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Research article

Global exponential stability conditions for quaternion-valued neural networks with leakage, transmission and distribution delays

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Abstract: This paper studies the global exponential stability problem of quaternion-valued neural networks (QVNNs) with leakage, transmission, and distribution delays. To address this issue, a direct method based on system solutions is proposed to ensure the global exponential stability of the considered network models. In addition, this method does not need to construct any Lyapunov-Krasovskii functional, which greatly reduces the amount of computation. Finally, a numerical example is given to demonstrate the effectiveness of the proposed results.

Keywords: global exponential stability; distribution delays; transmission time-varying delays; quaternion-valued neural networks; leakage delay
Mathematics Subject Classification: 93D20

1. Introduction

Over the past two decades, neural networks (NNs) have been used in real-world applications including fault diagnosis, image processing, speech synthesis, wheeled mobile robot and engineering optimization (see [1–8] and the references therein). It is well known that the stability of NNs plays an important role in the process of solving practical problems. However, some practical problems that cannot be solved by real-valued NNs in existing research can be solved by complex-valued neural networks (CVNNs) [9–11]. As a result, the research on the stability of CVNNs has gradually increased. On the other hand, due to the limited signal switching and transmission speed in the network, delay occurs frequently in the NNs, and it often leads to a key source of divergence and instability in NNs [12–16]. This is why more and more researchers were interested in the problem of stability analysis of CVNNs with time delays over the past decade [17, 18]. Generally speaking, the methods for dealing with CVNNs with delays include Lyapunov–Krasovskii functional (LKF) method (see [11, 19–22] and the references therein), matrix measure method [23], etc.

Quaternions are hyper-complex numbers. They consist of three independent imaginary components and a real component, which perfectly correspond to three and four dimensional feature vectors. Therefore, quaternion-valued NNs (QVNNs) are regarded as an extension of CVNNs. Since QVNNs possess the merits of both quaternions and NNs, the performance of QVNNs is more preferable than that of CVNNs, which is convenient to practical applications related to complex signals. Recently, QVNNs have attracted a lot of attentions from different fields, more and more attention is focused on establishing sufficient conditions of various dynamics behaviors for QVNNs, such as Lagrange stability [24–26], dissipativity [27], μ-stability [28, 29], multi-stability [30], exponential stability [31–35], and robust stability [36–38]. However, quaternion multiplication does not meet the commutative law, so the dynamical analysis of QVNNs is much harder than one of CVNNs.

In recent studies, we have learned that leakage delay can affect the stability of dynamic networks. Despite this fact, we do not see much existing work on QVNNs with leakage delays. This may be due to some theoretical and technical difficulties, as shown in [39]. In [36], the robust stability of QVNNs with leakage and transmission delays is studied, by using Homeomorphic mapping theorem, LKF method and LMI technique. The global Lagrange exponential stability of QVNNs with transmission, leakage and distribution delays has been studied in [24]. By employing the LKF method associated with the inequality technique and free-weighting-matrix technique, we got a sufficient condition in appropriate LMI to assure the global Lagrange exponential stability. Meantime, the domain of attraction is estimated. In [40], they used a quaternion-valued inequality, a LKF and a homeomorphic mapping to solve the problem of globally robust stability of delayed QVNNs with the leakage delay and transmission delay. In [41], an impulsive QVNN model with leakage, transmission and distribution delays is considered. Based on the homeomorphic mapping method, by using Lyapunov stability theorem and LMI approach, sufficient conditions for the global robust stability of the unique equilibrium point are given. In [33], the discrete-time QVNNs are studied directly rather than through real decomposition method or plural decomposition method. Then, by using homeomorphic mapping theorem and Cauchy-Schwarz inequality, based on LKF and matrix inequality, we consider the issue of exponential stability analysis for discrete-time QVNNs with transmission and leakage delays. However, these obtained stability criteria need to solve multiple nonlinear inequalities or LMIs, that greatly increases the computational complexity.

Based on the above analysis, in this article, we investigate global exponential stability criteria for QVNNs with leakage, distribution, and transmission delays by proposing a direct method based on system solutions. First, a useful property of system solutions is given. On this basis, sufficient conditions for global exponential stability are investigated. The resulting stability conditions consist of only a few simple linear scalar inequalities that can be solved by standard software tools. Finally, the effectiveness of the theoretical results is clearly demonstrated by a numerical example. The advantages are as follows:

(i) A direct method based on system solutions is proposed for establishing stability criteria of QVNNs under consideration;

(ii) The obtained stability criterion is only composed of several simple linear scalar inequalities;

(iii) The proposed method can be applied to real-value NNs and CVNNs.

Notations: Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Q}^n$ be the field of real numbers, real quaternion division algebra, and the linear space of all n-dimensional quaternion vectors respectively. The quaternion is represented by $q = q^R + iq^I + jq^J + kq^K$, where $q^R, q^I, q^J, q^K \in \mathbb{R}$, i, j and k are imaginary units and obey the following rules: $ij = k$, $jk = i$, $ki = j$ and $i^2 = j^2 = k^2 = -1$. Set $|q| = \sqrt{(q^R)^2 + (q^I)^2 + (q^J)^2 + (q^K)^2}$. Let $C(\mathbb{H}_1, \mathbb{H}_2)$ represent to all bounded continuous functions from $\mathbb{H}_1$ to $\mathbb{H}_2$.

2. Preliminaries

This paper will study QVNNs with distribution, transmission and leakage delays below:

$$\dot{p}_i(t) = -d_i p_i(t-\sigma) + \sum_{j=1}^{n} a_{ij} f_j(p_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(p_j(t-\delta_{ij}(t))) + \sum_{j=1}^{n} c_{ij} \int_{t-\tau_{ij}}^{t} f_j(p_j(s))\mathrm{d}s + L_i, \ \forall t \geq 0, i = 1, 2, \ldots, n, \tag{2.1a}$$

$$p_i(\varpi) = \phi_i(\varpi), \forall \varpi \in [-\vartheta, 0], i = 1, 2, \ldots, n, \tag{2.1b}$$

which variable $p_i(t)$ refers to the state of the ith neuron, the positive integer n is the number of neurons, $f_j(\cdot)$ is the nonlinear activation function, $d_i > 0$ is the self-feedback connection weight coefficient, σ is the leakage delay, L_i represents the external input, a_{ij}, b_{ij} and c_{ij} represent the constants of neuronal interconnections, $\delta_{ij}(t)$ is the time-varying transmission delay subject to $0 \leq \delta_{ij}(t) \leq \bar{\delta}_{ij}$, τ_{ij} is the distribution delay, $\phi_i \in C^1([-\vartheta, 0], \mathbb{Q})$ is the initial function, and

$$\vartheta = \max\left\{\sigma, \max_{1\leq i,j\leq n} \tau_{ij}, \max_{1\leq i,j\leq n} \bar{\delta}_{ij}\right\}.$$

Remark 1. *[1, 39] In some neural networks, there are a large number of synapses with different sizes and parallel paths with different lengths, which limits the space range. Therefore, there are transmission velocity distribution delays on these paths. Under these circumstances, the spread of signal is not momentary, a more appropriate method is to add distribution delays, which has cumulative influence on the system. Moreover, there is always a representative time delay, which is essentially different from the conventional delays, and it broadly exists in the negative feedback terms of the system which are identified as leakage terms, named leakage delay. The leakage delay is usually incorporated in the study of network modeling, such a type of time delay often has a tendency to destabilize the neural networks and is difficult to handle. Therefore, it is of great practical significance to study the stability of NNs with leakage delays.*

Next we determine the properties of $f_i(\cdot)$, which usually require the following assumption:

A_1: The function $f_i(\cdot)$ is continuous, and there exist positive scalars ℓ_i satisfying the following condition:

$$|f_i(\theta_1) - f_i(\theta_2)| \leq \ell_i |\theta_1 - \theta_2|, \ \forall \theta_1, \theta_2 \in \mathbb{Q}, i = 1, 2, \ldots, n.$$

Under the assumption A_1, there exists a unique equilibrium point $\tilde{p}$ of (2.1). By setting $q_i(t) = p_i(t) - \tilde{p}_i$, we shift the equilibrium point to the origin, the QVNN (2.1) can be transformed into:

$$\begin{aligned}\dot{q}_i(t) =& - d_i q_i(t-\sigma) + \sum_{j=1}^{n} a_{ij} g_j(q_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(q_j(t-\delta_{ij}(t))) \\ &+ \sum_{j=1}^{n} c_{ij} \int_{t-\tau_{ij}}^{t} g_j(q_j(s))\mathrm{d}s,\ \forall t \geq 0, i = 1, 2, \ldots, n,\end{aligned} \tag{2.2a}$$

$$q_i(\varpi) = \psi_i(\varpi), \forall \varpi \in [-\vartheta, 0], i = 1, 2, \ldots, n, \tag{2.2b}$$

where $\psi_i(\varpi) = \phi_i(\varpi) - \tilde{p}_i$ and $g_i(\cdot) = f_i(\cdot + \tilde{p}_i) - f_i(\tilde{p}_i)$.

Obviously, the transformed neuronal activation function $g_i(\cdot)$ satisfies the assumption followed:

A_2: The function $g_i(\cdot)$ is continuous, and there exist positive scalars ℓ_i satisfying the following condition:

$$|g_i(\theta)| \leq \ell_i |\theta|,\ \forall \theta \in \mathbb{Q}, i = 1, 2, \ldots, n.$$

Definition 1. *If there are $H \geq 1$ and $\lambda > 0$ such that every solution of QVNN (2.2) satisfies*

$$\|q(t)\| \leq H \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda t}, \forall t \geq 0,$$

then QVNN (2.2) is called globally exponentially stable (GES), where

$q(t) = \mathrm{col}(q_1(t), \ldots, q_n(t))$,

$\|\psi\|_{\vartheta} = \sup\limits_{\varpi \in [-\vartheta, 0]} \|\psi(\varpi)\|$,

$\psi(\varpi) = \mathrm{col}(\psi_1(\varpi), \ldots, \psi_n(\varpi))$.

3. Global exponential stability analysis

We will investigate the following main result related to global exponential stability of QVNN (2.2).

Theorem 1. *Under the assumption A_2, if there are $0 < \lambda < d_i$ and $\tilde{h}_i > 0$ such that*

$$(1 - \aleph_i(\lambda))\tilde{h}_i - \sum_{j=1}^{n} \Delta_{ij}(\lambda)\tilde{h}_j > 0, i = 1, 2, \ldots, n, \tag{3.1}$$

where

$\aleph_i(\lambda) = \frac{\frac{1}{\lambda}\mathrm{e}^{\lambda\sigma}(\mathrm{e}^{\lambda\sigma}-1)d_i^2}{d_i-\lambda}$, $\Delta_{ij}(\lambda) = \frac{\pounds_{ij}(\lambda)\left(\frac{(\mathrm{e}^{\lambda\sigma}-1)d_i}{\lambda}+1\right)}{d_i-\lambda}$,

$\pounds_{ij}(\lambda) = \ell_j \left(|a_{ij}| + |b_{ij}|\mathrm{e}^{\lambda\bar{\delta}_{ij}} + |c_{ij}|\frac{\mathrm{e}^{\lambda\tau_{ij}}-1}{\lambda}\right)$,

then the following insertions are true:

(i) *There exists $h_i > 1$ such that $|q_i(t)| \leq h_i \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda t}$ for all $t \in \mathbb{R}$ and $i = 1, 2, \ldots, n$;*

(ii) *QVNN (2.2) is GES with the decay rate λ.*

Proof. For any given initial function $\varpi \in C^1([-\vartheta, 0], \mathbb{Q})$, let $\{q_1(t), q_2(t), \ldots, q_n(t)\}$ be the solution of (2.2). It follows from (3.1) that

$$\aleph_i(\lambda) + \frac{\sum_{j=1}^n \Delta_{ij}(\lambda)\tilde{h}_j}{\tilde{h}_i} < 1,\ i = 1, 2, \ldots, n,$$

so, there are scalars $h_i > 1$ such that

$$\Theta_i(\lambda) := \frac{\max\left\{2\mathrm{e}^{\lambda\sigma} - 1, \frac{|q_i(\sigma)|\mathrm{e}^{d_i\sigma}}{\|\psi\|_\vartheta}\right\}}{h_i} + \aleph_i(\lambda) + \frac{\sum_{j=1}^n \Delta_{ij}(\lambda)h_j}{h_i} < 1,\ i = 1, 2, \ldots, n. \tag{3.2}$$

From (2.2a), we obtained

$$\begin{aligned}\dot{q}_i(t) =& - d_i q_i(t) + d_i \int_{t-\sigma}^t \dot{q}_i(\beta)\mathrm{d}\beta + \sum_{j=1}^n a_{ij} g_j(q_j(t)) + \sum_{j=1}^n b_{ij} g_j(q_j(t - \delta_{ij}(t)))\\ &+ \sum_{j=1}^n c_{ij} \int_{t-\tau_{ij}}^t g_j(q_j(s))\mathrm{d}s, t \geq 0, i = 1, 2, \ldots, n.\end{aligned} \tag{3.3}$$

Case 1: $t \leq \sigma$**.** Then $t - \sigma \leq 0$. We derive

$$\begin{aligned}\int_{t-\sigma}^t \dot{q}_i(\zeta)\mathrm{d}\zeta &= \int_0^t \dot{q}_i(\zeta)\mathrm{d}\zeta + \int_{t-\sigma}^0 \dot{q}_i(\zeta)\mathrm{d}\zeta\\ &= \int_0^t \dot{q}_i(\zeta)\mathrm{d}\zeta + q_i(0) - q_i(t - \sigma).\end{aligned}$$

This, together with (2.2) and (3.3), gives

$$\begin{aligned}\dot{q}_i(t) + d_i q_i(t) =& d_i\psi_i(0) - d_i\psi_i(t - \sigma)\\ &+ d_i \int_0^t \Big[- d_i q_i(\zeta - \sigma) + \sum_{j=1}^n a_{ij} g_j(q_j(\zeta))\\ &+ \sum_{j=1}^n b_{ij} g_j(q_j(\zeta - \delta_{ij}(\zeta))) + \sum_{j=1}^n c_{ij} \int_{\zeta-\tau_{ij}}^{\zeta} g_j(q_j(s))\mathrm{d}s\Big]\mathrm{d}\zeta\\ &+ \sum_{j=1}^n a_{ij} g_j(q_j(t)) + \sum_{j=1}^n b_{ij} g_j(q_j(t - \delta_{ij}(t)))\\ &+ \sum_{j=1}^n c_{ij} \int_{t-\tau_{ij}}^t g_j(q_j(s))\mathrm{d}s,\ 0 \leq t \leq \sigma, i = 1, 2, \ldots, n.\end{aligned} \tag{3.4}$$

Multiplying by $\mathrm{e}^{d_i t}$ on the both sides, and then taking the integrals from 0 to t, we derive

$$q_i(t) = \sum_{k=0}^7 \mathcal{I}_{ik}(t),\ 0 \leq t \leq \sigma, i = 1, 2, \ldots, n,$$

where

$\mathcal{I}_{i0}(t) = \mathrm{e}^{-d_i t}\psi_i(0) + d_i \int_0^t \mathrm{e}^{d_i(\beta - t)}[\psi_i(0) - \psi_i(\beta - \sigma)]\mathrm{d}\beta,$

$\mathcal{I}_{i1}(t) = -d_i^2 \int_0^t e^{d_i(\beta-t)} \int_0^\beta q_i(\zeta-\sigma)d\zeta d\beta,$
$\mathcal{I}_{i2}(t) = d_i \sum_{j=1}^n a_{ij} \int_0^t e^{d_i(\beta-t)} \int_0^\beta g_j(q_j(\zeta))d\zeta d\beta,$
$\mathcal{I}_{i3}(t) = d_i \sum_{j=1}^n b_{ij} \int_0^t e^{d_i(\beta-t)} \int_0^\beta g_j(q_j(\zeta-\delta_{ij}(\zeta)))d\zeta d\beta,$
$\mathcal{I}_{i4}(t) = d_i \sum_{j=1}^n c_{ij} \int_0^t e^{d_i(\beta-t)} \int_0^\beta \int_{\zeta-\tau_{ij}}^{\zeta} g_j(q_j(s))ds d\zeta d\beta,$
$\mathcal{I}_{i5}(t) = \sum_{j=1}^n a_{ij} \int_0^t e^{d_i(\beta-t)} g_j(q_j(\beta))d\beta,$
$\mathcal{I}_{i6}(t) = \sum_{j=1}^n b_{ij} \int_0^t e^{d_i(\beta-t)} g_j(q_j(\beta-\delta_{ij}(\beta)))d\beta,$
$\mathcal{I}_{i7}(t) = \sum_{j=1}^n c_{ij} \int_0^t e^{d_i(\beta-t)} \int_{\beta-\tau_{ij}}^{\beta} g_j(q_j(s))ds d\beta.$

Furthermore,

$$|q_i(t)| \le \sum_{k=0}^{7} |\mathcal{I}_{ik}(t)| \le \sum_{k=0}^{7} \mathfrak{I}_{ik}(t),\ 0 \le t \le \sigma, i = 1, 2, \dots, n, \tag{3.5}$$

where

$\mathfrak{I}_{i0}(t) = e^{-\lambda t}(2e^{\lambda\sigma}-1)\|\psi\|_\vartheta,$
$\mathfrak{I}_{i1}(t) = d_i^2 \int_0^t e^{d_i(\beta-t)} \int_0^\beta |q_i(\zeta-\sigma)|d\zeta d\beta,$
$\mathfrak{I}_{i2}(t) = d_i \sum_{j=1}^n |a_{ij}|\ell_j \int_0^t e^{d_i(\beta-t)} \int_0^\beta |q_j(\zeta)|d\zeta d\beta,$
$\mathfrak{I}_{i3}(t) = d_i \sum_{j=1}^n |b_{ij}|\ell_j \int_0^t e^{d_i(\beta-t)} \int_0^\beta |q_j(\zeta-\delta_{ij}(\zeta))|d\zeta d\beta,$
$\mathfrak{I}_{i4}(t) = d_i \sum_{j=1}^n |c_{ij}|\ell_j \int_0^t e^{d_i(\beta-t)} \int_0^\beta \int_{\zeta-\tau_{ij}}^{\zeta} |q_j(s)|ds d\zeta d\beta,$
$\mathfrak{I}_{i5}(t) = \sum_{j=1}^n |a_{ij}|\ell_j \int_0^t e^{d_i(\beta-t)}|q_j(\beta)|d\beta,$
$\mathfrak{I}_{i6}(t) = \sum_{j=1}^n |b_{ij}|\ell_j \int_0^t e^{d_i(\beta-t)}|q_j(\beta-\delta_{ij}(\beta))|d\beta,$
$\mathfrak{I}_{i7}(t) = \sum_{j=1}^n |c_{ij}|\ell_j \int_0^t e^{d_i(\beta-t)} \int_{\beta-\tau_{ij}}^{\beta} |q_j(s)|ds d\beta.$

Obviously, (i) is true for $t \le 0$.

We assert that (i) holds when $t \le \sigma$; otherwise, due to the continuity of $q_i(t)$, there exist $\sigma \ge \hat{t} > 0$ and $1 \le r \le n$ such that

$$|q_i(t)| \le h_i\|\psi\|_\vartheta e^{-\lambda t},\ \forall\ t \le \hat{t},\ i = 1, 2, \dots, n, \tag{3.6}$$

$$|q_r(\hat{t})| = h_r\|\psi\|_\vartheta e^{-\lambda\hat{t}}. \tag{3.7}$$

Using (3.6), one has

$$\begin{aligned}
\mathfrak{I}_{r1}(\hat{t}) =& d_r^2 \int_0^{\hat{t}} e^{d_r(\beta-\hat{t})} \int_0^\beta |q_r(\zeta-\sigma)|d\zeta d\beta. \\
\le& d_r^2 h_r\|\psi\|_\vartheta e^{\lambda\sigma} \int_0^{\hat{t}} e^{d_r(\beta-\hat{t})} \int_0^\beta e^{-\lambda\zeta}d\zeta d\beta. \\
\le& \frac{d_r^2}{\lambda} h_r\|\psi\|_\vartheta e^{\lambda\sigma}(e^{\lambda\sigma}-1) \int_0^{\hat{t}} e^{d_r(\beta-\hat{t})-\lambda\beta}d\beta \\
\le& h_r\|\psi\|_\vartheta e^{-\lambda\hat{t}} \frac{\frac{d_r^2}{\lambda}e^{\lambda\sigma}(e^{\lambda\sigma}-1)}{d_r-\lambda},
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
\mathfrak{T}_{r2}(\hat{t}) =& d_r \sum_{j=1}^{n} |a_{rj}|\ell_j \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_0^{\beta} |q_j(\zeta)|\mathrm{d}\zeta\mathrm{d}\beta \\
\leq& d_r \|\psi\|_{\vartheta} \sum_{j=1}^{n} |a_{rj}|\ell_j h_j \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_0^{\beta} \mathrm{e}^{-\lambda\zeta}\mathrm{d}\zeta\mathrm{d}\beta \\
\leq& \frac{d_r}{\lambda} \|\psi\|_{\vartheta} \sum_{j=1}^{n} |a_{rj}|\ell_j h_j (\mathrm{e}^{\lambda\sigma} - 1) \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta}\mathrm{d}\beta \\
\leq& \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \frac{\frac{d_r}{\lambda}\sum_{j=1}^{n} |a_{rj}|\ell_j h_j (\mathrm{e}^{\lambda\sigma} - 1)}{d_r - \lambda},
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
\mathfrak{T}_{r3}(\hat{t}) =& d_r \sum_{j=1}^{n} |b_{rj}|\ell_j \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_0^{\beta} |q_j(\zeta - \delta_{rj}(\zeta))|\mathrm{d}\zeta\mathrm{d}\beta \\
\leq& d_r \|\psi\|_{\vartheta} \sum_{j=1}^{n} |b_{rj}|\ell_j h_j \mathrm{e}^{\lambda\bar{\delta}_{rj}} \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_0^{\beta} \mathrm{e}^{-\lambda\zeta}\mathrm{d}\zeta\mathrm{d}\beta \\
\leq& \frac{d_r}{\lambda} \|\psi\|_{\vartheta} \sum_{j=1}^{n} |b_{rj}|\ell_j h_j (\mathrm{e}^{\lambda\sigma} - 1)\mathrm{e}^{\lambda\bar{\delta}_{rj}} \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta}\mathrm{d}\beta \\
\leq& \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \frac{\frac{d_r}{\lambda}\sum_{j=1}^{n} |b_{rj}|\ell_j h_j (\mathrm{e}^{\lambda\sigma} - 1)\mathrm{e}^{\lambda\bar{\delta}_{rj}}}{d_r - \lambda},
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
\mathfrak{T}_{r4}(\hat{t}) =& d_r \sum_{j=1}^{n} |c_{rj}|\ell_j \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_0^{\beta} \int_{\zeta-\tau_{rj}}^{\zeta} |q_j(s)|\mathrm{d}s\mathrm{d}\zeta\mathrm{d}\beta \\
\leq& d_r \|\psi\|_{\vartheta} \sum_{j=1}^{n} |c_{rj}|\ell_j h_j \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_0^{\beta} \int_{\zeta-\tau_{rj}}^{\zeta} \mathrm{e}^{-\lambda s}\mathrm{d}s\mathrm{d}\zeta\mathrm{d}\beta \\
=& \frac{d_r}{\lambda} \|\psi\|_{\vartheta} \sum_{j=1}^{n} |c_{rj}|\ell_j h_j (\mathrm{e}^{\lambda\tau_{rj}} - 1) \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_0^{\beta} \mathrm{e}^{-\lambda\zeta}\mathrm{d}\zeta\mathrm{d}\beta \\
\leq& \frac{d_r}{\lambda^2} \|\psi\|_{\vartheta} \sum_{j=1}^{n} |c_{rj}|\ell_j h_j (\mathrm{e}^{\lambda\tau_{rj}} - 1)(\mathrm{e}^{\lambda\sigma} - 1) \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta}\mathrm{d}\beta \\
\leq& \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \frac{\frac{d_r}{\lambda^2}\sum_{j=1}^{n} |c_{rj}|\ell_j h_j (\mathrm{e}^{\lambda\tau_{rj}} - 1)(\mathrm{e}^{\lambda\sigma} - 1)}{d_r - \lambda},
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\mathfrak{T}_{r5}(\hat{t}) =& \sum_{j=1}^{n} |a_{rj}|\ell_j \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})}|q_j(\beta)|\mathrm{d}\beta \\
\leq& \|\psi\|_{\vartheta} \sum_{j=1}^{n} |a_{rj}|\ell_j h_j \int_0^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta}\mathrm{d}\beta \\
=& \|\psi\|_{\vartheta} \sum_{j=1}^{n} |a_{rj}|\ell_j h_j \mathrm{e}^{-\lambda\hat{t}} \int_0^{\hat{t}} \mathrm{e}^{(d_r-\lambda)(\beta-\hat{t})}\mathrm{d}\beta
\end{aligned}
$$

$$\leq\|\psi\|_{\vartheta}\mathrm{e}^{-\lambda\hat{t}}\frac{\sum_{j=1}^{n}|a_{rj}|\ell_j h_j}{d_r-\lambda}, \tag{3.12}$$

$$\begin{aligned}\mathfrak{T}_{r6}(\hat{t})=&\sum_{j=1}^{n}|b_{rj}|\ell_j\int_0^{\hat{t}}\mathrm{e}^{d_r(\beta-\hat{t})}|q_j(\beta-\delta_{rj}(\beta))|\mathrm{d}\beta\\ \leq&\|\psi\|_{\vartheta}\sum_{j=1}^{n}|b_{rj}|\ell_j h_j\mathrm{e}^{\lambda\bar{\delta}_{rj}}\int_0^{\hat{t}}\mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta}\mathrm{d}\beta\\ =&\|\psi\|_{\vartheta}\sum_{j=1}^{n}|b_{rj}|\ell_j h_j\mathrm{e}^{\lambda\bar{\delta}_{rj}}\mathrm{e}^{-\lambda\hat{t}}\int_0^{\hat{t}}\mathrm{e}^{(d_r-\lambda)(\beta-\hat{t})}\mathrm{d}\beta\\ \leq&\|\psi\|_{\vartheta}\mathrm{e}^{-\lambda\hat{t}}\frac{\sum_{j=1}^{n}|b_{rj}|\ell_j h_j\mathrm{e}^{\lambda\bar{\delta}_{rj}}}{d_r-\lambda},\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathfrak{T}_{r7}(\hat{t})=&\sum_{j=1}^{n}|c_{rj}|\ell_j\int_0^{\hat{t}}\mathrm{e}^{d_r(\beta-\hat{t})}\int_{\beta-\tau_{rj}}^{\beta}|q_j(s)|\mathrm{d}s\mathrm{d}\beta\\ \leq&\|\psi\|_{\vartheta}\sum_{j=1}^{n}|c_{rj}|\ell_j h_j\int_0^{\hat{t}}\mathrm{e}^{d_r(\beta-\hat{t})}\int_{\beta-\tau_{rj}}^{\beta}\mathrm{e}^{-\lambda s}\mathrm{d}s\mathrm{d}\beta\\ =&\frac{1}{\lambda}\|\psi\|_{\vartheta}\sum_{j=1}^{n}|c_{rj}|\ell_j h_j(\mathrm{e}^{\lambda\tau_{rj}}-1)\int_0^{\hat{t}}\mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta}\mathrm{d}\beta\\ \leq&\|\psi\|_{\vartheta}\mathrm{e}^{-\lambda\hat{t}}\frac{\frac{1}{\lambda}\sum_{j=1}^{n}|c_{rj}|\ell_j h_j(\mathrm{e}^{\lambda\tau_{rj}}-1)}{d_r-\lambda}.\end{aligned} \tag{3.14}$$

Substituting (3.8)–(3.14) into (3.5) to get

$$\begin{aligned}|q_r(\hat{t})|\leq&(2\mathrm{e}^{\lambda\sigma}-1)\mathrm{e}^{-\lambda\hat{t}}\|\psi\|_{\vartheta}+\|\psi\|_{\vartheta}\mathrm{e}^{-\lambda\hat{t}}\aleph_r(\lambda)h_r\\ &+\|\psi\|_{\vartheta}\mathrm{e}^{-\lambda\hat{t}}\sum_{j=1}^{n}h_j\frac{£_{rj}(\lambda)\left(\frac{(\mathrm{e}^{\lambda\sigma}-1)d_r}{\lambda}+1\right)}{d_r-\lambda}\\ =&h_r\|\phi\|_{\nu}\mathrm{e}^{-\lambda\hat{t}}\Theta_r(\lambda).\end{aligned}$$

It follows from (3.2) that $|q_r(\hat{t})|<h_r\|\psi\|_{\vartheta}\mathrm{e}^{-\lambda\hat{t}}$, which contradicts to (3.7). Consequently, (i) holds when $t\leq\sigma$.

Case 2: $t>\sigma$**.** Clearly, $t-\sigma>0$. We substitute (2.2a) into the right of (3.3) to obtain

$$\begin{aligned}\dot{q}_i(t)+d_iq_i(t)=&d_i\int_{t-\sigma}^{t}\Bigg[-d_iq_i(\beta-\sigma)+\sum_{j=1}^{n}a_{ij}g_j(q_j(\beta))\\ &+\sum_{j=1}^{n}b_{ij}g_j(q_j(\beta-\delta_{ij}(\beta)))+\sum_{j=1}^{n}c_{ij}\int_{\beta-\tau_{ij}}^{\beta}g_j(q_j(s))\mathrm{d}s\Bigg]\mathrm{d}\beta\\ &+\sum_{j=1}^{n}a_{ij}g_j(q_j(t))+\sum_{j=1}^{n}b_{ij}g_j(q_j(t-\delta_{ij}(t)))\end{aligned}$$

$$+\sum_{j=1}^{n} c_{ij}\int_{t-\tau_{ij}}^{t} g_j(q_j(s))\mathrm{d}s,\ \forall t\geq\sigma, i=1,2,\ldots,n. \tag{3.15}$$

Multiplying by $\mathrm{e}^{d_i(t-\sigma)}$ on the both sides, and then taking the integrals from σ to t, we derive

$$\begin{aligned} q_i(t) =& \mathrm{e}^{-d_i(t-\sigma)}q_i(\sigma)+\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}\Bigg(d_i\int_{\beta-\sigma}^{\beta}\Bigg[-d_iq_i(\zeta-\sigma)+\sum_{j=1}^{n}a_{ij}g_j(q_j(\zeta)) \\ &+\sum_{j=1}^{n}b_{ij}g_j(q_j(\zeta-\delta_{ij}(\zeta)))+\sum_{j=1}^{n}c_{ij}\int_{\zeta-\tau_{ij}}^{\zeta}g_j(q_j(s))\mathrm{d}s\Bigg]\mathrm{d}\zeta \\ &+\sum_{j=1}^{n}a_{ij}g_j(q_j(\beta))+\sum_{j=1}^{n}b_{ij}g_j(q_j(\beta-\delta_{ij}(\beta))) \\ &+\sum_{j=1}^{n}c_{ij}\int_{\beta-\tau_{ij}}^{\beta}g_j(q_j(s))\mathrm{d}s\Bigg)\mathrm{d}\beta,\ \forall t\geq\sigma, i=1,2,\ldots,n. \end{aligned} \tag{3.16}$$

In the light of the assumption A_2, one derive

$$\begin{aligned} |q_i(t)| \leq& \mathrm{e}^{-d_i(t-\sigma)}|q_i(\sigma)|+d_i^2\mathfrak{L}_{i1}(t)+d_i\mathfrak{L}_{i2}(t)+d_i\mathfrak{L}_{i3}(t)+d_i\mathfrak{L}_{i4}(t) \\ &+\mathfrak{L}_{i5}(t)+\mathfrak{L}_{i6}(t)+\mathfrak{L}_{i7}(t),\ \forall t\geq\sigma, i=1,2,\ldots,n, \end{aligned} \tag{3.17}$$

where

$\mathfrak{L}_{i1}(t)=\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}\int_{\beta-\sigma}^{\beta}|q_i(\zeta-\sigma)|\mathrm{d}\zeta\mathrm{d}\beta$,

$\mathfrak{L}_{i2}(t)=\sum_{j=1}^{n}|a_{ij}|\ell_j\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}\int_{\beta-\sigma}^{\beta}|q_j(\zeta)|\mathrm{d}\zeta\mathrm{d}\beta$,

$\mathfrak{L}_{i3}(t)=\sum_{j=1}^{n}|b_{ij}|\ell_j\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}\int_{\beta-\sigma}^{\beta}|q_j(\zeta-\delta_{ij}(\zeta))|\mathrm{d}\zeta\mathrm{d}\beta$,

$\mathfrak{L}_{i4}(t)=\sum_{j=1}^{n}|c_{ij}|\ell_j\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}\int_{\beta-\sigma}^{\beta}\int_{\zeta-\tau_{ij}}^{\zeta}|q_j(s)|\mathrm{d}s\mathrm{d}\zeta\mathrm{d}\beta$,

$\mathfrak{L}_{i5}(t)=\sum_{j=1}^{n}|a_{ij}|\ell_j\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}|q_j(\beta)|\mathrm{d}\beta$,

$\mathfrak{L}_{i6}(t)=\sum_{j=1}^{n}|b_{ij}|\ell_j\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}|q_j(\beta-\delta_{ij}(\beta))|\mathrm{d}\beta$,

$\mathfrak{L}_{i7}(t)=\sum_{j=1}^{n}|c_{ij}|\ell_j\int_\sigma^t \mathrm{e}^{d_i(\beta-t)}\int_{\beta-\tau_{ij}}^{\beta}|q_j(s)|\mathrm{d}s\mathrm{d}\beta$.

Now, we assert that (i) holds for all $t\in\mathbb{R}$; otherwise, due to Case 1 and the continuity of $q_i(t)$, there exist $\hat{t}>\sigma$ and $1\leq r\leq n$ such that

$$|q_i(t)|\leq h_i\|\psi\|_\vartheta \mathrm{e}^{-\lambda t},\ \forall\ t\leq\hat{t},\ i=1,2,\ldots,n, \tag{3.18}$$

$$|q_r(\hat{t})|=h_r\|\psi\|_\vartheta \mathrm{e}^{-\lambda\hat{t}}. \tag{3.19}$$

According to (3.18)

$$\begin{aligned} \mathfrak{L}_{r1}(\hat{t}) =& \int_\sigma^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}|q_r(\zeta-\sigma)|\mathrm{d}\zeta\mathrm{d}\beta \\ \leq& h_r\|\psi\|_\vartheta \mathrm{e}^{\lambda\sigma}\int_\sigma^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}\mathrm{e}^{-\lambda\zeta}\mathrm{d}\zeta\mathrm{d}\beta \end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{\lambda}h_r\|\psi\|_\vartheta e^{\lambda\sigma}(e^{\lambda\sigma}-1)\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})-\lambda\beta}d\beta\\
&\leq h_r\|\psi\|_\vartheta e^{-\lambda\hat{t}}\frac{\frac{1}{\lambda}e^{\lambda\sigma}(e^{\lambda\sigma}-1)}{d_r-\lambda},
\end{aligned}\tag{3.20}
$$

$$
\begin{aligned}
\mathfrak{L}_{r2}(\hat{t})=&\sum_{j=1}^n |a_{rj}|\ell_j\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}|q_j(\zeta)|d\zeta d\beta\\
\leq&\|\psi\|_\vartheta\sum_{j=1}^n |a_{rj}|\ell_j h_j\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}e^{-\lambda\zeta}d\zeta d\beta\\
=&\frac{1}{\lambda}\|\psi\|_\vartheta\sum_{j=1}^n |a_{rj}|\ell_j h_j(e^{\lambda\sigma}-1)\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})-\lambda\beta}d\beta\\
\leq&\|\psi\|_\vartheta e^{-\lambda\hat{t}}\frac{\frac{1}{\lambda}\sum_{j=1}^n |a_{rj}|\ell_j h_j(e^{\lambda\sigma}-1)}{d_r-\lambda},
\end{aligned}\tag{3.21}
$$

$$
\begin{aligned}
\mathfrak{L}_{r3}(\hat{t})=&\sum_{j=1}^n |b_{rj}|\ell_j\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}|q_j(\zeta-\delta_{rj}(\zeta))|d\zeta d\beta\\
\leq&\|\psi\|_\vartheta\sum_{j=1}^n |b_{rj}|\ell_j h_j e^{\lambda\bar{\delta}_{rj}}\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}e^{-\lambda\zeta}d\zeta d\beta\\
=&\frac{1}{\lambda}\|\psi\|_\vartheta\sum_{j=1}^n |b_{rj}|\ell_j h_j(e^{\lambda\sigma}-1)e^{\lambda\bar{\delta}_{rj}}\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})-\lambda\beta}d\beta\\
\leq&\|\psi\|_\vartheta e^{-\lambda\hat{t}}\frac{\frac{1}{\lambda}\sum_{j=1}^n |b_{rj}|\ell_j h_j(e^{\lambda\sigma}-1)e^{\lambda\bar{\delta}_{rj}}}{d_r-\lambda},
\end{aligned}\tag{3.22}
$$

$$
\begin{aligned}
\mathfrak{L}_{r4}(\hat{t})=&\sum_{j=1}^n |c_{rj}|\ell_j\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}\int_{\zeta-\tau_{rj}}^{\zeta}|q_j(s)|ds d\zeta d\beta\\
\leq&\|\psi\|_\vartheta\sum_{j=1}^n |c_{rj}|\ell_j h_j\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}\int_{\zeta-\tau_{rj}}^{\zeta}e^{-\lambda s}ds d\zeta d\beta\\
=&\frac{1}{\lambda}\|\psi\|_\vartheta\sum_{j=1}^n |c_{rj}|\ell_j h_j(e^{\lambda\tau_{rj}}-1)\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}\int_{\beta-\sigma}^{\beta}e^{-\lambda\zeta}d\zeta d\beta\\
=&\frac{1}{\lambda^2}\|\psi\|_\vartheta\sum_{j=1}^n |c_{rj}|\ell_j h_j(e^{\lambda\tau_{rj}}-1)(e^{\lambda\sigma}-1)\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})-\lambda\beta}d\beta\\
\leq&\|\psi\|_\vartheta e^{-\lambda\hat{t}}\frac{\frac{1}{\lambda^2}\sum_{j=1}^n |c_{rj}|\ell_j h_j(e^{\lambda\tau_{rj}}-1)(e^{\lambda\sigma}-1)}{d_r-\lambda},
\end{aligned}\tag{3.23}
$$

$$
\mathfrak{L}_{r5}(\hat{t})=\sum_{j=1}^n |a_{rj}|\ell_j\int_\sigma^{\hat{t}} e^{d_r(\beta-\hat{t})}|q_j(\beta)|d\beta
$$

$$
\begin{aligned}
&\leq \|\psi\|_{\vartheta} \sum_{j=1}^{n} |a_{rj}| \ell_j h_j \int_{\sigma}^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta} \mathrm{d}\beta \\
&\leq \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \frac{\sum_{j=1}^{n} |a_{rj}| \ell_j h_j}{d_r-\lambda},
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
\mathfrak{L}_{r6}(\hat{t}) &= \sum_{j=1}^{n} |b_{rj}| \ell_j \int_{\sigma}^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} |q_j(\beta-\delta_{rj}(\beta))| \mathrm{d}\beta \\
&\leq \|\psi\|_{\vartheta} \sum_{j=1}^{n} |b_{rj}| \ell_j h_j \mathrm{e}^{\lambda\bar{\delta}_{rj}} \int_{\sigma}^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta} \mathrm{d}\beta \\
&\leq \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \frac{\sum_{j=1}^{n} |b_{rj}| \ell_j h_j \mathrm{e}^{\lambda\bar{\delta}_{rj}}}{d_r-\lambda},
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
\mathfrak{L}_{r7}(\hat{t}) &= \sum_{j=1}^{n} |c_{rj}| \ell_j \int_{\sigma}^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_{\beta-\tau_{rj}}^{\beta} |q_j(s)| \mathrm{d}s \mathrm{d}\beta \\
&\leq \|\psi\|_{\vartheta} \sum_{j=1}^{n} |c_{rj}| \ell_j h_j \int_{\sigma}^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})} \int_{\beta-\tau_{rj}}^{\beta} \mathrm{e}^{-\lambda s} \mathrm{d}s \mathrm{d}\beta \\
&= \frac{1}{\lambda} \|\psi\|_{\vartheta} \sum_{j=1}^{n} |c_{rj}| \ell_j h_j (\mathrm{e}^{\lambda\tau_{rj}}-1) \int_{\sigma}^{\hat{t}} \mathrm{e}^{d_r(\beta-\hat{t})-\lambda\beta} \mathrm{d}\beta \\
&\leq \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \frac{\frac{1}{\lambda} \sum_{j=1}^{n} |c_{rj}| \ell_j h_j (\mathrm{e}^{\lambda\tau_{rj}}-1)}{d_r-\lambda}.
\end{aligned} \tag{3.26}
$$

Insert (3.20)–(3.26) into (3.17) to get

$$
\begin{aligned}
|q_r(\hat{t})| \leq &\mathrm{e}^{-d_r(\hat{t}-\sigma)} |q_r(\sigma)| + \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \aleph_r(\lambda) h_r \\
&+ \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}} \sum_{j=1}^{n} h_j \frac{£_{rj}(\lambda) \left(\frac{(\mathrm{e}^{\lambda\sigma}-1) d_r}{\lambda} + 1 \right)}{d_r-\lambda}.
\end{aligned}
$$

This, together with Case 1 and (3.2), implies that $|q_r(\hat{t})| \leq h_r \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda\hat{t}}$, which contradicts to (3.19). Consequently, (i) holds.

(ii) From (i), the following inequality can be obtained:

$$
\|q(t)\| = \left(\sum_{i=1}^{n} |q_i(t)|^2 \right)^{\frac{1}{2}} \leq H \|\psi\|_{\vartheta} \mathrm{e}^{-\lambda t}, \ \forall t \geq \sigma,
$$

where $H = \left(\sum_{i=1}^{n} h_i^2 \right)^{\frac{1}{2}}$, that is, QVNN (2.2) is GES which the decay rate is λ. □

Remark 2. *One of the first tasks in the study of QVNNs with time delays is the stability analysis. Many existing results on stability analysis are derived by transforming QVNNs into two complex-valued* *systems or four real-valued parts systems, then LMI stability conditions are obtained by employing the so-called LKF method that is difficult and will greatly increase the computational complexity (see [28, 29, 31, 32, 42] and the references therein). When the number of delays increases, the amount of calculations will be very large. To overcome the trouble, for the problem of global exponential stability analysis of QVNN with multiple time-varying delays, in this paper a new method based on system solutions is proposed, which does not involve any LKF. Computationally, the stability criteria obtained in this paper are actually to determine the existence of a positive vector of simple linear scalar inequalities, which is easier to verify by using the software YALMIP (similar to LMI toolbox). What is important: LMIs contain more decision variables, while simple linear scalar inequalities involves few decision variables, and hence simple linear scalar inequalities in this paper have lower computational complexity than LMIs.*

Remark 3. *The global exponential stability problem of QVNNs with leakage, transmission, and distribution delays is studied in this article. The delay-dependent and decay-rate dependent GES criterion of the QVNNs* (2.2) *is given in Theorem 1. It can be used to CVNNs or real-valued NNs. The obtained stability criterion can used to characterize the relation between the decay rate and the time-varying delays's upper-bounds.*

Remark 4. *Let* $\sigma = 0$ *and* $c_{ij} = 0, i, j = 1, 2, \ldots, n$, *then the QVNN* (2.2) *is transformed into the following form:*

$$\dot{q}_i(t) = -d_i q_i(t) + \sum_{j=1}^{n} a_{ij} g_j(q_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(q_j(t - \delta_{ij}(t))), \ \forall t \geq 0, i = 1, 2, \ldots, n, \tag{3.27a}$$

$$q_i(\varpi) = \psi_i(\varpi), \forall \varpi \in [-\vartheta, 0], i = 1, 2, \ldots, n. \tag{3.27b}$$

Corollary 1. *Under the assumption* A_2, *if there are* $0 < \lambda < d_i$ *and* $\tilde{h}_i > 0$ *such that*

$$\tilde{h}_i - \sum_{j=1}^{n} \Delta_{ij}(\lambda) \tilde{h}_j > 0, i = 1, 2, \ldots, n, \tag{3.28}$$

where $\Delta_{ij}(\lambda) = \frac{\pounds_{ij}(\lambda)}{d_i - \lambda}$, $\pounds_{ij}(\lambda) = \ell_j \left(|a_{ij}| + |b_{ij}| e^{\lambda \bar{\delta}_{ij}} \right)$, *then the following insertions are true:*

(i) *There exists* $h_i > 1$ *such that* $|q_i(t)| \leq h_i \|\psi\|_{\vartheta} e^{-\lambda t}$ *for all* $t \in \mathbb{R}$ *and* $i = 1, 2, \ldots, n$;

(ii) *QVNN* (3.27) *is GES with the decay rate* λ.

The proof of Corollary 1 for QVNN (3.27) is similar to Theorem 1.

4. Numerical examples

We will proof the effectiveness of the theoretical results that obtained in this article via the following example.

Example 1. *In QVNN* (2.2) *with leakage, transmission, and distribution delays, we choose* $n = 2$,

$$d_1 = d_2 = 5,$$

$$a_{11} = 1 - i - 0.1j - 0.5k,\ a_{12} = -1 - i - 0.4j + 0.2k,$$

$$a_{21} = 2 - i - 0.1j + 0.2k,\ a_{22} = 2 - 5i - 0.3j - 0.1k,$$

$$b_{11} = 1 - i - 0.1j - 0.5k,\ b_{12} = 1 - i - 0.2j + 0.1k,$$

$$b_{21} = 1 + i - 0.2j + 0.5k,\ b_{22} = 1 - i - 0.4j - 0.1k,$$

$$c_{11} = 1 - i - 0.2j - 0.5k,\ c_{12} = -1 - i - 0.4j + 0.3k,$$

$$c_{21} = 2 - i - 0.3j + 0.3k,\ c_{22} = 2 - 5i - 0.1j - 0.5k,$$

and the activation functions are $g_p(q_p) = 0.01\tanh(q_p^R) + 0.01\tanh(q_p^I)i + 0.01\tanh(q_p^J)j + 0.01\tanh(q_p^K)k$, *where* $q_p \in \mathbb{Q}$, $q_p = q_p^R + q_p^I i + q_p^J j + q_p^K k$, $p = 1, 2$.

We choose $\sigma = 0.1$, $\tau_{ij} = 0.1$, $\delta_{ij}(t) = 0.5\sin t + 0.5$, $t > 0$, $i, j = 1, 2$. *Clearly,* $\bar{\delta}_{ij} = 1$ *for* $i, j = 1, 2$. *Set* $\ell_i = 0.02$, $i = 1, 2$. *Then the assumption* A_2 *is satisfied. When* $\lambda = 0.6$, *by employing the software tool YALMIP, we obtain a feasible solution of inequalities in* (3.1)*:* $\tilde{h}_1 = 81.5264$ *and* $\tilde{h}_2 = 86.5244$. *For 20 stochastically taken initial functions, Figures 1–4 show that the state trajectories of the considered QVNN converge to the zero equilibrium point, which verifies the result that the considered QVNN is GES.*

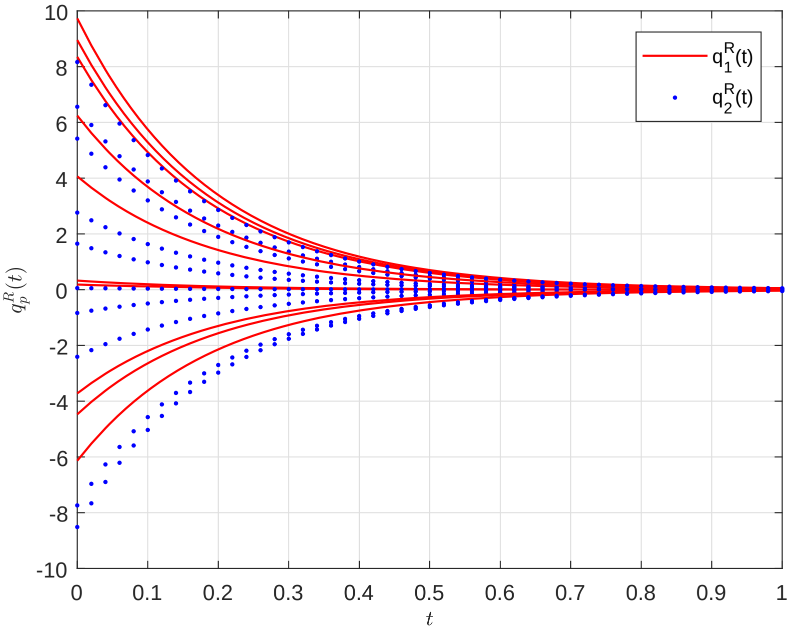

Figure 1. Trajectories of $q_p^R(t)$, $p = 1, 2$.

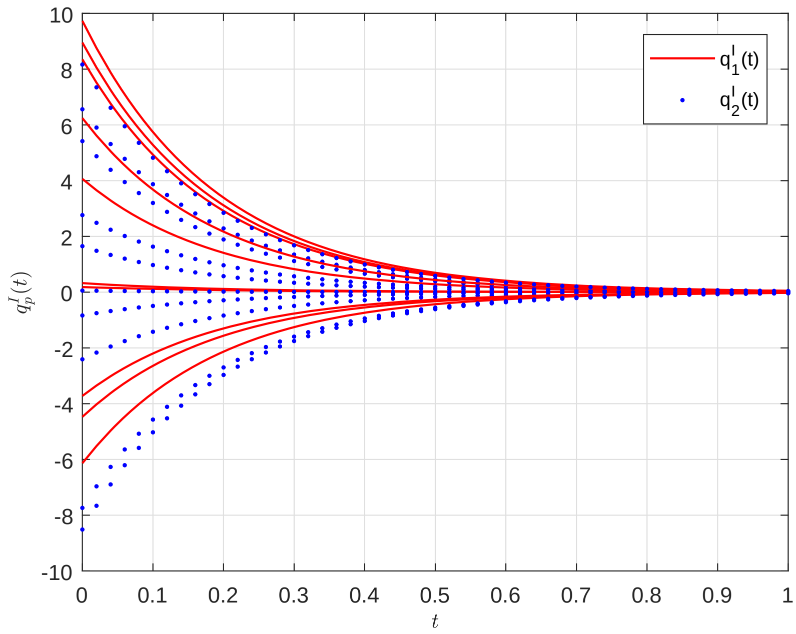

Figure 2. Trajectories of $q_p^I(t)$, $p = 1, 2$.

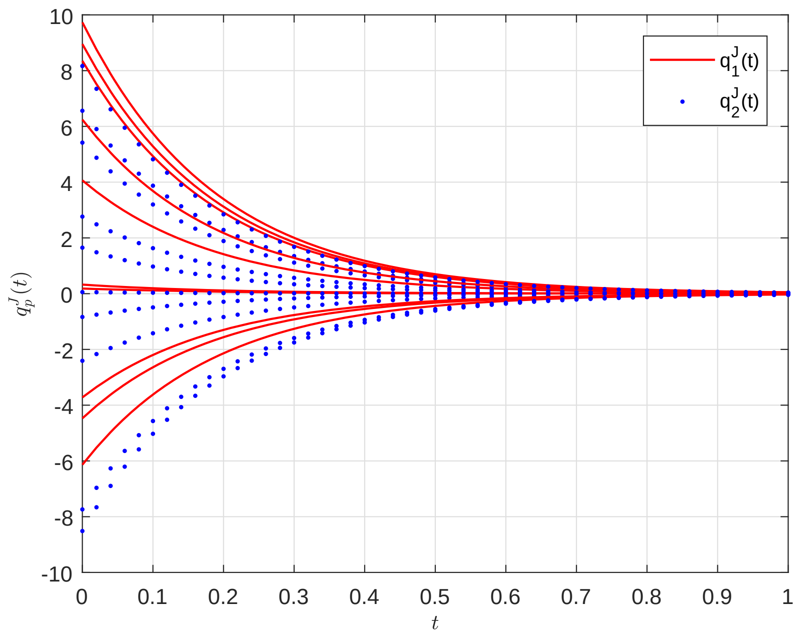

Figure 3. Trajectories of $q_p^J(t)$, $p = 1, 2$.

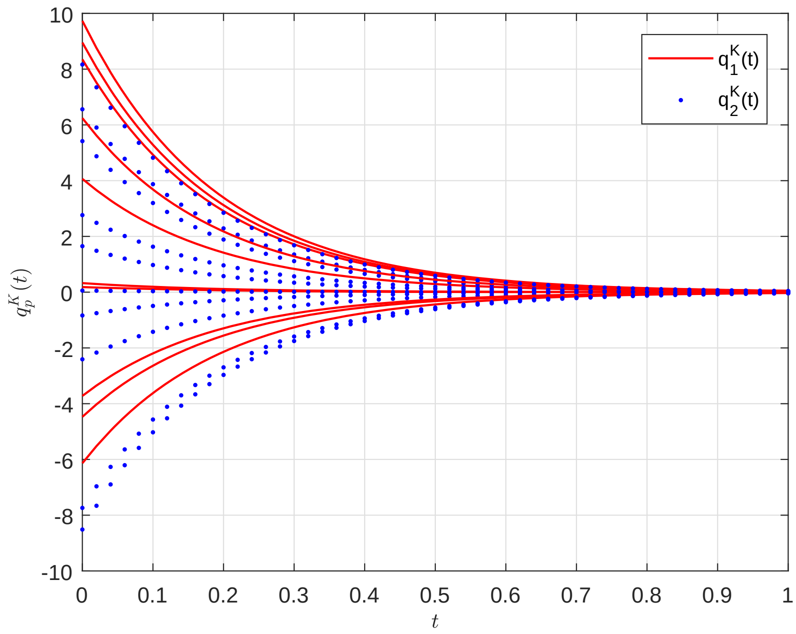

Figure 4. Trajectories of $q_p^K(t)$, $p = 1, 2$.

Example 2. *With the application and development of neural network theory, neural network can be applied not only to biological neurons, but also to some practical systems [43]. In the example in [44], the quadruple-tank process system (QTPS) can be regarded as a practical problem, which is shown in Figure 5. The differential equation model of mass balance in QTPS is as follows:*

$$\dot{\tilde{x}}(t) = \tilde{A}_0\tilde{x}(t) + \tilde{A}_1\tilde{x}(t) + \tilde{B}_0u(t) + \tilde{B}_1u(t-\tau(t)), t \geq 0, \tag{4.1}$$

where $\tilde{x}$ *is the state vector,* $u(t)$ *is the control input,* $\tau : [0, +\infty) \to [0, \bar{\tau}]$ *is the time-varying delay, and*

$$\tilde{A}_0 = -diag(0.0021, 0.0021, 0.0424, 0.0424), \tilde{A}_1 = \begin{bmatrix} 0 & 0 & 0.0424 & 0 \\ 0 & 0 & 0 & 0.0424 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{B}_0 = \begin{bmatrix} 0.1113\gamma_1 & 0 \\ 0 & 0.1042\gamma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.1113(1-\gamma_1) \\ 0.1042(1-\gamma_2) & 0 \end{bmatrix},$$

$$\gamma_1 = 0.333, \gamma_2 = 0.307.$$

$$\bar{K} = \begin{bmatrix} -0.1609 & -0.1765 & -0.0795 & -0.2073 \\ -0.1977 & -0.1579 & -0.2288 & -0.0772 \end{bmatrix}.$$

Based on the actual context of QTPS, it is natural to treat control input $u(t)$ as a non-linear function as follows:

$$u(t) = \bar{K}\hat{f}(\tilde{x}(t)),\ \hat{f}(\tilde{x}(t)) = (\hat{f}_1(\tilde{x}_1(t)), \hat{f}_2(\tilde{x}_2(t)), \hat{f}_3(\tilde{x}_3(t)), \hat{f}_4(\tilde{x}_4(t)))^T,$$

where $\bar{K}$ is the control gain matrix. Then the QTPS (4.1) *can be rewritten to the form:*

$$\dot{\tilde{x}}(t) = \tilde{A_0}\tilde{x}(t) + \tilde{A_1}\tilde{x}(t) + \hat{B}\hat{f}(\tilde{x}(t)) + \hat{C}\hat{f}(\tilde{x}(t-\tau(t))), t \geq 0, \tag{4.2}$$

where $\hat{B} = \tilde{B}_0\bar{K}$, $\hat{C} = \tilde{B}_1\bar{K}$.

Let $\hat{f}_i(\tilde{x}_i(t)) = 0.01^2\tilde{x}_i(t), i = 1, 2, 3, 4$, the QTPS (4.2) *is convert to the following form:*

$$\dot{\tilde{x}}(t) = \tilde{A_0}\tilde{x}(t) + (\tilde{A_1} + 0.01^2\hat{B})\tilde{x}(t) + 0.01^2\hat{C}\tilde{x}(t-\tau(t)), t \geq 0, \tag{4.3}$$

where $\hat{B} = \tilde{B}_0\bar{K}$, $\hat{C} = \tilde{B}_1\bar{K}$.

Clearly, (4.3) *can be reviewed as a special form of QVNN* (3.27)*, where $d_1 = d_2 = 0.0021, d_3 = d_4 = 0.0424$, $[a_{ij}]_{4\times4} = 100\tilde{A_1} + 0.01\hat{B}$, $[b_{ij}]_{4\times4} = 0.01\hat{C}$, $g(\tilde{x}(t)) = 0.01\tilde{x}(t)$.*

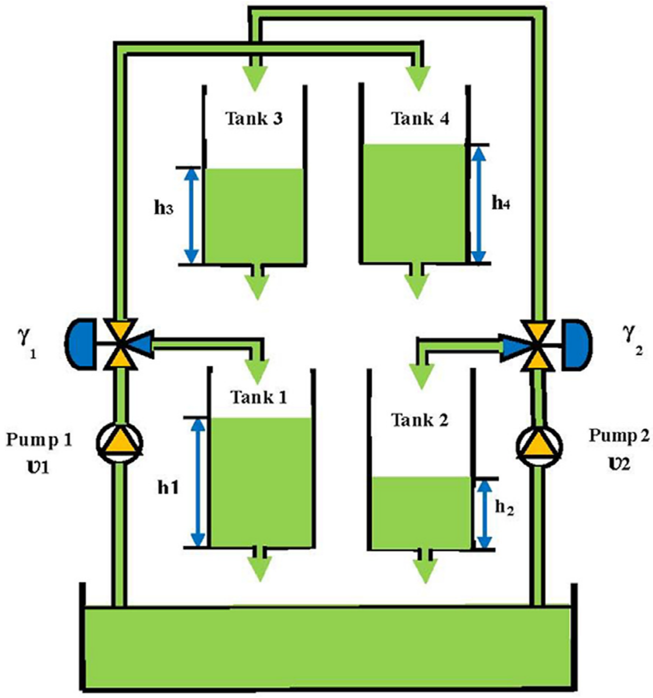

Figure 5. Schematic representation of the QTPS [44].

We choose $\delta_{ij}(t) = 0.25sin(0.5t) + 0.25$, $t > 0$, $i, j = 1, 2, 3, 4$. Clearly, $\bar{\delta}_{ij} = 0.5$ for $i, j = 1, 2, 3, 4$. Set $\ell_i = 0.01$, $i = 1, 2, 3, 4$. Then the assumption A_2 is satisfied. When $\lambda = 0.002$, by employing the software tool YALMIP, we obtain a feasible solution of inequalities in (3.28)*: $\tilde{h}_1 = 88.4901$,$\tilde{h}_2 = 88.4901$, $\tilde{h}_3 = 88.4899$, $\tilde{h}_4 = 88.4899$. The conditions of the Corollary 1 are satisfied easily, which verifies the result that the considered QTPS is GES. For 20 stochastically taken initial functions, Figure 6 show that the state trajectories of the considered QVNN converge to the zero equilibrium point, which verifies the result that the considered QVNN is GES.*

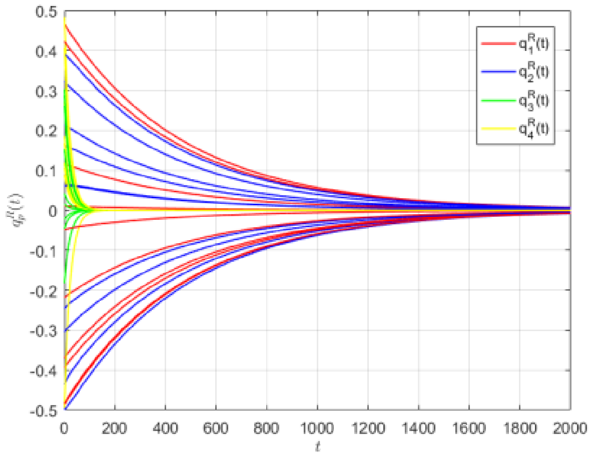

Figure 6. Trajectories of $q_p^R(t)$, $p = 1, 2, 3, 4$.

5. Conclusions

In this article, we study the global exponential stability of QVNNs with leakage, distribution and transmission delays. By proposing a direct method based on system solutions, the global exponential stability criteria are established of linear scalar inequalities form. Especially, the proposed method doesn't need both LKF construction and model decomposition, which reduces the calculation amount to a certain extent.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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