;
r=2;
p=1;
nc=(gamma/6)^.5;
K=(((25*tau*gamma/(alpha^2))+1)/D)^.5;
omega=(5/alpha)*(gamma/6)^.5;

%Space discretization
n=200;
m=100;
n0=m;
Du=zeros(n,m);
```

```matlab
x0=0;
xn=10;
t0=0;
tm=1.5;


%Vectorizactions and space-time meshgrid creation
X=linspace(x0,xn,n);
T=linspace(t0,tm,m);
[tt,xx]=meshgrid(T,X);
dx=(xn-x0)/(n);
dt=(tm-t0)/(m);


%Initial conditions for the Telegraph equation
Mi0=.9*(1+mc*exp(nc*K*X)).^(-r);
dMi0=.9*-r*(nc*mc*exp(nc*K*X)).*(1+mc*exp(nc*K*X)).^(-r-1);


Mv0=.1*(1+mc*exp(nc*K*X)).^(-r);
dMv0=.1*-r*(nc*mc*exp(nc*K*X)).*(1+mc*exp(nc*K*X)).^(-r-1);


%SIR initial conditions
S0=.6;
%Another test funnction
%S0xt=S0*exppdf(X,2)';
S0xt=S0*Scaled_Beta(X, 10, 1, x0, xn)';
%S0xt=S0*Scaled_Beta(X,1, 2, x0, xn)';
trapz(X,S0xt);
```

```matlab
I0 =.3;
R0=1 -(S0+I0 ) ;


%Vectorization of all initial conditions
y0 =[ Mi0 ' ; dMi0 ' ; Mv0 ' ; dMv0 ' ; S0xt ; I0 ; R0 ] ;


%Main solver ODE45
options = odeset ( 'AbsTol' , 1.0 e -3, 'RelTol' , 1.0 e -3);
[ t , y ] = ode45 (@( t , y ) odefcn ( t , y ,X, dx ) , T, y0 , options ) ;


%Reasignation of main result
Mi=y ( : ,1 : n ) ';
Mv=y ( : ,( 2 * n +1):3 * n ) ';
Stx=y ( : ,( 4 * n +1):5 * n ) ';
I=y ( : , end -1) ;
R=y ( : , end ) ;
S=1 -(I+R) ;
hold on
%Graph result section
figure (1)
surf ( xx , tt ,Mi-Mv, 'edgecolor' , 'none' ) ;
set ( gcf , 'Renderer' , 'ZBuffer' ) ;
shading interp
xlabel ( 'x' ) ;
ylabel ( 't' ) ;
zlabel ( '$M_{i}(x,t)-M_{v}(x,t)$' , 'Interpreter' , 'LaTex' ) ;
```

51

```matlab
title('Ode45 Mi-Mv Solution');
rotate3d;
view(3)
grid on
drawnow;

figure(2)
surf(xx,tt,Stx,'edgecolor','none');
set(gcf, 'Renderer', 'ZBuffer');
shading interp
xlabel('x');
ylabel('t');
zlabel('$S(x,t)$','Interpreter','LaTex');
title('Ode45 S Solution');
rotate3d;
drawnow

figure(3)
plot(t,S,'b',t,I,'r--',t,R,'b-.');
xlabel('t');
ylabel('$S(t),I(t),R(t)$','Interpreter','LaTex');
title('Ode45 SIR Solution');
legend('Susceptible','Infected','Recovered')
rotate3d;
drawnow
```

```matlab
%Custom scaled function for a scaled beta distribution
function PDF = Scaled_Beta(x,a, b, x0, xn)
PDF = ((x-x0).^(a-1).*(xn-x).^(b-1))./...
    ((xn-x0).^(a+b-1).*beta(a,b) );
end
```

ODE solver function program MATLAB file:

```matlab
function dydt=odefcn(t,y,X,dx)

%Reassignation of the input variables
n=(length(y)-2)/5;
Mi=y(1:n);
dMi=y(n+1:2*n);
Mv=y(2*n+1:3*n);
dMv=y(3*n+1:4*n);
S=y(4*n+1:5*n);
I=y(5*n+1);
R=y(5*n+2);
taui=1;
tauv=1;
alphai=1;
alphav=1;
Di=1;
Dv=1;
gammai=1;
gammav=1;
c=50;
beta=1;
gamma=1;

%Calculation of telegraph points in the space dimension
%Refer to the numerical method for explanation of this
%part of the code and how it was constructed
```

```
for k=1:n

    if k==1

        dMit(k,1) = (Di/taui)*((1)*Mi(k)+...
            (-2)*Mi(k+1)+...
            (1)*Mi(k+2))/dx^2+ ...
            (gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
            (alphai/taui)*dMi(k);
        dMvt(k,1) = (Dv/tauv)*((1)*Mv(k)+...
            (-2)*Mv(k+1)+...
            (1)*Mv(k+2))/dx^2  ...
            +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
            (alphav/tauv)*dMv(k);

    elseif k==2 || k==n-1

        dMit(k,1) = (Di/taui)*(Mi(k-1) -...
            2*Mi(k)+Mi(k+1))/dx^2  ...
            +(gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
            (alphai/taui)*dMi(k);
        dMvt(k,1) = (Dv/tauv)*(Mv(k-1) -...
            2*Mv(k)+Mv(k+1))/dx^2  ...
            +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
            (alphav/tauv)*dMv(k);
```

55

```matlab
elseif k==3 || k==4 ||  k==n-2 || k==n-3


    dMit(k,1) = (Di/taui)*((-1/12)*Mi(k-2)+...
        (4/3)*Mi(k-1)+(-5/2)*Mi(k)+...
        (4/3)*Mi(k+1)+(-1/12)*Mi(k+2))/dx^2  ...
        +(gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
        (alphai/taui)*dMi(k);
    dMvt(k,1) = (Dv/tauv)*((-1/12)*Mv(k-2)+...
        (4/3)*Mv(k-1)+(-5/2)*Mv(k)+...
        (4/3)*Mv(k+1)+(-1/12)*Mv(k+2))/dx^2  ...
        +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
        (alphav/tauv)*dMv(k);


elseif k==n


    dMit(k,1) = (Di/taui)*((1)*Mi(k)+...
        (-2)*Mi(k-1)+...
        (1)*Mi(k-2))/dx^2+  ...
        (gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
        (alphai/taui)*dMi(k);
    dMvt(k,1) = (Dv/tauv)*((1)*Mv(k)+...
        (-2)*Mv(k-1)+...
        (1)*Mv(k-2))/dx^2  ...
        +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
        (alphav/tauv)*dMv(k);
```

```
else

    dMit(k,1) = (Di/taui)*((-1/560)*Mi(k-4)+...
        (8/315)*Mi(k-3)+...
        (-1/5)*Mi(k-2)+...
        (8/5)*Mi(k-1)+...
        (-205/72)*Mi(k)...
        +(-1/560)*Mi(k+4)+...
        (8/315)*Mi(k+3)+...
        (-1/5)*Mi(k+2)+...
        (8/5)*Mi(k+1))/dx^2 ...
        +(gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
        (alphai/taui)*dMi(k);
    dMvt(k,1) = (Dv/tauv)*((-1/560)*Mv(k-4)+...
        (8/315)*Mv(k-3)+...
        (-1/5)*Mv(k-2)+...
        (8/5)*Mv(k-1)+...
        (-205/72)*Mv(k)...
        +(-1/560)*Mv(k+4)+...
        (8/315)*Mv(k+3)+...
        (-1/5)*Mv(k+2)+...
        (8/5)*Mv(k+1))/dx^2 ...
        +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
        (alphav/tauv)*dMv(k);

end
```

```
    end


P=1./(1+exp(-c*(Mi-Mv)));


%Validation of the value of I for further calculations
%of the derivatives of the SIR model


  if I>0


      dSxt=-beta*(1-P).*S.*I-P.*S;
      dI=beta*I.*trapz(X,(1-P).*S)-gamma.*I;
      dR=trapz(X,P.*S)+gamma.*I;


  else


      dSxt=-P.*S;
      dI=0;
      dR=trapz(X,P.*S);


  end

%Assignation of the time differential
t
dydt=[dMi;dMit;dMv;dMvt;dSxt;dI;dR];
end
```

Main program for testing solutions MATLAB file:

```matlab
function SIRTel1COS()
close all
%Constants initializations
tau=3;
alpha=2;
D=2;
gamma=.5;
mc=2;
r=2;
p=1;
nc=(gamma/6)^.5;
K=(((25*tau*gamma/(alpha^2))+1)/D)^.5;
omega=(5/alpha)*(gamma/6)^.5;

%Space discretization
n=200;
m=50;
n0=m;
Du=zeros(n,m);
x0=0;
xn=2;
t0=0;
tm=2;
D=501;
c1=1;
c2=10;
```

```matlab
% c1 =1/20;
% c2 =.8;
%Vectorizactions and space-time meshgrid creation
X=linspace(x0,xn,n);
T=linspace(t0,tm,m);
[tt,xx]=meshgrid(T,X);
dx=(xn-x0)/(n);
dt=(tm-t0)/(m);


%Initial conditions for the Telegraph equation
Mi0=.9*(1+mc*exp(nc*K*X)).^(-r);
dMi0=.9*-r*(nc*mc*exp(nc*K*X)).*(1+mc*exp(nc*K*X)).^(-r-1);


Mv0=.1*(1+mc*exp(nc*K*X)).^(-r);
dMv0=.1*-r*(nc*mc*exp(nc*K*X)).*(1+mc*exp(nc*K*X)).^(-r-1);


%SIR initial conditions
S0=.5;
%Another test funnction
%S0xt=S0*exppdf(X,2)';
%S0xt=S0* Scaled_Beta(X, 10, 1, x0, xn)';
S0xt=S0* Scaled_Beta(X,1, 2, x0, xn)';
I0=.3;
R0=1-(S0+I0);


%Vectorization of all initial conditions
```

60

```matlab
y0=[Mi0';dMi0';Mv0';dMv0';S0xt;I0;R0];


%Main solver ODE45
options = odeset('AbsTol', 1.0e-4, 'RelTol', 1.0e-4);
[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,D,c1,c1), T,...
    y0,options);


%Reasignation of main result
I(:,1)=y(:,end-1)';
R(:,1)=y(:,end)';
S(:,1)=1-(I(:,1)+R(:,1));


[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,0,c1,c1), T,...
    y0,options);


%Reasignation of main result
I(:,2)=y(:,end-1)';
R(:,2)=y(:,end)';
S(:,2)=1-(I(:,2)+R(:,2));



[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,-D,c1,c1), T,...
    y0,options);


%Reasignation of main result
I(:,3)=y(:,end-1)';
```

```matlab
R(:,3)=y(:,end)';
S(:,3)=1-(I(:,3)+R(:,3));


figure(1)
plot(t,S(:,1),'b',t,S(:,2),'r--',t,S(:,3),'b-.');
xlabel('t');
ylabel('$S(t)$','Interpreter','LaTex');
title('Ode45 SIR Solution');
axis([0 tm 0 .5])
legend({'$D_{I}=500D_{V}$','$D_{I}=D_{V}$',...
    '$500D_{I}=D_{V}$'},'Interpreter','LaTex')
hold on


figure(2)
plot(t,I(:,1),'b',t,I(:,2),'r--',t,I(:,3),'b-.');
xlabel('t');
ylabel('$I(t)$','Interpreter','LaTex');
title('Ode45 SIR Solution');
axis([0 tm 0 .5])
legend({'$D_{I}=500D_{V}$','$D_{I}=D_{V}$',...
    '$500D_{I}=D_{V}$'},'Interpreter','LaTex')
hold on


%Main solver ODE45
[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,D,c2,c1), T,...
    y0,options);
```

```matlab
%Reasignation of main result
I(:,1)=y(:,end-1)';
R(:,1)=y(:,end)';
S(:,1)=1-(I(:,1)+R(:,1));


[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,0,c2,c1), T,...
    y0,options);


%Reasignation of main result
I(:,2)=y(:,end-1)';
R(:,2)=y(:,end)';
S(:,2)=1-(I(:,2)+R(:,2));


[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,-D,c2,c1), T,...
    y0,options);


%Reasignation of main result
I(:,3)=y(:,end-1)';
R(:,3)=y(:,end)';
S(:,3)=1-(I(:,3)+R(:,3));


hold on


figure(3)
plot(t,S(:,1),'b',t,S(:,2),'r--',t,S(:,3),'b-.');
```

```matlab
xlabel('t');
ylabel('$S(t)$','Interpreter','LaTex');
title('Ode45 SIR Solution');
legend({'$D_{I}=500D_{V}$','$D_{I}=D_{V}$',...
    '$500D_{I}=D_{V}$'},'Interpreter','LaTex')
axis([0 tm 0 .5])
hold on

figure(4)
plot(t,I(:,1),'b',t,I(:,2),'r--',t,I(:,3),'b-.');
xlabel('t');
ylabel('$I(t)$','Interpreter','LaTex');
title('Ode45 SIR Solution');
axis([0 tm 0 .5])
legend({'$D_{I}=500D_{V}$','$D_{I}=D_{V}$',...
    '$500D_{I}=D_{V}$'},'Interpreter','LaTex')
hold on

%Main solver ODE45
[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,D,c1,c2), T,...
    y0,options);

%Reasignation of main result
I(:,1)=y(:,end-1)';
R(:,1)=y(:,end)';
S(:,1)=1-(I(:,1)+R(:,1));
```

```matlab
[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,0,c1,c2), T,...
    y0,options);


%Reasignation of main result
I(:,2)=y(:,end-1)';
R(:,2)=y(:,end)';
S(:,2)=1-(I(:,2)+R(:,2));



[t,y] = ode45(@(t,y) odefcnCOS(t,y,X,dx,-D,c1,c2), T,...
    y0,options);


%Reasignation of main result
I(:,3)=y(:,end-1)';
R(:,3)=y(:,end)';
S(:,3)=1-(I(:,3)+R(:,3));


figure(5)
plot(t,S(:,1),'b',t,S(:,2),'r--',t,S(:,3),'b-.');
xlabel('t');
ylabel('$S(t)$','Interpreter','LaTex');
title('Ode45 SIR Solution');
legend({'$D_{I}=500D_{V}$','$D_{I}=D_{V}$',...
    '$500D_{I}=D_{V}$'},'Interpreter','LaTex')
axis([0 tm 0 .5])
```

```matlab
hold on

figure (6)
plot(t,I(:,1),'b',t,I(:,2),'r--',t,I(:,3),'b-.');
xlabel('t');
ylabel('$I(t)$','Interpreter','LaTex');
title('Ode45 SIR Solution');
legend({'$D_{I}=500D_{V}$','$D_{I}=D_{V}$',...
    '$500D_{I}=D_{V}$'},'Interpreter','LaTex')
axis([0 tm 0 .5])
hold on




%Custom scaled function for a scaled beta distribution
function PDF = Scaled_Beta(x,a, b, x0, xn)
PDF = ((x-x0).^(a-1).*(xn-x).^(b-1))./...
    ((xn-x0).^(a+b-1).*beta(a,b) );
end
end
```

ODE solver function program for testing solutions MATLAB file:

```matlab
function dydt=odefcn(t,y,X,dx,diff,beta,gamma)


%Reassignation of the input variables
n=(length(y)-2)/5;
Mi=y(1:n);
dMi=y(n+1:2*n);
Mv=y(2*n+1:3*n);
dMv=y(3*n+1:4*n);
S=y(4*n+1:5*n);
I=y(5*n+1);
R=y(5*n+2);
taui=1;
tauv=1;
alphai=1;
alphav=1;
if diff>0
    Di=diff-1;
    Dv=1;
elseif diff<0
    Di=1;
    Dv=abs(diff)-1;
else
    Di=1;
    Dv=1;
end
gammai=1;
```

```matlab
gammav=1;
c=50;


%Calculation of telegraph points in the space dimension
%Refer to the numerical method for explanation of this
%part of the code and how it was constructed
for k=1:n


    if k==1


        dMit(k,1) = (Di/taui)*((1)*Mi(k)+...
            (-2)*Mi(k+1)+...
            (1)*Mi(k+2))/dx^2+ ...
            (gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
            (alphai/taui)*dMi(k);
        dMvt(k,1) = (Dv/tauv)*((1)*Mv(k)+...
            (-2)*Mv(k+1)+...
            (1)*Mv(k+2))/dx^2 ...
            +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
            (alphav/tauv)*dMv(k);


    elseif k==2 || k==n-1


        dMit(k,1) = (Di/taui)*(Mi(k-1) -...
            2*Mi(k)+Mi(k+1))/dx^2 ...
```

```
        +(gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
        (alphai/taui)*dMi(k);
    dMvt(k,1) = (Dv/tauv)*(Mv(k-1) -...
        2*Mv(k)+Mv(k+1))/dx^2  ...
        +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
        (alphav/tauv)*dMv(k);


elseif k==3 || k==4 ||  k==n-2 || k==n-3


    dMit(k,1) = (Di/taui)*((-1/12)*Mi(k-2)+...
        (4/3)*Mi(k-1)+(-5/2)*Mi(k) +...
        (4/3)*Mi(k+1)+(-1/12)*Mi(k+2))/dx^2  ...
        +(gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
        (alphai/taui)*dMi(k);
    dMvt(k,1) = (Dv/tauv)*((-1/12)*Mv(k-2)+...
        (4/3)*Mv(k-1)+(-5/2)*Mv(k) +...
        (4/3)*Mv(k+1)+(-1/12)*Mv(k+2))/dx^2  ...
        +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
        (alphav/tauv)*dMv(k);


elseif k==n


    dMit(k,1) = (Di/taui)*((1)*Mi(k)+...
        (-2)*Mi(k-1)+...
        (1)*Mi(k-2))/dx^2+ ...
        (gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
```

```
        ( alphai / taui )*dMi(k);
dMvt(k,1)  =  (Dv/tauv)*((1)*Mv(k)+...
        (-2)*Mv(k-1)+...
        (1)*Mv(k-2))/dx^2  ...
        +(gammav/tauv)*(1-Mi(k)-Mv(k))*Mv(k) -...
        (alphav/tauv)*dMv(k);
```

else

```
dMit(k,1)  =  (Di/taui)*((-1/560)*Mi(k-4)+...
        (8/315)*Mi(k-3)+...
        (-1/5)*Mi(k-2)+...
        (8/5)*Mi(k-1)+...
        (-205/72)*Mi(k) ...
        +(-1/560)*Mi(k+4)+...
        (8/315)*Mi(k+3)+...
        (-1/5)*Mi(k+2)+...
        (8/5)*Mi(k+1))/dx^2  ...
        +(gammai/taui)*(1-Mi(k)-Mv(k))*Mi(k) -...
        (alphai/taui)*dMi(k);
dMvt(k,1)  =  (Dv/tauv)*((-1/560)*Mv(k-4)+...
        (8/315)*Mv(k-3)+...
        (-1/5)*Mv(k-2)+...
        (8/5)*Mv(k-1)+...
        (-205/72)*Mv(k) ...
        +(-1/560)*Mv(k+4)+...
```

```matlab
            ( 8 / 3 1 5 ) *Mv( k + 3 ) + . . .

            ( − 1 / 5 ) *Mv( k + 2 ) + . . .

            ( 8 / 5 ) *Mv( k + 1 ) ) / dx ^2   . . .

            +(gammav / t a u v ) *( 1 −Mi( k )−Mv( k ) ) *Mv( k )  − . . .

            ( a l p h a v / t a u v ) *dMv( k ) ;

    end


end


%Probability calculation
P = 1 . / ( 1 + exp ( − c *(Mi−Mv) ) ) ;


%Validation of the value of I for further calculations
%of the derivatives of the SIR model


if  I >0

    dSxt=− beta *( 1 −P) . * S . * I−P . * S ;
    dI=beta * I . * trapz (X,( 1 −P) . * S )−gamma . * I ;
    dR= trapz (X, P . * S )+gamma . * I ;


else

    dSxt=−P . * S ;
    dI =0;
    dR= trapz (X, P . * S ) ;
```

```
end


%Assignation of the time differential
t
dydt=[dMi;dMit;dMv;dMvt;dSxt;dI;dR];
end
```

BIOGRAPHICAL SKETCH

Jose de Jesus Galarza was born in Brownsville, Texas. He was raised in Matamoros, Tamaulipas and went to Monterrey, Nuevo Leon, Mexico at the age of 18 in 2004. He studied the Bachelor in Science in Mechatronics Engineering in the Universidad de Monterrey, from which he graduated in 2009. He certified in programming from the state of Nuevo Leon in 2009. He worked for a software company Infosys as Business Intelligence consultant from 2010 to 2013. He then worked in Ruhrpumpen as an analysis engineer 2013-2015. In the summer of 2015 he started the Masters in Science in Mathematics and Mechanical Engineering at The University of Texas-Pan American, now The University of Texas Rio Grande Valley. He received the Masters in Science in Mathematics in May 2017.

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