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By Otto A. Spies

In the May number of The Journal of Accountancy, on pages 350–353, Arthur S. Little has shown with great skill the way in which the present worth of future coupon payments, subject to a periodical and regular reduction of bond principal, may be accurately ascertained with the help of "Makeham's formula," which was recently published in Toronto by M. A. Mackenzie.

This solution, remarkable as it is, requires, however, first, the calculation of the present worth of the periodical bond payments, and is, therefore, not a direct computation of the present worth of the coupon payments.

It has been stated by Mr. Little, and he is also supported by Charles Ezra Sprague, on page 238 of the 1914 edition of his *Accountancy of Investment,* that a direct solution for the sum of the present value of the future coupon payments would require an algebraic expression so formidable as to be of the *n*th degree of difficulty and that it would not be practicable.

It appears to be taken for granted that both Mr. Little and Mr. Sprague have never made a successful mathematical computation of this problem. It is accepted that the computation of the present value of the periodical equal bond payments is not difficult. It is also accepted that the coupon payments are a direct function of the principal payments and their computation must, therefore, be also an easy problem for the mathematician. There is no question about this point and the solution of the problem is only a matter of mathematical skill and research.

Suppose the problem consisted in computing the total value of the twenty members of the irregular progression, as stated on page 352 of volume xxi of this magazine, namely:

$$\frac{6000}{1.01} + \frac{5700}{1.01^2} + \frac{5400}{1.01^3} \ldots \frac{600}{1.01^{19}} + \frac{300}{1.01^{20}}$$

Without information of any relationship to bond payments, would it be necessary to compute in 20 separate calculations the value of each member?

An analysis of this problem will convince anyone familiar with algebra that the direct solution is just as simple as the remarkable discovery of Mr. Makeham.

Analysis: If the unknown value of the sum of the above progression is designated by Y, there would exist the fundamental equation:

(1)

$$Y = \frac{6000}{1.01} + \frac{5700}{1.01^2} + \frac{5400}{1.01^3} \ldots \frac{600}{1.01^{19}} + \frac{300}{1.01^{20}}$$

It is evident this progression has not a constant ratio—that is, if one number is divided by a previous one, the ratio will always be different.

The first endeavor, therefore, should be to transform equation (1) into one with a constant ratio, if possible, for all members of the progression. This can be attained by certain transactions which do not change the equality of the equation and depend greatly upon the skill and experience of the student.

If equation (1) is multiplied on both sides by (1.01^{21}) the result would be:

(2)

$$Y\ (1.01^{21}) = 6000\ (1.01^{20}) + 5700\ (1.01^{19}) + 5400\ (1.01^{18}) \ldots 300\ (1.01)$$

If equation (1) is also multiplied by (1.01^{20}), the result is:

(3)

$$Y\ (1.01^{20}) = 6000\ (1.01^{19}) + 5700\ (1.01^{18}) + 5400\ (1.01^{17}) \ldots 300.$$

If equation (3) is deducted from equation (2), there results a new equation (4), in which 18 members of the progression have the same constant ratio (1.01) and the same factor (300).

Namely:

(4)

$$Y\ (1.01^{21} - 1.01^{20}) = 6000\ (1.01^{20}) - 300\ (1.01^{19}) - 300\ (1.01^{18}) \ldots 300\ (1.01) - 300$$

or transformed:

(5)

$$Y\ (1.01^{20})\ (1.01\text{-}1) = 6000\ (1.01^{20}) - 300\ (1.01^{19} + 1.01^{18}\ \ .\ \ .\ \ .\ \ 1.01) - 300.$$

It is now obvious, that the second member on the right side of equation (5) is a regular geometrical progression, the sum of which can be easily ascertained by known rules and equals:

$$300\ (1.01^{19} + 1.01^{18}\ \ .\ \ .\ \ .\ \ 1.01^{2} + 1.01) = \frac{300\ (1.01)\ (1.01^{19} - 1)}{0.01}$$

If this value is inserted in equation (5), the result will be that all except two members of the original progression will be represented by one algebraic expression, consisting of three factors:

This being done, equation (5) will be transformed in equation (6) as follows:

(6)

$$Y\ (1.01^{20})\ (0.01) = 6000\ (1.01^{20}) - \frac{300\ (1.01)\ (1.01^{19} - 1)}{0.01} - 300.$$

or, if equation (6) be divided by the factors of (Y), there results for (Y) equation (7), which would be:

(7)

$$Y = \frac{6000}{0.01} - \frac{300}{0.01^{2}}\ \frac{(1.01^{19} - 1)}{(1.01^{19}\qquad)} - \frac{300}{(0.01)\ (1.01^{20})}$$

The values of 1.01^{19} and 1.01^{20} are best computed by logarithms, and are:

$$1.01^{19} = 1.2081089$$
$$1.01^{20} = 1.2201902$$

These values for 1.01^{19} and 1.01^{20} inserted in equation (7) would determine (Y) to be as:

$$Y = 600000 - 3000000 \frac{(0.2081089)}{(1.2081089)} - \frac{30000}{1.2201902}$$

$$\text{or } Y = 600000 - 516780.26 - 24586.33$$

$$\text{or } Y = 58633.41.$$

In equation (7), which determines exactly the value of the sum of all members of the irregular geometrical progression, are recognized three algebraic expressions which have a specific meaning.

In the first expression: $\frac{(6000)}{(0.01)}$ is found the numerator

(6000), the numerator of the first member of the progression. This numeration is divided by the interest factor (1.01), diminished by one.

In the last or third algebraic expression: $\frac{300}{(0.01)\ (1.01^{20})}$ is recognized the last member of the original progression: $\frac{(300)}{(1.01^{20})}$ divided also by the interest factor (1.01) diminished by one.

In the second algebraic expression: $\frac{300}{0.01^2} \left(\frac{1.01^{19} - 1}{1.01^{19}}\right)$ is

found as one factor the difference of the numerators of the progression (300), divided by the square of the interest factor diminished by one (0.01^2).

The other factor consists of the interest factor at an exponent equalling the number of members of the progression diminished by one minus one, (1.01^{19}—1) divided by the same interest factor at the same power (1.01^{19}).

Thus it is demonstrated that the elements necessary for the solution are:

(1) Amount of the first coupon (6000)
(2) Interest factor of period (1.01)
(3) Amount of last coupon (300)
(4) Difference of amount of periodical coupon (300)
(5) Number of coupons or periods.

With the help of equation (7) it is possible to ascertain directly and precisely the sum of any irregular but symmetrical geometrical progression, as required in the computation of the present worth of future coupon payments.

If the coupon payments do not vary, that is, if there does not take place any bond reduction during the life of the bond issue, the present worth of the coupons (Y A) may be computed also by equation (7), which in that case would be remarkably simple, because the second member of equation (7) would be reduced to zero.

(9)

$$Y\ A = \frac{6000}{0.01}\ 1 - \frac{1}{1.01^{20}}$$

$$\text{or } Y\ A = \frac{6000}{0.01}\ \frac{(1.01^{20}-1)}{(1.01^{20})}$$

$$\text{or } Y\ A = 600000\ \frac{(0.2201902)}{(1.2201902)}$$

$$Y\ A = 600000\ (0.1804556)$$

$$Y\ A = \$108{,}273.33$$

The application of the developed theorem may be demonstrated in the following problem:

Required, the sum of following irregular but symmetrical geometrical progression:

$$\frac{120}{1.01} + \frac{119}{1.01^2} + \frac{118}{1.01^3}\ \ldots\ \frac{102}{1.01^{19}} + \frac{101}{1.01^{20}} = Y$$

according to equation (7).

$$Y = \frac{120}{0.01} - \frac{1}{0.01^2} \quad \frac{(1.01^{19}-1)}{(1.01^{19})} - \frac{101}{(0.01)\ (1.01^{20})}$$

or if inserted,

$$1.01^{19} = 1.2081089$$

$$1.01^{20} = 1.2201902$$

$$Y = 12000 - 10000 \ \frac{(0.2081089)}{(1.2081089)} - \frac{10100}{1.2201902}$$

or $Y = 12000 - 1722.60 - 8277.40$

or $Y = 2000$,

The correctness of this can be proven by separate calculations.