TWMS J. App. and Eng. Math. V.13, N.2, 2023, pp. 626-634

# DECOMPOSITION OF TENSOR PRODUCT OF COMPLETE GRAPHS INTO CYCLES AND STARS WITH FOUR EDGES

A. P. EZHILARASI[1], M. ILAYARAJA[2], A. MUTHUSAMY[1*], §

Abstract. In this paper, we prove that the necessary conditions are sufficient for the existence of a decomposition of tensor product of complete graphs into cycles and stars with four edges.

Keywords: Graph decomposition, Cycle, Star, Product graph.

AMS Subject Classification: 05C51, 05C70

## 1. Introduction

All graphs considered here are finite and simple. Let $P_n$, $C_n$, $S_n$ and $K_n$ denote a path, cycle, star and complete graph on $n$ vertices and $K_{m,n}$ denotes a complete bipartite graph with $m$ and $n$ vertices in the parts. Let $K_{m(n)}$ denote a *complete m-partite* graph with $n$ vertices in each part. We denote the cycle $C_k$ with vertices $x_1, x_2, \cdots, x_k$ and edges $x_1x_2, \cdots, x_{k-1}x_k, x_kx_1$ as $(x_1x_2\cdots x_k)$ and a star $S_{k+1}$ consists of a center vertex $x_0$ and $k$ end vertices $x_1, x_2, \cdots, x_k$ as $(x_0; x_1, x_2, \cdots, x_k)$.

For two graphs $G$ and $H$, we define their *tensor product*, denoted by $G \times H$, as follows: the vertex set is $V(G) \times V(H)$ and the edge set is

$$E(G \times H) \;\; = \;\; \{(g,h)(g', h') \,|\, gg' \in E(G) \,\text{and}\, hh' \in E(H)\}.$$

If $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_k$ such that the subgraph induced by $E_i$ is $H_i$, for all $i$, $1 \leq i \leq k$, then we say that $H_1, \ldots, H_k$ decompose $G$ and we write $G = H_1 \oplus \cdots \oplus H_k$. For $1 \leq i \leq k$, if $H_i \cong H$, we say that $G$ has a $H$-*decomposition* and it is denoted by $H|G$. If $G$ can be decomposed into $q$ copies of $H_1$ and $r$ copies of $H_2$, then we say that $G$ has a $\{qH_1, rH_2\}$-*decomposition.* If such a decomposition exits for all

[1] Department of Mathematics, Periyar University, Salem-11, Tamil Nadu, India.
e-mail: post2pauline@gmail.com; ORCID: https://orcid.org/0000-0003-0853-9492.
e-mail: ambdu@yahoo.com; ORCID: https://orcid.org/0000-0001-9014-6916.
[*] Corresponding author.
[2] Department of Mathematics, Karpagam Academy of Higher Education, Coimbatore-21, India.
e-mail: ilaya@yahoo.com; ORCID: https://orcid.org/0000-0002-7635-1400.
§ Manuscript received: February 03, 2021; accepted: July 2, 2021.
TWMS Journal of Applied and Engineering Mathematics, Vol.13, No.2 
A. Muthusamy is partially supported by the University Grant Commission, Government of India, New Delhi and the Department of Science and Technology, New Delhi.
(Grant No.F.510/7/DRS-I/2016(SAP-I) and Grant No. SR/FIST/MSI-115/2016(Level-I)).

$q$ and $r$ satisfying trivial necessary conditions, then we say that $G$ has a $\{H_1, H_2\}_{\{q,r\}}$-*decomposition* or *complete* $\{H_1, H_2\}$*-decomposition.*

The study of $\{H_1, H_2\}$ -decomposition has been introduced by Abueida and Daven [1]. Moreover, Abueida and O'Neil [2] have settled the existence of $\{H_1, H_2\}$ -decomposition of $K_m(\lambda)$ when $\{H_1, H_2\} = \{S_n, C_n\}$ for $n = 3, 4, 5$. Priyadharsini and Muthusamy [10] established necessary and sufficient conditions for the existence of $\{H_1, H_2\}$-multidecomposition of $\lambda K_n$ where $H_1,\ H_2 \in \{C_n, P_n, S_n\}$ . Lee [6], gave necessary and sufficient conditions for the decomposition of $K_{m,n}$ into at least one copy of each $C_k$ and $S_{k+1}$. Jeevadoss and Muthusamy [5] have obtained necessary and sufficient conditions for the existence of a decomposition of product graphs into paths and cycles with four edges. Pauline Ezhilarasi and Muthusamy [8] have obtained the necessary and sufficient conditions for the existence of a decomposition of product graphs into paths and stars with three edges. Ilayaraja et.al, [4] and Pauline Ezhilarasi and Muthusamy, [9] proved the existence of $\{P_5, S_5\}$-decomposition of product graphs. Many other results on decomposition of graphs into distinct subgraphs involving cycles and stars have been proved in [6, 7, 11–13]. In this paper, we establish necessary and sufficient conditions for the existence of a complete $\{C_4, S_5\}$-decomposition of $K_m \times K_n$.

To prove our results we state the following:

**Theorem 1.1.** *[3] Let $q$ and $r$ be non-negative integers and $n \geq m > 0$. Then there exists a complete $\{C_4, S_5\}$-decomposition of $K_{m,n}$ if and only if one of the following holds:*

(1) *$q \neq 0, 2$ and $r \neq 1$ when $m, n \equiv 2 \pmod 4$;*
(2) *$q, r \neq 1$ when $m, n \equiv 0 \pmod 2$;*
(3) *$q \neq 1$ and $q \geq \frac{n}{4}$(or $\frac{m}{4}$) when $m$(or $n$) is odd and $n$(or $m$) $\equiv 0 \pmod 4$.*

**Theorem 1.2.** *[5] If $m \equiv 0 \pmod 4$, then $K_m$ has a $\{(m/4)K_4, ((m^2 - 4m)/8)C_4\}$-decomposition.*

**Remark 1.1.** *If $G$ and $H$ has a complete $\{C_4, S_5\}$-decomposition, then $G \cup H = G \oplus H$ has a complete $\{C_4, S_5\}$-decomposition.*

**Remark 1.2.** *If two stars with four edges have same end vertices, then they can be decomposed into two cycles on four edges. i.e., $\{(a_0; x_1, \cdots, x_4),\ (a_1; x_1, \cdots, x_4)\}$ gives $\{(x_1a_0x_2a_1x_1),\ (x_3a_0x_4a_1x_3)\}$. We denote such pair of stars as $(a_0, a_1; x_1, \cdots, x_4)$.*

## 2. Base Constructions

In this section we prove some basic Lemmas which are required to prove our main result.

**Lemma 2.1.** *Let $q$ and $r$ be non-negative integers. Then there exists a complete $\{C_4, S_5\}$-decomposition of $K_{5,5} - I$, with $q = 0$ or $r = 0$.*

*Proof.* Let $V(G) = \{x_1, \cdots, x_5\} \cup \{y_1, \cdots, y_5\}$. Now, $\{(x_1; y_2, y_3, y_4, y_5), (x_2; y_1, y_3, y_4, y_5), (x_3; y_1, y_2, y_4, y_5), (x_4; y_1, y_2, y_3, y_5), (x_5; y_1, y_2, y_3, y_4)\}$ and $\{(y_2x_1y_3x_4), (y_4x_1y_5x_2), (y_1x_2y_3x_5), (y_1x_3y_5x_4), (y_2x_3y_4x_5)\}$ respectively gives required stars and cycles. □

**Lemma 2.2.** *There exists a complete $\{C_4, S_5\}$-decomposition of $K_{9,9} - I$, for all non-negative integers $q, r$ with $r \neq 1$.*

*Proof.* Let $V(G) = \{x_1, \cdots, x_9\} \cup \{y_1, \cdots, y_9\}$. We can write $K_{9,9} - I = 2(K_{5,5} - I) \oplus 2K_{4,4}$. By Lemma 2.1 and Theorem 1.1, $K_{5,5} - I$ and $K_{4,4}$ have a complete $\{C_4, S_5\}$-decomposition and hence $G$ has a complete $\{C_4, S_5\}$-decomposition except $(q, r) \in \{(1, 17), (3, 15), (15, 3)\}$. Now, by Remark 1.2, the cycles and stars
$\{(x_3y_5x_4y_6), (x_1, x_3; y_2, y_4, y_7, y_8), (x_5; y_3, y_4, y_6, y_7), (x_6; y_2, y_4, y_5, y_9), (x_7; y_4, y_5, y_8, y_9),$

$(x_8; y_2, y_3, y_4, y_5)$, $(x_9; y_3, y_4, y_5, y_8)$, $(y_1; x_2, x_3, x_6, x_8)$, $(y_1; x_4, x_5, x_7, x_9)$, $(y_2; x_4, x_5, x_7, x_9)$, $(y_3; x_2, x_4, x_6, x_7)$, $(y_6; x_2, x_7, x_8, x_9)$, $(y_7; x_4, x_6, x_8, x_9)$, $(y_8; x_2, x_4, x_5, x_6)$, $(y_9; x_3, x_4, x_5, x_8)$, $(x_1; y_3, y_5, y_6, y_9)$, $(x_2; y_4, y_5, y_7, y_9)\}$ gives a required decomposition for $(q, r) \in \{(1, 17), (3, 15)\}$.

For $(q, r) = (15, 3)$, the required decomposition is $\{(x_1; y_2, y_4, y_5, y_6)$, $(x_2; y_1, y_5, y_6, y_7)$, $(x_3; y_1, y_2, y_4, y_7)$, $(x_2y_3x_5y_4)$, $(x_4y_2x_5y_1)$, $(x_1y_3x_4y_9)$, $(x_3y_8x_5y_9)$, $(x_3y_6x_4y_5)$, $(x_1y_8x_4y_7)$, $(x_2y_8x_6y_9)$, $(x_7y_5x_8y_9)$, $(x_5y_6x_8y_7)$, $(x_7y_6x_9y_8)$, $(x_6y_5x_9y_7)$, $(x_6y_1x_7y_2)$, $(x_8y_3x_9y_4)$, $(x_6y_3x_7y_4)$, $(x_8y_1x_9y_2)\}$. Hence $K_{9,9} - I$ has a complete $\{C_4, S_5\}$-decomposition. □

**Theorem 2.1.** *Let $q$ and $r$ be non-negative integers. There exists a complete $\{C_4, S_5\}$-decomposition of $K_{n,n} - I$ for $n \equiv 1 \pmod 4$ with $r \neq 1$ and $q = 0$ or $r = 0$ when $n = 5$.*

*Proof.* When $n = 5, 9$, the proof follows from Lemmas 2.1 and 2.2.

When $n > 9$, let $n = 4k + 9$, $k \in \mathbb{Z}^+$ and $V(G) = V(K_{n,n} - I) = X \cup Y$, where $X = \{x_0, x_1, \cdots, x_{4k+8}\}$ and $Y = \{y_0, y_1, \cdots, y_{4k+8}\}$. Partition the sets $\{x_1, x_2, \cdots, x_{4k}\}$ and $\{y_1, y_2, \cdots, y_{4k}\}$ into 4-subsets $X_i$ and $Y_i$, where $i = 1, 2, \cdots, k$ respectively. Then $G[X_i \cup \{x_0\}, Y_i \cup \{y_0\}] \cong K_{5,5} - I$ and $G[X_i, Y_j] \cong K_{4,4}$ for all $i \neq j$. Therefore, $K_{n,n} - I = k(K_{5,5} - I) \oplus k(k-1)K_{4,4} \oplus (K_{9,9} - I) \oplus 2kK_{8,4}$. By Theorem 1.1, Lemma 2.2 and Remark 1.1, $k(k-1)K_{4,4} \oplus (K_{9,9} - I) \oplus 2kK_{8,4}$ can be decomposed into $\alpha$ copies of $C_4$ and $4k^2 + 12k + 18 - \alpha$ copies of $S_5$, where $0 \leq \alpha \leq 4k^2 + 12k + 18$. By Lemma 2.1, $k(K_{5,5} - I)$ can be decomposed into $5\beta$ copies of $C_4$ and $5(k - \beta)$ copies of $S_5$ with $0 \leq \beta \leq k$. Hence by Remark 1.1, $K_{n,n} - I$ can be decomposed into $q$ copies of $C_4$ and $r(= n(k + 2) - q)$ copies of $S_5$ with $0 \leq q \leq n(k + 2)$. Thus $K_{n,n} - I$ has a complete $\{C_4, S_5\}$-decomposition. □

**Lemma 2.3.** *Let $q$ and $r$ be non-negative integers. Then there exists a complete $\{C_4, S_5\}$-decomposition of $P_3 \times K_3$, with $q = 0$ or $r = 0$.*

*Proof.* Let $V(P_3 \times K_3) = \{x_{i,j} : 1 \leq i, j \leq 3\}$. Now, the cycles and stars $\{(x_{1,1}x_{2,2}x_{3,1}x_{2,3})$, $(x_{1,2}x_{2,1}x_{3,2}x_{2,3})$, $(x_{1,3}x_{2,1}x_{3,3}x_{2,2})\}$ and $\{(x_{2,1}; x_{1,2}, x_{1,3}, x_{3,2}, x_{3,3})$, $(x_{2,2}; x_{1,1}, x_{1,3}, x_{3,1}, x_{3,3})$, $(x_{2,3}; x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2})\}$ respectively gives the required decomposition of $P_3 \times K_3$. □

**Lemma 2.4.** *Let $q$ and $r$ be non-negative integers. Then there exists a complete $\{C_4, S_5\}$-decomposition of $P_3 \times K_5$ with $r \neq 1$.*

*Proof.* Let $V(P_3 \times K_5) = \{x_{i,j} : 1 \leq i \leq 3, 1 \leq j \leq 5\}$. Then the required complete $\{C_4, S_5\}$-decomposition is given below:

(1) $q = 10$ and $r = 0$. The required cycles are
$(x_{1,1}x_{2,4}x_{3,3}x_{2,5})$, $(x_{1,2}x_{2,5}x_{3,4}x_{2,1})$, $(x_{1,3}x_{2,4}x_{3,1}x_{2,5})$, $(x_{1,4}x_{2,5}x_{3,2}x_{2,1})$, $(x_{1,1}x_{2,2}x_{3,4}x_{2,3})$, $(x_{1,2}x_{2,3}x_{3,5}x_{2,4})$, $(x_{1,5}x_{2,1}x_{3,3}x_{2,2})$, $(x_{1,3}x_{2,1}x_{3,5}x_{2,2})$, $(x_{1,4}x_{2,2}x_{3,1}x_{2,3})$, $(x_{1,5}x_{2,3}x_{3,2}x_{2,4})$.

(2) $q = 8$ and $r = 2$. The required cycles and stars are
$(x_{1,1}x_{2,4}x_{3,3}x_{2,5})$, $(x_{1,2}x_{2,5}x_{3,4}x_{2,1})$, $(x_{1,3}x_{2,4}x_{3,1}x_{2,5})$, $(x_{1,4}x_{2,5}x_{3,2}x_{2,1})$, $(x_{1,1}x_{2,2}x_{3,4}x_{2,3})$, $(x_{1,2}x_{2,3}x_{3,5}x_{2,4})$, $(x_{1,4}x_{2,2}x_{3,1}x_{2,3})$, $(x_{1,5}x_{2,3}x_{3,2}x_{2,5})$, $(x_{2,1}; x_{1,3}, x_{1,5}, x_{3,3}, x_{3,5})$, $(x_{2,2}; x_{1,3}, x_{1,5}, x_{3,3}, x_{3,5})$.

(3) $q = 7$ and $r = 3$. The required cycles and stars are
$(x_{1,1}x_{2,4}x_{3,1}x_{2,5})$, $(x_{1,2}x_{2,5}x_{3,4}x_{2,1})$, $(x_{1,3}x_{2,5}x_{3,3}x_{2,1})$, $(x_{1,4}x_{2,5}x_{3,2}x_{2,1})$, $(x_{1,4}x_{2,2}x_{3,1}x_{2,3})$, $(x_{1,5}x_{2,3}x_{3,2}x_{2,4})$, $(x_{1,5}x_{2,1}x_{3,5}x_{2,2})$, $(x_{2,2}; x_{1,1}, x_{1,3}, x_{3,3}, x_{3,4})$, $(x_{2,3}; x_{1,1}, x_{1,2}, x_{2,4}, x_{2,5})$, $(x_{2,4}; x_{1,2}, x_{1,3}, x_{3,3}, x_{3,5})$.

(4) $q = 6$ and $r = 4$. The required cycles and stars are
$(x_{1,1}x_{2,4}x_{3,3}x_{2,5})$, $(x_{1,2}x_{2,5}x_{3,4}x_{2,1})$, $(x_{1,3}x_{2,4}x_{3,1}x_{2,5})$, $(x_{1,4}x_{2,5}x_{3,2}x_{2,1})$,

$(x_{1,3}x_{2,1}x_{3,5}x_{2,2})$, $(x_{1,5}x_{2,1}x_{3,3}x_{2,2})$, $(x_{2,2}; x_{1,1}, x_{1,4}, x_{3,1}, x_{3,4})$, $(x_{2,3}; x_{1,1}, x_{1,2}, x_{1,4}, x_{55})$, $(x_{2,4}; x_{1,2}, x_{1,5}, x_{3,2}, x_{3,5})$, $(x_{2,3}; x_{3,1}, x_{3,2}, x_{3,4}, x_{3,5})$.

(5) $q = 5$ and $r = 5$. The required cycles and stars are $(x_{1,1}x_{2,4}x_{3,3}x_{2,5})$, $(x_{1,2}x_{2,5}x_{3,4}x_{2,1}, x_{1,2})$, $(x_{1,3}x_{2,4}x_{3,1}x_{2,5})$, $(x_{1,4}x_{2,5}x_{3,2}x_{2,1})$, $(x_{1,5}x_{2,1}x_{3,3}x_{2,2})$, $(x_{2,1}; x_{1,2}, x_{1,3}, x_{3,4}, x_{3,5})$, $(x_{2,2}; x_{1,3}, x_{1,4}, x_{3,1}, x_{3,5})$, $(x_{2,3}; x_{1,4}, x_{1,5}, x_{3,1}, x_{3,2})$, $(x_{2,4}; x_{1,1}, x_{1,5}, x_{3,2}, x_{3,3})$, $(x_{2,5}; x_{1,1}, x_{1,2}, x_{3,3}, x_{3,4})$.

(6) $q = 4$ and $r = 6$. The required cycles and stars are $(x_{1,1}x_{2,4}x_{3,3}x_{2,5})$, $(x_{1,2}x_{2,5}x_{3,4}x_{2,1})$, $(x_{1,3}x_{2,4}x_{3,1}x_{2,5})$, $(x_{1,4}x_{2,5}x_{3,2}x_{2,1})$, $(x_{2,1}; x_{1,3}, x_{1,5}, x_{3,3}, x_{3,5})$, $(x_{2,2}; x_{1,3}, x_{1,5}, x_{3,3}, x_{3,5})$, $(x_{2,2}; x_{1,1}, x_{1,4}, x_{3,1}, x_{3,4})$, $(x_{2,3}; x_{1,1}, x_{1,2}, x_{1,4}, x_{1,5})$, $(x_{2,4}; x_{1,2}, x_{1,5}, x_{3,2}, x_{3,5})$, $(x_{2,3}; x_{3,1}, x_{3,2}, x_{3,4}, x_{3,5})$.

(7) $q = 3$ and $r = 7$. The required cycles and stars are $(x_{1,1}x_{2,2}x_{3,4}x_{2,3})$, $(x_{1,2}x_{2,3}x_{3,5}x_{2,4})$, $(x_{1,3}x_{2,4}x_{3,3}x_{2,2})$, $(x_{2,1}; x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5})$, $(x_{2,1}; x_{3,2}, x_{3,3}, x_{3,4}, x_{3,5})$, $(x_{2,2}; x_{1,4}, x_{1,5}, x_{3,1}, x_{3,5})$, $(x_{2,3}; x_{1,4}, x_{1,5}, x_{3,1}, x_{3,2})$, $(x_{2,4}; x_{1,1}, x_{1,5}, x_{3,1}, x_{3,2})$, $(x_{2,5}; x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4})$, $(x_{2,5}; x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4})$.

(8) $q = 2$ and $r = 8$. The required cycles and stars are $(x_{1,3}x_{2,1}x_{3,5}x_{2,2})$, $(x_{1,5}x_{2,1}x_{3,3}x_{2,2})$, $(x_{2,1}; x_{1,2}, x_{1,4}, x_{3,2}, x_{3,4})$, $(x_{2,2}; x_{1,1}, x_{1,4}, x_{3,1}, x_{3,4})$, $(x_{2,3}; x_{1,1}, x_{1,2}, x_{1,4}, x_{1,5})$, $(x_{2,3}; x_{3,1}, x_{3,2}, x_{3,4}, x_{3,5})$, $(x_{2,4}; x_{1,1}, x_{1,2}, x_{1,3}, x_{1,5})$, $(x_{2,4}; x_{3,1}, x_{3,2}, x_{3,3}, x_{3,5})$, $(x_{2,5}; x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4})$, $(x_{2,5}; x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4})$.

(9) $q = 0$ and $r = 10$. The required stars are $(x_{2,1}; x_{1,2}, x_{1,3}, x_{1,4}, x_{1,5})$, $(x_{2,1}; x_{3,2}, x_{3,3}, x_{3,4}, x_{3,5})$, $(x_{2,2}; x_{1,1}, x_{1,3}, x_{1,4}, x_{1,5})$, $(x_{2,2}; x_{3,1}, x_{3,3}, x_{3,4}, x_{3,5})$, $(x_{2,3}; x_{1,1}, x_{1,2}, x_{1,4}, x_{1,5})$, $(x_{2,3}; x_{3,1}, x_{3,2}, x_{3,4}, x_{3,5})$, $(x_{2,4}; x_{1,1}, x_{1,2}, x_{1,3}, x_{1,5})$, $(x_{2,4}; x_{3,1}, x_{3,2}, x_{3,3}, x_{3,5})$, $(x_{2,5}; x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4})$, $(x_{2,5}; x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4})$.

Thus the Lemma holds. □

**Lemma 2.5.** *Let $q$ and $r$ be non-negative integers. Then there exists a complete $\{C_4, S_5\}$-decomposition of $P_3 \times K_n$, for all odd $n \geq 3$ with $r \neq 1$.*

*Proof.* When $n = 3, 5$, the proof follows from Lammas 2.3 and 2.4. For $n > 5$,

$$P_3 \times K_n = \left(\frac{n-5}{2}\right)(P_3 \times K_3) \oplus \left(\frac{n-3}{2}\right) K_{2,4} \ \oplus \ (P_3 \times K_5)$$
$$\oplus \left\{ \bigoplus_{i=4}^{n-3} K_{i,4} \right\}, i \equiv 0 \pmod 2 \geq 4.$$

By Lemmas 2.3 and 2.4, $P_3 \times K_3$ and $P_3 \times K_5$ have a complete $\{C_4, S_5\}$-decomposition. Also, by Theorem 1.1, $K_{2,4}$ and $K_{i,4}$ have a complete $\{C_4, S_5\}$-decomposition. Hence, by the remark 1.1, the graph $P_3 \times K_n$ has the desired decomposition. □

**Lemma 2.6.** *There exists a complete $\{C_4, S_5\}$-decomposition of $K_4 \times K_4$, for all non-negative integers $q, r$ with $r \geq 4$.*

*Proof.* Let $V(K_4 \times K_4) = \{x_{i,j} : 1 \leq i, j \leq 4\}$. We prove this in two cases as follows:
Case 1: $q$ even.
First we decompose $K_4 \times K_4$ into $18S_5$ as follows:
$(x_{1,1}, x_{3,1}; x_{2,2}, x_{2,4}, x_{2,3}, x_{4,3})$, $(x_{1,2}, x_{3,2}; x_{2,1}, x_{2,3}, x_{2,4}, x_{4,4})$, $(x_{1,3}, x_{3,3}; x_{2,1}, x_{2,2}, x_{2,4}, x_{4,1})$, $(x_{1,4}, x_{3,4}; x_{2,1}, x_{2,2}, x_{2,3}, x_{4,2})$, $\{(x_{4,1}; x_{2,2}, x_{2,3}, x_{2,4}, x_{3,4})$, $(x_{4,2}; x_{2,1}, x_{2,3}, x_{2,4}, x_{3,3})$, $(x_{4,3}; x_{2,2}, x_{2,1}, x_{2,4}, x_{3,4})$, $(x_{4,4}; x_{2,2}, x_{2,3}, x_{2,1}, x_{3,3})$, $(x_{1,1}; x_{3,3}, x_{3,4}, x_{4,2}, x_{4,4})$, $(x_{1,2}; x_{3,3}, x_{3,4}, x_{4,1}, x_{4,3})$, $(x_{1,3}; x_{3,1}, x_{3,4}, x_{4,2}, x_{4,4})$, $(x_{1,4}; x_{3,3}, x_{3,2}, x_{4,1}, x_{4,3})$, $(x_{3,1}; x_{1,2}, x_{1,4}, x_{4,2}, x_{4,4})$, $(x_{3,2}; x_{1,1}, x_{1,3}, x_{4,1}, x_{4,3})\}$. By Remark 1.2, the above stars give required even number of cycles with $q \leq 8$.

By applying Remark 1.2 in the following decomposition, we get the required number of cycles and stars for $q > 8$.
$\{(x_{3,1}x_{1,2}x_{3,4}x_{1,3}), (x_{4,1}x_{1,3}x_{3,2}x_{1,4}), (x_{4,1}x_{2,2}x_{4,4}x_{3,2}), (x_{4,1}x_{2,4}x_{4,2}x_{3,4}), (x_{4,2}x_{1,4}x_{3,1}x_{2,3}),$
$(x_{4,4}x_{1,2}x_{4,1}x_{2,3}), (x_{1,1}, x_{3,1}; x_{2,2}, x_{2,4}, x_{4,2}, x_{4,4}), (x_{1,2}, x_{3,2}; x_{2,1}, x_{2,3}, x_{4,1}, x_{4,3}),$
$(x_{1,3}, x_{3,3}; x_{2,2}, x_{2,4}, x_{4,2}, x_{4,4}), (x_{1,4}, x_{3,4}; x_{2,1}, x_{2,3}, x_{4,1}, x_{4,3}), (x_{1,1}; x_{2,3}x_{3,2}, x_{3,4}, x_{4,3}),$
$(x_{2,1}; x_{1,3}, x_{3,3}, x_{4,2}, x_{4,4}), (x_{3,3}; x_{1,1}, x_{1,2}, x_{1,4}, x_{4,1}), (x_{4,3}; x_{2,1}, x_{2,2}x_{2,4}, x_{3,1})\}$.
Case 2: $q$ odd.
By applying Remark 1.2 in the following decomposition, we get the required number of cycles and stars for $q = 1, 3, 5$.
$\{(x_{2,1}x_{4,2}x_{2,4}x_{4,3}), (x_{1,2}, x_{2,2}; x_{3,3}, x_{3,4}, x_{4,1}, x_{4,3}), (x_{1,4}, x_{2,4}; x_{3,1}, x_{3,2}, x_{3,3}, x_{4,1}),$
$(x_{1,1}; x_{2,2}, x_{2,4}, x_{3,2}, x_{4,3}), (x_{1,2}; x_{2,1}, x_{2,3}, x_{2,4}, x_{3,1}), (x_{1,3}; x_{2,1}, x_{2,2}, x_{2,4}, x_{3,4}),$
$(x_{1,4}; x_{2,1}, x_{2,2}, x_{2,3}, x_{4,3}), (x_{2,1}; x_{3,2}, x_{3,3}, x_{3,4}, x_{4,4}), (x_{2,3}; x_{1,1}, x_{3,1}, x_{4,2}, x_{4,4}),$
$(x_{3,1}; x_{1,3}, x_{2,2}, x_{4,3}, x_{4,4}), (x_{3,2}; x_{1,3}, x_{2,3}, x_{4,3}, x_{4,4}), (x_{3,3}; x_{1,1}, x_{4,1}, x_{4,2}, x_{4,4}),$
$(x_{3,4}; x_{1,1}, x_{2,3}, x_{4,2}, x_{4,3}), (x_{4,1}; x_{1,3}, x_{2,3}, x_{3,2}, x_{3,4}), (x_{4,2}; x_{1,1}, x_{1,3}, x_{1,4}, x_{3,1}),$
$(x_{4,4}; x_{1,1}, x_{1,2}, x_{1,3}, x_{2,2})\}$.
By applying Remark 1.2 in the following decomposition, we get the required number of cycles and stars for $q = 7, 9, 11$.
$\{(x_{1,2}x_{2,1}x_{3,2}x_{2,3}), (x_{1,3}x_{2,2}x_{3,3}x_{2,4}), (x_{1,1}x_{4,2}x_{1,3}x_{3,2}), (x_{1,2}x_{4,3}x_{1,4}x_{3,3}), (x_{2,2}x_{4,1}x_{3,2}x_{4,3}),$
$(x_{2,1}x_{4,4}x_{3,1}x_{4,2}), (x_{2,4}x_{4,2}x_{3,4}x_{4,3}), (x_{2,2}, x_{2,3}; x_{1,1}, x_{1,4}, x_{3,1}, x_{3,4}),$
$(x_{1,2}, x_{1,3}; x_{3,1}, x_{3,4}, x_{4,1}, x_{4,4}), (x_{1,1}; x_{3,3}, x_{3,4}, x_{4,3}, x_{4,4}), (x_{1,4}; x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}),$
$(x_{2,1}; x_{1,3}, x_{1,4}, x_{4,3}, x_{4,4}), (x_{2,4}; x_{1,1}, x_{1,2}, x_{4,1}, x_{4,2}), (x_{4,1}; x_{2,3}, x_{2,4}, x_{3,3}, x_{3,4}),$
$(x_{4,2}; x_{2,1}, x_{3,1}, x_{2,3}, x_{3,3}), (x_{4,4}; x_{2,2}, x_{2,3}, x_{3,2}, x_{3,3})\}$.
For $q = 13$, the required decomposition is given below.
$\{(x_{4,4}x_{1,3}x_{3,4}x_{1,1}), (x_{4,2}x_{1,1}x_{3,3}x_{2,1}), (x_{4,3}x_{1,4}x_{2,3}x_{3,4}), (x_{4,4}x_{1,2}x_{4,1}x_{2,3}), (x_{2,3}x_{1,2}x_{2,4}x_{4,2}),$
$(x_{3,3}x_{2,4}x_{4,1}x_{2,2}), (x_{3,4}x_{4,1}x_{3,2}x_{2,1}), (x_{4,2}x_{3,4}x_{2,2}x_{3,1}), (x_{4,4}x_{3,2}x_{2,3}x_{3,1}), (x_{2,1}x_{1,3}x_{2,2}x_{1,4}),$
$(x_{3,1}x_{1,3}x_{3,2}x_{1,4}), (x_{4,1}x_{1,3}x_{4,2}x_{1,4}), (x_{4,3}x_{2,1}x_{4,4}x_{2,2}), (x_{1,1}; x_{2,2}, x_{2,3}, x_{2,4}, x_{3,2}),$
$(x_{1,2}; x_{2,1}, x_{3,1}, x_{3,3}, x_{3,4}), (x_{2,4}; x_{1,3}, x_{3,1}, x_{3,2}, x_{4,3}), (x_{3,3}; x_{1,4}, x_{4,1}, x_{4,2}, x_{4,4}),$
$(x_{4,3}; x_{1,1}, x_{1,2}, x_{3,1}, x_{3,2})\}$. □

**Lemma 2.7.** *There exists a complete $\{C_4, S_5\}$-decomposition of $C_4 \times C_4$, for all even integer $q \geq 0$.*

*Proof.* Let $V(C_4 \times C_4) = \{x_{i,j} : 1 \leq i, j \leq 4\}$. The $S_5$-decomposition of $C_4 \times C_4$ is given below.
$(x_{1,1}, x_{3,1}; x_{2,2}, x_{2,4}, x_{4,2}, x_{4,4}), (x_{1,2}, x_{3,2}; x_{2,1}, x_{2,3}, x_{4,1}, x_{4,3}),$
$(x_{1,3}, x_{3,3}; x_{2,2}, x_{2,4}, x_{4,2}, x_{4,4}), (x_{1,4}, x_{3,4}; x_{2,1}, x_{2,3}, x_{4,1}, x_{4,3})$.
By Remark 1.2, the pair of stars given above gives the required decomposition. □

**Lemma 2.8.** *There exists a complete $\{C_4, S_5\}$-decomposition of $C_4 \times K_4$, for all even integer $q \geq 0$.*

*Proof.* Let $V(C_4 \times K_4) = \{x_{i,j} : 1 \leq i, j \leq 4\}$. The $S_5$-decomposition of $C_4 \times K_4$ is given below.
$(x_{1,1}, x_{3,1}; x_{2,2}, x_{2,4}, x_{2,3}, x_{4,3}), (x_{1,2}, x_{3,2}; x_{2,1}, x_{2,3}, x_{2,4}, x_{4,4}),$
$(x_{1,3}, x_{3,3}; x_{2,1}, x_{2,2}, x_{2,4}, x_{4,1}), (x_{1,4}, x_{3,4}; x_{2,1}, x_{2,2}, x_{2,3}, x_{4,2}),$
$(x_{4,2}, x_{4,4}; x_{1,1}, x_{1,3}, x_{3,1}, x_{3,3}), (x_{4,1}, x_{4,3}; x_{1,2}, x_{1,4}, x_{3,2}, x_{3,4})$.
By Remark 1.2, the pair of stars given above gives the required decomposition. □

**Lemma 2.9.** *There exists a complete $\{C_4, S_5\}$-decomposition of $P_3 \times K_6$, for all non-negative integers $q, r$ with $r \geq 3$.*

*Proof.* Let $V(P_3 \times K_6) = \{x_{i,j} : 1 \leq i \leq 3,\ 1 \leq j \leq 6\}$. First we decompose $P_3 \times K_6$ into $15S_5$ as follows:

$(x_{2,2}, x_{2,4}; x_{1,1}, x_{1,3}, x_{3,1}, x_{3,3})$, $(x_{2,5}, x_{2,6}; x_{1,1}, x_{1,3}, x_{3,1}, x_{3,3})$,
$(x_{1,6}, x_{3,6}; x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4})$, $(x_{1,5}, x_{3,5}; x_{2,1}, x_{2,2}, x_{2,4}, x_{2,6})$,
$(x_{1,4}, x_{3,4}; x_{2,1}, x_{2,2}, x_{2,3}, x_{2,6})$, $(x_{1,2}, x_{3,2}; x_{2,3}, x_{2,4}, x_{2,5}, x_{2,6})$,
$\{(x_{2,1}; x_{1,2}, x_{1,3}, x_{3,2}, x_{3,3})$, $(x_{2,3}; x_{1,1}, x_{1,5}, x_{3,1}, x_{3,5})$, $(x_{2,5}; x_{1,4}, x_{1,6}, x_{3,4}, x_{3,6})\}$.
The above pairs of stars gives required even number of cycles and the following set of cycle and stars gives the required decomposition for the remaining choices of $q$.
$\{(x_{1,2}x_{2,5}x_{3,2}x_{2,3})$, $(x_{2,1}, x_{2,6}; x_{1,2}, x_{1,4}, x_{3,2}, x_{3,4})$, $(x_{2,1}, x_{2,2}; x_{1,3}, x_{1,6}, x_{3,3}, x_{3,6})$,
$(x_{2,2}, x_{2,3}; x_{1,1}, x_{1,4}, x_{3,1}, x_{3,4})$, $(x_{2,4}, x_{2,5}; x_{1,1}, x_{1,3}, x_{3,1}, x_{3,3})$,
$(x_{1,5}, x_{3,5}; x_{2,1}, x_{2,2}, x_{2,4}, x_{2,6})$, $(x_{2,3}; x_{1,5}, x_{1,6}, x_{3,5}, x_{3,6})$, $(x_{2,4}; x_{1,2}, x_{1,6}, x_{3,2}, x_{3,6})$,
$(x_{2,5}; x_{1,4}, x_{1,6}, x_{3,4}, x_{3,6})$, $(x_{2,6}; x_{1,1}, x_{1,3}, x_{3,1}, x_{3,3})\}$. □

**Lemma 2.10.** *There exists a complete $\{C_4, S_5\}$-decomposition of $K_4 \times K_6$, for all non-negative integers $q, r$ with $r \geq 9$.*

*Proof.* Since $K_4 \times K_6 = 3(P_3 \times K_6)$ and $P_3 \times K_6$ has a complete $\{C_4, S_5\}$-decomposition (by Lemma 2.9), by Remark 1.1, $K_4 \times K_6$ has a complete $\{C_4, S_5\}$-decomposition. □

**Lemma 2.11.** *There exists a complete $\{C_4, S_5\}$-decomposition of $K_5 \times K_6$, for all non-negative integers $q, r$ with $r \neq 1$.*

*Proof.* Since $K_5 \times K_6 = 5((P_3 \backslash E(3S_5) \oplus 3S_5) \times K_6)$, as in Lemma 2.9 we have a required decomposition of $5((P_3 \backslash E(3S_5)) \times K_6)$. Now, we decompose $3S_5 \times K_6$ into $15S_5$-decomposition as follows:
$(x_{1,1}, x_{4,1}; x_{3,2}, x_{3,3}, x_{5,2}, x_{5,3})$, $(x_{1,3}, x_{4,3}; x_{3,1}, x_{3,5}, x_{5,1}, x_{5,5})$, $(x_{1,5}, x_{4,5}; x_{3,4}, x_{3,6}, x_{5,4}, x_{5,6})$,
$\{(x_{2,1}; x_{1,2}, x_{1,3}, x_{3,2}, x_{3,3})$, $(x_{4,1}; x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3})$, $(x_{5,1}; x_{2,2}, x_{2,3}, x_{3,2}, x_{3,3})\}$,
$\{(x_{2,3}; x_{1,1}, x_{1,5}, x_{3,1}, x_{3,5})$, $(x_{4,3}; x_{1,1}, x_{1,5}, x_{2,1}, x_{2,5})$, $(x_{5,3}; x_{2,1}, x_{2,5}, x_{3,1}, x_{3,5})\}$,
$\{(x_{2,5}; x_{1,4}, x_{1,6}, x_{3,4}, x_{3,6})$, $(x_{4,5}; x_{1,4}, x_{1,6}, x_{2,4}, x_{2,6})$, $(x_{5,5}; x_{2,4}, x_{2,6}, x_{3,4}, x_{3,6})\}$. From the stars $\{(x_{2,1}; x_{1,2}, x_{1,3}, x_{3,2}, x_{3,3})$, $(x_{4,1}; x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3})$, $(x_{5,1}; x_{2,2}, x_{2,3}, x_{3,2}, x_{3,3})\}$ we have the cycles $\{(x_{1,2}x_{2,1}x_{1,3}x_{4,1})$, $(x_{2,2}x_{4,1}x_{2,3}x_{5,1})$, $(x_{3,2}x_{5,1}x_{3,3}x_{2,1})\}$.
So from the above pair of stars and 3-sets of stars we can get a complete $\{C_4, S_5\}$-decomposition of $3S_5 \times K_6$ (Remark 1.2). Hence by Remark 1.1, $K_5 \times K_6$ has a complete $\{C_4, S_5\}$-decomposition. □

**Lemma 2.12.** *Let $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 4$. There exists a complete $\{C_4, S_5\}$-decomposition of $K_m \times K_n$, where $q$ and $r$ are non-negative integers with $r \neq 1$.*

*Proof.* When $n = 5$, if $m = 4, 6$ the proof follows from Lemmas 2.10 and 2.11. So, let $m > 6$ and $m = 2k$. Now,

$$\begin{aligned} K_m \times K_n = K_{2k} \times K_5 &= (K_4 \times K_5) \ \oplus \ K_{2(k-2)} \times K_5 \ \oplus \ K_{4,2(k-2)} \times K_5 \\ &= K_4 \times K_5 \ \oplus \ K_{2(k-2)} \times K_5 \ \oplus \ 5K_{4,8(k-2)}. \end{aligned}$$

By Theorem 1.2 and Lemma 2.10, $K_4 \times K_5$ and $K_{4,8(k-2)}$ have a complete $\{C_4, S_5\}$-decomposition. By applying the above recursive relation to $K_{2(k-2)} \times K_5$, we have a complete $\{C_4, S_5\}$-decomposition of $K_{2(k-2)} \times K_5$. Hence by Remark 1.1, $K_m \times K_n$ has a complete $\{C_4, S_5\}$-decomposition.
When $n > 5$, $K_m \times K_n = \frac{m(m-1)}{2}(K_{n,n} - I)$. By Theorem 2.1 and Remark 1.1, $K_m \times K_n$ has a complete $\{C_4, S_5\}$-decomposition. □

**Lemma 2.13.** *Let $m \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$. There exists a complete $\{C_4, S_5\}$-decomposition of $K_m \times K_n$, where $q$ and $r$ are non-negative integers with $r \neq 1$.*

*Proof.* We can write, $K_m \times K_n = \frac{m(m-1)}{4}(P_3 \times K_n)$. By Theorem 1.1, $P_3 \times K_n$ has a complete $\{C_4, S_5\}$-decomposition and hence by Remark 1.1, $K_m \times K_n$ has a complete $\{C_4, S_5\}$-decomposition. □

**Lemma 2.14.** *Let $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 2$. There exists a complete $\{C_4, S_5\}$-decomposition of $K_m \times K_n$, where $q$ and $r$ are non-negative integers with $r \neq 1$.*

*Proof.* Since $K_m \times K_n = K_n \times K_m$, by applying similar proof of Lemma 2.12, we get a required decomposition for $m > 5$.
When $m = 5$, $K_m \times K_n$ can be written as $5(P_3 \times K_n)$. By Theorem 1.1 and Remark 1.1, $K_m \times K_n$ has a complete $\{C_4, S_5\}$-decomposition. □

**Lemma 2.15.** *Let $m, n \equiv 0 \pmod 4$. There exists a complete $\{C_4, S_5\}$-decomposition of $K_m \times K_n$, where $q$ and $r$ are non-negative integers with $r \neq 1$.*

*Proof.* By Theorem 1.2, $K_m$ can be viewed as $(\frac{m}{4})K_4 \oplus (\frac{m^2-4m}{8})C_4$ and $K_n$ can be viewed as $(\frac{n}{4})K_4 \oplus (\frac{n^2-4n}{8})C_4$. So,

$$K_m \times K_n = \frac{mn}{16}(K_4 \times K_4) \oplus \frac{mn(m-4)(n-4)}{64}(C_4 \times C_4) \oplus \frac{mn(m+n-8)}{32}(C_4 \times K_4).$$

Now, by Lemmas 2.6 to 2.8, $K_4 \times K_4$, $C_4 \times C_4$ and $C_4 \times K_4$ have a complete $\{C_4, S_5\}$-decomposition. Hence by Remark 1.1, $K_m \times K_n$ has a complete $\{C_4, S_5\}$-decomposition. □

**Lemma 2.16.** *Let $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$. There exists a complete $\{C_4, S_5\}$-decomposition of $K_m \times K_n$, where $q$ and $r$ are non-negative integers with $r \neq 1$.*

*Proof.* Let $m = 4k$ and $n = 4l + 2$. When $l = 1$,

$$\begin{aligned} K_m \times K_n = K_{4k} \times K_6 &= (K_4 \times K_6) \oplus (K_{4(k-1)} \times K_6) \oplus (K_{4,4(k-1)} \times K_6) \\ &= (K_4 \times K_6) \oplus (K_{4(k-1)} \times K_6) \oplus 6K_{4,20(k-1)}. \end{aligned}$$

By Theorem 1.1 and Lemma 2.10, $K_{4,20(k-1)}$ and $K_4 \times K_6$ have a complete $\{C_4, S_5\}$-decomposition. By applying the above recursive relation to $K_{4(k-1)} \times K_6$, we have a complete $\{C_4, S_5\}$-decomposition of $K_{4(k-1)} \times K_6$. Hence by Remark 1.1, $K_m \times K_6$ has a complete $\{C_4, S_5\}$-decomposition.
When $l > 1$,

$$\begin{aligned} K_m \times K_n &= K_m \times K_4 \oplus K_m \times K_{4(l-1)+2} \oplus K_m \times K_{4(l-1)+2,4} \\ &= K_m \times K_4 \oplus K_m \times K_{4(l-1)+2} \oplus mK_{(m-1)(4l-2),4}. \end{aligned}$$

By Theorem 1.1 and Lemma 2.15, $K_{(m-1)(4l-2),4}$ and $K_m \times K_4$ have a complete $\{C_4, S_5\}$-decomposition. Also, by applying the above recursive relation to $K_m \times K_{4(l-1)+2}$, we have a complete $\{C_4, S_5\}$-decomposition of $K_m \times K_{4(l-1)+2}$. Hence by Remark 1.1, $K_m \times K_n$ has a complete $\{C_4, S_5\}$-decomposition. □

## 3. Main Result

In this section we prove our main result as follows.

**Theorem 3.1.** *Let $q$ and $r$ be non-negative integers. Then $K_m \times K_n$ has a complete $\{C_4, S_5\}$-decomposition if and only if one of the following holds.*

1. $m \equiv 0 \pmod 2$ *and* $n \equiv 1 \pmod 4$;
2. $m \equiv 0 \pmod 4$ *and* $n \equiv 3 \pmod 4$;
3. $m \equiv 1 \pmod 4$ *and* $n \equiv 1 \pmod 2$;
4. $m, n \equiv 0 \pmod 4$;
5. $m \equiv 0 \pmod 4$ *and* $n \equiv 2 \pmod 4$.

*Proof.* **Necessity.** Since $K_m \times K_n$ is $(n-1)(m-1)$-regular with $mn$ vertices, $4|\frac{mn}{2}(m-1)(n-1)$. The values of $m$ and $n$ satisfying the above condition fallen in one of the following:

(1) $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 4$,
(2) $m \equiv 0 \pmod 4$ and $n \equiv 3 \pmod 4$,
(3) $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 2$,
(4) $m, n \equiv 0 \pmod 4$,
(5) $m \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$.

**Sufficiency.** Sufficiency follows by Lemmas 2.12 to 2.16. □

## 4. Conclusion

In this paper, we proved that the necessary condition $mn(m-1)(n-1) \equiv 0 \pmod 8$ is sufficient for the existence of a decomposition of tensor product of complete graphs into cycles and stars with four edges. Further, research on the existence of such decomposition of product graphs into cycles and stars of higher length $l > 4$ is under progress.

## References


[1] Abueida, A.A. and Daven, M., (2003), Multidesigns for graph-pairs of order 4 and 5, Graphs and Combin., 19 , pp. 433 – 447.

[2] Abueida, A.A. and O'Neil, T., (2007), Multidecomposition of $\lambda K_m$ into small cycles and claws, Bull. Inst. Comb. Appl., 49, pp. 32 – 40.

[3] Ilayaraja, M. and Muthusamy, A., (2020), Decomposition of complete bipartite graphs into cycles and stars with four edges, AKCE Int. J. Graphs and Combin., 17, pp. 697 – 702.

[4] Ilayaraja, M., Sowndhariya, K. and Muthusamy, A., (2020), Decomposition of product graphs into paths and stars on five vertices, AKCE Int. J. Graphs and Combin., 17, pp. 777 – 783.

[5] Jeevadoss, S. and Muthusamy, A., (2016), Decomposition of product graphs into paths and cycles of length four, Graphs and Combin., 32, pp. 199 – 223.

[6] Lee, H.-C., (2013), Multidecompositions of complete bipartite graphs into cycles and stars, Ars Combin., 108, pp. 355 – 364.

[7] Lee, H.-C., (2013), Decomposition of complete bipartite graphs with a 1-factor removed into cycles and stars, Discrete Math., 313, pp. 2354 – 2358.

[8] Pauline Ezhilarasi, A. and Muthusamy, A., (2019), Decomposition of product graphs into paths and stars with Three Edges, Bull. Inst. Combin. Appl., 87, pp. 47 – 74.

[9] Pauline Ezhilarasi, A. and Muthusamy, A., (2021), Decomposition of Cartesian Product of Complete Graphs into Paths and Stars with Four Edges, Comment. Math. Univ. Carolin. in press.

[10] Priyadharsini, H.M. and Muthusamy, A., (2009), $(G_m, H_m)$-Multifactorization of $\lambda K_m$, J. Combin. Math. Combin. Comput., 69, pp. 145 – 150.

[11] Priyadharsini, H.M. and Muthusamy, A., (2012), $(G_m, H_m)$-multidecomposition of $K_{m,m}(\lambda)$, Bull. Inst. Combin. Appl., 66, pp. 42 – 48.

[12] Shyu, T.-W., (2013), Decomposition of complete graphs into cycles and stars, Graphs and Combin., 29, pp. 301 – 313.

[13] Shyu, T.-W., (2021), Decomposition of complete bipartite graphs and complete graphs into paths, stars and cycles with four edges each, Discuss. Math. Graph Theory, 41, pp. 451 – 468.

**A. Pauline Ezhilarasi** is a research scholar in the Department of Mathematics, Periyar University, Salem. Her research areas include Graph Decomposition and Combinatorial Designs.

**M. Ilayaraja** is an assistant professor in the Department of Mathematics, Karpagam Academy of Higher Education, Coimbatore, Tamil Nadu, India. His research areas include Graph Decomposition and Combinatorial Designs.

**A. Muthusamy** is a professor in the Department of Mathematics, Periyar University, Salem. His research areas include Graph Decomposition and Combinatorial Designs.