An Investigation into the Physical Properties Governing the Coupled Harmonic Oscillator

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Introduction

Objective

Investigate the properties of a coupled harmonic oscillator.

Methodology

Using a two-mass system connected with 3 springs, we wanted to determine the properties of the system, namely the frequency at which the system oscillates by comparing it to a simple harmonic oscillator and utilizing a Fourier transform to better understand our data and determine, experimentally, the resonant frequencies of the system.

Simply put, we wanted to know how a system of this nature behaves, and we utilized mathematical tricks and a simpler model to do so.

Theory: The Simple Harmonic Oscillator

- To best understand the coupled harmonic oscillator, it is necessary to first understand the simple harmonic oscillator. The simple harmonic oscillator is model of systems with oscillatory behavior, described by Hooke's Law. [1]

$$F_s = -kx$$

Figure 1. Hooke's Law. F is the force, k is a constant of proportionality known as the spring constant, and x the displacement.

- You may have seen simple harmonic oscillators in the real world in the form of pendulums and springs. They are also used in most clocks.

- The solution to the simple harmonic oscillator is derived from Newton's second law, force equals mass times acceleration, F=ma. The solution is, x = Acos(ωt), A representing the amplitude of the wave and ω = (k/m)^(1/2).

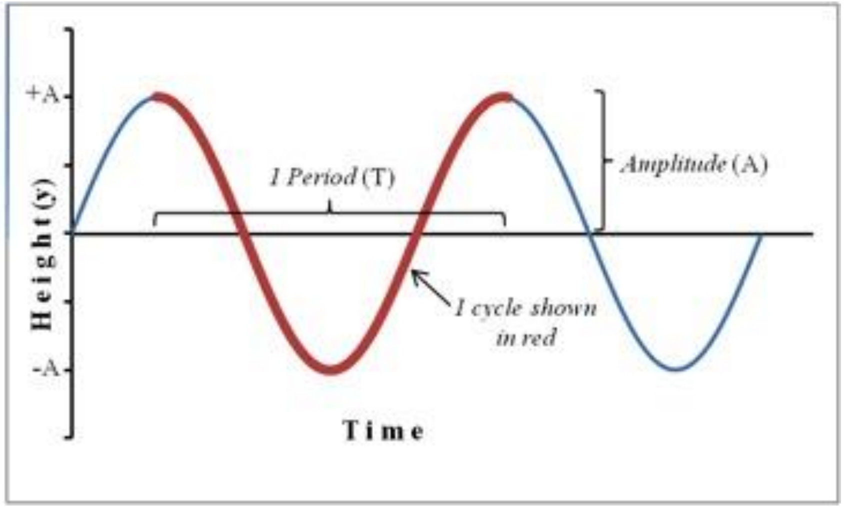

Figure 2. Position plot showing sinusoidal motion of an object in SHM [2]

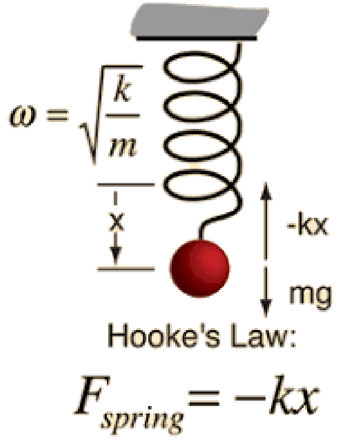

Figure 3. A diagram showing a spring undergoing SHM. [3]

Theory: The Coupled Harmonic Oscillator

- We can construct a coupled harmonic oscillator by attaching, in some way, two simple harmonic oscillators, a spring system like the one we used can be seen in figure 4.
- The equations which govern the motion will be found in an analogous way to the simple harmonic oscillator up to a point.
- We will find a system of equations which will yield two solutions, the overall difference being ω . We will find a positive an

$$\omega = \sqrt{\frac{(k + k' \pm k')}{m}}$$

$$\omega_- = \sqrt{\frac{k}{m}}$$

$$\omega_+ = \sqrt{\frac{(k + 2k')}{m}}$$

Figure 5. The solutions to a couped harmonic oscillator.

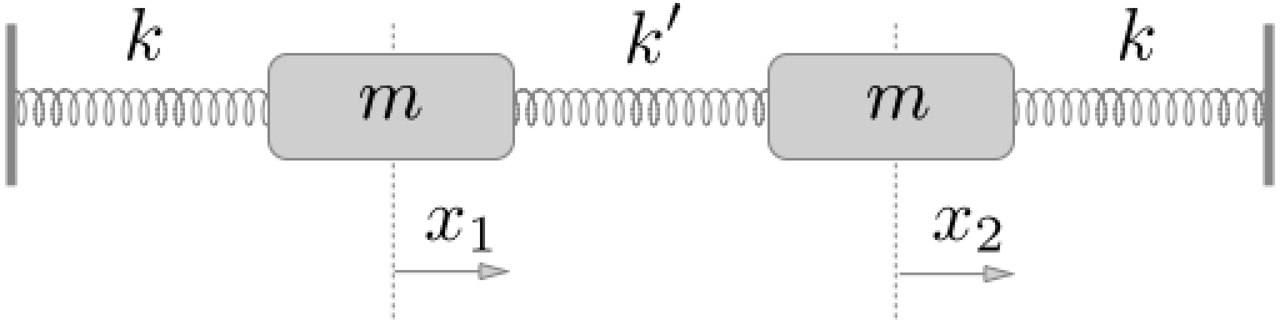

Figure 4. A coupled spring system. [4]

Figure 6. The symmetric (ω (-)) and antisymmetric (ω (+)) modes of oscillation. [5]

Theory: Mixed Modes

- We have three modes: Symmetric, Antisymmetric and mixed.
- A mixed mode is exactly as it sounds, it is some mix of the antisymmetric and symmetric modes.
- Physically, we can achieve a mixed mode by simply displacing our masses differently from each other, in a nonsymmetric way.
- The mixed mode may look chaotic, but it too has a resonant frequency which is simply a linear combination of the two natural frequencies.

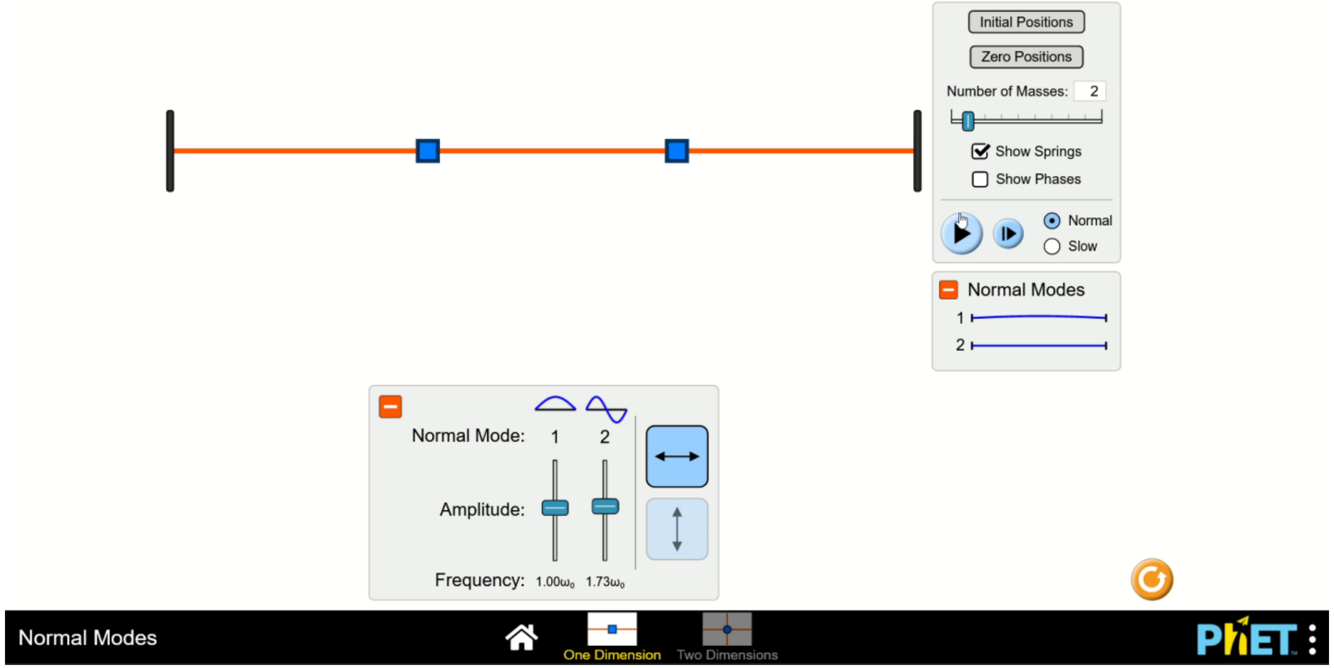

Figure 7. A mixed mode of oscillation. [5]

Apparatus

- Consists of a PASCO motion detector, 2 cardboard screens, 3 springs, 2 arts, and an air track

- The screen is sat atop the cart closest to the motion detector, such that the motion of the carts are trackable.

- 3 springs tether the carts to each other, as well as to the edge of the air track

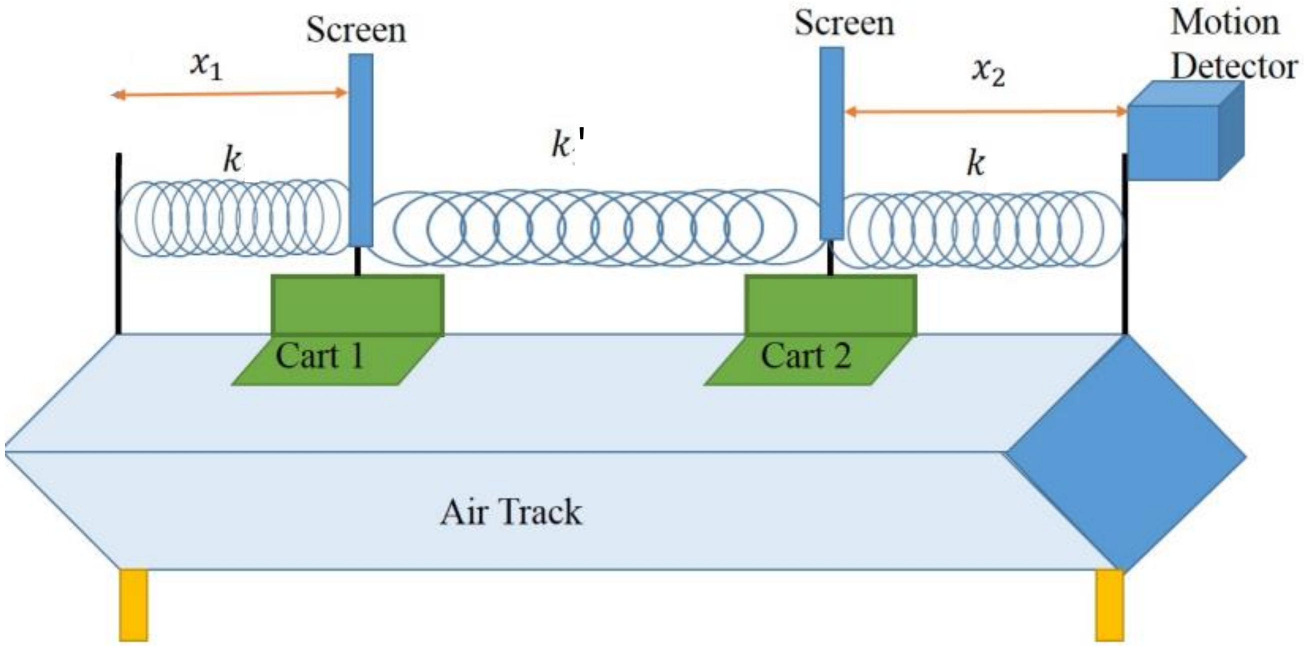

Figure 8. The Experimental Apparatus

Experiment

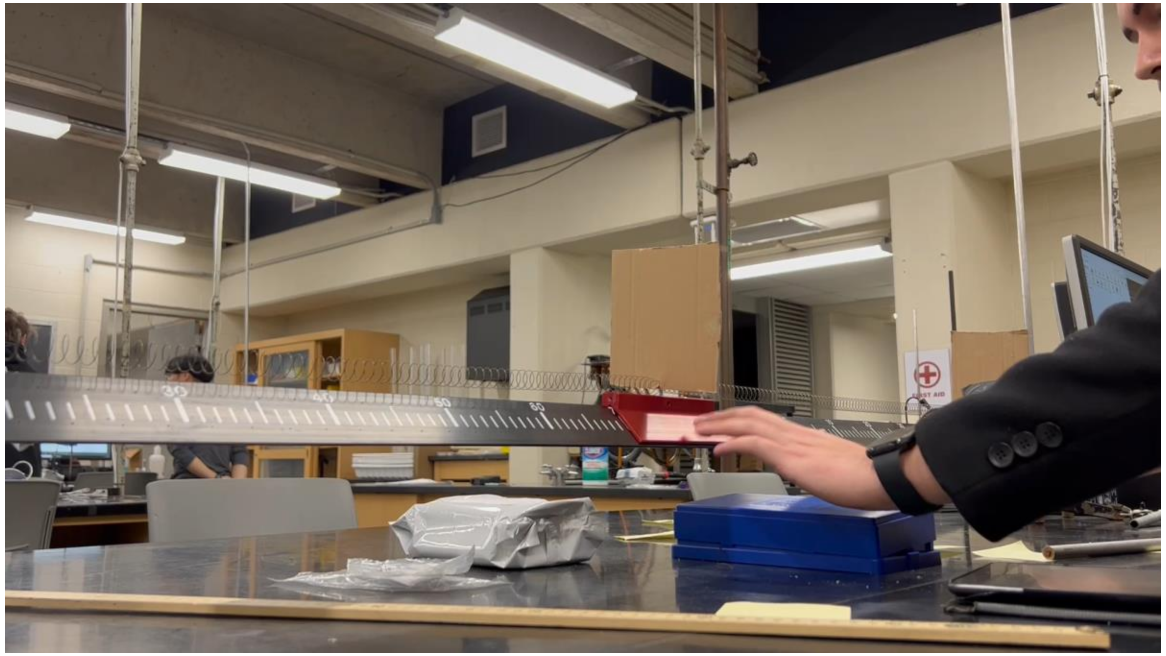

Figure 9. The Experimental Process

- First, we measured the spring constants of all springs
  - Measured initial lengths of hanging springs
  - Measured displaced length of hanging springs
  - Plotted Force (weight = mg) as a function of Displacement, the slope of which tells us our spring constant
- Next, we weighed the carts and mathematically determined the normal frequency of oscillation. These are our theoretical values.
- Finally, we used the PASCO motion detector to plot 4 different oscillatory modes: Symmetric, Anti-Symmetric, Mixed 1, and Mixed 2

Results/Analysis – Spring Constant and Normal Frequencies

- The spring constants for k2 and k3 were within 0.011 N/m of one another, while k1 remained an outlier. As such, k2 and k3 were the outer springs in our apparatus.

- Our normal oscillatory frequencies were calculated to be ω- = 2.745 Hz and ω+ = 2.745 Hz

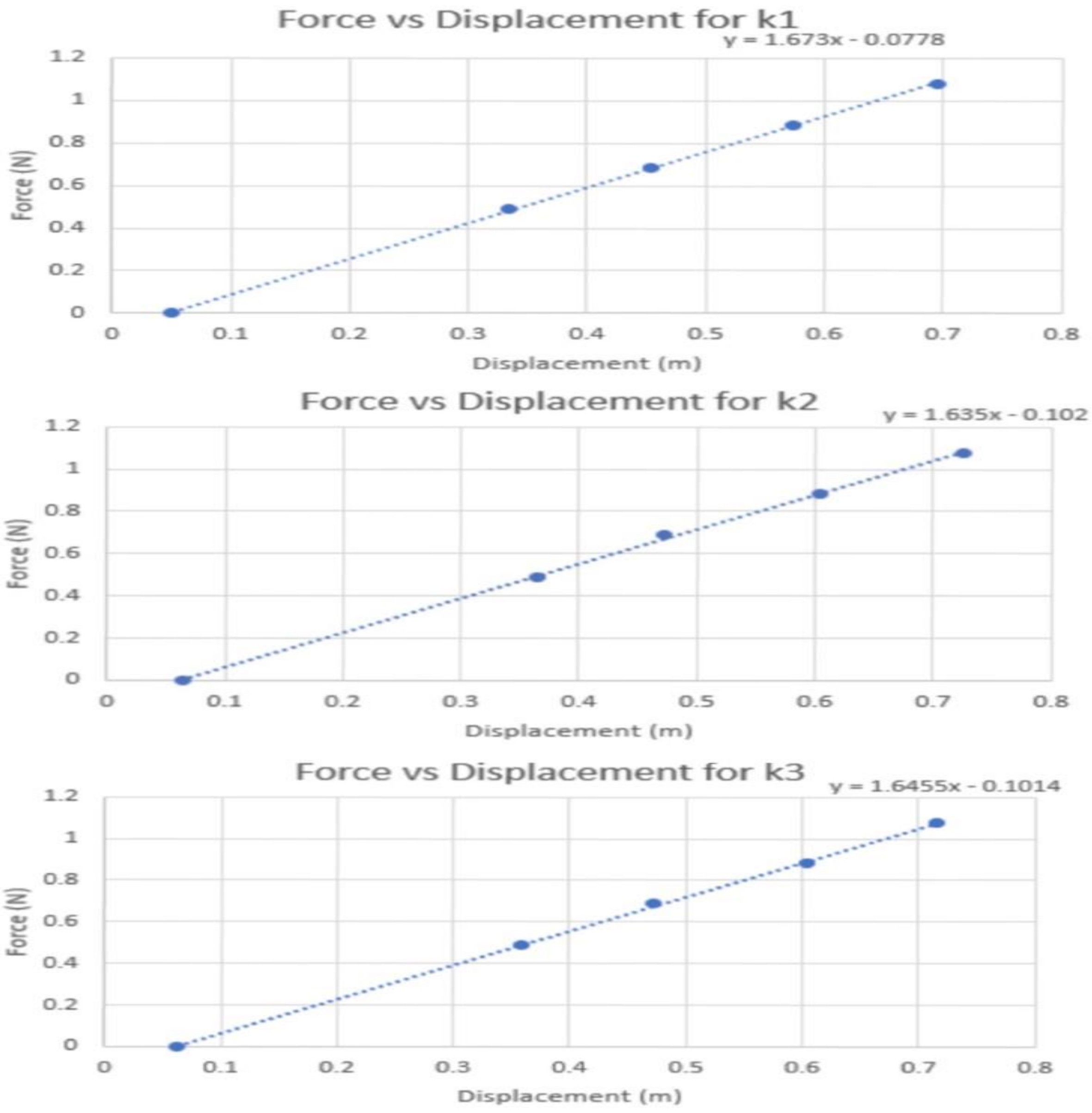

Figure 10 . The Force vs Displacement Plots for our Springs

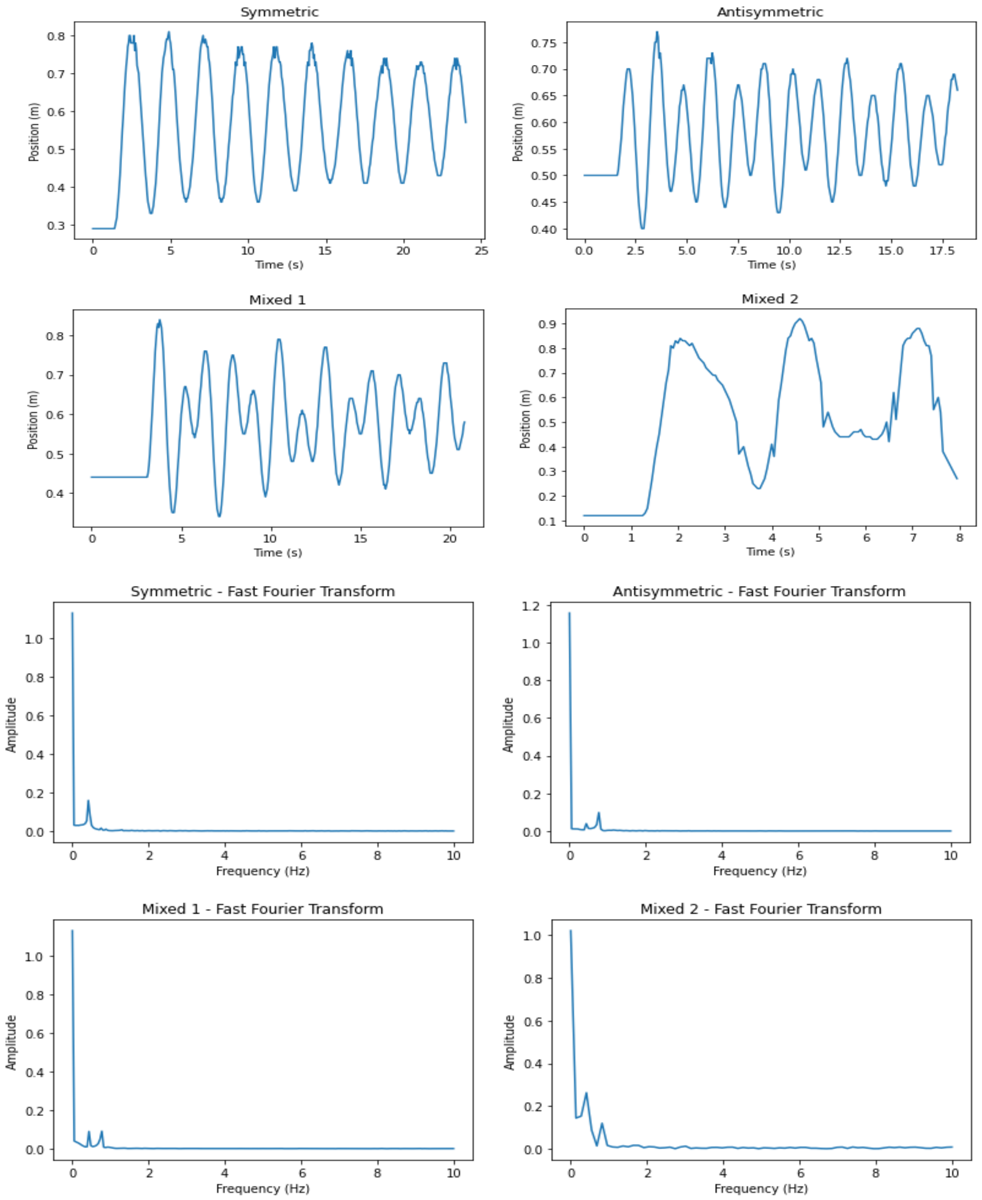

Figure 11. The Position vs Time Plots for our Oscillatory Modes, as well as the FFT Plots

Results/Analysis: Oscillatory Modes

To better analyze our data, we performed a Fourier Transform* which utilizes the equation:

$$\mathcal{F} = \int_{-\infty}^{\infty} f(x)e^{-2\pi i k x}\,dx$$

Figure 12 . The Fourier Transform

This allows us to transition from the time domain to the frequency domain, the Amplitude-Frequency plots of which show us the linear normal frequencies in the form of amplitude spikes. To convert from linear normal frequencies to angular frequencies, we utilize the relation:

$$\omega = 2\pi f$$

Figure 13. The relation between angular frequency, ω, and linear frequency, f

Which gives us angular frequencies around 2.576 Hz and 4.775 Hz for all 4 trials, which - when compared to the theoretical values from earlier using a statistical percent error measurement - give us 0.251% and 6.56% error, respectively.

* Our data is discrete, so technically we performed a "Fast Fourier Transform"

Summary

- The Experimental and Theoretical values for angular frequency were within 7% of each other
- Peak frequencies in the mixed mode oscillations slightly differ from that of the symmetric and anti-symmetric oscillations
- The mixed mode trials displayed direct proportionality between the displacement of the carts before release and the magnitude of the amplitude spikes in the FFT plots.

Questions?

References:

[1] https://www.gstatic.com/education/formulas2/472522532/en/hooke_s_law.svg

[2] https://www.webassign.net/question_assets/ncsucalcphysmechl3/lab_7_1/manual.html

[3] http://hyperphysics.phy-astr.gsu.edu/hbase/shm.html

[4] https://www.entropy.energy/scholar/node/coupled-oscillators

[5] https://phet.colorado.edu/en/simulations/normal-modes

[6]

[7]