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Research article

Some common fixed point theorems in bipolar metric spaces and applications

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Abstract: In this article, we prove some common fixed point theorems for generalized rational type contractions in bipolar metric spaces. These theorems also generalize and extend several interesting results of metric fixed point theory to the bipolar metric context. In addition, we provide some examples to illustrate our theorems, and applications are obtained in areas of homotopy theory and integral equations by using iterative methods for mathematical operators on a bipolar metric space.

Keywords: common fixed point; rational type contraction; bipolar metric spaces; iterative methods; mathematical operators
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1. Introduction and preliminaries

Fixed point theory is a powerful and fruitful tool in the study of non-linear phenomena. It is an interdisciplinary branch of mathematical sciences which can be applied in several areas of mathematics and other fields, viz., game theory, mathematical economics, optimization problems, approximation theory, initial and boundary value problems in ordinary and partial differential equations, variational inequalities, biology, chemistry, physics, engineering and others. The most fundamental result in fixed point theory that influenced several researchers was due to the Polish mathematician Stefan Banach [1] in 1922. Indeed, he proved a theorem which guarantees a unique fixed point of any contraction mapping in complete metric space. This result is popularly known as the Banach fixed point theorem or Banach contraction principle. Also, a constructive proof of the Banach fixed point theorem is very interesting because it yields one of the iterative methods for computing a fixed point. Because new discoveries of space and their properties are always interesting to researchers in mathematics, several researchers have generalized the metric space structure by either weakening the properties of the metric or modifying the domain and range of the metric: for example, b-metric by S. Czerwik [2] in 1993, partial metric by S. G. Matthews [13] in 1994, partial rectangular metric by S. Shukla [22] in 2014 and many others (see [3,5,6,17–19,21] and references therein). Interestingly, Matthews explored applications of partial metric spaces in the field of computer science, especially in the study of denotational semantics of programming languages and algorithms. Recently, in 2016, Mutlu et al. [14] generalized the metric space structure by changing the domain of the function in which they considered the distance between points of two different sets instead of a single set. The concept is known as a bipolar metric space, and various fixed point theorems including the Banach contraction principle, of metric spaces are also extended to the settings of the notion (see [9, 11, 12, 15, 20] and references therein). Furthermore, Kishore et al. [10] proved some common fixed point theorems in a bipolar metric space with important applications, while Mutlu et al. [15, 16] proved the coupled fixed point results and principle of locally and weakly contractive mappings in bipolar metric spaces. Hence, fixed point theory of bipolar metric space is an active research area, and it is capturing a lot of attention for further work.

In this article we will present some common fixed point theorems for generalized rational type contractions in bipolar metric spaces. The approach is based on some fixed point results for contraction type mappings of Kishore et al. [10] and others. Our results are the extensions of Banach's contraction, Kannan's contraction, Jaggi's contraction and Khan's contraction of the metric space to a bipolar metric space. Also, we show that our conclusions improve, generalize and extend comparable conclusions of the literature in bipolar metric spaces. Also, we have given some non-trivial examples to demonstrate our conclusions, and applications are obtained for homotopy theory and integral equations.

Throughout this paper $\mathbb{N}$ and $\mathbb{R}$ stand for the set of all positive integers and the set of all real numbers, respectively. In particular we write $\mathbb{R}^+ = [0, +\infty)$, the set of all non-negative reals. Moreover, we recall some definitions which are needed for our study.

Definition 1.1. *[14] Let X, Y $\neq \phi$ and $\varrho : X \times Y \to [0, +\infty)$ be a mapping satisfying the following properties:*

(B1) $\mu = \nu$, *if* $\varrho(\mu, \nu) = 0$;
(B2) $\varrho(\mu, \nu) = 0$, *if* $\mu = \nu$;
(B3) $\varrho(\mu, \nu) = \varrho(\nu, \mu)$, *if* $\mu, \nu \in X \cap Y$;
(B4) $\varrho(\mu_1, \nu_2) \leq \varrho(\mu_1, \nu_1) + \varrho(\mu_2, \nu_1) + \varrho(\mu_2, \nu_2)$ *for all* $\mu_1, \mu_2 \in X$ *and* $\nu_1, \nu_2 \in Y$.

Then, the mapping ϱ is called a bipolar metric on the pair (X,Y), and the triple (X, Y, ϱ) is called a bipolar metric space.

Example 1.1. *[14] Let X be the class of all singleton subsets of $\mathbb{R}$ and Y be the class of all nonempty compact subsets of $\mathbb{R}$. We define $\varrho : X \times Y \to \mathbb{R}$ as $\varrho(\mu, A) =\mid \mu - \inf(A) \mid + \mid \mu - \sup(A) \mid$. Then, the triple (X, Y, ϱ) is a complete bipolar metric space.*

Definition 1.2. *[14] Let (X_1, Y_1) and (X_2, Y_2) be two pair of sets. A map $S : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called*

(i) covariant if $S(X_1) \subseteq X_2$ and $S(Y_1) \subseteq Y_2$, and it is denoted as $S : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$;
(ii) contravariant if $S(X_1) \subseteq Y_2$ and $S(Y_1) \subseteq X_2$, and it is denoted as $S : (X_1, Y_1) \rightleftarrows (X_2, Y_2)$.

Definition 1.3. *[14] Let (X, Y, ϱ) be a bipolar metric space. Then,*

(1) *X = set of left points; Y = set of right points; X∩Y = set of central points.*
In particular, if $X \cap Y = \phi$, the space is called disjoint, and otherwise it is called joint. Unless otherwise stated, we shall work with joint spaces.
(2) *A sequence (μ_n) on the set X is called a left sequence, and a sequence (ν_n) on Y is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence.*
(3) *A sequence (μ_n) is said to be convergent to a point μ if and only if (μ_n) is a left sequence, $\lim_{n\to+\infty} \varrho(\mu_n, \mu) = 0$ and $\mu \in Y$, or (μ_n) is a right sequence, $\lim_{n\to+\infty} \varrho(\mu, \mu_n) = 0$ and $\mu \in X$.*
(4) *A bisequence (μ_n, ν_n) on (X, Y, ϱ) is a sequence on the set $X \times Y$. Furthermore, if the sequences (μ_n) and (ν_n) are convergent, then the bisequence (μ_n, ν_n) is said to be convergent. In addition, if (μ_n) and (ν_n) converge to a common point $t \in X\cap Y$, then (μ_n, ν_n) is called biconvergent.*
(5) *A bisequence (μ_n, ν_n) is a Cauchy bisequence if $\lim_{n\to+\infty} \varrho(\mu_n, \nu_n) = 0$.*

Remark 1.1. *In a bipolar metric space, every convergent Cauchy bisequence is biconvergent.*

Definition 1.4. *[14] A bipolar metric space is called complete if every Cauchy bisequence is convergent, hence biconvergent.*

Definition 1.5. *[14] A covariant or a contravariant map S from the bipolar metric space (X_1, Y_1, ϱ_1) to the bipolar metric space (X_2, Y_2, ϱ_2) is continuous, if and only if $(u_n) \to v$ on (X_1, Y_1, ϱ_1) implies $S(u_n) \to S(v)$ on (X_2, Y_2, ϱ_2).*

Now, we have the following fixed point theorems on metric spaces.

Theorem 1.1 (**Banach's contraction (1922)**). *[1] Let (X, ϱ) be a complete metric space, and $T : X \to X$ satisfying $\varrho(T\mu, T\nu) \leq \mu_1 \varrho(\mu, \nu)$, for all $\mu, \nu \in X$, with $0 \leq \mu_1 < 1$. Then, T has a unique fixed point.*

Theorem 1.2 (**Kannan's contraction (1969)**). *[7] Let (X, ϱ) be a complete metric space, and $T : X \to X$ satisfying $\varrho(T\mu, T\nu) \leq \mu_1[\varrho(\mu, T\mu) + \varrho(\nu, T\nu)]$, for all $\mu, \nu \in X$, with $0 \leq \mu_1 < \frac{1}{2}$. Then, T has a unique fixed point.*

In addition, two other fixed point theorems have been obtained under some new contractive conditions, which are the following.

Theorem 1.3 (**Khan's contraction (1975)**). *[8] Let (X, ϱ) be a complete metric space, and $T : X \to X$ satisfying*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(\mu, T\nu) + \varrho(T\nu, \nu)\varrho(\nu, T\mu)}{\varrho(\mu, T\nu) + \varrho(\nu, T\mu)},$$

for all $\mu, \nu \in X$, with $0 \leq \mu_1 < 1$. Then, T has a unique fixed point.

Theorem 1.4 (**Jaggi's contraction (1977)**). *[4] Let T be a continuous self map defined on a complete metric space (X, ϱ). Suppose that T satisfies the following contractive condition:*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(T\nu, \nu)}{\varrho(\mu, \nu)} + \mu_2 \varrho(\mu, \nu),$$

for all $\mu, \nu \in X$, $\mu \neq \nu$, and for some $\mu_1, \mu_2 \in [0, 1)$ with $\mu_1 + \mu_2 < 1$. Then, T has a unique fixed point in X.

2. Main results

Now, we establish two common fixed point results for mappings satisfying the generalized rational type contractive conditions which extend Theorems 1.1–1.4 to bipolar metric spaces. First we prove the following theorem.

Theorem 2.1. *Let (X, Y, ϱ) be a complete bipolar metric space, and let T,S : $(X, Y, \varrho) \rightleftarrows (X, Y, \varrho)$ be contravariant mappings satisfying*

$$\varrho(S\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(S\nu, \nu)}{\varrho(\mu, \nu)} + \mu_2\varrho(\mu, \nu) + \mu_3[\varrho(\mu, T\mu) + \varrho(S\nu, \nu)], \tag{2.1}$$

for all $(\mu, \nu) \in X \times Y$, with $\mu \neq \nu$ and $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$, where $\mu_1, \mu_2, \mu_3 \geq 0$. Then, T,S : $X \cup Y \longrightarrow X \cup Y$ have a unique common fixed point, provided that T and S are continuous in (X,Y).

Proof. Let $\mu_0 \in$ X and $\nu_0 \in Y$. For each $n \in \mathbb{N} \cup \{0\}$, we use the following one of the iterative methods to define sequences (μ_n) and (ν_n):

$$S\mu_{2n} = \nu_{2n}\ ,\ T\mu_{2n+1} = \nu_{2n+1}\ ,\ S\nu_{2n} = \mu_{2n+1}\ ,\ T\nu_{2n+1} = \mu_{2n+2}.$$

Now, by (2.1), we get

$$\begin{aligned}
\varrho(\mu_{2n+1}, \nu_{2n+1}) &= \varrho(S\nu_{2n}, T\mu_{2n+1}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n+1}, T\mu_{2n+1})\varrho(S\nu_{2n}, \nu_{2n})}{\varrho(\mu_{2n+1}, \nu_{2n})} + \mu_2\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n+1}, T\mu_{2n+1}) + \varrho(S\nu_{2n}, \nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n+1}, \nu_{2n+1})\varrho(\mu_{2n+1}, \nu_{2n})}{\varrho(\mu_{2n+1}, \nu_{2n})} + \mu_2\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n+1}, \nu_{2n+1}) + \varrho(\mu_{2n+1}, \nu_{2n})] \\
&= \mu_1\varrho(\mu_{2n+1}, \nu_{2n+1}) + \mu_2\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3\varrho(\mu_{2n+1}, \nu_{2n+1}) + \mu_3\varrho(\mu_{2n+1}, \nu_{2n}),
\end{aligned}$$

$$\varrho(\mu_{2n+1}, \nu_{2n+1}) \leq \frac{\mu_2 + \mu_3}{1 - \mu_1 - \mu_3}\varrho(\mu_{2n+1}, \nu_{2n}). \tag{2.2}$$

We obtain also

$$\begin{aligned}
\varrho(\mu_{2n+1}, \nu_{2n}) &= \varrho(S\nu_{2n}, S\mu_{2n}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n}, S\mu_{2n})\varrho(S\nu_{2n}, \nu_{2n})}{\varrho(\mu_{2n}, \nu_{2n})} + \mu_2\varrho(\mu_{2n}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n}, S\mu_{2n}) + \varrho(S\nu_{2n}, \nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n}, \nu_{2n})\varrho(\mu_{2n+1}, \nu_{2n})}{\varrho(\mu_{2n}, \nu_{2n})} + \mu_2\varrho(\mu_{2n}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n}, \nu_{2n}) + \varrho(\mu_{2n+1}, \nu_{2n})] \\
&= \mu_1\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_2\varrho(\mu_{2n}, \nu_{2n}) + \mu_3\varrho(\mu_{2n}, \nu_{2n}) + \mu_3\varrho(\mu_{2n+1}, \nu_{2n}),
\end{aligned}$$

$$\varrho(\mu_{2n+1}, \nu_{2n}) \leq \frac{\mu_2 + \mu_3}{1 - \mu_1 - \mu_3}\varrho(\mu_{2n}, \nu_{2n}). \tag{2.3}$$

Take $\lambda = \dfrac{\mu_2 + \mu_3}{1 - \mu_1 - \mu_3}$, and then $\lambda \in [0, 1)$ because $\mu_1 + \mu_2 + 2\mu_3 \in [0, 1)$. Hence, from (2.2) and (2.3), we get

$$\varrho(\mu_{2n+1}, \nu_{2n+1}) \leq \lambda^{4n+2}\varrho(\mu_0, \nu_0) \qquad \text{and} \qquad \varrho(\mu_{2n+1}, \nu_{2n}) \leq \lambda^{4n+1}\varrho(\mu_0, \nu_0). \tag{2.4}$$

Now, we can get that for any $n \in \mathbb{N}$,

$$\varrho(\mu_{n+1}, \nu_{n+1}) \leq \lambda^{2n+2}\varrho(\mu_0, \nu_0),$$

$$\varrho(\mu_{n+1}, \nu_n) \leq \lambda^{2n+1}\varrho(\mu_0, \nu_0),$$

and

$$\varrho(\mu_n, \nu_n) \leq \lambda^{2n}\varrho(\mu_0, \nu_0).$$

Further, for all $m, n \in \mathbb{N}$, we consider the following cases:

Case 1. If $m > n$, we have

$$\begin{aligned}
\varrho(\mu_n, \nu_m) &\leq \varrho(\mu_n, \nu_n) + \varrho(\mu_{n+1}, \nu_n) + \varrho(\mu_{n+1}, \nu_m) \\
&\leq \lambda^{2n}\varrho(\mu_0, \nu_0) + \lambda^{2n+1}\varrho(\mu_0, \nu_0) + \varrho(\mu_{n+1}, \nu_m) \\
&\leq (\lambda^{2n} + \lambda^{2n+1})\varrho(\mu_0, \nu_0) + \varrho(\mu_{n+1}, \nu_{n+1}) + \varrho(\mu_{n+2}, \nu_{n+1}) + \varrho(\mu_{n+2}, \nu_m) \\
&\leq (\lambda^{2n} + \lambda^{2n+1})\varrho(\mu_0, \nu_0) + \lambda^{2n+2}\varrho(\mu_0, \nu_0) + \lambda^{2n+3}\varrho(\mu_0, \nu_0) + \varrho(\mu_{n+2}, \nu_m) \\
&\vdots \\
&\leq \lambda^{2n}(1 + \lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^{2(m-n)})\varrho(\mu_0, \nu_0) \\
&\leq \lambda^{2n}\left(\frac{1 - \lambda^{2(m-n)+1}}{1 - \lambda}\right)\varrho(\mu_0, \nu_0).
\end{aligned}$$

Since $\lambda < 1$, $\lim_{n,m\to+\infty} \varrho(\mu_n, \nu_m) = 0$.

Case 2. If $m < n$, we have

$$\begin{aligned}
\varrho(\mu_n, \nu_m) &\leq \varrho(\mu_{m+1}, \nu_m) + \varrho(\mu_{m+1}, \nu_{m+1}) + \varrho(\mu_n, \nu_{m+1}) \\
&\leq \lambda^{2m+1}\varrho(\mu_0, \nu_0) + \lambda^{2m+2}\varrho(\mu_0, \nu_0) + \varrho(\mu_n, \nu_{m+1}) \\
&\leq (\lambda^{2m+1} + \lambda^{2m+2})\varrho(\mu_0, \nu_0) + \varrho(\mu_{m+2}, \nu_{m+1}) + \varrho(\mu_{m+2}, \nu_{m+2}) + \varrho(\mu_n, \nu_{m+2}) \\
&\vdots \\
&\leq \lambda^{2m+1}(1 + \lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^{2(n-m-1)})\varrho(\mu_0, \nu_0) \\
&\leq \lambda^{2m+1}\left(\frac{1 - \lambda^{2(n-m)-1}}{1 - \lambda}\right)\varrho(\mu_0, \nu_0).
\end{aligned}$$

Again, since $\lambda < 1$, $\lim_{n,m\to+\infty} \varrho(\mu_n, \nu_m) = 0$.

This means that $\varrho(\mu_n, \nu_m)$ can be made arbitrarily small by large m and n, and hence (μ_n, ν_m) is a Cauchy bisequence in (X, Y). By the completeness of (X,Y,ϱ), the bisequence (μ_n, ν_n) biconverges to some $\mu^* \in$ X∩Y such that $\lim_{n\to+\infty}(\mu_n) = \lim_{n\to+\infty}(\nu_n) = \mu^*$. Also, $S(\mu_{2n}) = (\nu_{2n}) \longrightarrow \mu^* \in$ X∩Y implies that $S(\mu_{2n})$ has a unique limit μ^*, and $(\mu_n) \longrightarrow \mu^*$ implies that $(\mu_{2n}) \longrightarrow \mu^*$. Now, the continuity of S implies that $S(\mu_{2n}) \longrightarrow S\mu^*$. Therefore, S$\mu^* = \mu^*$.

Similarly, $T(\nu_{2n+1}) = (\mu_{2n+2}) \longrightarrow \mu^* \in$ X∩Y implies that $T(\nu_{2n+1})$ has a unique limit μ^*, and $(\nu_n) \longrightarrow \mu^*$ implies that $(\nu_{2n+1}) \longrightarrow \mu^*$. Now, the continuity of T implies that $T(\nu_{2n+1}) \longrightarrow T\mu^*$. Therefore, T$\mu^* = \mu^*$. Thus, S$\mu^*$ = T$\mu^* = \mu^*$, i.e., T and S have a common fixed point.

Now, we will prove the uniqueness of the common fixed point. If $\nu^* \in$ X∩Y is another common fixed point of S and T, that is, Sν^* = Tν^* = $\nu^* \in$ X∩Y, then we get

$$\begin{aligned}\varrho(\nu^*, \mu^*) &= \varrho(S\nu^*, T\mu^*)\\ &\leq \mu_1 \frac{\varrho(\mu^*, T\mu^*)\varrho(S\nu^*, \nu^*)}{\varrho(\mu^*, \nu^*)} + \mu_2\varrho(\mu^*, \nu^*) + \mu_3[\varrho(\mu^*, T\mu^*) + \varrho(S\nu^*, \nu^*)]\\ &= \mu_1 \frac{\varrho(\mu^*, \mu^*)\varrho(\nu^*, \nu^*)}{\varrho(\mu^*, \nu^*)} + \mu_2\varrho(\mu^*, \nu^*) + \mu_3[\varrho(\mu^*, \mu^*) + \varrho(\nu^*, \nu^*)].\end{aligned}$$

Therefore, $\varrho(\nu^*, \mu^*) \leq \mu_2\varrho(\mu^*, \nu^*)$, which is a contradiction, and hence, $\mu^* = \nu^*$. This completes the theorem. □

Now, we have the following example to validate Theorem 2.1.

Example 2.1. *Let $X = \{7, 8, 11, 17\}$ and $Y = \{2, 4, 17, 18\}$. Define $\varrho : X \times Y \to [0, +\infty)$ as the usual metric, $\varrho(\mu, \nu) = |\mu - \nu|$. Then, the triple (X,Y, ϱ) is a bipolar metric space. The contravariant mappings $T, S : X \cup Y \rightleftarrows X \cup Y$, defined by*

$$T\alpha = \begin{cases} 17, & \alpha \in X \cup \{18\},\\ 18, & otherwise,\end{cases} \qquad and \qquad S\alpha = \begin{cases} 17, & \alpha \in \{17, 18\},\\ 18, & otherwise,\end{cases}$$

satisfy the inequality of Theorem 2.1 for $\mu_1 = \frac{1}{3}$, $\mu_2 = \frac{1}{4}$, $\mu_3 = \frac{1}{5}$, and 17 ∈ X∩Y is the only common fixed point of T and S.

Moreover, by taking T = S in Theorem 2.1, we get the following result, which is a generalization of Theorems 1.1, 1.2 and 1.4 in the context of a bipolar metric space.

Corollary 2.1. *Let (X, Y, ϱ) be a complete bipolar metric space and $T : (X, Y, \varrho) \rightleftarrows (X, Y, \varrho)$ be a contravariant mapping satisfying*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(T\nu, \nu)}{\varrho(\mu, \nu)} + \mu_2\varrho(\mu, \nu) + \mu_3[\varrho(\mu, T\mu) + \varrho(T\nu, \nu)],$$

for all $(\mu, \nu) \in X \times Y$, with $\mu \neq \nu$ and $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$, where $\mu_1, \mu_2, \mu_3 \geq 0$. Then, $T : X\cup Y \longrightarrow X\cup Y$ has a unique fixed point, provided that T is continuous in (X,Y).

Remark 2.1. *The results obtained by Mutlu et al. [14] are special cases of Corollary 2.1.*

Now, we prove our second result in a bipolar metric space as follows.

Theorem 2.2. *Let (X, Y, ϱ) be a complete bipolar metric space and T,S : $(X, Y, \varrho) \rightleftarrows (X, Y, \varrho)$ be contravariant mappings satisfying*

$$\varrho(S\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(\mu, S\nu) + \varrho(S\nu, \nu)\varrho(\nu, T\mu)}{\varrho(\mu, S\nu) + \varrho(\nu, T\mu)} + \mu_2\varrho(\mu, \nu) + \mu_3[\varrho(\mu, T\mu) + \varrho(S\nu, \nu)], \tag{2.5}$$

for all $(\mu, \nu) \in X \times Y$, with $\mu \neq \nu$ and $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$, where $\mu_1, \mu_2, \mu_3 \geq 0$. Then, T,S : $X\cup Y \longrightarrow X\cup Y$ have a unique common fixed point, provided that T,S are continuous in (X,Y).

Proof. Let $\mu_0 \in X$ and $\nu_0 \in Y$. For each $n \in \mathbb{N} \cup \{0\}$, we use the following iterative method to define sequences (μ_n) and (ν_n):

$$S\mu_{2n} = \nu_{2n}\ ,\ T\mu_{2n+1} = \nu_{2n+1}\ ,\ S\nu_{2n} = \mu_{2n+1}\ ,\ T\nu_{2n+1} = \mu_{2n+2}.$$

Now, from (2.5), we get

$$\begin{aligned}
\varrho(\mu_{2n+1}, \nu_{2n+1}) &= \varrho(S\nu_{2n}, T\mu_{2n+1}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n+1}, T\mu_{2n+1})\varrho(\mu_{2n+1}, S\nu_{2n}) + \varrho(S\nu_{2n}, \nu_{2n})\varrho(\nu_{2n}, T\mu_{2n+1})}{\varrho(\mu_{2n+1}, S\nu_{2n}) + \varrho(\nu_{2n}, T\mu_{2n+1})} \\
&\quad + \mu_2\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n+1}, T\mu_{2n+1}) + \varrho(S\nu_{2n}, \nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n+1}, \nu_{2n+1})\varrho(\mu_{2n+1}, \mu_{2n+1}) + \varrho(\mu_{2n+1}, \nu_{2n})\varrho(\nu_{2n}, \nu_{2n+1})}{\varrho(\mu_{2n+1}, \mu_{2n+1}) + \varrho(\nu_{2n}, \nu_{2n+1})} \\
&\quad + \mu_2\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n+1}, \nu_{2n+1}) + \varrho(\mu_{2n+1}, \nu_{2n})] \\
&= \mu_1\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_2\varrho(\mu_{2n+1}, \nu_{2n}) + \mu_3\varrho(\mu_{2n+1}, \nu_{2n+1}) + \mu_3\varrho(\mu_{2n+1}, \nu_{2n}),
\end{aligned}$$

which implies

$$\varrho(\mu_{2n+1}, \nu_{2n+1}) \leq \frac{\mu_1 + \mu_2 + \mu_3}{1 - \mu_3}\varrho(\mu_{2n+1}, \nu_{2n}). \tag{2.6}$$

We obtain also

$$\begin{aligned}
\varrho(\mu_{2n+1}, \nu_{2n}) &= \varrho(S\nu_{2n}, S\mu_{2n}) \\
&\leq \mu_1 \frac{\varrho(\mu_{2n}, S\mu_{2n})\varrho(\mu_{2n}, S\nu_{2n}) + \varrho(S\nu_{2n}, \nu_{2n})\varrho(\nu_{2n}, S\mu_{2n})}{\varrho(\mu_{2n}, S\nu_{2n}) + \varrho(\nu_{2n}, S\mu_{2n})} \\
&\quad + \mu_2\varrho(\mu_{2n}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n}, S\mu_{2n}) + \varrho(S\nu_{2n}, \nu_{2n})] \\
&= \mu_1 \frac{\varrho(\mu_{2n}, \nu_{2n})\varrho(\mu_{2n}, \mu_{2n+1}) + \varrho(\mu_{2n+1}, \nu_{2n})\varrho(\nu_{2n}, \nu_{2n})}{\varrho(\mu_{2n}, \mu_{2n+1}) + \varrho(\nu_{2n}, \nu_{2n})} \\
&\quad + \mu_2\varrho(\mu_{2n}, \nu_{2n}) + \mu_3[\varrho(\mu_{2n}, \nu_{2n}) + \varrho(\mu_{2n+1}, \nu_{2n})] \\
&= \mu_1\varrho(\mu_{2n}, \nu_{2n}) + \mu_2\varrho(\mu_{2n}, \nu_{2n}) + \mu_3\varrho(\mu_{2n}, \nu_{2n}) + \mu_3\varrho(\mu_{2n+1}, \nu_{2n}),
\end{aligned}$$

which implies

$$\varrho(\mu_{2n+1}, \nu_{2n}) \leq \frac{\mu_1 + \mu_2 + \mu_3}{1 - \mu_3}\varrho(\mu_{2n}, \nu_{2n}). \tag{2.7}$$

Take $\lambda' = \dfrac{\mu_1 + \mu_2 + \mu_3}{1 - \mu_3}$, and then $\lambda' \in [0, 1)$ because $\mu_1 + \mu_2 + 2\mu_3 \in [0, 1)$. Hence, from the previous two inequalities (2.6) and (2.7), we get

$$\varrho(\mu_{2n+1}, \nu_{2n+1}) \leq \lambda'^{4n+2}\varrho(\mu_0, \nu_0) \qquad \text{and} \qquad \varrho(\mu_{2n+1}, \nu_{2n}) \leq \lambda'^{4n+1}\varrho(\mu_0, \nu_0). \tag{2.8}$$

Now, we can get that for any $n \in \mathbb{N}$,

$$\varrho(\mu_{n+1}, \nu_{n+1}) \leq \lambda'^{2n+2}\varrho(\mu_0, \nu_0),$$

$$\varrho(\mu_{n+1}, \nu_n) \leq \lambda'^{2n+1}\varrho(\mu_0, \nu_0)$$

and

$$\varrho(\mu_n, \nu_n) \leq \lambda'^{2n}\varrho(\mu_0, \nu_0).$$

Hence, for all $m, n \in \mathbb{N}$, we consider the following cases:

Case 1. If $m > n$, we have

$$\begin{aligned}
\varrho(\mu_n, \nu_m) &\leq \varrho(\mu_n, \nu_n) + \varrho(\mu_{n+1}, \nu_n) + \varrho(\mu_{n+1}, \nu_m)\\
&\leq \lambda'^{2n}\varrho(\mu_0, \nu_0) + \lambda'^{2n+1}\varrho(\mu_0, \nu_0) + \varrho(\mu_{n+1}, \nu_m)\\
&\leq (\lambda'^{2n} + \lambda'^{2n+1})\varrho(\mu_0, \nu_0) + \varrho(\mu_{n+1}, \nu_{n+1}) + \varrho(\mu_{n+2}, \nu_{n+1}) + \varrho(\mu_{n+2}, \nu_m)\\
&\leq (\lambda'^{2n} + \lambda'^{2n+1})\varrho(\mu_0, \nu_0) + \lambda'^{2n+2}\varrho(\mu_0, \nu_0) + \lambda'^{2n+3}\varrho(\mu_0, \nu_0) + \varrho(\mu_{n+2}, \nu_m)\\
&\vdots\\
&\leq \lambda'^{2n}(1 + \lambda' + \lambda'^2 + \lambda'^3 + \ldots + \lambda'^{2(m-n)})\varrho(\mu_0, \nu_0)\\
&\leq \lambda'^{2n}\left(\frac{1 - \lambda'^{2(m-n)+1}}{1 - \lambda'}\right)\varrho(\mu_0, \nu_0).
\end{aligned}$$

Since $\lambda' < 1$, $\lim_{n,m\to+\infty} \varrho(\mu_n, \nu_m) = 0$.

Case 2. If $m < n$,

$$\begin{aligned}
\varrho(\mu_n, \nu_m) &\leq \varrho(\mu_{m+1}, \nu_m) + \varrho(\mu_{m+1}, \nu_{m+1}) + \varrho(\mu_n, \nu_{m+1})\\
&\leq \lambda'^{2m+1}\varrho(\mu_0, \nu_0) + \lambda'^{2m+2}\varrho(\mu_0, \nu_0) + \varrho(\mu_n, \nu_{m+1})\\
&\leq (\lambda'^{2m+1} + \lambda'^{2m+2})\varrho(\mu_0, \nu_0) + \varrho(\mu_{m+2}, \nu_{m+1}) + \varrho(\mu_{m+2}, \nu_{m+2}) + \varrho(\mu_n, \nu_{m+2})\\
&\vdots\\
&\leq \lambda'^{2m+1}(1 + \lambda' + \lambda'^2 + \lambda'^3 + \ldots + \lambda'^{2(n-m-1)})\varrho(\mu_0, \nu_0)\\
&\leq \lambda'^{2m+1}\left(\frac{1 - \lambda'^{2(n-m)-1}}{1 - \lambda'}\right)\varrho(\mu_0, \nu_0).
\end{aligned}$$

Again, since $\lambda' < 1$, $\lim_{n,m\to+\infty} \varrho(\mu_n, \nu_m) = 0$.

This means that $\varrho(\mu_n, \nu_m)$ can be made arbitrarily small by large m and n, and hence (μ_n, ν_m) is a Cauchy bisequence in (X,Y). By the completeness of (X,Y, ϱ), the bisequence (μ_n, ν_n) biconverges to some $\mu^* \in X\cap Y$ such that $\lim_{n\to+\infty}(\mu_n) = \lim_{n\to+\infty}(\nu_n) = \mu^*$. Also, $S(\mu_{2n}) = (\nu_{2n}) \longrightarrow \mu^* \in X\cap Y$ implies that $S(\mu_{2n})$ has a unique limit μ^*, and $(\mu_n) \longrightarrow \mu^*$ implies that $(\mu_{2n}) \longrightarrow \mu^*$. Now, the continuity of S implies that $S(\mu_{2n}) \longrightarrow S\mu^*$. Therefore, $S\mu^* = \mu^*$.

Similarly, $T(\nu_{2n+1}) = (\mu_{2n+2}) \longrightarrow \mu^* \in X\cap Y$ implies that $T(\nu_{2n+1})$ has a unique limit μ^*, and $(\nu_n) \longrightarrow \mu^*$ implies that $(\nu_{2n+1}) \longrightarrow \mu^*$. Now, the continuity of T implies that $T(\nu_{2n+1}) \longrightarrow T\mu^*$. Therefore, $T\mu^* = \mu^*$. Thus, $S\mu^* = T\mu^* = \mu^*$, i.e., T and S have a common fixed point.

Now, we will prove the uniqueness of the common fixed point. If $\nu^* \in X\cap Y$ is another common fixed point of S and T, that is, $S\nu^* = T\nu^* = \nu^* \in X\cap Y$, then we get

$$\begin{aligned}
\varrho(\nu^*, \mu^*) &= \varrho(S\nu^*, T\mu^*)\\
&\leq \mu_1\frac{\varrho(\mu^*, T\mu^*)\varrho(\mu^*, S\nu^*) + \varrho(S\nu^*, \nu^*)\varrho(\nu^*, T\mu^*)}{\varrho(\mu^*, S\nu^*) + \varrho(\nu^*, T\mu^*)} + \mu_2\varrho(\mu^*, \nu^*) + \mu_3[\varrho(\mu^*, T\mu^*) + \varrho(S\nu^*, \nu^*)]\\
&= \mu_1\frac{\varrho(\mu^*, \mu^*)\varrho(\mu^*, \nu^*) + \varrho(\nu^*, \nu^*)\varrho(\nu^*, \mu^*)}{\varrho(\mu^*, \nu^*) + \varrho(\nu^*, \mu^*)} + \mu_2\varrho(\mu^*, \nu^*) + \mu_3[\varrho(\mu^*, \mu^*) + \varrho(\nu^*, \nu^*)].
\end{aligned}$$

Therefore, $\varrho(\nu^*, \mu^*) \leq \mu_2\varrho(\mu^*, \nu^*)$, which is a contradiction, and hence, $\mu^* = \nu^*$. This completes the theorem. □

The following example shows the validity of our Theorem 2.2.

Example 2.2. *Let* $X = \{7, 8, 11, 17\}$ *and* $Y = \{2, 4, 17, 18\}$. *Define* $\varrho : X \times Y \to [0, +\infty)$ *as the usual metric,* $\varrho(\mu, \nu) = |\ \mu - \nu\ |$. *Then, the triple (X,Y,* ϱ*) is a bipolar metric space. Moreover, contravariant mappings* $T, S : X \cup Y \to X \cup Y$*, defined as in Example 2.1, satisfy also the inequality of Theorem 2.2 with* $\mu_1 = \dfrac{1}{3}$, $\mu_2 = \dfrac{1}{4}$, $\mu_3 = \dfrac{1}{5}$. *Nevertheless,* $17 \in X \cap Y$ *is the only common fixed point of* T *and* S.

However, by taking T = S in Theorem 2.2, we get the following corollary, which is a generalization of Theorems 1.1–1.3 in the context of bipolar metric spaces.

Corollary 2.2. *Let* (X, Y, ϱ) *be a complete bipolar metric space and* $T : (X, Y, \varrho) \rightleftarrows (X, Y, \varrho)$ *be a contravariant mapping satisfying*

$$\varrho(T\nu, T\mu) \leq \mu_1 \frac{\varrho(\mu, T\mu)\varrho(\mu, T\nu) + \varrho(T\nu, \nu)\varrho(\nu, T\mu)}{\varrho(\mu, T\nu) + \varrho(\nu, T\mu)} + \mu_2 \varrho(\mu, \nu) + \mu_3[\varrho(\mu, T\mu) + \varrho(T\nu, \nu)],$$

for all $(\mu, \nu) \in X \times Y$*, with* $\mu \neq \nu$ *and* $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$*, where* $\mu_1, \mu_2, \mu_3 \geq 0$. *Then,* $T : X \cup Y \longrightarrow X \cup Y$ *has a unique fixed point, provided that T is continuous in (X,Y).*

3. Applications

Now, we study the following application for the existence of a solution in homotopy theory.

Theorem 3.1. *Let (S,T,*ϱ*) be a complete bipolar metric space, and let (A,B) be an open subset of* (S, T) *so that* $(\overline{A}, \overline{B})$ *is a closed subset of* (S, T) *and* $(A, B) \subseteq (\overline{A}, \overline{B})$. *Suppose* $L : (\overline{A} \cup \overline{B}) \times [0, 1] \to S \cup T$*, is a mathematical operator satisfying the following conditions:*

(i) $\mu \neq L(\mu, k)$ *for each* $\mu \in \partial A \cup \partial B$ *and* $k \in [0, 1]$*, where (*$\partial A \cup \partial B$ *stands for the boundary of* $A \cup B$ *in* $S \cup T$*).*

(ii) $\varrho(L(\nu, k), L(\mu, k)) \leq \mu_1 \dfrac{\varrho(\mu, L(\mu, k)\varrho(L(\nu, k), \nu))}{\varrho(\mu, \nu)} + \mu_2 \varrho(\mu, \nu) + \mu_3[\varrho(\mu, L(\mu, k) + \varrho(L(\nu, k), \nu)]$*, for all* $\mu \in \overline{A}, \nu \in \overline{B}, k \in [0, 1]$*, and* $0 \leq \mu_1 + \mu_2 + 2\mu_3 < 1$*, where* $\mu_1, \mu_2, \mu_3 \geq 0$.

(iii) There exists an $M > 1$ *such that* $\varrho(L(\mu, \rho), L(\nu, \sigma)) \leq M\ |\ \rho - \sigma\ |$*, for all* $\mu \in \overline{A}, \nu \in \overline{B}$ *and* $\rho, \sigma \in [0, 1]$.

Then, $L(., 0)$ *has a fixed point if and only if* $L(., 1)$ *has a fixed point.*

Proof. Let $C = \{\rho \in [0, 1] : \mu = L(\mu, \rho), \mu \in A\}$, $D = \{\sigma \in [0, 1] : \nu = L(\nu, \sigma), \nu \in B\}$. Since $L(., 0)$ has a fixed point in $A \cup B$, we have $0 \in C \cap D$. Thus, $C \cap D$ is a non-empty set. Now, we shall show that $C \cap D$ is both closed and open in [0, 1], and so, by connectedness, $C = D = [0, 1]$. Let $(\{\rho_n\}_{n=1}^{+\infty})$, $(\{\sigma_n\}_{n=1}^{+\infty}) \subseteq$ (C,D) with $(\rho_n, \sigma_n) \to (\lambda, \lambda) \in [0, 1]$ as $n \to +\infty$. We also claim that $\lambda \in C \cap D$. Since $(\rho_n, \sigma_n) \in (C, D)$ for $n = 0, 1, 2, 3, \ldots$, there exists a bisequence $(\mu_n, \nu_n) \in (A, B)$ such that $\nu_n = L(\mu_n, \rho_n)$ and $\mu_{n+1} = L(\nu_n, \sigma_n)$ are iteratively defined. Also, we get

$$\begin{aligned} \varrho(\mu_{n+1}, \nu_n) &= \varrho(L(\nu_n, \sigma_n), L(\mu_n, \rho_n)) \\ &\leq \mu_1 \frac{\varrho(\mu_n, L(\mu_n, \rho_n))\varrho(L(\nu_n, \sigma_n), \nu_n)}{\varrho(\mu_n, \nu_n)} + \mu_2 \varrho(\mu_n, \nu_n) + \mu_3[\varrho(\mu_n, L(\mu_n, \rho_n) + \varrho(L(\nu_n, \sigma_n), \nu_n)] \end{aligned}$$

$$= \mu_1 \frac{\varrho(\mu_n, \nu_n)\varrho(\mu_{n+1}, \nu_n)}{\varrho(\mu_n, \nu_n)} + \mu_2\varrho(\mu_n, \nu_n) + \mu_3[\varrho(\mu_n, \nu_n) + \varrho(\mu_{n+1}, \nu_n)],$$

which implies

$$\varrho(\mu_{n+1}, \nu_n) \leq \frac{\mu_2 + \mu_3}{1 - \mu_1 - \mu_3}\varrho(\mu_n, \nu_n).$$

Also,

$$\begin{aligned}
&\varrho(\mu_n, \nu_n) \\
=\ & \varrho(L(\nu_{n-1}, \sigma_{n-1}), L(\mu_n, \rho_n)) \\
\leq\ & \mu_1 \frac{\varrho(\mu_n, L(\mu_n, \rho_n))\varrho(L(\nu_{n-1}, \sigma_{n-1}), \nu_{n-1})}{\varrho(\mu_n, \nu_{n-1})} + \mu_2\varrho(\mu_n, \nu_{n-1}) + \mu_3[\varrho(\mu_n, L(\mu_n, \rho_n) + \varrho(L(\nu_{n-1}, \sigma_{n-1}), \nu_{n-1})] \\
=\ & \mu_1 \frac{\varrho(\mu_n, \nu_n)\varrho(\mu_n, \nu_{n-1})}{\varrho(\mu_n, \nu_{n-1})} + \mu_2\varrho(\mu_n, \nu_{n-1}) + \mu_3[\varrho(\mu_n, \nu_n) + \varrho(\mu_n, \nu_{n-1})],
\end{aligned}$$

which implies

$$\varrho(\mu_n, \nu_n) \leq \frac{\mu_2 + \mu_3}{1 - \mu_1 - \mu_3}\varrho(\mu_n, \nu_{n-1}).$$

By a similar process as used in Theorem 2.1, we can easily prove that (μ_n, ν_n) is a Cauchy bisequence in (A, B). By completeness, there exist $\lambda_1 \in A \cap B$ such that $\lim_{n\to+\infty}(\mu_n) = \lim_{n\to+\infty}(\nu_n) = \lambda_1$. Now, we have

$$\begin{aligned}
\varrho(L(\lambda_1, \sigma), \nu_n) &= \varrho(L(\lambda_1, \sigma), L(\mu_n, \rho_n)) \\
&\leq \mu_1 \frac{\varrho(\mu_n, L(\mu_n, \rho_n)\varrho(L(\lambda_1, \sigma), \lambda_1)}{\varrho(\mu_n, \lambda_1)} + \mu_2\varrho(\mu_n, \lambda_1) + \mu_3[\varrho(\mu_n, L(\mu_n, \rho_n)) + \varrho(L(\lambda_1, \sigma), \lambda_1)] \\
&= \mu_1 \frac{\varrho(\mu_n, \nu_n)\varrho(L(\lambda_1, \sigma), \lambda_1)}{\varrho(\mu_n, \lambda_1)} + \mu_2\varrho(\mu_n, \lambda_1) + \mu_3[\varrho(\mu_n, \nu_n) + \varrho(L(\lambda_1, \sigma), \lambda_1)].
\end{aligned}$$

Applying the limit as $n \to +\infty$, we get $\varrho(L(\lambda_1, \sigma), \lambda_1) \leq \mu_3\varrho(L(\lambda_1, \sigma), \lambda_1)$, which is a contradiction. Hence, $\varrho(L(\lambda_1, \sigma), \lambda_1) = 0$, which implies $L(\lambda_1, \sigma) = \lambda_1$. Similarly, $L(\lambda_1, \rho) = \lambda_1$. Therefore, $\rho = \sigma \in C \cap D$, and clearly $C \cap D$ is closed in $[0, 1]$.

Next, we have to prove that $C \cap D$ is open in $[0, 1]$. Suppose $(\rho_0, \sigma_0) \in (C, D)$, and then there is a bisequence (μ_0, ν_0) so that $\mu_0 {=} L(\mu_0, \rho_0)$, $\nu_0 {=} L(\nu_0, \sigma_0)$. Since $A \cup B$ is open, there is $r{>}0$ so that $B\varrho(\mu_0, r) {\subseteq} A \cup B$ and $B\varrho(r, \nu_0) \subseteq A \cup B$. Choose $\varrho \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$ and $\sigma \in (\rho_0 - \epsilon, \rho_0 + \epsilon)$ such that $| \rho - \sigma_0 | \leq \frac{1}{M^n} < \frac{\epsilon}{2}$, $| \sigma - \rho_0 | \leq \frac{1}{M^n} < \frac{\epsilon}{2}$ and $| \rho_0 - \sigma_0 | \leq \frac{1}{M^n} < \frac{\epsilon}{2}$. Thus, we have $\nu \in \overline{B_{C\cup D}(\mu_0, r)} = \{\nu, \nu_0 \in B | \varrho(\mu_0, \nu) \leq r + \varrho(\mu_0, \nu_0)\}$ and $\mu \in \overline{B_{C\cup D}(r, \nu_0)} = \{\mu, \mu_0 \in A | \varrho(\mu, \nu_0) \leq r + \varrho(\mu_0, \nu_0)\}$. Additionally, we have

$$\begin{aligned}
\varrho(L(\mu, \rho), \nu_0) &= \varrho(L(\mu, \rho), L(\nu_0, \sigma_0)) \\
&\leq \varrho(L(\mu, \rho), L(\nu, \sigma_0)) + \varrho(L(\mu_0, \rho), L(\nu, \sigma_0)) + \varrho(L(\mu_0, \rho), L(\nu_0, \sigma_0)) \\
&\leq 2M \mid \rho - \sigma_0 \mid + \varrho(L(\mu_0, \rho), L(\nu, \sigma_0)) \\
&\leq \frac{2}{M^{n-1}} + \varrho(L(\mu_0, \rho), L(\nu, \sigma_0)).
\end{aligned}$$

Letting $n \to +\infty$, we get $\varrho(L(\mu, \rho), \nu_0) \leq \varrho(L(\mu_0, \rho), L(\nu, \sigma_0))$. By (ii), we have

$$\varrho(L(\mu, \rho), \nu_0) \leq \varrho(L(\mu_0, \rho), L(\nu, \sigma_0))$$

$$\leq \mu_1 \frac{\varrho(\mu_0, L(\mu_0,\rho)\varrho(L(\nu,\sigma_0),\nu))}{\varrho(\mu_0,\nu)} + \mu_2\varrho(\mu_0,\nu) + \mu_3[\varrho(\mu_0, L(\mu_0,\rho) + \varrho(L(\nu,\sigma_0),\nu)]$$
$$\leq \mu_1 \frac{\varrho(\mu_0,\mu_0)(\varrho((\nu,\nu))}{\varrho(\mu_0,\nu)} + \mu_2\varrho(\mu_0,\nu) + \mu_3[\varrho(\mu_0,\mu_0) + \varrho(\nu,\nu)]$$
$$\leq \mu_2\varrho(\mu_0,\nu),$$

which implies
$$\varrho(L(\mu,\rho),\nu_0) \leq \varrho(\mu_0,\nu) \leq r + \varrho(\mu_0,\nu_0).$$

In an analogous manner, we get $\varrho(\mu_0, L(\nu,\sigma)) \leq \varrho(\mu,\nu_0) \leq r + \varrho(\mu_0,\nu_0)$. However, as $n \to +\infty$, we have
$$\varrho(\mu_0,\nu_0) = \varrho(L(\mu_0,\rho_0), L(\nu_0,\sigma_0)) \leq M \mid \rho_0 - \sigma_0 \mid \leq \frac{1}{M^{n-1}} \to 0,$$

which implies $\mu_0 = \nu_0$.

Thus, for each fixed σ, $\sigma = \rho \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$, and $L(.,\rho) : \overline{B_{C\cup D}(\mu_0, r)} \to \overline{B_{C\cup D}(\mu_0, r)}$. Since all the hypotheses of Corollary 2.1 hold, $L(.,\rho)$ has a fixed point in $\overline{A} \cap \overline{B}$, which must be in $A \cap B$. Then, $\rho = \sigma \in C \cap D$ for each $\sigma \in (\sigma_0 - \epsilon, \sigma_0 + \epsilon)$. Hence, $(\sigma_0 - \epsilon, \sigma_0 + \epsilon) \in C \cap D$, which gives that C ∩ D is open in [0,1]. We can use a similar process for the converse. □

Next, we discuss the existence and uniqueness of the solution of an integral equation as an application of Corollary 2.1.

Theorem 3.2. *We consider the integral equation*
$$\gamma(\mu) = f(\mu) + \int_{X\cup Y} p(\mu,\nu,\gamma(\nu))d\nu \ \text{ for } \mu \in X \cup Y,$$

where $X \cup Y$ is a Lebesgue measurable set.

Now, suppose the following:

(i) $P : (X^2 \cup Y^2) \times [0,+\infty) \to [0,+\infty)$ *and* $f \in L^\infty(X) \cup L^\infty(Y)$.

(ii) There is a continuous function $\tau : (X^2 \cup Y^2) \to [0,+\infty)$ *such that*
$$\mid P(\mu,\nu,\gamma(\nu)) - P(\mu,\nu,\beta(\nu)) \mid$$
$$\leq \tau(\mu,\nu)\Big\{\mu_1 \frac{\mid \beta(\nu) - T\beta(\nu) \mid\mid T\gamma(\nu) - \gamma(\nu) \mid}{\mid \beta(\nu) - \gamma(\nu) \mid} + \mu_2 \mid \beta(\nu) - \gamma(\nu) \mid + \mu_3[\mid \beta(\nu) - T\beta(\nu) \mid + \mid T\gamma(\nu) - \gamma(\nu) \mid]\Big\},$$

for $\mu,\nu \in (X^2 \cup Y^2)$.

(iii) $\| \int_{X\cup Y} \tau(\mu,\nu)d\nu \| \leq 1$, *that is,* $\sup_{\mu\in X\cup Y} \int_{X\cup Y} \mid \tau(\mu,\nu) \mid d\nu \leq 1$.

Then, the integral equation has a unique solution in $L^\infty(X) \cup L^\infty(Y)$.

Proof. Let A = $L^\infty(X)$ and B = $L^\infty(Y)$ be two normed linear spaces, where X, Y are Lebesgue measurable sets, and $m(X\cup Y) < \infty$. Consider $\varrho : A \times B \to [0,+\infty)$ to be defined by $\varrho(g,h) = \| g - h \|_\infty$, for all $g, h \in A \times B$. Then, (A, B, ϱ) is a complete bipolar metric space. Define the contravariant mapping (mathematical operator) $I : L^\infty(X) \cup L^\infty(Y) \to L^\infty(X) \cup L^\infty(Y)$ by
$$I(\gamma(\mu)) = \int_{X\cup Y} p(\mu,\nu,\gamma(\nu))d\nu + f(\mu),$$

where $\mu \in X \cup Y$.

Now, we have

$$\begin{aligned}
\varrho(I(\gamma(\mu)), I(\beta(\mu))) &= \| I(\gamma(\mu)) - I(\beta(\mu)) \| \\
&= \Big| \int_{X\cup Y} p(\mu, \nu, \gamma(\nu)) d\nu - \int_{X\cup Y} p(\mu, \nu, \beta(\nu)) d\nu \Big| \\
&\leq \int_{X\cup Y} | p(\mu, \nu, \gamma(\nu)) - p(\mu, \nu, \beta(\nu)) | \, d\nu \\
&\leq \int_{X\cup Y} \tau(\mu, \nu) \Big\{ \mu_1 \frac{| \beta(\nu) - T\beta(\nu) \| T\gamma(\nu) - \gamma(\nu) |}{| \beta(\nu) - \gamma(\nu) |} + \mu_2 | \beta(\nu) - \gamma(\nu) | \\
&\quad + \mu_3(| \beta(\nu) - T\beta(\nu) | + | T\gamma(\nu) - \gamma(\nu) |) \Big\} d\nu \\
&\leq \Big\{ \mu_1 \frac{\| \beta(\nu) - T\beta(\nu) \| \| T\gamma(\nu) - \gamma(\nu) \|}{\| \beta(\nu) - \gamma(\nu) \|} + \mu_2 \| \beta(\nu) - \gamma(\nu) \| \\
&\quad + \mu_3(\| \beta(\nu) - T\beta(\nu) \| + \| T\gamma(\nu) - \gamma(\nu) \|) \Big\} \int_{X\cup Y} |\tau(\mu, \nu)| d\nu \\
&\leq \Big\{ \mu_1 \frac{\| \beta - T\beta \| \| T\gamma - \gamma \|}{\| \beta - \gamma \|} + \mu_2 \| \beta - \gamma \| \\
&\quad + \mu_3(\| \beta - T\beta \| + \| T\gamma - \gamma \|) \Big\} \sup_{\mu \in X\cup Y} \int_{X\cup Y} | \tau(\mu, \nu) | \, d\nu \\
&\leq \mu_1 \frac{\| \beta - T\beta \| \| T\gamma - \gamma \|}{\| \beta - \gamma \|} + \mu_2 \| \beta - \gamma \| + \mu_3(\| \beta - T\beta \| + \| T\gamma - \gamma \|) \\
&= \mu_1 \frac{\varrho(\beta, I(\beta)) \varrho(I(\gamma), \gamma)}{\varrho(\beta, \gamma)} + \mu_2 \varrho(\beta, \gamma) + \mu_3(\varrho(\beta, I(\beta) + \varrho(I(\gamma), \gamma)).
\end{aligned}$$

It follows from Corollary 2.1 that the mathematical operator I has a unique fixed point in $A \cup B$. □

4. Conclusions

In the present paper, we have proved some common fixed point theorems for generalized rational type contractions in bipolar metric spaces. Established theorems extend Theorems 1.1–1.4 to bipolar metric spaces. Also, our results generalize several fixed point theorems due to various researchers in the literature of bipolar metric spaces. Moreover, we have given some examples for justification of our results and provided applications for our obtained results. Henceforth, our theorems open a direction to new fixed point results and applications in a bipolar metric space.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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