7-2022

Relaxed Wythoff has all Beatty Solutions

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RELAXED WYTHOFF HAS ALL BEATTY SOLUTIONS

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Abstract. We find conditions under which the P-positions of three subtraction games arise as pairs of complementary Beatty sequences. The first game is due to Fraenkel and the second is an extension of the first game to non-monotone settings. We show that the P-positions of the second game can be inferred from the recurrence of Fraenkel's paper if a certain inequality is satisfied. This inequality is shown to be necessary if the P-positions are known to be pairs of complementary Beatty sequences, and the family of irrationals for which this inequality holds is explicitly given. We highlight several games in the literature that have P-positions as pairs of complementary Beatty sequences with slope in this family. The third game we present is novel, and we show that the P-positions can be inferred from the same recurrence in any setting. It is shown that any pair of complementary Beatty sequences arises as the P-positions of some game in this family. We also provide background on some inverse problems which have appeared in the field over the last several years, in particular the Duchêne-Rigo conjecture. This paper presents a solution to the Fraenkel problem posed at the 2011 BIRS workshop, a modification of the Duchêne-Rigo conjecture.

1. Introduction

Wythoff's game starts with two piles, labeled A and B, of finitely many tokens. Two players alternate, and at each turn the current player may either

(a) remove any positive number of tokens from a single pile, possibly the entire pile,
(b) or remove an equal amount of tokens from piles A and B simultaneously.

Under normal gameplay[1], the player who leaves both piles empty wins. The game is itself a modification of Nim (under normal gameplay). In Nim, there may be more than two piles, and players may make only the first move (a) listed above. It is for this reason that the moves of type (a) are sometimes called *Nim* moves. Moves of type (b) are sometimes called *diagonal* moves or *bishop* moves, as indicated by the representation of a game position by a piece on a chessboard [2] [14].

If piles A and B contain x and y tokens, respectively, then label the game position as (x, y). It is customary to make the assumption with no loss of generality that $x \leq y$. Label a position (x, y) as an N-position if any player moving *from* (x, y) has a winning strategy. Otherwise, label (x, y) as a P-position. Let $\mathcal{N}$ denote the set of N-positions and let $\mathcal{P}$ denote the set of P-positions. It is well known that the set of all game positions is partitioned by $\mathcal{P}$ and $\mathcal{N}$ because Wythoff's game is an acyclic combinatorial game with no ties [21].

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Date: April 13, 2023.

[1]In the literature, one also encounters misère gameplay under which the player who leaves both piles empty loses. Misère play is the typical setting of Nim, which we do not adopt here.

Wythoff computed the P-positions of his game as the set

$$\left\{\left([n\phi], [n\phi^2]\right) \mid n \in \mathbb{Z}_{\geq 0}\right\} \tag{1}$$

where $[\cdot]$ denotes the floor function and $\phi = \left(1+\sqrt{5}\right)/2$. Projecting onto the components of these pairs yields the sequences of integers $\{[n\phi]\}$ and $\{[n\phi^2]\}$ called the Beatty sequences of slope ϕ and ϕ^2, respectively. It is no coincidence that $1/\phi + 1/\phi^2 = 1$, for Lord Rayleigh discovered in 1894 this condition precisely characterizes when the sequences $\{[n\phi]\}$ and $\{[n\phi^2]\}$ exactly cover the positive integers [22], a property which is intimately connected to the study of two-player subtraction games. Beatty sequences have a rich literature that touches upon many topics in number theory and combinatorics (see for instance [1], [20], and the references therein). Formally they can be defined as

Definition 1 (Homogeneous Beatty Sequence). *Let $\alpha > 0$ be irrational. The sequence*

$$\{a_n\}_{n=0}^{\infty} = \{[n\alpha]\}_{n=0}^{\infty} \tag{2}$$

is called the (homogeneous) Beatty sequence *with slope α. In this paper, we are concerned only with homogeneous Beatty sequences, so the first adjective will be omitted throughout.*

It is for the next theorem, mentioned in the previous paragraph, that we may refer to a pair of *complementary* Beatty sequences:

Theorem 1 (Rayleigh-Beatty). *Let $\alpha < \beta$ be positive irrationals and define*

$$A = \{[n\alpha]\}_{n=0}^{\infty} \quad B = \{[n\beta]\}_{n=0}^{\infty}. \tag{3}$$

Then $A \cup B = \mathbb{Z}_{\geq 0}$ and $A \cap B = \{0\}$ if and only if $1/\alpha + 1/\beta = 1$. In this case, A and B are said to be the complementary Beatty sequencess parameterized by α or β.

In [5] and [10], a countable family of Wythoff games is defined by adjoining valid moves. For a fixed positive integer t, the game t-Wythoff is played with standard Nim moves and an extended diagonal move in which a player who removes k and ℓ tokens from both piles may do so if and only if $|k-\ell| < t$. The P-positions of games in this family are given by the complementary Beatty sequences parameterized by α where

$$\alpha = \frac{2-t+\sqrt{t^2+4}}{2}. \tag{4}$$

Note that $\alpha = \phi$ is the special case of $t = 1$, which corresponds to the original Wythoff's game with golden P-positions. This family of irrationals will appear repeatedly and its significance lies in the following.

Lemma 1. *If $\alpha = (2-t+\sqrt{t^2+4})/2$ for some integer $t > 0$ and β satisfies $1/\alpha + 1/\beta = 1$, then $\beta = \alpha + t$ and, in particular, $\{\alpha\} = \{\beta\}$.*

In the history of combinatorial games, it has always been attractive to study the P-positions of a given game. This is the forward problem. Lately, it has become attractive to study the *inverse* problem: where one starts with pairs from a given complementary sequence or family thereof and finds one or more games which have pairs from those sequences as P-positions. This has been discussed in [6], [8], [9], [11], [12], [13], and [18]. A particular setting of this inverse problem was formalized by Duchêne and Rigo in their conjecture, stated in [3].

Conjecture 1 (Duchêne-Rigo). *Given a pair of complementary Beatty sequences $S = (A_n, B_n)_{n\geq 1}$, there exists an invariant game having $S \cup (0,0)$ as its set of P-positions.*

This conjecture was proven by Larsson, Hegarty, and Fraenkel in [17]. Though the existence is indeed satisfactory for the resolution of the conjecture, it was noted in [4], [12], [13], and [18] that the general solution is complicated for an ordinary person to play.

For a basic explanation: start with desired complementary sequences; the P-positions of an auxiliary game G^* must first be determined to obtain the set of legal moves for a second game $(\mathrm{G}^*)^*$. Only does this second game have P-positions which are the desired complementary sequences. This means a player solving the inverse problem accumulates additional information to obtain their desired P-positions. Figure 1 depicts valid moves for the second game when the known complementary sequences are the complementary Beatty sequences with parameters $\alpha = 5/4+\pi/100$ and β satisfying $1/\alpha+1/\beta = 1$. The complicated structure in the figure indicates the player, in fact, accumulates much additional information.

As such, an inverse problem without the requirement of invariance and with the additional requirement of having "nice" rules has been formulated since then. Indeed, it was posed by Fraenkel at the 2011 BIRS workshop in combinatorial games

> "Find nice rules for a 2-player combinatorial game for which the P-positions are obtained from a pair of complementary Beatty sequences," [4]

In this paper, we are concerned with Fraenkel's inverse problem, which has led to the games of [4], [11], [18] and others.

The main results of this paper are two-fold. First, an extension of the game family of [6] is presented and its Beatty P-positions are explicated, from which we recover an *uncountable* family of complementary Beatty sequences. Inspired by the derivation of the previous result, a novel game family is introduced with simple play rules for which *any* complementary Beatty sequences are the P-positions of some game in the family, solving the inverse problem posed by Fraenkel.

This paper is structured as follows. First a study of Definition 2 is made, determining necessary and sufficient condition under which games in this family have P-positions which

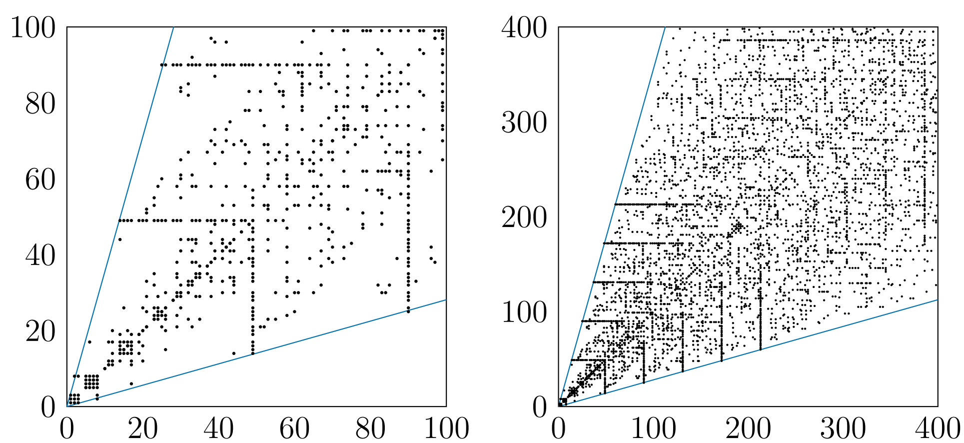

FIGURE 1. Legal moves in $(\mathrm{G}^*)^*$ depicted as ordered pairs. Lines with slopes $1/(\beta-1)$ and $\beta-1$. Adapted with an alternate parameter from "Wythoff's Star" so-named in [16].

are complementary Beatty sequences. Games in the family of Definition 5 are introduced and Theorem 3 indicates a sufficient criterion for when one of these games has P-positions arising in terms of a distinguished recurrence formula. Theorem 4 provides necessary and sufficient conditions so that a game in this family has P-positions which are complementary Beatty sequences. Definition 6 is presented and the P-positions of games in this family are computed by Theorem 5. Corollary 1 states that any desired pair of complementary Beatty sequences arises as the P-positions of some game in the family of Definition 6.

2. General 2-pile subtraction games

In [6], an infinite family of Wythoff's game variations is defined, again altering only the diagonal move. A game in this family is parameterized by a suitable function f of the present and next game positions in the following sense: a player who removes k and ℓ tokens from both piles may do so if and only if $|k-\ell|$ is bounded by f (made precise in the following definition). Setting $f \equiv 1$ or $f \equiv t$ for some integer $t > 1$ retrieve the special cases of the original Wythoff's game and t-Wythoff, respectively. Precisely, the family is introduced as

Definition 2 (General 2-pile subtraction games)**.** *Given two piles of tokens (x, y) of sizes x, y, with $0 \leq x \leq y < \infty$. Two players alternate removing tokens from the piles:*

- *(aa) Remove any positive number of tokens from a single pile, possibly the entire pile.*
- *(bb) Remove a positive number of tokens from each pile by sending (x_0, y_0) to (x_1, y_1) where*

$$|(y_0 - y_1) - (x_0 - x_1)| < f(x_1, y_1, x_0), \tag{5}$$

where the constraint function $f(x_1, y_1, x_0)$ is integer-valued, positive, monotone, and semi-additive.

The following theorem of the same paper determines the P-positions of any game defined above.

Theorem 2 (Fraenkel)**.** *Let $\mathcal{S} = \{(a_n, b_n)\}_{n=0}^{\infty}$ where $a_0 = b_0 = 0$ and for all $n \in \mathbb{Z}_{>0}$,*

$$a_n = \operatorname{mex}\{a_k, b_k \mid k < n\} \tag{6}$$

$$b_n = f(a_{n-1}, b_{n-1}, a_n) + b_{n-1} + a_n - a_{n-1}. \tag{7}$$

If f is positive, monotone, and semi-additive, then $\mathcal{S}$ is the set of P-positions of a general 2-pile subtraction game with constraint function f.

One can see (proven in Lemma 4) that the function defined by

$$f(x_1, y_1, [n\alpha]) = ([n\beta] - y_1) - ([n\alpha] - x_1) \tag{8}$$

has P-positions which are pairs of the complementary Beatty sequences parameterized by α. However, this function is many-valued because, in particular, it depends on the position moved to as well as from. The next section gives a way to tell how many-valued the function is.

3. Variance, Invariance, and k-Invariance

There exists a classification of subtraction games according how a player's allowed moves depend on the game state. In an *invariant* subtraction game, the move $(x, y) \to (x-k, y-\ell)$ is allowed if and only if for all positions (x_0, y_0), the move $(x_0, y_0) \to (x_0-k, y_0-\ell)$ is allowed (provided the pair is non-negative in both components.) Subtraction games not satisfying this property are called *variant*. The following definition introduced in [11] is weaker than invariance:

Definition 3 (k-Invariance)**.** *A subtraction game is called k-invariant if its set of positions can be partitioned into k disjoint subsets such that, within a subset, each allowed move is invariant in that subset.*

The game played with the function in Equation (8) is not k-invariant for any $k \in \mathbb{Z}_{>0}$, because the number of distinct values of f increases unboundedly. If the P-positions $\{(a_n, b_n)\}$ are known and the conclusion of Theorem 2 holds, one can see that a minimal requirement for such a function f is that

$$f(a_{n-1}, b_{n-1}, a_n) = (b_n - a_n) - (b_{n-1} - a_{n-1}). \tag{9}$$

As was observed in Example 3 of [11], this expression achieves at most three values when sequences $\{a_n\}$ and $\{b_n\}$ are complementary Beatty sequences. Therefore, if a constraint function depends only of the position moved *from*, then there is a chance to bound the number of distinct values of f. For example, consider the following family satisfying the minimal requirement:

Definition 4 (Beatty constraint function)**.** *Let $1 < \alpha < 2$ be irrational. If β satisfies $1/\alpha + 1/\beta = 1$, then*

$$f([n\alpha]) = f([n\beta]) = ([n\beta] - [(n-1)\beta]) - ([n\alpha] - [(n-1)\alpha]). \tag{10}$$

is said to be the Beatty constraint function parameterized by α.

Since this expression achieves at most three values, we can deduce that a Beatty constraint function induces a game which is k-invariant for some $k \leq 3$. For this reason, games played with these functions are the object of central study in this paper. The next lemma tells us when a Beatty constraint function may be used to play a game of Definition 2 by checking that it satisfies the hypotheses required by that definition. When such a function may be used, the P-positions are known to be the complementary Beatty sequences parameterized by α, according to Theorem 2.

Lemma 2. *Let irrationals $1 < \alpha < 2 < \beta$ satisfy $1/\alpha + 1/\beta = 1$. Define the sequence*

$$\Delta^2_{n-1} = ([n\beta] - [(n-1)\beta]) - ([n\alpha] - [(n-1)\alpha]). \tag{11}$$

The following are equivalent:

(i) Δ^2 is monotone
(ii) Δ^2 is constant and equals $[\beta] - 1$
(iii) $\alpha = (2-t+\sqrt{t^2+4})/2$ where $t = [\beta] - 1$.

If one of the above is false, then the range of Δ^2 lies within a subset of $\{[\beta]-2, [\beta]-1, [\beta]\}$.

Proof. The implication (ii) $\implies$ (i) is trivial, so let us prove (i) $\implies$ (ii). Afterwards, we prove (ii) $\iff$ (iii). Suppose the function is monotone. Applying the decomposition $\beta = [\beta] + \{\beta\}$, we can rewrite

$$
\begin{aligned}
[n\beta] &= [n([\beta] + \{\beta\})] && (12)\\
&= [n[\beta] + n\{\beta\}] && (13)\\
&= n[\beta] + [n\{\beta\}]. && (14)
\end{aligned}
$$

The first difference in parentheses in Equation (11) becomes

$$
\begin{aligned}
[n\beta] - [(n-1)\beta] &= n[\beta] + [n\{\beta\}] - ((n-1)[\beta] + [(n-1)\{\beta\}]) && (15)\\
&= [\beta] + g_{1/\{\beta\}}(n-1). && (16)
\end{aligned}
$$

where $g_{1/\rho}(n) = [(n+1)\rho] - [n\rho]$ for real ρ. After deriving a similar expression for the second difference in parentheses in Equation (11) involving α we have,

$$\Delta^2 = [\beta] - [\alpha] + g_{1/\{\beta\}}(n-1) - g_{1/\{\alpha\}}(n-1) = [\beta] - 1 + d_\alpha(n-1), \tag{17}$$

where the term

$$d_\alpha(n) = g_{1/\{\beta\}}(n) - g_{1/\{\alpha\}}(n) \tag{18}$$

is the sole part of the expression which may vary. To determine precisely when d_α is monotone, start by defining the sets $X = \{[n/\{\alpha\}]\}_{n\geq 0}$ and $Y = \{[n/\{\beta\}]\}_{n\geq 0}$. From Chapter 9 of [1], recall that

$$g_{1/\{\alpha\}}(n) = \begin{cases} 1 & n \in X \\ 0 & n \notin X \end{cases} \qquad g_{1/\{\beta\}}(n) = \begin{cases} 1 & n \in Y \\ 0 & n \notin Y. \end{cases} \tag{19}$$

Studying the possible cases, one obtains that d_α and Δ^2 achieve at most three of the following values:

$$d_\alpha(n) = \begin{cases} 0 & n \in (X\cap Y)\cup(X^c \cap Y^c) \\ 1 & n \in X^c \cap Y \\ -1 & n \in X \cap Y^c. \end{cases} \implies \Delta_n^2 = \begin{cases} [\beta] - 1 \\ [\beta] \\ [\beta] - 2. \end{cases} \tag{20}$$

Theorem 3.11 of [19] states that if an intersection of Beatty sequences is non-empty and non-zero, then the intersection in fact has infinite size. This means if d_α achieves any value, then it achieves that value infinitely many times, indicating the function d_α oscillates if it achieves more than one value. Therefore, if d_α is monotone, then it must achieve only one value. There are now three possibilities to examine, and we rule out the second two: that $d_\alpha \equiv 1$ or $d_\alpha \equiv -1$.

If $d_\alpha \equiv 1$, then $X^c \cap Y = \mathbb{Z}_{>0}$, so that $X = \emptyset$ and $Y = \mathbb{Z}_{\geq 0}$, a contradiction. A symmetric argument can be applied to eliminate the case $d_\alpha \equiv -1$, leaving possible only the remaining case $d_\alpha \equiv 0$, from which it follows that $\Delta_{n-1}^2 \equiv [\beta] - 1$.

To finish, a formula for α is derived, also providing a converse to Lemma 1. The argument in the previous paragraph shows $(X \cap Y) \cup (X^c \cap Y^c) = \mathbb{Z}_{\geq 0}$. Since both X and Y are non-empty, this implies $X = Y$. Proceed by elementary algebra. Let $t = [\beta] - 1$. Because $\{\alpha\} = \alpha - [\alpha]$ and $\{\beta\} = \beta - [\beta]$, it follows that

$$\alpha - [\alpha] = \beta - [\beta], \tag{21}$$

Substitute the values $\beta = \alpha/(\alpha - 1)$, $[\alpha] = 1$, and $[\beta] = t + 1$ to obtain the equation

$$\alpha - 1 = \frac{\alpha}{\alpha - 1} - (t + 1) \tag{22}$$

which can be re-arranged into a quadratic equation. The positive solution from the quadratic formula equals

$$\alpha = \frac{2 - t + \sqrt{t^2 + 4}}{2}, \tag{23}$$

completing the proof. $\square$

Remark 1. *The previous proof indicates that $\Delta^2_{n-1} \geq 0$, for all n because*

$$\Delta^2 \geq [\beta] - 2 \geq 2 - 2 = 0. \tag{24}$$

The preceding lemma and its proof constitute a refinement of Proposition 1 in [11], which studies the same difference, by additionally finding the pre-image of each value $[\beta]-2$,$[\beta]-1$, and $[\beta]$. The paper [18] also makes heavy study of this difference in its Section 3. The lemma (and the preceding remark) will be used in the next section also, particularly in the proof of Theorem 4. The preceding lemma also showed that a game played with a Beatty constraint function f is either invariant, 2-invariant, or 3-invariant. In the latter cases, the game positions decompose into the disjoint union

$$\{(a_n, y), (b_n, y) \mid n - 1 \in (X \cap Y) \cup (X^c \cap Y^c)\} \cup \tag{25}$$

$$\{(a_n, y), (b_n, y) \mid n - 1 \in X^c \cap Y\} \cup \tag{26}$$

$$\{(a_n, y), (b_n, y) \mid n - 1 \in X \cap Y^c\}. \tag{27}$$

where if the game is 2-invariant, one of the preceding sets is empty. In the next section, we expand Theorem 2 to be compatible with non-monotone constraint functions. That some *non-monotone* boolean constraint function enjoys the conclusion of Theorem 2 was observed in Section 4 of [7]. That the same conclusion is enjoyed by certain Beatty constraint functions is what Section 5 is devoted to.

4. Modified 2-pile subtraction games

By the preceding discussion, it is of interest to consider a game family including constraint functions lacking monotonicity. Continuing to require $f \geq 1$ is mandated by a complementary setting, for if f is vanishing, then certain Wythoff moves are blocked entailing that P-positions may not be complementary. This can be found in Table 6[2] of [6], where a vanishing constraint function has the P-position $(1, 1)$. By the proof of Lemma 3, a game of Definition 5 will have the P-position (a_n, a_n) if and only if $f(a_k, b_k, a_n) = 0$ for all $k < n$. If the constraint function depends only on the position moved from, the latter condition is simply $f(a_n) = 0$.

The main result of this section is a classification of the P-positions of games in Definition 5 which are pairs of complementary Beatty sequences. To remove the built-in compatibility of Definition 2 with Theorem 2, we will work with the following game family, particularly with functions in the family of Definition 4.

[2]This example is presented by its author as a counterexample, but it appears to instead be a genuine example. The main lemma of this section can be used to prove this.

Definition 5 (Modified 2-pile subtraction games). *Given two piles of tokens (x, y) of sizes x, y, with $0 \le x \le y < \infty$. Two players alternate, removing tokens from the piles as follows*

(aa) Remove any positive number of tokens from a single pile, possibly the entire pile.

(bb) Remove a positive number of tokens from each pile by moving from (x_0, y_0) to (x_1, y_1) where

$$|(y_0 - y_1) - (x_0 - x_1)| < f(x_1, y_1, x_0), \tag{28}$$

where the constraint function f is integer-valued.

Remark 2. *Because we are interested in k-invariant games (as discussed in the previous section), constraint functions which depend only on the position moved* from *are prioritized. For this reason, it is understood (unless otherwise specified) that a constraint function f in this family satisfies $f(x_1, y_1, x_0) = f(x_0)$. Setting the next lemma within this definition is convenient because it permits two motivating examples at the end of the section.*

Remark 3. *For the sake of consistent equation indexing, the following recurrence starting with $(a_0, b_0) = (0, 0)$ and proceeding by*

$$\begin{cases} a_n = \operatorname{mex}\{a_k, b_k \mid k < n\} \\ b_n = f(a_{n-1}, b_{n-1}, a_n) + b_{n-1} + a_n - a_{n-1} \end{cases} \quad \textit{for all } n > 0 \tag{29}$$

is the recurrence of Equation (7) adapted to the games of Definition 5.

By playing a game in Definition 5 with the Beatty constraint function parameterized by $\alpha = 1 + \sqrt{5}/5$, one can see the P-positions do not coincide with pairs of the recurrence in Equation (29) (see Figure 2 for a list of P-positions and recurrence pairs written as Beatty pairs). One could have decided that the failure of Theorem 2 to apply is because the constraint function was non-monotone, but this assumption is not necessary (except that it was required by Definition 2). Indeed, consider the non-monotone Beatty constraint function with parameter $\alpha = 1 + (2\sqrt{19} - 8)/2$. Playing a game with this constraint function, the P-positions can be verified to coincide with pairs of the recurrence in Equation (29). See Figure 3 for several of the first P-positions.

In the next lemma, we solve the forward problem of finding the P-positions of a particular game in the family of Definition 5, allowing us to designate our specific *inverse problem* in the next section.

Now that f has been assumed to be merely integer-valued, the next lemma is a generalization of Theorem 2. Algorithms like the one exposed in the statement and proof of this lemma are know as Minimum Excluded algorithms and variations of this proof strategy have

n	0	1	2	3	4	5	6	7	8	9
a_n	0	1	2	4	7	8	9	10	11	12
b_n	0	3	6	5	13	16	19	15	23	26
$[n\alpha]$	0	1	2	4	5	7	8	10	11	13
$[n\beta]$	0	3	6	9	12	16	19	22	25	29

FIGURE 2. Top: P-positions of the game played in Definition 5 with the constraint function in Equation (10) parameterized by $\alpha = 1 + \sqrt{5}/5$. Bottom: Complementary Beatty sequences parameterized by α. Disagreement at $n = 3$.

n	0	1	2	3	4	5	6	7	8	9
a_n	0	1	2	4	5	6	8	9	10	12
b_n	0	3	7	11	15	18	22	26	30	34
$[n\alpha]$	0	1	2	4	5	6	8	9	10	12
$[n\beta]$	0	3	7	11	15	18	22	26	30	34

FIGURE 3. Top: P-positions of the game played in Definition 5 with the constraint function in Equation (10) parameterized by $\alpha = 1 + (2\sqrt{19} - 8)/2$. Bottom: Complementary Beatty sequences parameterized by α.

their origins in Wythoff's original paper [2] [23]. Contemporary references include [3], [5], [6], [14], and many others. Later on, we adapt the lemma to analyze the game in Definition 6, exposing its P-positions in Theorem 5.

The statement of this lemma shows how to compute the set of P-positions. The proof shows that a player in a P-position may never move to a lower P-position, and that a player in an N-position may always move to a P-position, so that we may apply the P- and N-position structure theorem.

Lemma 3. *Let $\mathcal{S} = \{(a_n, b_n)\}_{n=0}^{\infty}$ where $a_0 = b_0 = 0$ and for all $n \in \mathbb{Z}_{>0}$*

$$a_n = \operatorname{mex}\{a_k, b_k \mid k < n\} \tag{30}$$

$$b_n = \min\{b \geq a_n \mid b \neq b_k \text{ and } |(b - b_k) - (a_n - a_k)| \geq f(a_k, b_k, a_n) \forall k < n\}. \tag{31}$$

Then $\mathcal{S} = \{(a_n, b_n)\}_{n=0}^{\infty}$ is the set of P-positions of a game in Definition 5 with constraint function f.

Remark 4. *The minimum above can be rewritten as the mex:*

$$b_n = \operatorname*{mex}_{\geq a_n} \left(\{b_k\}_{0<k<n} \cup \{b \geq a_n \mid |(b - b_k) - (a_n - a_k)| < f(a_k, b_k, a_n)\}_{k<n}\right). \tag{32}$$

Proof. In the setting of a subtraction game ending at $(0,0)$ the P-positions can be generated inductively, starting with $(0,0)$. Label the first non-zero P-position (a_1, b_1). The component $a_1 = \operatorname{mex}\{0\}$ is indicated because $(0, b)$ can be moved to $(0,0)$ for any $b > 0$. The component b_1 is found by taking the minimum b for which (a_1, b) has no move to lower P-positions:

$$b_1 = \min\{b \geq 1 \mid |(b - 0) - (1 - 0)| \geq f(0, 0, 1)\} \tag{33}$$

$$= 1 + f(0, 0, 1). \tag{34}$$

At the nth step, the position (a_n, b_n) is computed similarly by taking a mex for the first component

$$a_n = \operatorname{mex}\{a_k, b_k \mid k < n\}. \tag{35}$$

If a_n were not equal to this mex, then $a_n = a_k$ or $a_n = b_k$ for some $k < n$. In the first case, $(a_n, b_n) = (a_k, b_n)$. If $b_n > b_k$, then moving to (a_k, b_k) is possible by a Nim move, a contradiction. Otherwise, if $b_n < b_k$, then the P-position (a_k, b_k) has a move to the P-position (a_n, b_n) via a Nim move, which is also a contradiction. Finally, if $a_n = b_k$, then the P-position $(a_n, b_n) = (b_k, b_n)$ has a move to (b_k, a_k) which is relabelled as the P-position (a_k, b_k), also a contradiction.

For the second component, now that $n > 1$, we need to add a quantifier and restrict $b \neq b_k$:

$$b_n = \min\{b \geq a_n \mid b \neq b_k \text{ and } |(b - b_k) - (a_n - a_k)| \geq f(a_k, b_k, a_n) \forall k < n\}. \tag{36}$$

Note that we can be sure $b_k \geq a_k$ because that is the convention we adopted for denoting game positions. Moreover, $b_n = a_n$ if and only if $f(a_k, b_k, a_n) = 0$ for all $k < n$.

The above paragraph indicates how to compute the set $\mathcal{S}$. Now we show that if a player is in any game position in the complement of $\mathcal{S}$, then they can always move directly to a game position in $\mathcal{S}$. Once this is proven, the structure theorem of P- and N-positions implies that $\mathcal{S}$ is the set of P-positions.

It suffices to consider the following game positions, because the sequence $\{a_n\}$ consists of minimum excludants:

$$(a_n, a_m) \quad (a_n, b_m) \quad (b_n, a_m) \quad (b_n, b_m) \tag{37}$$

for distinct m and n. If $f(a_k, b_k, a_n) = 0$ for all $k < n$, then $b_n = a_n$ and only the first two cases need to be inspected. For the first position, note that $a_m > a_n = b_n$, so that moving to (a_n, b_n) is possible. For the second position, note that $b_m > a_n = b_n$, so a player can move to (a_n, b_n).

Otherwise, suppose $f(a_k, b_k, a_n) \neq 0$ for some $k < n$. To handle the first case, suppose $a_m > b_n$. Then a player may move to (a_n, b_n). Otherwise we have $a_m < b_n$. Interpreting b_n as the minimizer

$$b_n = \min\{b \geq a_n \mid b \neq b_k \text{ and } |(b - b_k) - (a_n - a_k)| \geq f(a_k, b_k, a_n) \forall k < n\}, \tag{38}$$

one can see that there exists $k < n$ such that

$$|(a_m - b_k) - (a_n - a_k)| < f(a_k, b_k, a_n) \tag{39}$$

which indicates a player may make the diagonal move to (a_k, b_k). For the second case, if $b_m > b_n$, then a player may make a Nim move to (a_n, b_n). If $b_m < b_n$, then interpreting b_n as a minimizer again reveals that a player may make a diagonal move from (a_n, b_m) to some (a_k, b_k).

For the third case, if $n < m$ then a player may make a Nim move to (b_n, a_n) which is relabeled as (a_n, b_n). Otherwise we shall have $m < n$, thus obtaining the strict inequality

$$a_m < a_n \leq b_n \leq a_m, \tag{40}$$

which is a contradiction, because the case $a_n = b_n$ was already addressed. Finally, suppose a player is in the fourth position-type above. If $b_m > a_n$, then a player can make a Nim move to (b_n, a_n). Otherwise, $b_m < a_n$ implies the strict inequality

$$b_n \leq b_m < a_n \leq b_n, \tag{41}$$

but we already handled the case $a_n = b_n$.

Therefore, $\mathcal{S}$ is the set of P-positions for a game played with the constraint function f. $\square$

The next two examples show how the previous lemma can be used to deduce P-positions. First, let us prove the observation surrounding Equation (8).

Lemma 4. *If a game in Definition 5 is played with the constraint function*

$$f(x_1, y_1, [n\alpha]) = ([n\beta] - y_1) - ([n\alpha] - x_1) \tag{42}$$

then the P-positions are pairs of the complementary Beatty sequences parameterized by α.

Proof. By induction, suppose $(a_k, b_k) = ([k\alpha], [k\beta])$ for all $k < n-1$. This is true in the base case $(0,0)$. At the nth step, the P-position (a_n, b_n) is computed by Lemma 3 as follows:

$$a_n = \operatorname{mex}\{a_k, b_k \mid k < n\} \tag{43}$$

$$b_n = \operatorname{mex}(\{b_k\} \cup [a_n + b_k - a_k - f(a_k, b_k, a_n) + 1, a_n + b_k - a_k + f(a_k, b_k, a_n) - 1]). \tag{44}$$

The upper endpoint of any of the intervals in Equation (44) can be simplified as

$$a_n + b_k - a_k + f(a_k, b_k, a_n) - 1 = a_n + b_k - a_k + [n\beta] - a_n - (b_k - a_k) - 1 \tag{45}$$

$$= [n\beta] - 1. \tag{46}$$

The lower endpoint the intervals can be simplified as

$$a_n + b_k - a_k - f(a_k, b_k, a_n) + 1 = a_n + b_k - a_k - [n\beta] + a_n + (b_k - a_k) + 1 \tag{47}$$

$$= 2a_n - [n\beta] + 2(b_k - a_k) + 1. \tag{48}$$

This indicates the intervals expand. After taking the entire union, we are left evaluating $\operatorname{mex}_{\geq a_n}([2a_n - [n\beta] + 1, [n\beta] - 1])$. Since $[n\beta] > a_n$, we can see that $2a_n - [n\beta] < a_n$, and so $2a_n - [n\beta] + 1 \leq a_n$, which indicates the mex equals $[n\beta]$. □

Let us also prove that the third counterexample in Proposition 2 of [6] is a actually a genuine example. The constraint function in question is

$$f(x_1, y_1, x_0) = (1 + (-1)^{y_1+1})x_1/2. \tag{49}$$

Table 6 of the same paper contains several $\mathcal{S}$-positions according to the recurrence formula of Equation (7). The author claims these are not P-positions, because "$(10, 29)$ cannot be moved to any of the lower $\mathcal{S}$-positions", suggesting that the pair $(10, 31)$ from the table is not a P-position, because $(10, 29)$ ought to be. However, a closer look shows that moving from $(10, 29)$ to the $\mathcal{S}$-position $(8, 21)$ *is* allowed, because $|(29-21)-(10-8)| = 6 < f(8, 21, 10) = (1 + (-1)^{22})8/2 = 8$. The remaining P-positions can be computed using Lemma 3, showing the same recurrence formula holds.

Lemma 5. *If a game in Definition 5 is played with the constraint function in Equation* (49)*, the set of $\mathcal{S}$-positions starting with $(a_0, b_0) = (0, 0)$ and $(a_1, b_1) = (1, 1)$, followed by*

$$a_n = \operatorname{mex}\{a_k, b_k \mid k < n\} \tag{50}$$

$$b_n = a_n + b_{n-1} - a_{n-1} + f(a_{n-1}, b_{n-1}, a_n) \tag{51}$$

$$= \begin{cases} a_n + b_{n-1} & b_{n-1} \text{ odd} \\ a_n + b_{n-1} - a_{n-1} & b_{n-1} \text{ even.} \end{cases} \tag{52}$$

for all $n \geq 2$ are P-positions.

Proof. Base cases at $n = 0, 1$ can be verified by hand. For an induction step, the next P-position is found by applying Lemma 3

$$a_n = \operatorname{mex}\{a_k, b_k \mid k < n\} \tag{53}$$

$$b_n = \operatorname{mex}(\{b_k\} \cup [a_n + b_k - a_k - f(a_k, b_k) + 1, a_n + b_k - a_k + f(a_k, b_k) - 1]_{k<n}). \tag{54}$$

Note that the intervals

$$[a_{n-1} + b_k - a_k - f(a_k, b_k) + 1, a_{n-1} + b_k - a_k + f(a_k, b_k) - 1]_{k<n-1} \tag{55}$$

cover the set $\{a_{n-1}, a_{n-1}+1, \ldots, b_{n-1}-1\}$ by the inductive hypothesis. Therefore it can be seen that the shifted intervals

$$[a_n + b_k - a_k - f(a_k, b_k) + 1, a_n + b_k - a_k + f(a_k, b_k) - 1]_{k<n-1} \tag{56}$$

cover the set $\{a_n, a_n + 1, \ldots, b_{n-1} - 1 + (a_n - a_{n-1})\}$.

If b_{n-1} is even, then $f(a_{n-1}, b_{n-1}) = 0$ and the last interval of Equation (54) is empty, so we simply take $b_n = a_n + b_{n-1} - a_{n-1}$. If b_{n-1} is odd, then $f(a_{n-1}, b_{n-1}) = a_{n-1}$ and the last interval considered is

$$[a_n + b_{n-1} - 2a_{n-1} + 1, a_n + b_{n-1} - 1]. \tag{57}$$

There are no integers in-between the set $\{a_n, a_n + 1, \ldots, b_{n-1} - 1 + (a_n - a_{n-1})\}$ and this interval, because we can see that

$$a_n + b_{n-1} - 2a_{n-1} + 1 \leq b_{n-1} - 1 + (a_n - a_{n-1}) + 1 \iff \tag{58}$$

$$1 \leq a_{n-1} \tag{59}$$

which holds for all $n \geq 2$. Therefore, we can determine that $b_n = a_n + b_{n-1}$. □

Now that we have seen two examples of how Lemma 3 can be used to determine a game's P-positions, let us move on to the main result characterizing when the P-positions in games of Definition 5 are pairs of complementary Beatty sequences.

5. The Inverse Problem

In this section, we pose our inverse problem: if the P-positions of a game in Definition 5 are pairs of complementary Beatty sequences, what is the constraint function of that game? Is it the Beatty constraint function? From which functions arise games whose P-positions are pairs of complementary Beatty sequences? These three questions are answered in this section, per the proviso in Remark 2.

The next theorem refines the minimum for a class of constraint functions which satisfy a certain inequality.

Theorem 3. *Let $f \geq 1$ be a function of integers into integers and let $n > 1$ be fixed. Suppose a game in Definition 5 has P-positions (a_k, b_k) which satisfy*

$$a_k = \text{mex}\{a_j, b_j \mid j < k\} \tag{60}$$
$$b_k = f(a_k) + b_{k-1} + a_k - a_{k-1}$$

for all $0 < k < n$. If $2f(a_n) - f(a_k) \geq 1$ is true for all $0 < k < n$, then

$$b_n = f(a_n) + b_{n-1} + a_n - a_{n-1}. \tag{61}$$

Proof. First off, if $f(a_n) = 1$, we can deduce the inequality

$$2 - f(a_k) \geq 1 \implies 1 \geq f(a_k) \implies f(a_k) = 1 \forall k < n \tag{62}$$

from which we obtain the classical Wythoff game. The recurrence formula is known in this case. Otherwise, suppose $f(a_n) \not\equiv 1$.

By combining the second line of Equation (60) and the hypothesis $f \geq 1$, one can deduce the sequence $\{b_k - a_k\}$ is monotone, because

$$b_k - a_k = b_{k-1} - a_{k-1} + f(a_k) \tag{63}$$

$$\geq b_{k-1} - a_{k-1} + 1 \tag{64}$$

As in the remark following Lemma 3, we can re-write the minimum for b_n of the same lemma as a minimum excludant:

$$b_n = \operatorname*{mex}_{\geq a_n} \left(\{b_k\}_{k<n} \cup \{b \geq a_n \mid |(b-b_k)-(a_n-a_k)| < f(a_n)\}_{k<n}\right). \tag{65}$$

By re-writing the inequality $|(b-b_k)-(a_n-a_k)| < f(a_n)$ we conclude that b belongs to the interval of integers

$$I_k = [a_n+b_k-a_k-f(a_n)+1, a_n+b_k-a_k+f(a_n)-1] \cap \mathbb{Z}_{\geq a_n}. \tag{66}$$

Therefore, the mex in equation (65) is an integer lying outside any interval I_k. With the assumption $2f(a_n)-f(a_k) \geq 1$ for all $0 < k < n$, we can deduce that the union of the intervals $\{I_k\}$ is also an interval. To see this, recall that $\{b_k - a_k\}$ is monotone. This indicates that the intervals can be ordered monotonically also. Now, count the number of integers lying between the right-endpoint of the interval I_k and the left-endpoint of the interval I_{k+1}:

$$(a_n + b_{k+1} - a_{k+1} - f(a_n)) - (a_n + b_{k+1} - a_{k+1} + f(a_n)) + 1 \tag{67}$$

$$= f(a_{k+1}) - 2f(a_n) + 1 \tag{68}$$

$$\leq 0 \tag{69}$$

where the last inequality follows from the assumption. Thus the number of integers between consecutive intervals is less than or equal to zero. In other words, we arrive at the *important conclusion* that there is no gaps between interval, i.e., consecutive intervals are either overlapping or the starting point of the next interval is exactly one integer away from the end point of the previous interval. This permits an ansatz for the minimum excluded b_n by simply evaluating $\max I_n + 1$.

$$\widetilde{b_n} = a_n + b_{n-1} - a_{n-1} + f(a_n). \tag{70}$$

Now that we have a candidate for b_n, we need to just be sure that $\widetilde{b_n} \neq b_j$ for all $j < n$, which can be demonstrated by contradiction. Suppose

$$b_j = a_n + b_{n-1} - a_{n-1} + f(a_n) \tag{71}$$

for some $j < n$. Since the sequence $\{a_k\}$ is monotone and $j < n$ we can deduce that

$$b_j > a_j + b_{n-1} - a_{n-1} + f(a_n) \tag{72}$$

$$\implies (b_j - a_j) - (b_{n-1} - a_{n-1}) > f(a_n). \tag{73}$$

But since $\{b_k - a_k\}_{k<n}$ is monotone, this implies that $f(a_n) < 0$, contradictory to our assumption that $f \geq 1$. □

Applying the previous result, we can deduce the following special case.

Lemma 6. *Let a game in Definition 5 be played with the Beatty constraint function parameterized by some $\alpha < 5/4$. Then the P-positions are pairs of the complementary Beatty sequences parameterized by α.*

Proof. The assumption $\alpha < 5/4$ and the relation $\alpha^{-1} + \beta^{-1} = 1$ imply that $[\beta] \geq 5$. The study of the difference function in Lemma 2 indicates that the constraint function may take values only from the set $\{[\beta]-1, [\beta], [\beta]-2\}$ which implies the constraint function is bounded from below, *i.e.*, $f \geq 3$. Suppose the function f is three-valued. Then

$$2 \min f - \max f = 2([\beta] - 2) - [\beta] = [\beta] - 4 \geq 1. \tag{74}$$

If f is two-valued, then there are two other possibilities to exhaust:

$$2 \min f - \max f = 2([\beta] - 1) - [\beta] = [\beta] - 2 \geq 3 \tag{75}$$

$$2 \min f - \max f = 2([\beta] - 2) - ([\beta] - 1) = [\beta] - 3 \geq 2. \tag{76}$$

If f is constant, $2 \min f - \max f \equiv f \geq 3$. In all of these cases, we see that $2 \min f - \max f \geq 1$, so that Theorem 3 may be applied. To explicate that the P-positions are indeed complementary Beatty sequences, an inductive argument is sufficient. Assume that $(a_n, b_n) = ([n\alpha], [n\beta])$ for all $n < N$. This is true for $N = 1$. For the inductive step, note that the formula of Equation 61 holds for all $n > 0$. We can thus deduce that

$$b_N = f(a_N) + b_{N-1} + a_N - a_{N-1} \tag{77}$$

$$= ([N\beta] - [N\alpha]) - ([(N-1)\beta] - [(N-1)\alpha]) + b_{N-1} + a_N - a_{N-1} \tag{78}$$

$$= ([N\beta] - [N\alpha]) - ([(N-1)\beta] - [(N-1)\alpha]) + [(N-1)\beta] + a_N - [(N-1)\alpha] \tag{79}$$

$$= [N\beta] + a_N - [N\alpha] \tag{80}$$

where the inductive hypothesis was applied to substitute values for b_{N-1} and a_{N-1}. It is a property of complementary Beatty sequences that $a_N = \text{mex}\{a_k, b_k\} = [N\alpha]$, which then implies $b_N = [N\beta]$, completing the induction. □

The next lemma demonstrates that the inequality derived in the proof of Theorem 3 is a *necessary* property of a constraint function whose game has P-positions which are complementary Beatty sequences.

Lemma 7. *Suppose a game in Definition 5 is played with the Beatty constraint function f parameterized by α. Then the P-positions are the complementary Beatty sequences if and only if* $2 \min f - \max f \geq 1$.

Proof. Theorem 3 shows that $2 \min f - \max f \geq 1$ implies the P-positions are pairs of the complementary Beatty sequences parameterized by α after expanding the recurrence.

For the reverse direction, we prove its contrapositive, that $2f \min f - \max f < 1$ implies the P-positions are not complementary Beatty sequences. Suppose $2 \min f - \max f < 1$. Continue as in the proof Theorem 3, where a sequence of intervals is considered. Select $n > 1$ and $0 < k < n$ such that there is a gap between the endpoints of the intervals I_{k-1} and I_k of maximal size. The size of the gap equals $f(a_k) - 2f(a_n) + 1$. If this is greater than one, then we can be sure the gap is not filled by the sequence $\{b_k\}$, because this sequence grows by at least two at a time. If every gap has size one, apply the next lemma, which shows not all gaps are filled in this case. In both of these situations, the fact that a gap appears in the intervals means that some b_n is less than $a_n + b_{n-1} - a_{n-1} - f(a_n)$.

$$b_n < a_n + b_{n-1} - a_{n-1} - f(a_n). \tag{81}$$

If n is the least value for which a gap in the intervals is left unfilled, then the lower P-positions are segments of the complementary Beatty sequences parameterized by α. The inequality above implies

$$\begin{aligned} b_n &\leq a_n + b_{n-1} - a_{n-1} - f(a_n) && (82)\\ &= a_n + b_{n-1} - a_{n-1} - (([n\beta] - a_n) - (b_{n-1} - a_{n-1})) && (83)\end{aligned}$$

The Beatty element $[n\beta]$ can be written $[n\beta] = b_n + d$ where $d > 0$, since $a_n + b_{n-1} - a_{n-1} + f(a_n) = [n\beta]$. (We have assumed n is the least value for which a gap appears.) Substituting into the previous inequality shows

$$\begin{aligned} b_n &\leq a_n + b_{n-1} - a_{n-1} - b_n - d + a_n + b_{n-1} - a_{n-1} && (84)\\ 2b_n - 2a_n &\leq 2(b_{n-1} - a_{n-1}) - d && (85)\\ 2(b_n - a_n) - 2(b_{n-1} - a_{n-1}) &\leq -d. && (86)\end{aligned}$$

However, the study of the difference function as in Lemma 2 reveals that this quantity is non-negative for complementary Beatty sequence $\{a_k\}$ and $\{b_k\}$. Therefore the sequence are not complementary Beatty sequences. □

Lemma 8 (Gap lemma). *Let f be a Beatty constraint function satisfying the property that any gap between the intervals*

$$\begin{aligned} I_{k-1} &= [a_n + b_{k-1} - a_{k-1} - f(a_n) + 1, a_n + b_{k-1} - a_{k-1} + f(a_n) - 1] && (87)\\ I_k &= [a_n + b_k - a_k - f(a_n) + 1, a_n + b_k - a_k + f(a_n) - 1] && (88)\end{aligned}$$

has size $f(a_k) - 2f(a_n) + 1$ exactly equal to one. Then there exists $n > 1$ and $0 < k < n$ such that the gap at I_{k-1} is left unfilled by any b_j. In symbols:

$$a_n + b_{k-1} - a_{k-1} + f(a_n) \notin \{b_j\}_{j<n}. \tag{89}$$

Proof. Suppose conversely that for every $n > 1$ and $0 < k < n$ satisfying $f(a_k) - 2f(a_n) + 1 = 1$, there exists b_j such that $a_n + b_{k-1} - a_{k-1} + f(a_n) = b_j$. Then for any $n > 1$, the cardinality of the set

$$G = \{0 < k < n \mid 2f(a_n) = f(a_k)\} \tag{90}$$

equals that of the set

$$H = \{0 < k < n \mid 2f(a_n) = f(a_k) \text{ and } a_n + b_{k-1} - a_{k-1} + f(a_n) = b_j\}. \tag{91}$$

By Lemma 6, we can assume $[\beta] = 2, 3, 4$. Recall f achieves the the following values:

$$f(a_n) = \begin{cases} [\beta] & n - 1 \in X^c \cap Y \\ [\beta] - 2 & n - 1 \in X \cap Y^c \\ [\beta] - 1 & \text{otherwise} \end{cases} \tag{92}$$

where $X = \{[n/\{\alpha\}]\}$ and $Y = \{[n/\{\beta\}]\}$. By performing casework, if $2f(a) = f(a')$ and $f \neq 0$, then one of the following is true:

$$\begin{cases} 2[\beta] = [\beta] - 1 & \implies \beta < 0 \\ 2[\beta] = [\beta] - 2 & \implies \beta < 0 \\ 2([\beta] - 1) = [\beta] & \implies [\beta] = 2 \\ 2([\beta] - 1) = [\beta] - 2 & \implies [\beta] = 0 \\ 2([\beta] - 2) = [\beta] & \implies [\beta] = 4 \\ 2([\beta] - 2) = [\beta] - 1 & \implies [\beta] = 3. \end{cases} \tag{93}$$

This shows that we need only consider the cases $[\beta] = 2, 3, 4$. It also shows for a given Beatty constraint function f that the equality $2f(a) = f(a')$ is achieved for a unique pair of values $f(a)$ and $f(a')$. Assume $[\beta] = 2$. The casework showed we require $f(a_n) = 1$ and $f(a_k) = 2$. The value $f(a_k) = 2$ is achieved if and only if $k - 1$ lies in the sequence $X^c \cap Y$. Note that this sequence has density $(1 - \{\alpha\})\{\beta\}$. The sequence $\{b_{k-1} - a_{k-1}\}_{\geq 1}$ has density $1/(\beta - \alpha)$, which means for fixed n that the sequence $\{b_{k-1} - a_{k-1} + a_n + f(a_n)\}_{k \geq 1}$ has the same density. After taking into account the density of the sequence $\{b_n\}$, one deduces

$$\lim_{\substack{n\to\infty \\ f(a_n)=1}} \frac{|H|}{n-1} \leq (1 - \{\alpha\})\{\beta\}/((\beta - \alpha)\beta) \tag{94}$$

$$< (1 - \{\alpha\})\{\beta\} = \lim_{\substack{n\to\infty \\ f(a_n)=1}} \frac{|G|}{n-1} \tag{95}$$

which indicates $|H| < |G|$ for sufficiently large n.

For $[\beta] = 4$, we require $f(a_n) = 2$ and $f(a_k) = 4$, so that $k - 1 \in X^c \cap Y$, which permits the same proof. For $[\beta] = 3$, we require $f(a_n) = 2$ and $f(a_k) = 1$ so that $k - 1 \in X \cap Y^c$. To adapt the argument in this case, replace the density $(1 - \{\alpha\})\{\beta\}$ with $\{\alpha\}(1 - \{\beta\})$ after considering the pre-images of each value f achieves.

Because $|H| < |G|$, for sufficiently large n in any case, we know that the sets H and G are not equal. Therefore, there is a gap of size one between two of the intervals which is left unfilled by any b_j. $\square$

The next lemma demonstrates that Beatty P-positions necessarily arise from Beatty constraint functions; i.e., the inverse problem has a unique solution which, if it exists, coincides with a candidate function.

Lemma 9. *Suppose a game in Definition 5 is played with the constraint function f and moreover that the P-positions agree with pairs of the complementary Beatty sequences $\{a_n\}$ and $\{b_n\}$ parameterized by α. Then f agrees with the function $\widetilde{f}$ in Equation* (10) *over all a_n with n positive.*

Proof. Let $1 < \alpha < 2 < \beta$ satisfy $1/\alpha + 1/\beta = 1$. Denote the P-positions by

$$\{(a_n, b_n)\} = \{([n\alpha], [n\beta])\}. \tag{96}$$

To proceed by induction, suppose $f(a_n) = \widetilde{f}(a_n)$ for all $n < N$. The Nth P-position is found be evaluating

$$a_N = \operatorname{mex}\{a_k, b_k\}_{k<N} \tag{97}$$

$$b_N = \operatorname*{mex}_{\geq a_N}\{b_k\}_{k<N} \cup [a_N + b_k - a_k - f(a_N)+1, a_N + b_k - a_k + f(a_N)-1]_{k<N}. \tag{98}$$

If there is no gap in the intervals, then the mex can be computed by adding one to the greatest interval's right-endpoint, to obtain

$$b_N = a_N + b_{N-1} - a_{N-1} + f(a_N). \tag{99}$$

This implies $f(a_N) = (b_N - a_N) - (b_{N-1} - a_{N-1}) = \widetilde{f}(a_N)$, which is the desired result. Otherwise, there is a gap between the intervals I_k and I_{k+1}, which implies that

$$b_N = a_N + b_k - a_k + f(a_N) \tag{100}$$

and

$$f(a_N) = (b_N - a_N) - (b_k - a_k) = \sum_{j=k+1}^{N} \widetilde{f}(a_j) \geq f(a_{k+1}). \tag{101}$$

But the existence of a gap implies that $2f(a_N) - f(a_{k+1}) \leq 0$ or that $f(a_N) \leq f(a_{k+1})/2$. The joint inequality $f(a_{k+1}) \leq f(a_N) \leq f(a_{k+1})/2$ implies $f(a_{k+1}) = f(a_N) = 0$, which contradicts the definition of f (recall that if $f = 0$ at any point, then the P-positions are not complementary). Therefore, there is no gap in the intervals, which implies $f(a_n) = \widetilde{f}(a_n)$ for all positive n. □

The next theorem extends the discussion preceding Lemma 2 to games in the family of Definition 5 and determines the range of possible Beatty sequences for the forward problem. Figure 4 illustrates the set of permitted α.

Theorem 4 (Unifying Theorem)**.** *Let a game in Definition 5 have P-positions which are the complementary Beatty sequences parameterized by α. Then one of the following is true:*

(i) $\alpha = (2-t+\sqrt{t^2+4})/2$ *for some integer* $t > 0$*,*

(ii) $[\beta] = 3$ *or* $[\beta] = 4$ *and there exist integers,* $p, q > 0$ *such that*

$$\alpha = \frac{\sqrt{4pq+([\beta]p-1)^2} + 2q-([\beta]p-1)}{2q}, \tag{102}$$

(iii) $[\beta] = 4$ *and there exist integers* $p, q > 0$ *such that*

$$\alpha = \frac{\sqrt{4pq + (q-3p-1)^2} + 3q-3p-1}{2q}, \tag{103}$$

(iv) $[\beta] \geq 5$ *or, equivalently,* $\alpha < 5/4$*.*

Proof. If $f \equiv t$ for some $t > 0$, then α is as in case (i) by Lemma 2. Otherwise, the previous lemmata showed we require that f is a Beatty constraint function as in Definition 10 and that $2 \min f - \max f \geq 1$.

To proceed, we inspect the possible values of f as α values, some of which indicate this inequality is true. Recall the trichotomy Equation (20), which says that if $X = \{[n/\{\alpha\}]\}$ and $Y = \{[n/\{\beta\}]\}$, then

$$d_\alpha(n) = \begin{cases} 0 & n \in (X \cap Y) \cup (X^c \cap Y^c) \\ 1 & n \in X^c \cap Y \\ -1 & n \in X \cap Y^c \end{cases} \implies f = \begin{cases} [\beta] - 1 \\ [\beta] \\ [\beta] - 2. \end{cases} \tag{104}$$

If f achieves more than one value, then f achieves each of those values infinitely many times as discussed in Lemma 2. Therefore we can determine the minimum and maximum by looking at the three values in Equation (104).

Suppose now that f is three-valued. We then require

$$2([\beta] - 2) - [\beta] \geq 1 \iff [\beta] \geq 5. \tag{105}$$

This case was handled in Lemma 6. Therefore suppose f achieves only two values. If those two values are $[\beta]-2$ and $[\beta]$, then we require the same inequality, which, again, was already handled. If f achieves only the values $[\beta] - 2$ and $[\beta] - 1$, then we require

$$2([\beta] - 2) - ([\beta] - 1) \geq 1 \iff [\beta] \geq 4. \tag{106}$$

If f achieves only the values $[\beta] - 1$ and $[\beta]$, then we require

$$2([\beta] - 1) - [\beta] \geq 1 \iff [\beta] \geq 3. \tag{107}$$

Therefore, if $[\beta] = 3$ or $[\beta] = 4$, then the inequality $2 \min f - \max f \geq 1$ is satisfied only when f excludes one or more of its possible values from its range.

To handle these cases in an explicit fashion, start by letting $[\beta] = 3$ or $[\beta] = 4$ and exclude the value $f = [\beta] - 2$ by setting $X \cap Y^c = \emptyset$. Theorem 3.13 of [19] states this is equivalent to the existence of integers $p, q > 0$ solving $p(1 - \{\beta\}) + q\{\alpha\} = 1$. Note that

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 \tag{108}$$

implies $\beta = \alpha/(\alpha - 1) = (1 + \{\alpha\})/\{\alpha\}$. Therefore

$$\{\beta\} = (1 + \{\alpha\})/\{\alpha\} - [\beta] \quad \text{and} \tag{109}$$

$$1 - \{\beta\} = [\beta] - \frac{1}{\{\alpha\}}. \tag{110}$$

Manipulating the equation $p\left([\beta] - 1/\{\alpha\}\right) + q\{\alpha\} = 1$, one arrives at the quadratic equation

$$q\alpha^2 + ([\beta]p-1-2q)\alpha + q-p+([\beta]p-1) = 0. \tag{111}$$

The quadratic formula yields a solution in the interval $(1, 2)$

$$\alpha = \frac{\sqrt{4pq + ([\beta]p-1)^2} + 2q - ([\beta]p-1)}{2q}. \tag{112}$$

Finally, we let $[\beta] = 4$ and exclude the value $f = [\beta]$ by setting $X^c \cap Y = \emptyset$. Equivalently, there exist integers $p, q > 0$ such that

$$p\{\beta\} + q(1 - \{\alpha\}) = 1. \tag{113}$$

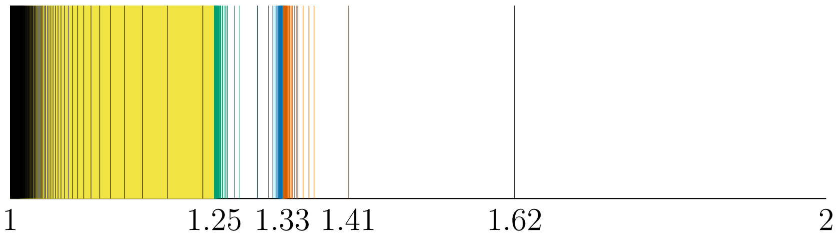

FIGURE 4. Black: irrationals in case (i) of Theorem 4. Red: irrationals in case (ii) with $[\beta] = 3$. Green: irrationals in case (ii) with $[\beta] = 4$. Blue: irrationals in case (iii). Yellow: irrationals in case (iv).

Similar to the previous derivation, the quadratic equation

$$q\alpha^2 + (3p - 3q + 1)\alpha + (2q - 4p - 1) = 0 \tag{114}$$

arises and the quadratic formula yields the solution

$$\alpha = \frac{\sqrt{4pq + (q-3p-1)^2} + 3q-3p-1}{2q}. \tag{115}$$

In each of the above cases $2 \min f - \max f \geq 2$, and Theorem 3 implies the equality $b_n = a_n + b_{n-1} - a_{n-1} + f(a_n)$ is always true.

The preceding casework is exhaustive, yielding the statement. Figure 4 displays visually the family of irrationals derived. □

If $p = q = 1$ and $t = [\beta] - 1$ are substituted into Equation (102), then the special case of the formula in Equation (4) appears. Moreover, setting $q = p$, $m = [\beta]p-1$, and $[\beta] = 3, 4$ in Equation (102), one can deduce a family of Definition 5 games whose P-positions are pairs of the complementary Beatty sequences parameterized by

$$\alpha = \frac{\sqrt{m^2+4p^2}+2p-m}{2p}. \tag{116}$$

This is precisely the irrational family mentioned in Section 1.6 of [14] and the family of Section 1.4.5 in [15]. In those papers, the family of irrationals is made in reference to a game whose P-positions are "close" to pairs of the complementary Beatty sequences parameterized by α. The preceding theorem shows that if $m = 3p - 1$ or $m = 4p - 1$, then the P-positions may be brought even closer (to a distance of zero) by playing a game in Definition 5 with the Beatty constraint function parameterized by α. This is in contrast with the statement of the appendix in the same paper, which states that for $p > 1$, there exists no game of a certain type with P-positions which are the complementary Beatty pairs. The appendix also makes reference to a special family of irrationals

$$r = \frac{\sqrt{4p^2 + 1} + 2p - 1}{2p}. \tag{117}$$

This family can be deduced by the same method the family Equation (103) was derived by instead setting $[\beta] = 2$ and letting $q = p$.

In [18], an invariant family of games is defined whose P-positions arise as pairs of the complementary Beatty sequences parameterized by

$$\alpha_k = [1; k, 1, k, 1, k, \dots] = \frac{1 + \sqrt{1 + 4/k}}{2} = \frac{\sqrt{4k + k^2} + k}{2k} \tag{118}$$

for all $k \geq 1$. For any $[\beta_k] \geq 2$, we can substitute $p = 1$ and $q = [\beta_k] - 1 = k$ into Equation (102) to arrive at the same formula.

It is for the previous examples that we view the family of quadratic irrationals in the previous theorem statement as a kind of unifying family. A special property is that Δ^2 is two valued for these α.

The key to our solution to Fraenkel's inverse problem is to observe that the disagreement of b_n with the recurrence formula depends (in a sufficient way) on the intervals I_k mentioned in the proof of Theorem 3 having gaps between them. To instead be sure that the intervals always overlap, we can replace each interval's lower endpoint with $-\infty$. This has the advantage of also removing the problematic absolute value symbol which yields non-complementary P-positions for vanishing constraint functions. The next game family is introduced as Definition 6. As a corollary, we deduce a family of games whose P-positions are any desired complementary Beatty sequences.

6. Relaxed Wythoff

In this section, a third, novel family of games is defined. This family accepts a constraint function and has three possible move types. It is shown that some game in this family has P-positions which are pairs of any desired complementary Beatty sequences.

Definition 6 (Relaxed Wythoff)**.** *Given two piles of tokens (x, y) of sizes x, y, with $0 \leq x \leq y < \infty$. Two players alternate removing tokens from the piles in one of the following ways:*

- *(aa) Remove any positive number of tokens from a single pile, possibly the entire pile.*
- *(bb) Remove more tokens from the smaller pile than from the larger pile. (A special case of the next move type.)*
- *(cc) Remove a positive number of tokens from each pile by sending (x_0, y_0) to (x_1, y_1) where*

$$(y_0 - y_1) - (x_0 - x_1) < f(x_0), \tag{119}$$

where the constraint function f is integer-valued and non-negative.

Lemma 3 provides a strategy to prove the following theorem, which is analogous to Theorem 2

Theorem 5 (P-positions of Relaxed Wythoff)**.** *Let $\mathcal{S} = \{(a_n, b_n)\}_{n=0}^{\infty}$ where $a_0 = b_0 = 0$ and for all $n \in \mathbb{Z}_{>0}$ set*

$$a_n = \operatorname{mex}\{a_k, b_k \mid k < n\} \tag{120}$$

$$b_n = f(a_n) + b_{n-1} + a_n - a_{n-1}. \tag{121}$$

If $f \geq 0$ and $f(1) \geq 1$, then $\mathcal{S} = \{(a_n, b_n)\}_{n\geq 0}$ is the set of P-positions of a game in Definition 6 played with the constraint function f. Moreover, the sets $\{a_n\}$ and $\{b_n\}$ are monotone and complementary sequences of integers covering the natural numbers and $\{b_n - a_n\}$ is a monotone sequence.

Proof. We proceed by inductively constructing the P-positions similar to the proof of Lemma 3. Let f be as given. The trivial P-position is $(0, 0)$. Similar to Lemma 3, the first non-trivial P-position has left-component $a_1 = \text{mex}\{0\} = 1$ and

$$b_1 = \min\{b \geq 1 \mid (b - b_0) - (1 - a_0) \geq f(1)\} \tag{122}$$

where we have omitted the absolute value signs. Note that any b in this set has the property that

$$b \geq 1 + b_0 - a_0 + f(1) = 1 + f(1) = a_1 + f(1). \tag{123}$$

The least such b equals $b_1 = 1 + f(1)$. By the assumption on f, this implies $b_1 \geq 2$. Note that $b_1 - a_1 = f(1) > 0 = b_0 - a_0$. Therefore, an induction hypothesis that

$$\begin{cases} b_k = a_k + b_{k-1} - a_{k-1} + f(a_k) \quad \text{and} \\ b_k - a_k > b_{k-1} - a_{k-1} \end{cases} \tag{124}$$

is true for all $1 < k < n$.

For an inductive step, we proceed similar to the derivation of the first non-trivial P-position. After adding a quantifier and restricting $b \neq b_k$, we obtain

$$a_n = \text{mex}\{a_k, b_k \mid k < n\} \tag{125}$$

$$b_n = \min\{b \geq a_n \mid b \neq b_k \text{ and } (b - b_k) - (a_n - a_k) \geq f(a_n) \forall k < n\}. \tag{126}$$

If $b \neq b_k$ lies in the set restriction, then rearranging shows

$$b \geq a_n + f(a_n) + b_k - a_k \quad \forall k < n. \tag{127}$$

Since the inequality holds for all k and $\{b_k - a_k\}$ is increasing, we may as well set $k = n - 1$ and find the minimal such b at the endpoint of the inequality:

$$b_n = a_n + f(a_n) + b_{n-1} - a_{n-1}, \tag{128}$$

completing the proof of the recurrence formula. Re-arranging Equation (128) reveals

$$b_n = b_{n-1} + (a_n - a_{n-1}) + f(a_n), \tag{129}$$

from which we can deduce that $\{b_n\}$ is strictly monotone, because $\{a_n\}$ is strictly monotone and $f \geq 0$. A similar re-arrangement shows $\{b_n - a_n\}$ is monotone. For a final remark on the P-positions, notice that $\{a_n, b_n \in \mathcal{S}\} = \mathbb{Z}_{\geq 0}$ since the first component consists of minimum excludants.

To prove that a player in an N-position may always move to a P-position, apply the same argument of Lemma 3 with the substitution of the minimum

$$\min\{b \geq a_n \mid b \neq b_k \text{ and } (b - b_k) - (a_n - a_k) \geq f(a_n) \forall k < n\} \tag{130}$$

instead of

$$\min\{b \geq a_n \mid b \neq b_k \text{ and } |(b - b_k) - (a_n - a_k)| \geq f(a_n) \forall k < n\}. \tag{131}$$

$\square$

Re-using some of the results from the proof of Theorem 4, we obtain the following corollary to Theorem 5.

n	0	1	2	3	4	5	6	7	8	9
a_n	0	1	2	4	5	7	8	10	11	13
b_n	0	3	6	9	12	16	19	22	25	29
$[n\alpha]$	0	1	2	4	5	7	8	10	11	13
$[n\beta]$	0	3	6	9	12	16	19	22	25	29

FIGURE 5. Top: P-positions of the game played by Definition 6 with the constraint function in Equation (10) parameterized by $\alpha = 1+\sqrt{5}/5$. Bottom: Beatty sequences parameterized by α. Compare to Figure 2.

Corollary 1. *If $\{a_n\} = \{[n\alpha]\}$ and $\{b_n\} = \{[n\beta]\}$ are complementary Beatty sequences with $\alpha < \beta$, then there is a function f such that if a game in Definition 6 is played with f as its constraint function, then the P-positions are precisely the pairs $\{(a_n, b_n)\}_{n=0}^{\infty}$.*

Proof. Theorem 5 shows that the game in Definition 6 played with the Beatty constraint function f parameterized by α should yield the desired Beatty P-positions. It remains only to prove $f(1) \neq 0$ and that $f(a_n) \geq 0$ for all $a_n > 0$ to apply Theorem 5.

First,

$$f(1) = [\beta] - [\alpha] = [\beta] - 1 \geq 2 - 1 = 1, \tag{132}$$

so the first criterion is satisfied. For the second criterion, recall the proof of Theorem 4, which showed that $[\beta] - 2 \leq f \leq [\beta]$. Since $[\beta] \geq 2$, we can deduce that $f \geq 0$. Therefore, the hypotheses on f are satisfied, so the P-positions of the game in Definition 6 with the function f may now be evaluated using the recurrence in the statement of this theorem. It is seen directly that the P-positions are the complementary Beatty sequences parameterized by α. □

Therefore, Relaxed Wythoff gives simple rules for which the P-positions are pairs of given complementary Beatty sequences. In Figure 5, we return to the example in Figure 2. Playing the game as in Definition 6 with the same constraint function yields P-positions which coincide with the associated Beatty pairs.

7. Discussion and Conclusion

The forward problem of finding the P-positions of a given game has existed since game theory began. The inverse problem of finding a game for given P-positions has recently been developed and offers a fresh perspective on classical problems. In this paper, we precisely characterized the Beatty sequences arising from the P-positions of a known game family. The analysis of the sequence Δ^2_{n-1} was crucial and yielded insight to finding simple rules for another game with Beatty P-positions. A solution was proposed to the inverse problem posed by Fraenkel at the 2011 BIRS workshop via the definition of Relaxed Wythoff played with a Beatty constraint function. Note that this inverse problem was further refined at the end of [11]:

> Is it possible to find short 2-invariant game rules, without disclosing any part of the P-positions?

Our proof of Theorem 4 showed that if a game is played with the Beatty constraint function parameterized by some α in the statement of the theorem, then the game is k-invariant for some $k \leq 3$. Each of the irrationals in the countable family of Theorem 3 corresponds to a Beatty constraint function inducing a 2-invariant game. Supplying the players with the formula for the Beatty constraint function automatically reveals the irrationals which parameterize the Beatty P-positions. Revealing some aspect of the irrationals is a common feature of the games in [11] and [13]. In spite of this, each game is indeed simple. A move can be validated with finite calculations. Moreover, the study of the game family in Definition 5 over Beatty constraint functions is significant on its own because it expands the study of non-monotone constraint functions, also seen in [7]. A striking feature of this analysis revealed that the *uncountable* family of complementary Beatty sequences parameterized by $\alpha \in (1, 5/4)$ arise as P-positions of games in the family of Definition 5.

8. Acknowledgments

This work was completed under the care and supervision of several institutions and advisors. Gratefully, we acknowledge Hampshire College and The Five College Consortium for providing the authors with a place to meet each other and explore mathematics in an experimental fashion while the first author was an undergraduate student. Amherst College and later Smith College provided new academic homes for the second author after leaving Hampshire College. Once the first author completed his studies at Hampshire College, he was given a host of new ideas at The University of California, Santa Cruz, particularly from Dr. François Monard. The first author also thanks The Ross Program and Dr. Jim Fowler for providing a motivating and hospitable work environment. The authors also thank a reviewer for helpful references and suggestions.

References

[1] J-P. Allouche and J. Shallit, *Automatic Sequences*. Cambridge University Press, 2003. 2, 6

[2] E. Duchêne, A. S. Fraenkel, V. Gurvich, N. B. Ho, C. Kimberling, and U. Larsson, Wythoff Wisdom. *Games of No Chance 5*. MSRI Publications, 70:321–330. Cambridge University Press, 2019. 1, 9

[3] E. Duchêne and M. Rigo, Invariant games. *Theoretical Computer Science*, 411(34–36):3169–3180, 2010. 2, 9

[4] M. Fisher and U. Larsson, Chromatic Nim finds a game for your solution. *Games of No Chance 5*. MSRI Publications, 70:321–330. Cambridge University Press, 2019. 3

[5] A. S. Fraenkel, How to Beat Your Wythoff Games* Opponent on Three Fronts. *The American Mathematical Monthly*, 89(6):353–361, 1982. 2, 9

[6] A. S. Fraenkel, New Games Related to Old and New Sequences. *Advances in Computer Games*. The International Federation for Information Processing, 135:367–382, 2003. 2, 3, 4, 7, 9, 11

[7] A. S. Fraenkel, Games Played by Boole and Galois. *Discrete Appl. Math.*, 156:420–427, 2008. 7, 23

[8] A. S. Fraenkel, Martin Gardner and Wythoff's Game. Preprint, 2011. 2

[9] A. S. Fraenkel, Beating Your Fractional Beatty Game Opponent and: What's the Question to Your Answer? *Advances in Combinatorics, in Memory of Herbert S. Wilf, Proc. Waterloo Workshop on Computer Algebra*, I. S. Kotsireas and E. V. Zima, eds., May 2011, 175–186 , Springer, 2013. 2

[10] A. S. Fraenkel and I. Borosh, A Generalization of Wythoff's Game. *Journal of Combinatorial Game Theory (A)* 15(2):175–191, 1973. 2

[11] A. S. Fraenkel and U. Larsson, Take-away games on Beatty's Theorem and the notion of k-invariance. *Games of No Chance 5*. MSRI Publications, 70:333–342. Cambridge University Press, 2019. 2, 3, 5, 7, 22, 23

[12] L. Goldberg and A. S. Fraenkel, Extensions of Wythoff's Game. Preprint, 2013. 2, 3

[13] L. Goldberg and A. S. Fraenkel, Rulesets for Beatty Games. 2017. *Intern. J. Game Theory*, special issue on combinatorial games. 47(2):577–594, 2018. 2, 3, 23

[14] U. Larsson, Restrictions of m-Wythoff Nim and p-complementary Beatty sequences. *Games of No Chance 4*. MSRI Publications 63:137–160, 2015. 1, 9, 19

[15] U. Larsson, 2-Pile Nim with a Restricted Number of Move-size Imitations. *Integers* 9(6):671–690, 2009. 19

[16] U. Larsson, CombinArtorial Games. *Handbook of the Mathematics of the Arts and Sciences*. Springer, Cham; 2020. 3

[17] U. Larsson, P. Hegarty, and A. S. Fraenkel, Invariant and dual subtraction games resolving the Duchêne–Rigo conjecture. *Theoretical Computer Science*, 412(8–10):729–735, 2011. 3

[18] U. Larsson and Mike Weimerskirch, Impartial games whose rulesets produce given continued fractions. *Games of No Chance 5*. MSRI Publications, 70:403–419. Cambridge University Press, 2019. 2, 3, 7, 20

[19] I. Niven, Diophantine Approximations. *Interscience Tracts in Pure and Applied Mathematics*, vol. 14. Interscience Publishers, 1963. 6, 18

[20] G. Polanco, The logarithm of irrational numbers and Beatty sequences. *Acta Arithmetica*, 179:101–123, 2017. 2

[21] A. N. Siegel, Combinatorial Game Theory. *Graduate Studies in Mathematics* 146. American Mathematical Society, 2013. 1

[22] I. J. Schoenberg, The location of frets on a guitar and theorem of Lord Rayleigh. In *Mathematical Time Exposures*, page 40. The Mathematical Association of America, 1982. 2

[23] W. A. Wythoff, A Modification of the game of Nim. *Nieuw Archiefvoor Wiskunde* 7(2):199–202, 1907. 9