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Cohen-Macaulay Properties of Closed Neighborhood Ideals

A Thesis
Presented to
the Graduate School of
Clemson University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science
Mathematical Sciences

by
Jackson Leaman
May 2023

Accepted by:
Dr. Keri Ann Sather-Wagstaff, Committee Chair
Dr. Michael Burr
Dr. Beth Novick

Abstract

This thesis investigates Cohen-Macaulay properties of squarefree monomial ideals, which is an important line of inquiry in the field of combinatorial commutative algebra. A famous example of this is Villareal's edge ideal [11]: given a finite simple graph G with vertices $x_1, \ldots, x_n$, the edge ideal of G is generated by all the monomials of the form $x_i x_j$ where x_i and x_j are adjacent in G. Villareal's characterization of Cohen-Macaulay edge ideals associated to trees is an often-cited result in the literature. This was extended to chordal and bipartite graphs by Herzog, Hibi, and Zheng in [7] and by Herzog and Hibi in [6].

In 2020, Sharifan and Moradi [10] introduced a related construction called the closed neighborhood ideal of a graph. Whereas an edge ideal of a graph G is generated by monomials associated to each edge in G, the closed neighborhood ideal is generated by monomials associated to its closed neighborhoods. In 2021, Sather-Wagstaff and Honeycutt [8] characterized trees whose closed neighborhood ideals are Cohen-Macaulay. We will provide a generalization of this characterization to chordal graphs and bipartite graphs. Additionally, we will survey the behavior of the depth of closed neighborhood ideals under certain graph operations.

Acknowledgments

First, I would like to acknowledge the tremendous amount of time and effort Dr. Keri Ann Sather-Wagstaff has devoted to guiding me over the course of our research. Her patience, understanding, tutelage, and clear-headedness have made this project possible, and for that I am deeply grateful.

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Finally, I wish to thank my significant other, who has provided much stability, and whose companionship has proved an endless source of joy.

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Chapter 1

Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field. Unless otherwise specified, we will always use this ring. Likewise, throughout this thesis, G will always be a finite simple graph with a vertex set $V(G) = \{x_1, \ldots, x_n\}$. The recurring theme of this thesis is to associate squarefree monomial ideals in R to G, and to use properties of G to deduce algebraic properties of the ideal.

This introduction will provide two examples of ideals associated to graphs. First, edge ideals, which are very well-studied, then closed neighborhood ideals, which are rather new. The novel results in this thesis deal with closed neighborhood ideals. The purpose of discussing edge ideals is to supply the appropriate motivation for closed neighborhood ideals, as many basic results about closed neighborhood ideals are analogous to results for edge ideals.

Chapters 2–4 will provide all the necessary algebraic and graph theoretic background for the reader, while Chapter 5 will discuss our new results.

1.1 Edge Ideals

Edge ideals provide a 1-1 correspondence between squarefree quadratic monomial ideals (ideals generated by monomials of the form $x_i x_j$ with $i \neq j$) and finite simple graphs. Let us start with a definition.

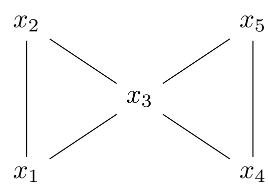
Definition 1.1.1. *The* **edge ideal** *of G, denoted $I(G)$, is the squarefree monomial ideal in R,*

$$I(G) = \langle x_i x_j : \quad x_i \sim x_j \rangle$$

where $x_i \sim x_j$ if x_i and x_j are adjacent in G, i.e., $x_i x_j$ is an edge in G.

We provide a simple example, which will become a running example throughout this thesis.

Example 1.1.2. *Let G be the butterfly graph with vertices as labeled below.*



Then $I(G) = \langle x_1x_2, x_1x_3, x_2x_3, x_3x_4, x_3x_5, x_4x_5 \rangle$ because the edges of G are x_1x_2, x_1x_3, and so on.

Edge ideals were introduced in [11] by Villarreal. Much of the theory of edge ideals has been concerned with conditions under which the quotient ring $R/I(G)$ is Cohen-Macaulay, in which case G is called a Cohen-Macaulay graph.

Cohen-Macaulayness is an important and largely studied property of a ring in many fields, including algebraic geometry. It is a niceness condition like smoothness and unmixedness that is nicer in the sense that it is preserved under taking general hypersurface sections.

Villarreal found in the aforementioned paper that a tree T is Cohen-Macaulay if and only if every minimal vertex cover of T has the same size [11, Corollary 2.5]. We will define the Cohen-Macaulay property in Chapter 3, but we will address vertex covers now.

Definition 1.1.3. *A* **vertex cover** *of G is a set of vertices $C \subset V(G)$ such that for every edge $e \in E(G)$, there is a vertex $v \in C$ such that v is incident with e. A vertex cover is said to be* **minimal** *if it does not properly contain any vertex cover.*

Minimal vertex covers have a strong connection to special primes in R called minimal primes, which we define as follows.

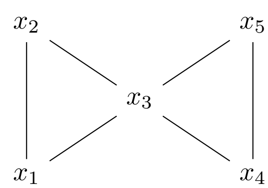
Definition 1.1.4. *Let $I \subset R$ be an ideal. A* **minimal prime** *of I is a prime ideal P containing I that does not properly contain any other prime ideal containing I.*

Minimal primes provide the main connection between vertex covers and edge ideals. We state this connection in the following theorem.

Theorem 1.1.5 ([11], Page 279)**.** *The minimal primes of $I(G)$ are exactly the prime ideals generated by the minimal vertex covers of G. In other words, if $\mathcal{P}$ is the set of minimal vertex covers of G, then $I(G)$ has the following irreducible decomposition.*

$$I(G) = \bigcap_{P \in \mathcal{P}} \langle P \rangle$$

Example 1.1.6. *Consider the graph from Example 1.1.2.*



Then $\{x_1, x_3, x_4\}$ is a minimal vertex cover of G: indeed, every edge of G is incident with x_1, x_3, or x_4, and no proper subset of $\{x_1, x_3, x_4\}$ vertices is a vertex cover. It is straightforward to show that the remaining minimal vertex covers are $\{x_2, x_3, x_4\}$, $\{x_1, x_3, x_5\}$, $\{x_2, x_3, x_5\}$, $\{x_1, x_2, x_4, x_5\}$. Thus, Theorem 1.1.5 implies that $I(G)$ has the following irreducible decomposition.

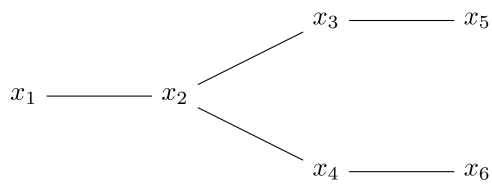
$$I(G) = \langle x_1, x_3, x_4 \rangle \cap \langle x_2, x_3, x_4 \rangle \cap \langle x_1, x_3, x_5 \rangle \cap \langle x_2, x_3, x_5 \rangle \cap \langle x_1, x_2, x_4, x_5 \rangle$$

We can already read certain algebraic properties off of this decomposition theorem: for example, the height of the ideal $I(G)$ is simply the number of variables that generate its smallest minimal prime, in this case 3. Hence $R/I(G)$ has (Krull) dimension $5 - 3 = 2$. We will define height and dimension in Chapter 3.

A graph G is said to be ***well-covered*** if all of its minimal vertex covers have the same size; in this event, the ideal $I(G)$ is ***unmixed***. We will see in Theorem 3.3.4 below that if $R/I(G)$ is Cohen-Macaulay, then $I(G)$ is necessarily unmixed. Hence, if G is a Cohen-Macaulay graph, then it is well-covered. In particular, this shows that the butterfly graph in Example 1.1.2 is not Cohen-Macaulay. Moreover, for trees, Villarreal proves the following.

Theorem 1.1.7 ([11], Corollary 2.5)**.** *Let G be a tree. Then G is Cohen-Macaulay if and only if it is well-covered.*

Example 1.1.8. *The following graph is Cohen-Macaulay because it is a tree and one checks readily that its minimal vertex covers all have size* 3*.*



1.2 Closed Neighborhood Ideals

Closed neighborhood ideals are constructed similarly to edge ideals, but the monomials we add correspond to closed neighborhoods of vertices in a graph G. We will start by defining closed neighborhoods.

Definition 1.2.1. *The* **closed neighborhood** *of a vertex $v \in G$ is the set*

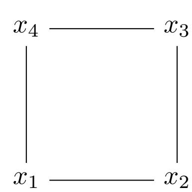
$$N[v] = \{v \in G : y \sim x \text{ or } v = x\}.$$

Example 1.2.2. *Consider the butterfly graph in Example 1.1.2. Then $N[x_1] = \{x_1, x_2, x_3\}$.*

Definition 1.2.3. *The* **closed neighborhood ideal** *of G, denoted $\mathrm{CNI}(G)$, is the squarefree monomial ideal*

$$\mathrm{CNI}(G) = \left\langle \prod_{x_i \in N[x_j]} x_i : \quad x_j \in G \right\rangle.$$

Example 1.2.4. *Let C_4 be the 4-cycle with vertices as labeled below.*



Then $\mathrm{CNI}(C_4) = \langle x_1x_2x_3, x_2x_3x_4, x_3x_4x_1, x_4x_1x_2 \rangle$ *because* $N[x_1] = \{x_1, x_2, x_4\}, N[x_2] = \{x_1, x_2, x_3\}$*, and so on.*

Closed neighborhood ideals are a more recent construction than edge ideals. They were introduced by Sharifan and Moradi in [10] in 2020. They study the projective dimension and regularity of closed neighborhood ideals, and provide a decomposition theorem similar to that in the case of edge ideals. To state this theorem, we first define dominating sets.

Definition 1.2.5. *A* **dominating set** *of G is a set of vertices $D \subset V(G)$ such that for every vertex $x \in V(G)$, there is a vertex $y \in D$ such that $x \in N[y]$, i.e. for every vertex $x \in V(G)$, x has a neighbor in D or x is itself in D. A dominating set is* **minimal** *if it does not properly contain any dominating set of G.*

Example 1.2.6. *Consider the graph C_4 in Example 1.2.4.*

$$\begin{tikzcd} x_4 \arrow[r, dash] \arrow[d, dash] & x_3 \arrow[d, dash] \\ x_1 \arrow[r, dash] & x_2 \end{tikzcd}$$

Then $\{x_1, x_2\}$ is a minimal dominating set, because $x_3 \sim x_2$ and $x_4 \sim x_1$, and neither $\{x_1\}$ nor $\{x_2\}$ are dominating sets of G. It is straightforward to show that the remaining minimal dominating are exactly $\{x_1, x_3\}$, $\{x_1, x_4\}$, $\{x_2, x_3\}$, $\{x_2, x_4\}$, and $\{x_3, x_4\}$.

The next result says that minimal dominating sets play the same role in decomposing closed neighborhood ideals as vertex covers do for edge ideals.

Theorem 1.2.7 ([10], Lemma 2.2)**.** *The minimal primes of* $\mathrm{CNI}(G)$ *are exactly the prime ideals generated by the minimal dominating sets of G. In other words, if $\mathcal{D}$ is the set of minimal dominating sets of G, then* $\mathrm{CNI}(G)$ *has the following irredundant prime decomposition.*

$$\mathrm{CNI}(G) = \bigcap_{D \in \mathcal{D}} \langle D \rangle$$

Example 1.2.8. *We have enumerated the minimal dominating sets in Example 1.2.4, so Theorem 1.2.7 gives the following decomposition of* $\mathrm{CNI}(C_4)$.

$$\mathrm{CNI}(C_4) = \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_1, x_4 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle$$

Again, this decomposition allows us to immediately compute the height of a closed neighborhood ideal of a graph G, and therefore the dimension of $R/\mathrm{CNI}(G)$. The 4-cycle in Example 1.2.4, for instance, has height 2 and dimension 2. If all minimal dominating sets of G have the same size, then G is said to be ***well-dominated***, and correspondingly the ideal $\mathrm{CNI}(G)$ is ***unmixed***.

Much of this thesis is following the work of Honeycutt and Sather-Wagstaff, who published a similar theorem to Villarreal's in [8]. To state this theorem, we need only assert one more definition.

Definition 1.2.9. *A graph G is called a* **whisker graph** *if every vertex of G has degree 1, or has a unique degree-1 neighbor. Equivalently, G is a whisker graph if it is obtained by the following process: start with a graph H, and append exactly one new vertex to each vertex in H. These new vertices are called the* **whiskers** *of G.*

Example 1.2.10. *Example 1.1.8 is a whisker graph. Its whiskers are x_1, x_5, and x_6.*

Theorem 1.2.11 ([8], Corollary 3.12)**.** *For a tree T, the following are equivalent.*

(i) *T is well-dominated.*

(ii) *T is a whisker graph.*

(iii) *$R/\mathrm{CNI}(T)$ is a complete intersection.*

(iv) *$\mathrm{CNI}(T)$ is Cohen-Macaulay.*

In particular, the Cohen-Macaulay condition for T is independent of the field k.

We will define a complete intersection ideal along with Cohen-Macaulayness in Chapter 3. In Chapter 5, two of our main results will be the following two generalizations of this theorem to bipartite graphs and to chordal graphs; see Theorems 5.3.6 and 5.4.9.

Theorem 1.2.12. *Let B be a connected bipartite graph with more than 4 vertices. Then the following are equivalent.*

(i) *B is well-dominated.*

(ii) *B is a whisker graph.*

(iii) *$\mathrm{CNI}(B)$ is a complete intersection.*

(iv) *$R/\mathrm{CNI}(B)$ is Cohen-Macaulay.*

In particular, the Cohen-Macaulay condition for B is independent of the field k.

Theorem 1.2.13. *Let G be a chordal graph. Then the following are equivalent.*

(i) G *is well-dominated.*

(ii) $\operatorname{CNI}(G)$ *is a complete intersection.*

(iii) $R/\operatorname{CNI}(G)$ *is Cohen-Macaulay.*

In particular, the Cohen-Macaulay condition for G is independent of the field k.

Chapter 2

Monomial Ideals - Background

The edge ideals and closed neighborhood ideals discussed in Section 1 belong to a special class of ideals called squarefree monomial ideals.

2.1 Definitions and Examples

Let us start with a few definitions.

Definition 2.1.1. *A* **monomial** *in the ring $R = k[x_1, \ldots, x_n]$ is an element of the form $\prod_{1 \le i \le n} x_i^{k_i}$ where each $k_i \in \mathbb{N} = \{0, 1, 2, \ldots\}$. A monomial is said to be* **squarefree** *if $k_i \le 1$ for every i.*

A monomial, in other words, is a product of powers of variables. For example $x_1 x_2^2$ is a monomial in $k[x_1, x_2]$, though it is not squarefree.

Definition 2.1.2. *If m is a monomial, then the* **support** *of m (denoted supp(m)) is the set of variables in m with nonzero exponent.*

Example 2.1.3. *Let $R = k[x_1, x_2, x_3, x_4]$. The support of $x_1 x_2^2 x_4$ is $\{x_1, x_2, x_4\}$.*

Definition 2.1.4. *A* **monomial ideal** *$I \subset R$ is an ideal with a generating set consisting entirely of monomials. A monomial ideal is* **squarefree** *if it has a generating set consisting entirely of squarefree monomials.*

Example 2.1.5. *Let $I = \langle x_1 x_2, x_2 x_3^2, x_3^3 x_4 \rangle$ and $J = \langle x_1 x_2, x_2 x_3, x_3 x_4 \rangle$. Then I is a monomial ideal and J is a squarefree monomial ideal.*

Example 2.1.6. *Any closed neighborhood ideal or edge ideal is a squarefree monomial ideal.*

Monomial ideals have a number of useful combinatorial properties, and squarefree monomial ideals even more so, as we will see after the assertion of a few more theorems, starting with Dickson's Lemma, a special case of Hilbert's Basis Theorem.

Theorem 2.1.7 ([9], Theorem 1.3.1)**.** *Let $I \subset R$ be a monomial ideal. Then I is generated by a finite set of monomials.*

Theorem 2.1.8 ([9], Theorem 1.1.9)**.** *Let I be a monomial ideal with monomial generators $m_1, m_2, \ldots, m_k$. Then for each monomial $f \in R$, we have $f \in I$ if and only if $m_i \mid f$ for some i.*

These theorems tell us that ideal membership is very easy to test for monomial ideals given a monomial generating set. It also allows us to quickly spot *redundant* and *irredundant* generators.

Definition 2.1.9. *A generating set of monomials $S = \{m_1, \ldots, m_k\}$ for a monomial ideal I is* **redundant** *if I is generated by some proper subset of S. Otherwise, it is* **irredundant***.*

Example 2.1.10. *The sets $\{x_1x_2x_3, x_1^2x_2x_3\}$ and $\{x_1x_2x_3\}$ are both generating sets for the monomial ideal $I = \langle x_1x_2x_3 \rangle$. The first is redundant whereas the latter is irredundant.*

Corollary 2.1.11. *Each monomial ideal I has a unique irredundant monomial generating set.*

This corollary and the previous few theorems all together assert that each monomial ideal I is generated by a unique, finite, irredundant set of monomials.

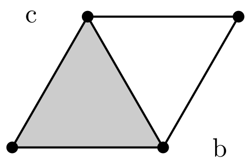
This thesis will restrict its discussion to squarefree monomial ideals, as they arrive most naturally from graphs. Hence, the remaining sections of this chapter will primarily be concerned with computations for squarefree monomial ideals.

2.2 Irreducible Decompositions

We stated in Chapter 1 that closed neighborhood ideals and edge ideals can be decomposed into intersections of prime ideals that are generated by sets which somehow cover the graph. In the case of edge ideals, these were vertex covers, which "cover" every edge; in the case of closed neighborhood ideals, these were dominating sets, which "cover" each closed neighborhood. In this section, we will make this more precise by establishing the connection between squarefree monomial ideals and simplicial complexes; see Examples 2.2.11 and 2.2.12.

Definition 2.2.1. *A* **simplicial complex** Δ *with vertex set* $V = \{x_1, \ldots, x_n\}$ *is a downward-closed collection of subsets of* V*: that is,* $\Delta \in P(V)$ *is such that whenever* $G \subseteq F \in \Delta$*, we have* $G \in \Delta$*. The elements of* Δ *are called* **faces** *of* Δ *and the maximal faces (those not properly contained in another face) are called* **facets**. *If* Δ *has facets* $F_1, \ldots, F_n$*, then we will sometimes write* $\Delta = \langle F_1, \ldots, F_n \rangle$ *and say that* Δ *is* **generated by** $F_1, \ldots, F_n$*. Additionally, if* $F = \{f_1, \ldots, f_n\}$ *is a collection of faces of* Δ *such that every facet of* Δ *is in* F*, we may also write* $\Delta = \langle f_1, \ldots, f_n \rangle$*.*

Example 2.2.2. *Let* $V = \{a, b, c, d\}$*. Then* Δ *as below is a simplicial complex on* V *with facets* $\{a, b, c\}, \{b, d\}$*, and* $\{c, d\}$*.*



Definition 2.2.3. *Let* Δ *be a simplicial complex. The* **dimension** *of a face* $F \in \Delta$ *is* $dim(F) = |F| - 1$*. The* **dimension** *of* Δ*, abbreviated* $dim(\Delta)$*, is the maximum dimension of its faces.*

Example 2.2.4. *The simplicial complex in Example 2.2.2 has dimension* 2*.*

Simplicial complexes on vertices $x_1, \ldots, x_n$ turn out to be in a $1 - 1$ correspondence with squarefree monomial ideals in $k[x_1, \ldots, x_n]$.

Definition 2.2.5. *The* **facet ideal** *of a simplicial complex* Δ *on* $x_1, \ldots, x_n$*, denoted* $\mathcal{F}(\Delta)$*, is the ideal in* $k[x_1, \ldots, x_n]$ *generated by squarefree monomials whose supports are the facets of* Δ*.*

Example 2.2.6. *Consider again the simplicial complex in Example 2.2.2. The facets of* Δ *are* $\{a, b, c\}, \{b, d\}$*, and* $\{b, c\}$*. Then* $\mathcal{F}(\Delta) = \langle abc, bd, bc \rangle$*.*

Definition 2.2.5 provides our main 1-1 correspondence between squarefree monomial ideals and simplicial complexes.

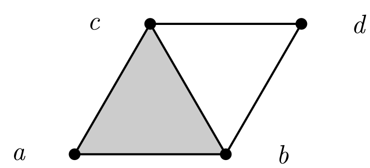
Definition 2.2.7. *Let* $I \subset k[x_1, \ldots, x_n]$ *be a squarefree monomial ideal generated by squarefree monomials* $m_1, \ldots m_k$*. Then the* **facet complex** *of* I *denoted* $\delta(I)$ *is the simplicial complex on* $x_1, \ldots, x_n$ *generated by the facets supp*(m_1)*, …, supp*(m_k)*.*

Example 2.2.8. *Let* $I = \langle abc, bd, bc \rangle \subset k[a, b, c, d]$*. Then* $\delta(I)$ *is the simplicial complex given in Example 2.2.2.*

It is easy to see that the maps δ and $\mathcal{F}$ are inverses of each other, and consequently define a bijection between simplicial complexes on $x_1, \ldots, x_n$ and squarefree monomial ideals in $k[x_1, \ldots, x_n]$. Through this bijection, combinatorial data about simplicial complexes can reveal algebraic data about their corresponding ideals, and vice versa. To establish this, we need the following definition.

Definition 2.2.9. *Let Δ be a simplicial complex on V. A* **vertex cover** *of Δ is a set $C \subset V$ such that for every facet $F \in \Delta$, there is some vertex $v \in C$ such that $v \in F$. A vertex cover is called* **minimal** *if it does not properly contain any vertex cover.*

Example 2.2.10. *Consider the simplicial complex Δ in Example 2.2.2.*



The set $\{a, d\}$ is a vertex cover because the facets of Δ are $\{a, b, c\}, \{b, d\}$, and $\{c, d\}$, each of which contains a or d. Moreover, $\{a, d\}$ is a minimal vertex cover, since neither $\{a\}$ nor $\{d\}$ are vertex covers. One readily checks that the remaining minimal vertex covers are $\{b, c\}, \{b, d\}$, and $\{c, d\}$.

Example 2.2.11. *Set $\Delta = \langle \{x, y\} : x, y \in V(G), x \sim y \rangle$. In other words Δ is the simplicial complex generated by the edges of our graph G. Then the vertex covers of Δ are precisely the vertex covers of G as a graph.*

Example 2.2.12. *Set $\Delta = \langle N[x] : x \in V(G) \rangle$. In other words Δ is the simplicial complex whose faces are the subsets of closed neighborhoods in our graph G. Then the vertex covers of Δ as a simplicial complex are precisely the dominating sets of G.*

One might wonder if there's a decomposition theorem for the facet ideal of a simplicial complex similar to Theorems 1.1.5 and 1.2.7. In fact, the facet ideal of a simplicial complex is the intersection of prime ideals generated by its minimal vertex covers.

Theorem 2.2.13 ([4], Proposition 1.8)**.** *Let Δ be a simplicial complex on $x_1, \ldots, x_n$, and let $\mathcal{V}$ be the set of minimal vertex covers of Δ. Then $\mathcal{F}(\Delta)$ has the following irreducible decomposition.*

$$\mathcal{F}(\Delta) = \bigcap_{V \in \mathcal{V}} \langle V \rangle$$

In the case of closed neighborhood ideals and edge ideals, this theorem specializes to the decomposition theorems discussed in the introduction. We close with a simple theorem about the dimension of a facet ideal. We will define and discuss dimension in more depth in Chapter 3.

Theorem 2.2.14. *Let Δ be a simplicial complex on $x_1, \ldots, x_n$. If k is the size of the smallest minimal vertex cover of Δ, then dim$\big(R/\mathcal{F}(\Delta)\big) = n - k$.*

Proof. By Theorem 2.2.13, $\operatorname{ht}(\mathcal{F}) = k$. So, Corollary 3.1.8 implies $\dim(R/\mathcal{F}(\Delta)) = n - k$. □

Example 2.2.15. *Theorem 2.2.13 recovers the fact from Section 1.2. that dim$(R/\operatorname{CNI}(G))$ is the size of the smallest minimal dominating set of G.*

2.3 Colon Ideals

Colon ideals, also known as ideal quotients, play a special role in the study of monomial ideals and simplicial complexes.

Definition 2.3.1. *Let R be a commutative ring and $I, J \subseteq R$ ideals. The* **colon ideal** *or* **ideal quotient***, denoted $(I : J)$, is the set $\{r \in R : rJ \subseteq I\}$. If J is cyclic and generated by the element f, then we will sometimes write $(I : J)$ as $(I : f)$ instead. The colon ideal $(I : J)$ is itself an ideal that contains I.*

Example 2.3.2. *Let $R = \mathbb{Z}$. Let $I = 6\mathbb{Z}$ and $J = 2\mathbb{Z}$. Then $(I : J) = 3\mathbb{Z}$.*

Example 2.3.3. *Suppose $I \subset R$ and that $f \in I$. Then $(I : f) = R$.*

In the case of monomial ideals, computing colon ideals is very simple.

Theorem 2.3.4 ([9], Theorem 2.5.4)**.** *Suppose that $I = \langle f_1, \ldots, f_m \rangle$ and $J = \langle g_1, \ldots, g_n \rangle$ are squarefree monomial ideals, and moreover that the generating sets $\{f_1, \ldots, f_m\}$ and $\{g_1, \ldots, g_n\}$ are irredundant. Then*

$$(I : J) = \bigcap_{i=1}^{n} \bigg(\sum_{j=1}^{m} (\langle f_j \rangle : g_i) \bigg).$$

Note that when taking a colon of a squarefree monomial ideal I by a cyclic ideal generated by a squarefree monomial g, this amounts to "deleting" the variables of g from the generators of I.

Corollary 2.3.5. *Let $I \subset k[x_1, \ldots, x_n]$ be a squarefree monomial ideal with minimal monomial generators $f_1, \ldots, f_m$ and let g be a monomial. Then $(I : f) = \langle \frac{f_1}{gcd(f_1,g)}, \ldots, \frac{f_m}{gcd(f_m,g)} \rangle$.*

Proof. It is straightforward to see that $(\langle f_i \rangle : g) = \langle \frac{f_i}{\gcd(f_i,g)} \rangle$ for each monomial f_i. Then by Theorem 2.3.4, we have $(I : g) = \sum_{i=1}^{m} \langle \frac{f_i}{\gcd(f_i,g)} \rangle = \langle \frac{f_1}{\gcd(f_1,g)}, \ldots, \frac{f_m}{\gcd(f_m,g)} \rangle$. □

We give an example of this computation next.

Example 2.3.6. *Let $R = k[x_1, x_2, x_3, x_4]$, and let $I = \langle x_1x_2x_3, x_2x_4, x_1x_3x_4 \rangle$ and $J = \langle x_1x_3 \rangle$. Then we have*

$$(I : J) = (\langle x_1x_2x_3 \rangle : x_1x_3) + (\langle x_2x_4 \rangle : x_1x_3) + (\langle x_1x_3x_4 \rangle : x_1x_3)$$

by Theorem 2.3.4, and Corollary 2.3.4 gives

$$\begin{aligned} (\langle x_1x_2x_3 \rangle : x_1x_3) &= \langle x_2 \rangle \\ (\langle x_2x_4 \rangle : x_1x_3) &= \langle x_2x_4 \rangle \\ (\langle x_1x_3x_4 \rangle : x_1x_3) &= \langle x_4 \rangle \end{aligned}$$

hence $(I : J) = \langle x_2, x_2x_4, x_4 \rangle = \langle x_2, x_4 \rangle$.

Example 2.3.7. *Let $\Delta = \langle F_1, \ldots, F_n \rangle$ be a simplicial complex on V. Let $x \in V$, and define $\Delta - x = \langle F_1 - \{x\}, \ldots, F_n - \{x\} \rangle$. Then $(\mathcal{F}(\Delta) : x) = \mathcal{F}(\Delta - x)$.*

We can also compute colon ideals via irreducible decompositions.

Theorem 2.3.8 ([9], Theorem 7.4.4)**.** *Let I and $J = \langle f_1, \ldots, f_k \rangle$ be squarefree monomial ideals and suppose that I has minimal primes $P_1, \ldots, P_n$. Then $(I : J)$ has the following decomposition.*

$$(I : J) = \bigcap_{f_1,\ldots,f_k \notin P_i} P_i$$

Colon ideals yield a special exact sequence that will play an important role Chapter 5.

Theorem 2.3.9. *Let R be a commutative ring, and let $I \subseteq R$ be an ideal. If $x \in R$, then there is an exact sequences of R-modules*

$$0 \longrightarrow R/(I : x) \longrightarrow R/I \longrightarrow R/(I, x) \longrightarrow 0$$

where $(I, x) = I + (x)$.

We will see special applications of this exact sequence near the end of the following chapter.

Chapter 3

Cohen-Macaulayness - Background

Cohen-Macaulay rings and modules are often studied in combinatorial commutative algebra. Throughout this thesis, our main concern will be characterizing Cohen-Macaulay rings that arise from closed neighborhood ideals. Throughout this chapter, we will assume that k is a field, $J \subset k[x_1, \ldots, x_n]$ is an ideal, and set $R = k[x_1, \ldots, x_n]/J$.

3.1 Dimension

The dimension and depth of rings and modules provide two different measures of algebraic information. We will start by defining dimension for both modules and rings. First, we introduce the following notation: if A is a commutative ring with identity, then $\mathrm{Spec}(A)$ denotes the set of prime ideals in A.

Definition 3.1.1. *Let $P \subset R$ be a prime ideal. The* **height** *of P, denoted $ht(P)$, is defined as follows.*

$$ht(P) = max\{k : \exists P_0, \ldots, P_k \in Spec(R), P = P_0 \supsetneq \ldots \supsetneq P_k\}$$

In other words, it is the maximal length of a strictly descending sequence of prime ideals contained in P. Such a maximal chain exists due to a theorem of Krull [1, Theorem A.1] because R is Noetherian. If I is not a prime ideal, then we define

$$ht(I) = min\{ht(P) : P \in Spec(R), I \subset P\}.$$

Equivalently, $ht(I)$ is the minimum height of its minimal primes.

Example 3.1.2. *Let $I = \langle x_{n_1}, \ldots, x_{n_p} \rangle \subset k[x_1, \ldots, x_n]$ where $x_{n_i} \neq x_{n_j}$ for all i, j. In other words, I is generated by p distinct variables. Then $ht(I) = p$.*

Example 3.1.3. *Let $R = k[x_1, x_2, x_3, x_4]$ and set $I = \langle x_1x_2x_3, x_2x_3x_4, x_3x_4x_1, x_4x_1x_2 \rangle$. Then it is straigtforward to show that I has irreducible decomposition*

$$I = \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_1, x_4 \rangle \cap \langle x_2, x_3 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_3, x_4 \rangle.$$

so $ht(I) = 2$.

Definition 3.1.4. *The* **dimension** *of R, denoted dim(R), is defined as follows.*

$$dim(R) = sup\{ht(P) : P \in Spec(R)\}$$

The following theorem allows us to compute the dimension of a polynomial ring over a field.

Theorem 3.1.5 ([1], Theorem A.12)**.** *Let A be a Noetherian ring. Then $dim(A[x]) = dim(A) + 1$.*

Example 3.1.6. *Let $R = k[x_1, \ldots, x_n]$. Then $dim(R) = n$. This follows from induction via Theorem 3.1.5.*

Theorem 3.1.7 ([1], Theorem A.16)**.** *Let $P \in Spec(R)$. Then*

$$ht(P) = dim(R) - dim(R/P).$$

Corollary 3.1.8. *Let $R = k[x_1, \ldots, x_n]$ and let $I \subset R$ be an ideal. Then $ht(I) = dim(R) - dim(R/I)$.*

It is important to note that dimension can also be defined for A-modules.

Definition 3.1.9. *Let A be Noetherian and let M be a finitely generated A-module. The dimension of M is defined as follows,*

$$dim_A(M) = dim(A/Ann(M))$$

where $Ann(M) = \{r \in A : rm = 0$ for all $m \in M\}$.

Note that if $I \subset A$ is an ideal, then $\dim_A(A/I) = \dim(A/I)$. That is, the dimension of A/I as an A-module is the same as its dimension as a ring. Since our usual ring R is Noetherian and we are primarily concerned with R-modules of the form R/I where I is an ideal in R, which are finitely generated, we will simply write $\dim(R/I)$ in place of $\dim_R(R/I)$.

Next, let us discuss briefly the combinatorial aspects of dimension we will use in this thesis.

Example 3.1.10. *Let G be a graph on $\{x_1, \dots x_n\}$ and consider the closed neighborhood ideal $I = \mathrm{CNI}(G) \subset k[x_1, \dots, x_n]$. Recall that I is an intersection of primes generated by minimal dominating sets of G (Theorem 1.2.7). Let $\gamma(G)$ be the size the smallest minimal dominating set of G. Then $ht(\mathrm{CNI}(G)) = \gamma(G)$ so $dim(R/\mathrm{CNI}(G)) = n - \gamma(G)$.*

The final definition of this section is an algebraic version of well covered and well dominated graphs called unmixedness.

Definition 3.1.11. *Let $I \subset R$ be a squarefree monomial ideal with minimal primes $P_1, \dots, P_m$. Then I is said to be* **unmixed** *if $ht(P_1) = ht(P_2) = \dots = ht(P_m)$.*

3.2 Depth

Depth is the other parameter with which we will be concerned. Roughly speaking, depth, as we will define it, is another measure of the size of an R-module. We make this idea precise with the following definitions. For this section and the following section, we will assume that k is a field and $R = k[x_1, \dots, x_n]/J$ where J is generated by homogeneous elements, and that $M = R/I$ where I is generated by homogeneous elements. Moreover, $\mathfrak{m}$ will denote the maximal ideal $\mathfrak{m} = \langle x_1, \dots, x_n \rangle R$.

Definition 3.2.1. *An element $r \in R$ is said to be an* **M-regular** *element of R if $rM \neq M$ and whenever $m \in M$ and $rm = 0$, then $m = 0$.*

Example 3.2.2. *Let $R = k[x_1, x_2, x_3]$ and let $I = \langle x_1 \rangle$. Now, set $M = R/I$. Then x_2, x_3 are both M-regular elements, because $M \cong k[x_2, x_3]$ is an integral domain.*

Definition 3.2.3. *Consider a sequence of homogeneous elements $f = f_1, \dots, f_m \in \mathfrak{m}$ and define $N_i = M/\langle f_1, \dots, f_i \rangle M$. Then f is called an* **M-regular sequence** *if for every $i = 0, \dots, m-1$, the element f_{i+1} is N_i-regular.*

Example 3.2.4. *The sequence of variables $x_1, \dots, x_n$ is an R-regular sequence.*

We are now ready to define depth.

Definition 3.2.5. *The* **depth** *of M, denoted $depth_R(M)$, is the length of the longest homogeneous M-regular sequence in $\mathfrak{m}$.*

Let us now assert a theorem of Rees about regular sequences of M.

Theorem 3.2.6 ([1], Theorem 1.2.5)**.** *All maximal homogeneous M-regular sequences in $\mathfrak{m}$ have the same length.*

This leads to the following theorem about the depth of R modulo a regular sequence.

Theorem 3.2.7 ([1], Proposition 1.2.10 (d))**.** *Suppose that $f_1, \ldots, f_m$ is a homogeneous R-regular sequence. Then $depth(R/\langle f_1, \ldots, f_m\rangle R) = depth(R) - m$.*

We conclude this section with one last theorem about the relationship between depth and dimension of M.

Theorem 3.2.8 ([1], Theorem 1.2.12)**.** *$depth(M) \leq dim(M)$.*

3.3 The Cohen-Macaulay Property

Now that we have established a few theorems about dimension and depth, we are now poised to define Cohen-Macaulayness and establish a few crucial theorems about Cohen-Macaulay rings and Cohen-Macaulay modules.

Definition 3.3.1. *M is said to be a* **Cohen-Macaulay module** *if $dim(M) = depth(M)$.*

Definition 3.3.2. *R is a* **Cohen-Macaulay ring** *if it is a Cohen-Macaulay as a module over itself.*

Example 3.3.3. *The ring $R = k[x_1, \ldots, x_n]$ is a Cohen-Macaulay ring, since $x_1, x_2, \ldots, x_n$ is a maximal homogeneous R-regular sequence, so*

$$n \leq depth(R) \leq dim(R) = n$$

by Theorem 3.2.8 and Example 3.1.6.

Cohen-Macaulay rings and modules are studied for a number of nice algebro-geometric and combinatorial properties. One "nice" property of Cohen-Macaulay in the setting of quotients of polynomial rings is unmixedness.

Theorem 3.3.4 ([9], Theorem 5.3.16)**.** *Let $R = k[x_1, \ldots, x_n]$ and suppose $M = R/I$ is a Cohen-Macaulay module over R. Then I is an unmixed ideal.*

We will now discuss a special class of ideals $I \subset R$ which are Cohen-Macaulay. The following definition can be stated in a more general fashion, but we will specialize to the case of polynomial rings here.

Definition 3.3.5. *An ideal $I \subset R = k[x_1, \ldots, x_n]$ is said to be a* **complete intersection** *if it is generated by an R-regular sequence.*

Example 3.3.6. *Let $R = k[x_1, x_2, x_3]$. Then $\langle x_1x_2, x_3 \rangle$ is a complete intersection ideal.*

Theorem 3.3.7. *Let $I \subset R = k[x_1, \ldots, x_n]$ be a squarefree monomial ideal. Then I is a complete intersection if and only if its generating monomials have disjoint support.*

Theorem 3.3.8. *If $I \subset R = k[x_1, \ldots, x_n]$ is a complete intersection ideal, then R/I is a Cohen-Macaulay ring.*

Proof. Note that $\text{depth}(R) = \dim(R) = n$, so if I is generated by a regular sequence of length r, then R/I has depth $n - r$ by Theorem 3.2.7. Moreover, Proposition A.4 from [1] implies that $\dim(R/I) = n - r$. Hence, R/I is Cohen-Macaulay over R (and consequently, over itself). $\square$

3.4 Depth and Exact Sequences

We will conclude this chapter by discussing a few results regarding the behavior of depth for modules that form exact sequences. In particular, we will use homological information to bound (and in some cases, determine) the depth of quotient rings. In this section, we will take R to be the polynomial ring $k[x_1, \cdots, x_n]$ where k is a field.

Theorem 3.4.1 ([1], Proposition 1.2.9)**.** *Let $M = R/I_1$, $N = R/I_2$, and $U = R/I_3$ where $I_1, I_2, I_3 \subset R$ are generated by homogeneous elements. Suppose further there is an exact sequence as follows.*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0$$

Then we have the following inequality.

$$depth(M) \geq min\{depth(U), depth(N)\}$$

This theorem, part of the Depth Lemma, is frequently used in more general cases. In the case of monomial ideals, we have the following strengthening.

Theorem 3.4.2 ([2], Theorem 4.3)**.** *Let $I \subset R$ be a monomial ideal, and f a monomial. Then*

$$\mathit{depth}(R/I) \in \{\mathit{depth}(R/(I:f))), \mathit{depth}(R/(I,f))\}$$

Moreover, if $\mathit{depth}(R/(I,f)) \geq \mathit{depth}(R/(I:f))$, then $\mathit{depth}(R/I) = \mathit{depth}(R/(I:f))$

This theorem can be seen as a strengthening of Theorem 3.4.1 by considering the following exact sequence from Theorem 2.3.9.

$$0 \longrightarrow R/(I:f) \longrightarrow R/I \longrightarrow R/(I,f) \longrightarrow 0$$

We will conclude this chapter by stating an important related result that we will make use of in Chapter 5.

Theorem 3.4.3 ([2], Lemma 4.1)**.** *Let $I \subset R$ be a monomial ideal and f a monomial. Then $\mathit{depth}(R/I) \leq \mathit{depth}(R/(I:f))$.*

It is worth noting that this theorem actually supplies the second part of Theorem 3.4.2 after establishing that $\mathrm{depth}(R/I) \in \{\mathrm{depth}(R/(I:f)), \mathrm{depth}(R/(I,f))\}$.

Chapter 4

Graph Theory - Background

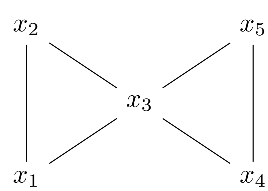
The main results of this thesis are restricted to squarefree monomial ideals that arise from graphs. Hence, we will cover some preliminaries of graph theory and notation that we will use frequently. Throughout this chapter, G will always be a finite simple graph.

4.1 Definitions and Notation

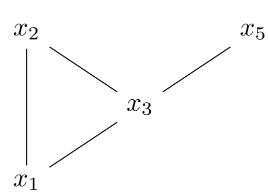
Notation 4.1.1. *The vertex set of G will be abbreviated as $V(G)$.*

Definition 4.1.2. *Let $H \subseteq V(G)$. Then the* **subgraph induced by H**, *denoted $G[H]$, us the graph with vertices in H and an edge xy whenever xy is an edge in G.*

Example 4.1.3. *Consider the butterfly graph below, G.*



If $H = \{x_1, x_2, x_3, x_5\}$, then $G[H]$ is the following graph.



One interesting characteristic of the butterfly graph is the presence of a vertex adjacent to all other vertices in the graph.

Definition 4.1.4. *A vertex $v \in V(G)$ is a* **universal vertex** *if every other vertex in G is adjacent to v.*

Example 4.1.5. *Consider the graph G in Example 4.1.3. In this case, x_3 is a universal vertex.*

Notation 4.1.6. *Let G be a graph and S a set of vertices in G. Then we define $G - S = G[S^c]$. That is, $G - S$ is the induced subgraph on vertices not in S.*

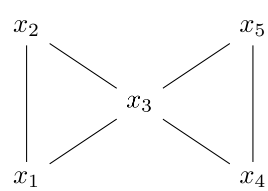
Notation 4.1.7. *If x, y are adjacent in G, then we will use the notation $x \sim_G y$. When the graph is unambiguous, we will simply write $x \sim y$.*

Definition 4.1.8. *Let $x \in V(G)$. The* **open neighborhood** *of x, $N(x)$, is defined as follows.*

$$N(x) = \{y \in V(G) : x \sim y\}$$

Definition 4.1.9. *Let G be a graph and $x \in V(G)$. The* **closed neighborhood** *of x, denoted $N[x]$, is the set $N(x) \cup \{x\}$.*

Example 4.1.10. *Consider the butterfly graph from Example 4.1.3.*



Then $N(x_2) = \{x_1, x_3\}$ and $N[x_2] = \{x_1, x_2, x_3\}$.

4.2 Vertex Covers and Independent Sets

Recall from Chapter 1 that edge ideals decompose into intersections of primes generated by vertex covers, via Theorem 1.1.5. Here, we will discuss vertex covers and their complements, independent sets.

Definition 4.2.1. *A set $V \subseteq V(G)$ is called a* **vertex cover** *of G if every edge in G is incident with a vertex in V. A vertex cover is* **minimal** *if it does not properly contain any vertex cover.*

Definition 4.2.2. *The* **vertex covering number** *of G, denoted $\tau(G)$, is the size of the smallest vertex cover of G.*

Example 4.2.3. *Consider the butterfly graph in Example 4.1.3. The minimal vertex covers of G are $\{x_1, x_3, x_4\}$, $\{x_1, x_3, x_5\}$, $\{x_2, x_3, x_4\}$, $\{x_2, x_3, x_5\}$, $\{x_1, x_2, x_4, x_5\}$, so $\tau(G) = 3$.*

Definition 4.2.4. *An* **independent set** *for G is a set M of vertices of G which are pairwise non-adjacent. An independent set is* **maximal** *if it isn't properly contained in any other independent set.*

Definition 4.2.5. *The* **independence number** *of G, denoted $\alpha(G)$, is the size of the largest independent set in G.*

Example 4.2.6. *Consider the butterfly graph from Example 4.1.3. Then $\{x_2, x_5\}$ is an independent set since x_2 and x_5 are not adjacent. Moreover, it is maximal since x_1, x_3, and x_4 are all adjacent to x_2 or x_5. It is straightforward to show that the remaining maximal independent sets in G are $\{x_2, x_4\}, \{x_1, x_5\}, \{x_1, x_4\}$, and $\{x_3\}$. Hence, $\alpha(G) = 2$.*

Vertex covers and independent sets are connected in the following manner.

Theorem 4.2.7. *A subset $V \subseteq V(G)$ is a vertex cover if and only if V^c is an independent set. Moreover, V is a minimal vertex cover if and only if V^c is a maximal independent set.*

Proof. Suppose V is a vertex cover. Let $x, y \in V^c$. Then if $x \sim y$ is an edge, then the edge xy is not covered by V, which is a contradiction. Thus any two $x, y \in V^c$ are not adjacent. Conversely, suppose M is an independent set. Then if xy is an edge in G, then at least one of $x, y \notin M$. Hence M^c contains a vertex incident with any given edge of G.

Now, suppose V is a minimal vertex cover. Note that V^c is an independent set by the previous discussion. Then if $V^c \subsetneq N \subseteq V(G)$, then $N^c \subsetneq V$, so N^c is not a vertex cover by minimality of V. This implies that N is not an independent set, so V^c is a maximal independent set.

Conversely, suppose M is a maximal independent set. Note that M^c is a vertex cover by the previous discussion. Then if $U \subsetneq M^c$, then $M \subsetneq U^c$, so U^c is not an independent set by maximality of M, and therefore U is not vertex cover. Thus M^c is a minimal vertex cover. $\square$

Now, we will define a well-covered graph.

Definition 4.2.8. *A graph G is said to be* **well-covered** *if all of its minimal vertex covers have the same size.*

Example 4.2.9. *The butterfly graph G from Example 4.1.3 is not well covered, since $\{x_3\}$ and $\{x_2, x_5\}$ are both maximal independent sets.*

An easy connection between well-covered graphs and maximal independent sets is given by the following theorem.

Theorem 4.2.10. *G is well-covered if and only if every maximal independent set for G has the same size.*

Proof. Recall from Theorem 4.2.7 that minimal vertex covers are exactly the complements of maximal independent sets. If all the minimal vertex covers of G have the same size then all of the maximal independent sets of G have the same size, and vice versa. □

4.3 Dominating Sets

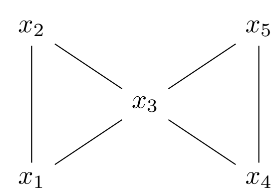
Dominating sets are another way of covering graphs. Rather than covering the edges of G, they cover the closed neighborhoods of G. We may define them as follows.

Definition 4.3.1. *A* **dominating set** *of G is a set of vertices D such that every vertex $v \in V(G)$ is either in D or adjacent to a vertex in D. A dominating set is* **minimal** *if it does not properly contain any dominating set.*

Observation 4.3.2. *Let H be a spanning subgraph of G and suppose D is a dominating set of H. Then D is also a dominating set of G.*

Definition 4.3.3. *The* **domination number** *of G, denoted $\gamma(G)$, is the size of the smallest dominating set of G. The size of the largest minimal dominating set is denoted $\Gamma(G)$.*

Example 4.3.4. *Consider the butterfly graph G from Example 4.1.3.*



Then $\{x_3\}$ is a dominating set since every vertex in G is in the closed neighborhood of x_3. Moreover, it is minimal, since its only proper subset is empty and hence not a dominating set. It is straightforward to show that the remaining minimal dominating sets are $\{x_1, x_4\}$, $\{x_1, x_5\}$, $\{x_2, x_4\}$, and $\{x_2, x_5\}$. Hence, $\gamma(G) = 1$ and $\Gamma(G) = 2$.

Definition 4.3.5. *The graph G is* **well-dominated** *if all of its minimal dominating sets have the same size.*

We now establish a simple proposition about minimal dominating sets that will become useful in Chapter 5.

Observation 4.3.6. *Let G be a connected graph with two or more vertices. If D is a minimal dominating set in G, then its complement D^c is a dominating set.*

Proof. Let $x \in G$. We wish to show that $x \in D^c$ or there is a vertex $v \in D^c$ such that $x \sim v$. If $x \in D^c$, we are done, so suppose that $x \in D$ instead. If x has no neighbors in D^c, then all of its neighbors are in D. But then D is not minimal as $D - \{x\}$ is a dominating set, which contradicts our assumption. □

There is an important connection between dominating sets and independent sets, which we describe in the following proposition.

Proposition 4.3.7. *Let M be a maximal independent set of G. Then M is also a minimal dominating set of G.*

Proof. Suppose for contradiction that M is not a dominating set. Then, there is a vertex $v \in V(G)$ that is not in M and not adjacent to any vertex in M, so $M \cup \{v\}$ is an independent set, contradicting maximality. It follows that M is a dominating set.

Moreover, M is a minimal dominating set, since for every set of vertices $N \subsetneq M$, there is a vertex $v' \in M - N$ that is not in N nor is it adjacent to any vertex in N. Hence N is not a dominating set, so M is minimal. □

Corollary 4.3.8. *Let M be a maximal independent set of G. Then we have the following inequalities.*

$$\gamma(G) \leq |M| \leq \Gamma(G)$$

Proof. By Proposition 4.3.7, M is a minimal dominating set. □

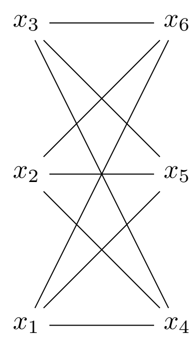
We conclude with the following theorem.

Theorem 4.3.9. *If G is well-dominated, then it is well-covered.*

Proof. Suppose G is well-dominated. Then $\gamma(G) = \Gamma(G)$. By Corollary 4.3.8, if M is a maximal independent set of G, then $|M| = \gamma(G)$. Hence all maximal independent sets of G have the same size, so by Theorem 4.2.10, G is well-covered. □

Note that the converse of Theorem 4.3.9 does not hold, as shown in the following example.

Example 4.3.10. *Consider the utility graph G below.*



It is straightforward to see that the minimal vertex covers G are precisely $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$. Hence, G is well-covered. On the other hand, G is not well-dominated, as both $\{x_2, x_5\}$ and $\{x_1, x_2, x_3\}$ are minimal dominating sets.

4.4 Special Types of Graphs

This section will serve as a miscellany of graph families that are important or even central to this thesis. We will begin with bipartite graphs, which play a major role in Chapter 5.

Definition 4.4.1. *A* **bipartite graph** *G is a graph whose vertex set $V(G)$ can be partitioned into two nonempty sets A and B so that every vertex in A is adjacent only to vertices in B, and vice versa.*

Example 4.4.2. *The utility graph from Example 4.3.10, otherwise known as $K_{3,3}$ is bipartite.*

Bipartite graphs can also be characterized by their cycles. This characterization will be important in Chapter 5.

Theorem 4.4.3 ([3], 1.6.1)**.** *A graph on at least two vertices is bipartite if and only if it contains no odd cycle.*

Example 4.4.4. *Any tree T on at least two vertices is a bipartite graph. This is because a tree is acyclic, and thus contains no odd cycle.*

Let us now introduce a special family of bipartite graphs that generalizes Example 4.4.2.

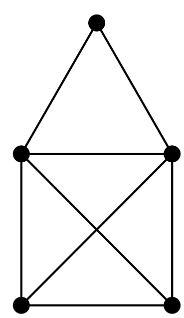
Definition 4.4.5. *Let s, t be two positive integers. The* **complete bipartite graph** $K_{s,t}$, *is the bipartite graph with partite sets A and B such that $|A| = s$ and $|B| = t$, where every vertex in A is adjacent to every vertex in B.*

Chordal graphs are another important family of graphs in combinatorial commutative algebra. In [7], Herzog, Hibi, and Zheng characterize the chordal graphs with Cohen-Macaulay edge ideals. We will likewise characterize the chordal graphs with Cohen-Macaulay closed neighborhood ideals in Chapter 5.

Definition 4.4.6. *A graph G is* **chordal** *if every n-cycle of G with $n > 3$ has a chord.*

Example 4.4.7. *Any tree is vacuously a chordal graph.*

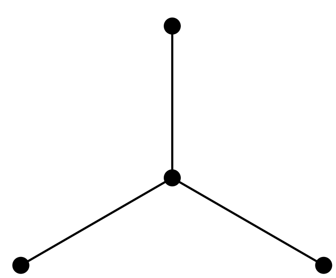
Example 4.4.8. *The following graph is a chordal graph.*



We close this chapter by introducing two more graphs that will feature in Chapter 5.

Definition 4.4.9. *Let n be a positive integer. The n-star is the graph $K_{1,n}$.*

Example 4.4.10. *The following graph is a 3-star.*



Definition 4.4.11. *The* **null graph** *on k vertices, denoted N_k, is the graph with k vertices and no edges.*

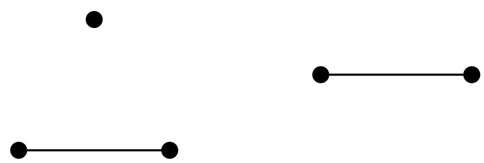
Definition 4.4.12. *Let G and H be graphs with disjoint vertex sets. The* **join** *of G and H, denoted $G+H$, is the graph with vertices $V(G) \cup V(H)$, such that there is an edge between x and y in $G+H$ provided any of the following are true.*

(i) $x \sim_G y$.

(ii) $x \sim_H y$.

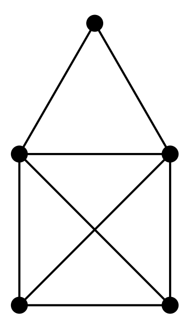
(iii) $x \in G$ *and* $y \in H$.

Example 4.4.13. *Let G and H be the following graphs.*



G $\quad$ H

Then $G+H$ is the graph from Example 4.4.8.



G+H

Example 4.4.14. $K_{s,t} \cong N_s + N_t$.

Chapter 5

Results

In this chapter we will introduce novel results about closed neighborhood ideals with an emphasis on decomposition theorems, the behavior of closed neighborhood ideals under graph operations, calculations of depth, and, of course, Cohen-Macaulay properties. Unless otherwise stated R will always be the a polynomial ring $k[x_1, \ldots, x_n]$ where k is a field, and G will always be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_n\}$.

Before beginning, it is important to discuss the following vocabulary. We will often refer to the depth and dimension of $R/\operatorname{CNI}(G)$ as simply the depth of G or the dimension of G. Likewise, we call call G a ***Cohen-Macaulay graph*** if $R/\operatorname{CNI}(G)$ is Cohen-Macaulay as an R-module, and a ***complete intersection graph*** if $\operatorname{CNI}(G)$ is a complete intersection ideal.

5.1 Definitions and Notation

Before proving our new results, let us first reassert a number of important definitions and theorems. We will begin by defining the closed neighborhood ideal itself.

Definition 5.1.1. *The* **closed neighborhood ideal** *of G, denoted $\operatorname{CNI}(G)$, is the squarefree monomial ideal*

$$\operatorname{CNI}(G) = \left\langle \prod_{x_i \in N[x_j]} x_i : \ x_j \in G \right\rangle.$$

This chapter will restrict its attention to closed neighborhood ideals and the algebraic properties thereof. One of the main tools in studying these ideals is the following theorem.

Theorem 5.1.2 ([10], Lemma 2.2)**.** *The minimal primes of* $\mathrm{CNI}(G)$ *are exactly the prime ideals generated by the minimal dominating sets of* G*. In other words, if* $\mathcal{D}$ *is the set of minimal dominating sets of* G*, then* $\mathrm{CNI}(G)$ *has the following irredundant prime decomposition.*

$$\mathrm{CNI}(G) = \bigcap_{D \in \mathcal{D}} \langle D \rangle$$

This allows us to relate algebraic information about closed neighborhood ideals to their dominating sets. The following theorems provide this connection.

Theorem 5.1.3. *Suppose that* G *has* n *vertices. Then we have* $ht(\mathrm{CNI}(G)) = \gamma(G)$*. Additionally,* $dim(R/\mathrm{CNI}(G)) = n - \gamma(G)$

Proof. By definition, the height of an ideal in a Noetherian ring is the minimal height of its minimal primes. Since by Theorem 5.1.2 the minimal primes of $\mathrm{CNI}(G)$ are precisely the ideals generated by its minimal dominating sets $\langle D_1 \rangle, \ldots, \langle D_k \rangle$. But the height of these minimal primes are simply $|D_1|, \ldots, |D_k|$ respectively, so the height of $\mathrm{CNI}(G)$ is the size of its smallest minimal dominating set, i.e., $\gamma(G)$. Additionally, by Corollary 3.1.8, $\dim(R/\mathrm{CNI}(G)) = n - \gamma(G)$. □

Theorem 5.1.4. $\mathrm{CNI}(G)$ *is unmixed if and only if* G *is well dominated.*

Proof. By Theorem 5.1.2 the minimal primes of $\mathrm{CNI}(G)$ are precisely the primes generated by its minimal dominating sets. □

5.2 Spanning Trees and Dominating Sets

In this section we prove that every closed neighborhood ideal of a connected graph G can be decomposed into closed neighborhood ideals of its spanning trees; see Theorem 5.2.3. To prove this, we will first establish the following lemma.

Lemma 5.2.1. *Suppose* G *is connected, and let* D *be a minimal dominating set of* G*. Then there is a spanning tree* T *of* G *such that* D *is a minimal dominating set of* T*.*

Proof. Let C be any cycle in G, if one exists. We will show that there exists an edge $e \in C$ such that D is a minimal dominating set of $G - e$.

Case 1: $C \subseteq D$. In this case, we may delete any edge e in C and $C - e$ is also dominated by D.

Similarly, $G - e$ is dominated by D. Indeed, suppose $x \in G - e$. If $x \in D$ then x is dominated by D in $G - e$. On the other hand, if $x \notin D$, then there exists some $y \in D$ such that xy is an edge in G. Since $x \notin D$, we have $xy \neq e$ so $xy \in G - e$ and x remains dominated by D. Additionally, D still dominates $G - e$ minimally; if otherwise, then there is a set $D' \subsetneq D$ that dominates $G - e$, and therefore dominates G, which contradicts the minimality of D. Finally, $G - e$ remains connected, as we may replace e in any path containing e with the path $C - e$.

Case 2: $C \nsubseteq D$. In this case, there exists a vertex $v \in C$ such that $v \notin D$. Then v has two distinct neighbors in C, call them u and u'. If at least one of u, u' is not in D, then let our deleted edge e be the edge uv or $u'v$ respectively. Then D dominates $G - e$, since e was not an edge between $v \notin D$ and a vertex in D. Moreover, D still dominates $G - e$ minimally by the same reasoning as in Case 1. If instead both $u, u' \in D$, then let e be the edge uv. Then D still dominates $C - e$ since $u' \sim v$ in $G - e$. Moreover, D still dominates $G - e$ minimally by the same reasoning as in Case 1.

By sequentially deleting an edge for every cycle in G according to either case above, we construct a spanning tree of G that has D as a minimal dominating set. $\square$

This helpful lemma allows us to construct spanning trees of G with a given minimal dominating set of G. Throughout this section, we will establish results that allow us to use facts about the spanning trees of a graph G to determine algebraic properties of $\mathrm{CNI}(G)$.

Observation 5.2.2. *Let H be a spanning subgraph of G. Then $\mathrm{CNI}(G) \subseteq \mathrm{CNI}(H)$.*

Proof. This observation follows from the simple fact that $N_H[x] \subseteq N_G[x]$ for all $x \in V(G)$. Hence the monomial generators of $\mathrm{CNI}(H)$ all divide the monomial generators of $\mathrm{CNI}(G)$. $\square$

This observation will give us one containment in the proof of the following theorem.

Theorem 5.2.3. *Let G be connected and let $T_1, \ldots, T_k$ be the spanning trees of G. Then*

$$\mathrm{CNI}(G) = \cap_{i=1}^{k} \mathrm{CNI}(T_i).$$

Proof. Each T_i is a spanning subgraph, so by Observation 5.2.2, we have that $\mathrm{CNI}(G) \subseteq \cap_{i=1}^{k} \mathrm{CNI}(T_i)$. Now, note that by Theorem 5.1.2, we have $\mathrm{CNI}(G) = \cap_{j=1}^{m} \langle D_j \rangle$ where the D_j's are the minimal dominating sets of G. But each D_j is similarly a minimal dominating set of some T_j so that $\mathrm{CNI}(T_j) \subseteq \langle D_j \rangle$. Hence, $\cap_{i=1}^{k} \mathrm{CNI}(T_i) \subseteq \cap_{j=1}^{m} \langle D_j \rangle = \mathrm{CNI}(G)$. $\square$

We will think of this theorem as a decomposition theorem, similar to Theorem 1.2.7. In fact, we can use the irreducible decompositions of spanning trees to relate the dimension of $\operatorname{CNI}(G)$ to the dimension of its spanning trees, in the same way we relate the height of a squarefree monomial ideal I to its minimal primes.

Corollary 5.2.4. *Let G be connected with spanning trees $\{T_1, \ldots, T_k\}$. Then we have the following.*

$$dim(R/\operatorname{CNI}(G)) = max\{dim(R/\operatorname{CNI}(T_i))\}_{i=1}^{k}$$

Proof. For each spanning tree T_i we have an irreducible decomposition $\operatorname{CNI}(T_i) = \cap_{j=1}^{n_i}\langle D_{i,j}\rangle$. By Theorem 5.2.3, we have that $\operatorname{CNI}(G) = \cap_{i=1}^{k}\cap_{j=1}^{n_i}\langle D_{i,j}\rangle$, which is an intersection of prime ideals. So, $ht(\text{CNI(G)})$ is the minimal height of the $D'_{i,j}s$, which is also the minimal height of the tree ideals $\operatorname{CNI}(T_i)$. Hence, $ht(\operatorname{CNI}(G)) = \min\{ht(\operatorname{CNI}(T_i)) : 1 \leq i \leq k\}$. Then it follows that $\dim(R/\operatorname{CNI}(G)) = \max\{\dim(R/\operatorname{CNI}(T_i)) : 1 \leq i \leq k\}$. □

Corollary 5.2.5. *Let G be a connected graph whose every spanning tree is well-dominated. Then G is well dominated and $R/\operatorname{CNI}(G)$ has dimension $|G|/2$.*

Proof. Note that if T is any spanning tree of G, it must be well dominated by assumption, and hence a whisker graph by Theorem 1.2.11; so $\dim(\operatorname{CNI}(T)) = |T|/2 = |G|/2$, as the whiskers of T form a minimal dominating set. Then if $T_1, \ldots, T_k$ are the spanning trees of G, we have an irreducible decomposition $\operatorname{CNI}(T_i) = \cap_{j=1}^{n_i}\langle D_{i,j}\rangle$ where $|D_{i,j}| = |G|/2$. By Theorem 5.2.3, $\operatorname{CNI}(G) = \cap_{i=1}^{k}\cap_{j=1}^{n_i}\langle D_{i,j}\rangle$, which is an intersection over prime ideals each with dimension $|G|/2$. Then $\operatorname{CNI}(G)$ is unmixed so G is well-dominated, and by Corollary 5.2.4 we have $\dim(R/\operatorname{CNI}(G)) = |G|/2$. □

5.3 Cohen-Macaulay Bipartite Graphs

Here, we will give our main characterization of Cohen-Macaulay bipartite graphs. We start by observing a simple fact about the dimension of closed neighborhood ideals. It complements Corollary 5.2.5 by considering graphs whose spanning trees are not necessarily well dominated.

Observation 5.3.1. *For any connected graph G, we have $dim(R/\operatorname{CNI}(G)) \geq |G|/2$.*

Proof. Recall that $\dim(R/\operatorname{CNI}(G)) = |G| - \gamma(G)$. Let D be a minimal dominating set of G. If $|D| < |G|/2$, then $\gamma(G) \leq |G|/2$ so $|G| - \gamma(G) > |G|/2$ and we are done. If $|D| \geq |G|/2$, then

D^c is a dominating set by Observation 4.3.6 such that $|D^c| \leq |G|/2$. Then D^c contains a minimal dominating set D' such that $|D'| \leq |G|/2$, so $\gamma(G) \leq |G|/2$ hence $|G| - \gamma(G) \geq |G|/2$. $\square$

This lower bound on dimension will allow us to lift results from trees to graphs. Moreover, this is tight for a well-dominated bipartite graph, as shown in the following observation.

Observation 5.3.2. *Let G be a connected bipartite graph with partite sets A and B. Then A and B are minimal dominating sets. Moreover, if G is well-dominated, then dim$(R/\operatorname{CNI}(G)) = |G|/2 = |A| = |B|$.*

Proof. We will show without loss of generality that A is a minimal dominating set. So, suppose $x \in G$. Then $x \in A$ or $x \in B$. If $x \in A$, then x is dominated by A. If $x \in B$, then x has a neighbor in A since G is connected and bipartite, so x is dominated by A. On the other hand, if $A' \subsetneq A$, then there is a vertex $y \in A$ so that $y \notin A'$. But $y \in A$, so y has no neighbor in A' and is therefore not dominated by A'. So A is a minimal dominating set.

If G is well-dominated, then A and B are minimal dominating sets. Hence $|A| = |B|$. But $|G| = |A| + |B|$ so $|A| = |B| = |G|/2$, so $\gamma(G) = |G|/2$, so we have by Theorem 5.1.3 that $\dim(R/\operatorname{CNI}(G)) = |G| - |G|/2 = |G|/2$. $\square$

Following this observation, we see that well-dominated connected graphs of dimension $|G|/2$ admit the following characterization in terms of their spanning trees.

Proposition 5.3.3. *Let G be a connected graph such that dim$(R/\operatorname{CNI}(G)) = |G|/2$. Then G is well-dominated if and only if every spanning tree of G is a whisker graph.*

Proof. The first direction follows immediately from Theorem 1.2.11 and Corollary 5.2.5. For the converse, assume that G is well-dominated and that $\dim(G) = |G|/2$. Let T be a spanning tree of G and suppose for contradiction that T is not a whisker graph, i.e., not well-dominated by Theorem 1.2.11. Then T contains a minimal dominating set D of size $|D| \neq |G|/2$. If $|D| < |G|/2$, then since D is also a dominating set of G, there is a minimal dominating set of $G, D' \subseteq D$ so that $|D'| < |G|/2$, which contradicts our assumption that G is well-dominated. On the other hand, if $|D| \geq |G|/2$, then $|D^c| < |G|/2$ since $|D| \neq |G|/2$. Moreover, D^c is a dominating set of G by Observation 4.3.6, so there is a minimal dominating set $D' \subseteq D^c$ of G such that $|D'| < |G|/2$, so again G is not well-dominated as assumed. Thus, T must be well-dominated, and therefore a whisker graph. $\square$

Corollary 5.3.4. *Let G be a connected bipartite graph. Then G is well-dominated if and only if its every spanning tree is a whisker graph.*

Proof. Note that G has partitions A, B which are both minimal dominating sets. Suppose that G is well-dominated; then $|A| = |B| = |G|/2$, so $\dim(R/\operatorname{CNI}(G)) = |G|/2$. Then applying Proposition 5.3.3, every spanning tree G is whisker graph. The other direction follows from Corollary 5.2.5. □

Before getting to our main result of this section, we make one more remark about whisker graphs.

Remark 5.3.5. *If G is a whisker graph, then* $\operatorname{CNI}(G)$ *is a complete intersection ideal.*

Proof. Suppose G is a whisker graph. Then the irredundant generators of B are precisely the squarefree quadratic monomials of the form $x_i x_j$ where x_j is a whisker adjacent to x_i. This is because if x_i is not a whisker, then it has a unique neighbor x_j that is a whisker, and $N[x_j] = \{x_i, x_j\} \subseteq N[x_i]$. Moreover, the fact that G is whiskered implies that the monomial generators are given by whiskers, and these monomials have disjoint support. Then by Theorem 3.3.7, $\operatorname{CNI}(G)$ is a complete intersection ideal. □

We will provide a generalization of this remark in Theorem 5.4.2 in the following section. We are now ready to state and prove the main theorem of this section, which is Theorem 1.2.12 from the introduction.

Theorem 5.3.6. *Let B be a connected bipartite graph with more than 4 vertices. Then the following are equivalent.*

(i) *B is well-dominated.*

(ii) *B is a whisker graph.*

(iii) $\operatorname{CNI}(B)$ *is a complete intersection.*

(iv) $R/\operatorname{CNI}(B)$ *is Cohen-Macaulay.*

In particular, the Cohen-Macaulay condition for B is independent of the field k.

Proof. *(iii)* $\Longrightarrow$ *(iv)* is Theorem 3.3.8, *(iv)* $\Longrightarrow$ *(i)* is just Theorems 3.3.4 and 5.1.4, and *(ii)* $\Longrightarrow$ *(iii)* follows from Remark 5.3.5. All that remains is to show that *(i)* $\Longrightarrow$ *(ii)*.

($i \implies ii$) We show this by contradiction. Suppose that B is well-dominated, but not a whisker graph. Observation 5.3.2 implies that $\dim(B) = |B|/2$. Let T be a spanning tree of B; by Proposition 5.3.3, T is a whisker graph.

Let H be the set of vertices so that T is obtained by whiskering $T[H]$. Because B is not a whisker graph of $B[H]$, there must be some vertex $v \in B \setminus H$ that is degree 1 (i.e., a leaf) in T, but degree greater than 1 in B. Let e be an edge not in T that is incident with v in B. Then $T + e$ has a cycle C that contains v and the edge e, that is also a cycle in B. Now, Let u be the other vertex incident with edge e. Further, note that since C is a cycle in a bipartite graph, C must be an even cycle. Now, we write $C = ux_1 \ldots x_n v$, and since C is an even cycle, $n \geq 2$.

Let d be the edge ux_1 and consider the graph $S = T + e - d$. Corollary 1.5.2 in [3] asserts that a connected graph G is a tree if and only if it has $|G| - 1$ edges. Note that S is connected since C is a cycle in $T + e$ which is connected, and any path from x to y in $T + e$, if it contains the edge d, can be substituted for a path going through the rest of C instead. Certainly $T + e - d$ has $|S| - 1$ edges, so S is a tree, and a spanning tree of B. Then by assumption, S is a whisker graph.

Since v has exactly two neighbors in S, namely u and x_n, and x_n is not a whisker (it has neighbors v and x_{n-1}), we see that u must the whisker of v in S. But then u has exactly two neighbors in $T + e$, namely v and x_1. Hence u has only one neighbor in T, namely x_1, so u is a leaf in T.

Now, let d' be the edge x_1x_2 and consider the graph $S' = T + e - d'$. Then S' is a spanning tree of B by the same reasoning as before, so in particular S' is a whisker graph by assumption. Now, since u is a leaf in T, its unique neighbor in T is just x_1. Similarly, v has a unique neighbor x_n in T. Hence, the neighbors of u in S' are exactly v and x_1, and the neighbors of v in S' are exactly u and x_n. Then u and v both have degree 2 in S'. However, both cannot have a degree 1 neighbor in S'; if they did, then S would have the 3-path x_1uvx_n as a connected component. However, S' is connected and has more than 4 vertices, so this is impossible. Then S' is not a whisker graph, as every vertex in a whisker graph is either degree 1, or has a unique neighbor of degree 1; but one of u or v has no neighbor of degree 1. On the other hand, we assumed every spanning tree of B is a whisker graph, so this is a contradiction. $\square$

5.4 Complete Intersection Closed Neighborhood Ideals and Cohen-Macaulay Chordal Graphs

We have found that, like trees, bipartite graphs are Cohen-Macaulay if and only if they are whisker graphs. In this section, we will generalize this and characterize the graphs that are complete intersections. To this end, we assert our first definition.

Definition 5.4.1. *A* **simplicial clique** K *in* G *is a clique containing a vertex* v *such that* $N[v] = K$*. In this event,* v *is called a* **simplicial vertex** *of* K*.*

We may now state and prove the following theorem.

Theorem 5.4.2. *A graph* G *is a complete intersection graph if and only if its vertex set can be partitioned into disjoint simplicial cliques.*

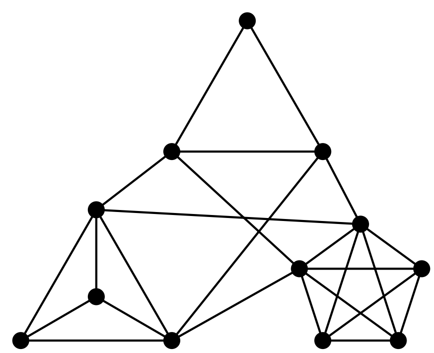
Proof. ($\Leftarrow$) Suppose the vertex set of G can be partitioned into simplicial cliques $S_1, \ldots, S_k$, where each S_i contains a simplicial vertex s_i. As every vertex in G falls in some S_i, every closed neighborhood in G contains some S_i. Thus, the non-redundant generators of CNI(G) are precisely the monomials m_{S_i}, which have pairwise disjoint support. Then by Theorem 3.3.7, CNI(G) is a complete intersection.

($\Rightarrow$) Suppose CNI(G) is a complete intersection. Then its generators $M_1, \ldots, M_k$ corresponding to the closed neighborhoods of vertices $v_1, \ldots, v_k$ have disjoint support. Moreover, if $x \in V(G)$, then $M_i | M_{N[x]}$ for some i, so $N[v_i] \subseteq N[x]$. Hence $v_i \in N[x]$, so $v_i \sim x$, implying that $x \in N[v_i]$ so $x|M_i$. Hence every vertex of G appears in some monomial M_i. Then by Theorem 3.3.7, not only do the generators of CNI(G) all have disjoint support, but they also partition the vertices of G. Moreover, for any vertex $v \in G$, there is a unique generator M_i s.t. supp$(M_i) \subseteq N[v]$; otherwise, $N[v]$ corresponds to a non-redundant generator that is not among the M_i's. It follows that each set of vertices supp(M_i) is a clique, and moreover, that each v_i is a simplicial vertex in M_i. Thus the set $\{\text{supp}(M_i)\}_{i=1}^{k}$ partitions G into simplicial cliques. $\square$

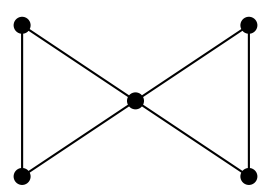
We give the property described in Theorem 5.4.2 a name in the following definition. The name HHZ refers to the authors of [7], Herzog, Hibi, and Zheng, who use this property in [7, Theorem, 2.1] but do not name it.

Definition 5.4.3. *If a graph G can be partitioned into disjoint simplicial cliques, then G is said to be an* **HHZ graph***.*

Example 5.4.4. *The following graph is an HHZ graph. Its simplicial cliques are the apparent 3-clique, the 4-clique, and 5-clique.*



Example 5.4.5. *The butterfly graph from Example 1.1.2 is* **not** *an HHZ graph. It is indeed a union of its simplicial cliques, but cannot be partitioned into them.*



Note that the HHZ condition can be seen as a generalization of a whisker graph; every whisker graph can be partitioned into simplicial cliques – specifically, its whiskers. We may now use Theorem 5.4.2 to characterize complete intersections in various graph families.

Corollary 5.4.6. *Let G be an n-regular graph with $n \geq 1$. Then* $\mathrm{CNI}(G)$ *is a complete intersection if and only if G is a disjoint union of K_{n+1} graphs.*

Proof. By Theorem 5.4.2, $\mathrm{CNI}(G)$ is a complete intersection if and only if G is HHZ. Then let s be a simplicial vertex in a maximal simplicial clique $S \subseteq G$. Since G is n-regular, s has exactly n neighbors so $|S| = n + 1$. Since S is a clique, and every vertex v in S also has exactly n neighbors, those n neighbors of v must all be in S, so the maximal simplicial cliques in G are disconnected from each other. Conversely, every disjoint union of K_{n+1} graphs is straightforwardly partitionable into simplicial cliques. $\square$

Corollary 5.4.7. *Let G be a triangle-free connected graph with more than one vertex. Then* $\mathrm{CNI}(G)$ *is a complete intersection if and only if G is a whisker graph.*

Proof. Suppose $\mathrm{CNI}(G)$ is a complete intersection, so by Theorem 5.4.2 G is an HHZ graph. Since G contains no triangles, its simplicial cliques can only be 2-cliques or 1-cliques. A connected graph with a simplicial 1-clique, however, can only be the graph containing a single vertex, as a simplicial 1-clique is an isolated vertex. G has more than one vertex, however, so its maximal simplicial cliques must all be 2-cliques, but a simplicial 2-clique is just a whisker. The converse holds by Remark 5.3.5. □

Corollary 5.4.8. *Let G be an HHZ graph with with simplicial cliques $S_1, \ldots, S_k$. Then G is well-dominated with minimal dominating sets of size k.*

Proof. Suppose that G is an HHZ graph. Then Theorem 5.4.2 implies that $\mathrm{CNI}(G)$ is a complete intersection ideal, and in particular, Theorem 3.3.8 shows that $R/\mathrm{CNI}(G)$ is Cohen-Macaulay. By Theorem 3.3.4, $\mathrm{CNI}(G)$ is unmixed, so G is well-dominated. Moreover, if $s_1 \in S_1, \ldots, s_k \in S_k$ are simplicial vertices, then $\{s_1, \ldots, s_k\}$ is a minimal dominating set. □

Here is the main result of this section. It is Theorem 1.2.13 from the introduction.

Theorem 5.4.9. *Let G be a chordal graph. Then the following are equivalent.*

(i) G is well-dominated.

(ii) G is an HHZ graph.

(iii) $\mathrm{CNI}(G)$ is a complete intersection.

(iv) $R/\mathrm{CNI}(G)$ is Cohen-Macaulay.

In particular, the Cohen-Macaulay condition for G is independent of the field k.

Proof. $(i) \implies (ii) \implies (iii) \implies (iv)$. By Theorem 4.3.9, a well-dominated graph is well-covered. Moreover, a chordal graph is well-covered if and only if it is HHZ [7, Theorem 2.1]. Therefore, if G is a well dominated chordal graph, then it is HHZ, so G is a complete intersection by Theorem 5.4.2 and therefore Cohen-Macaulay by Theorem 3.3.8.

$(iv) \implies (i)$. If G is Cohen-Macaulay, then $\mathrm{CNI}(G)$ is unmixed so G is well-dominated. □

We close this discussion with a simple theorem about the depth of complete intersections that will become helpful for computing depth in subsequent sections.

Lemma 5.4.10. *Let G be an HHZ graph with n vertices and simplicial cliques $S_1, \dots, S_k$. Then we have depth$(R/\operatorname{CNI}(G)) = n - k$.*

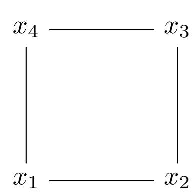
Proof. Note that the generating monomials of G are exactly $m_{S_1}, \dots, m_{S_k}$, which forms a regular sequence. Then by Theorem 3.2.7, we have $\operatorname{depth}(R/\operatorname{CNI}(G)) = n - k$. $\square$

5.5 Restricted Closed Neighborhood Ideals and Redundancy

We now turn our attention towards the behavior of depth under certain redundancy conditions. To this end, we consider restricted closed neighborhood ideals, which provide a notion of closed neighborhood ideals restricted to certain vertices in G. We will use restricted closed neighborhood ideals to better understand the closed neighborhood ideals of induced subgraphs of G.

Definition 5.5.1. *Let H be a subset of $V(G)$. Then the* **restricted closed neighborhood ideal at H**, *denoted $\operatorname{CNI}_G(H)$, is the squarefree monomial ideal in R generated by closed neighborhoods of vertices in H. In other words, $\operatorname{CNI}_G(H) = \langle m_{N[x]} : x \in H \rangle$.*

Example 5.5.2. *Let G be the 4-cycle as labeled below, and let $H = \{x_1, x_2\}$.*



Then $\operatorname{CNI}_G(H) = \langle x_1x_2x_4, x_1x_2x_3 \rangle$. Note that this is not the same as the closed neighborhood ideal of the induced subgraph $G[H]$, which is $\operatorname{CNI}(G[H]) = \langle x_1x_2 \rangle$.

Like closed neighborhood ideals, restricted closed neighborhood ideals have their own notion of domination and their own irreducible decomposition theorem, similar to Theorem 1.2.7.

Definition 5.5.3. *Let G be a graph and $H \subseteq V(G)$. An* **H-dominating set** *D is a set of vertices of G such that for all $x \in H$, there exists $y \in D$ such that $x \sim y$ or $x = y$. An H-dominating set is called* **minimal** *if it does not properly contain any other H-dominating set.*

Definition 5.5.4. *Let $H \subseteq V(G)$. Then the* **H-domination number**, *denoted $\gamma_G(H)$, is the size of the smallest H-dominating set.*

Example 5.5.5. *The minimal H-dominating sets in Example 5.5.2 are the sets $\{x_1\}, \{x_2\}, \{x_3, x_4\}$. Hence, $\gamma_G(H) = 1$.*

The notion of a minimal H-dominating set provides the following decomposition theorem for restricted closed neighborhood ideals.

Theorem 5.5.6. *Suppose G is a graph and $H \subseteq V(G)$. Let $\mathcal{D}$ be the set of minimal H-dominating sets in G. Then $\mathrm{CNI}_G(H) = \cap_{D\in\mathcal{D}}\langle D\rangle$. In other words, the minimal primes of $\mathrm{CNI}_G(H)$ are precisely the primes generated by minimal H-dominating sets of G.*

Proof. Let D be a set of vertices of G. If D is a minimal H-dominating set, then every minimal monomial generator of $\mathrm{CNI}_G(H)$ contains some vertex in D, so $\mathrm{CNI}_G(H) \subseteq \langle D\rangle$.

Conversely, suppose $\mathrm{CNI}_G(H) \subseteq \langle D\rangle$. For every minimal monomial generator $m = m_{N[x]} \in \mathrm{CNI}_G(H)$, there is a vertex $v \in D$ such that $v|m$, in other words, $v \in \mathrm{supp(m)} = N[x]$. But then every closed neighborhood of vertices of H must contain some $v \in D$, so D is an H-dominating set.

Next, let $P_1, \ldots, P_r$ be the prime ideals in the irredundant irreducible decomposition of $\mathrm{CNI}_G(H)$. It follows that for each P_i, the set $\mathrm{supp}(P_i)$ is H-dominating. Moreover, $\mathrm{supp}(P_i)$ is minimal, or else P_i is redundant. It follows that $\mathcal{D} = \{\mathrm{supp}(P_1), \ldots, \mathrm{supp}(P_r)\}$, so we have $\mathrm{CNI}_G(H) = \cap_{1\le i\le r}P_i = \cap_{D\in\mathcal{D}}\langle D\rangle$. □

Next, we apply restricted closed neighborhood ideals to realize closed neighborhood ideals of induced subgraphs as colon ideals.

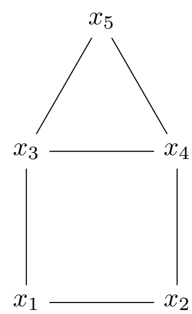
Corollary 5.5.7. *Let G be a graph and $H \subseteq V(G)$. Then $\mathrm{CNI}_G(H) : m_{H^c} = \mathrm{CNI}(G[H])$.*

Proof. Let $\mathcal{D}$ be the set of minimal H-dominating sets D of G such that $D \cap H^c = \emptyset$. Then $\mathrm{CNI}_G(H) : m_{H^c} = \cap_{D\in\mathcal{D}}\langle D\rangle$ by Theorem 5.5.6 and Theorem 2.3.8. But $\mathcal{D}$ consists precisely of the minimal dominating sets of H, so $\cap_{D\in\mathcal{D}}\langle D\rangle = \mathrm{CNI}(G[H])$. □

Corollary 5.5.7 allows us to "remove" certain vertices by a natural algebraic operation on the restricted neighborhood ideal of a graph. In certain cases, we can lift this result to regular closed neighborhood ideals, via the following definitions and propositions.

Definition 5.5.8. *A set of vertices S is said to be* **redundant** *if for all $x \in S$, there exists some $y \notin S$ such that $N[y] \subseteq N[x]$, or equivalently, $m_{N[y]} \mid m_{N[x]}$.*

Example 5.5.9. *Let $R = k[x_1, x_2, x_3, x_4, x_5]$ and consider the house graph with vertices labeled as follows.*



Then the set $\{x_3, x_4\}$ is redundant.

Observation 5.5.10. *If S is a redundant set of vertices in G, then $\mathrm{CNI}_G(G - S) = \mathrm{CNI}(G)$.*

The following proposition provides a relationship between redundant sets of vertices and colon ideals.

Proposition 5.5.11. *Let S be a redundant set of vertices in G. Then $\mathrm{CNI}(G) : m_S = \mathrm{CNI}(G - S)$.*

Proof. Let $H = S^c$. By Observation 5.5.10 we have that $\mathrm{CNI}_G(G[H]) = \mathrm{CNI}_G(G - S) = \mathrm{CNI}(G)$. Then we have that

$$\mathrm{CNI(G)} : m_S = \mathrm{CNI}_G(H) : m_{H^c} = \mathrm{CNI}(G[H]) = \mathrm{CNI}(G - S)$$

by Corollary 5.5.7. $\square$

Observation 5.5.12. *Let S be a redundant set of vertices. Then*

$$\mathit{depth}(R/(\mathrm{CNI}(G - S)) \geq \mathit{depth}(R/\,\mathrm{CNI}(G)).$$

Proof. This a direct application of Theorem 3.4.3 to $\mathrm{CNI}(G - S) = \mathrm{CNI}(G) : m_S$. $\square$

To conclude our discussion of redundant sets of vertices, we first define two parameters of an ideal that allow us to perform inductive arguments. The first one is nonstandard.

Definition 5.5.13. *The* **total degree** *of a monomial ideal I, denoted $\tau(I)$, is the sum of the degrees of its minimal monomial generators.*

Example 5.5.14. *Let $R = k[x_1, x_2, x_3, x_4]$ The ideal $\langle x_1x_2, x_2x_3x_4\rangle$ has total degree $2 + 3 = 5$.*

Definition 5.5.15. *The* **minimal number of generators** *of a monomial I, denoted $\mu(I)$, is the size of the smallest monomial generating set of I.*

Example 5.5.16. *In Example 5.5.9, we have $\mu(\mathrm{CNI}(G)) = 3$. This is because the smallest monomial generating set is $\{x_1x_2x_3, x_1x_2x_4, x_3x_4x_5\}$.*

Theorem 5.5.17. *Let I be a squarefree monomial ideal in the ring $R = k[x_1, \ldots, k_n]$. Then depth$(R/I) \geq n - \mu(I)$.*

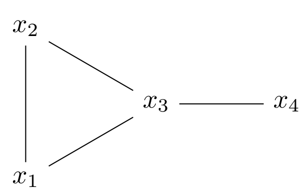
Proof. We show this by induction on the total degree of I. Suppose the theorem holds for all J such that $\tau(J) < \tau(I)$. If I is a prime ideal, i.e. all of its minimal monomial generators have degree 1, then $\operatorname{depth}\big(R/\operatorname{CNI}(G)\big) = |G| - \mu(\operatorname{CNI}(G))$ by Theorem 3.2.7 and we are done. Note that this also covers the base case $\tau(I) = 1$. Otherwise, choose some variable x_i such that $x_i \notin I$, but x_i divides some minimal monomial generator of I. Let $J_1 = I : x_i$ and let $J_2 = I + x_i$. Now, by Corollary 2.3.5, it is easy to see that that $\tau(J_1), \tau(J_2) < \tau(I)$, so J_1, J_2 satisfy the induction hypothesis. Moreover, by choice of x_i, by Corollary 2.3.5 we have that $\mu(J_1), \mu(J_2) \leq \mu(I)$. By Theorem 3.4.2, either $\operatorname{depth}(R/I) = \operatorname{depth}(R/J_1) \geq n - \mu(J_1) \geq n - \mu(I)$ or $\operatorname{depth}(R/I) = \operatorname{depth}(R/J_2) \geq n - \mu(J_2) \geq n - \mu(I)$, as desired. □

Corollary 5.5.18. *Let S be a redundant set of vertices in G. Then depth$(R/\operatorname{CNI}(G)) \geq |S|$.*

Proof. Note that $|S| + \mu(G) \leq |G|$. Then Theorem 5.5.17 implies that $|S| \leq |G| - \mu(G) \leq \operatorname{depth}(R/\operatorname{CNI}(G))$. □

In certain cases, deletion of a redundant set of vertices allows for a quick computation of the depth of a closed neighborhood ideal.

Example 5.5.19. *Let $R = k[x_1, x_2, x_3, x_4]$. Consider the following graph G.*



Then $\gamma(G) = 1$, because $\{x_3\}$ is a minimal dominating set, so dim$R/\operatorname{CNI}(G) = 3$. Since G is not well-dominated, it is not Cohen-Macaulay, so depth$(R/\operatorname{CNI}(G)) < 3$. On the other hand, *the set* $\{x_1, x_2\}$ *is redundant, so Corollary 5.5.18 implies that* $2 \leq \mathit{depth}(R/\operatorname{CNI}(G)) < 3$*, so* $\mathit{depth}(R/\operatorname{CNI}(G)) = 2$*.*

We may also apply Corollary 5.5.18 towards the following theorem about the depth of trees.

Proposition 5.5.20. *Assume* G *is a tree with* $n \geq 3$*. Let* L *be its set of leaves. Then* $\gamma_G(L) \leq \mathit{depth}(R/\operatorname{CNI}(G)) \leq n - |L|$*.*

Proof. Let S be a smallest L-dominating set. We may assume that S itself contains no leaves: if $x \in S$ is a leaf, then let y be its neighbor. Since $n \geq 3$, y is not a leaf. Certainly, the set $S' = (S \cup \{y\}) \setminus \{x\}$ is still an L-dominating set, since the only leaf dominated by x is x itself, and x is dominated by $y \in S'$. Moreover, $|S'| \leq |S|$, so S' must also be a smallest L-dominating set.

With S as above containing no leaves, S is a redundant set. This is because each $x \in S$ has a neighbor y that is a leaf, and $N[y] \subseteq N[x]$. Hence we have by Corollary 5.5.18 that $\operatorname{depth}(R/\operatorname{CNI}(G)) \geq |S| = \gamma_G(L)$. On the other hand, $\operatorname{depth}(R/\operatorname{CNI}(G)) \leq \operatorname{depth}(R/(\operatorname{CNI}(G-S))$ by Observation 5.5.12. But $\operatorname{depth}(R/(\operatorname{CNI}(G-S)) \leq n-|L|$ since $G-S$ has $|L|$ isolated vertices. $\square$

In the following theorem, we specialize these results to the discussion of deleting simplicial cliques and non-simplicial vertices of simplicial cliques.

Theorem 5.5.21. *Let* G *be a graph and let* K *be a simplicial clique in* G*. Then let* S *be a set of non-simplicial vertices in* K*. Then*

1. $\mathit{depth}(R/(\operatorname{CNI}(G - S)) \geq \mathit{depth}(R/\operatorname{CNI}(G))$*, and*
2. $\mathit{depth}(R/(\operatorname{CNI}(G - K)) > \mathit{depth}(R/\operatorname{CNI}(G))$*.*

Proof. The set S is redundant since if $v \in K$ is a simplicial vertex and $s \in S$, then $N[v] \subseteq N[s]$. Hence $\operatorname{CNI}(G - S) = (\operatorname{CNI}(G) : m_S)$ by Proposition 5.5.11. Next, Theorem 3.4.2 implies that $\operatorname{depth}(R/\operatorname{CNI}(G : m_S)) \geq \operatorname{depth}(R/\operatorname{CNI}(G))$, so part 1 follows.

For 2, let V be the set of simplicial vertices in K and set $M = R/\operatorname{CNI}(G - K)$. Note that that m_V has disjoint support from all the monomial generators of $\operatorname{CNI}(G - K)$. Hence, m_V is an M-regular element, and we have the following.

$$M/m_V M \cong R/(\operatorname{CNI}(G - K) + \langle m_V \rangle) \cong R/(\operatorname{CNI}(G - S))$$

Thus, Theorem 3.2.7 explains the equality in the following.

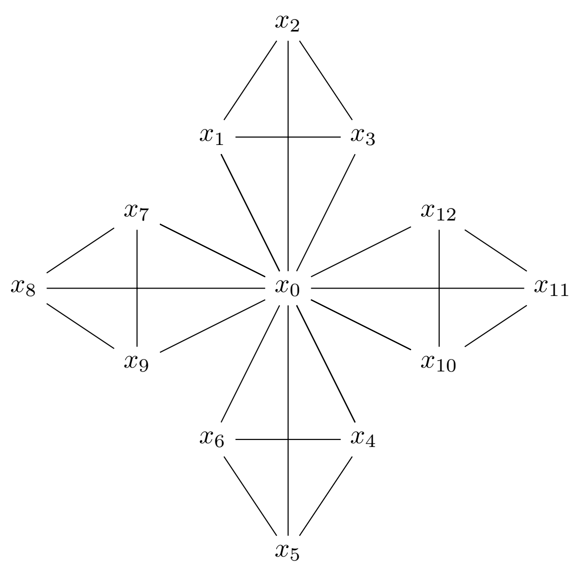
$$\operatorname{depth}(R/(\operatorname{CNI}(G-K)) = \operatorname{depth}(R/(\operatorname{CNI}(G-S)) + 1 > \operatorname{depth}(R/\operatorname{CNI}(G))$$

The inequality is by Part 1. □

We put these theorems to use by computing the depth of the following family of graphs.

Definition 5.5.22. *The* **windmill graph** $\operatorname{Wd}(k,n)$ *is the graph obtained by attaching n many k-cliques at a single shared universal vertex, where $n, k \geq 2$.*

Example 5.5.23. *Here is* $\operatorname{Wd}(4,4)$.



Before asserting our theorem about windmill graphs, let us first note that $\operatorname{Wd}(k,n)$ has exactly $n(k-1)+1$ vertices.

Theorem 5.5.24. *Let $n, k \geq 2$. Let $r = n(k-1)$ and let $R = k[x_0, \ldots, x_r]$. Let $G = \operatorname{Wd}(k,n)$ on vertices $x_0, \ldots, x_r$. Then depth$(R/\operatorname{CNI}(G)) = n(k-2)+1$*

Proof. Without loss of generality, let x_0 be the universal vertex of G. Note that x_0 is a non-simplicial vertex in a simplicial clique. Then by Theorem 5.5.21

$$\operatorname{depth}(R/\operatorname{CNI}(G)) \leq \operatorname{depth}(R/(\operatorname{CNI}(G-x_0)).$$

But $G - x_0$ is a disjoint union of n cliques, hence a complete intersection with n generators, which has depth $n(k-1)+1-n = n(k-2)+1$, so $\operatorname{depth}(R/\operatorname{CNI}(G)) \leq n(k-2)+1$. On the other hand, by selecting x_0 and choosing $k-2$ vertices from each clique in G other than x_0, we construct a redundant set containing $n(k-2)+1$ vertices, hence, by Corollary 5.5.18, $\operatorname{depth}(R/\operatorname{CNI}(G)) \geq n(k-2)+1$. □

5.6 Depths of Joins of Graphs

Recall that the join of two graphs G and H is a graph $G+H$ whose underlying vertex set is $V(G) \cup V(H)$ (we assume that $V(G)$ and $V(H)$ are disjoint), and whose edges are exactly the edges in G, the edges in H, and all edges between $x \in G$ and $y \in H$. Throughout this section, we will always take R to be the polynomial ring in $n = |G+H| = |G| + |H|$ variables, and we will consider $\operatorname{CNI}(G)$ and $\operatorname{CNI}(H)$ to both be squarefree monomial ideals in R, even if G and H do not have as many vertices as R has variables, unless otherwise specified.

Lemma 5.6.1. *Given graphs G, H, we have*

$$\mathit{depth}(R/(\operatorname{CNI}(G+H)) \in \{n-2, \mathit{depth}(R/\operatorname{CNI}(G)), \mathit{depth}(R/(\operatorname{CNI}(H))\}.$$

Moreover, if $\mathit{depth}(R/\operatorname{CNI}(G)), \mathit{depth}(R/(\operatorname{CNI}(H)) \leq n-2$, then

$$\mathit{depth}(R/(\operatorname{CNI}(G+H)) = \mathit{min}\{\mathit{depth}(R/\operatorname{CNI}(G)), \mathit{depth}(R/(\operatorname{CNI}(H)))\}.$$

Proof. This follows from repeated application of Theorem 3.4.2. Note that

$$(\operatorname{CNI}(G+H) : m_G) = (\operatorname{CNI}_{G+H}(G) : m_G) + (\operatorname{CNI}_{G+H}(H) : m_G) = \langle m_H \rangle + \operatorname{CNI}(H) = \operatorname{CNI}(H).$$

Now, let

$$J = \operatorname{CNI}(G+H) + \langle m_G \rangle = m_H \cdot \operatorname{CNI}(G) + \langle m_G \rangle,$$

so that

$$\operatorname{depth}(R/(\operatorname{CNI}(G+H)) \in \{\operatorname{depth}(R/(\operatorname{CNI}(H)), \operatorname{depth}(R/J)\} \tag{5.6.1.1}$$

by Theorem 3.4.2. Then $J + \langle m_H \rangle = \langle m_G, m_H \rangle$ and $(J : m_H) = \mathrm{CNI}(G)$. Hence,

$$\operatorname{depth}(R/J) \in \{n-2, \operatorname{depth}(R/\operatorname{CNI}(G))\} \tag{5.6.1.2}$$

again by Theorem 3.4.2. By combining (5.6.1.1) and (5.6.1.2), we have that

$$\operatorname{depth}(R/(\operatorname{CNI}(G+H)) \in \{n-2, \operatorname{depth}(R/\operatorname{CNI}(G)), \operatorname{depth}(R/(\operatorname{CNI}(H))\}.$$

For the second part, suppose that $\operatorname{depth}(R/\operatorname{CNI}(G)), \operatorname{depth}(R/(\operatorname{CNI}(H)) \leq n-2$, and set $J_2 = J_1 + \langle m_H \rangle$. Recall from the second part of Theorem 3.4.2 that if

$$\operatorname{depth}(R/(J : m_H)) \leq \operatorname{depth}(R/(J + \langle m_H \rangle))$$

then $\operatorname{depth}(R/J) = \operatorname{depth}(R/(J : m_H)) = \operatorname{depth}(R/\operatorname{CNI}(G))$. On the other hand, we have that

$$\operatorname{depth}(R/(J : m_H)) = \operatorname{depth}(R/\operatorname{CNI}(G)) \leq n-2 = \operatorname{depth}(R/\langle m_G, m_H \rangle) = \operatorname{depth}(R/(J + \langle m_H \rangle))$$

so it follows that $\operatorname{depth}(R/J) = \operatorname{depth}(R/\operatorname{CNI}(G))$. Therefore, (5.6.1.1) implies that

$$\operatorname{depth}(R/(\operatorname{CNI}(G+H)) \in \{\operatorname{depth}(R/(\operatorname{CNI}(H)), \operatorname{depth}(R/(\operatorname{CNI}(G))\}.$$

Again, by the second part of Theorem 3.4.2, if $\operatorname{depth}(R/(\operatorname{CNI}(H)) \leq \operatorname{depth}(R/(\operatorname{CNI}(G))$, then we have that $\operatorname{depth}(R/(\operatorname{CNI}(G+H)) = \operatorname{depth}(R/(\operatorname{CNI}(H))$. The exact same argument while swapping the roles of G and H reveals that if $\operatorname{depth}(R/\operatorname{CNI}(G)) \leq \operatorname{depth}(R/(\operatorname{CNI}(H))$, then $\operatorname{depth}(R/(\operatorname{CNI}(G+H)) = \operatorname{depth}(R/\operatorname{CNI}(G))$. Hence, $\operatorname{depth}(R/(\operatorname{CNI}(G+H))$ is the minimum of the other depths. □

We now use Lemma 5.6.1 to compute the depth of the closed neighborhood ideals of complete bipartite graphs.

Theorem 5.6.2. *Let $G = K_{s,t}$ and $n = s+t$ where $s, t \geq 1$. Then depth$(R/\operatorname{CNI}(G)) =$ min$\{s,t\}$.*

Proof. First, suppose s or t is 1, then G is just a star. But a closed neighborhood ideal of a star with n vertices is an ideal of the form $I = \langle x_1x_2, \ldots, x_1x_n \rangle$. Theorem 3.4.2 implies that $\operatorname{depth}(R/I)$ is either

$\operatorname{depth}(R/(I : x_1))$ or $\operatorname{depth}(R/(I, x_1))$. But $(I : x_1) = \langle x_2, \dots, x_n \rangle$ and, in addition, $I + \langle x_1 \rangle = \langle x_1 \rangle$, which have depth 1 and $n-1$ respectively. Since in this case $\operatorname{depth}(R/(I : x_1)) \leq \operatorname{depth}(R/(I + \langle x_1 \rangle))$, by the second part of Theorem 3.4.2, we have $\operatorname{depth}(R/\operatorname{CNI}(G)) = \operatorname{depth}(R/(I : x_1)) = 1$.

If $s, t \geq 2$ then note that G can be seen as $N_s + N_t$. Moreover, $\operatorname{depth}(R/\operatorname{CNI}(N_s)) = t \leq n-2$ and $\operatorname{depth}(R/\operatorname{CNI}(N_t)) = s \leq n-2$, so by Lemma 5.6.1, $\operatorname{depth}(R/\operatorname{CNI}(G)) = \min\{s, t\}$. □

Lemma 5.6.1 also allows us to immediately generate a new family of Cohen-Macaulay graphs. First, we establish the a new definition.

Definition 5.6.3. *The* **codepth** *of a graph G in the polynomial R, denoted $\text{codepth}_R(G)$, is $\dim(R) - \operatorname{depth}(R/\operatorname{CNI}(G))$.*

Codepth does not depend on the number of variables in the ring R, as implied by the following proposition.

Proposition 5.6.4. *Let $m = |G|$, $n > m$, and set $k = n - m$, $R = k[x_1, \dots, x_n]$, and $Q = k[x_1, \dots, x_m]$. Then $\operatorname{depth}(R/\operatorname{CNI(G)}) - k = \operatorname{depth}(Q/\operatorname{CNI(G)})$. In particular, we have that $\text{codepth}_R(G) = \text{codepth}_Q(G)$.*

Proof. Let $I = \langle x_{m+1}, \dots, x_n \rangle$ and note that $Q \cong R/I$. It follows that $(R/\operatorname{CNI}(G))/I \cong Q/\operatorname{CNI}(G)$. But it is straightforward that $x_{m+1}, \dots, x_n$ is a regular sequence of length k in $R/\operatorname{CNI}(G)$, so Theorem 3.2.7 implies that $\operatorname{depth}(R/\operatorname{CNI(G)}) - k = \operatorname{depth}(Q/\operatorname{CNI(G)})$, as desired. This implies straightforwardly that $\text{codepth}_R(G) = \text{codepth}_Q(G)$. □

Hence, from here on out we will write $\text{codepth}(G)$ rather than $\text{codepth}_R(G)$. We now state the following easy corollary of Lemma 5.6.1.

Corollary 5.6.5. *Let G, H be graphs such with codepth 2. Then $\text{codepth}(G + H) = 2$.*

Proof. This follows quickly from Lemma 5.6.1. □

One might wonder what happens when G has codepth 1 rather than codepth 2. Such graphs have the following property.

Observation 5.6.6. *Suppose that $\text{codepth}(G) = 1$. Then G has a universal vertex.*

Proof. Since G has codepth 1, we have that $\operatorname{depth}(R/\operatorname{CNI}(G)) = n - 1$. Theorem 3.2.8 implies that $n - 1 \leq \dim(R/\operatorname{CNI}(G))$. Hence, $\gamma(G) = 1$, meaning there exists a vertex $x \in V(G)$ such that $\{x\}$ is a dominating set of G. Then $\{x\}$ is a universal vertex. $\square$

Universal vertices also have a predictable behavior under graph joins.

Observation 5.6.7. *$G + H$ has a universal vertex if and only if G or H has a universal vertex.*

Proof. For the first direction, suppose without loss of generality that G has a universal vertex x. Then every vertex in $V(G)$ is adjacent to x in $G + H$. Moreover, every vertex in $V(H)$ is adjacent to x in $G + H$ by the definition of the join of two graphs.

Conversely, suppose $G + H$ has a universal vertex. Then there is a vertex $x \in V(G + H)$ such that every vertex in $G + H$ is adjacent to x. Since $V(G + H) = V(G) \cup V(H)$, we may assume without loss of generality that $x \in V(G)$. Then it is straightforward to see that x is a universal vertex in G as well. $\square$

On the other hand, if neither G nor H have a universal vertex, then we have the folowing.

Observation 5.6.8. *Suppose neither G nor H has a universal vertex. Then $\gamma(G + H) = 2$.*

Proof. By Observation 5.6.7, $\gamma(G + H) > 1$. On the other hand, if $x \in V(G)$ and $y \in V(H)$, then $\{x, y\}$ is a minimal dominating set of $G + H$. $\square$

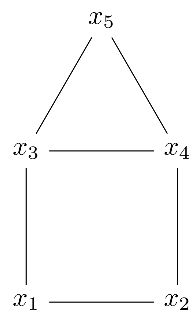
Theorem 5.6.9. *Suppose G, H are graphs with codepth 2, and furthermore that neither G nor H has a universal vertex. Then $G + H$ is Cohen-Macaulay.*

Proof. If neither G nor H has a universal vertex, then $\dim(R/\operatorname{CNI}(G + H)) = n - 2$ by Observation 5.6.8. Theorem 5.6.5 implies that $\operatorname{depth}(R/\operatorname{CNI}(G + H)) = n - 2$, so $G + H$ is Cohen-Macaulay. $\square$

Using this theorem, we can generate a large family of Cohen-Macaulay graphs. To make this simpler, we assert the following fact.

From Proposition 5.6.4 and Observation 5.6.7, it is straightforward to see that if G and H satisfy the hypotheses of Theorem 5.6.9, then so does $G + H$. Hence we can build larger and larger Cohen-Macaulay graphs by joining together graphs of depth $n - 2$ in an inductive fashion. To this end, we close with a few examples of such graphs.

Example 5.6.10. *Consider the house graph G from Example 5.5.9.*



Note that $\mathrm{CNI}(G) = \langle x_1x_2x_3, x_1x_2x_4, x_3x_4x_5\rangle$. *Let* $J_1 = (\mathrm{CNI}(G) : x_1x_2) = \langle x_3, x_4\rangle$ *and let* $J_2 = \mathrm{CNI}(G) + \langle x_1, x_2\rangle = \langle x_1x_2, x_3x_4x_5\rangle$. *Both J_1 and J_2 are complete intersection ideals with* 2 *generators, so R/J_1 and R/J_2 both have depth* 3. *Theorem 3.4.2 then implies that $R/CNI(G)$ also has depth* 3, *so G has codepth* 2. *Also, note that G does not have a universal vertex. Therefore G satisfies the hypotheses of Theorem 5.6.9.*

Proposition 5.6.11. *Let G be an HHZ graph with exactly two maximal simplicial cliques S_1, S_2. Then G satisfies the hypotheses of Theorem 5.6.9.*

Proof. Since $\mathrm{CNI}(G) = \langle m_{S_1}, m_{S_2}\rangle$, Theorem 3.2.7 implies that $\mathrm{depth}(R/CNI(G)) = n - 2$. Moreover, G has no universal vertex, since if $v \in V(G)$ were universal, then setting s_1, s_2 to be simplicial vertices in S_1 and S_2 respectively, we must have $v \sim s_1$ and $v \sim s_2$, meaning $v \in S_1$ and $v \in S_2$. But G is an HHZ graph, so S_1 and S_2 are disjoint. Therefore, G satisfies the hypotheses of Theorem 5.6.9, as desired. $\square$

By recursively joining graphs such as those in Example 5.6.10 and Proposition 5.6.11, we obtain a large family of Cohen-Macaulay graphs.

Example 5.6.12. *Consider the house graph G from Example 5.5.9, and define kG to be graph obtained by joining G to itself k-times. For example, $3G = (G+G)+G$. Note that Proposition 5.6.7 implies that kG does not have a universal vertex. Then kG is Cohen-Macaulay by Theorem 5.6.9 and induction.*

More generally, we have the following family of Cohen-Macaulay graphs.

Example 5.6.13. *Let F be a finite collection of graphs with codepth* 2, *none of which has a universal vertex. Then any graph obtained by joining any number of the graphs in F in any order is Cohen-Macaulay.*

Chapter 6

Future Questions

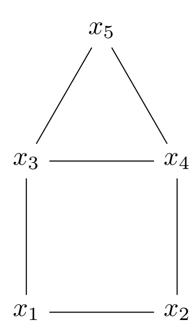
6.1 Perfect Graphs

Our characterization of chordal and bipartite graphs in Theorems 5.4.9 and 5.3.6 might leave one wondering if these theorems can be further generalized. In particular, chordal and bipartite graphs are both special cases of a large family of graphs called perfect graphs. Let us briefly define the parameters that characterize a perfect graph.

Definition 6.1.1. *An n-coloring of a graph G by the set $X = \{1, 2, \ldots, n\}$ is a map $f : V(G) \mapsto X$. A coloring f is called* **proper** *if $x \sim y \implies f(x) \neq f(y)$, i.e., adjacent vertices are assigned different colors. The* **chromatic number** *of G, denoted $\chi(G)$, is smallest number k such that there is a proper k-coloring of G.*

Example 6.1.2. *A complete graph K_n has chromatic number n.*

Example 6.1.3. *The house graph below has chromatic number 3. For instance, let $f(x_1) = f(x_4) = 1, f(x_2) = f(x_3) = 2$, and $f(x_4) = 3$.*



Definition 6.1.4. *The* **maximum clique number** *of G, denoted $\omega(G)$, is the size of the largest clique in G.*

Example 6.1.5. *Let G be the house graph in Example 6.1.3. Then $\omega(G) = 3$.*

We are now ready to define perfect graphs.

Definition 6.1.6. *G is called a* **perfect graph** *if $\chi(G) = \omega(G)$, and, moreover, $\chi(H) = \omega(H)$ for every induced subgraph H of G.*

Both bipartite graphs [3, Page 135] and chordal graphs [3, Proposition 5.5.2] are standard examples of perfect graphs. It is therefore tempting to generalize Theorems 5.3.6 and 5.4.9 by conjecturing that all well-dominated perfect graphs are Cohen-Macaulay. However, this is not the case, as seen in the following smallest counterexample over $\mathbb{Q}$.

Example 6.1.7. *Let G be the complement of the cycle C_8 and let $R = \mathbb{Q}[x_1, \ldots, x_8]$. Then a calculation with Macaulay2 [5] reveals that G is well-dominated and perfect, but dim$(R/\operatorname{CNI}(G)) = 6$ and depth$(R/\operatorname{CNI}(G)) = 5$.*

This leads us to the following question.

Question 6.1.8. *Which perfect graphs are Cohen-Macaulay?*

6.2 Inductive Depth Computations

Much of Section 5.5 centered around inductive arguments using Theorem 3.4.2, which allows us to bound, and in some cases compute, the depths of graphs. The crux of each such computation was to make a "smart" choice of monomial m so that $(\operatorname{CNI}(G) : m)$ and $\operatorname{CNI}(G) + \langle m \rangle$ have known depth. For instance, in Example 5.6.10, we computed the depth of the house graph in one step by choosing the monomial $m = x_1x_2$; the resulting ideals $(\operatorname{CNI}(G) : m)$ and $\operatorname{CNI}(G) + \langle m \rangle$ were just complete intersection ideals. On the other hand, our proof of Theorem 5.6.1 requires a two-step argument by choosing the monomials m_H and then m_G, and performing the same argument in the opposite order. This leads us to the following two questions.

Question 6.2.1. *Under what conditions on G is there a monomial m such that $(\operatorname{CNI}(G) : m)$ and $\operatorname{CNI}(G) + \langle m \rangle$ are complete intersections?*

Question 6.2.2. *Let $k < n$. Can Theorem 5.6.1 be generalized to graphs with depth $n - k$ by an inductive argument?*

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