Trigonometric Polynomials Methods to Simulate Oscillating Chaotic Systems

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INTRODUCTION

Numerical simulations are an important tool in the understanding of chaotic dynamic systems. Limitations inherent to the simulations may lead to erroneous interpretations of the behavior of these systems. Among these limitations are superstability (suppression of the chaos of chaotic systems), computational chaos (induction of chaos in non-chaotic systems) and apparent contradictions between theoretical predictions and observed responses in the simulations. These issues are present whether the solver is fixed-step or variable-step. If we are to trust the output of numerical approaches when solving dynamic systems, it is of utmost importance to know whether any simulations are yielding the correct answers, or whether such answers have been affected by errors. Such errors are either inherent to the numerical approach or due to rounding. In both cases, they may alter the outcome to the point of not representing the true dynamics of the system. Simulations ought to reflect the system's true behavior. The outcomes of the simulations must be coherent with the simulated system. If one wants to avoid numerical approaches that may induce computational chaos, superstability or theoretical discrepancies, *it is necessary to choose the right numerical method*.

This research investigates the potential selection of a numerical method capitalizing on known characteristics of the dynamic system. The use of special methods that tap on some known features of the chaotic dynamic system at hand has received little attention thus far. In this work, we propose the use of trigonometric polynomials methods (TPM's) derived by Gautschi [1]. Gautschi's method leverages on the oscillatory response inherent to chaotic systems. We *hypothesize* that using Gautschi's method should result in simulations that adhere more accurately to the real solution. This is in contrast to the use of traditional methods commonly applied for these kinds of simulations. We give evidence that the adequate choice of an special method tapping on prior knowledge should outperformed an otherwise default choice. Empirical evidence is given in the form of simulations over a set of chaotic systems. As additional advantages, superstability is avoided and choosing of the integration step is straight-forward as it is based on the system frequency.

RELATED WORK

Often, chaotic systems are simulated using traditional fixed-step approaches such as explicit Euler, implicit Euler or Runge-Kutta (R-K) often of order 4. However, there are many examples in literature evidencing instabilities such as chaos suppression or induction when using fixed-step methods on chaotic systems [2, 3, 4, 5]. In general, the misbehavior is associated to the choice of the integration step size [5]. Although, a methodology has been put forth for choosing the integration step size [5], this is neither generic i.e. it is only for Euler methods, or free of hassles, e.g. lack of implementation detail may result in replication discrepancies, and use in large systems may be impractical.

When variable-step solvers are chosen, a popular choice for chaotic system simulation is Runge-Kutta-Fehlberg (4,5) (RKF45), but literature has also seen other examples. Variable-step, although palliative, are insufficient to eliminate improper behaviors in simulations. For instance, some non-chaotic continuous dynamic systems of order two, when simulated with RKF45, exhibit behaviors that can be interpreted as chaotic [6]. Further, hardly ever, the choice of the approach is discussed or justified in terms of the properties of the chaotic system.

In order to ensure trustable numerical simulations of chaotic dynamic systems, literature presents several attempts. One of first attempts was Corless' *control defect* [7]. With this control, if the norm of defect is equal or minus to 1, then the exact solution to a nearby problem is obtained. However, the defect control for numerical methods to higher order, such as the R-K, is computationally expensive. Moreover, it is necessary to choose the appropriate norm for the defect which is dependent on the problem. Later, the Clean Numerical Simulation (CNS) capitalized on high performance computing and used high order Taylor series e.g. at least 50 terms, and data in the multiple precision with large enough number of digits e.g. at least 300 digits, plus a convergence check with smaller numerical noises. A shortcoming of CNS is the excessive computational demand required. Also, a strategy based on interval extension exists to detect suspicious simulations for chaotic mappings characterized by recursive functions. The rationale involves estimating an lower bound for the error between two different pseudo-orbits produced from two equivalent mathematical models on distinct floating point representations. The simulation is rejected if the lower bound for the error is bigger than the required precision. This strategy has been successfully applied to logistic mapping, whereby the Lyapunov exponent was calculated with high precision [8]. Because the extension by intervals uses recursive functions, its application is computationally expensive. Also, each combination of numerical method and chaotic system requires its own solution. Pano et al. [9] suggested a numerical method based on trigonometric polynomials [1] that exploits the oscillatory property of the chaotic systems. They chose the variable-step method with local error control based on RKF(4,5) used for comparison against the fixed-steps methods, even though the chosen method may present improper behavior [6]. Notwithstanding, it represents a departing point for our research here.

METHODS

We empirically test our hypothesis by showing simulations over four known chaotic systems. We simulated the chaotic systems of Chen [10], Chua [11, 12], Lorenz [13] and Rossler [14]. These are all recurrently found in the domain literature, and further, they are known to exhibit the aforementioned numerical issues of computational chaos and superstability when simulated with regular environments, e.g. Mathematica or Matlab with default parameterization. Regarding the numerical methods, the performance of a TPM, in this case Gautschi's method, was compared to several traditional fixed-step alternatives. The TPM used was Gautschi [1] of algebraic order 2. The frequency of each of the chaotic systems was numerically estimated before the simulations by means of the characteristic values (eigenvalues) of the respective Jacobian matrix. For fixed step we considered Forward-Euler (FE), Back-Euler (BE) and Runge-Kutta-4 (RK-4) for comparison. Both FE, BE and RK-4 are well studied. BE was implemented using the Predict-Evaluate-Correct-Evaluate (PECE) strategy with two corrections [15]. We used Varsakelis methodology [5] to choose the integration step in the case of the Eulers method. For the other cases, numerical stability theory, both absolute and relative, was used to derive an upper bound for the integration step. For each chaotic system, the integration steps of each fixed step method were calculated, and the shortest one was chosen for further simulations. In the case of TPM's it is convenient to know the frequency of the system beforehand. Otherwise, it is advisable to underestimate it [1]. We opted for underestimation in order to more realistically exemplify scenarios faced by users where the precise frequency of the system is unknown. This permitted us to evaluate the impact of such uncertainty. To facilitate comparisons of results across the simulations of the different systems between fixed-step and variable-step strategies, we choose a referential method that can keep simulations without losing chaos for over 50,000 time units. Our references was a variable step method with local error control. Among our candidates, we have a RKF(4,5) as implemented by [16], an extrapolation method (ODEX) as implemented by [17], and an Adams Bashforth Adams Moulton (AB-AM) that incorporates variable order as implemented by [18]. This type of methods necessitates the user to provide the absolute and relative errors to keep the numerical precision within the required intervals, as well as indicating the output time period of the results to reach the output frequency rate. The output frequency rate was made equal to the shortest step of the fixed methods to guarantee that all simulations have the same number of observations.

The numerical results were evaluated using both quantitative and qualitative measures. The quantitative measures were the maximum Lyapunov exponent, the Kolmogorov-Sinai (K-S) entropy, and the computational costs. The maximum Lyapunov exponent and the K-S entropy were calculated using Tisean's routines [19]. The qualitative measures was observation of the phase space around attractors of the chaotic systems for verification purposes. For variable-step solvers, the maximum Lyapunov exponent and the K-S entropy were calculated for the starting and ending intervals of the simulations. Each interval was taken to have $100,000$ iterations (data) on the simulations. For fixed-step solvers, these measures were estimated only for the ending interval. These metrics provide an immediate evaluation of superstability for fixed-step solvers. Finally, the efficiency of the fixed methods, was established in terms of the number of evaluations (calls to the subroutine containing the model) executed by the numerical method during the whole simulations. This is a good proxy of the computational cost associated to the method. All simulations were carried out using Fortran PGI.

RESULTS

For the variable-step methods, simulations were carried out considering absolute and relative errors between 10^{-8} and 10^{-2}. The only method that succeeded in avoiding superstability was AB-AM with absolute and relative error equal to 10^{-8}. Table I reports the integration steps estimated for the fixed-step numerical methods. As expected, despite the use of Varsakelis methodology [5], the integration step employed satisfies this methodology, BE exhibited superstability for Chen system. This emphasizes the word of caution when employing BE for chaotic systems. Table II reports the maximum Lyapunov exponents and the K-S entropies of the simulations that used the AB-AM integrating method. These values were further used as reference to compare the performance of the fixed-step methods.

TABLE I. Integration steps estimated for the fixed-step numerical methods for each chaotic system. The letter bold indicates the step chosen for the simulations. The parameterization and initial conditions of the systems were as follows: Chen: $a = 35$, $b = 3$, $c = 28$ and $(-10, 0, 37)$ [10], Chua: $\alpha = 10$, $\beta = 11.5$, $m_1 = -0.276$, $m_2 = -3.036$, $m_1 = -0.276$, $b_2 = 0.8$, $b_3 = 1.4$ y $(0.1, 0, 0)$ [12], Lorenz: $\sigma = 10$, $r = 28$, $b = 8/3$ with $(-15.8, -17.48, 35.64)$ [13], and Rossler: $a = b = 0.2$, $c = 5.7$ y $(-9, 0, 0)$ [14].

Numerical method/Chaotic system	Chen	Chua	Lorenz	Rossler
FE	**0.0020**	**0.0010**	0.0200	**0.0050**
BE	0.0250	0.0100	**0.0020**	0.0100
RK-4	0.0500	0.1250	0.1000	0.2000
TPM	0.0125	0.0910	0.0230	0.1110
Frequency [a]	20	3	20	1.5

[a] Subestimated frequency used by the TPM.

TABLE II. Performance for the tested chaotic systems simulated with AB-AM. Statistics are only reported for AB-AM since this was the only method the succeeded in avoiding superstability.

Chaotic system	Evaluation time	Max Lyapunov exponent	Standard error	Confidence interval 95%	KS Entropy	Standard error	Confidence interval 95%
Chen	At Start[a]	0.105	0.004	(0.097, 0.112)	0.0247	0.0005	(0.0239, 0.0256)
	At final	0.105	0.005	(0.097, 0.112)	0.0273	0.0005	(0.0265, 0.0281)
Chua	At Start[a]	0.059	0.003	(0.054, 0.064)	0.0142	0.0004	(0.0135, 0.0148)
	At final	0.067	0.002	(0.062, 0.071)	0.0156	0.0004	(0.0150, 0.0162)
Lorenz	At Start[a]	0.072	0.003	(0.067, 0.077)	0.0700	0.0009	(0.0685, 0.0716)
	At final	0.070	0.003	(0.065, 0.075)	0.0650	0.0010	(0.0634, 0.0667)
Rossler	At Start[a]	0.062	0.002	(0.058, 0.066)	0.0137	0.0003	(0.0132, 0.0142)
	At final	0.052	0.002	(0.048, 0.056)	0.0152	0.0003	(0.0147, 0.0157)

[a] At start: Calculated at the simulation onset; At final: Calculated at the simulation offset.

The phase space of the Chen system for some of the methods are shown in Figure 1. Figure 2 shows the relative errors of the quantitative metrics against the reference and the computational cost of the approaches as proxied by the number of evaluations. The TMP permitted the preservation of chaos in large simulations (of at least 50,000 time units). The choice of integration step is easy being based on the system frequency. And, moreover, the computational cost is three times smaller than compared alternatives. Since the TPM (Gautschi) is a method that exploits the oscillatory characteristics of the systems, with this work it is conjectured that it outperformes the other fixed-step methods. However, more evidence is needed to support this claim. In this research, we did not considered chaotic system with large Lyapunov exponents. Systems with this particularity are even more sensitive to initial conditions.

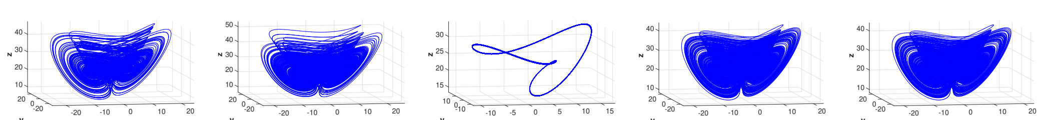

FIGURE 1. Phase space of the Chen system for some of the methods. Left to right: AB-AM, FE, BE, RR-4 and TPM. BE failed to show the expected chaotic attractor and exhibited superstability.

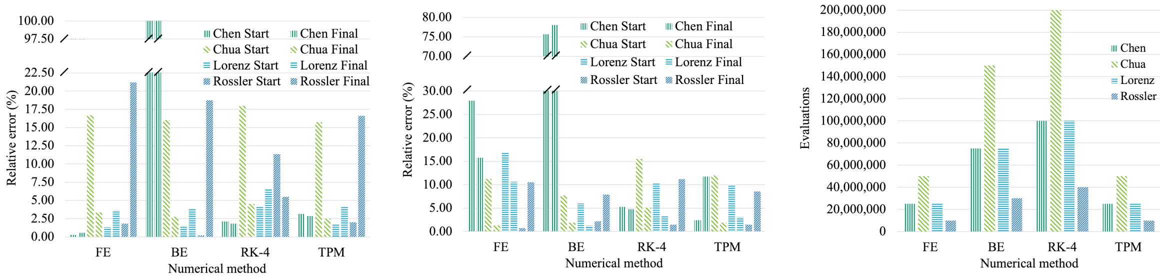

FIGURE 2. (Left) Relative errors on the maximum Lyapunov exponents for the fixed-step methods across the different systems. (Middle) Relative errors on the K-S entropy for the fixed-step methods across the different systems. (Right) Number of evaluations for the fixed-step methods across the different systems.

CONCLUSIONS

We have provided evidence regarding how the use of a TPM that exploit a feature of the chaotic system permitted trustable simulations of chaotic systems, and further show that such simulations are computationally more economic than traditional methods. Our results suggest that ignoring the particularities of the system being simulated is unwise; the adequate choice of the numerical method is fundamental to provide the best guarantees in the outcomes of the simulation, thus supporting our hypothesis. We expect this research to bring further attention to special numerical methods. As future research, we intend to carry out an analysis of stability of the proposed method, and to evaluate the performance of the numerical methods based on trigonometric polynomials with algebraic order bigger than 2.

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