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METRIC DIMENSION OF LINE GRAPH OF THE SUBDIVISION OF THE GRAPHS OF CONVEX POLYTOPES

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Abstract. The metric generator for the simple connected graph Γ is the set of vertices $\mathfrak{Y} \subseteq \mathbb{V}(\Gamma)$ with the property that every pair of vertices $u, v(u \neq v) \in \mathbb{V}$ are determined (or resolved) by some vertex of $\mathfrak{Y}$. The minimum possible cardinality of this metric generator is called the metric dimension of Γ, denoted by $dim(\Gamma)$or $\beta(\Gamma)$. In this article, we determine the exact metric dimension and some other properties of the line graph of the subdivision graph of the graph of convex polytope D_n (exists in the literature).

Keywords: Subdivision graph, resolving set, line graph, metric dimension.

AMS Subject Classification: 05C12, 05C76.

1. Introduction

The idea of the locating set (or resolving set) was presented independently by Slater in 1975 [11] and Harary and Melter in 1976 [5]. After these two important initial papers, several works regarding theoretical properties, as well as applications, of this graph invariant were published. Initially, Slater considered special acknowledgment of a thief in the network, while others noticed problems in picture preparing (or image processing) and design acknowledgment (or pattern recognition) [8], applications to science are given in [4], to the route of exploring specialist (navigating agent or robots) in systems (or networks) are examined in [7], to issues of check and system revelation (or network discovery) in [3], application to combinatorial enhancement (or optimization) is yielded in [9], and for more work see [10, 12].

The distance between two vertices α and β in the simple connected graph $\Gamma = \Gamma(\mathbb{V}, \mathbb{E})$, denoted by $\partial_\Gamma(\alpha, \beta)$, defined as the length of a shortest $\alpha - \beta$ path in Γ. A single vertex a in Γ is said to determine (distinguish or resolve) a pair of vertices $\alpha, \beta \in \mathbb{V}$ if $\partial_\Gamma(\alpha, a) \neq \partial_\Gamma(\beta, b)$. A set of vertices $\mathfrak{Y} \subseteq \mathbb{V}(\Gamma)$ is a metric generator (or resolving set) for Γ, if every pair of distinct vertices of Γ can be determined (or resolved) by some vertex of $\mathfrak{Y}$.

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Equivalently, for an ordered subset of vertices $\mathfrak{Y} = \{\jmath_1, \jmath_2, \jmath_3, ..., \jmath_k\}$ of Γ, any vertex $v \in \mathbb{V}$ may be represented uniquely in the form of the vector $\zeta(v|\mathfrak{Y}) = (\partial_\Gamma(\jmath_1, v), \partial_\Gamma(\jmath_2, v), \partial_\Gamma(\jmath_3, v), ..., \partial_\Gamma(\jmath_k, v))$. Then $\mathfrak{Y}$ is the metric generator of Γ if for every two different vertices $\alpha, \beta \in \mathbb{V}$, we have $\zeta(\alpha|\mathfrak{Y}) \neq \zeta(\beta|\mathfrak{Y})$. The metric generator $\mathfrak{Y}$ with the minimum possible cardinality is the metric basis for Γ, and this minimum cardinality is known as the metric dimension of Γ, denoted by $dim(\Gamma)$ or $\beta(\Gamma)$. A set Y consisting of vertices of the graph Γ is said to be an independent resolving set for Γ, if Y is both resolving (metric generator) and independent.

For an undirected graph Γ, the line graph of the graph Γ is a graph $L(\Gamma)$ with vertex set $\mathbb{V}(L(\Gamma)) = \mathbb{E}(\Gamma)$ and two different nodes are adjacent in $L(\Gamma)$ iff they have a common end vertex in Γ. Sometimes a line graph is also termed as edge graph, derived graph, or interchange graph. When every edge of the given undirected graph Γ is replaced by a path of length two, the graph so obtained is known as the subdivision graph of the graph Γ, denoted by $S(\Gamma)$.

The graph of a convex polytope D_n consisting of $2n$ 5-sided faces and a pair of n-sided faces were defined by Baca in [2]. For this family of the plane graph, Imran et al. in [6], prove the following result regarding its metric dimension as:

Theorem 1.1. [6] *Let n be the positive integer such that $n \geq 6$ and D_n be the plane graph on $4n$ vertices and $6n$ edges. Then, we have $dim(D_n) = 3$ i.e., it has location number 3.*

Now, simply for this graph D_n, two question arises: (1) *what should be the metric dimension of the subdivision graph of the graph of convex polytope D_n?* and (2) *what should be the metric dimension of the line graph of the subdivision graph of the graph of convex polytope D_n?* Now, working in this direction we obtain an interesting result regarding the metric dimension of the line graph of the subdivision graph of the graph of convex polytope D_n.

In this article, we determine the exact metric dimension of the line graph of the subdivision graph of the graph of convex polytope D_n [2], denoted by $L(S(D_n))$. We also prove that the line graph $L(S(D_n))$ possesses an independent minimum resolving set of cardinality three i.e., just 3 vertices properly chosen are adequate to resolve all the vertices of the graph $L(S(D_n))$. In the accompanying section, we acquire the metric dimension of the line graph of the subdivision graph of the graph of convex polytope D_n (see Figure 1), and for each positive integer n; $n \geq 6$ we demonstrate that $\beta(L(S(D_n))) = 3$.

2. The plane graph $L(S(D_n))$

The plane graph consisting of $2n$ 5-sided faces and a pair of n-sided faces were defined by Baca in [2], and is denoted by D_n. The subdivision of the plane graph D_n (for $n = 8$) and the line graph of this subdivision was shown in [1]. We denote this so obtained line graph from the subdivision graph of the plane graph D_n by $L(S(D_n))$. The radially symmetrical plane graph $L(S(D_n))$ comprises the vertex set and an edge set of cardinality $12n$ and $18n$ respectively. It has $4n$ 3-sided faces, $2n$ 10-sided faces, and a pair of $2n$-sided faces (see Figure 1). By $\mathbb{E}(L(S(D_n)))$ and $\mathbb{V}(L(S(D_n)))$, we signify the arrangement of edges and vertices of the plane graph $L(S(D_n))$ separately. Thus, we have

$$\mathbb{V}(L(S(D_n))) = \{p_t, q_t, r_t, s_t, t_t, u_t, v_t, w_t, x_t, y_t, z_t, a_t : 1 \leq t \leq n\}$$

and

$$\mathbb{E}(L(S(D_n))) = \{p_tq_t, p_tr_t, q_tr_t, r_ts_t, s_tt_t, s_tu_t, t_tu_t, u_tv_t, v_tx_t, v_tw_t, w_tx_t, x_ty_t, y_tz_t, y_ta_t, z_ta_t : 1 \leq t \leq n\} \cup \{q_tp_{t+1}, w_tt_{t+1}, a_tz_{t+1} : 1 \leq t \leq n\}$$

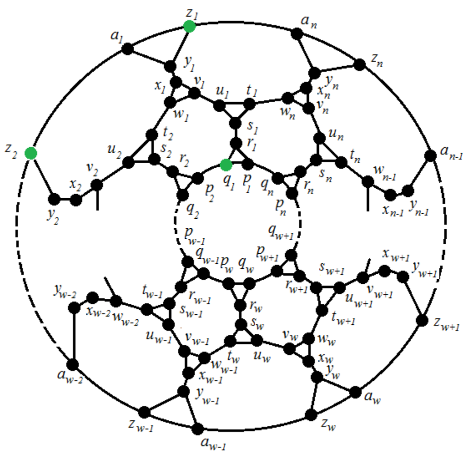

FIGURE 1. The Plane Graph $L(S(D_n))$, for $n \geq 6$.

For our gentle purpose, we call the cycle brought forth by the arrangement of vertices $\{p_t, q_t : 1 \leq t \leq n\}$ in the graph, $L(S(D_n)$ as the pq-cycle, the arrangement of vertices $\{r_t : 1 \leq t \leq n\}$ and $\{s_t : 1 \leq t \leq n\}$ in the graph, $L(S(D_n)$ as the set of inward and interior vertices, the cycle brought forth by the arrangement of vertices $\{t_t, u_t, v_t, w_t : 1 \leq t \leq n\}$ in the graph, $L(S(D_n)$ as the $tuvw$-cycle, the arrangement of vertices $\{x_t : 1 \leq t \leq n\}$ and $\{y_t : 1 \leq t \leq n\}$ in the graph, $L(S(D_n)$ as the set of exterior and outward vertices, and the cycle brought forth by the arrangement of vertices $\{z_t, a_t : 1 \leq t \leq n\}$ as the za-cycle. In the present section, we obtain that the minimum cardinality for the metric generator of the line graph of the subdivision graph of the graph of convex polytope D_n is 3. We also see that the resolving set for the line graph of the subdivision graph of the graph of convex polytope D_n is independent. For the metric dimension of the graph $L(S(D_n))$, we have the following result:

Theorem 2.1. *Let n be the positive integer such that $n \geq 6$ and $L(S(D_n))$ be the planar graph on $12n$ vertices as defined above. Then, we have $dim(L(S(D_n))) = 3$ i.e., it has location number 3.*

Proof. Note that for $6 \leq n \leq 10$, one can see that the set $\mathfrak{Y} = \{z_1, z_2, q_1\}$ is the metric basis set for the graph $L(S(D_n))$ by total enumeration. Now, for $n \geq 11$, we consider the resulting two cases relying on the positive integer n i.e., when the positive whole number n is even and when it is odd.

Case(I) When the integer n is even.

In this case, the integer n can be written as $n = 2w$, where $w \in \mathbb{N}$ and $w \geq 3$. Let $\mathfrak{Y} = \{z_1, z_2, q_1\} \subset \mathbb{V}(L(S(D_n)))$ (one can find that the location of these metric basis vertices in green color, as shown in Figure 1). Now, in order to show that $\mathfrak{Y}$ is a locating set for the plane graph $L(S(D_n))$, we consign the metric codes for each vertex of $\mathbb{V}(L(S(D_n)))$ regarding the set $\mathfrak{Y}$.

Now, the metric codes for the nodes of pq-cycle $\{v = p_t, q_t : 1 \leq t \leq n\}$ are

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(p_t\|\mathfrak{Y})$: $(t=1)$	$(7,8,1)$
$\zeta_M(p_t\|\mathfrak{Y})$: $(t=2)$	$(7,7,1)$
$\zeta_M(p_t\|\mathfrak{Y})$: $(t=3)$	$(8,7,3)$
$\zeta_M(p_t\|\mathfrak{Y})$: $(4\le t\le w+1)$	$(2t+2,2t,2t-3)$
$\zeta_M(p_t\|\mathfrak{Y})$: $(t=w+2)$	$(2w+4,2w+4,2w-1)$
$\zeta_M(p_t\|\mathfrak{Y})$: $(w+3\le t\le 2w)$	$(4w-2t+8,4w-2t+10,4w-2t+3)$

and

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(q_t\|\mathfrak{Y})$: $(t=1)$	$(7,8,0)$
$\zeta_M(q_t\|\mathfrak{Y})$: $(t=2)$	$(7,7,2)$
$\zeta_M(q_t\|\mathfrak{Y})$: $(3\le t\le w+1)$	$(2t+3,2t+1,2t-2)$
$\zeta_M(q_t\|\mathfrak{Y})$: $(w+2\le t\le 2w)$	$(4w-2t+7,4w-2t+9,4w-2t+2)$

The metric codes for the set of inward nodes $\{\upsilon=r_t:1\le t\le n\}$ are

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(r_t\|\mathfrak{Y})$: $(t=1)$	$(6,7,1)$
$\zeta_M(r_t\|\mathfrak{Y})$: $(t=2)$	$(6,6,2)$
$\zeta_M(r_t\|\mathfrak{Y})$: $(3\le t\le w+1)$	$(2t+2,2t,2t-2)$
$\zeta_M(r_t\|\mathfrak{Y})$: $(t=w+2)$	$(2w+3,2w+4,2w-1)$
$\zeta_M(r_t\|\mathfrak{Y})$: $(w+3\le t\le 2w)$	$(4w-2t+7,4w-2t+9,4w-2t+3)$

The metric codes for the set of interior nodes $\{\upsilon=s_t:1\le t\le n\}$ are

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(s_t\|\mathfrak{Y})$: $(t=1)$	$(5,6,2)$
$\zeta_M(s_t\|\mathfrak{Y})$: $(t=2)$	$(5,5,3)$
$\zeta_M(s_t\|\mathfrak{Y})$: $(3\le t\le w+1)$	$(2t+1,2t-1,2t-1)$
$\zeta_M(s_t\|\mathfrak{Y})$: $(t=w+2)$	$(2w+2,2w+3,2w)$
$\zeta_M(s_t\|\mathfrak{Y})$: $(w+3\le t\le 2w)$	$(4w-2t+6,4w-2t+8,4w-2t+4)$

The metric codes for the nodes of $tuvw$-cycle $\{\upsilon=t_t,u_t,v_t,w_t:1\le t\le n\}$ are

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(t_t\|\mathfrak{Y})$: $(t=1)$	$(5,6,3)$
$\zeta_M(t_t\|\mathfrak{Y})$: $(t=2)$	$(4,5,4)$
$\zeta_M(t_t\|\mathfrak{Y})$: $(3\le t\le w+1)$	$(2t,2t-2,2t)$
$\zeta_M(t_t\|\mathfrak{Y})$: $(t=w+2)$	$(2w+2,2w+2,2w+1)$
$\zeta_M(t_t\|\mathfrak{Y})$: $(w+3\le t\le 2w)$	$(4w-2t+6,4w-2t+8,4w-2t+5)$

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(u_t\|\mathfrak{Y})$: $(t=1)$	$(4,5,3)$
$\zeta_M(u_t\|\mathfrak{Y})$: $(t=2)$	$(5,4,4)$
$\zeta_M(u_t\|\mathfrak{Y})$: $(3\le t\le w+1)$	$(2t+1,2t-1,2t)$
$\zeta_M(u_t\|\mathfrak{Y})$: $(w+2\le t\le 2w)$	$(4w-2t+5,4w-2t+7,4w-2t+5)$

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(v_t\|\mathfrak{Y})$: $(t=1)$	$(3,4,4)$
$\zeta_M(v_t\|\mathfrak{Y})$: $(2\le t\le w)$	$(2t+1,2t-1,2t+1)$
$\zeta_M(v_t\|\mathfrak{Y})$: $(t=w+1)$	$(2w+2,2w+1,2w+3)$
$\zeta_M(v_t\|\mathfrak{Y})$: $(w+2\le t\le 2w)$	$(4w-2t+4,4w-2t+6,4w-2t+5)$

and

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(w_t\|\mathfrak{Y})$: $(t=1)$	$(3,4,5)$
$\zeta_M(w_t\|\mathfrak{Y})$: $(2\le t\le w)$	$(2t+1,2t-1,2t+2)$
$\zeta_M(w_t\|\mathfrak{Y})$: $(t=w+1)$	$(2w+2,2w+1,2w+2)$
$\zeta_M(w_t\|\mathfrak{Y})$: $(w+2\le t\le 2w)$	$(4w-2t+4,4w-2t+6,4w-2t+4)$

The metric codes for the set of exterior nodes $\{v=x_t:1\le t\le n\}$ are

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(x_t\|\mathfrak{Y})$: $(t=1)$	$(2,3,5)$
$\zeta_M(x_t\|\mathfrak{Y})$: $(2\le t\le w)$	$(2t,2t-2,2t+2)$
$\zeta_M(x_t\|\mathfrak{Y})$: $(t=w+1)$	$(2w+1,2w,2w+3)$
$\zeta_M(x_t\|\mathfrak{Y})$: $(w+2\le t\le 2w)$	$(4w-2t+3,4w-2t+5,4w-2t+5)$

The metric codes for the set of outward nodes $\{v=y_t:1\le t\le n\}$ are

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(y_t\|\mathfrak{Y})$: $(t=1)$	$(1,2,6)$
$\zeta_M(y_t\|\mathfrak{Y})$: $(2\le t\le w)$	$(2t-1,2t-3,2t+3)$
$\zeta_M(y_t\|\mathfrak{Y})$: $(t=w+1)$	$(2w,2w-1,2w+4)$
$\zeta_M(y_t\|\mathfrak{Y})$: $(w+2\le t\le 2w)$	$(4w-2t+2,4w-2t+4,4w-2t+6)$

At last, the metric codes for the nodes of za-cycle $\{v=z_t,a_t:1\le t\le n\}$ are

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(z_t\|\mathfrak{Y})$: $(t=1)$	$(0,2,7)$
$\zeta_M(z_t\|\mathfrak{Y})$: $(t=2)$	$(2,0,8)$
$\zeta_M(z_t\|\mathfrak{Y})$: $(3\le t\le w+1)$	$(2t-2,2t-4,2t+3)$
$\zeta_M(z_t\|\mathfrak{Y})$: $(w+2\le t\le 2w)$	$(4w-2t+2,4w-2t+4,4w-2t+7)$

and

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(a_t\|\mathfrak{Y})$: $(t=1)$	$(1,1,7)$
$\zeta_M(a_t\|\mathfrak{Y})$: $(2\le t\le w)$	$(2t-1,2t-3,2t+4)$
$\zeta_M(a_t\|\mathfrak{Y})$: $(t=w+1)$	$(2w-1,2w-1,2w+4)$
$\zeta_M(a_t\|\mathfrak{Y})$: $(w+2\le t\le 2w-1)$	$(4w-2t+1,4w-2t+3,4w-2t+6)$
$\zeta_M(a_t\|\mathfrak{Y})$: $(t=2w)$	$(1,3,7)$

We notice that no two vertices are having indistinguishable metric codes, suggesting that $\beta(L(S(D_n)))\le 3$. Now, so as to finish the evidence for this case, we show that $\beta(L(S(D_n)))\ge 3$ by working out that there is no resolving set $\mathfrak{Y}$ such that $|\mathfrak{Y}|=2$. Despite what might be expected, we guess that $\beta(L(S(D_n)))=2$. Now, by A_1, A_2, A_3,...,A_{12}, we denote the set of vertices as $A_1=\{p_t:1\le t\le n\}$, $A_2=\{q_t:1\le t\le n\}$,...,$A_{12}=\{a_t:1\le t\le n\}$. At that point, we have the accompanying prospects to be talked about.

- When one, as well as the other node, is in the set A_l; $l=1,2,3,...,12$.

Resolving sets	Contradictions
$\mathfrak{Y} = \{p_1, p_g\}$, p_g $(2 \leq g \leq n)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w$ and $\zeta(q_1\|\mathfrak{Y}) = \zeta(q_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{q_1, q_g\}$, q_g $(2 \leq g \leq n)$	$\zeta(p_1\|\mathfrak{Y}) = \zeta(r_1\|\mathfrak{Y})$, for $2 \leq g \leq w$ and $\zeta(p_1\|\mathfrak{Y}) = \zeta(p_2\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{r_1, r_g\}$, r_g $(2 \leq g \leq n)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ and $\zeta(u_1\|\mathfrak{Y}) = \zeta(t_1\|\mathfrak{Y})$, for $w \leq g \leq w+1$.
$\mathfrak{Y} = \{s_1, s_g\}$, s_g $(2 \leq g \leq n)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$, and $\zeta(u_1\|\mathfrak{Y}) = \zeta(t_1\|\mathfrak{Y})$, for $w \leq g \leq w+1$.
$\mathfrak{Y} = \{z_1, z_g\}$, z_g $(2 \leq g \leq n)$	$\zeta(w_1\|\mathfrak{Y}) = \zeta(v_1\|\mathfrak{Y})$, for $2 \leq g \leq w+1$.

Resolving sets	Contradictions
$\mathfrak{Y} = \{t_1, t_g\}$, t_g $(2 \leq g \leq w)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ $\zeta(s_2\|\mathfrak{Y}) = \zeta(z_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(p_1\|\mathfrak{Y}) = \zeta(q_1\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{u_1, u_g\}$, u_g $(2 \leq g \leq w)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ $\zeta(t_2\|\mathfrak{Y}) = \zeta(p_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(p_1\|\mathfrak{Y}) = \zeta(q_1\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{v_1, v_g\}$, v_g $(2 \leq g \leq n)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ $\zeta(u_2\|\mathfrak{Y}) = \zeta(r_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(p_1\|\mathfrak{Y}) = \zeta(r_2\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{w_1, w_g\}$, w_g $(2 \leq g \leq n)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ $\zeta(z_1\|\mathfrak{Y}) = \zeta(r_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(s_1\|\mathfrak{Y}) = \zeta(r_2\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{x_1, x_g\}$, x_g $(2 \leq g \leq n)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ $\zeta(v_2\|\mathfrak{Y}) = \zeta(a_n\|\mathfrak{Y})$, for $g = w$ and $\zeta(p_1\|\mathfrak{Y}) = \zeta(q_2\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{y_1, y_g\}$, y_g $(2 \leq g \leq n)$	$\zeta(p_n\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ $\zeta(w_1\|\mathfrak{Y}) = \zeta(v_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(a_1\|\mathfrak{Y}) = \zeta(z_1\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{a_1, a_g\}$, a_g $(2 \leq g \leq n)$	$\zeta(w_1\|\mathfrak{Y}) = \zeta(v_1\|\mathfrak{Y})$, for $2 \leq g \leq w$ and $\zeta(z_1\|\mathfrak{Y}) = \zeta(z_2\|\mathfrak{Y})$, for $g = w+1$.

- When one node is in the set A_1 and other lies in the set A_l; $l = 2, 3, ..., 12$.

Resolving sets	Contradictions
$\mathfrak{Y}=\{p_1,q_g\}$, q_g $(1\le g\le n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\le g\le w-1$ $\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(s_n\|\mathfrak{Y})=\zeta(r_2\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{p_1,r_g\}$, r_g $(1\le g\le n)$	$\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $1\le g\le w$ and $\zeta(r_n\|\mathfrak{Y})=\zeta(p_2\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{p_1,s_g\}$, s_g $(1\le g\le n)$	$\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $1\le g\le w$ and $\zeta(r_n\|\mathfrak{Y})=\zeta(p_2\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{p_1,t_g\}$, t_g $(1\le g\le n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\le g\le w-1$ $\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(r_n\|\mathfrak{Y})=\zeta(p_2\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{p_1,u_g\}$, u_g $(1\le g\le n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\le g\le w-1$ $\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(r_n\|\mathfrak{Y})=\zeta(p_2\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{p_1,v_g\}$, v_g $(1\le g\le n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\le g\le w-1$ $\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(r_2\|\mathfrak{Y})=\zeta(s_n\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{p_1,w_g\}$, w_g $(1\le g\le n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\le g\le w-1$ and $\zeta(u_2\|\mathfrak{Y})=\zeta(x_1\|\mathfrak{Y})$, for $w\le g\le w+1$.
$\mathfrak{Y}=\{p_1,x_g\}$, x_g $(1\le g\le n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\le g\le w-1$ $\zeta(u_2\|\mathfrak{Y})=\zeta(x_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(r_3\|\mathfrak{Y})=\zeta(w_{n-1}\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{p_1,y_g\}$, y_g $(1\le g\le n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\le g\le w-1$ and $\zeta(u_2\|\mathfrak{Y})=\zeta(x_1\|\mathfrak{Y})$, for $w\le g\le w+1$.

Resolving sets	Contradictions
$\mathfrak{Y}=\{p_1,z_g\}$, z_g $(1\le g\le n)$	$\zeta(a_n\|\mathfrak{Y})=\zeta(a_1\|\mathfrak{Y})$, for $g=1$ $\zeta(x_2\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=2$ and $\zeta(x_1\|\mathfrak{Y})=\zeta(u_2\|\mathfrak{Y})$, for $3\le g\le w+1$.
$\mathfrak{Y}=\{p_1,a_g\}$, a_g $(1\le g\le n)$	$\zeta(v_n\|\mathfrak{Y})=\zeta(u_2\|\mathfrak{Y})$, for $g=1$ $\zeta(x_1\|\mathfrak{Y})=\zeta(u_2\|\mathfrak{Y})$, for $2\le g\le w$ and $\zeta(t_n\|\mathfrak{Y})=\zeta(r_{n-1}\|\mathfrak{Y})$, for $g=w+1$.

- When one node is in the set A_2 and other lies in the set A_l; $l=3,...,12$.

Resolving sets	Contradictions
$\mathfrak{Y} = \{q_1, r_g\}$, r_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $g = 1$ $\zeta(r_1\|\mathfrak{Y}) = \zeta(p_1\|\mathfrak{Y})$, for $2 \leq g \leq w$ and $\zeta(q_n\|\mathfrak{Y}) = \zeta(r_2\|\mathfrak{Y})$, for $g = w + 1$.
$\mathfrak{Y} = \{q_1, s_g\}$, s_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ $\zeta(u_1\|\mathfrak{Y}) = \zeta(t_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(q_n\|\mathfrak{Y}) = \zeta(r_2\|\mathfrak{Y})$, for $g = w + 1$.
$\mathfrak{Y} = \{q_1, t_g\}$, t_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ $\zeta(r_1\|\mathfrak{Y}) = \zeta(p_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(q_n\|\mathfrak{Y}) = \zeta(r_2\|\mathfrak{Y})$, for $g = w + 1$.
$\mathfrak{Y} = \{q_1, u_g\}$, u_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ $\zeta(r_1\|\mathfrak{Y}) = \zeta(p_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(q_n\|\mathfrak{Y}) = \zeta(r_2\|\mathfrak{Y})$, for $g = w + 1$.
$\mathfrak{Y} = \{q_1, v_g\}$, v_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y} = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ $\zeta(q_n\|\mathfrak{Y}) = \zeta(q_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(v_1\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $g = w + 1$.
$\mathfrak{Y} = \{q_1, w_g\}$, w_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ $\zeta(x_1\|\mathfrak{Y}) = \zeta(u_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(r_3\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w + 1$.
$\mathfrak{Y} = \{q_1, x_g\}$, $x_g(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ and $\zeta(v_1\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $w \leq g \leq w + 1$.
$\mathfrak{Y} = \{q_1, y_g\}$, y_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ and $\zeta(v_1\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $w \leq g \leq w + 1$.
$\mathfrak{Y} = \{q_1, z_g\}$, z_g $(1 \leq g \leq n)$	$\zeta(p_2\|\mathfrak{Y}) = \zeta(p_1\|\mathfrak{Y})$, for $g = 1$ $\zeta(y_2\|\mathfrak{Y}) = \zeta(a_1\|\mathfrak{Y})$, for $g = 2$ and $\zeta(v_1\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $3 \leq g \leq w + 1$.
$\mathfrak{Y} = \{q_1, a_g\}$, a_g $(1 \leq g \leq n)$	$\zeta(y_2\|\mathfrak{Y}) = \zeta(a_n\|\mathfrak{Y})$, for $g = 1$ $\zeta(t_2\|\mathfrak{Y}) = \zeta(v_1\|\mathfrak{Y})$, for $2 \leq g \leq w$ and $\zeta(z_1\|\mathfrak{Y}) = \zeta(y_2\|\mathfrak{Y})$, for $g = w + 1$.

- When one node is in the set A_3 and other lies in the set A_l; $l = 4, ..., 12$.

Resolving sets	Contradictions
$\mathfrak{Y} = \{r_1, s_g\}$, s_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ $\zeta(u_1\|\mathfrak{Y}) = \zeta(t_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(q_1\|\mathfrak{Y}) = \zeta(p_1\|\mathfrak{Y})$, for $g = w + 1$.
$\mathfrak{Y} = \{r_1, t_g\}$, t_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ and $\zeta(u_1\|\mathfrak{Y}) = \zeta(t_1\|\mathfrak{Y})$, for $w \leq g \leq w + 1$.
$\mathfrak{Y} = \{r_1, u_g\}$, u_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w - 1$ and $\zeta(u_1\|\mathfrak{Y}) = \zeta(t_1\|\mathfrak{Y})$, for $w \leq g \leq w + 1$.

Resolving sets	Contradictions
$\mathfrak{Y}=\{r_1,v_g\}$, v_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(a_1\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{r_1,w_g\}$, w_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $g=w$ and $\zeta(a_1\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{r_1,x_g\}$, x_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $g=w$ and $\zeta(a_1\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{r_1,y_g\}$, y_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $g=w$ and $\zeta(a_1\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{r_1,z_g\}$, z_g $(1\leq g\leq n)$	$\zeta(a_n\|\mathfrak{Y})=\zeta(a_1\|\mathfrak{Y})$, for $g=1,w+1$ $\zeta(a_n\|\mathfrak{Y})=\zeta(v_2\|\mathfrak{Y})$, for $g=2$ and $\zeta(t_2\|\mathfrak{Y})=\zeta(y_n\|\mathfrak{Y})$, for $3\leq g\leq w$.
$\mathfrak{Y}=\{r_1,a_g\}$, a_g $(1\leq g\leq n)$	$\zeta(a_{n-1}\|\mathfrak{Y})=\zeta(w_2\|\mathfrak{Y})$, for $g=1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $2\leq g\leq w$ and $\zeta(u_1\|\mathfrak{Y})=\zeta(q_n\|\mathfrak{Y})$, for $g=w+1$.

- When one node is in the set A_4 and other lies in the set A_l; $l=5,6,...,12$.

Resolving sets	Contradictions
$\mathfrak{Y}=\{s_1,t_g\}$, t_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(p_1\|\mathfrak{Y})=\zeta(q_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{s_1,u_g\}$, u_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(u_1\|\mathfrak{Y})=\zeta(t_1\|\mathfrak{Y})$, for $g=w$ and $\zeta(p_1\|\mathfrak{Y})=\zeta(q_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{s_1,v_g\}$, v_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ and $\zeta(t_1\|\mathfrak{Y})=\zeta(u_1\|\mathfrak{Y})$, for $w\leq g\leq w+1$.
$\mathfrak{Y}=\{s_1,w_g\}$, w_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $g=w$ and $\zeta(a_1\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{s_1,x_g\}$, x_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $g=w$ and $\zeta(a_1\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{s_1,y_g\}$, y_g $(1\leq g\leq n)$	$\zeta(r_n\|\mathfrak{Y})=\zeta(p_n\|\mathfrak{Y})$, for $1\leq g\leq w-1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $g=w$ and $\zeta(a_1\|\mathfrak{Y})=\zeta(z_1\|\mathfrak{Y})$, for $g=w+1$.
$\mathfrak{Y}=\{s_1,z_g\}$, z_g $(1\leq g\leq n)$	$\zeta(a_n\|\mathfrak{Y})=\zeta(a_1\|\mathfrak{Y})$, for $g=1,w+1$ $\zeta(z_n\|\mathfrak{Y})=\zeta(u_2\|\mathfrak{Y})$, for $g=2$ and $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $3\leq g\leq w$.
$\mathfrak{Y}=\{s_1,a_g\}$, a_g $(1\leq g\leq n)$	$\zeta(q_1\|\mathfrak{Y})=\zeta(p_1\|\mathfrak{Y})$, for $g=1$ $\zeta(y_n\|\mathfrak{Y})=\zeta(t_2\|\mathfrak{Y})$, for $2\leq g\leq w$ and $\zeta(p_n\|\mathfrak{Y})=\zeta(r_n\|\mathfrak{Y})$, for $g=w+1$.

- When one node is in the set A_5 and other lies in the set A_l; $l=6,7,...,12$.

Resolving sets	Contradictions
$\mathfrak{Y} = \{t_1, u_g\}$, u_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(z_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(y_1\|\mathfrak{Y}) = \zeta(t_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{t_1, v_g\}$, v_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(a_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(y_1\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{t_1, w_g\}$, w_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(a_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(y_1\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{t_1, x_g\}$, x_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(a_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(y_1\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{t_1, y_g\}$, y_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(a_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(y_1\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{t_1, z_g\}$, z_g $(1 \leq g \leq n)$	$\zeta(a_n\|\mathfrak{Y}) = \zeta(y_1\|\mathfrak{Y})$, for $g = 1$ $\zeta(y_n\|\mathfrak{Y}) = \zeta(w_1\|\mathfrak{Y})$, for $g = 2$, we have , $\zeta(z_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $3 \leq g \leq w$ and $\zeta(y_1\|\mathfrak{Y}) = \zeta(t_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{t_1, a_g\}$, a_g $(1 \leq g \leq n)$	$\zeta(y_n\|\mathfrak{Y}) = \zeta(w_1\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(z_n\|\mathfrak{Y}) = \zeta(y_1\|\mathfrak{Y})$, for $g = w$ and $\zeta(z_1\|\mathfrak{Y}) = \zeta(r_n\|\mathfrak{Y})$, for $g = w+1$.

- When one node is in the set A_6 and other lies in the set A_l; $l = 7, 8, ..., 12$.

Resolving sets	Contradictions
$\mathfrak{Y} = \{u_1, v_g\}$, v_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(y_n\|\mathfrak{Y}) = \zeta(s_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(v_2\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{u_1, w_g\}$, w_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(y_n\|\mathfrak{Y}) = \zeta(s_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(v_2\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{u_1, x_g\}$, x_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(y_n\|\mathfrak{Y}) = \zeta(s_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(v_2\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{u_1, y_g\}$, y_g $(1 \leq g \leq n)$	$\zeta(r_n\|\mathfrak{Y}) = \zeta(p_n\|\mathfrak{Y})$, for $1 \leq g \leq w-1$ $\zeta(y_n\|\mathfrak{Y}) = \zeta(s_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(v_2\|\mathfrak{Y}) = \zeta(s_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{u_1, z_g\}$, z_g $(1 \leq g \leq n)$	$\zeta(p_1\|\mathfrak{Y}) = \zeta(q_1\|\mathfrak{Y})$, for $1 \leq g \leq 2$, $\zeta(y_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $3 \leq g \leq w$ and $\zeta(z_2\|\mathfrak{Y}) = \zeta(z_n\|\mathfrak{Y})$, for $g = w+1$.
$\mathfrak{Y} = \{u_1, a_g\}$, a_g $(1 \leq g \leq n)$	$\zeta(p_1\|\mathfrak{Y}) = \zeta(q_1\|\mathfrak{Y})$, for $g = 1$ $\zeta(y_n\|\mathfrak{Y}) = \zeta(t_2\|\mathfrak{Y})$, for $2 \leq g \leq w-1$ $\zeta(z_n\|\mathfrak{Y}) = \zeta(v_2\|\mathfrak{Y})$, for $g = w$ and $\zeta(z_2\|\mathfrak{Y}) = \zeta(t_n\|\mathfrak{Y})$, for $g = w+1$.

Similarly, we get contradictions, when one vertex is in the set A_l $(7 \leq l \leq 11)$ and the other lies in the set A_l $(8 \leq l \leq 12)$. In this manner, the above conversation explains that there is no resolving set comprising of two vertices for $\mathbb{V}(L(S(D_n)))$ implying that

$\beta(L(S(D_n))) = 3$ in this case.

Case(II) When the integer n is odd.
In this case, the integer n can be written as $n = 2w + 1$, where $w \in \mathbb{N}$ and $w \geq 3$. Let $\mathfrak{Y} = \{z_1, z_2, q_1\} \subset \mathbb{V}(L(S(D_n)))$. Now, in order to show that $\mathfrak{Y}$ is a resolving set for the plane graph $L(S(D_n))$, we consign the metric codes for each vertex of $\mathbb{V}(L(S(D_n)))$ regarding the set $\mathfrak{Y}$.

Now, the metric codes for the nodes of pq-cycle $\{v = p_t, q_t : 1 \leq t \leq n\}$ are

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y} = \{z_1, z_2, q_1\}$
$\zeta_M(p_t\|\mathfrak{Y})$:$(t = 1)$	$(7, 8, 1)$
$\zeta_M(p_t\|\mathfrak{Y})$:$(t = 2)$	$(7, 7, 1)$
$\zeta_M(p_t\|\mathfrak{Y})$:$(t = 3)$	$(8, 7, 3)$
$\zeta_M(p_t\|\mathfrak{Y})$:$(4 \leq t \leq w + 2)$	$(2t + 2, 2t, 2t - 3)$
$\zeta_M(p_t\|\mathfrak{Y})$:$(w+3 \leq t \leq 2w+1)$	$(4w - 2t + 10, 4w - 2t + 12, 4w - 2t + 5)$

and

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y} = \{z_1, z_2, q_1\}$
$\zeta_M(q_t\|\mathfrak{Y})$:$(t = 1)$	$(7, 8, 0)$
$\zeta_M(q_t\|\mathfrak{Y})$:$(t = 2)$	$(7, 7, 2)$
$\zeta_M(q_t\|\mathfrak{Y})$:$(3 \leq t \leq w + 1)$	$(2t + 3, 2t + 1, 2t - 2)$
$\zeta_M(q_t\|\mathfrak{Y})$:$(t = w + 2)$	$(2w + 5, 2w + 5, 2w)$
$\zeta_M(q_t\|\mathfrak{Y})$:$(w+3 \leq t \leq 2w+1)$	$(4w - 2t + 9, 4w - 2t + 11, 4w - 2t + 4)$

The metric codes for the set of inward nodes $\{v = r_t : 1 \leq t \leq n\}$ are

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y} = \{z_1, z_2, q_1\}$
$\zeta_M(r_t\|\mathfrak{Y})$:$(t = 1)$	$(6, 7, 1)$
$\zeta_M(r_t\|\mathfrak{Y})$:$(t = 2)$	$(6, 6, 2)$
$\zeta_M(r_t\|\mathfrak{Y})$:$(3 \leq t \leq w + 1)$	$(2t + 2, 2t, 2t - 2)$
$\zeta_M(r_t\|\mathfrak{Y})$:$(t = w + 2)$	$(2w + 5, 2w + 4, 2w + 1)$
$\zeta_M(r_t\|\mathfrak{Y})$:$(w+3 \leq t \leq 2w+1)$	$(4w - 2t + 9, 4w - 2t + 11, 4w - 2t + 5)$

The metric codes for the set of interior nodes $\{v = s_t : 1 \leq t \leq n\}$ are

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y} = \{z_1, z_2, q_1\}$
$\zeta_M(s_t\|\mathfrak{Y})$:$(t = 1)$	$(5, 6, 2)$
$\zeta_M(s_t\|\mathfrak{Y})$:$(t = 2)$	$(5, 5, 3)$
$\zeta_M(s_t\|\mathfrak{Y})$:$(3 \leq t \leq w + 1)$	$(2t + 1, 2t - 1, 2t - 1)$
$\zeta_M(s_t\|\mathfrak{Y})$:$(t = w + 2)$	$(2w + 4, 2w + 3, 2w + 2)$
$\zeta_M(s_t\|\mathfrak{Y})$:$(w+3 \leq t \leq 2w+1)$	$(4w - 2t + 8, 4w - 2t + 10, 4w - 2t + 6)$

The metric codes for the nodes of $tuvw$-cycle $\{v = t_t, u_t, v_t, w_t : 1 \leq t \leq n\}$ are

$\zeta_M(v\|\mathfrak{Y})$	$\mathfrak{Y} = \{z_1, z_2, q_1\}$
$\zeta_M(t_t\|\mathfrak{Y})$:$(t = 1)$	$(5, 6, 3)$
$\zeta_M(t_t\|\mathfrak{Y})$:$(t = 2)$	$(4, 5, 4)$
$\zeta_M(t_t\|\mathfrak{Y})$:$(3 \leq t \leq w + 1)$	$(2t, 2t - 2, 2t)$
$\zeta_M(t_t\|\mathfrak{Y})$:$(t = w + 2)$	$(2w + 4, 2w + 2, 2w + 3)$
$\zeta_M(t_t\|\mathfrak{Y})$:$(w+3 \leq t \leq 2w+1)$	$(4w - 2t + 8, 4w - 2t + 10, 4w - 2t + 7)$

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(u_t\|\mathfrak{Y})$:$(t=1)$	$(4,5,3)$
$\zeta_M(u_t\|\mathfrak{Y})$:$(t=2)$	$(5,4,4)$
$\zeta_M(u_t\|\mathfrak{Y})$:$(3\le t\le w+1)$	$(2t+1,2t-1,2t)$
$\zeta_M(u_t\|\mathfrak{Y})$:$(t=w+2)$	$(2w+3,2w+3,2w+3)$
$\zeta_M(u_t\|\mathfrak{Y})$:$(w+3\le t\le 2w+1)$	$(4w-2t+7,4w-2t+9,4w-2t+7)$

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(v_t\|\mathfrak{Y})$:$(t=1)$	$(3,4,4)$
$\zeta_M(v_t\|\mathfrak{Y})$:$(2\le t\le w+1)$	$(2t+1,2t-1,2t+1)$
$\zeta_M(v_t\|\mathfrak{Y})$:$(t=w+2)$	$(2w+2,2w+3,2w+3)$
$\zeta_M(v_t\|\mathfrak{Y})$:$(w+3\le t\le 2w+1)$	$(4w-2t+6,4w-2t+8,4w-2t+7)$

and

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(w_t\|\mathfrak{Y})$:$(t=1)$	$(3,4,5)$
$\zeta_M(w_t\|\mathfrak{Y})$:$(2\le t\le w+1)$	$(2t+1,2t-1,2t+2)$
$\zeta_M(w_t\|\mathfrak{Y})$:$(t=w+2)$	$(2w+2,2w+3,2w+2)$
$\zeta_M(w_t\|\mathfrak{Y})$:$(w+3\le t\le 2w+1)$	$(4w-2t+6,4w-2t+8,4w-2t+6)$

The metric codes for the set of exterior nodes $\{\upsilon=x_t:1\le t\le n\}$ are

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(x_t\|\mathfrak{Y})$:$(t=1)$	$(2,3,5)$
$\zeta_M(x_t\|\mathfrak{Y})$:$(2\le t\le w+1)$	$(2t,2t-2,2t+2)$
$\zeta_M(x_t\|\mathfrak{Y})$:$(t=w+2)$	$(2w+1,2w+2,2w+3)$
$\zeta_M(x_t\|\mathfrak{Y})$:$(w+3\le t\le 2w+1)$	$(4w-2t+5,4w-2t+7,4w-2t+7)$

The metric codes for the set of outward nodes $\{\upsilon=y_t:1\le t\le n\}$ are

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(y_t\|\mathfrak{Y})$:$(t=1)$	$(1,2,6)$
$\zeta_M(y_t\|\mathfrak{Y})$:$(2\le t\le w+1)$	$(2t-1,2t-3,2t+3)$
$\zeta_M(y_t\|\mathfrak{Y})$:$(t=w+2)$	$(2w,2w+1,2w+4)$
$\zeta_M(y_t\|\mathfrak{Y})$:$(w+3\le t\le 2w+1)$	$(4w-2t+4,4w-2t+6,4w-2t+8)$

At last, the metric codes for the nodes of za-cycle $\{\upsilon=z_t,a_t:1\le t\le n\}$ are

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(z_t\|\mathfrak{Y})$:$(t=1)$	$(0,2,7)$
$\zeta_M(z_t\|\mathfrak{Y})$:$(t=2)$	$(2,0,8)$
$\zeta_M(z_t\|\mathfrak{Y})$:$(3\le t\le w+1)$	$(2t-2,2t-4,2t+3)$
$\zeta_M(z_t\|\mathfrak{Y})$:$(t=w+2)$	$(2w,2w,2w+5)$
$\zeta_M(z_t\|\mathfrak{Y})$:$(w+3\le t\le 2w+1)$	$(4w-2t+4,4w-2t+6,4w-2t+9)$

and

$\zeta_M(\upsilon\|\mathfrak{Y})$	$\mathfrak{Y}=\{z_1,z_2,q_1\}$
$\zeta_M(a_t\|\mathfrak{Y})$:$(t=1)$	$(1,1,7)$
$\zeta_M(a_t\|\mathfrak{Y})$:$(2\le t\le w+1)$	$(2t-1,2t-3,2t+4)$
$\zeta_M(a_t\|\mathfrak{Y})$:$(w+2\le t\le 2w)$	$(4w-2t+3,4w-2t+5,4w-2t+8)$
$\zeta_M(a_t\|\mathfrak{Y})$:$(t=2w+1)$	$(1,3,7)$

Again we see that no two vertices are having indistinguishable metric codes, suggesting that $dim(L(S(D_n))) \leq 3$. Now, on assuming that $dim(L(S(D_n))) = 2$, we consider that to be are parallel prospects as talked about in Case(I) and logical inconsistency can be inferred correspondingly. Consequently, $dim(L(S(D_n))) = 3$ for this situation too, which completes the proof of the theorem.

□

This result can also be written as:

Theorem 2.2. *Let n be the positive integer such that $n \geq 6$ and $L(S(D_n))$ be the planar graph on $12n$ vertices as defined above. Then, its independent resolving number is* 3.

Proof. For proof, refer to Theorem 2.1. □

3. Conclusions

In this study, we obtained the exact metric dimension of the line graph of the subdivision graph of the graph of convex polytope D_n. We found that the metric dimension of the line graph $L(S(D_n))$ is the same as the metric dimension of the graph of convex polytope D_n i.e., $\beta(D_n) = \beta(L(S(D_n)))$. We also observed that the basis set $\mathfrak{Y}$ is independent for the graph $L(S(D_n))$. We close this section by raising an open problem that naturally arises from the text.

Open Problem 1: Characterise those families of the graphs of convex polytopes (say A_n) with $\beta(A_n) = \beta(L(S(A_n)))$.

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