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A New Beta Power Generator for Continuous Random Variable: Features and Inference to Model Asymmetric Data

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Abstract: Statistical methodologies have broad applications in sports and other exercise sciences. These methods can be used to predict the winning probability of a team or individual in a match. Due to the applicability of the statistical methods in sports, this paper introduces a new method of obtaining statistical distributions. The new method is called a novel beta power-*L* family of distributions. Some mathematical characteristics of the new family are obtained. Based on the novel beta power-*L* family, a special model, namely, a novel beta power Weibull model is studied. Finally, the applicability/usefulness of the novel beta power Weibull distribution is shown by analyzing the time-to-even data taken from different football matches during 1964-2018. The data consist of seventy-eight observations and is representing the waiting time duration of the fastest goal scored ever in the history of football. The fitting results of the novel beta power Weibull distribution are compared with other models. Based on three model selection criteria, it is observed that the proposed novel beta power Weibull model provides a close fit to the waiting time data.

Keywords: Power transformation; Statistical model; Hazard function; Aging; Statistics and numerical data; Simulation; Comparative study.

1 Introduction

Sports refer to the competitive physical activity or game in which an individual or whole team competes against others. Among the sports games, football is an interesting game with a rich and long history. It is believed that the development of this game dates back to the 12^{th} century, which originated from England. In this game, two teams, each with 11 players, play against each other with a spherical ball. Football is controlled and governed by FIFA (International Federation of Association Football) which conducts world cups for women and men after every four years. For more information about football and FIFA, we refer to [1] and [2]. The theory and practice of statistics are widely used in sports, particularly, the statistical analyses of sport have become popular in the professional setting. Appropriate implementation of statistical methods is becoming more and more important in sports. There is consensus that the inappropriate implementation of the statistical methodology and study design leads to incorrect results, poor interpretation of the research/study findings, and wrong conclusions. For more information about the applications of statistical methods in sports, we refer to [3,4,5,6,7,8]. There are now numerous peer-reviewed journals that welcome innovative statistical methodologies for dealing with or analyzing data in sports and other exercise fields. Furthermore, a number of high-profile conferences are organized each year for presenting statistical methodologies in sports research. In addition, a new growing Section on statistics in sports within the ASA (American Statistical Association) encouraging researchers to look for new statistical methodologies [9]. In statistical distributions theory, the two-parameter Weibull

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model is an interesting model. It has been widely implemented to handle time-to-event data. The Weibull CDF, say $A(x;\boldsymbol{\phi})$, can be proposed as

$$A(x;\boldsymbol{\phi}) = 1 - e^{-\eta x^{\lambda}}; \qquad x \geq 0, \tag{1}$$

where $\boldsymbol{\phi} = (\lambda, \eta)$, $\lambda > 0$ and $\eta > 0$. This model can be applied to discuss various data in different sectors such as: Medical sciences [10,11], reliability engineering [12,13,14], and finance sector [15]. For more detailed information about the modifications and usefulness of the Weibull distribution [16,17,18,19]. The PDF, say $a(x;\boldsymbol{\phi})$, of the Weibull model can be formulated as

$$a(x;\boldsymbol{\phi}) = \eta\lambda x^{\lambda-1} e^{-\eta x^{\lambda}}; \qquad x > 0, \tag{2}$$

with failure/hazard function (HF) $h(x;\boldsymbol{\phi})$ given by

$$h(x;\boldsymbol{\phi}) = \eta\lambda x^{\lambda-1}; \qquad x > 0. \tag{3}$$

The HF can be increasing for $\lambda > 1$, decreasing for $\lambda < 1$, or constant for $\lambda = 1$. Thus, the Weibull model is an impressive model for analyzing data whose HF is either increasing, decreasing, or constant. However, in most cases, the HF of the data behaves non-monotonically [20]. Therefore, in such cases, the Weibull is not a suitable candidate model to apply [21]. To obtain more flexible forms of the Weibull distribution, numerous contributions have been done in the literature. For example, [22] introduced the BE-Weibull (beta extended Weibull) family, [23] proposed the Weibull-G family, [24] proposed the (TW-G) transmuted Weibull-G family, [25] studied the ExW-G (extended Weibull-G) family, [26] listed odd flexible Weibull-H class, [27] introduced the bivariate odd Weibull-G family, [28] proposed a discrete analogue of odd Weibull-G family, odd LoL Weibull-G family by [29], and [30] introduced the additive odd log-logistic (LoL) odd Weibull-G family. On the other hand, various probability classes based on different models were proposed and discussed in-detail, for instance, odd LoL and Burr-X transmuted-G families by[31,32], LoL tan generalized family by [33], Gudermannian generated family by [34], exponentiated-G family [35], and generalized heavy-tailed family by [36].

In this paper, we introduce a new method for obtaining modified and flexible versions of the available distributions in the so-called NBP-L (novel beta power-L) family. Let X have the NBP-L family, then its CDF $K(x;\theta,\boldsymbol{\phi})$ is given by

$$K(x;\theta,\beta,\boldsymbol{\phi}) = 1 - \left(1 - \frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta}; \ x \in \mathbb{R}, \tag{4}$$

where $\theta > 0, \beta > 0, \boldsymbol{\phi} \in \mathbb{R}$, and $A(x;\boldsymbol{\phi})$ is a CDF of any sub-model. In order to prove that the CDF is a valid CDF, we have two propositions.

Proposition 1. For the expression $K(x;\theta,\beta,\boldsymbol{\phi})$ obtained in Eq. (4), we must prove that

$$\lim_{x\to-\infty} K(x;\theta,\beta,\boldsymbol{\phi}) = 0 \quad \text{and} \quad \lim_{x\to\infty} K(x;\theta,\beta,\boldsymbol{\phi}) = 1.$$

Proof.

$$\begin{aligned}\lim_{x\to-\infty} K(x;\theta,\beta,\boldsymbol{\phi}) &= \lim_{x\to-\infty}\left\{1 - \left(1 - \frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta}\right\} \\ &= 1 - \left(1 - \frac{A(-\infty;\boldsymbol{\phi})}{\beta^{1-A(-\infty;\boldsymbol{\phi})}}\right)^{\theta} = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\infty} K(x;\theta,\beta,\boldsymbol{\phi}) &= \lim_{x\to\infty}\left\{1 - \left(1 - \frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta}\right\} \\ &= 1 - \left(1 - \frac{A(\infty;\boldsymbol{\phi})}{\beta^{1-A(\infty;\boldsymbol{\phi})}}\right)^{\theta} = 1.\end{aligned}$$

Proposition 2. The CDF $K(x;\theta,\beta,\boldsymbol{\phi})$ is differentiable and RC (right continuous).

Proof.

$$\frac{d}{dx}K(x;\theta,\beta,\boldsymbol{\phi}) = k(x;\theta,\beta,\boldsymbol{\phi}).$$

From propositions 1 and 2, we observe that the function provided in Eq. (4) is a valid CDF. For $\theta > 0, \beta > 0, x \in \mathbb{R}, \boldsymbol{\phi} \in \mathbb{R}$, the PDF $k(x;\theta,\boldsymbol{\phi})$ and HF $h(x;\theta,\boldsymbol{\phi})$ associating to Eq. (4), are respectively, given by

$$k(x;\theta,\beta,\boldsymbol{\phi}) = V(x;\theta,\beta,\boldsymbol{\phi})\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta-1} \tag{5}$$

and

$$h(x;\theta,\beta,\boldsymbol{\phi}) = V(x;\theta,\beta,\boldsymbol{\phi})\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{-1},$$

where $\frac{d}{dx}A(x;\boldsymbol{\phi}) = a(x;\boldsymbol{\phi})$ and

$$V(x;\theta,\beta,\boldsymbol{\phi}) = \frac{\theta a(x;\boldsymbol{\phi})\left[1+(\log\beta)A(x;\boldsymbol{\phi})\right]}{\beta^{1-A(x;\boldsymbol{\phi})}}.$$

Our motivations to report the NBP-*L* family are: To propose asymmetric and symmetric distributions which are capable of modeling skewed and normal data. Further, it can also be utilized in a variety of applied problems in many fields like sports, survival analysis, astronomy, agriculture, medicine, economics, among others; to generate special distributions including heavy-tailed which are capable of analyzing different kinds of failure rates; and to create consistently better fits than other created distributions under the same baseline distribution.

The outline of the reported paper can be listed as follows: In Section 2, some statistical features of the NBP-*L* class are derived including the identifiability, rth moment, lifecycle predictions, and order statistics. In Section 3, a special model of the new class, NBP-Weibull, is discussed in detail. In Section 4, the maximum likelihood technique is derived to report the class estimators, and the behavior of these estimators is tested and evaluated in Section 5 based on the NBP-Weibull model as an example. In Section 6, the popularity of the NBP-Weibull model is tested. Finally, some concluding remarks are listed in Section 7.

2 Mathematical Properties

2.1 Identifiability property (IP)

Here, we derive the identifiability properties of the NBP-*X* family using the additional parameters θ and β. Let θ_1 has the CDF $K(x;\theta_1,\beta,\boldsymbol{\phi})$ and θ_2 has the CDF $K(x;\theta_2,\beta,\boldsymbol{\phi})$. Then, the parameter θ is identifiable due to $\theta_1 = \theta_2$ for

$$K(x;\theta_1,\beta,\boldsymbol{\phi}) = K(x;\theta_2,\beta,\boldsymbol{\phi}), \tag{6}$$

where

$$1-\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta_1} = 1-\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta_2},$$

$$\theta_1\log\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right) = \theta_2\log\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right),$$

after some algebraic simplifications, we find that $\theta_1 = \theta_2$. Similarly, the derivation of IP using the parameter β can be reported as follows: Assume β_1 has the CDF $K(x;\theta,\beta_1,\boldsymbol{\phi})$ and β_2 has the CDF $K(x;\theta,\beta_2,\boldsymbol{\phi})$. Based on the IP concept, let

$$K(x;\theta,\beta_1,\boldsymbol{\phi}) = K(x;\theta,\beta_2,\boldsymbol{\phi}), \tag{7}$$

then $\beta_1^{1-A(x;\boldsymbol{\phi})} = \beta_2^{1-A(x;\boldsymbol{\phi})}$, after some simplifications, we get $e^{\log(\beta_1)} = e^{\log(\beta_2)}$ which is lead to $\beta_1 = \beta_2$.

2.2 Moments

In this passage, rth moment is derived. Depending on these moments, different statistical concepts such as mean, variance, skewness, kurtosis, and index of dispersion can be reported. These measures can be used as a statistical tool for discussing and evaluating the data under study. Determining the shape of the data requires at least three metrics, including skewness ”positively skewed, negatively skewed, symmetric”, kurtosis ”leptokurtic, platykurtic, mesokurtic”, and scattering/dispersion index ”over-dispersed, under-dispersed, equi-dispersed” coefficients. Based on the previous reasons, the authors discussed the rth moment of the NBP-L class. Let X follows the NBP-L family with parameters $(\theta,\beta,\boldsymbol{\phi})$, then the r^{th} moment of X is derived as

$$\mu_r^{/}=\int_{\Omega}x^r\frac{\theta a(x;\boldsymbol{\phi})\left[1+(\log\beta)A(x;\boldsymbol{\phi})\right]}{\beta^{1-A(x;\boldsymbol{\phi})}}\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta-1}dx,$$

after using the binomial expansion, we have

$$\begin{aligned}\mu_r^{/}&=\sum_{i=0}^{\infty}(-1)^i\binom{\theta-1}{i}\int_{\Omega}x^n\frac{V^*(x;\theta,\beta,\boldsymbol{\phi})\left[A(x;\boldsymbol{\phi})\right]^i}{\left[\beta^{1-A(x;\boldsymbol{\phi})}\right]^{i+1}}dx\\&=\sum_{i,j=0}^{\infty}\Upsilon_{(\theta,\beta)}^{(i,j)}\int_{\Omega}x^nV^*(x;\theta,\beta,\boldsymbol{\phi})\left[A(x;\boldsymbol{\phi})\right]^i\left[1-A(x;\boldsymbol{\phi})\right]^j dx\\&=\sum_{i,j,k=0}^{\infty}\Upsilon_{(\theta,\beta)}^{(i,j,u)}\int_{\Omega}x^n\frac{V^*(x;\theta,\beta,\boldsymbol{\phi})}{\theta}\left[A(x;\boldsymbol{\phi})\right]^{i+u}dx\\&=\sum_{i,j,k=0}^{\infty}\Upsilon_{(\theta,\beta)}^{(i,j,u)}\begin{pmatrix}\int_{\Omega}x^na(x;\boldsymbol{\phi})\left[A(x;\boldsymbol{\phi})\right]^{i+u}dx\\+(\log\beta)\int_{\Omega}x^na(x;\boldsymbol{\phi})\left[A(x;\boldsymbol{\phi})\right]^{i+u+1}dx\end{pmatrix}\\&=\sum_{i,j,k=0}^{\infty}\Upsilon_{(\theta,\beta)}^{(i,j,u)}\begin{pmatrix}\frac{1}{(i+u+1)}\int_{\Omega}x^nG_{i+u+1}(x;\boldsymbol{\phi})dx\\+\frac{\log\beta}{(i+u+2)}\int_{\Omega}x^nG_{i+u+2}(x;\boldsymbol{\phi})dx\end{pmatrix},\end{aligned}\qquad(8)$$

where

$$V^*(x;\theta,\beta,\boldsymbol{\phi})=\theta a(x;\boldsymbol{\phi})\left[1+(\log\beta)A(x;\boldsymbol{\phi})\right],$$

$$\Upsilon_{(\theta,\beta)}^{(i,j)}=\frac{(-1)^{i+j}(i+1)^j(\log\beta)^j}{j!}\binom{\theta-1}{i},$$

$$\Upsilon_{(\theta,\beta)}^{(i,j,u)}=\frac{(-1)^{i+j+u}(i+1)^j(\log\beta)^j\theta}{j!}\binom{\theta-1}{i}\binom{j}{u}$$

and

$$g_{i+u+s}(x;\boldsymbol{\phi})=(i+u+s)a(x;\boldsymbol{\phi})\left[A(x;\boldsymbol{\phi})\right]^{(i+u+s)-1};\ s=1,2,$$

is the PDF of the exponentiated random variable with exponentiated parameter $(i+u+s)$. To obtain the mean, variance, skewness, kurtosis, and index of dispersion, the statistical relationships known in the literature can be used “no need to mention these laws again”.

2.3 Lifecycle predictions

In this passage, some reliability concepts are derived including mean time between failure (MBF), mean time to failure (MTF), and availability (AVB). For any component/system/element, MTF, MBF, and AVB must be calculated and discussed in detail. This is due to the maintenance process for this component/system/item. Based on the first moment around zero, say $\mu_1^{/}$, some of these measures can be derived. If X follows the NBP-L family with parameters $\theta_1(\theta_2)$ and $\beta_1(\beta_2)$ then the MBF and MTF can be formulated as

$$\text{MBF}=\frac{-x}{\ln(1-K(x;\theta_1,\beta_1,\phi_1))};\ x>0\qquad(9)$$

and

$$\text{MTF}=\mu_1^{/}|_{(\theta_2,\beta_2,\phi_2)},\qquad(10)$$

respectively. Further, the probability that the component/system/item is successful at time x can be expressed as

$$\text{AVB} = -\mu_1^{/}|_{(\theta_2,\beta_2,\phi_2)} \frac{\ln(1-K(x;\theta_1,\beta_1,\phi_1))}{x};\ x>0. \tag{11}$$

2.4 Order statistics (OS)

Consider $X_1,\ldots,X_n$ be a random sample from the NBP-L family, and $X_{1:n},\ldots,X_{n:n}$ denotes the corresponding OS. Then, the CDF of the i^{th} OS can be expressed as follows

$$\begin{aligned} K_{i:n}(x;\theta,\beta,\boldsymbol{\phi}) &= \sum_{u=i}^{n}\binom{n}{u}\left[K(x;\theta,\beta,\boldsymbol{\phi})\right]^{u}\left[1-K(x;\theta,\beta,\boldsymbol{\phi})\right]^{n-u} \\ &= \sum_{u=i}^{n}\sum_{j=0}^{u}\binom{n}{u}\binom{u}{j}\left[1-K(x;\theta,\beta,\boldsymbol{\phi})\right]^{n-u+j} \\ &= \sum_{u=i}^{n}\sum_{j=0}^{u}\binom{n}{u}\binom{u}{j}\left(1-\frac{A(x;\boldsymbol{\phi})}{\beta^{1-A(x;\boldsymbol{\phi})}}\right)^{\theta(n-u+j)}. \end{aligned}$$

By taking the differentiation $K_{i:n}(x;\theta,\beta,\boldsymbol{\phi})$, we get the PDF of the i^{th} OS $k_{i:n}(x;\theta,\beta,\boldsymbol{\phi})$. Based on $k_{i:n}(x;\theta,\beta,\boldsymbol{\phi})$, we can get the moments of $X_{i:n}$ for the NBP-L family.

3 A Novel Beta Power-Weibull (Nbp-W) Distribution

Using $A(x;\boldsymbol{\phi}) = 1-e^{-\eta x^{\lambda}}$ in Eq. (4), we obtain a new modified version of the model defined in Eq. (1) . The new model is called a novel beta power-Weibull (NBP-W) distribution. Let X follows the proposed NBP-W model with shape parameters (θ,λ) and scale parameters (η,β), its CDF is given by

$$K(x;\theta,\beta,\boldsymbol{\phi}) = 1-\left(1-\frac{1-e^{-\eta x^{\lambda}}}{\beta^{e^{-\eta x^{\lambda}}}}\right)^{\theta}, \qquad x\geq 0. \tag{12}$$

Based on Eq. (12), a random sample cab be generated using the formula

$$Q(u) = 1+\frac{1-e^{-\eta x^{\lambda}}}{\beta^{e^{-\eta x^{\lambda}}}}+(1-u_q)^{\frac{1}{\theta}}, \tag{13}$$

where q represents the q^{th} quantile of u and $u\in(0,1)$. For $x>0$, the PDF of the NBP-W distribution is given by

$$k(x;\theta,\beta,\boldsymbol{\phi}) = \Delta(x;\theta,\beta,\eta,\lambda)\left(1-\frac{1-e^{-\eta x^{\lambda}}}{\beta^{e^{-\eta x^{\lambda}}}}\right)^{\theta-1}, \tag{14}$$

where

$$\Delta(x;\theta,\beta,\eta,\lambda) = \frac{\theta\eta\lambda x^{\lambda-1}e^{-\eta x^{\lambda}}\left[1+(\log\beta)\left(1-e^{-\eta x^{\lambda}}\right)\right]}{\beta^{e^{-\eta x^{\lambda}}}}.$$

Some plots of $k(x;\theta,\beta,\boldsymbol{\phi})$ are obtained in Figure 1. These plots are sketched for various values of the model parameters as follows: (A) $\theta=1.5,\lambda=0.6,\eta=1,\beta=1.5$ (Red-line), (B) $\theta=1.2,\lambda=1.2,\eta=1,\beta=0.5$ (Blue-line), (C) $\theta=1.2,\lambda=3.2,\eta=1,\beta=2$ (Green-line), and (D) $\theta=0.4,\lambda=4.2,\eta=1,\beta=2.5$ (Black-line). From Figure 1, we can observe that the PDF shape of the NBP-W can used as a probability tool to analyze asymmetric and symmetric data. For $x>0$, the HF $h(x;\theta,\beta,\boldsymbol{\phi})$ of the NBP-W distribution is given by

$$h(x;\theta,\beta,\boldsymbol{\phi}) = \Delta(x;\theta,\beta,\eta,\lambda)\left(1-\frac{1-e^{-\eta x^{\lambda}}}{\beta^{e^{-\eta x^{\lambda}}}}\right)^{-1}.$$

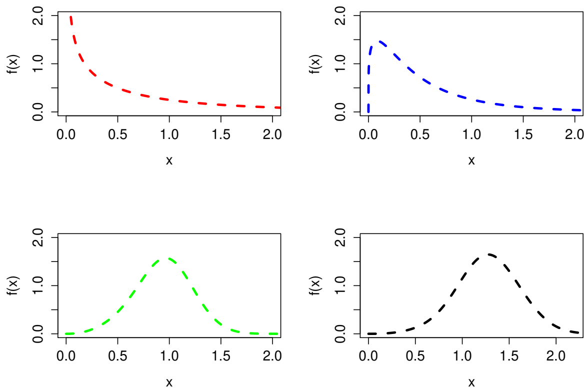

Fig. 1: Some plots of $k(x;\theta,\beta,\phi)$.

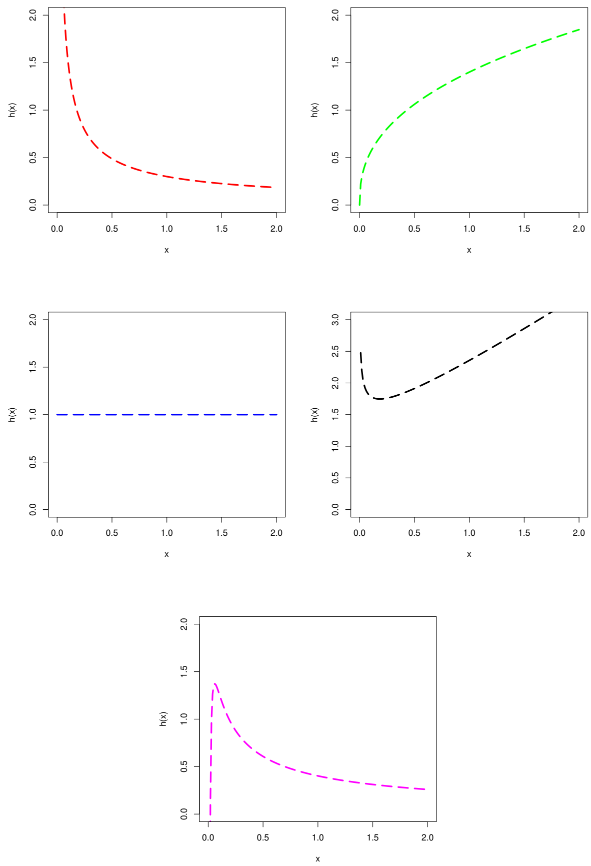

Fig. 2: Some plots of $h(x;\theta,\beta,\phi)$.

Some plots of $h(x;\theta,\beta,\phi)$ are sketched in Figure 2. These plots are sketched as follows: (A) $\theta=1,\lambda=0.3,\eta=1,\beta=1$ (Red-line), (B) $\theta=1,\lambda=1.4,\eta=1,\beta=1$ (Green-line), (C) $\theta=1,\lambda=1,\eta=1,\beta=1$ (Blue-line), (D) $\theta=0.8,\lambda=0.8,\eta=0.5,\beta=0.3$ (Black-line), and (E) $\theta=0.5,\lambda=0.3,\eta=0.5,\beta=0.08$ (Pink-line). From Figure 2, we can observe that the HF shape of the NBP-W can be either be decreasing, increasing, constant, bathtub, or unimodal.

The following models can be derived as special distributions from the NBP-W distribution:

1.NBP-W$(1,1,\lambda,\eta)$ follows the Weibull model with parameters λ and η.
2.NBP-W$(1,1,2,\eta)$ is the Rayleigh model with parameter $\eta>0$.
3.NBP-W$(1,1,1,\eta)$ refers to the exponential model with parameter $\eta>0$.

4 Maximum likelihood (MLH) technique

Consider a sample $X_1,X_2,...,X_m$ of size m taken from the NBP-L family with PDF $k(x;\theta,\beta,\phi)$. Corresponding to $k(x;\theta,\beta,\phi)$, the likelihood function (LHF) can be expressed as

$$L(\theta,\beta,\phi|x_1,x_2,...,x_m)=\prod_{r=1}^{m}k(x_r;\theta,\beta,\phi). \tag{15}$$

Using the PDF $k(x;\theta,\beta,\phi)$ in Eq. (15), we get

$$L(\theta,\beta,\phi|x_1,x_2,...,x_m)=\prod_{r=1}^{m}V(x_r;\theta,\beta,\phi)\left(1-\frac{A(x_r;\phi)}{\beta^{1-A(x_r;\phi)}}\right)^{\theta-1}. \tag{16}$$

The log LHF (LLHF) can be formulated as

$$\begin{aligned}\ell(\theta,\beta,\phi)=&\,m\log\theta+\sum_{r=1}^{m}\log a(x_r;\phi)+\sum_{r=1}^{m}\log[1+(\log\beta)A(x_r;\phi)]\\&-\sum_{r=1}^{m}[1-A(x_r;\phi)]\log\beta+(\theta-1)\sum_{r=1}^{m}\log\left(1-\frac{A(x_r;\phi)}{\beta^{1-A(x_r;\phi)}}\right).\end{aligned}$$

Thus, the partial derivatives of the LLHF are given by

$$\frac{\partial}{\partial\theta}\ell(\theta,\beta,\phi)=\frac{m}{\theta}+\sum_{r=1}^{m}\log\left(1-\frac{A(x_r;\phi)}{\beta^{1-A(x_r;\phi)}}\right),$$

$$\begin{aligned}\frac{\partial}{\partial\beta}\ell(\theta,\beta,\phi)=&\,\frac{1}{\beta}\sum_{r=1}^{m}\frac{A(x_r;\phi)}{[1+(\log\beta)A(x_r;\phi)]}-\frac{1}{\beta}\sum_{r=1}^{m}[1-A(x_r;\phi)]\\&+(\theta-1)\sum_{r=1}^{m}\frac{(A(x_r;\phi)-1)\beta^{A(x_r;\phi)-2}A(x_r;\phi)}{\left(1-\frac{A(x_r;\phi)}{\beta^{1-A(x_r;\phi)}}\right)}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial\phi}\ell(\theta,\beta,\phi)=&\sum_{r=1}^{m}\frac{\frac{\partial}{\partial\phi}a(x_r;\phi)}{a(x_r;\phi)}+\sum_{r=1}^{m}\frac{(\log\beta)\frac{\partial}{\partial\phi}A(x_r;\phi)}{[1+(\log\beta)A(x_r;\phi)]}\\&+(\log\beta)\sum_{r=1}^{m}\frac{\partial}{\partial\phi}A(x_r;\phi)-(\theta-1)\sum_{r=1}^{m}\frac{\beta^{1-A(x_r;\phi)}\Xi(x_r;\beta,\phi)}{\left(1-\frac{A(x_r;\phi)}{\beta^{1-A(x_r;\phi)}}\right)},\end{aligned}$$

where

$$\Xi(x_r;\beta,\phi)=\left[\left(\frac{\partial}{\partial\phi}A(x_r;\phi)\right)-A(x_r;\phi)(\log\beta)\right].$$

Solving $\frac{\partial}{\partial\theta}\ell(\theta,\beta,\phi)=0$, $\frac{\partial}{\partial\beta}\ell(\theta,\beta,\phi)=0$, and $\frac{\partial}{\partial\phi}\ell(\theta,\beta,\phi)=0$, provide the MLEs $\left(\hat{\theta},\hat{\beta},\hat{\phi}\right)$ of (θ,β,ϕ).

Table 1: Simulation results based on set I.

n	parameters	MLEs	MSEs	Biass
	θ	3.951200	10.454560	2.751200
25	λ	0.498070	4.411e-03	-1.929e-03
	η	0.576563	0.2527386	-0.423436
	β	3.868400	12.170440	2.968400
	θ	3.259600	7.826480	2.059600
50	λ	0.491456	1.522e-03	-8.543e-03
	η	0.678954	0.192016	-0.321045
	β	3.122200	9.111020	2.222200
	θ	2.834000	6.209200	1.634000
75	λ	0.490513	9.957e-04	-9.486e-03
	η	0.745762	0.151108	-0.254237
	β	2.663000	7.228300	1.763000
	θ	2.256400	4.014320	1.056400
100	λ	0.491252	6.338e-04	-8.747e-03
	η	0.834881	0.098496	-0.165118
	β	2.039800	4.673180	1.139800
	θ	1.511600	1.184080	0.311600
200	λ	0.497286	1.449e-04	-2.713e-03
	η	0.950945	0.029404	-0.049054
	β	1.236200	1.378420	0.336200
	θ	1.260800	0.231040	0.060800
300	λ	0.499466	2.134e-05	-5.331e-04
	η	0.990607	0.005518	-0.009392
	β	0.965600	0.268960	0.065600
	θ	1.222800	0.086640	0.022800
400	λ	0.499858	4.982e-06	-1.410e-04
	η	0.996300	0.002280	-0.003699
	β	0.924600	0.100860	0.024600
	θ	1.207600	0.028880	0.007600
500	λ	0.499951	1.199e-06	-4.897e-05
	η	0.998751	0.000779	-0.001248
	β	0.908200	0.033620	0.008200

5 Simulation: Performance Of The Estimators

In this subsection, we test the performances of $\hat{\theta}, \hat{\lambda}, \hat{\eta}$, and $\hat{\beta}$ of the NBP-W model by organizing a brief simulation study. To conduct this study, we use the inverse CDF technique to obtain the RNs (random numbers) from the NBP-W model. The simulation study was carried out for three different combination sets of the model parameters as follows: schema I: $\theta = 1.2, \lambda = 0.5, \eta = 1, \beta = 0.9$, schema II: $\theta = 1.4, \lambda = 0.5, \eta = 1, \beta = 1.2$, and schema III: $\theta = 0.8, \lambda = 0.5, \eta = 1.2, \beta = 1.5$. For all the three sets of parameters (I-III), a random sample of sizes $n = 25, 50, 75, ..., 500$ was generated using Eq. (13). By performing the simulation study, the values of the MLEs of θ, λ, η, and β, are obtained for each set of I, II, and III. Two statistical approaches, including bias and mean squared errors (MSEs) are considered to discuss the performances of $\hat{\theta}, \hat{\lambda}, \hat{\eta}$, and $\hat{\beta}$ where

$$MSE\left(\hat{\boldsymbol{\kappa}}\right) = \frac{1}{500}\sum_{r=1}^{500}\left(\hat{\boldsymbol{\kappa}}_r - \boldsymbol{\kappa}\right)^2$$

and

$$Bias\left(\hat{\boldsymbol{\kappa}}\right) = \frac{1}{500}\sum_{r=1}^{500}\left(\hat{\boldsymbol{\kappa}}_r - \boldsymbol{\kappa}\right),$$

where $\boldsymbol{\kappa} = (\theta, \lambda, \eta, \beta)$. The results of this section can be listed in Tables 1-3 and visually displayed in Figures 3-5.

According to these results, the MLH can be used effectively to estimate the NBP-W parameters because when the value of n increases, the magnitude of the bias and MSEs decrease toward zero.

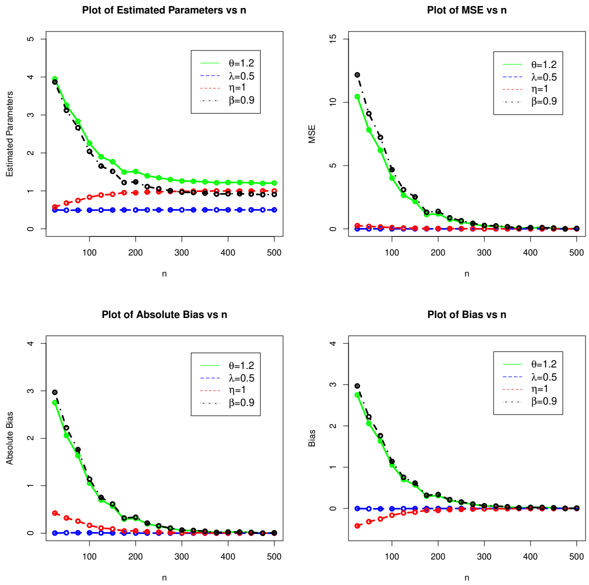

Fig. 3: Visually display of the results presented in Table 1.

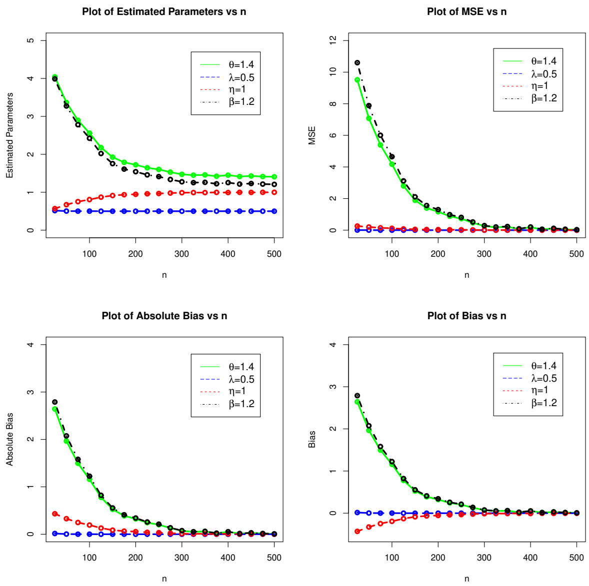

Fig. 4: Visually display of the results presented in Table 2.

6 Analyzing The Football Data

This section lists a practical elucidation of the NBP-W model by discussing and analyzing the time-to-event data. The data is from seventy-seven various football matches played between 1964-2018. The data are: 2.1, 2.2, 2.56, 2.8, 2.8, 3, 3, 3.17, 3.55, 3.57, 3.6, 3.69, 3.9, 4, 4, 4, 5, 5, 6, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7.22, 7.27, 7.3, 7.42, 7.66, 7.69, 7.7, 7.8, 8, 8, 8, 8, 8, 8, 8, 8.1, 8.1, 8.3, 8.7, 9, 9, 9, 9, 9.55, 9.6, 9.9, 10, 10, 10, 10, 10.12, 10.69, 10.8, 11, 11, 11, 12, 12, 12, 12, 12, 12, 12, 13, 13, 14, 14. Figure 6 shows some nonparametric plots for this data. Some observations of the data set represent the time waiting of the fastest goal ever scored in the history of football. For example, in 2017, Gavin Stokes scored the quickest goal in the history of football. He scored this goal in a match between Mary hill vs Clyde bank in 2.1 seconds.

Table 2: Simulation results based on set II.

n	parameters	MLEs	MSEs	Biass
25	θ	4.042400	9.512640	2.642400
	λ	0.515444	5.107e-03	1.544e-02
	η	0.567914	0.259840	-0.432085
	β	3.989200	10.598960	2.789200
50	θ	3.365600	7.076160	1.965600
	λ	0.503581	1.803e-03	3.581e-03
	η	0.671478	0.199025	-0.328521
	β	3.274800	7.884240	2.074800
75	θ	2.897600	5.391360	1.497600
	λ	0.501250	96.238e-04	1.250e-03
	η	0.752346	0.148293	-0.247653
	β	2.780800	6.007040	1.580800
100	θ	2.559200	4.173120	1.159200
	λ	0.501561	5.695e-04	1.561e-03
	η	0.806398	0.116935	-0.193601
	β	2.423600	4.649680	1.223600
200	θ	1.724000	1.166400	0.324000
	λ	0.499841	7.232e-05	-1.584e-04
	η	0.944875	0.033822	-0.055124
	β	1.542000	1.299600	0.342000
300	θ	1.472000	0.259200	0.0720
	λ	0.499998	5.606e-06	-1.969e-06
	η	0.987911	0.007310	-0.012088
	β	1.276000	0.288800	0.076000
400	θ	1.450400	0.181440	0.050400
	λ	0.500053	3.188e-06	5.337e-05
	η	0.991671	0.004960	-0.008328
	β	1.253200	0.202160	0.053200
500	θ	1.407200	0.025920	0.007200
	λ	0.500024	2.931e-07	2.421e-05
	η	0.998748	0.000783	-0.001251
	β	1.207600	0.028880	0.007600

In 2003, Frederico Chaves (a Brazilian footballer) scored the fastest goal in 3.17 seconds in the match between Amrica Minerio vs Vila Nova. It was the fastest goal scored in the history of Brazilian football. In 1979, Colin Cowperthwaite (an English footballer) scored a goal in 3.55 seconds in a match between Barrow v Kettering Town. At that time, it was believed to be the fastest goal in AF (association football). In 1979, Carlos DantAn Seppaquercia (an Argentinian footballer) scored a goal in 5 seconds in a match between Gimnasia y Esgrima LP vs HuracAn. It is considered the fastest goal at the APD (Argentine Primera DivisiAn). In 2017, Ghazi Ayadi (a Tunisian footballer) scored a goal in a match between Club Africain vs CS Sfaxien. He scored this goal in 6 seconds, which is the fastest goal in the TSL (Tunisian Super Ligue). In 2012, Ji Xiang (a Chinese footballer) scored a goal in 7 seconds in a match between Jiangsu Sainty v Guangzhou Evergrande. It was the fastest goal in CSL (Chinese Super League). In 2018, AntAnio Xavier (a Portuguese footballer) scored a goal in 14 seconds in a match between Tondela vs Portimonense. It was the fastest goal in the history of PL (Primeira Liga).

Using the data related to the waiting time till the first goal in different football games, we compare the performance of the NBP-Weibull distribution with the Weibull model with parameters $\lambda > 0$ and $\eta > 0$, exponentiated Weibull (E-Weibull) distribution with parameters $a > 0, \lambda > 0$ and $\eta > 0$, and Kumaraswamy Weibull (K-Weibull) distribution with parameters $a > 0, b > 0, \lambda > 0$, and $\eta > 0$. The SFs (survival functions) of the competing distributions are

–The Weibull distribution

$$S(x;\lambda,\eta) = e^{-\eta x^{\lambda}}, \qquad x \geq 0.$$

–The E-Weibull distribution

$$S(x;a,\lambda,\eta) = 1 - \left(1 - e^{-\eta x^{\lambda}}\right)^{a}, \qquad x \geq 0.$$

Table 3: Simulation results based on set III.

n	parameters	MLEs	MSEs	Biass
	θ	4.084400	13.794480	3.284400
25	λ	0.536848	9.089e-03	0.036848
	η	0.463453	0.696151	-0.736546
	β	4.237000	9.579500	2.737000
	θ	3.647600	11.959920	2.847600
50	λ	0.525351	3.150e-03	0.025351
	η	0.558370	0.6083025	-0.641629
	β	3.873000	8.305500	2.373000
	θ	3.168800	9.948960	2.368800
75	λ	0.515343	1.739e-03	0.015343
	η	0.667927	0.50253187	-0.532072
	β	3.474000	6.909000	1.974000
	θ	2.505200	7.161840	1.705200
100	λ	0.509171	9.871e-04	0.009171
	η	.8153475	0.364714	-0.384652
	β	2.921000	4.973500	1.421000
	θ	1.446800	2.716560	0.646800
200	λ	0.501845	1.185e-04	0.001845
	η	1.054672	0.137187	-0.145327
	β	2.039000	1.886500	0.539000
	θ	1.119200	1.340640	0.319200
300	λ	0.501050	7.319e-05	0.001053
	η	1.128365	0.067545	-0.0716347
	β	1.766000	0.931000	0.266000
	θ	1.035200	0.987840	0.235200
400	λ	0.500853	3.060e-05	0.000853
	η	1.146966	0.050235	-0.053033
	β	1.696000	0.686000	0.196000
	θ	0.875600	0.317520	0.075600
500	λ	0.500388	1.328e-05	0.000388
	η	1.182921	0.016207	-0.017078
	β	1.563000	0.220500	0.063000

–The K-Weibull distribution

$$S(x;a,b,\lambda,\eta) = \left(1-\left[1-e^{-\eta x^{\lambda}}\right]^{a}\right)^{b}, \qquad x \geq 0.$$

Next, to see which model provides the close fit to data, three analytical quantities "Anderson-Darling (AD), Cramer-von Mises (CM), Kolmogorov-Smirnov (KS)" with its p-value are discussed. The word "close fit" means that a model whose values for the analytical quantities are smaller. The values of these quantities can be derived as

$$AD = -m - \frac{1}{m}\sum_{r=1}^{m}(2r-1)\Phi(r,m),$$

$$CM = \frac{1}{12m} + \sum_{r=1}^{m}\left[\frac{2r-1}{2m} - K(x_r)\right]^2,$$

$$KS = sup_x\left[K_r(x) - K(x)\right],$$

where

$$\Phi(r,m) = \left[\log K(x_r) + \log\left\{1 - K(x_{m-r+1})\right\}\right].$$

We use the R software with the "method = SANN" algorithm for the analysis of the data. For the time-to-event data, the MLEs of the NBP-Weibull and other models are presented in Table 4, and values of the analytical quantities are reported in Table 5.

Based on the results reported in Table 5, we can observe that the NBP-Weibull distribution provides a close fit to the data. To support the close fit capability of the NBP-Weibull distribution, a visual display based on the numerical results

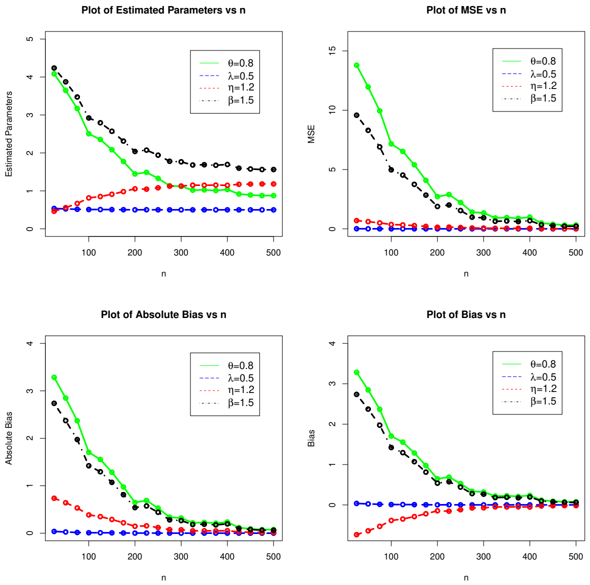

Fig. 5: Visually display of the results presented in Table 3.

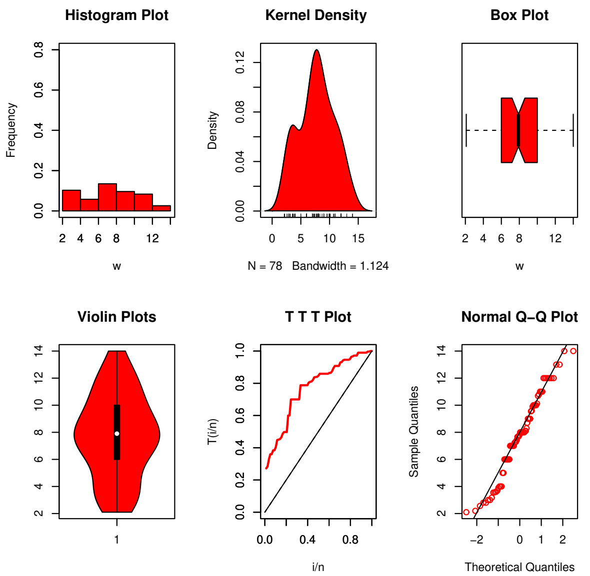

Fig. 6: The nonparametric plots for real data.

Table 4: The MLEs of the models.

Models	θ	η	λ	β	a	b
NBP-Weibull	2.1869	0.0050	2.3365	2.5972	-	-
Weibull	-	0.0023	2.8122	-	-	-
E-Weibull	-	0.0064	2.4055	-	1.1416	-
K-Weibull	-	0.0075	1.9164	-	1.6756	4.1447

Table 5: The goodness of fit measures of the models.

Models	CM	AD	KS	p-value
NBP-Weibull	0.0850	0.5972	0.0953	0.4773
Weibull	0.1038	0.6947	0.1042	0.3653
E-Weibull	0.1227	0.8021	0.1331	0.1259
K-Weibull	0.1307	0.8496	0.1173	0.2333

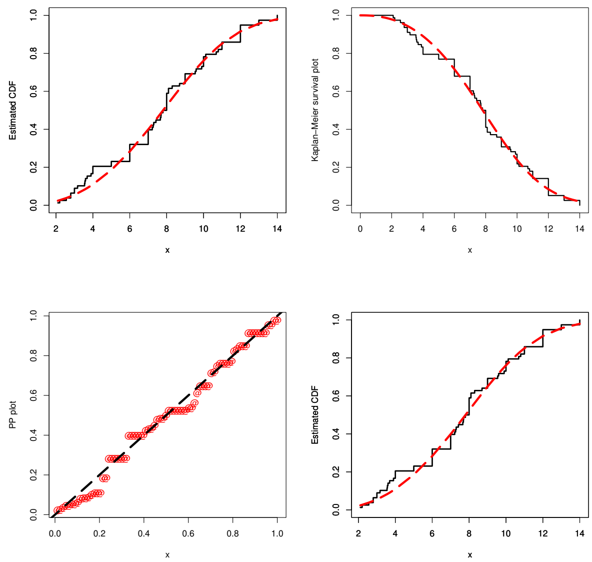

Fig. 7: The estimated CDF, KM, and PP plots of the NBP-Weibull distribution.

is provided in Figure 6. For this purpose, we consider the fitted CDF, SF, and PP (probability-probability) plots of the NBP-Weibull distribution. Figure 7 shows that the NBP-Weibull follows the empirical CDF and (KM) Kaplan-Meier SF very closely.

7 Concluding Remarks

Statistical methodologies are very crucial for modeling and analyzing time-to-event data. With the help of statistical methods, we can predict the performance of a team and its winning chances. This paper introduced a new approach to generate new statistical models, namely, a NBP-L family of distributions. Some mathematical and statistical properties were derived and discussed. After listing the general generator, a special member of the NBP-L family called, a NBP-Weibull distribution was reported. It was found that the NBP-Weibull model can be utilized to discuss asymmetric data under bi-modal shape. Moreover, the NBP-Weibull distribution can be applied to model different shapes/kinds of failure rates including increasing, decreasing, unimodal, bathtub, and constant. Thus, the generated model is flexible enough to read different types of data. The MLH approach was used to estimate the parameters of the NBP-Weibull model. Finally, the proposed NBP-Weibull distribution was implemented to model real-time data, and it was observed that the NBP-Weibull was best model for analyzing the waiting time data. In the future, the bivariate extension of this family will be discussed. Moreover, the discrete analogue will be derived with its features.

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