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On the Solutions of Linear, Maxilinear and Reciprocal Type Difference Equations

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ON THE SOLUTIONS OF LINEAR, MAXILINEAR AND RECIPROCAL TYPE DIFFERENCE EQUATIONS

THESIS

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THESIS

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CHAPTER 1

INTRODUCTION

Let N denote the set of nonnegative integers. The set of all real-valued functions whose domain is in N, along with the usual definition of addition of functions and multiplication of a functions by a real number, is a vector space and is denoted by V_∞. The elements of the vector space V_∞ are called sequences. If x is an element of V_∞, then we will use the notation x_n instead of x(n) to denote the general term of the sequence and use $\{x_n\}$ to represent the sequence x_0 , x_1,x_2 ,….

Now consider a given real-valued function f: D → D, where D⊆ R.

Using f, we can construct a first order difference equation by writing

$$x_{n+1} = f(x_n),\ n= 0,1,2,\ldots$$

Similarly, a second order difference equation can be constructed by writing

$$x_{n+1} = f(x_n, x_{n-1}),\ n= 0,1,2,\ldots$$

where f: DxD→ D

Generally, a difference equation of order k can be constructed by letting

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k+1}),\ n= 0,1,2,..$$

where f: D^k → D

If the function f has the form

$$f(x) = ax + b$$

where a and b are any real constants, then we obtain an equation of the form

$$x_{n+1} = ax_n + b. \tag{1}$$

Definition: The general linear difference equation of order 1 is given by

$$x_{n+1} = a_n x_n + b_n \tag{2}$$

where a_n and b_n are sequences. When an initial value of x_0 is given, a unique solution exists. An equation of the form:

$$x_{n+1} = ax_n + b \tag{3}$$

is called a linear equation of the first order with constant coefficients. When b= 0, this turns the problem into one of the form

$$x_{n+1} = ax_n. \tag{4}$$

Definition: If a linear equation can be written in the form (4), it is described as homogeneous. Otherwise it is non-homogeneous.

To solve the homogeneous equation

$$x_{n+1} = ax_n$$

we assume x_0 is an initial value, then

$$x_1 = ax_0$$

$$x_2 = ax_1 = a^2 x_0$$

$$x_3 = ax_2 = a^3 x_0$$

$$\vdots$$

$$x_k = ax_{k-1} = a^k x_0$$

Theorem:

The general homogeneous equation $x_{n+1}= ax_n$ has the exact solution $x_n=a^n x_0$ for all $n\geq 0$ and x_0 is an initial value.

In the event that f is a function of two variables, we can also construct a difference equation by writing

$$x_{n+1} = f(x_n , x_{n-1})$$

Note that if

$$f(x,y) = ax + by + c$$

then our difference equation becomes

$$x_{n+1} = ax_n + bx_{n-1} + c, \quad b\neq 0. \tag{5}$$

This is a linear difference equation of order 2. When c= 0, we refer to (5) as homogeneous and denote it by

$$x_{n+1} = ax_n + bx \tag{6}$$

Note that (6) can also be written as

$$x_{n+1} + px_n + qx_{n-1} = 0 \tag{7}$$

where p = -a and q = -b.

A difference equation will be called maxi- linear if it has the form

$$x_{n+1} = \max\{f(x_n), g(x_{n-1})\}$$

where f and g are linear functions of a single variable. A difference equation will be called reciprocal type if it has the form

$$x_{n+1} = \{ \frac{A}{x_n} , x_{n-1} \}$$

or

$$x_{n+1} = \{ \frac{1}{x_n} , Ax_{n-1} \}$$

Generally, a solution of a first order difference equation is a sequence of real numbers generated recursively from an initial value x_0. Similarly, solutions of a second order equation are generated from two initial values x_{-1} and x_0.

CHAPTER 2

HISTORY AND PRELIMINARY RESULTS

Within the past twenty-five years, the study of difference equations has a new significance. This came about largely from increased usage of computers and the formulation and analysis of discrete-time systems. The numerical integration of differential equations and the study of deterministic chaos also have played a role in bringing difference equation to the forefront of mathematical analysis.

Preliminary results will focus on the behavior of solutions to homogeneous linear difference equations of orders one and two that have constant coefficients.

Definition: Bounded sequence, periodic sequence, oscillatory sequence

Definition 2.1: Periodic Sequence: A sequence $\{x_n\}$ is said to be periodic with period p if and only if it satisfies $x_{n+p} = x_n$ for all n and is said to be eventually periodic if $x_{n+p} = x_n$ for all $n \geq k$, for some $k \in N$

Definition 2.2: Bounded Sequence: A sequence $\{x_n\}$ is said to be bounded if and only if there is some K such that $|x_n| \leq K$ for all n.

Definition 2.3 : Oscillatory sequence: A sequence $\{x_n\}$ is said to oscillate about zero or simply oscillate if the terms x_n are neither eventually all positive nor eventually all negative

Theorem 2.1:

Consider the equation

$$x_{n+1}=ax_n \tag{8}$$

and let x_0 be an initial value. Then

(i) If $|a| < 1$, then all solutions of (8) tend to zero as n→ ∞

(ii) If $|a|=1$, then all solutions of (8) are periodic

(iii) If $|a| > 1$, then all nontrivial solutions of (8) are unbounded

(iv) If $a < 0$, then all nontrivial solutions of (8) oscillate about zero.

Theorem 2.2:

Consider the equation

$$x_{n+2}+ax_{n+1}+bx_n=0. \tag{9}$$

Then $x_n=t^n$, $t\neq 0$ is a solution to (9) if

$$t^2+a\,t+b=0 \tag{10}$$

Note: Equation (10) is called the characteristic equation of equation (9).

Theorem 2.3:

If t_1 and t_2 are real solutions of (10) and $t_1 \neq t_2$, then the general solution of (9) is $x_n=c_1t_1^n+c_2t_2^n$. If $t_1=t_2=t$, then the general solution of (9) is $x_n=c_1t^n+c_2\,t^n\,n=t^n\,(c_1+c_2\,n)$.

Theorem 2.4:

If t_1 and t_2 are non-real solutions of (10), say $t_{1,2}$= p$\pm iq$, then general solution of (9) is x_n=$(\sqrt{p^2+q^2})^n$($c_1\cos(n\theta) + c_2\sin(n\theta)$) where $\theta = \tan^{-1}(\frac{q}{p})$.

Theorem 2.5:

If (10) has a negative root, then (9) has oscillating solutions.

Theorem 2.6:

If (10) has a root whose absolute value is greater than one, then (9) has an unbounded solution.

Theorem 2.7:

The equations $x_{n+2} + x_{n+1} + x_n$ =0, x_{n+2} +x_n=0, and x_{n+2} -x_n =0 have periodic solutions.

Examples: Consider the following equations

i) $x_{n+2} - 3x_{n+1} - 4x_n = 0$

Let $x_n = t^n$, then our equation becomes $t^{n+2} - 3t^{n+1} - 4t^n = 0$.

Factoring we obtain

$t^n\,(t^2 - 3\,t - 4) = 0$ and since $t \neq 0$, then $t^2 - 3t - 4 = 0$; therefore, the solutions of our characteristic equation are $t_1 = -1, t_2 = 4$. Let $u_n = (-1)^n$ and $v_n = 4^n$.

Then the general solution is $x_n = c_1(-1)^n + c_2 4^n = c_1 u_n + c_2 v_n$. When $c_2 = 0$, the solution is periodic and oscillatory. When $c_2 \neq 0$, then solution is nonoscillatory and unbounded.

ii) $x_{n+2} - 4x_{n+1} + 4x_n = 0$

Let $x_n = t^n$, then our equation becomes $t^{n+2} - 4t^{n+1} + 4t^n = 0$. This implies

$t^n \left(t^2 - 4t + 4\right) = 0$ and since t$\neq$ 0, then $t^2 - 4\,t + 4 = 0$; therefore, the solutions are $t_1 = t_2 = 2$.

Hence the general solution is $x_n = c_1\, 2^n + nc_2\, 2^n = 2^n\ (c_1 + c_2 n)$.

All nontrivial solutions are unbounded.

iii) $x_{n+2} + x_n = 0$

Let $x_n = t^n$, then $t^{n+2} + t^n = 0$ implies $t^n \left(t^2 + 1\right) = 0$.

$t^2 + 1 = 0,$ then t=$\pm\, i$

Since $i^n = r^n\ (\cos(n\theta) + \sin(n\theta))$ and r=$|i| = 1$, $\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$,

then $i^n = \cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)$. Thus the general solution will be

$x_n = c_1 \cos\left(\frac{n\,\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right)$.

The solution is periodic of period 4.

iv) $x_{n+2} - 2rx_{n+1} + r^2\, x_n = 0$

Again let $x_n = t^n$ then $t^{n+2} - 2rt^{n+1} + r^2\, t^n = 0$.

Thus $t^n\ (t^2 - 2rt + r^2\ ) = 0$, since t$\neq$ 0, then $t^2 - 2r\,t + r^2 = 0$. Hence the solutions are $t_1 = t_2 = r$.Therefore, the general solution is

$x_n = c_1\ \ r^n + c_2\, nr^n$.

When $|r| < 1$, solutions tend to 0 as n$\to\infty$.

When$|r| > 1$, solutions are unbounded.

When r< 0, solutions are oscillatory.

When r > 0, solutions are nonoscillatory.

When r = -1, $c_1 \neq 0$, $c_2 = 0$, solutions are periodic.

CHAPTER 3

OVERVIEW AND MAIN RESULTS

This study will focus on the behavior of the solutions to the equations

$$x_{n+1} = \max\{\frac{1}{x_n}, Ax_{n-1}\} \tag{11}$$

and

$$x_{n+1} = \max\{1-x_n, Bx_{n-1}\} \tag{12}$$

where A and B are constants.

The solution behaviors of (11) will be studied under conditions where $A< 0$ and $A> 0$. The solution behaviors of (12) will be considered with the cases where $0< B\leq 1$ and $B>1$.

Our main interest in (11) and (12) will be on the periodic properties, boundedness, and asymptotic properties of solutions.

Multiple theorems and propositions will be generated during the study based on our findings. The results for (11) will consist of excerpts from [3] while the results for (12) will be new since equation (12) has not appeared in the literature search.

We will consider different cases for the constant "A" and the initial values in (11). We also consider numerous cases for the constant "B" and the initial values in (12). We will compare the behaviors of the solutions of (12) to those of (11). New theorems will be derived on (12), and most of the theorems will be proved by induction.

Main Results

Excerpts From [3] on $x_{n+1}= \max\{\frac{1}{x_n}, Ax_{n-1}\}$

Theorem 3.1.1: Assume A< 0 and let x_{-1}, x_0 denote initial values.

a) If $x_{-1} < 0$, $x_0 > 0$, and $x_1 = \frac{1}{x_0}$, then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots)$$

b) If $x_{-1} < 0$, $x_0 > 0$, and $x_1 = Ax_{-1}$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots).$$

In both (a) and (b) solutions are eventually 2- periodic.

Proof: Assume $x_{-1} < 0$ and $x_0 > 0$, then

$x_1 = \max\{\frac{1}{x_0}, Ax_{-1}\} > 0$. Using induction it is easy to see that $x_n > 0$ for every n≥ 0,

and consequently

$$x_{n+1} = \max\{\frac{1}{x_n}, A\, x_{n-1}\} = \frac{1}{x_n}, \qquad n \geq 1.$$

Hence, if $x_1 = \frac{1}{x_0} \geq Ax_{-1}$, then every solution is eventually 2- periodic, moreover (x_n) can be written as follows:

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots)$$

If $x_1 = Ax_{-1}$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots). \qquad \text{QED.}$$

Theorem 3.1.2: Consider x_{n+1}= max$\{\frac{1}{x_n}$,$Ax_{n-1}\}$, where A< 0

a) If $x_{-1} > 0$, $x_0 < 0$, $x_1 = \frac{1}{x_0}$, and A$\in$ (-∞, -1], then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \frac{x_0}{A}, \ldots, \frac{A}{x_0}, \frac{x_0}{A}, \ldots).$$

b) If $x_{-1} > 0$, $x_0 < 0$, $x_1 = \frac{1}{x_0}$, and A $\in$ (-1, 0), then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots).$$

c) If $x_{-1} > 0$, $x_0 < 0$, $x_1 = Ax_{-1}$, and $\frac{1}{Ax_0} \geq A^2x_{-1}$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots).$$

d) If $x_{-1} > 0$, $x_0 < 0$, $x_1 = Ax_{-1}$, and $\frac{1}{Ax_0} < A^2x_{-1}$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2x_{-1}, \frac{1}{A^2x_{-1}}, \ldots, A^2x_{-1}, \frac{1}{A^2x_{-1}}, \ldots).$$

In all cases solutions are eventually 2- periodic.

Proof: If $x_{-1} > 0$ and $x_0 < 0$, then x_1 = max $\{\frac{1}{x_0}, Ax_{-1}\} < 0$,

x_2= max $\{\frac{1}{x_1}, Ax_0\} = Ax_0 > 0$, and x_3= max $\{\frac{1}{x_2}, Ax_1\} > 0$ because $\frac{1}{x_2} > 0$.

By induction we obtain $x_n > 0$, for all n≥ 2. Hence

$x_{n+1} = \frac{1}{x_n}$ for all n≥ 3. Consequently, in this case, every solution is

eventually 2- periodic.

a) If $x_1 = \frac{1}{x_0}$ and A $\in (-\infty, -1]$, then $Ax_{-1} \leq \frac{1}{x_0}$, consequently,

x_3= max $\{\frac{1}{Ax_0}, \frac{A}{x_0}\} = \frac{A}{x_0}$ because A≤ 1 implies A≤$\frac{1}{A}$ and this implies $\frac{A}{x_0} \geq \frac{1}{Ax_0}$ since

$x_0 < 0$. Hence $(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \frac{x_0}{A}, \ldots, \frac{A}{x_0}, \frac{x_0}{A}, \ldots).$

b) If $x_1 = \frac{1}{x_0}$ and A ∈ (-1,0), then $x_2 = Ax_0$ and $x_3 = \max\{\frac{1}{Ax_0}, \frac{A}{x_0}\} = \frac{1}{Ax_0}$

since $\frac{1}{A} < A$. Thus

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots).$$

c) If $x_1 = Ax_{-1} \geq \frac{1}{x_0}$, then $A^2x_{-1} \leq \frac{A}{x_0}$; and if $\frac{1}{Ax_0} \geq A^2x_{-1}$, then

$x_3 = \max\{\frac{1}{Ax_0}, A^2x_{-1}\} = \frac{1}{Ax_0}$, so that

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots).$$

d) If $x_1 = Ax_{-1}$ and $\frac{1}{Ax_0} < A^2x_{-1}$, then $x_3 = A^2x_{-1}$ and

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2x_{-1}, \frac{1}{A^2x_{-1}}, \ldots, A^2x_{-1}, \frac{1}{A^2x_{-1}}, \ldots). \quad \text{Q.E.D}$$

Theorem 3.1.3: Consider $x_{n+1} = \max\{\frac{1}{x_n}, Ax_{n-1}\}$, where A< 0

a) If $x_{-1}, x_0 > 0$, then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots).$$

b) If x_{-1}, $x_0 < 0$, and $\frac{1}{Ax_{-1}} \leq Ax_0$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots).$$

c) If $x_{-1}, x_0 < 0$, and $\frac{1}{Ax_{-1}} > Ax_0$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, Ax_{-1}, \ldots, \frac{1}{Ax_{-1}}, Ax_{-1}, \ldots).$$

All solutions are eventually 2-periodic.

Proof: a) If $x_{-1}, x_0 > 0$, then $x_1 = \frac{1}{x_0} > 0$. By induction we have $x_n > 0$,

for all n≥ −1 and, consequently, $x_{n+1} = \frac{1}{x_n}$, for all n≥ 0. Thus in this case

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots).$$

b) If $x_{-1}, x_0 < 0$, then $x_1 = Ax_{-1} > 0$, and $x_2 = \max\{\frac{1}{x_1}, Ax_0\} = \max\{\frac{1}{Ax_{-1}}, Ax_0\} > 0$.

Using induction we have $x_n > 0$, for all n$\geq$ 1, which implies $x_{n+1} = \frac{1}{x_n}$ for all n $\geq$ 2.

Therefore, if $\frac{1}{Ax_{-1}} \leq Ax_0$, we have

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, \frac{1}{Ax_0}, \ldots, Ax_0, \frac{1}{Ax_0}, \ldots).$$

c) On the other hand, if $\frac{1}{Ax_{-1}} > Ax_0$, we have

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, Ax_{-1}, \ldots, \frac{1}{Ax_{-1}}, Ax_{-1}, \ldots).$$ Q.E.D

Theorem 3.1.4: Consider $x_{n+1} = \max\{\frac{1}{x_n}, Ax_{n-1}\}$, where A > 0.

a) If $x_{-1} < 0$, $x_0 > 0$, and A $\in$ (0,1], then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, \ldots x_0, \frac{1}{x_0}, \ldots).$$

b) If $x_{-1} < 0$, $x_0 > 0$, and A > 1, then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \ldots, A^n x_0, \frac{A^n}{x_0}, \ldots).$$

Proof:

a) Let $x_{-1} < 0$, $x_0 > 0$, then $x_1 = \frac{1}{x_0} > 0$ and

$$x_2 = \max\{x_0, Ax_0\} = x_0 \max\{1, A\} > 0.$$

Hence if A $\in$ (0,1], then $x_2 = x_0$, and $x_3 = \max\{\frac{1}{x_0}, \frac{A}{x_0}\} = \frac{1}{x_0}$. That is

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots).$$

b)If A > 1, then $x_2 = Ax_0$ and $x_3 = \max\{\frac{1}{x_2}, A x_1\} = \max\{\frac{1}{Ax_0}, \frac{A}{x_0}\} = \frac{A}{x_0} > 0$.

By induction we obtain $x_{2n} = A^n x_0$, $x_{2n+1} = \frac{A^n}{x_0}$ for all n$\geq$ 1, that is,

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \ldots, A^n x_0, \frac{A^n}{x_0}, \ldots).$$ Q.E.D

Theorem 3.1.5: Consider $x_{n+1}= \max\{\frac{1}{x_n}, Ax_{n-1}\}$,

a) If $x_{-1} > 0$, $x_0 < 0$, and $A \in (0,1]$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots).$$

b) If $x_{-1} > 0$, $x_0 < 0$, and $A > 1$, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, A^2x_{-1}, \ldots, A^{n+1}x_{-1}, \frac{A^{n-1}}{x_{-1}} \ldots).$$

Proof: a) If $x_{-1} > 0$ and $x_0 < 0$, then $x_1 = Ax_{-1} > 0$, $x_2 = \frac{1}{Ax_{-1}} > 0$, and

$$x_3 = \max\{\frac{1}{x_2}, Ax_1\} = \max\{Ax_{-1}, A^2x_{-1}\} = Ax_{-1}\max\{1, A\}.$$

Clearly $x_n > 0$, for all $n \geq 1$.

If $A \in (0,1]$, then $x_3 = A^2x_{-1} = x_1$ and $x_4 = \frac{1}{Ax_{-1}} = x_2$, that is,

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots, Ax_{-1}, \frac{1}{Ax_{-1}}, \ldots).$$

b) If $A > 1$, then $x_3 = Ax_{-1}$ and $x_4 = \frac{1}{x_{-1}}$. By induction we obtain that $x_{2n} = \frac{A^{n-2}}{x_{-1}}$, $x_{2n-1} = A^n x_{-1}$, for all $n \geq 2$. Hence

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, \frac{1}{Ax_{-1}}, A^2x_{-1}, \ldots, A^{n+1}x_{-1}, \frac{A^{n-1}}{x_{-1}} \ldots). \qquad \text{Q.E.D}$$

The case where $A > 0$ and both x_{-1} and x_0 are positive will now be considered. Note that if x_n is a positive solution of (11), then multiplying (11) by x_n and letting $y_n = x_n x_{n-1}$ transforms (11) into

$$y_{n+1} = \max\{Ay_n, 1\}, \text{ where } y_0 > 0. \qquad (11^*)$$

Lemma 3.1: If $A \in (0, 1]$, then each solution of (11^*) is eventually constant, in fact, either $y_n = 1$ eventually, or $y_n = y_0$ eventually.

Remark: Note that y_n eventually constant implies solutions of (11) are eventually 2-periodic. A dual result to Lemma 3.1 is the following.

Lemma 3.2: Consider the difference equation

$$y_{n+1} = \min\{ Ay_n, 1\} \qquad (11^{**})$$

where $y_0 > 0$. Then for A$\geq$ 1, each solution of (11^{**}) is eventually constant.

The proof of Lemma 3.1 and Lemma 3.2 appear in [3].

Theorem 3.1.6: Consider $x_{n+1} = \max\{\frac{1}{x_n}, Ax_{n-1}\}$

a) If $x_{-1}, x_0 > 0$, $x_1 = \frac{1}{x_0}$, and $A > 1$

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \ldots, A^n x_0, \frac{A^n}{x_0}, \ldots).$$

b) If $x_{-1}, x_0 > 0$, $x_1 = Ax_{-1}$, and A$\geq$ 1, then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2 x_{-1}, A^2 x_0, \ldots, A^n x_{-1}, A^n x_0, \ldots).$$

c) If $x_{-1}, x_0 > 0$, $x_1 = Ax_{-1}$, and A $\in$ (0,1), then (x_n) is eventually 2- periodic.

d) If $x_{-1}, x_0 > 0$, $x_1 = \frac{1}{x_0}$, and A $\in$ (0,1] , then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, \ldots, x_0, \frac{1}{x_0}, \ldots).$$

Proof:

a) If A $\in (1, \infty)$, and $x_1 = \frac{1}{x_0}$, then we get $x_2 = Ax_0$, $x_3 = \frac{A}{x_0}$, and by induction it follows that $x_{2n} = A^n x_0$, and $x_{2n+1} = \frac{A^n}{x_0}$ for all n$\geq$ 1, that is,

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \ldots, A^n x_0, \frac{A^n}{x_0}, \ldots).$$

b) If $x_1 = Ax_{-1} \geq \frac{1}{x_0}$ and A $\in [1, \infty)$; therefore, $\frac{1}{x_0} \leq Ax_{-1}$. Hence $\frac{1}{Ax_{-1}} \leq x_0$, and thus $\frac{1}{x_{-1}} \leq Ax_0$. Thus $\frac{1}{Ax_{-1}} \leq \frac{1}{x_{-1}} \leq Ax_0$ because $A > 1$; therefore,

$x_2 = \max\{\frac{1}{Ax_{-1}}, Ax_0\} = Ax_0$ and $x_3 = \max\{\frac{1}{Ax_0}, A^2 x_{-1}\} = A^2 x_{-1}$.

By induction we get $x_{2n-1}=A^n x_{-1}$, $x_{2n}=A^n x_0$, for all n≥ 1. Thus

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2x_{-1}, A^2x_0, \ldots, A^n x_{-1}, A^n x_0, \ldots).$$

c) Follows from remarks after Lemma 3.1, and the proof is similar to (a)

d) Follows from remarks after Lemma 3.1. Q.E.D

Theorem 3.1.7: Consider $x_{n+1}= \max\{\frac{1}{x_n}, Ax_{n-1}\}$,

a) If x_{-1}, $x_0 < 0$, $x_1 = \frac{1}{x_0}$, and A ∈ (0,1], then

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \ldots, A^n x_0, \frac{A^n}{x_0}, \ldots).$$

b) If x_{-1}, $x_0 < 0$, $x_1 = Ax_{-1}$, and A ∈ (0,1], then

$$(x_n) = (x_{-1}, x_0, Ax_{-1}, Ax_0, A^2x_{-1}, \ldots, A^n x_0, A^{n+1}x_{-1}, \ldots).$$

c) If x_{-1}, $x_0 < 0$, $x_1 = \frac{1}{x_0}$, and A> 1 , then

(x_n) is eventually 2-periodic.

d) If x_{-1}, $x_0 < 0$, $x_1 = Ax_{-1}$, and A > 1, then

(x_n) is eventually 2- periodic.

Proof:

a) If x_{-1} ,$x_0 < 0$, then $x_n < 0$, for all n≥ −1. If $x_1= \max\{\frac{1}{x_0}, Ax_{-1}\}= \frac{1}{x_0}$ and

A∈ (0,1], we have $x_2= \max\{x_0, Ax_0\}=Ax_0$ since A ≤ 1 and $x_0 < 0$ and

$x_3= \max\{\frac{1}{Ax_0}, \frac{A}{x_0}\}= \frac{A}{x_0}$. By induction we have $x_{2n} = A^n x_0$, $x_{2n+1}= \frac{A^n}{x_0}$, for all n≥ 1,

that is,

$$(x_n) = (x_{-1}, x_0, \frac{1}{x_0}, Ax_0, \frac{A}{x_0}, \ldots, A^n x_0, \frac{A^n}{x_0}, \ldots).$$

b) The proof is similar to 3.1.7 (a)

c) Apply Lemma 3.2, where x_{-1},$x_0 < 0$, and A > 1.

d) Apply Lemma 3.2, where x_{-1} ,$x_0 < 0$, and A> 1.

New Results on $x_{n+1} = \max\{1-x_n, Bx_{n-1}\}$.

We begin our study of (12) with

Lemma 3.3: If $a < 0$ and $0 < B < 1$ and two consecutive terms of a solution of (12) have the form $B^k a$, $1-B^k a$ for some $k \in N$, then the solution continues…,$B^{k+n}a$, $1-B^{k+n}a$,..

Proof is by induction and is omitted.

Theorem 3.2.1: Consider $x_{n+1} = \max\{1-x_n, Bx_{n-1}\}$

a): If $x_{-1} < 0 < 1 < x_0$, and $0 < B < 1$, and $x_1 = 1-x_0$, then

$$(x_n) = (x_{-1}, x_0, 1-x_0, x_0, B(1-x_0), 1-B(1-x_0), \ldots, B^n(1-x_0), 1-B^n(1-x_0), \ldots).$$

b): If $x_{-1} < 0 < 1 < x_0$, and $0 < B < 1$, and $x_1 = Bx_{-1}$, and $1-Bx_{-1} > Bx_0$, then

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, 1-Bx_{-1}, B^2x_{-1}, 1-B^2x_{-1}, \ldots, B^nx_{-1}, 1-B^nx_{-1}, \ldots).$$

c): If $x_{-1} < 0 < 1 < x_0$, and $B > 1$, and $x_1 = Bx_{-1}$, then

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, Bx_0, \ldots, B^nx_{-1}, B^nx_0, \ldots).$$

d): If $x_{-1} < 0 < 1 < x_0$, and $B > 1$, and assume $x_1 = 1-x_0$, then

$$(x_n) = (x_{-1}, x_0, 1-x_0, Bx_0, B(1-x_0), B^2x_0, B^2(1-x_0), \ldots, B^nx_0, B^n(1-x_0), \ldots).$$

Proof:

a) Let $x_{-1} < 0 < 1 < x_0$, and $0 < B < 1$, if $x_1 = 1-x_0$, then $x_2 = \max\{x_0, Bx_0\} = x_0$ because $x_0 > 0$ and $B < 1$. $x_3 = \max\{1-x_0, B(1-x_0)\} = B(1-x_0)$ because $1-x_0 < 0$ and $B < 1$, $x_4 = \max\{1-B(1-x_0), Bx_0\} = 1-B(1-x_0)$ because $1-B(1-x_0) = 1-B+Bx_0 > Bx_0$ since $B < 1$. Now assume, we have $a = 1-x_0$, and using Lemma 3.3, then the results follow.

$$(x_n) = (x_{-1}, x_0, 1-x_0, x_0, B(1-x_0), 1-B(1-x_0), \ldots, B^n(1-x_0), 1-B^n(1-x_0), \ldots).$$

b) Let $x_{-1} < 0 < 1 < x_0$, $0 < B < 1$, $x_1 = Bx_{-1}$, and assume $1 - Bx_{-1} \geq Bx_0$, then

$x_2 = \max\{1 - x_1, Bx_0\} = \max\{1 - Bx_{-1}, Bx_0\} = 1 - Bx_{-1}$.

At this point we apply the Lemma 3.3 where a= x_{-1}. Thus,

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, 1-Bx_{-1}, B^2x_{-1}, 1-B^2x_{-1} \ldots B^n x_{-1}, 1-B^n x_{-1}, \ldots).$$

c) Assume for n=0,1,2,…, $x_{2n-1} = B^n x_{-1}$, and $x_{2n} = B^n x_0$, then $x_1 = Bx_{-1}$

implies $1-x_0 \leq Bx_{-1}$, and thus $1-Bx_{-1} \leq x_0$. Then $x_{2n+1} = \max\{1-B^n x_0, B^{n+1}x_{-1}\} =$

$B^{n+1}x_{-1}$ because $1-B^n x_0 < B^n - B^n x_0 = B^n(1-x_0) \leq B^{n+1}x_{-1}$,

$x_{2n+2} = \max\{1-B^{n+1}x_{-1}, B^{n+1}x_0\} = B^{n+1}x_0$ because

$1-B^{n+1}x_{-1} < B^n - B^{n+1}x_{-1} = B^n(1-Bx_{-1}) \leq B^n x_0 < B^{n+1}x_0$. Hence

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, Bx_0, \ldots, B^n x_{-1}, B^n x_0, \ldots).$$

d)Let $x_{-1} < 0 < 1 < x_0$, and B> 1, and assume $x_1 = 1-x_0$, $x_2 = \max\{1-x_1, Bx_0\} =$

$\max\{x_0, Bx_0\} = Bx_0$, then $x_3 = \max\{1-x_2, Bx_1\} = \max\{1-Bx_0, B-Bx_0\} = B(1-x_0)$.

Therefore, $x_4 = \max\{1-x_3, Bx_2\} = \max\{1-B(1-x_0), B^2x_0\} = B^2x_0$, because

$1-B(1-x_0) < B - B(1-x_0) = Bx_0 < B^2x_0$.By induction we see that $x_{2n} = B^n x_0$,

and $x_{2n-1} = B^{n-1}(1-x_0)$. Therefore

$$(x_n) = (x_{-1}, x_0, 1-x_0, Bx_0, B(1-x_0), B^2x_0, B^2(1-x_0), \ldots, B^n x_0, B^n(1-x_0), \ldots). \text{ Q.E.D}$$

Note: The solutions in (a),(b), and in (c) are oscillating around zero, but the solutions in (c) and (d) are unbounded.

Theorem 3.2.2: Consider $x_{n+1} = \max\{1\text{-}x_n, Bx_{n-1}\}$

a): If $0< x_{-1} < 1 < x_0$, and $0< B< 1$, and assume $Bx_0 \leq 1\text{-}Bx_{-1}$, then

$(x_n) = (x_{-1}, x_0, Bx_{-1}, 1\text{-}Bx_{-1}, Bx_{-1}, 1\text{-}B x_{-1}, \ldots, Bx_{-1}, 1\text{-}Bx_{-1}, \ldots)$.

b): If $0< x_{-1} < 1 < x_0$, and $B> 1$, then

$(x_n) = (x_{-1}, x_0, Bx_{-1}, Bx_0, \ldots, B^n x_{-1}, B^n x_0, \ldots)$.

Proof:

a) Let $0< x_{-1} < 1 < x_0$, and $0<B< 1$, and assume

$Bx_0 \leq 1\text{-}Bx_{-1}$, then $x_1 = \max\{1\text{-}x_0, Bx_{-1}\} = Bx_{-1}$, $x_2 = \max\{1\text{-}x_1, Bx_0\} =$ $\max\{1\text{-}Bx_{-1}, Bx_0\} = 1\text{-}Bx_{-1}$, and $x_3 = \max\{1\text{-}x_2, Bx_1\} = \max\{Bx_{-1}, B^2x_{-1}\} = Bx_{-1}$, and $x_4 = \max\{1\text{-}x_3, Bx_2\} = \max\{1\text{-}Bx_{-1}, B(1\text{-}Bx_{-1})\} = 1\text{-}Bx_{-1}$. Using induction it is easy to see that $x_{2n} = 1\text{-}Bx_{-1}$, and $x_{2n-1} = Bx_{-1}$ for $n \geq 1$. Hence

$(x_n) = (x_{-1}, x_0, Bx_{-1}, 1\text{-}Bx_{-1}, Bx_{-1}, 1\text{-}B x_{-1}, \ldots, Bx_{-1}, 1\text{-}Bx_{-1}, \ldots)$. Therefore, x_n is eventually 2 periodic.

b) Let $0< x_{-1} < 1 < x_0$, $B> 1$. Then $x_1 = Bx_{-1}$ and

$x_2 = \max\{1\text{-}x_1, B x_0\} = \max\{1\text{-}Bx_{-1}, Bx_0\} = Bx_0$ since $1\text{-}Bx_{-1} < 1 < Bx_0$, and $x_3 =$ $\max\{1\text{-}x_2, Bx_1\} = \max\{1\text{-}Bx_0, B^2x_{-1}\} = B^2x_{-1}$, and

$x_4 = \max\{1\text{-}x_3, Bx_2\} = \max\{1\text{-}B^2x_{-1}, B^2x_0\} = B^2x_0$. By induction we obtain

$x_{2n} = B^n x_0$, and $x_{2n-1} = B^n x_{-1}$ for $n \geq 1$. Thus

$(x_n) = (x_{-1}, x_0, Bx_{-1}, Bx_0, \ldots, B^n x_{-1}, B^n x_0, \ldots)$. Q.E.D.

Note: The solutions in (a) are bounded; however, the solutions in (b) are unbounded.

Theorem 3.2.3: Consider $x_{n+1} = \max\{1\text{-}x_n, Bx_{n-1}\}$

a) If $0< x_{-1}, x_0 < 1$, and $0< B < 1$, and assume x_1=1-x_0, then

$$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, \ldots, 1\text{-}x_0, x_0, \ldots).$$

b) If $0< x_{-1}, x_0 < 1$, $0< B < 1$, and x_1=Bx_{-1}, and 1-Bx_{-1} >Bx_0 , then

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, 1\text{-}Bx_{-1}, \ldots Bx_{-1}, 1\text{-}Bx_{-1}, \ldots).$$

c) If $0< x_{-1}, x_0 < 1$, and B> 1, and assume x_1= 1-x_0, then

$$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, Bx_0, B(1\text{-}x_0), B^2x_0, B^2(1\text{-}x_0), \ldots, B^n x_0, B^n(1\text{-}x_0), \ldots).$$

d) If $0< x_{-1}, x_0 < 1$, and B> 1, and assume x_1= Bx_{-1}, then

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, Bx_0, \ldots, B^n x_{-1}, B^n x_0, \ldots).$$

Proof:

a) Let $0< x_{-1}, x_0 < 1$, and $0< B < 1$, and assume x_1=1-$x_0 > 0$, then x_2= $\max\{x_0, Bx_0\}$= x_0, and x_3= $\max\{1\text{-}x_0, B(1\text{-}x_0)\}$= 1-$x_0$. Using induction it is easy to see that $x_n > 0$ for every n≥ 0, consequently, x_{2n}= x_0, and x_{2n-1}= 1-x_0. Thus the solutions are 2-periodic and can be written as follows,

$$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, \ldots, 1\text{-}x_0, x_0, \ldots).$$

c) If $0< x_{-1}, x_0 < 1$, and $0< B < 1$, and assume x_1=Bx_{-1}, and 1-Bx_{-1} > Bx_0, then

$$x_2 = \max\{1\text{-}x_1, Bx_0\} = \max\{1\text{-}Bx_{-1}, Bx_0\} = 1\text{-}Bx_{-1}$$

$$x_3 = \max\{1\text{-}x_2, Bx_1\} = \max\{Bx_{-1}, B^2x_{-1}\} = Bx_{-1}$$

$$x_4 = \max\{ 1\text{-}x_3, Bx_2\} = \max\{ 1\text{-}Bx_{-1}, B(1\text{-}Bx_{-1})\} = 1\text{-}Bx_{-1} \text{ because } 1\text{-}Bx_{-1} > 0.$$

By induction x_{2n}= 1-Bx_{-1}, and x_{2n-1}= Bx_{-1}. Hence

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, 1\text{-}Bx_{-1}, Bx_{-1}, 1\text{-}Bx_{-1}, \ldots Bx_{-1}, 1\text{-}Bx_{-1}, \ldots).$$

c) The proof is similar to 3.2.1 (d)

d) Let $0< x_{-1},x_0 <1$, B> 1 and x_1=Bx_{-1} ; therefore, 1-x_0 ≤Bx_{-1}. Then

x_2= max{1-Bx_{-1}, Bx_0}= Bx_0 because 1-B$x_{-1} \le x_0$ <Bx_0,

x_3= max{ 1-Bx_0 ,B$^2x_{-1}$}= B$^2x_{-1}$ because 1-B$x_0 < 1 - x_0 \le$ Bx_{-1} < B$^2x_{-1}$

x_4 = max{1-B^2x_{-1},B^2x_0}= B^2x_0 because 1-B$^2x_{-1} < 1 -$B$x_{-1} \le x_0 <$ B$^2x_0$.

By induction, we have x_{2n-1}= B$^nx_{-1}$, x_{2n}= B$^nx_0$. Hence

$$(x_n) = (x_{-1}, x_0, Bx_{-1}, Bx_0, \ldots, B^nx_{-1}, B^nx_0, \ldots).$$ Q.E.D.

Note: The solutions in (a) and (b) are periodic with period two, but the solutions in (c) and (d) are unbounded.

Theorem 3.2.4: Consider x_{n+1} = max{ 1-x_n, Bx_{n-1}}

a) $x_{-1}, x_0 < 0$, and 0< B < 1, then

$$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, Bx_0, 1\text{-}Bx_0, \ldots, 1\text{-} B^{n-1}x_0, B^nx_0, \ldots).$$

b) x_{-1}, $x_0 < 0$, and B> 1, then

$$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, x_0, \ldots, B^n(1\text{-}x_0), B^nx_0, \ldots).$$

Proof:

a) Let $x_{-1} < x_0 < 0$, and 0< B < 1, then x_1= max {1-x_0, Bx_{-1}}= 1-x_0 , and x_2= max{1-x_1, Bx_0}= max{x_0, Bx_0}= Bx_0, and x_3= max{1-x_2, Bx_1}= max{1-Bx_0,B-Bx_0}= 1-Bx_0, and x_4= max{ 1-x_3, Bx_2}= max{ Bx_0, B$^2x_0$}= B$^2x_0$. Using induction it is easy to see that x_{2n-1}=1-B$^{n-1}x_0$, x_{2n}= B$^nx_0$. Hence

$$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, Bx_0, 1\text{-}Bx_0, \ldots 1\text{-} B^{n-1}x_0, B^nx_0, \ldots).$$

b) If $x_{-1} < x_0 < 0$, and B> 1, then x_1= max{1-x_0, Bx_{-1}}=1-x_0,

x_2= max{x_0, Bx_0}= x_0, and x_3= max{1-x_0, B(1-x_0)}= B(1-x_0), and

x_4= max{1-x_3,Bx_2}= max{1-B(1-x_0),Bx_0}= Bx_0, and x_5 = max{1-x_4, Bx_3}= B^2(1-x_0).

By induction we obtain that x_{2n} = $B^{n-1}x_0$, and x_{2n-1}= $B^{n-1}(1-x_0)$ for all n≥ 1.

Therefore, the solutions are unbounded. The even subsequences are divergent to negative infinity ($-\infty$), and the odd subsequences are divergent to infinity (∞). Hence it can be written as follows.

$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, x_0, \ldots B^n(1\text{-}x_0), B^n x_0, \ldots).$ Q.E.D

Theorem 3.2.5: Consider $x_{n+1}= \max\{1\text{-}x_n, Bx_{n-1}\}$

a) If $x_{-1}< 0 < x_0 \leq 1$, B=1, then
$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, x_0, 1\text{-}x_0, x_0, \ldots ,1\text{-}x_0, x_0,..).$

b) If $x_{-1}< 0 < 1< x_0$, B=1, and assume that x_1= 1-x_0, then
$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, x_0, 1\text{-}x_0, \ldots ,1\text{-}x_0, x_0,..).$

c) If $x_{-1} < 0 < 1< x_0$, B=1, and assume that x_1= x_{-1}, then
$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, \ldots, x_{-1}, x_0,\ldots).$

d) If $0< x_{-1} < 1< x_0$, and B =1, then
$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, \ldots ,x_{-1}, x_0,\ldots).$

e) If $0 < x_{-1}$, x_0 < 1, and B =1, and assume x_1= 1-x_0, then
$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, x_0, 1\text{-}x_0, \ldots, 1\text{-}x_0, x_0,..).$

f) If $0 < x_{-1}$, x_0 < 1, and B =1, and assume x_1= x_{-1}, then
$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, \ldots, x_{-1}, x_0, \ldots)$

g) If x_{-1}, x_0 < 0 and B =1, then
$(x_n) = (x_{-1}, x_0, 1\text{-}x_0, x_0, 1\text{-}x_0, \ldots ,1\text{-}x_0, x_0,..).$

h) If x_{-1}, x_0 > 1, and B =1 , then
$(x_n) = (x_{-1}, x_0, x_{-1}, x_0, \ldots ,x_{-1}, x_0,\ldots).$

Proof:

a) If $x_{-1} < 0 < x_0 \leq 1$ and B =1 ,then x_1= max { 1-x_0, x_{-1}} = 1-x_0,

x_2= max{ 1-x_1, x_0}= max{x_0, x_0} = x_0, x_3= max{ 1-x_2, x_1} =1-x_0, and

x_4= max{1-x_3, x_2}= x_0. By induction, we easily see that x_{2n-1}= 1-x_0, x_{2n}=x_0.

Hence (x_n) = (x_{-1}, x_0, 1-x_0, x_0, 1-x_0, x_0, … 1-x_0, x_0,..).

b) The proof is similar to 3.2.1 (a).

c) If $x_{-1} < 1 < x_0$, and B=1, and assume that $x_{-1} > 1 - x_0$ which implies

$x_0 > 1 - x_{-1}$, then x_1= max{1-x_0, x_{-1}}= x_{-1},

x_2= max{ 1-x_1, x_0} = max{ 1-x_{-1}, x_0}= x_0, because

if 1-x_{-1} > x_0, then x_0+x_{-1} < 1, contradicts our assumption, and

x_3= max{ 1-x_2, x_1}= max{ 1-x_0, x_{-1}}= x_{-1}. By induction x_{2n-1}= x_{-1}, x_{2n}= x_0. Thus

(x_n) = (x_{-1}, x_0, x_{-1}, x_0, … x_{-1}, x_0,…).

d) The proof is similar to 3.2.2 (b)

e) The proof is similar to 3.2.3 (a)

f) The proof is similar to 3.2.3 (d)

g) The proof is similar to 3.2.4

h) Assume $1 < x_{-1}$, x_0, then x_1= max{1-x_0, x_{-1}}= x_{-1}, x_2= max{1-x_1, x_0}=

max{ 1-x_{-1}, x_0}= x_0. By induction x_{2n-1}= x_{-1} and x_{2n}= x_0. Q.E.D

CHAPTER 4

SUMMARY, CONCLUSION AND RECOMMENDATION

This study is focused on investigating the solutions of reciprocal type difference equation and max-linear difference equations. The solutions of these difference equations exhibited various properties, such as periodicity , boundedness,unboundness, oscillation, and non oscillation. The research also shed light on discrete dynamical system theory since difference equations are examples of discrete dynamical systems.

The investigation of max-type difference equations has attracted much attention recently because solutions behaviors can differ greatly from solutions of linear difference equations.The max-type difference equations of higher order remain to be studied and will be a source of much research in the future. Additional research with the maximum operator being replaced with the minimum operator or a median operator may also lead to new equations to study.

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