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Research paper

Pricing life settlements in the secondary market using fuzzy internal rate of return

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Abstract:
In this paper, fuzzy set theory is implemented to model internal rate of return for calculating the price of life settlements. Deterministic, probabilistic and stochastic approaches is used to price life settlements in the secondary market for the Iranian insurance industry. Research findings were presented and analyzed for whole life insurance policies using the interest rates announced in the supplement of Regulation No. 68 and Iranian life table, which recently has been issued to be used by insurance companies. Also, the results of three approaches were compared with surrender value, which indicates the surrender value is lower than the fuzzy price calculated based on the probabilistic and stochastic approaches and it is higher than the price calculated based on the deterministic approach. Therefore, selling life settlements in the secondary market in Iran based on calculated fuzzy price using probabilistic and stochastic approaches will benefit the policyholder. Also, the price is obtained in the form of an interval using the fuzzy sets theory and the investor can decide which price is suitable for this policy based on financial knowledge. Furthermore, in order to show validity of the proposed fuzzy method, the findings are compared to the results of using the random internal rate of return.

Keywords: Life settlements, Fuzzy random variables, life expectancy, Secondary market, Adjustment multiplier.
JEL Classifications: C02, G12, G22.

1 Introduction

A life settlement (LS) is a financial transaction by which an existing life insurance policy is sold to an investor generally for a greater price than its cash surrender value. In this way, the investor undertakes to pay, if any, the outstanding premiums and has the right to receive the death benefit when the insured dies, [13]. Secondary life insurance markets are growing rapidly. Therefore, pricing LSs is one of the most important problems involved investors and policyholders. Some countries, in a bid to guarantee sellers a fair price, have passed regulations setting a price floor on secondary life insurance market transactions, and more are considering doing the same, [4]. Sanches and Puchades discussed the main quantitative aspects of LS transaction and analyzed the sensitivity of the price of a LS to changes in the insureds life expectancy [13]. Dedes discussed how transactions in the secondary

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market for life insurance policies can be fairly priced for both policyholders and life settlement companies and presented deterministic, probabilistic and stochastic pricing approaches.
In recent years, fuzzy random variables have been used to consider uncertainty of economics parameters in insurance industry, [1, 10–12, 14]. Also, in capital market, we usually deal with projects for investment that taking a long time for its realization. In such cases, a description of uncertainty is usually impossible due to the absence of objective information about the probabilities of future events. Therefore, the fuzzy methods are used to describe uncertainty sources such as internal rate of return (IRR). In other words, the theory of fuzzy sets has been used to model problems for which the available information is scarce or ambiguous and requires judgment. Although in calculations, the internal rate of return may be considered constant over time (see e.g. [13]), but in reality, it changes during different years. The fuzzy internal rate of return, instead of a crisp price, provides the actuary with a range of all possible values to consider the appropriate price based on economic information. Therefore, in this article, the fuzzy methods are used to model this uncertainty. According to our best knowledge, fuzzy methods have been widely applied to financial problems [5, 9, 16]. But evaluating the fuzzy price of LS based on deterministic, probabilistic and stochastic pricing approaches has been investigated for the first time in this paper.
The remainder of this article is organized as follows. In section 2, the definitions and basic concepts such as fuzzy random variables, the single premium at the start of the insurance policy, the adjustment multiplier for the mortality probability and life expectancy are provided. In section 3, fuzzy deterministic, probabilistic and stochastic pricing approaches for pricing LSs are presented. Numerical test results for fuzzy pricing LSs are given in Section 4. Finally, our conclusions are given in Section 5.

2 Basic concepts

In this section, we provide the definitions and basic concepts. Suppose a x^*-year-old person has purchased a whole life insurance policy with M monetary unit benefits. A few years later, due to economic pressure or health problems, he tends to sell his insurance policy in the secondary market. In the following, the single premium at the start of the insurance policy, the adjustment multiplier for the mortality probability and life expectancy (LE) of this person at the age of x is discussed. In the whole life insurance contract, if the insured dies, the insurance company will pay M monetary unit to the beneficiaries at the end of year of his death. The single premium of life insurance is:

$$SP = M \sum_{k=1}^{\omega - x^*} (1+r)^{-k} \times {}_{k-1|}\, q_{x^*}, \tag{1}$$

where ${}_{k-1|}q_{x^*}$ is the probability that the insured aged x^* dies the kth year, r is the interest rate and ω is the maximum attainable age in the considered mortality table.
In order to calculate the mortality probability according to the health status of insured, an adjustment multiplier is used. Because the insured with bad health conditions has a different LE than ordinary people, and using the standard life table shows the LE more than it is. Therefore, using the adjustment multiplier makes the mortality probability and life expectancy (LE) closer to reality for these insureds. Then, from the adjusted mortality probabilities q^*_{x+t} , $t = 1, 2, \cdots, \omega - x$, we can find the adjusted probability that the insured of age x survives k years as

$$ {}_kp^*_x = \prod_{t=0}^{k-1}(1 - q^*_{x+t}). \tag{2} $$

Consequently, the LE of the insured can be calculated by:

$$ e^*_x = \sum_{k=1}^{\omega-x} {}_kp^*_x = \sum_{k=1}^{\omega-x}\prod_{t=0}^{k-1}(1 - q^*_{x+t}) \tag{3} $$

Therefore, we should determine how to calculate the adjustment multiplier. For this aim, some methods have been proposed. In [13], the adjustment multiplier is calculated as follows :

$$ \beta = 1 + \sum_{j=1}^{m}\rho_j $$

where coefficient ρ_j supposes as a measure of negative factors for the lifestyle of the insured or positive factors detected in the medical in the mortality as explained in [2, 8, 13, 15]. The modified mortality probability q^*_x is considered as a linear function of the standard probability q_x and is calculated by:

$$ q^*_x = \beta q_x = (1 + \sum_{j=1}^{m}\rho_j)q_x. \tag{4} $$

Since $0 \leq q^*_x \leq 1$, we should have:

$$ -1 < \sum_{j=1}^{m}\rho_j < \frac{1}{q_x} - 1. $$

For this purpose, the following equation is proposed in [15]

$$ q^*_{x+t} = min\{1, (1 + \sum_{j=1}^{m}\rho_j)q_x\}, t = 1, 2, \cdots, \omega - x. \tag{5} $$

Furthermore, in real world, uncertainty has many sources, such as randomness, ambiguity, and inaccuracy. In this paper, using fuzzy random variables presented

in paper [1,12], we examine the effect of changes in one of the sources of uncertainty, such as the internal rate of return on life settlement pricing.
A fuzzy set denoted by $\tilde{A}$ is defined as

$$\tilde{A} = \left\{(x, \mu_{\tilde{A}(x)})|x \in X\right\}$$

where $\mu_{\tilde{A}}$ is the membership function as follows:

$$\mu_{\tilde{A}} : X \to [0,1] \, .$$

Also, a fuzzy set can be represented by its α-cuts which is a set A_α as follows:

$$A_\alpha = \left\{x \in X|\mu_{\tilde{A}(x)} \geq \alpha\right\}, \forall \alpha \in (0,1]$$

An infima random variable $\underline{A}_\alpha$ and a suprema random variable $\overline{A}_\alpha$ are defined as the lower and upper extremes of α-cuts of $\tilde{A}$. Therefore, we can write A_α as follows (for more details see [12]):

$$A_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha] = \left[\inf_{x\in X} \{\mu_{\tilde{A}}(x) \geq \alpha\}, \sup_{x\in X} \{\mu_{\tilde{A}}(x) \geq \alpha\}\right],$$

Assume that the internal rate of return is a fuzzy number with the α cuts, $IRR_\alpha = [\underline{IRR}_\alpha, \overline{IRR}_\alpha]$. As a result, the fuzzy discount rate $\tilde{d}_t$ is defined as $\tilde{d}_{t_\alpha} = [\underline{d_t}_\alpha, \overline{d_t}_\alpha]$ for 1 monetary unit payable in t years, with α cuts for each $\alpha \in [0,1]$. Considering it is a decreasing function of internal rate of return, we have:

$$\tilde{d}_{t_\alpha} = [\underline{d_t}_\alpha, \overline{d_t}_\alpha] = [(1 + \overline{IRR}_\alpha)^{-t}, (1 + \underline{IRR}_\alpha)^{-t}]. \tag{6}$$

3 Fuzzy pricing approaches

In the following, fuzzy set theory implemented to model internal rate of return is used to calculate the price of life settlements based on deterministic, probabilistic and stochastic approaches in the secondary market.

The deterministic value of LS for person of age x using [13] is:

$$VED_x = \frac{M}{(1 + IRR)^{e_x^*}} = Md_{e_x^*} \tag{7}$$

Therefore, to apply 7, it is enough to know the LE indicated in the medical underwriter report and IRR. Furthermore, using fuzzy IRR in Eq. 6 and considering the Eq. 7 , the fuzzy deterministic value $V\tilde{E}D$ is calculated as follows:

$$V\tilde{E}D = (\underline{VED}, \overline{VED}) = (M\overline{d}_{e_x^*}, M\underline{d}_{e_x^*}).$$

The second approach for pricing LS is called probabilistic which is used more than other approaches. In this case, all the cash flows that can be paid are considered, being weighted by the probability of their occurrence. Therefore, the probabilistic value for an insured of age x based on [13] is:

$$VEP_x = \sum_{k=1}^{\omega-x} \frac{M}{(1+IRR)^k}\ _{k-1|}q_x^* = M \sum_{k=1}^{\omega-x} d^k\ _{k-1|}q_x^*. \tag{8}$$

Considering fuzzy IRR in Eq. 6 and the Eq. 8 together, the fuzzy deterministic value $V\tilde{E}P$ will be concluded:

$$V\tilde{E}P = (\underline{VEP}, \overline{VEP}) = (M \sum_{k=1}^{\omega-x} \overline{d}^k\ _{k-1|}q_x^*, M \sum_{k=1}^{\omega-x} \underline{d}^k\ _{k-1|}q_x^*).$$

The third valuation method is stochastic approach. In this approach, the random variable future lifetime of the insured with current age x, T_x^*, is used which can be obtained from the modified mortality table corresponding to the insured. The possible outcomes of T_x^* are $0, 1, 2, , \omega - x - 1$ with respective probabilities $q_x^*, _{1|}q_x^*, _{2|}q_x^*, \cdots, _{\omega-x-1|}q_x^*$. So, the stochastic economic value is calculated as follows based on [13]:

$$VEE_x = \frac{M}{(1+IRR)^{T_x^*+1}} = Md^{T_x^*+1} \tag{9}$$

It should be noted that based on actuarial mathematics concepts (see e.g. [7]), it can be proved that the mathematical expectation of the random variable VEE_x will result the probabilistic economic value, VEP_x. The results obtained using the random method are different each time the program is run. Therefore, we repeat the program an acceptable number of times and average the results. In this approach, also, considering fuzzy IRR in Eq. 6 and the Eq. 9 together will conclude the fuzzy deterministic value $V\tilde{E}E$ as follows:

$$V\tilde{E}E = (\underline{VEE}, \overline{VEE}) = (M\overline{d}^{T_x^*+1}, M\underline{d}^{T_x^*+1}).$$

4 Numerical test results

In this section, the results have been analyzed based on the standard mortality probabilities obtained from Iranian life table and the adjusted mortality probabilities similar to the article [13]. Suppose a 45-year-old person has purchased a whole life insurance policy with 1000 monetary unit benefits and after 20 years, he tends to sell his insurance policy in the secondary market. In the following, the single premium at the start of the insurance policy, the adjustment multiplier for the mortality probability and life expectancy of this person at the age of 65 is discussed.

The single premium at the start of the insurance policy is obtained 64.41. Also, Two risk factors are considered for a 65-year-old person and the adjustment multiplier β is calculated for this person.

The first risk factor considered for this person is hypopharyngeal cancer, which, according to the American Cancer Society (2019), has a 5-year relative survival rate of 52%, [3]. That is, the survival probability of this person is 52% of what is calculated based on the standard table. Based on Iranian life table, ${}_5P_{65} = 0.9040$. Therefore, ${}_5P^*_{65} = 0.52 \times 0.9040 = 0.4701$ and using equations (5) and (2), the value of ρ_1 is obtained 6.00.
The second factor is this person's interest in base jumping, which indicates the mortality probability 0.0167. Considering that according to the standard tables $q_{65} = 0.0162$, the increase in the mortality probability due to interest in this sport is obtained $\rho_2 = 0.0309$. Therefore, the estimation of β multiplier in Eq.4 will be equal to $\beta = 7.03$. For simplicity, we consider $\beta = 7.03$ for all ages.
Figure 1 shows the standard and adjusted mortality probability with $\beta = 7.03$ for a 65-year-old insured.

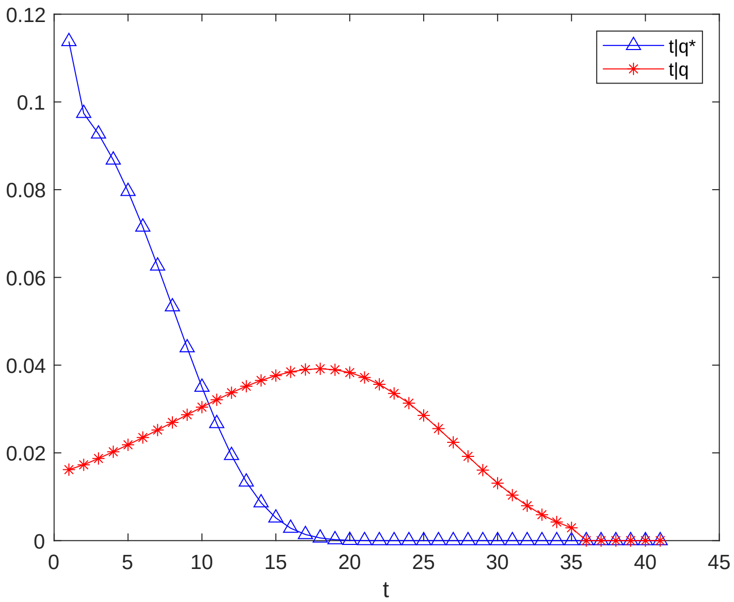

Figure 1: The standard and adjusted mortality probability with $\beta = 7.03$ for a 65-year-old insured.

The mathematical reserves (V), surrender value (VR) , life expectancy with standard mortality probability (e_x) and modified life expectancy with $\beta = 7.03$

According to Insurance Regulation No. 68, the surrender value of the insurance policy is at least 90% of the mathematical reserve of the insurance policy. In this article, 10% of the mathematical reserve is considered as a cost or penalty of surrender and the rest of the amount is considered as the surrender value.

(e_x^*) for different ages are shown in table 1.

Table 1: The mathematical reserves, surrender values, standard and modified life expectancies for different ages.

x	e_x	e_x^*	V	VR
60	20.09	6.08	187.30	168.57
65	16.21	3.91	246.20	221.58
70	12.70	2.25	313.74	282.37
75	9.63	1.07	385.36	346.82

As it is clear in table 1, e_x^* is obtained less than e_x for this insured. Calculating the value of LS using fuzzy numbers allow the investor to have a full range of values that the LS can take. The fuzzy deterministic, probabilistic and stochastic value of LS with its α-cuts are shown for different ages in table 2. Comparing the results of three fuzzy approaches with surrender value indicates the surrender value is lower than the fuzzy price calculated based on the probabilistic and stochastic approaches and it is higher than the fuzzy price calculated based on the deterministic approach. Therefore, selling life settlements in the secondary market based on calculated fuzzy price using probabilistic and stochastic approaches will benefit the policyholder. Also, Our findings in table 2 shows that all possible values for a 65-year-old insured are in the interval $[353.67, 396.83]$ using probabilistic approach and in the interval $[348.92, 406.58]$ using stochastic approach. It means that the fuzzy price obtained by probabilistic and stochastic approaches are approximately close together.
Furthermore, in order to show validity of the proposed fuzzy method, the research findings is compared to the results of the random internal rate of return with normal distribution used in [7] as $r \sim N(0.22, 0.02)$.
After simulating the technical interest rate for 5000 samples, we obtained a 95% confidence interval (CI) for the single premium and we reflect its results in table 2 to compare with the fuzzy method which indicates the validity of the findings of the article. Also, the comparison of the two methods shows that the fuzzy single premium requires fewer assumptions than the random method, and its calculations are simpler due to the absence of the need to generate random numbers.

5 Conclusions

The modeling of uncertainty resources in financial transactions is very important. In this paper, we used fuzzy set theory to model internal rate of return as a source of uncertainty, and calculated the price of LS based on three approaches. In this way, we implemented the fuzzy discount factor and calculated the random fuzzy value of LS. For this purpose, we explained how the fuzzy deterministic, probabilistic and

Table 2: The comparison of fuzzy deterministic, probabilistic and stochastic price of LS with the simulation in [7]

		Deterministic		Probabilistic		Stochastic		Simulation in [7]
x	α	$\underline{VED}(\alpha)$	$\overline{VED}(\alpha)$	$\underline{VEP}(\alpha)$	$\overline{VEP}(\alpha)$	$\underline{VEE}(\alpha)$	$\overline{VEE}(\alpha)$	%95 CI
	1	25.66	25.66	291.41	291.41	284.99	284.99	
	0.9	24.82	26.54	289.31	293.53	282.39	287.64	
	0.8	24.00	27.44	287.25	295.67	279.82	290.32	
	0.7	23.22	28.38	285.21	297.84	277.28	293.04	
	0.6	22.46	29.35	283.20	300.04	274.79	295.81	
60	0.5	21.72	30.36	281.21	302.28	272.32	298.61	[251.64,335.11]
	0.4	21.01	31.41	279.24	304.54	269.90	301.45	
	0.3	20.33	32.49	277.31	306.83	267.50	304.34	
	0.2	19.67	33.61	275.39	309.15	265.14	307.27	
	0.1	19.03	34.77	273.50	311.50	262.81	310.25	
	0	18.41	35.97	271.63	313.88	260.52	313.27	
	1	52.02	52.02	374.14	374.14	376.05	376.05	
	0.9	50.64	53.45	371.99	376.30	373.20	378.94	
	0.8	49.29	54.92	369.87	378.48	370.37	381.87	
	0.7	47.98	56.43	367.77	380.69	367.58	384.83	
	0.6	46.71	57.98	365.70	382.93	364.83	387.82	
65	0.5	45.47	59.58	363.64	385.18	362.10	390.85	[333.23,416.99]
	0.4	44.27	61.23	361.61	387.46	359.41	393.92	
	0.3	43.10	62.93	359.59	389.77	356.75	397.03	
	0.2	41.97	64.67	357.60	392.10	354.11	400.17	
	0.1	40.87	66.47	355.62	394.45	351.51	403.35	
	0	39.79	68.32	353.67	396.83	348.92	406.58	
	1	98.63	98.63	459.53	459.53	472.68	472.68	
	0.9	96.57	100.75	457.51	461.57	469.78	475.61	
	0.8	94.55	102.91	455.50	463.63	466.91	478.57	
	0.7	92.58	105.12	453.51	465.70	464.06	481.55	
	0.6	90.65	107.38	451.54	467.79	461.24	484.57	
70	0.5	88.76	109.70	449.58	469.90	458.45	487.61	[419.94,499.99]
	0.4	86.92	112.07	447.64	472.02	455.68	490.69	
	0.3	85.12	114.49	445.71	474.16	452.94	493.79	
	0.2	83.36	116.97	443.80	476.32	450.22	496.92	
	0.1	81.63	119.51	441.90	478.50	447.53	500.09	
	0	79.95	122.11	440.02	480.70	444.87	503.29	
	1	172.73	172.73	539.91	539.91	562.36	562.36	
	0.9	169.98	175.52	538.14	541.68	559.58	565.15	
	0.8	167.28	178.37	536.39	543.46	556.83	567.97	
	0.7	164.63	181.27	534.65	545.26	554.09	570.81	
	0.6	162.02	184.22	532.92	547.07	551.38	573.67	
75	0.5	159.46	187.23	531.20	548.89	548.69	576.56	[505.77,575.35]
	0.4	156.94	190.28	529.49	550.72	546.01	579.46	
	0.3	154.47	193.40	527.79	552.56	543.36	582.39	
	0.2	152.04	196.57	526.10	554.41	540.73	585.34	
	0.1	149.65	199.79	524.43	556.27	538.11	588.32	
	0	147.31	203.08	522.76	558.15	535.52	591.31	

stochastic price of LS are calculated. The prices based on these approaches obtained in the form of intervals using the fuzzy sets theory which contains all possible values and according to financial knowledge, the investor can judge which price is suitable for this policy. Furthermore, the results compare to the surrender value of policy which indicates that the fuzzy deterministic price is less than surrender value and the fuzzy probabilistic and stochatic prices are more than it.

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