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Research Article

Differential geometric approach of Betchov-Da Rios soliton equation

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Abstract

In the present paper, we investigate differential geometric properties the soliton surface M associated with Betchov-Da Rios equation. Then, we give derivative formulas of Frenet frame of unit speed curve $\Phi = \Phi(s,t)$ for all t. Also, we discuss the linear map of Weingarten type in the tangent space of the surface that generates two invariants: k and h. Moreover, we obtain the necessary and sufficient conditions for the soliton surface associated with Betchov-Da Rios equation to be a minimal surface. Finally, we examine a soliton surface associated with Betchov-Da Rios equation as an application.

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1. Introduction

Analytical solution of nonlinear partial differential equations plays an important role in mathematics and all applications of mathematics. Many methods have been developed to obtain solutions to these equations. However, due to the continuous renewal of science with developing technology, different solution methods are being investigated. Among these solutions, solitary wave solutions which attract the most attention especially due to their physical applications. For the solution of such partial differential equations, the term soliton is generally used in cases involving more than one wave. Soliton theory has an important place in mathematics, physics and biology, continues to be researched in scientific studies, it is also related to the field of differential geometry.

In classical differential geometry, the study of motion of a vortex filament is one of the important problems of mathematical physics. Hasimoto's work in 1972 is a pioneering study on this subject. The aforementioned work considered the self-induced motion of a thin isolated vortex filament travelling without stretching in an incompressible fluid. If the position vector of vortex filament is denoted by $\Phi = \Phi(s,t)$, then the equation

$$\Phi_t = \Phi_s \times \Phi_{ss}$$

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is hold [8]. This equation is known as smoke ring or vortex filament equation. Also, the relationship between the propagating wave solutions of Nonlinear Schrodinger (NLS) equation of these motions without form change is a well-known fact, [4–6, 8, 11, 13].

In the work [12], it has been obtained that the binormal motion of the constant curvature curves causes integrable extensions of Dym equations. It is also shown that the binormal motion of curves of constant torsion leads to integrable extensions of classical sine–Gordon equations. NLS equation is examined in a general intrinsic geometric setting as introduced earlier in a kinematic analysis of certain hydrodynamic motions by [9]. Furthermore, differential geometric properties of the soliton surfaces associated with NLS equation are obtained and the connection between Hasimoto derivation is stated in [10].

The thin filament is expressed, smooth and without self-intersection. The velocity induced by vortex line at an external point is expressed by Da Rios via so-called localized induction approximation (LIA). The movement of a thin vortex in a thin inviscid fluid by the motion of a curve propagating in $\mathbb{R}^4$ is described by the following equation

$$\Phi_t = \Phi_s \times \Phi_{ss} \times \Phi_{sss}. \tag{1}$$

This is called Betchov-Da Rios equation or localized induction equation, and can be viewed as a dynamical system on the space of curves in $\mathbb{R}^4$. In [2], the explicit solutions of Betchov-Da Rios soliton equation in three-dimensional Lorentzian space forms are investigated. Also, the explicit examples of surfaces are given in L^3, S^3_1 and H^3_1, where the solutions are lying in [2]. On the other hand, the parametrization of Hopf cylinders, which are solutions of Betchov-Da Rios soliton equation in $H^3_1(-1)$ are investigated in [1]. Moreover, it is obtained that the soliton solutions are the null geodesics of Lorentzian Hopf cylinders in [1]. Additionally, the solutions of Betchov-Da Rios soliton equation are investigated in the three-dimensional anti-De Sitter space and it is proved that the solutions are the helices which are sweeping out a B-scroll i.e., the null geodesics of that B-scroll in [3].

The paper is organized as follows. In section 2, we examine differential geometric properties the soliton surface $M : \Phi = \Phi(s, t)$ associated with Betchov-Da Rios equation. It is proved that s-parameter curve $\Phi(s, t)$ is parametrized by arclength for all t when $\Phi(s, 0)$ is a unit speed curve. Then, we give derivative formulas of Frenet frame $\{T, N, B_1, B_2\}$ of unit speed s-parameter curve $\Phi = \Phi(s, t)$ for all t. In section 3, we discuss the linear map of Weingarten type in the tangent space of the surface. In accordance with this scope, we obtain the geometric invariants k and h. Moreover, the necessary and sufficient conditions for the soliton surface associated with Betchov-Da Rios equation to be a minimal surface. And, a new result is obtained on existence of flat points of the soliton surface associated with Betchov-Da Rios equation. Furthermore, we obtain the mean curvature vector field and Gaussian curvature of soliton surface. Then, we investigate this kind of soliton surface as a numerical example.

2. Betchov-Da Rios soliton equation in four dimensional euclidean space

In this section, the soliton surface $M : \Phi = \Phi(s, t)$ associated with Betchov-Da Rios equation is investigated by using derivative formulas of Frenet frame of unit speed s-parameter curve $\Phi = \Phi(s, t)$ for all t.

Proposition 2.1. *Assume that $\Phi = \Phi(s, t)$ is a solution of Betchov-Da Rios equation in $\mathbb{R}^4$. If $\Phi(s, 0)$ is a unit speed curve then other s-parameter curve $\Phi = \Phi(s, t)$ is also unit speed for all t.*

Proof. We need to prove that $\langle \Phi_s, \Phi_s \rangle_t = 0$ for the solutions of equation (1). Then we have

$$\begin{aligned}
\langle \Phi_s, \Phi_s \rangle_t &= 2\left\langle (\Phi_s)_t, \Phi_s \right\rangle \\
&= 2\left\langle (\Phi_t)_s, \Phi_s \right\rangle \\
&= 2\left\langle \Phi_{ss} \times \Phi_{ss} \times \Phi_{sss} + \Phi_s \times \Phi_{sss} \times \Phi_{sss} + \Phi_s \times \Phi_{ss} \times \Phi_{ssss}, \Phi_s \right\rangle \\
&= 2\left\langle \Phi_s \times \Phi_{ss} \times \Phi_{ssss}, \Phi_s \right\rangle = 0.
\end{aligned}$$

Therefore, for a solution of Betchov-Da Rios equation, we obtain that $\Phi = \Phi(s,t)$ is parametrized by arclength for all t. ☐

Theorem 2.2. *Let $\Phi = \Phi(s,t)$ be a solution of Betchov-Da Rios equation such that s-parameter curve $\Phi = \Phi(s,t)$ is unit speed curve for all t. Then we have the following equations:*

(i)

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ -\kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2}$$

(ii)

$$\frac{d}{dt}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \xi_{13} & \xi_{14} \\ 0 & 0 & \xi_{23} & \xi_{24} \\ -\xi_{13} & -\xi_{23} & 0 & \xi_{34} \\ -\xi_{14} & -\xi_{24} & -\xi_{34} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{3}$$

where

$$\xi_{13}(s,t) = \kappa^2(s,t)\tau(s,t)\sigma(s,t), \tag{4}$$

$$\xi_{14}(s,t) = (-\kappa^2(s,t)\tau(s,t))_s, \tag{5}$$

$$\xi_{23}(s,t) = \frac{1}{\kappa(s,t)}[(\kappa^2(s,t)\tau(s,t)\sigma(s,t))_s + (\kappa^2(s,t)\tau(s,t))_s\sigma(s,t)], \tag{6}$$

$$\xi_{24}(s,t) = \frac{1}{\kappa(s,t)}[\kappa^2(s,t)\tau(s,t)\sigma^2(s,t) + (-\kappa^2(s,t)\tau(s,t))_{ss}], \tag{7}$$

$$\xi_{34}(s,t) = \frac{1}{\tau(s,t)}\begin{bmatrix} (-\kappa^2(s,t)\tau(s,t))_s\kappa(s,t) \\ +\dfrac{\sigma(s,t)}{\kappa(s,t)}[(\kappa^2(s,t)\tau(s,t)\sigma(s,t))_s + (\kappa^2(s,t)\tau(s,t))_s\sigma(s,t)] \\ +(\dfrac{1}{\kappa(s,t)}[\kappa^2(s,t)\tau(s,t)\sigma^2(s,t) + (-\kappa^2(s,t)\tau(s,t))_{ss}])_s \end{bmatrix}. \tag{8}$$

Here $\{T, N, B_1, B_2\}$ denote Frenet frame field, κ is the curvature function, τ is the first torsion function and σ is the second torsion of the s-parameter curve $\Phi = \Phi(s,t)$ for all t.

Proof. **(i)** This equation is Frenet formula of a unit speed curve in four dimensional Euclidean space.

(ii) We need to find smooth functions $\xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}$, and ξ_{34} such that

$$\frac{d}{dt}\begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & \xi_{12} & \xi_{13} & \xi_{14} \\ -\xi_{12} & 0 & \xi_{23} & \xi_{24} \\ -\xi_{13} & -\xi_{23} & 0 & \xi_{34} \\ -\xi_{14} & -\xi_{24} & -\xi_{34} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

in terms of κ, τ and σ. We know that

$$\begin{aligned}
\Phi_s(s,t) &= T(s,t),\\
\Phi_{ss}(s,t) &= \kappa(s,t)N(s,t),\\
\Phi_{sss}(s,t) &= \kappa_s(s,t)N(s,t) + \kappa(s,t)N_s(s,t),\\
&= \kappa_s(s,t)N(s,t) + \kappa(s,t)[-\kappa(s,t)T(s,t) + \tau(s,t)B_1(s,t)]\\
&= -\kappa^2(s,t)T(s,t) + \kappa_s(s,t)N(s,t) + \kappa(s,t)\tau(s,t)B_1(s,t).
\end{aligned}$$

By using Betchov-Da Rios equation, we get

$$\begin{aligned}
\Phi_t(s,t) &= \Phi_s(s,t) \times \Phi_{ss}(s,t) \times \Phi_{sss}(s,t)\\
&= \det \begin{bmatrix} T(s,t) & N(s,t) & B_1(s,t) & B_2(s,t)\\ 1 & 0 & 0 & 0\\ 0 & \kappa(s,t) & 0 & 0\\ -\kappa^2(s,t) & \kappa_s(s,t) & \kappa(s,t)\tau(s,t) & 0 \end{bmatrix}\\
&= -\kappa^2(s,t)\tau(s,t)B_2(s,t).
\end{aligned}$$

Then by compatibility condition $\Phi_{st} = \Phi_{ts}$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial s}(\Phi_t(s,t)) &= \frac{\partial}{\partial s}(-\kappa^2(s,t)\tau(s,t)B_2(s,t))\\
&= \kappa^2(s,t)\tau(s,t)\sigma(s,t)B_1(s,t)\\
&+ [-2\kappa(s,t)\kappa_s(s,t)\tau(s,t) - \kappa^2(s,t)\tau_s(s,t)]B_2(s,t).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{\partial}{\partial t}(\Phi_s(s,t)) &= \frac{\partial}{\partial t}(T(s,t))\\
&= \xi_{12}(s,t)N(s,t) + \xi_{13}(s,t)B_1(s,t) + \xi_{14}(s,t)B_2(s,t).
\end{aligned}$$

Therefore we get

$$\begin{aligned}
\xi_{12}(s,t) &= 0,\\
\xi_{13}(s,t) &= \kappa^2(s,t)\tau(s,t)\sigma(s,t),\\
\xi_{14}(s,t) &= -2\kappa(s,t)\kappa_s(s,t)\tau(s,t) - \kappa^2(s,t)\tau_s(s,t) = (-\kappa^2(s,t)\tau(s,t))_s.
\end{aligned}$$

Then by compatibility condition $T_{st} = T_{ts}$, we get

$$\begin{aligned}
\frac{\partial}{\partial s}(T_t(s,t)) &= \frac{\partial}{\partial s}(\xi_{13}(s,t)B_1(s,t) + \xi_{14}(s,t)B_2(s,t))\\
&= (\xi_{13})_s(s,t)B_1(s,t) + \xi_{13}(s,t)(-\tau(s,t)N(s,t) + \sigma(s,t)B_2(s,t))\\
&+ (\xi_{14})_s(s,t)B_2(s,t) + \xi_{14}(s,t)(-\sigma(s,t)B_1(s,t))\\
&= (-\xi_{13}(s,t)\kappa(s,t))N(s,t) + ((\xi_{13})_s(s,t) - \xi_{14}(s,t)\sigma(s,t))B_1(s,t)\\
&+ (\xi_{13}(s,t)\sigma(s,t) + (\xi_{14})_s(s,t))B_2(s,t).
\end{aligned}$$

Furthermore, we also have

$$\frac{\partial}{\partial t}(T_s(s,t)) = \frac{\partial}{\partial t}(\kappa(s,t)N(s,t))$$

$$= \kappa_t(s,t)N(s,t) + \kappa(s,t)(\xi_{23}(s,t)B_1(s,t) + \xi_{24}(s,t)B_2(s,t))$$

$$= \kappa_t(s,t)N(s,t) + \kappa(s,t)\xi_{23}(s,t)B_1(s,t) + \kappa(s,t)\xi_{24}(s,t)B_2(s,t).$$

This implies that

$$\kappa_t(s,t) = -\xi_{13}(s,t)\tau(s,t) = -\kappa^2(s,t)\tau^2(s,t)\sigma(s,t),$$
$$\kappa(s,t)\xi_{23}(s,t) = (\xi_{13})_s(s,t) - \xi_{14}(s,t)\sigma(s,t) = (\kappa^2(s,t)\tau(s,t)\sigma(s,t))_s + (\kappa^2(s,t)\tau(s,t))_s\sigma(s,t),$$
$$\kappa(s,t)\xi_{24}(s,t) = \xi_{13}(s,t)\sigma(s,t) + (\xi_{14})_s(s,t) = \kappa^2(s,t)\tau(s,t)\sigma^2(s,t) + (-\kappa^2(s,t)\tau(s,t))_{ss}.$$

By last two equations we obtain

$$\xi_{23}(s,t) = \frac{1}{\kappa(s,t)}[(\kappa^2(s,t)\tau(s,t)\sigma(s,t))_s + (\kappa^2(s,t)\tau(s,t))_s\sigma(s,t)],$$

$$\xi_{24}(s,t) = \frac{1}{\kappa(s,t)}[\kappa^2(s,t)\tau(s,t)\sigma^2(s,t) + (-\kappa^2(s,t)\tau(s,t))_{ss}].$$

From compatibility condition of $N_{st} = N_{ts}$, we find the following equalities

$$-\kappa(s,t)\xi_{13}(s,t) + \tau_t(s,t) = (\xi_{23})_s(s,t) - \sigma(s,t)\xi_{24}(s,t),$$

$$-\kappa(s,t)\xi_{14}(s,t) + \tau(s,t)\xi_{34}(s,t) = \xi_{23}(s,t)\sigma(s,t) + (\xi_{24})_s(s,t).$$

And we have

$$\begin{aligned}\tau_t(s,t) &= \kappa^3(s,t)\tau(s,t)\sigma(s,t)\\ &\quad + (\frac{1}{\kappa(s,t)}[(\kappa^2(s,t)\tau(s,t)\sigma(s,t))_s + (\kappa^2(s,t)\tau(s,t))_s\sigma(s,t)]_s)_s\\ &\quad - \frac{\sigma(s,t)}{\kappa(s,t)}[\kappa^2(s,t)\tau(s,t)\sigma^2(s,t) + (-\kappa^2(s,t)\tau(s,t))_{ss}],\end{aligned}$$

$$\xi_{34}(s,t) = \frac{1}{\tau(s,t)}\begin{bmatrix} (-\kappa^2(s,t)\tau(s,t))_s\kappa(s,t) \\ +\frac{\sigma(s,t)}{\kappa(s,t)}[(\kappa^2(s,t)\tau(s,t)\sigma(s,t))_s + (\kappa^2(s,t)\tau(s,t))_s\sigma(s,t)] \\ +(\frac{1}{\kappa(s,t)}[\kappa^2(s,t)\tau(s,t)\sigma^2(s,t) + (-\kappa^2(s,t)\tau(s,t))_{ss}])_s \end{bmatrix}.$$

Finally, by compatibility condition $B_{1st} = B_{1ts}$, we get

$$\sigma_t(s,t) = (\xi_{34})_s(s,t) + \tau(s,t)\xi_{24}(s,t).$$

□

3. Investigation of Betchov-Da Rios soliton surface

For a two-dimensional surface $M : \Phi = \Phi(s,t)$ in four dimensional Euclidean space, an invariant linear map of Weingarten type in the tangent space of the surface can be introduced that generates two invariants: k and h [7]. In this section, these invariants of the soliton surface are discussed. The conditions, where these surfaces are minimal and consist of flat points, are discussed.

Theorem 3.1. *Let $\Phi = \Phi(s,t)$ be a solution of Betchov-Da Rios equation. The linear map $\gamma : T_pM \to T_pM$ of Weingarten type in the tangent space of the soliton surface $M : \Phi = \Phi(s,t)$ is obtained as follows:*

$$\gamma\left(\Phi_s(s,t)\right) = -2\kappa(s,t)\sigma(s,t)T(s,t) + \frac{\xi_{34}(s,t)}{\kappa(s,t)\tau(s,t)}B_2(s,t),$$

$$\gamma\left(\Phi_t(s,t)\right) = -\kappa(s,t)\xi_{34}(s,t)T(s,t) - 2\sigma(s,t)\xi_{24}(s,t)B_2(s,t).$$

Here, the tangent space to M at an arbitrary point P of M is span $\{\Phi_s, \Phi_t\}$.

Proof. For a two-dimensional surface in four dimensional Euclidean space, we use the standard notations in [7]. First of all, we get

$$g_{11}(s,t) = \langle \Phi_s(s,t), \Phi_s(s,t)\rangle = \langle T(s,t), T(s,t)\rangle = 1,$$

$$g_{12}(s,t) = \langle \Phi_s(s,t), \Phi_t(s,t)\rangle = \left\langle T(s,t), -\kappa^2(s,t)\tau(s,t)B_2(s,t)\right\rangle = 0,$$

$$\begin{aligned} g_{22}(s,t) &= \langle \Phi_t(s,t), \Phi_t(s,t)\rangle = \left\langle -\kappa^2(s,t)\tau(s,t)B_2(s,t), -\kappa^2(s,t)\tau(s,t)B_2(s,t)\right\rangle \\ &= \kappa^4(s,t)\tau^2(s,t). \end{aligned}$$

Then we set

$$W = \sqrt{g_{11}g_{22} - g_{12}^2} = \kappa^2\tau.$$

For orthonormal normal frame field $\{N, B_1\}$ of M, we have the second derivative formulas

$$\begin{aligned} \Phi_{ss} &= \Gamma_{11}^1\Phi_s + \Gamma_{11}^2\Phi_t + c_{11}^1N + c_{11}^2B_1, \\ \Phi_{st} &= \Gamma_{12}^1\Phi_s + \Gamma_{12}^2\Phi_t + c_{12}^1N + c_{12}^2B_1, \\ \Phi_{tt} &= \Gamma_{22}^1\Phi_s + \Gamma_{22}^2\Phi_t + c_{22}^1N + c_{22}^2B_1 \end{aligned} \tag{9}$$

where Γ_{ij}^k are Christoffel's symbols and c_{ij}^k are functions on M for $i,j,k = 1,2$. On the other hand, we have

$$\Phi_{ss}(s,t) = \kappa(s,t)N(s,t),$$

$$\Phi_{st}(s,t) = \kappa^2(s,t)\tau(s,t)\sigma(s,t)B_1(s,t) + (-2\kappa(s,t)\kappa_s(s,t)\tau(s,t) - \kappa^2(s,t)\tau_s(s,t))B_2(s,t),$$

$$\begin{aligned} \Phi_{tt}(s,t) &= \kappa^2(s,t)\tau(s,t)\xi_{14}(s,t)T(s,t) + \kappa^2(s,t)\tau(s,t)\xi_{24}(s,t)N(s,t) \\ &\quad + \kappa^2(s,t)\tau(s,t)\xi_{34}(s,t)B_1(s,t) + (-2\kappa(s,t)\kappa_t(s,t)\tau(s,t) - \kappa^2(s,t)\tau_t(s,t))B_2(s,t). \end{aligned}$$

Then we calculate

$$c_{11}^1(s,t) = \langle \Phi_{ss}(s,t), N(s,t) \rangle = \kappa(s,t),$$

$$c_{11}^2(s,t) = \langle \Phi_{ss}(s,t), B_1(s,t) \rangle = 0,$$

$$c_{12}^1(s,t) = \langle \Phi_{st}(s,t), N(s,t) \rangle = 0,$$

$$c_{22}^1(s,t) = \langle \Phi_{tt}(s,t), B_1(s,t) \rangle = \kappa^2(s,t)\tau(s,t)\xi_{24}(s,t),$$

$$c_{12}^2(s,t) = \langle \Phi_{st}(s,t), N(s,t) \rangle = \kappa^2(s,t)\tau(s,t)\sigma(s,t),$$

$$c_{22}^2(s,t) = \langle \Phi_{tt}(s,t), B_1(s,t) \rangle = \kappa^2(s,t)\tau(s,t)\xi_{34}(s,t).$$

We introduce the functions as follows:

$$\Delta_1(s,t) = \kappa^3(s,t)\tau(s,t)\sigma(s,t),$$

$$\Delta_2(s,t) = \kappa^3(s,t)\tau(s,t)\xi_{34}(s,t),$$

$$\Delta_3(s,t) = -\kappa^4(s,t)\tau^2(s,t)\sigma(s,t)\xi_{24}(s,t).$$

After computations, one can easily obtain coefficients l_{11}, l_{12} and l_{22} of the second fundamental form as follows:

$$l_{11}(s,t) = 2\kappa(s,t)\sigma(s,t),$$

$$l_{12}(s,t) = \kappa(s,t)\xi_{34}(s,t),$$

$$l_{22}(s,t) = -2\kappa^2(s,t)\tau(s,t)\sigma(s,t)\xi_{24}(s,t).$$

Now, consider the linear map $\gamma : T_pM \to T_pM$ of Weingarten type in the tangent space of the soliton surface M determined by the conditions

$$\gamma(\Phi_s) = \gamma_1^1\Phi_s + \gamma_1^2\Phi_t,$$

$$\gamma(\Phi_t) = \gamma_2^1\Phi_s + \gamma_2^2\Phi_t$$

where

$$\gamma_1^1(s,t) = -2\kappa(s,t)\sigma(s,t),$$
$$\gamma_1^2(s,t) = -\frac{\xi_{34}(s,t)}{\kappa^3(s,t)\tau^2(s,t)},$$
$$\gamma_2^1(s,t) = -\kappa(s,t)\xi_{34}(s,t),$$

and

$$\gamma_2^2(s,t) = \frac{2\sigma(s,t)\xi_{24}(s,t)}{\kappa^2(s,t)\tau(s,t)}.$$

So we obtain following equalities

$$\gamma\left(\Phi_s(s,t)\right) = -2\kappa(s,t)\sigma(s,t)T(s,t) + \frac{\xi_{34}(s,t)}{\kappa(s,t)\tau(s,t)}B_2(s,t),$$

$$\gamma\left(\Phi_t(s,t)\right) = -\kappa(s,t)\xi_{34}(s,t)T(s,t) - 2\sigma(s,t)\xi_{24}(s,t)B_2(s,t).$$

Clearly, the soliton surface M lies in a $2-$plane if and only if M is totally geodesic, i.e., $c_{ij}^k = 0$. At least one of the coefficients c_{ij}^k is not zero, since the metric induced on M is nondegenerate, i.e., $\kappa(s,t) \neq 0$. The second fundamental tensor of the soliton surface M is expressed as follows:

$$\Pi(\Phi_s, \Phi_s) = \kappa N,$$

$$\Pi(\Phi_s, \Phi_t) = \kappa^2 \tau\sigma B_1,$$

$$\Pi(\Phi_t, \Phi_t) = \kappa^2 \tau \xi_{24} N + \kappa^2 \tau \xi_{34} B_1.$$

□

Theorem 3.2. *If $\Phi = \Phi(s,t)$ is a solution of Betchov-Da Rios equation, then the invariants of the soliton surface $M : \Phi = \Phi(s,t)$ are*

$$k(s,t) = \frac{-4\kappa(s,t)\sigma^2(s,t)\tau(s,t)\xi_{24}(s,t) - \xi_{34}^2(s,t)}{\kappa^2(s,t)\tau^2(s,t)},$$

$$h(s,t) = \frac{-\sigma(s,t)\xi_{24}(s,t) + \kappa^3(s,t)\tau(s,t)\sigma(s,t)}{\kappa^2(s,t)\tau(s,t)}$$

respectively.

Proof. We have found the coefficients of first fundamental form of M as $g_{11} = 1$, $g_{12} = 0$ and $g_{22} = \kappa^4(s,t)\tau^2(s,t)$. And the coefficients l_{11}, l_{12} and l_{22} of the second fundamental form of M are given as $l_{11} = 2\kappa(s,t)\sigma(s,t)$, $l_{12} = \kappa(s,t)\xi_{34}(s,t)$ and $l_{22} = -2\kappa^2(s,t)\tau(s,t)\sigma(s,t)\xi_{24}(s,t)$. Thus, the invariant k of the soliton surface M is given by

$$k(s,t) = \det\gamma(s,t) = \frac{-4\kappa(s,t)\sigma^2(s,t)\tau(s,t)\xi_{24}(s,t) - \xi_{34}^2(s,t)}{\kappa^2(s,t)\tau^2(s,t)},$$

and the invariant h of the soliton surface $\Phi(s,t)$ is obtain as follows:

$$h(s,t) = -\frac{1}{2} tr\ \gamma(s,t) = \frac{-\sigma(s,t)\xi_{24}(s,t) + \kappa^3(s,t)\tau(s,t)\sigma(s,t)}{\kappa^2(s,t)\tau(s,t)}.$$

The orthonormal frame field of the tangent space to M are obtained as

$$X = \frac{\Phi_s}{\|\Phi_s\|} = \frac{T}{\|T\|} = T,$$
$$Y = \frac{\Phi_t}{\|\Phi_t\|} = \frac{-\kappa^2\tau B_2}{\|-\kappa^2\tau B_2\|} = B_2.$$

Thus, we may write

$$\gamma(T(s,t)) = -2\kappa(s,t)\sigma(s,t)T(s,t) + \frac{\xi_{34}(s,t)}{\kappa(s,t)\tau(s,t)}B_2(s,t),$$

$$\gamma(B_2(s,t)) = \frac{\xi_{34}(s,t)}{\kappa(s,t)\tau(s,t)}T(s,t) + \frac{2\sigma(s,t)\xi_{24}(s,t)}{\kappa^2(s,t)\tau(s,t)}B_2(s,t).$$

According to orthonormal frame field $\{X, Y\}$ of the tangent space to M, we may rewrite

$$\gamma(X) = \overline{\gamma}_1^1 X + \overline{\gamma}_1^2 Y,$$

$$\gamma(Y) = \overline{\gamma}_2^1 X + \overline{\gamma}_2^2 Y$$

where

$$\overline{\gamma}_1^1(s,t) = -2\kappa(s,t)\sigma(s,t),\ \overline{\gamma}_1^2(s,t) = \frac{\xi_{34}(s,t)}{\kappa(s,t)\tau(s,t)},$$
$$\overline{\gamma}_2^1(s,t) = \frac{\xi_{34}(s,t)}{\kappa(s,t)\tau(s,t)}, \overline{\gamma}_2^2(s,t) = \frac{2\sigma(s,t)\xi_{24}(s,t)}{\kappa^2(s,t)\tau(s,t)}.$$

Then, we can easily see that $\gamma^T = \gamma$. This means that γ is a symmetric linear operator. If X and Y are two tangent vectors of the tangent space to M at an arbitrary point P, then we have

$$\langle \gamma(X), Y \rangle = \langle \gamma(Y), X \rangle.$$

Hence, with the help of the orthonormal tangent vectors X and Y, the second fundamental tensor of the soliton surface M is expressed as follows:

$$\Pi(X, X) = h_{11}^1 N + h_{11}^2 B_1,$$

$$\Pi(X, Y) = h_{12}^1 N + h_{12}^2 B_1,$$

$$\Pi(Y, Y) = h_{22}^1 N + h_{22}^2 B_1$$

where

$$h_{11}^k = c_{11}^k,$$
$$h_{12}^k = -\frac{c_{12}^k}{\kappa^2\tau},$$
$$h_{22}^k = \frac{c_{22}^k}{\kappa^4\tau^2}$$

for $k = 1, 2$. We obtain the characteristic equation of the linear map γ as follows:

$$v^2 + 2hv + k = 0.$$

Therefore, we get

$$h^2 - k \geqslant 0.$$

Then, it follows that

$$4(h^2 - k) = \frac{(2\sigma\xi_{24} + 2\kappa^3\tau\sigma)^2 + (\xi_{34}\kappa)^2}{\kappa^4\tau^2}.$$

This equality implies that the condition $h^2 - k = 0$ is equivalent to the equalities

$$2\sigma\xi_{24} + \kappa^3\tau\sigma = 0, \quad \xi_{34}\kappa = 0.$$

□

Corollary 3.3. *Suppose $\Phi = \Phi(s,t)$ is a solution of Betchov-Da Rios equation. The soliton surface $M : \Phi = \Phi(s,t)$ is minimal if and only if*

$$(\kappa^2\tau)_{ss} = \kappa^2\tau(\sigma^2 + \tau^2),$$

$$\left(-\kappa^2\tau\right)_s \kappa + \frac{\sigma}{\kappa}[\left(\kappa^2\tau\sigma\right)_s + \left(\kappa^2\tau\right)_s \sigma] + (\frac{1}{\kappa}\left(\kappa^2\tau\sigma^2 + (-\kappa^2\tau)_{ss}\right))_s = 0.$$

Proof. The necessary and sufficient condition of soliton surface M to be a minimal soliton surface is that $h^2 - k = 0$. Above remark shows that $h^2 - k = 0$ if and only if

$$\xi_{24} = -\kappa^3\tau, \quad \xi_{34} = 0.$$

By substituting the values of ξ_{24} and ξ_{34}, we get the proof. □

Corollary 3.4. *Suppose $\Phi = \Phi(s,t)$ is a solution of Betchov-Da Rios equation. If*

$$-4\kappa\sigma^2\tau\xi_{24} - \xi_{34}^2 = 0 \text{ and } -\sigma\xi_{24} + \kappa^3\tau\sigma = 0,$$

then the soliton surface $M : \Phi = \Phi(s,t)$ consists of flat points.

Proof. We know that if $h = k = 0$, then the soliton surface M consists of flat points. The equation $h = k = 0$ implies that

$$-4\kappa\sigma^2\tau\xi_{24} - \xi_{34}^2 = 0 \text{ and } -\sigma\xi_{24} + \kappa^3\tau\sigma = 0.$$

Thus, we get the proof. □

Remark 3.5. On the other hand, we also get the following equality

$$\Delta_1^2 + \Delta_2^2 + \Delta_3^2 = \kappa^6\tau^2\left(\sigma^2 + \xi_{34}^2 + \kappa^2\sigma^2\tau^2\xi_{24}^2\right).$$

If $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then these points are flat points of the soliton surface. This implies that if $\sigma = \xi_{34} = 0$, then the soliton surface $M : \Phi = \Phi(s,t)$ consists of flat points. If $\sigma = \xi_{34} = 0$, then we obtain the same results given in above corollary.

Theorem 3.6. *If $\Phi = \Phi(s,t)$ is a solution of Betchov-Da Rios equation, then the mean curvature vector field of the soliton surface $M : \Phi = \Phi(s,t)$ is obtained by*

$$H = \frac{1}{2}(\kappa + \frac{\xi_{24}}{\kappa^2\tau})N + \frac{\xi_{34}}{2\kappa^2\tau}B_1.$$

Proof. By using the equations in obtained results, we find the mean curvature vector field of the soliton surface M as follows:

$$\begin{aligned} H &= \frac{1}{2}(\Pi(X,X) + \Pi(Y,Y)) \\ &= \frac{1}{2}(\kappa N + \frac{\kappa^2\tau\xi_{24}}{\kappa^4\tau^2}N + \frac{\kappa^2\tau\xi_{34}}{\kappa^4\tau^2}B_1) \\ &= \frac{1}{2}(\kappa + \frac{\xi_{24}}{\kappa^2\tau})N + \frac{\xi_{34}}{2\kappa^2\tau}B_1. \end{aligned}$$

□

Theorem 3.7. *If $\Phi = \Phi(s,t)$ is a solution of Betchov-Da Rios equation, then Gaussian curvature of the soliton surface $M : \Phi = \Phi(s,t)$ is given by*

$$K = \frac{\xi_{24}}{\kappa\tau} - \sigma^2.$$

Proof. By using obtained values of h_{ij}^k, Gaussian curvature of the soliton surface M is derived as follows:

$$K = \sum_{k=1}^{2} h_{11}^k h_{22}^k - (h_{12}^k)^2 = \kappa\frac{\xi_{24}}{\kappa^2\tau} - (-\sigma)^2 = \frac{\xi_{24}}{\kappa\tau} - \sigma^2.$$

□

4. Application

In this section, we investigate an soliton surface associated with Betchov-Da Rios equation as an application. Consider the soliton surface with the following parametrization

$$\Phi(s,t) = (3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{4s}{5}, -\frac{36}{625}t + 5).$$

And Frenet frames of the curves $\Phi = \Phi(s,t)$ for all $t \in \mathbb{R}$ on Betchov-Da Rios soliton surface are given as follows:

$$T(s,t) = (-\frac{3}{5}\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5}, \frac{4}{5}, 0),$$

$$N(s,t) = (-\cos\frac{s}{5}, -\sin\frac{s}{5}, 0, 0),$$

$$B_1(s,t) = (\frac{4}{5}\sin\frac{s}{5}, -\frac{4}{5}\cos\frac{s}{5}, \frac{3}{5}, 0),$$

$$B_2(s,t) = (0,0,0,1).$$

Then, we obtain the curvature functions as

$$\kappa(s,t) = \frac{3}{25},\ \tau(s,t) = \frac{4}{25},\ \sigma(s,t) = 0.$$

Furthermore, we get the geometric invariants of the soliton surface

$$k = 0 \text{ and } h = 0.$$

Thus, the given soliton surface consists of flat points. Figure 1 illustrates Betchov-Da Rios soliton surface with projection on the plane $(x+y, z, w)$.

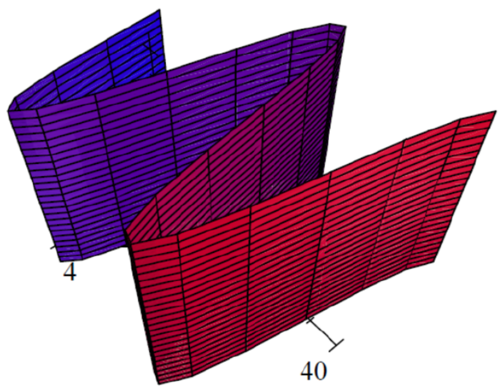

Figure 1. Betchov-Da Rios soliton surface with projection on plane $(x+y, z, w)$

Figure 2 illustrates Betchov-Da Rios soliton surface with projection on plane $(x, y, z+w)$.

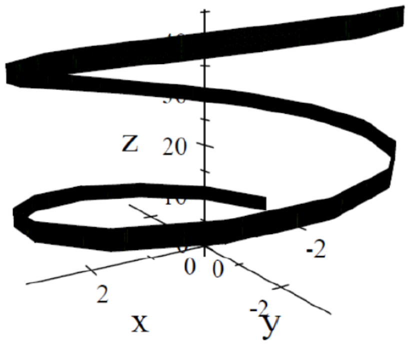

Figure 2. Betchov-Da Rios soliton surface with projection on plane $(x, y, z+w)$

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