TITLE: **PLANCK/LFI: Back of the envelope estimate for the effect of quantization on the sensitivity to primordial non gaussianities**

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Contents

LIST OF ABBREVIATIONS

acronym	Explanation
FSL	First Light
LFI	Low Frequency Instrument
NMQE	Noise Model for the Quantization Error
TBC	To Be Confirmed
TBD	To Be Defined
pdf	Probability Distribution Function

1 Applicable and Reference Documents

Applicable Documents

[AD-1] Maris et al 2004

Reference Documents

2 Scope of the document

This document is a Back–of–the–envelope calculation of the impact of REBA quantization on non–gaussianities.

2.1 Limits of Applicability

This document just analyzes the case in which the usual common *Noise Model for the Quantization Error* (NMQE) is valid.

In this case quantization mainly acts as an addictive source of noise whose standard deviation is ϵ_q.

Denoting with σ the standard deviation of the unquantized signal, this condition holds for

$$\frac{\epsilon_q}{\sigma} \precsim 1.$$

3 Introduction

In PLANCK each region of sky is sampled many times with a noisy detector, samples are piled up for well defined sky positions (usually defined as "pixels" of a map) and a weighted average is taken with weights proportional to the amount of contribution of each sample to the "pixel", a process named "Map Making".

Here we will make the assumption that Map Making could be approximated by a simple arithmetic averaging of $N_{\rm smp}$ independent samples with any sample having the same weigth.

So the whole process of sampling and quantization could be approximated by the averaging of $N_{\rm smp}$ independent measures of the same sky region obtained from a noisy and uniformly quantized detector, whose normally distributed noise has variance σ^2 and whose quantization step is q.

In this view the pdf for the samples on the map is the convolution of the pdf of the sky signal and the pdf for the sampling and Map Making process, which in turn is the convolution of the noise pdf and quantization pdf.

In this document we are interested in analyzing the level of perturbation introduced by quantization in the pdf of the map, whic is equivalent to compare the noise pdf with quantization with the noise pdf without quantization.

We considered quantization harmfull for the mission if significantly degrades the instrumental performances i.e. the quality of the unquantized signal. This is equivalent at looking for significant changes in the quantized noise pdf when compared to the unquantized one.

A convenient approach to this problem is to consider the effect on the central moments of the pdfs under examination, i.e. to look at their characteristic function for the central moments (the Fourier Transform of the pdf)

$$\phi_{q,N_{\rm smp},{\rm sky}}(\omega) = \phi_{q,N_{\rm smp}}(\omega)\phi_{\sigma,N_{\rm smp}}(\omega)\phi_{\rm sky}(\omega) \tag{1}$$

where $\phi_{\rm sky}(\omega)$ is the characteristic function for the sky signal pdf, and $\phi_{q,N_{\rm smp}}(\omega)$ is the characteristic function for $N_{\rm smp}$ averaging and quantization, $\phi_{\sigma,N_{\rm smp}}(\omega)$ is the characteristic function for the noise and averaging, $-\infty \le \omega \le +\infty$ is the argument of the characteristic function. It is convenient to consider also $\phi_{\sigma,q,N_{\rm smp}}(\omega)$ which is the product of $\phi_{q,N_{\rm smp}}(\omega)$ and $\phi_{\sigma,N_{\rm smp}}(\omega)$.

3.1 Combining Moments

For a pdf with characteristic function $\phi(\omega)$ the k–th central moment is obtained from the well know formula

$$\mu_k = (\sqrt{-1})^k \left. \frac{d^k\phi(\omega)}{d\omega^k}\right|_{\omega=0} \tag{2}$$

from this equation it is easy to derive how central moments combine for the product of two characteristic functions $\phi_{\rm A}$ and $\phi_{\rm B}$ [1]

$$\mu_{{\rm A,B},n} = \Sigma_{k=0}^{n} \frac{n!}{(n-k)!k!}\mu_{{\rm A},n-k}\mu_{{\rm B},k} \tag{3}$$

with $\mu_{{\rm A,B},n}$ the central moment of order n for the product of the two characteristic functions, and $\mu_{{\rm A},n-k}$, $\mu_{{\rm B},k}$ the central moments for the two characteristic functions.

[1] It is easy to see that replacing $\mu_{{\rm A},n-k}$ with X^{n-k} and $\mu_{{\rm B},k}$ with Y^k Eq. (3) formally reduces to $(X+Y)^n$.

In our case, from Eq. (3) and the fact that both noise and quantization have null central moments for k–odd the odd central moments for quantized noise are null.

A more interesting consequence of Eq. (3) occurs when combining two processes, one with null odd central moments and the other with non–null odd central moments. Let suppose to have the B process with non–null odd moments, and the A process to have null odd moments. In this case the minimum condition to have not null products of the kind $\mu_{\mathrm{A},n-k}\mu_{\mathrm{B},k}$ with k odd is to have $n-k$ even. This authomatically excludes $\mu_{\mathrm{A,B},n}$ for even order n to be affected by odd moments. While for odd order n $\mu_{\mathrm{A,B},n}$ are potentially affected by both even moments of process A and odd moments of process B.

3.2 Combining Moments for Partitions

We have also to consider the case in which we draw randomly samples from a set of mixed populations all with null expectation. This is as an example the case of map–making in which the rms of noise properties changes over the map, so that noise fluctuations has different rms from region to region. Let to denote with N_{pop} the number of independent populations from which a saple with value x is randomly drawn, and let be to denote with g_p, $p = 1,\ 2,\ \ldots,\ N_{\mathrm{pop}}$ the partition function for samples among each population, having $\sum_{p=1}^{N_{\mathrm{pop}}} g_p = 1$ and with $\mathcal{P}_{\mathrm{pop}}(x)$ the pdf for the population p. Then the pdf for the value x is

$$\mathcal{P}(x) = \sum_{p=1}^{N_{\mathrm{pop}}} g_p \mathcal{P}_{\mathrm{pop}}(x). \tag{4}$$

The Characteristic Function for $\mathcal{P}(x)$ is a linear combination of the characteristic function of each $\mathcal{P}_{\mathrm{pop}}(x)$ here denoted as $\phi_p(\omega)$.

$$\phi(\omega) = \sum_{p=1}^{N_{\mathrm{pop}}} g_p \phi_p; \tag{5}$$

so that from Eq. (8)

$$\mu_k = \sum_{p=1}^{N_{\mathrm{pop}}} g_p \mu_{p,k}, \tag{6}$$

where $\mu_{p,k}$ is the k–th central moment for the population p.

It easy to demonstrate that in general even if all the populations entering the partition are normal distributed with null expectation then the final population will be not necessarily normally distributed. Infact in case of a normal distributed population its central moments for k odd are null, while all the central moments for k even are $C_k\mu_2^{k/2}$, with C_k a constant uniquely defined by k, with $C_2 = 1$. From Eq. (8): $\mu_2 = \sum_{p=1}^{N_{\mathrm{pop}}} g_p\mu_{p,2}$ and $\mu_k = C_k \sum_{p=1}^{N_{\mathrm{pop}}} g_p\mu_{p,2}^{k/2}$. But to be normal distributed it would be also: $\mu_k = C_k\mu_2^{k/2}$, which is equivalent to ask $(\sum_{p=1}^{N_{\mathrm{pop}}} g_p\mu_{p,2})^{k/2} = \sum_{p=1}^{N_{\mathrm{pop}}} g_p\mu_{p,2}^{k/2}$ which has not general solution unless all the populations have the same $\mu_{p,2}$.

4 Method

Moments for the quantization error for the case of samples obtained from averages of N_{smp} independented measures are derived by using the characteristic function for the average of N_{smp}

k	$\mu^k_{q,N_{\rm smp}}/q^k$
2	$\frac{1}{12}\frac{1}{N_{\rm smp}}$
4	$\frac{1}{48N^2_{\rm smp}} - \frac{1}{120N^3_{\rm smp}}$
6	$\frac{5}{576N^3_{\rm smp}} - \frac{1}{96N^4_{\rm smp}} + \frac{1}{125N^5_{\rm smp}}$
8	$\frac{35}{6912N^4_{\rm smp}} - \frac{5}{576N^5_{\rm smp}} + \frac{101}{8640N^6_{\rm smp}} - \frac{61}{2304N^6_{\rm smp}}$
10	$\frac{35}{9216N^5_{\rm smp}} - \frac{35}{2304N^6_{\rm smp}} + \frac{61}{2304N^7_{\rm smp}} - \frac{13}{576N^8_{\rm smp}} + \frac{23}{9072N^9_{\rm smp}}$

Table 1: Central moments for averaged quantization errors for $k \le 10$. For these calculations the serie have been truncated at order 20.

drawn from the uniform distribution defined in the range $[-q/2, +q/2]$.

From [AD-1] the characteristic function for $N_{\rm smp} = 1$ is a sinc function and for the average of $N_{\rm smp}$ samples

$$\phi_{q,N_{\rm smp}}(\omega) = \left(\frac{\sin\frac{\omega}{2N_{\rm smp}}}{\frac{\omega}{2N_{\rm smp}}}\right)^{N_{\rm smp}} \tag{7}$$

the k–th central moment is then obtained from the well know formula

$$\mu^k_{q,N_{\rm smp}} = (\sqrt{-1})^k \left.\frac{d^k\phi_{q,N_{\rm smp},{\rm sky}}(\omega)}{d\omega^k}\right|_{\omega=0} \tag{8}$$

which is null for k odd. The calculation of the derivative in Eq. (8) is complicated but since we are interested its limit for $\omega \to 0$ it could be simplified by replacing $\sin\frac{\omega}{2N_{\rm smp}}/\frac{\omega}{2N_{\rm smp}}$ in the equation by its Taylor expansion about $\omega = 0$. followed by derivation and calculation of its value in the $\omega \to 0$ limit. In doing this operation, attention has to be taken at properly truncate the series. Tab. 1 gives the first 5 even moments computed by using this method and the `Maple` computer algebra system.

In [AD-1] it have been shown that the quantization error ... `problema della larghezza`

Some simple montecarlo realization have been used to verify the correcteness of the above expansion.

5 Leading Term Analysis

From Tab. 1 it is evident that the leading term in $\mu^k_{q,N_{\rm smp}}$ is proportional to $1/N^{k/2}_{\rm smp}$. Moreover, defining the variance for the quantization error as

$$\epsilon^2_q = \frac{q^2}{12}\frac{1}{N_{\rm smp}} \tag{9}$$

it is possible to see that the leading term of $\mu^k_{q,N_{\rm smp}}$ could be expressed in terms of ϵ_q being proportional to ϵ_q^k.

It is easy to see also that the leading term is of "gaussian" nature, i.e. it could be derived from the relation for the central moments of even order for a normal distribution with variance σ^2

$$M_k = \frac{k!}{2^{k/2}(k/2)!}\frac{\sigma^k}{N_{\rm smp}^{k/2}}; \tag{10}$$

by replacing σ with ϵ_q.

Since both the quantization effect and the noise scale in the same manner with $N_{\rm smp}$, their ratio will not depend on $N_{\rm smp}$ as a consequence the relative variation of moments is constant all over the map, it does not depend on the scanning strategy and it is just a function of ϵ_q/σ.

We may use these results to derive the central moments of a normal distributed noise quantized and averaged $N_{\rm smp}$ times from Eq. (3). Given the scaling property above it is easy to factorize μ_{2k} as

$$\mu_{2k} = M_{2k}\mathcal{P}_{2k}\left(\frac{\epsilon_q}{\sigma}\right);$$

where $\mu^{\rm G}_{2k}$ is the moment of order $2k$ for a normal distribution and $\mathcal{P}_{2k}(x)$ a polynomial of degree $2k$ from which

$$\mu_{2k} = M_{2k}\left[1+\left(\frac{\epsilon_q}{\sigma}\right)^2\right]^k. \tag{11}$$

Since $\epsilon_q/\sigma < 1$ the leading term of these expansions is allways the quadratic one so that

$$\mu_{2k} \approx M_{2k}\left[1+k\left(\frac{\epsilon_q}{\sigma}\right)^2\right], \tag{12}$$

which is characterized by a linear scaling of μ_{2k} with k so that higher order moments are those affected by quantization, However, having ϵ_q/σ in the range $3\times10^{-2}\div10^{-1}$ the typical perturbation for higher order moments is in the range $k10^{-3} \div k10^{-2}$;

In general tests of gaussianity are based on some kind of comparison of higher order moments with what expected for normal distributed data. So by defining

$$\gamma_{2k} = \frac{\mu_{2k}}{\mu_2},\ k = 2, 3, 4, \ldots; \tag{13}$$

it is possible to see that the expectation of γ_{2k} is allways 1 as expected from the gaussian case.

In conclusion at first order, for small quantization errors, large effects from quantization are not expected on the shape of noise pdf. If, with respect to non gaussianity tests, noise is gaussian, then even quantized noise will be for a large extent "gaussian" due to the central limit theorem.

6 Effect on Estimators Variance

Quantization acts as noise, at least in the extent quantization is small. If so it will affect the PLANCK sensitivity to non gaussianity as a further source of noise (so as an example it will reduce the sensitivity to deviations of odd order moments from the ideal zero value).

To estimate this effect it is sufficient to consider the variation of sampling RMS for odd moments. For these moments the sampling RMS is

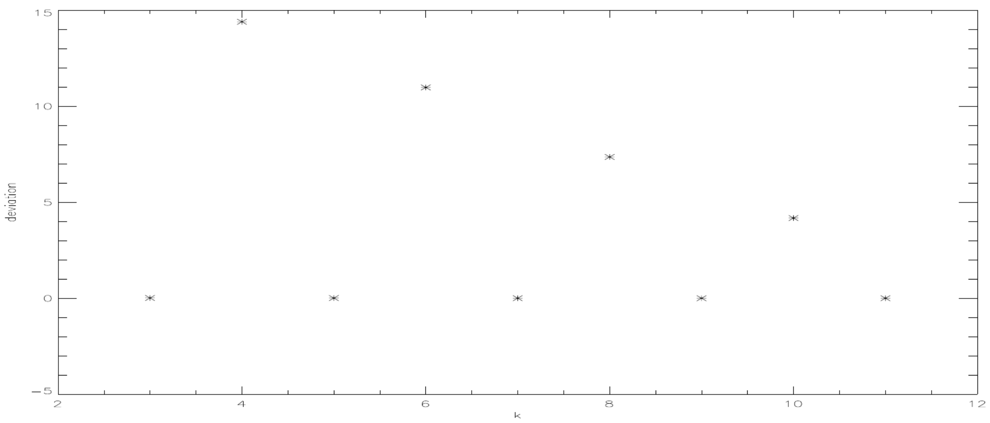

Figure 1: Deviation of expectation of sampling moments of order k from Eq. (10). Deviation is in units $s_{g,k}$, see Eq. (24).

$$\mathrm{RMS}_{2k+1} \propto \sqrt{\mu_{2(2k+1)}};$$

so an estimate of the relative variation of sensitivity for the case of quantized noise is

$$\frac{\mathrm{RMS}^q_{2k+1}}{\mathrm{RMS}_{2k+1}} \approx \sqrt{\frac{\mu_{2(2k+1)}}{M_{2(2k+1)}}};$$

and from Eq. (12)

$$\frac{\mathrm{RMS}^q_{2k+1}}{\mathrm{RMS}_{2k+1}} \approx 1 + (2k+1)\left(\frac{\epsilon_q}{\sigma}\right)^2. \tag{14}$$

Having ϵ_q/σ in the range $3 \times 10^{-2} \div 10^{-1}$ the typical loss in sensitivity for skeness is less than 3% and for the fifth moment it is less than 5%.

6.1 Next–to–Leading Term

The next–to–leading term in the expansions of Tab. 1 is proportional to $q^k/N_{\mathrm{smp}}^{k/2+1}$ which is small when compared to the sampling variance and can be neglected.

7 Simulations

In order to check the above results we perform a simple MonteCarlo simulation to assess the level of perturbations on the noise properties of a map.

The simulation is based on the following assumptions:

1. a map is organized according to the HEALpix ring scheme, with $N_{\rm side} = 256$, and it is oriented in the ecliptical coordinates;
2. there is no sky signal in the map;
3. noise in the map is just uncorrelated white noise with unit variance and null mean;
4. the simple scanning strategy is assumed, spin axis is kept constantly on the ecliptic, boresight angle is assumed to be 85°
5. to save computing time circles are scanned just 30 times per pointing period and it is assumed that at the ecliptic equator a circle is as wide as a pixel;

With these assumptions a pixel in a map is the average of a number of independent samples corresponding to independent measures of the same sky region. The number of samples per pixels is determined by the ecliptical colatitude of the pixel $\theta_{\rm pxl}$

$$N_{\rm smp} = 30 \sin \theta_{\rm pxl}. \tag{15}$$

We skipped circumpolar pixels i.e. pixels with a colatitude smaller than 5° or larger than 175°.

7.1 Method

The MonteCarlo scheme to generate a single map of quantized noise is the following

1. For each pixel in the map a number, $N_{\rm smp}$, of independent white noise realizations, x, with null mean and unit variance have been generated.
2. each white noise realization have been quantized according to the formula

$$x_q = q \,\mathrm{round}\left[\frac{x}{q}\right] \tag{16}$$

3. the pixel value is the average of the $N_{\rm smp}$ independent realizations.

while generating a map for a given q the map for unquantized noises have been also generated.

For each map central moments for all the pixels visited at least once have been computed. In particular we considered the first 11 moments defined as follow:

$$\begin{array}{rcl} \tilde{\mu}_{i,1} & = & \frac{1}{N_{\rm pxl}} \sum_p V_p \\ \tilde{\mu}_{i,2} & = & \frac{1}{N_{\rm pxl}} \sum_p (V_p - \tilde{\mu}_1)^2 \\ \tilde{\mu}_{i,3} & = & \frac{1}{N_{\rm pxl}} \frac{1}{\tilde{\mu}_2^{3/2}} \sum_p (V_p - \tilde{\mu}_1)^3 \\ & \ldots & \\ \tilde{\mu}_{i,11} & = & \frac{1}{N_{\rm pxl}} \frac{1}{\tilde{\mu}_2^{11/2}} \sum_p (V_p - \tilde{\mu}_1)^{11} \end{array} \tag{17}$$

with V_p the value of the $p-$th pixel on the map, either including or not quantization, i the map index. The moments for quantized maps are indicated with a subscripted q, e.g. $\tilde{\mu}_1$ is the average for unquantized map and $\tilde{\mu}_{q,1}$ is the corresponding average for the quantized map. With this definition $\tilde{\mu}_3$ and $\tilde{\mu}_4$ are the sampling skewness and kurtosis for the maps.

We generated 200 maps, quantized and unquantized, for each quantization step q. Quantization steps have been defined by fixing the quantization error at the level of the single sample: $q = \sqrt{12}\epsilon_q$;. We take $\epsilon_q/\sigma_{\rm wn} = 0.05$, 0.05, 0.1, 0.2, 0.4, 0.5, 0.7, 1, and 2, equivalent to $q/\sigma_{\rm wn} = 0.17$, 0.35, 0.69, 1.39, 1.73, 2.42, 3.46, 6.93. The first two values are typical of PLANCK/LFI.

7.2 Results

We are interested in quantifing the level of perturbations introduced by quantization in the central moments and the deviation from gaussianity. We derived different indicators of this based on the sampling average and sampling variance of moments from k=2 up to k=11 drawn from each subset of 200 maps.

In particular for each k we denoted with $m_{q,k}$ and $s_{q,k}$ the sampling average and standard deviation of moments quantized maps moments of order k $\tilde{\mu}_{q,i,k}$

$$m_{q,k} = \mathrm{mean}(\tilde{\mu}_{q,i,k}), \quad (18)$$
$$s_{q,k} = \mathrm{std\_dev}(\tilde{\mu}_{q,i,k}). \quad (19)$$
$$(20)$$

While with $m_{g,k}$ and $s^2_{g,k}$ we mean the sampling mean and variance of the same moments computed for maps without quantization

$$m_{g,k} = \mathrm{mean}(\tilde{\mu}_{g,i,k}), \quad (21)$$
$$s_{g,k} = \mathrm{std\_dev}(\tilde{\mu}_{g,i,k}). \quad (22)$$
$$(23)$$

7.3 Deviation of $m_{g,k}$ from Eq. (10)

It is important to stress that due to the map–making process, white noise is not evenly distributed over the map, being smaller near the ecliptic poles. This has the potential to introduce deviations from gaussianity. The analysis of this problem is a typical application of the case described in Sect. 3.2. Assuming the simple scanning strategy at the root of the current simulations, the different populations of pixels are simply pixels with different colatitude as suggested by Eq. (15). So N_{pop} is the number of bands of pixels with constant colatitude θ_p, g_p is the fraction of visited pixels with that colatitude.

Indeed, the expectation for central moments with k even for the simulated noise maps slightly deviates from the pure gaussian case Eq. (10). [2] We quantify this deviation as

$$\delta\mu_{g,k} = \frac{m_{g,k} - M_k}{s_{g,k}}; \quad (24)$$

which is shown in Fig. 1. Since for odd moments the expectation is zero, the deviation is marginal or null. But for moments with k even there is a clear deviation which decreases increasing k.

The deviation can be modelized by noting that each g_p is proportional to the number of map pixels with colatitude θ_p, so that $g_p \propto \sin\theta_p$, while if noise is normal distributed for noise moments of order k each pixel of class p will have $\sigma_p^2 \propto 1/(\sin\theta_p)^{k/2}$. So that from Eq. (3.2) in the limit $N_{\mathrm{pop}} \gg 1$ it is possible to write

$$\mu_{g,k} = \mu_{g,0,k}\Gamma_k \quad (25)$$
$$\Gamma_k \approx \frac{\int_{\theta_1}^{\theta_2} d\theta\,[\sin\theta]^{1-k/2}}{\int_{\theta_1}^{\theta_2} d\theta \sin\theta} \quad (26)$$

[2] This is also due to some unaccuracy of the random number generator.

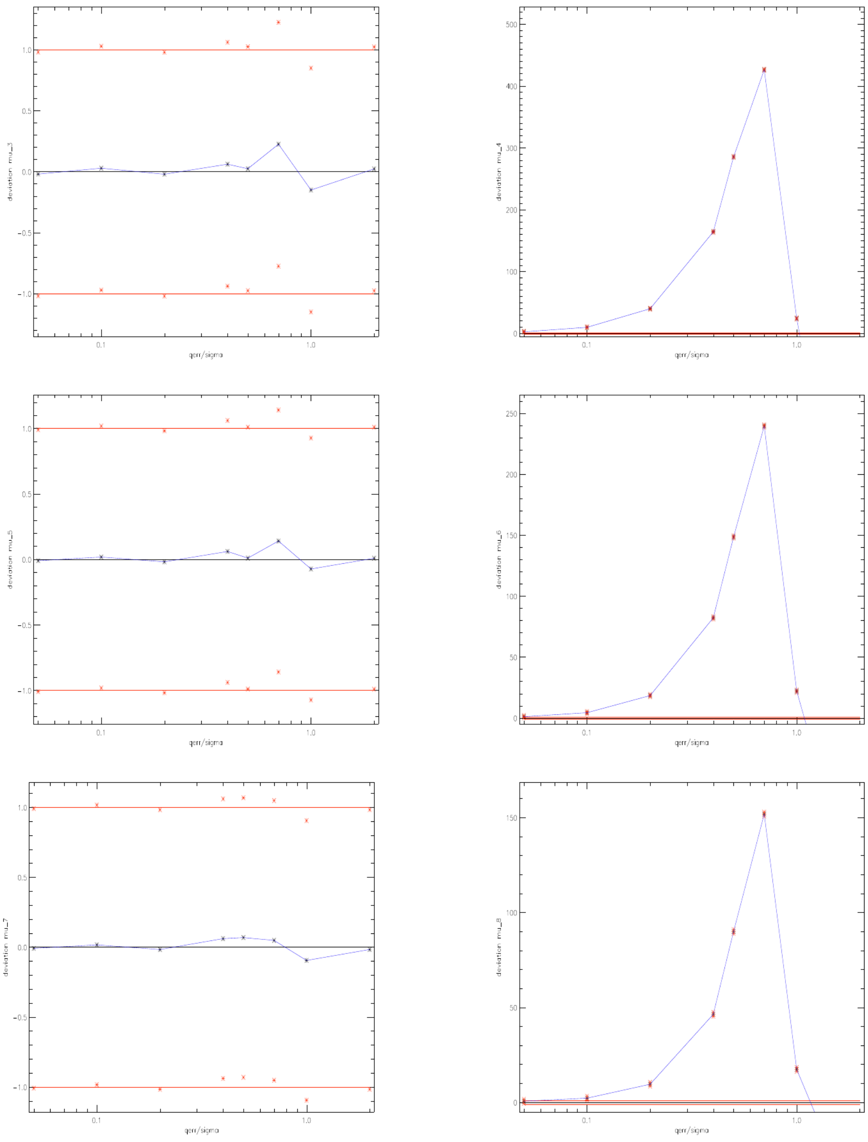

Figure 2: Variation of moments indexes (Eq. (27) for different ϵ_q/σ. Odd moments at left, even moments at right. From top to bottom and left to right $k = 3, 4, 5, 6, 7, 8$.

k	Γ_k
2	1.4892345
3	2.0394672
4	3.1466709
5	5.6061292
6	11.550095
7	26.911562
8	68.699306
9	186.86086
10	530.95984
11	1556.0707

Table 2: Table of Γ_k coefficients from Eq. (26).

where, θ_1 and θ_2 are the minimum and maximum colatitudes scanned by PLANCK in our case 5° and 85° respectively, while $\mu_{g,0,k}$ is the k–th central moment for a map pixel at the equator. Tab. 2 lists the Γ_k for a set of k values. The fact that Fig. 1 shows $\delta\mu_{g,k}$ decreasing while k increases is due to the fact that $\mu_{g,0,k} \propto \mu_{g,0,2}^{k/2}$ with $\mu_{g,0,2} < 1$ and that $s_{g,k}$ increases when k increases.

In conclusion to keep in account of map–making we will not take M_k as the refence value for moments of order k for the not–quantized case, but rather of $m_{g,k}$.

7.4 Deviation of sampling moments from quantized maps from the not quantized case.

The index of variation for sampling moments of order k due to the quantization is defined as

$$\delta m_{q,g,k} = \frac{m_{q,k}\left(\frac{m_{q,2}}{m_{g,2}}\right)^{k/2} - m_{g,k}}{s_{g,k}}. \quad (27)$$

Note the coefficient $(m_{q,2}/m_{g,2})^{k/2}$ after $m_{q,k}$ needed to normalize moments of quantized maps in the same manner of moments of not quantized maps.

Fig. 2 shows $\delta m_{q,g,k}$ for $3 \le k \le 8$ as a function of ϵ_q/σ. Black lines and black symbols denotes expectations, red lines and red symbols the $\pm 1\sigma$ variability from map to map. The continuous lines represents the range of variability of the equivalent $\delta m_{g,k}$ computed replacing $m_{q,k}$ with $m_{g,k}$ in Eq. (27). Of course in this case the expectation is zero and the 1σ rms variability is ± 1.

It is fairly evident as for odd moments the quantization does not introduce significant effects when compared to the sampling variance. Even if the figure seems to suggest a slight systematic positive bias, which is quite evident at $\epsilon_q/\sigma = 1$ and does not exceed $\approx 10\%$ of the sampling standard deviation.

A completely different picture emerges for even k. In that case $\delta m_{q,g,k} > 1$ occurs even at low ϵ_q/σ, especially for low k. It is also evident that at large ϵ_q/σ there is a drop in the deviation. This is due to the reduction of signal variance which occurs when the quantization is too large, so that the NMQE theory is no longer valid. To support this interpretation look at the $k = 2$ line in

Table 3: Deviations of expectations for even moments with for quantized maps. See Eq. (27) for the definition of $\delta m_{q,g,k}$ and Fig. 2. The case $k = 2$ is added just as a reference.

	$\delta m_{q,g,k}$			
	$\epsilon_q/\sigma = 0.05$	$\epsilon_q/\sigma = 0.1$	$\epsilon_q/\sigma = 0.5$	$\epsilon_q/\sigma = 1$
$k = 2$	1.4689324	5.9046804	148.10128	-0.43916934
$k = 4$	2.5737697	11.080605	308.75341	23.963025
$k = 6$	1.1250432	4.8399778	152.89782	21.971449
$k = 8$	0.55274363	2.2491733	87.840994	17.787774
$k = 10$	0.25960609	0.88828532	47.231529	12.893483

Tab. 3. If the NMQE theory would hold

$$\delta m_{q,g,2} = \left(\frac{\epsilon_q/\sigma}{s_{g,2}}\right)^2; \tag{28}$$

and having in our simulations $s_{g,2} \approx 2.6 \times 10^{-5}$ for $\epsilon_q/\sigma = 1$ we would have $\delta m_{q,g,2} = 3.9 \times 10^4$. The other cases are instead largely consistent with Eq. (28).

As discussed in Sect. 6.1 the measure of noise properties on the final data will include quantization since it is unlikely that white noise effects will be disentangled from quantization effects. So a better definition of Eq. (27) would be

$$\delta m_{q,k} = \frac{m_{q,k} - m_{g,k}}{s_{q,k}}. \tag{29}$$

The equivalent of Fig. 2 for this new measure of deviation from gaussianity is in Fig. 3. As exepected from Sect. 6.1, in the extent in which the quantization error is sufficiently small to assess validity of the NMQE theory, the effect of quantization on the newly defined deviation for the central moments is largely negligible.

7.5 χ^2 estimate of non gaussianity

An alternative approach to the analysis of single moments is to take some combination of them. A typical approach is to consider a χ^2 test of not gaussianity.

We defined a χ^2 estimator for a single unquantized map as follow:

$$\chi^2_i = \sum_{k>2} \left(\frac{\tilde{\mu}_{i,k} - m_{g,k}}{s_{g,k}}\right)^2 \tag{30}$$

where $m_{g,k}$ and $s_{g,k}$ are the expectation and standard deviation for moment $\tilde{\mu}_k$ estimated on the whole set of maps for the case of non gaussianity and without quantization. By using the estimators of moments for quantized maps, $\tilde{\mu}_{q,i,k}$, we defined two possible "quantized" χ^2

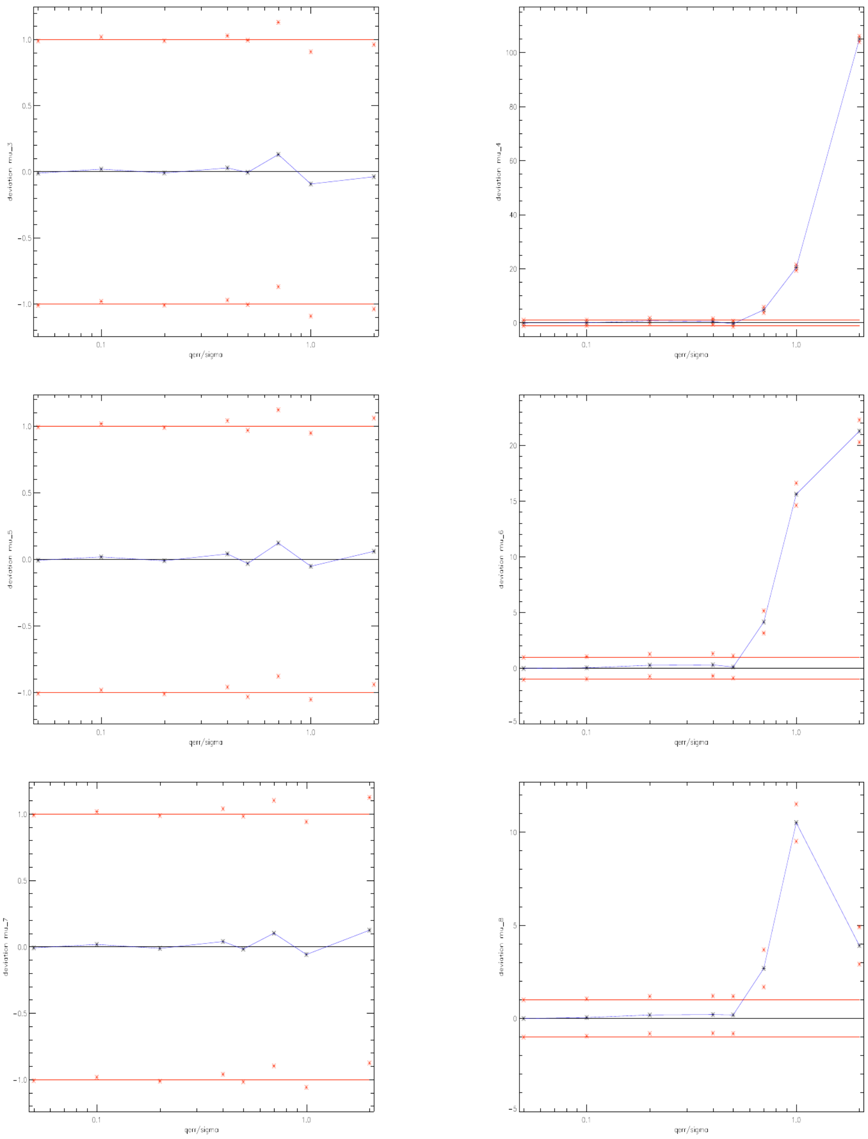

Figure 3: Variation of moments indexes (Eq. (29) for different ϵ_q/σ. Odd moments at left, even moments at right. From top to bottom and left to right $k = 3, 4, 5, 6, 7, 8$.

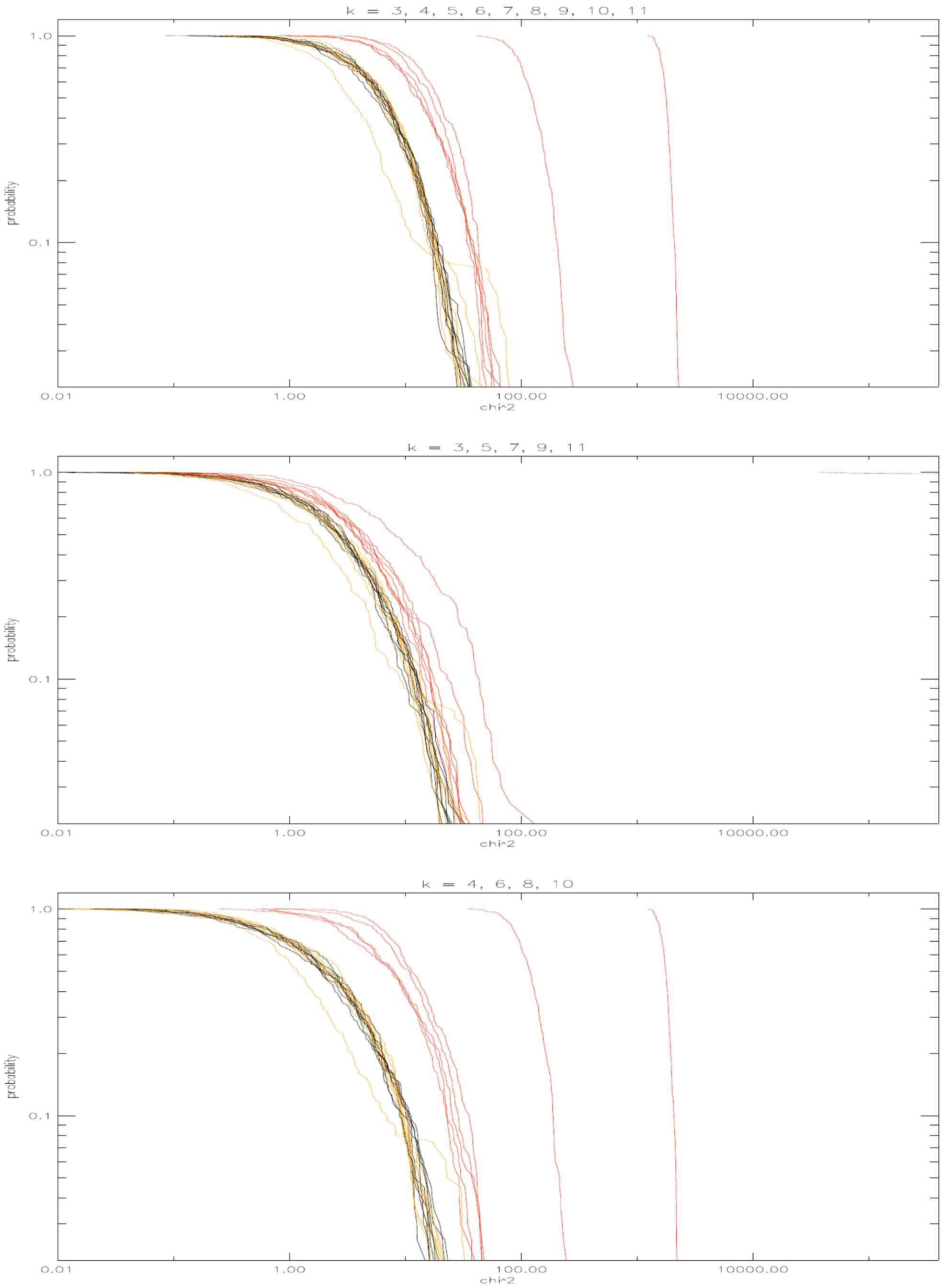

Figure 4: Cumulative distribution of χ_i^2 (black), $\chi_{q,0,i}^2$ (red), $\chi_{q,i}^2$ (yellow), for all the k (top), odd k (middle) and even k (bottom).

$$\tilde{\chi}^2_{q,0,i} = \sum_{k>2} \left(\frac{\tilde{\mu}_{q,i,k} \left(\frac{\mu_{q,i,2}}{\mu_{i,2}} \right)^{k/2} - m_{g,k}}{s_{g,k}} \right)^2 \quad (31)$$

$$\chi^2_{q,i} = \sum_{k>2} \left(\frac{\tilde{\mu}_{q,i,k} - m_k}{s_k} \right)^2 \quad (32)$$

while $\chi^2_{q,i}$ is a real analog of the χ^2_i for quantized data as it takes the quantization as a source of noise, $\tilde{\chi}^2_{q,0,i}$ represents a more sensitive estimator for the quantization effect on not gaussianities. The choice of which moments enters the above definitions is arbitrary.

Fig. 4 shows the cumulative distribution of χ^2_i (black), $\tilde{\chi}^2_{q,0,i}$ (red) and $\chi^2_{q,i}$ (yellow) for each set of 200 realizations and for the given set of q. The figures gives the probability that a random realization by chance has a χ^2 exceeding a given threeshold value. In the figures we considered a χ^2 defined by combining all the moments, or just the even or the odd ones.

The conclusion which can be drawn from these kind of plots are the following:

1. For $\epsilon_q/\sigma \le 0.5$ there is just a mild dependence of the quantization effect on q which becomes large as the break up limit for the NMQE theory is approached.

2. The sensitivity to the quantization increases increasing the order of k of the moment.

3. Moments for even k are more sensitive than moments for odd k. Indeed, most of the alteration of $\tilde{\chi}^2_{q,0,i}$ in the top frame is due to even moments while most of the alteration of $\tilde{\chi}^2_{q,0,i}$ in the bottom frame is due to the $k = 11$.

4. As in the previous sections, a self consistent definition of χ^2 including the quantization effects, remove the most part of the sensitivity to the quantization effect. As shown by the fact that χ^2_i (black), and $\chi^2_{q,i}$ (yellow) distributions largely overlaps.

8 Conclusions

To the extent in which the NMQE theory is valid, the quantization modifies the values of estimators of non gaussianity in a predictable way when compared to the case without quantization.

However, due to the central limit theorem, in the extent in which the common noise model for the quantization error (NMQE theory) is valid, the gaussian nature of quantized noise is largely maintained.

So when non gaussianity estimators which takes in account of quantization are considered, results similar to the unquantized case is obtained.

In conclusion at first order, for small quantization errors, large effects from quantization are not expected on the shape of noise pdf. If, with respect to non gaussianity tests, noise is gaussian, then even quantized noise will be for a large extent “gaussian” due to the central limit theorem.