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VALUATION OF ASIAN OPTIONS IN A HIGH VOLATILITY MARKET WITH JUMPS

Zeeshan Khalid

United Arab Emirates University

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College of Science

Department of Mathematical Sciences

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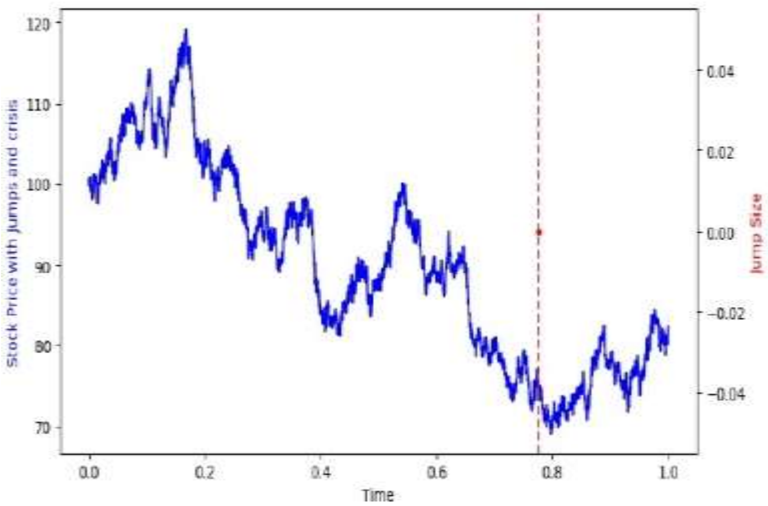

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Cover: Simulation image of Asian options with jumps and crisis

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Declaration of Original Work

I, Zeeshan Khalid, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Valuation of Asian Options in a High Volatility Market with Jumps"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Prof. Youssef El-Khatib, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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Abstract

The evaluation of financial derivatives represents a central part of financial risk management. There are many types of derivatives among other path-dependent options. In this study, we aim at valuing Asian options. They are path dependent and have several benefits. For instance, their values are habitually lower than European options. Also, an Asian option on a commodity drops the risk value close to maturity. Though, the disadvantage is that they are in general difficult to value since the distribution of the payoff is usually unknown. It is agreed in the literature that a stochastic process with a jumps model for the underlying asset provides a more precise value for the option price e.g. [1]. Moreover, the volatility is not constant, and it increases during a crisis see for instance the model of [2].

This work investigates the pricing of Asian options under a modified version of the pioneer Black Scholes model [3]. It aims at suggesting an alternate model that comprises jumps (as in [1] for instance) and increased volatility (see the model of [4]).

The study will propose to model the underlying asset with a new "hybrid" stochastic differential equation with jumps and high volatility. Then, under these settings, the valuation of Asian options will be investigated based on the works of [2,4]. Numerical techniques for finance will be used in this thesis to get a solution to the pricing problem. Several illustrations of the solution will be offered to demonstrate the efficiency of the used methods.

Keywords: Pricing, Asian Options, high Volatility, Jumps, Brownian motion.

Title and Abstract (in Arabic)

تقييم الخيارات الآسيوية في سوق عالية التقلب مع القفزات

الملخص

يمثل تقييم المشتقات المالية جزءًا أساسيًا من إدارة المخاطر المالية. هناك العديد من أنواع المشتقات من بين الخيارات الأخرى المعتمدة على المسار. في هذه الدراسة ، نهدف إلى تقييم الخيارات الآسيوية. فهي تعتمد على المسار ولها فوائد عديدة. على سبيل المثال ، قيمها عادة أقل من الخيارات الأوروبية. أيضًا ، ينخفض خيار آسيوي على سلعة ما قيمة المخاطرة بالقرب من الاستحقاق. رغم ذلك ، فإن العيب هو أنه يصعب تقييمها بشكل عام لأن توزيع المردود غير معروف عادة. من المتفق عليه في الأدبيات أن العملية العشوائية مع نموذج قفزات للأصل الأساسي توفر قيمة أكثر دقة لسعر الخيار [١]. علاوة على ذلك ، فإن التقلبات ليست ثابتة ، وتزداد أثناء الأزمة ، انظر على سبيل المثال نموذج [٢]. يبحث هذا العمل في تسعير الخيارات الآسيوية بموجب نسخة معدلة من نموذج ـلـﺌك صحـﮩلس الرائد [٣]. يهدف إلى اقتراح نموذج بديل يتضمن القفزات (كما في [١] على سبيل المثال) وزيادة التقلب (انظر نموذج [٤]). ستقترح الدراسة نمذجة الأصل الأساسي باستخدام معادلة تفاضلية عشوائية آهجينةْ جديدة مع قفزات وتقلبات عالية. بعد ذلك ، في ظل هذه الإعدادات ، سيتم التحقق من تقييم الخيارات الآسيوية بناءً على أعمال [٢ ، ٤]. سيتم استخدام التقنيات العددية للتمويل في هذه الأطروحة للحصول على حل لمشكلة التسعير. سيتم تقديم العديد من الرسوم التوضيحية للحل لإثبات كفاءة الطرق المستخدمة.

مفاهيم البحث الرئيسية : التسعير، الخيارات الآسيوية ، التقلبات العالية ، القفزات، الحركة البراونية

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I would like to thank my beloved family, beginning with my parents, who without their love and encouragement, I would have never been where I am today. Special thanks are extended to my friend Moath Bakour for his assistance and support.

Dedication

To my great parents and teachers

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List of Abbreviations

ADX	Abu Dhabi Security Exchange
DFM	Dubai Financial Market
PDE	Partial Differential Equation
SCA	Securities and Commodities Authority
SDE	Stochastic Differential Equation
VWAP	Volume Weighted Average Price

Chapter 1: Introduction

1.1 Financial Derivatives

A financial derivative is an agreement between two or more parties whose value is driven by an underlying asset. Derivatives give, in general, a right (or an obligation) to buy (or to sell) an underlying financial asset at a predetermined future time at an agreed price. The underlying asset includes currencies, commodities, stocks, indices, and even interest rates. Initially, these derivatives were designed for investors to reduce exchange trade risk, but lately, investors have been using them to explore more market opportunities. Derivatives are attracted by investors as they are exposed to price changes for different financial assets without actually owning the assets.

1.2 Asian Options

One of the most-known derivatives is the so-called option which gives the right but not the obligation to buy or sell a financial asset. There are several types of options such as Asian, American, or European options. The names do not have to do with the locations. Asian options are path-dependent options. They are popular since they allow the buyer to trade the underlying asset at an average price instead of the spot price. The average price of the underlying asset can either decide the option strike price (average-strike Asian options) or the underlying settlement price (average-price Asian options). Moreover, the average prices can be computed using the arithmetic or geometric mean. The type of option that will be examined throughout this thesis is the Asian option.

1.3 Thesis Objective

In this study, we aim at valuing Asian options. This problem has been widely investigated in the literature. Pricing of Asian options has been discussed for instance in these studies [5–9]. An essential question in the pricing of options is how to model the underlying asset price. It is agreed in the literature that a stochastic process with a jumps model for the underlying asset provides a more precise value for the option price. Moreover, the volatility is not constant and increases during a crisis. Thus combining high volatility and jumps in one hybrid model is interesting since it merges two important facts observed in the market in one model. This type of model has been already suggested in the work of [10] for European options. To our best knowledge, existing research works on pricing Asian options do not consider a hybrid model that combines jumps and high volatility.

This work investigates the pricing of Asian options in a high-volatility market with jumps, using a modified version of the pioneer Black Scholes model. Several models with jumps were investigated in the literature see for instance [1, 11, 12]. It aims at suggesting an alternate model that comprises jumps and increased volatility as in [4]. This thesis will propose to model the underlying asset using a stochastic differential equation with jumps and high volatility. The pricing problem will be discussed using stochastic calculus tools such as the Ito formula, moreover, numerical techniques for finance will be used in this thesis to get a solution to the pricing problem. Several illustrations of the solution will be offered to demonstrate the efficiency of the used methods. The Suggested model will be in the form of the equation

$$dS_t = rS_t dt + (\sigma S_t + \beta)(dB_t + dM_t), \quad t \in [0,T], \quad S_0 > 0, \tag{1.1}$$

where β is a positive constant representing the crisis impact in the volatility. Formerly the objective is to solve Equation (1.1), then calculate the price of Asian options given by the formula

$$C_0 = E\left[\left(\frac{1}{T}\int_o^T S_t dt - K, 0\right)^+\right] e^{-rT}. \tag{1.2}$$

Our objectives include the following

1. Suggest a new jump model for the underlying asset price with highly volatile situations.
2. Solve the SDE of the underlying asset price
3. Discuss the pricing problem for Asian options.
4. Comparing our results with the existing models in the literature

After this introduction chapter, the remaining structure of the thesis is as below.

Chapter 2: Financial products and derivatives: This chapter deals with financial markets and products and types of options contracts which include (American options, European Options, Path dependent option contracts, and Asian options). The references [13, 14] offer extensive presentations on financial derivatives and markets.

Chapter 3: Stochastic Tools: Some elementary tools from stochastic calculus helpful to our study are presented in Chapter 3. After defining stochastic processes, Brownian motion, and stochastic integration, we provide some details about their properties. Other concepts such as Stochastic Differential Equations (SDE), Poisson Process, the jump-diffusion process, and the Itô formula for the jump-diffusion process needed also in our thesis can be found in this chapter. For more details about stochastic calculus, the reader is referred to the books of [15–17].

Chapter 4: Pricing Asian options in a jump-diffusion model with high volatility: Main contribution of this thesis is given in Chapter 4 where our alternative model is introduced based on the model presented in [10] and the study of pricing Asian options under this model is discussed. In this chapter, the Itô formula is utilized to obtain the solution of the SDE of the underlying model. Moreover, the derivation of the PDE of the Asian option under our model using the same methodology of [9] is discussed. Numerical simulations are conducted on the underlying asset price trajectories. Several illustrations are given for the underlying asset prices.

Chapter 5: Conclusion: Some remarks and future directions are given to conclude.

Chapter 2: Financial Products and Derivatives

As the valuation of financial derivatives and their products are very important in stochastic finance, the purpose of this work is to model an SDE that can be used in the valuation of Asian options with high volatility and period jumps so that the price of an underlying asset can be investigated in high volatile situations while considering Asian options. For the desired SDE it is important to understand and explore the Asian option's price which can be done by approaching different theoretical stochastic tools and numerical methods. After the completion of this investigation, we shall have a new model which can be used to get the underlying asset price with jumps and highly volatile situations also one must be able to solve such SDE of the underlying asset and pricing problem of the Asian options.

2.1 Financial Markets and Products

The literature contains a huge number of books and references introducing financial markets and financial products as well as their ways of functionality. We can cite here the books of [13, 16]. In this thesis, we provide some information on financial markets and products related to our work. A type of marketplace that provides a channel for selling and purchasing assets is known as Financial Market. These assets include stocks, foreign exchange bonds, and derivatives. It can be classified as a place where investors and businesses meet up to expand.

In the United Arab Emirates, an important financial market is the Abu-Dhabi Securities Exchange-ADX located in Abu Dhabi. This market was founded on 15th November 2000. By December 2022 market cap of ADX was around 199 billion dollars. ADX is the second-largest market in the Arab region and deals with financial instruments which are approved by the UAE Securities and Commodities Authority (SCA). For more details about these markets we refer to the websites.

Another important financial market is the Dubai financial market-DFM located in Dubai and it was founded the same year as ADX but some months before, more precisely on 26 march 2000. By 2016 market cap of DFM was more than 89.18 billion dollars. Participants of DFM include listed companies (issuers), custodians, brokers, and investors, who play a very important role in market dynamics. Below are the link to the websites of the two markets.

2.1.1 Financial Assets

A financial asset is a type of asset that can be converted into cash in a very short period. The value of the financial asset depends upon the type of contract or ownership claim. Financial assets do not have any inherent physical worth even if they do not have a physical form like land or commodities but their value depends on a lot of supply and demand in the market place and it also depends upon the degree of risk associated with them.

Following are mentioned different types of financial assets

- Stock
- Bond
- Currency
- Commodity

2.1.1.1 Stock

Stock is defined as the financial claim which shows the comparable ownership of the investor or the holder towards the earnings and overall assets of the business for which the stocks are issued. Example: Etisalat

2.1.1.2 Bond

A bond is defined as a financial instrument that gives fixed interest payments to the holder. Corporations and government institutions use these bonds to raise funds which they can use in different projects. The holder of such instruments is called a creditor of debt. Example: Currency bonds

2.1.1.3 Currency

Currency is defined as an instrument for monetary exchange that has replaced the traditional barter system of exchanging goods whereas such a medium is accepted within a specific country. Different countries do have different currencies, but currently in the world one of the strongest currencies is United States Dollar.

2.1.1.4 Commodity

A commodity is defined as an instrument that is employed in commerce and business-related activities. These items act as input for general commerce and business production activities. Silver and gold are the most common commodities that are traded over the years in the commodities market.

2.1.2 Financial Derivatives

Derivatives are known as such financial instruments whose values are derived from underlying assets. Derivatives are termed financial contracts that are set between two or more two parties. The derivative can be traded in exchange also in the over-the-counter market. The price for these derivatives can be derived from the fluctuations that occur in the underlying asset. Usually, derivatives are high-proportioned instruments that increase their rewards and risk. These financial securities are often used to access certain markets and they also may be traded to hedge against risk. Derivatives are also used in certain cases to reduce risk factors (hedging) or to relate risk with the expectation of reward. Common financial derivatives include the following:

1. Futures
2. Forwards
3. Options
4. Swaps

2.1.2.1 Forward

A forward contract is a personalized contract between two parties, for which the settlement takes place on a predefined specific date in the future for a price that is settled today. Some important postulates of forward contracts are mentioned below.

- These are two-sided contracts and they are exposed to ask that one of the contractual parties may not fulfill transactions and default the contract.
- Forward contracts are custom design contracts. The price contracts so they are unique in terms of contract period or date of expiration or the quality or asset type.
- Forward contracts must be settled by delivering the assets on the set derivative date.

2.1.2.2 Futures

A future contract is a type of contract between two parties in such a way where both parties (buyer and the seller) agree on factors or sell the particular asset under consideration of a specific quantity and a pre-settled price at a date that is specified in the future. Some important postulates of future contracts are mentioned below:

- Future contract occurs only at a recognized stock exchange, where the exchange acts as a moderator and facilitates both parties.
- In future contracts in the beginning exchange needs both parties for a nominal account which that part of the contract and is known as the margin.

Definition 2.1.1. A swap is a type of derivative contract in which two parties exchange liabilities or cash flow from two different financial instruments.

Mostly cash flow involved in swaps is on a notional principal amount like a bond or loan, whereas instruments can be almost anything if it has a legal and financial value. In a swap contract, the principal amount usually does not change hands and stays with the original owner. Although one cash flow may be fixed but the other is variable and it depends upon floating currency exchange rate, index rate, and benchmark interest rate. Mostly at the start of the swap contract at least of these cash flows are set with a random or uncertain variable like equity price, foreign exchange rate, interest rate, or commodity price.

2.1.2.3 Options

An option contract is a type of contract that gives the right but not an obligation to sell or buy a financial asset. It is such a type of agreement between both parties that facilitates a potential transaction for an asset at a preset price and date. Using options is like a form of authority, so the buyer of the option contract pays an amount called a premium to buy the rights from the seller. These contracts can be traded in both exchange trad and OTC markets.

2.2 Types of Options Contract

Options contracts are termed call options or put options.Call option buyer has the rights, not an obligation to buy the financial asset at the strike price by the expiration date.A call option is more likely a leveraged bet which can be purchased on the appreciation of an asset when it is detected that the price of the asset will increase while Put option buyer has the rights, not an obligation to sell the financial asset at a strike price where aspiration date is more like benchmarked to profit if the price declines when it is expected that price of the asset can decrease.

2.2.1 American Option Contract

It is a type of options contract that gives the rights to the holder to exercise the contract at any time before or at the expiration date benchmark contract. American option contract style allows the investor to take profit anytime if the price goes up, so you do not have to wait for the maturity date of the contract. American options are usually exercised before an ex-dividend date which allows investors to own shares and also get the next dividend payment. Since the American option gives this right to the owner that he can exercise the contract anytime makes these contracts are very valuable, however for this right premium cost is high. Normally, the last day to exercise the weekly American option is Friday of the week in which the contract is expiring. Contrarily for the monthly American option, third Friday of the month is the last day to exercise the contract.

2.2.2 European Option Contract

A European option is a type of option that limits the execution of the contract only at the time of maturity, unlike American options this type of option does not allow the holder to execute the contract any time before or at the option expiration time. In simple words, if the underlying asset has moved up in price, an investor cannot exercise the contract earlier, instead, he has to wait for the time for maturity to execute the call or put option.

Even though American options contract can be exercised earlier but it comes with a very high premium as compared to European option. The premium price for the European options is not as high as for American options. Investors cannot exercise the European contract before the time of maturity but they can sell it back to the market before the expiry of the contract and they can receive the difference between the premiums earned and initial payments. Most of the indexes use European options because it reduces the load of accounting to be done by the brokerage so it can be said that investors sometimes do not have the choice between American or European options. The Black -Scholes model is mostly used to evaluate European Options.

The European index options stop trading when the business closes Thursday before the third Friday of the expiration date. This gap in trading gives brokers the capacity to price the assets of the underlying index.

2.2.2.1 Path Dependent Options

Exotic options are such a type of options contracts that are different than normal traditional options contracts with regards to their payment structures, strike prices, and expiration dates. A Path Dependent option is an exotic option whose value not only just depends upon the price of the underlying asset but also on the path that asset took during the life of the option.

All option contracts give this right to the holder to exercise its right before or at the maturity date of the option contract at a specific price known as a strike. Normally the trading price of the underlying asset is compared to the strike price to calculate profitability but in path-dependent options, the price used to determine profitability can vary. Profitability can be based on high or low prices or average prices. For example, there are two options in path-dependent options.

1. Path Dependent Option. The value of soft-dependent options is based on a single price event that occurred during the option life. This price can be the highest or lowest trading price of the underlying asset or it can be a triggering event like the underlying touching a specific price. This group of options includes Lock back options, chooser options, and barrier options.
2. Hard Path Depended Options. Hard-dependent options consider the entire trading history of the underlying asset. Some of these options take the average price, which is sampled at different time intervals. The type of options In this group is known as Asian options which are also called average options.

   Value of soft dependent options is based on a single price event that occurred during the option life. This price can be the highest or lowest trading price of the underlying asset or it can be a triggering event like the underlying touching a specific price. This group of options includes Lock back options, chooser options, and barrier options.

2.3 Asian Options

An Asian option is such a type of option in which the payoff value depends upon the average price of the underlying asset for a certain period which is against the standard options like American or European options in which the payoff depends upon the price of the underlying asset at a specific time known as maturity. Thus, the Asian option gives the buyer the to purchase or sell the underlying asset at an average price in place of the spot price.

The mean price of an asset that is observed over some specific period is known as the average price. It can be calculated simply by finding the simple arithmetic average of closing prices for a specific time. With the traded volume adjusted, (VWAP) volume weighted average price can be calculated on an intra-day basis.

As compared to other options Asian options have generally low volatility because of the average mechanism. They are preferred by investors who have underlying assets exposed to relatively more time, such as suppliers or consumers of commodities, etc.

Asian options are also a type of exotic options and are used in solving some particular business problems that can not be solved by ordinary options. These options are generated by tweaking other ordinary options in some minor ways. Asian options are also less expensive generally than their standard counterparts, as the volatility of the average price is less than the spot price volatility.

Asian options typically include the following:

- When a particular business is worried about the average exchange rate over time.
- When a single price at a point can bring manipulation.
- When a particular asset market is highly volatile.
- When due to thinly traded markets pricing becomes ineffective.
- This type of contract is very attractive as it cost low as compared to other options.

Chapter 3: Stochastic Tools

The introduction of the Brownian motion at the beginning of the 20th century can be seen as the start of Stochastic calculus. Numerous mathematical concepts and tools were established since then, and are utilized to investigate noisy systems. Stochastic processes are known to be a family of random variables indexed by time. They are applied to a variety of scientific areas such as Biology. Physics, Chemistry. But stochastic calculus is known more to be serving Finance and mainly pricing financial derivatives. In the following, we present some of the elementary concepts and essential tools from stochastic calculus that are needed for our study in this thesis. Nowadays, there are many books and references on stochastic calculus, we recommend the reader to read the books of [16–18] for comprehensive presentations on stochastic calculus for finance.

3.1 Stochastic Processes

In probability theory, a Stochastic process is defined as a process that involves the operation of chances. For example, in the process of radioactive decay, every atom breaks down with a fixed probability for any given time interval. In general, a stochastic process is referred to a family of random variables indexed against other variables or a set of variables. Stochastic processes are also termed as most general study objects in probability. Markov processes, Poisson process(radioactive decay), and times series with index variables are some common types of stochastic processes.

Definition 3.1.1. A probability space is a triple based on $(\Omega, \mathscr{F}, P)$ where Ω is the sample space, F is the sigma-algebra and P is the probability measure on $\mathscr{F}$.

The three elements of the probability space are mentioned below

1. Ω: It is the sample space of all the possible outcomes of the experiment.

2. $\mathscr{F}$: It is σ-algebra defined as the collection of all the subsets of Ω on which we are able to assign the possibilities. These are also known as events.

3. P: It is the probability measure of a function that associates the probability to each of the events belonging to the σ-algebra $\mathscr{F}$.

Definition 3.1.2. A filtration $(\mathscr{F}_t)_{t\geq 0}$ is defined as a non-decreasing family of the subsets of σ-algebra $\mathscr{F}$.

Let us consider an example of filtration if $(X_t)_{t\geq 0}$ is a stochastic process defined on $(\Omega, \mathscr{F}, \mathbb{P})$, then $\mathscr{F}_t = \sigma(X_s, s \leq t)$ is a filtration. This filtration is known as the natural filtration for the process X and it is noted as $(\mathscr{F}_t^X)_{t\geq 0}$.

Definition 3.1.3. A stochastic process $(X_t)_{t\geq 0}$ is known adapted to filtration $(\mathscr{F}_t)_{t\geq 0}$ if for every $t \geq 0$,there is a random variable X_t which is measurable with respect to $\mathscr{F}_t$.

Remark 3.1.1. A stochastic process is always redesigned with consideration to its natural filtration. It is observable that if the stochastic process $(X_t)_{t\geq 0}$ is being adapted to a filtration $(\mathscr{F}_t)_{t\geq 0}$ and also that if$\mathscr{F}_0$ do contains all the subsets of $\mathscr{F}$ which have a zero probability, then every process $(\tilde{X}_t)_{t\geq 0}$ that fulfills $\mathbb{P}(\tilde{X}_t = X_t) = 1, t \geq 0$, is adapted to filtration$(\mathscr{F}_t)_{t\geq 0}$.

To understand the dynamic aspects which are associated with filtration, we need to understand the notion of progressive measurability.

Definition 3.1.4. A stochastic process $(X_t)_{t\geq 0}$ that is being adapted to filtration $(\mathscr{F}_t)_{t\geq 0}$, is called progressively measurable with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if for every $t \geq 0$,

$$\forall A \in \mathscr{B}(\mathbb{R}), \{(s,\omega) \in [0,t] \times \Omega, X_s(\omega) \in A\} \in \mathscr{B}([0,t]) \otimes \mathscr{F}_t.$$

It is possible to construct an adapted measurable process with a diagonal method, but it cannot help in progressively measurable processes. However, the following mentioned proposition shows that a continuous and adapted stochastic process is progressively measurable.

Proposition 3.1.1. *If a continuous stochastic process $(X_t)_{t\geq 0}$ is adapted with respect to filtration $(\mathscr{F}_t)_{t\geq 0}$, then it is also progressively measurable with respect to it.*

3.2 Brownian Motion and Itô Formula

From now on we will be considering the terminal time as T. All the definitions are for the period $[0,T]$. Brownian Motion is one of the most important stochastic processes. Brownian

Motion serves as a model of the Gaussian process, Markov process, and continuous time martingales. Its definition is presented below:

Definition 3.2.1. The standard Brownian motion is a stochastic process $(B_t)_{t\in[0,T]}$ satisfying:

1. $B_0 = 0$.
2. The sample trajectories $t \longrightarrow B_t$ are continuous, with probability one.
3. For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments $B_{t_1} - B_{t_0}$, $B_{t_2} - B_{t_1}$,..., $B_{t_n} - B_{t_{n-1}}$ are mutually independent random variables.
4. For any given times $0 \leq s < t$, the increment $B_t - B_s$ has the Gaussian distribution with mean zero and variance $t - s$, we can write $B_t - B_s \sim N(0, t-s)$.

Notice that the Brownian motion is a continuous time stochastic process that has stationary and also independent Gaussian distributed increments and continuous paths. An Itô process $(X_t)_{t\in[0,T]}$ can have the following form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s, \qquad t \in [0,T]. \tag{3.1}$$

Remark 3.2.1. The construction of the stochastic integral $\int_0^t b_s dB_s$ and its definition can be found in many books on stochastic calculus such as in chapter. 4 of [16].

Theorem 3.2.1. ***Itô Formula for Itô Processes****. Let $(X_t)_{t\in[0,T]}$ be a process in of the form (3.1) and assume that $f \in \mathscr{C}_b^{1,2}([0,T]\times\mathbb{R})$ then the Itô formula is*

$$\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial f}{\partial s}(s,X_s)\,ds + \int_0^t a_s \frac{\partial f}{\partial x}(s,X_s)\,ds \\ &\quad + \int_0^t b_s \frac{\partial f}{\partial x}(s,X_s)\,dB_s + \frac{1}{2}\int_0^t |b_s|^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)\,ds. \end{aligned}$$

The previous theorem is very useful in solving stochastic differential equations, Notice that an Ordinary Differential Equation (ODE) of the form

$$dx(t) = f(t, x(t))dt$$

with the initial conditions $x(0) = x_0$ can be written as

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

where $x(t) = x(t, x_0, t_0)$ is the solution with initial condition $x(t_0) = x_0$. However, Stochastic Differential Equations (SDE) are usually written

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t, \quad t \in [0, T], \quad X_0 \text{ is given.} \tag{3.2}$$

Notice that, the process X_t does not depend only on the time t, but it is also depending on the $\omega \in \Omega$. It is common to write X_t instead of $X(t, \omega)$.

3.3 Poisson Process

The Poisson process is one of the most generally used processes with counting.This process is widely used in cases where event occurs at certain time but they are all random. It has many applications in various fields such as queuing theory, reliability engineering, telecommunications, and finance, among others. It can be defined as a stochastic process that models the coming of events over time where the number of events that happen in any time interval of fixed length is a random variable following a Poisson distribution.

It is mostly utilized in cases to model a random sequence of events that happen in continuous time, such as the arrival of customers at a store, the number of earthquakes that occur in a region over a certain period of time, or the number of photons detected by a sensor in a given time interval.The process is considered for random events. Let us consider an example from historical data, we understand that earthquakes occurred in a certain area at a rate of 3 per month. Furthermore, earthquake's timing is completely random,we do not have any regular information on earth quakes so we can conclude that Poisson Process can be a good model for earthquakes because it involved random. Below, we provide the definition of a Poisson process. Below, we provide the definition of a Poisson process.

Definition 3.3.1. Let $\lambda > 0$ be a positive constant. The counting process $(N_t)_{t\in[0,T]}$ is a Poisson process with rates λ if all the following statements are satisfied:

- $N_0 = 0$
- N_t has stationary and independent increments
- The number of events that occur in any time interval of length $h > 0$ follows a Poisson distribution with mean λh, i.e., for any $s < t$, the random variable $N_t - N_s$ has a Poisson distribution with mean $\lambda(t-s)$.

The above definition can be written as

$$\begin{aligned} N_0 &= 0 \\ N_t - N_s \quad \& \quad & N_u \text{ are independent for } \quad 0 \leq s < t \leq u \\ P(N_t - N_s = n) &= \frac{e^{-\lambda(t-s)}(\lambda(t-s))^n}{n!}, \quad n = 0, 1, 2, \ldots \end{aligned}$$

3.4 Jump Diffusion Process and Ito Formula for a Jump-Diffusion Process

A Jump diffusion is a stochastic process that involves continuous diffusion process and jumps. It is a generalization of the standard Brownian motion, which is a continuous-time stochastic process but with the added feature of allowing for sudden, discontinuous jumps in the process. More precisely, it combines a continuous diffusion process with a discrete jump process. that models the random motion of particles suspended in a fluid, If $(X_t)_{t\in[0,T]}$ is a jump-diffusion process then it can be written as the sum of a continuous diffusion process $(B_t)_{t\in[0,T]}$ and a pure-jump process $(J_t)_{t\in[0,T]}$, i.e.,

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s + \sum_{0\leq s\leq t} \Delta J_s,$$

$\mu(s, X_s)$ and $\sigma(s, X_s)$ are called the drift and volatility functions, respectively, ΔJ_s is the size of the jump that occurs at time s. In the case of a jump-diffusion process, the Itô formula provided in (3.2.1) takes the following form:

$$df(t, X_t) = \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\mu(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\sigma^2(t,x)\right] dt + \sum_{i=1}^{N_t} \Delta f(t_{i-1}, x_{i-1}, \Delta X_i)$$

where f is a function satisfying the regular conditions of continuity and differentiability, $\Delta X_i = X_{t_i} - X_{t_{i-1}}$ are the jump increments at the jump times t_i, and N_t is the total number of jumps up to time t. The term $\Delta f(x_{i-1}, t_{i-1}, \Delta X_i)$ represents the change in f due to the i-th jump.

Chapter 4: Pricing Asian Options in a Jump-Diffusion Model with High Volatility

4.1 Introduction

One of the fundamental problems in financial mathematics and risk management is the pricing of financial derivatives. A derivative product in finance represents a contract between buyer and seller. It must be honored according to an agreement which in general is to buy or sell a primary financial asset. The financial derivative's value is usually considered as a stochastic process evolving with time and denoted by $C = (C_t)_{t\in[0,T]}$. This value depends essentially on the underlying asset price denoted by $S = (S_t)_{t\in[0,T]}$. Studying a financial derivative means determining a solution to the pricing problem, which is to calculate the "fair" prices C.

Therefore, this problem consists primarily of finding a model for the underlying asset prices $S = (S_t)_{t\in[0,T]}$ and after calculating $C_t = f(t,S_t)$ for any $t \in [0,T]$. To evaluate an option, we need first to determine the price of the option at maturity, denoted by C_T, which is known according to the type of the contract. It is equal, in fact, to the so-called payoff. In the case of an arithmetic Asian option-type call, the payoff can be expressed as

$$C_T = \left(\frac{1}{T}\int_o^T S_t dt - K, 0\right)^+ \tag{4.1}$$

The goal is to find the value at the time of signature of the contract or the time of the agreement, in other words, the value of the option at $t = 0$. The option value at the initial time is called premium and denoted by C_0 which can be written as the expected value of the discounted payoff

$$C_0 = E\left[\left(\frac{1}{T}\int_o^T S_t dt - K, 0\right)^+\right] e^{-rT}. \tag{4.2}$$

It is clear from the previous equality that the pricing problem is depending basically on the model selected for $S = (S_t)_{t\in[0,T]}$. The literature contains an important amount of works on pricing financial derivatives such as Black and Scholes [3], Merton [1], and Heston [2] models. One of the most important parameters in stochastic modeling is the so-called volatility which predicts the extent to which an amount tends to vary. The volatility of a price is not visible from the market. In Black and Scholes, the underlying asset is driven by a geometric Brownian motion. Merton's model adds jumps to the process. The Heston model studies stochastic volatility. The model in [4] evaluates the option prices in a high-volatility model. Modeling financial assets with jumps permits us to capture the non-smooth and non-continuous comportment detected recurrently in the trajectory of financial assets. The continuous processes (such as the Black-Scholes model) cannot capture the incidence of big price movements in financial markets, which are frequently determined by unexpected news events, or fluctuations in market structure. By including jump processes in asset pricing models, we can improve the estimates of asset prices. A jump-diffusion model assumes that the trajectory of a financial asset price is driven by the below SDE:

$$dS_t = rS_t dt + \sigma S_t (dB_t + dM_t), \qquad t \in [0,T], \quad S_0 > 0.$$

S_0 is the spot price, r is the interest rate and σ is the volatility which is assumed constant. B_t is a standard Brownian motion and the pure jump process is denoted by M_t. The above model does not deal with crises. Several articles on modeling during crisis periods emerged in the literature e.g. [19] and [4]. On the other hand, many models with jumps are discussed in the literature such as the models of [12, 20]. The main objectives of this chapter can be summarized as follows:

1. Suggesting a jump model with high volatility for the underlying asset price

2. Solving the SDE of the underlying asset price

3. Deriving the PDE of the price of an Asian options

4. Numerical simulations to show the efficiency of the model.

The model we are treating in this thesis can be considered a hybrid model that combines jumps and crises. Yet, as far as we could tell, this is the first to attempt to price Asian options under such a model which is presented in the next section.

4.2 The Model

Let $(\Omega, \mathscr{F}, \mathscr{F}_{t\in[0.T]}, P)$ be a filtered probability space and consider a standard Brownian motion $(B_t)_{t\in[0.T]}$ and a Poisson process $(N_t)_{t\in[0.T]}$ with intensity λ. The compensated Poisson process $(M_t)_{t\in[0.T]}$ is defined as $M_t := N_t - \lambda t$. The two processes B and N are independent. As in [10] the stochastic differential equation of the underlying asset price is given by

$$dS_t = \mu S_t dt + (\sigma S_t + \beta) dB_t + b S_t dM_t, \quad t \in [0,T], \quad S_0 > 0, \tag{4.3}$$

where β is a positive constant representing the crisis impact in the volatility. The parameter μ represents the return of the underlying asset price and σ is its volatility when $\beta = 0$. Moreover, we assume the following condition $1 + b > 0$. We assume that there is an Asian option with continuous arithmetic call type built on the financial asset with prices governed by the equation (4.3). The strike is K and the maturity is K. The payoff is then as in (4.1) and the price of the option at time $t = 0$ expressed in (4.1) is to be determined. Formerly the objective is to solve Equation (4.3), then derive the PDE of the price of Asian options under the suggested model.

4.3 Solution of the Underlying Asset Price SDE

In this section, theoretical stochastic tools are employed to find a solution for the SDE (4.3). Our approach is based on the work of [4]. The next lemma is a version of the Itô formula for jump-diffusion processes. It can be found in [21].

Lemma 4.3.1. *Let $X = (X_t)_{t\in[0,T]}$ be the process defined by*

$$X_t = X_0 + \int_0^t f_u du + \int_0^t g_u dB_u + \int_0^t k_u dM_u$$

and assume that

$$\int_0^t |f_u| \, du < \infty, \quad \int_0^t |g_u|^2 \, du < \infty, \text{ and } \int_0^t \lambda \, |k_u| \, du < \infty$$

Then for any function $F \in \mathscr{C}^{1,2}([0,T]\times\mathbb{R})$

$$F(X_t,t) = F(X_0,0) + \int_0^t g_s \partial_x F(X_{s-},s)\,dB_s + \sum_{s\le t}\left(F(X_s,s) - F(X_s,s)\right)$$
$$+ \int_0^t \left(\partial_s F(X_s,s) + (f_s - k_s\lambda_s)\,\partial_x F(X_{s^-},s) + \frac{1}{2} g_s^2 \partial_{xx}^2 F(X_{s^-},s)\right) ds.$$

The previous formula can be expressed in the following way (see [22])

$$\begin{aligned} F(X_t,t) &= F(X_0,0) + \int_0^t g_s \partial_x F(X_{s^-},s)\,dB_s \\ &+ \int_0^t \left[F(X_{s-}+k_s,s) - F(X_{s^-},s)\right] dM_s \\ &+ \int_0^t \Big[\partial_s F(X_s,s) + (f_s - k_s\lambda_s)\,\partial_x F(X_{s^-},s) + \frac{1}{2} g_s^2 \partial_{xx}^2 F(X_{s^-},s) \\ &+ \lambda_s \left(F(X_{s-}+k_s,s) - F(X_{s^-}),s)\right] ds. \end{aligned} \tag{4.4}$$

In the following lemma we provide the solution of the SDE of our model where the coefficients of the model r, σ, a, b and λ are all constant.

Proposition 4.3.2. *The solution of the SDE (4.3) is given by*

$$S_t = \left(S_0 + \frac{\beta}{\sigma}\right)\xi_t - \frac{\beta}{\sigma}\left[1 + \left(\mu - \lambda\frac{b^2}{1+b}\right)\int_0^t \xi_t \xi_s^{-1} ds - \frac{b}{1+b}\int_0^t \xi_t \xi_s^{-1} dM_s\right] \tag{4.5}$$

where

$$\xi_t = exp\left[\left(\mu - \lambda b - \frac{\sigma^2}{2}\right)t + \sigma B_t\right](1+\lambda b)^{N_t}. \tag{4.6}$$

Proof. We generalize the proof in [4] from the Brownian motion case to jump-diffusion processes. Notice that the process $(\xi_t)_{0\leqslant t\leqslant T}$ given by (4.6) corresponds to the solution of the SDE resulting from the underlying asset price if $\beta = 0$. It satisfies the SDE

$$d\xi_t = \mu\xi_t dt + \sigma\xi_t dB_t + b\xi_t dM_t, \quad t\in[0,T], \quad \xi = 1 > 0, \tag{4.7}$$

To see this, we can obtain (4.6) by applying Itô formula to $F(\xi_t, t) = \ln \xi_t$ using the SDE of (4.7). Assuming now the solution of Equation (4.3) can be written as the product of two processes

$$S_t = Y_t \xi_t, \quad t \in [0, T], \tag{4.8}$$

where $Y_0 = S_0$. Then it is sufficient to find Y_t to obtain the solution of S_t. But we have $Y_t = S_t \xi_t^{-1}$ and thus

$$dY_t = d(S_t \xi_t^{-1}) = S_t d\xi_t^{-1} + \xi_t^{-1} dS_t + d[S_t, \xi_t^{-1}]. \tag{4.9}$$

But using Itô formula on ξ_t with the function $F(\xi_t, t) := \frac{1}{\xi_t}$ gives

$$\begin{aligned} d\xi_t^{-1} &= \sigma \xi_t (-\xi_t^{-2}) dB_t + \left(\frac{1}{\xi_t + b\xi_t} - \frac{1}{\xi_t} \right) dM_t \\ &\quad + \left[(\mu \xi_t - b\xi_t \lambda)(-\xi_t^{-2}) + \frac{1}{2} \sigma^2 \xi_t^2 (2\xi_t^{-3}) + \lambda \left(\frac{1}{\xi_t + b\xi_t} - \frac{1}{\xi_t} \right) \right] dt. \end{aligned}$$

The previous equation can be simplified to

$$d\xi^{-1}{}_t = \left(-\mu + \sigma^2 + \frac{\lambda b^2}{1+b} \right) \xi_t^{-1} dt - \sigma \xi^{-1}{}_t dB_t + \frac{-b}{1+b} \xi_t^{-1} dM_t \tag{4.10}$$

Incorporating Equations (4.3-4.10) into (4.9) leads to

$$\begin{aligned} dY_t &= S_t \left(\left(-\mu + \sigma^2 + \frac{\lambda b^2}{1+b} \right) \xi_t^{-1} dt - \sigma \xi_t^{-1} dB_t + \frac{-b}{1+b} \xi_t^{-1} dM_t \right) \\ &\quad + \xi_t^{-1} (\mu S_t dt + (\sigma S_t + \beta) dB_t + b S_t dM_t) + d[S_t, \xi_t^{-1}]. \end{aligned}$$

But

$$
\begin{aligned}
d[S_t, \xi_t^{-1}] &= d\left[\mu S_t dt + (\sigma S_t + \beta) dB_t + bS_t dM_t, \left(-\mu + \sigma^2 + \frac{\lambda b^2}{1+b}\right)\right. \\
&\quad \left. \xi_t^{-1} dt - \sigma \xi_t^{-1} dB_t + \frac{-b}{1+b} \xi_t^{-1} dM_t\right] \\
&= (\sigma S_t + \beta)(-\sigma \xi_t^{-1}) dt + bS_t \frac{-b}{1+b} \xi_t^{-1} dN_t,
\end{aligned}
$$

where we have used $dM_t = dN_t - \lambda dt$ and

$$
[dB_t, dB_t] = dt, \;\; [dN_t, dN_t] = dN_t, \quad \text{and} \quad [dt, dt] = [dB_t, dt] = [dt, dB_t] = 0.
$$

Then use the two precedent equations to get

$$
\begin{aligned}
dY_t &= \left(-\beta \sigma \xi_t^{-1} + S_t \xi_t^{-1} \frac{\lambda b^2}{1+b}\right) dt + \beta \xi_t^{-1} dB_t - S_t \xi_t^{-1} \frac{b^2}{1+b} dN_t \\
&\quad + \frac{-b}{1+b} \xi_t^{-1} S_t dM_t + \xi_t^{-1} bS_t dM_t.
\end{aligned}
$$

But

$$
\begin{aligned}
& -S_t \xi_t^{-1} \frac{b^2}{1+b} dN_t + \frac{-b}{1+b} \xi_t^{-1} S_t dM_t + \xi_t^{-1} bS_t dM_t = \\
& -S_t \xi_t^{-1} \frac{b^2}{1+b} dN_t + \frac{-b}{1+b} \xi_t^{-1} S_t (dN_t - \lambda dt) + \xi_t^{-1} bS_t (dN_t - \lambda dt) \\
= \; & -S_t \xi_t^{-1} \left[\frac{b^2 + b}{1+b} - b\right] dN_t + S_t \xi_t^{-1} \left[-\lambda \frac{-b}{1+b} - \lambda b\right] dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
dY_t &= (-\beta \sigma \xi_t^{-1}) dt + \beta \xi_t^{-1} dB_t \\
&= -\frac{\beta}{\sigma} \left[\sigma^2 \xi_t^{-1} dt - \sigma \xi_t^{-1} dB_t\right] \\
&= -\frac{\beta}{\sigma} \left[d(\xi_t^{-1}) + \left(\mu - \lambda \frac{b^2}{1+b}\right) \xi_t^{-1} dt - \frac{b}{1+b} \xi_t^{-1} dM_t\right]
\end{aligned}
$$

And

$$
\begin{aligned}
Y_t &= S_0 - \frac{\beta}{\sigma}\left[\xi_t^{-1} - \xi_0^{-1} + \left(\mu - \lambda\frac{b^2}{1+b}\right)\int_0^t \xi_s^{-1} ds \right. \\
&\quad \left. - \frac{b}{1+b}\int_0^t \xi_s^{-1} dM_s \right]. \qquad (4.11)
\end{aligned}
$$

Substituting (4.11) and (4.6) in (4.8) to get the solution of S_t as

$$
\begin{aligned}
S_t = \; & Y_t\xi_t = \left\{\left(S_0 + \frac{\beta}{\sigma}\right) - \frac{\beta}{\sigma}\left[\xi_t^{-1} + \left(\mu - \lambda\frac{b^2}{1+b}\right)\int_0^t \xi_s^{-1} ds \right.\right. \\
& \left.\left. - \frac{b}{1+b}\int_0^t \xi_s^{-1} dM_s\right]\right\}\xi_t \\
= \; & \left(S_0 + \frac{\beta}{\sigma}\right)\xi_t - \frac{\beta}{\sigma}\left[1 + \left(\mu - \lambda\frac{b^2}{1+b}\right)\int_0^t \xi_t\xi_s^{-1} ds \right. \\
& \left. - \frac{b}{1+b}\int_0^t \xi_t\xi_s^{-1} dM_s\right].
\end{aligned}
$$

This is the solution of S_t as expressed in (4.5). This ends the proof.

4.4 Pricing Jump-Diffusion Model During Crisis

This section discusses the PDE of the Asian option price with payoff given by Equation (1.2) under the combination crisis and jumps. The SDE of our model as in (4.3) is

$$dS_t = \mu S_t dt + (\sigma S_t + \beta)dB_t + bS_t dM_t$$

where we assume that the crisis impact is the parameter β and the jump part represented by the compounded Poisson process $(M_t)_{0 \leq t \leq T}$.

Using the methodologies of the derivation of the European options' PDE in the existing literature see for example [6, 9, 23], we obtain in the following proposition the PDE that characterizes the Asian option price of our crisis with jumps model.

Suppose that the price process for the Asian option is given by $C(t, S_t, Y_t)$. Here $Y_t = \int_0^t S_\tau d\tau$ and $C(t,x,y)$ is a function. Letting $S_t = x$ and $Y_t = y$, the Equation (4.3) becomes:

$$dx = \mu x dt + (\sigma x + \beta)dB_t + bx dM_t.$$

The Itô formula applied to $C(t,x,y)$ gives

$$
\begin{aligned}
dC(t,x,y) &= (\sigma x+\beta)C_x dB_t + [C(t,x(1+b),y) - C(t,x,y)]dM_t \\
&\quad + \Big(C_t + xC_y + (\mu - b\lambda)xC_x + \frac{1}{2}(\sigma x+\beta)^2 C_{xx} \\
&\quad + \lambda[C(t,x(1+b),y) - C(t,x,y)]\Big)\,dt. \qquad (4.12)
\end{aligned}
$$

If we assume that we invest the value of the option in a portfolio consisting of the underlying asset price S_t and a risky free asset with continuous interest rate r. Moreover, we impose the self-financing condition which means that no withdrawal from the portfolio and no external money is incorporated into it. Let $\Delta(t)$ denote the number of shares invested in S_t and $V(t,x,y)$ is the value of the portfolio, then $V(t,x,y) - \Delta(t)x$ is the number of shares invested in the risky free asset. Thus

$$
dV(t,x,y) = \Delta(t)dx + r(V(t,x,y) - \Delta(t)x)dt, \qquad (4.13)
$$

this is because the portfolio is self-financing.Thus, we have

$$
\begin{aligned}
dV(t,x,y) &= \Delta(t)(\mu x dt + (\sigma x+\beta)dB_t + bx dM_t) + r(V(t,x,y) - \Delta(t)x)dt \\
&= [\mu x\Delta(t) + r(V(t,x,y) - \Delta(t)x)]dt + (\sigma x+\beta)\Delta(t)dB_t \\
&\quad + \Delta(t)bx dM_t \qquad (4.14)
\end{aligned}
$$

The market is incomplete and we aim at minimizing the difference between the value of the portfolio V expressed in the Equation (4.14) and the price of the option C as given by Equation (4.12). We have then

$$
\begin{aligned}
E[(C-V)^2] &= E\Big[\Big(\int_0^T L^C - L^V dt + \int_0^T (\sigma x+\beta)(C_x - \Delta(t))dB_t \\
&\quad + \int_0^T C(t,x(1+b),y) - C(t,x,y) - \Delta(t)bx dM_t\Big)^2\Big],
\end{aligned}
$$

where L^C and L^V are the terms in dt in Equations (4.12) and (4.14) respectively. Assume $\Delta(t) = C_x$, then a suggested price for the Asian option of our model can be then taken as a solution of:

$$
\begin{aligned}
&C_t + xC_y + (\mu - b\lambda)xC_x + \frac{1}{2}(\sigma x + \beta)^2 C_{xx} \\
&+\lambda[C(t,x(1+b),y) - C(t,x,y)] = \mu x\Delta(t) + r(C(t,x,y) - \Delta(t)x)
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
&C_t(t,x,y) + xC_y(t,x,y) + (r - b\lambda)xC_x(t,x,y) + \frac{1}{2}(\sigma x + \beta)^2 C_{xx}(t,x,y) \\
&+\lambda[C(t,x(1+b),y) - C(t,x,y)] - rC(t,x,y) = 0.
\end{aligned}
$$

The above equation can be reduced to the PDE of the Asian option in the Geometric Brownian motion model case when $b = \beta = 0$.

4.5 Numerical Simulations

This section provides numerical simulations for the underlying asset price process S_T given by Equation (4.5). We use Euler Scheme to discretize the SDE of the model given by the Equation (4.3). First of all, the number Consider an integer $N > 0$, and discretise the time interval $[0,T]$ into steps $t_j = j\Delta t, j = 0,1,\ldots,N$ of identical size $\Delta t = \frac{T}{N}$. We start by simulating a trajectory $\left(W_{t_j}\right)_{t_j \in [0,T]}$ of the Brownian motion and a trajectory of the Poisson process $\left(N_{t_j}\right)_{t_j \in [0,T]}$ and then we use Equation (4.3) to find an approximation of S_{t_j}.Simulations are performed for the process. The details of the simulations of the Brownian motion and for the Poisson process are described in the following subsections.

4.5.1 Simulation of the Brownian Motion and the Poisson Process

We simulate $\left(W_{t_j}\right)_{t_j \in [0,T]}$ noting the fact that the Brownian motion fulfills:

$$
\begin{aligned}
&W_{t_0} = 0, \\
&W_{t_{j+1}} = t_{j+1} + \sqrt{\Delta t} Z_{t_j}, \quad j = 1,\ldots,N,
\end{aligned}
$$

where Z_{t_j} follows a normal distribution $N(0,1)$. Regarding the Poisson Process $\left(N_{t_j}\right)_{t_j \in [0,T]}$ with intensity λ. The jump times can be noted as $\left(T_{N_{t_j}}\right)_{j=0,\ldots,N}$. We are using the following properties of the Poisson process:

$$T_{N_{t_0}} = 0,$$
$$T_{N_{j+1}} = T_{N_{t_j}} + \text{ExpLaw}(\lambda), \quad j = 1, \ldots, N,$$

where ExpLaw is an exponential random variable. A Poisson process path $N_{t_j}, j = 0, \ldots, N$ is then determined by using:

$$N_{t_0} = 1,$$
$$N_{t_j} = \sum_{k=0}^{j} 1_{T_k < t_j}, \quad j = 1, 2, \ldots, N.$$

4.5.2 Illustrations

Below, we provide some figures of the underlying asset price at maturity. (Figures 4.1 - 4.7) show the simulations under varying conditions.

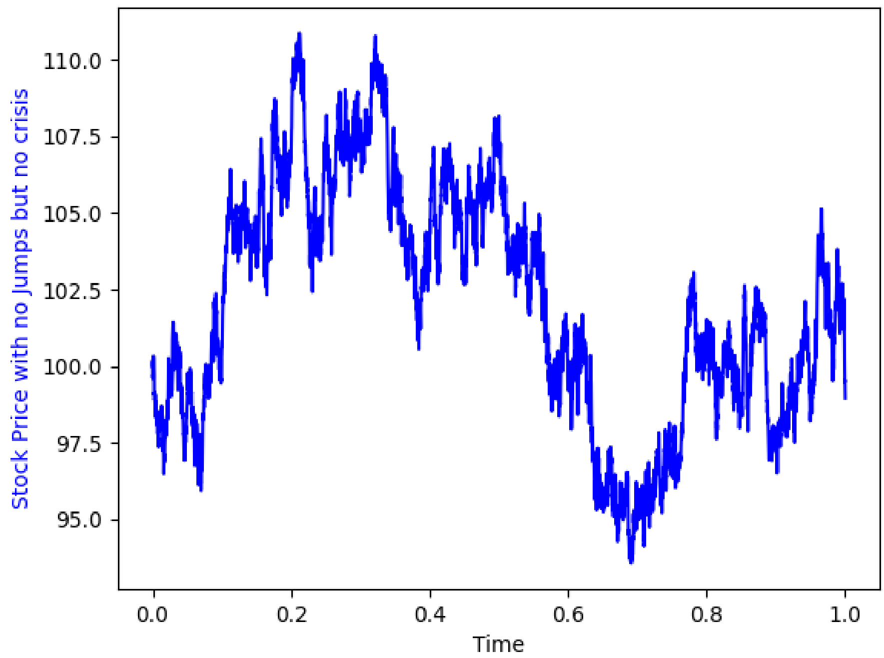

Figure 4.1: First run of the simulation where b=0 and $\beta = 0$. Spot price $S_0 = 100$, number of discretization $N = 5040$.

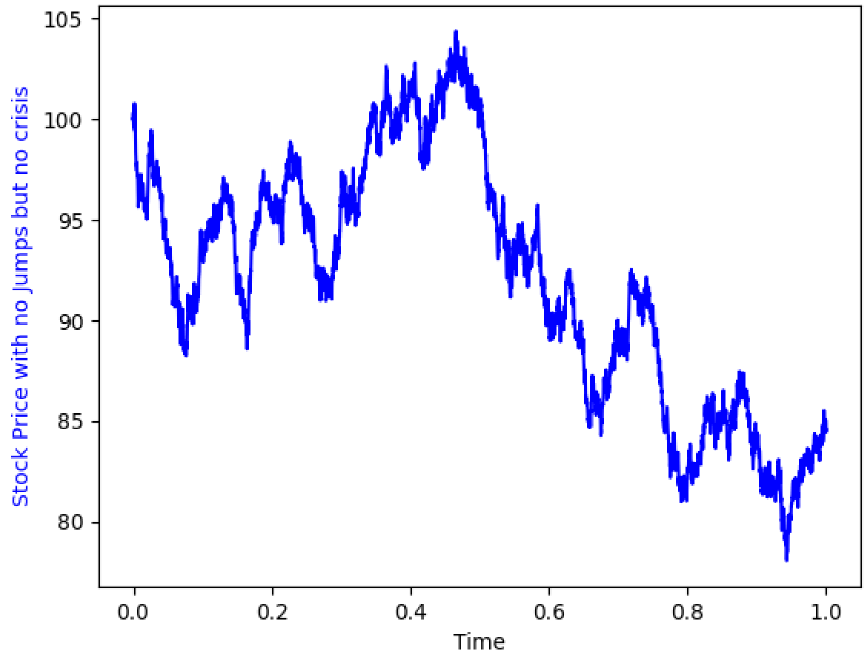

Figure 4.2: Second run of the simulation where b=0 and $\beta = 0$. Spot price $S_0 = 100$, number of discretization $N = 5040$

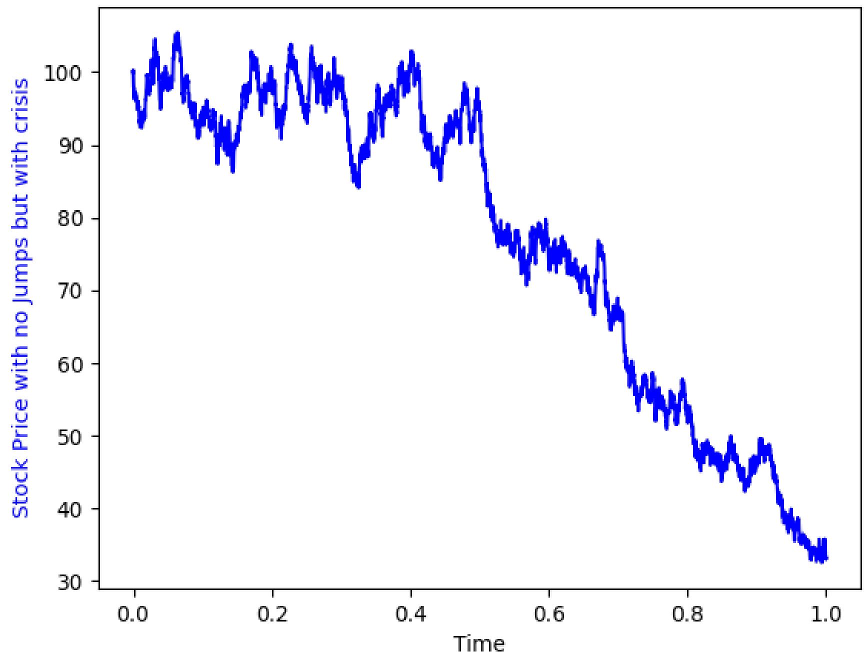

Figure 4.3: The trajectory of the asset price without jumps and with the crisis.

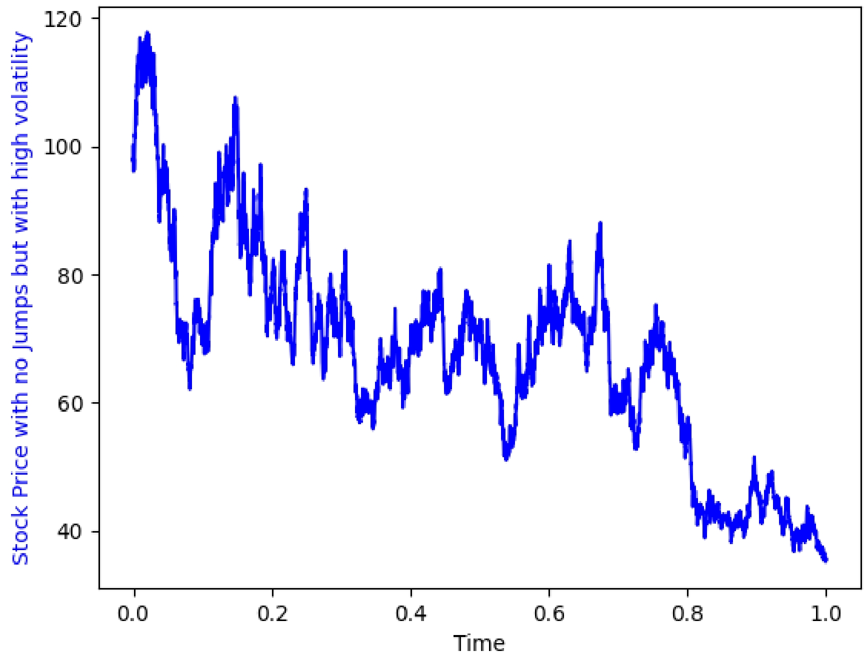

Figure 4.4: A second run of the simulation. Spot price $S_0 = 100$, number of discretization $N = 5040$, and $\beta = 2$.

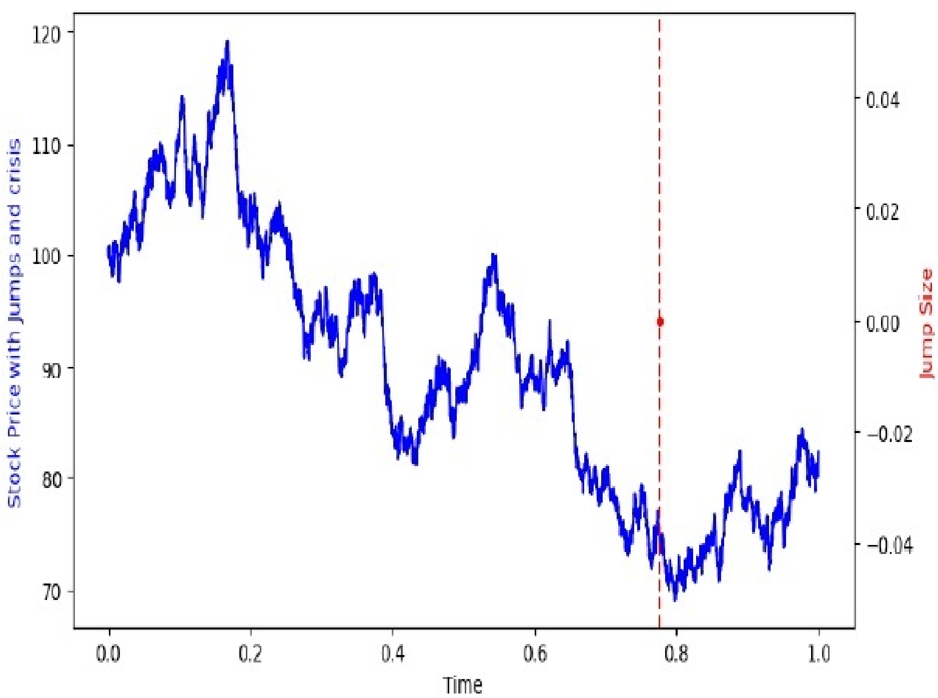

Figure 4.5: A run of the simulation. Spot price $S_0 = 100$, number of discretization $N = 5040$, and $\beta = 2$.

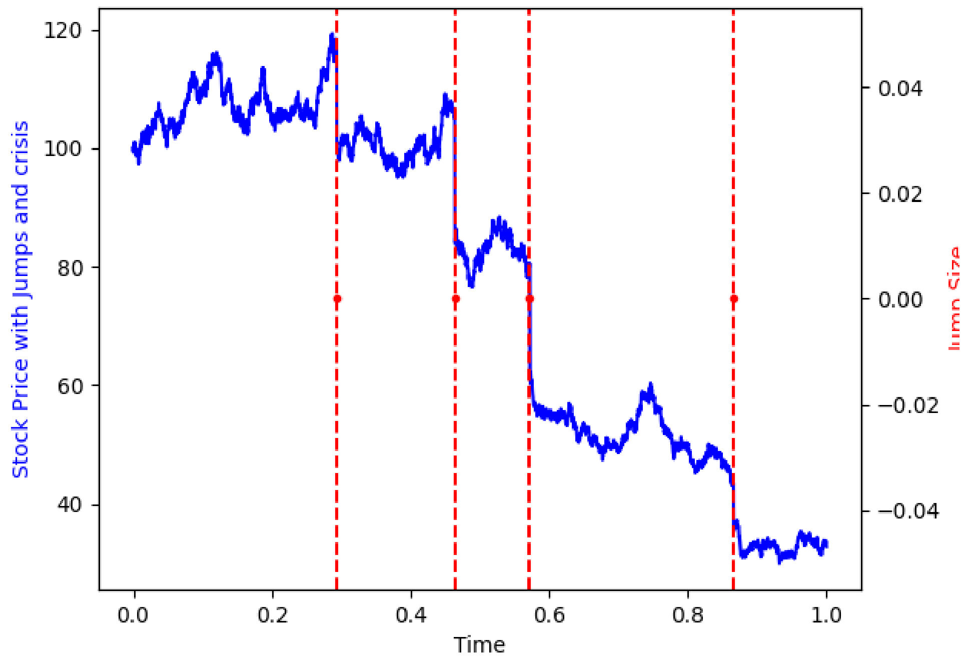

Figure 4.6: First run of the simulation. Spot price $S_0 = 100$, number of discretization $N = 5040$, and $\lambda = 3$.

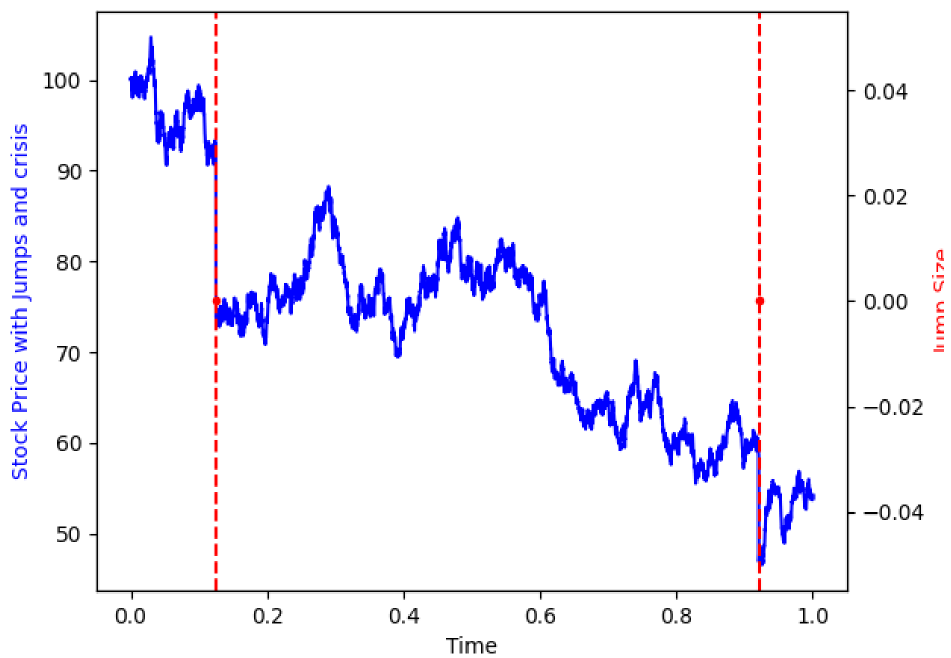

Figure 4.7: Run of simulation with spot price $S_0 = 100$, Number if discretization $N = 5040$, $\lambda = 3$.

Chapter 5: Conclusion

Financial derivatives are very popular products. They are traded broadly. "Derivative notional amounts increased in the first quarter of 2022 by 22.9 trillion dollars, or 12.9 percent, to 200.4 trillion dollars" according to the "Quarterly Report on Bank Trading and Derivatives Activities" of [24].

They are used mainly in reducing the risk arising from the fluctuations in the prices of a given financial asset. The valuation of financial derivatives products is then an important problem in risk management and can be seen as one of the most important problems in stochastic finance in general.

In this thesis, the pricing of Asian options is explored under a modified version of the pioneer Black Scholes model. The study investigated the solution of the SDE of asset prices. Moreover, it discusses the PDE of the Asian options price under a high volatility model with jumps. The model incorporates two important properties that can be observed in markets: jumps and high volatility.

Numerical simulations and figures are provided and are favorable for the suggested model. As a future direction of research, it would be interesting to investigate a numerical solution for the partial differential equation for the model.

References

[1] R. C. Merton, "On the pricing of corporate debt: The risk structure of interest rates," *The Journal of finance*, vol. 29, no. 2, pp. 449–470, 1974.

[2] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The review of financial studies*, vol. 6, no. 2, pp. 327–343, 1993.

[3] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of political economy*, vol. 81, no. 3, pp. 637–654, 1973.

[4] Y. El-Khatib and A. Hatemi-J, "Option valuation and hedging in markets with a crunch," *Journal of Economic Studies*, vol. 44, no. 5, pp. 801–815, 2017.

[5] H. Geman and M. Yor, "Bessel processes, asian options, and perpetuities," *Mathematical finance*, vol. 3, no. 4, pp. 349–375, 1993.

[6] J. Vecer, "A new pde approach for pricing arithmetic average asian options," *Journal of computational finance*, vol. 4, no. 4, pp. 105–113, 2001.

[7] M. Curran, "Valuing asian and portfolio options by conditioning on the geometric mean price," *Management science*, vol. 40, no. 12, pp. 1705–1711, 1994.

[8] R. Zvan, P. A. Forsyth, and K. R. Vetzal, "Robust numerical methods for pde models of asian options," Ph.D. dissertation, University of Waterloo, 1996.

[9] L. C. G. Rogers and Z. Shi, "The value of an asian option," *Journal of Applied Probability*, vol. 32, no. 4, pp. 1077–1088, 1995.

[10] Y. El-Khatib, M. A. Hajji, and M. Al-Refai, "Options pricing in jump diffusion markets during financial crisis," *Applied Mathematics & Information Sciences*, vol. 7, no. 6, p. 2319, 2013.

[11] R. Cont, P. Tankov, and E. Voltchkova, "Hedging with options in models with jumps," pp. 197–217, 2007.

[12] D. S. Bates, "Jumps and stochastic volatility: Exchange rate processes implicit in deutsche mark options," *The Review of Financial Studies*, vol. 9, no. 1, pp. 69–107, 1996.

[13] A. Martin, B. Candelas, Á. Rodríguez-Rozas, J. D. Martín-Guerrero, X. Chen, L. Lamata, R. Orús, E. Solano, and M. Sanz, "Toward pricing financial derivatives with an ibm quantum computer," *Physical Review Research*, vol. 3, no. 1, p. 013167, 2021.

[14] P. Wilmott, *Paul Wilmott introduces quantitative finance*. John Wiley & Sons, 2013.

[15] T. Rolski, H. Schmidli, V. Schmidt, and J. L. Teugels, *Stochastic processes for insurance and finance*. John Wiley & Sons, 2009.

[16] N. Privault, *Stochastic finance: An introduction with market examples*. CRC Press, 2013.

[17] B. Oksendal, *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[18] B. K. Øksendal and A. Sulem, *Applied stochastic control of jump diffusions*. Springer, 2007, vol. 498.

[19] G. Dibeh and H. M. Harmanani, "Option pricing during post-crash relaxation times," *Physica A: Statistical Mechanics and its Applications*, vol. 380, pp. 357–365, 2007.

[20] R. C. Merton, "Option pricing when underlying stock returns are discontinuous," *Journal of financial economics*, vol. 3, no. 1-2, pp. 125–144, 1976.

[21] D. J. Higham and P. E. Kloeden, "Numerical methods for nonlinear stochastic differential equations with jumps," *Numerische Mathematik*, vol. 101, no. 1, pp. 101–119, 2005.

[22] Y. El-Khatib and Q. M. Al-Mdallal, "Numerical simulations for the pricing of options in jump diffusion markets," *Arab Journal of Mathematical Sciences*, vol. 18, no. 2, pp. 199–208, 2012.

[23] N. Cai and S. Kou, "Pricing asian options under a general jump diffusion model," *Operations Research*, vol. 1241, no. 10.1287, 2010.

[24] "Quarterly report on bank trading and derivatives activities," Dec 2018. [Online]. Available: https://www.occ.gov/publications-and-resources/publications/quarterly-report-on-bank-trading-and-derivatives-activities/index-quarterly-report-on-bank-trading-and-derivatives-activities.html

Appendix

Project Simulation Code

```
# This is a sample Python script.

# Press R to execute it or replace it
with your code.
# Press Double  to search everywhere
for classes, files,
# tool windows, actions, and settings.

# See PyCharm help at
# https://www.jetbrains.com/help/pycharm/

import numpy as np
import matplotlib.pyplot as plt

# Define the parameters
S0 = 100    # Initial stock price
r = 0.05    # Risk-free interest rate
sigma = 0.2    # Volatility
mu = r - sigma**2/2    # Drift
lamda = 3    # Jump intensity
muJ = -0.2    # Mean jump size
sigmaJ = 0.3    # Jump size volatility
```

```
T = 1    # Time horizon
N = 5040    # Number of time steps

bb=0.2
betaa=10.7
# Define the time step
dt = T/N

# Set the random seed for reproducibility
np.random.seed(1)

# Simulate the stock price
t = np.linspace(0, T, N+1)
S = np.zeros(N+1)
S[0] = S0
Jtimes = []  # List to store the jump times
for i in range(N):
    dW = np.sqrt(dt) * np.random.normal()
    dN = np.random.poisson(lamda*dt)
    if dN > 0:
        Jtimes.append(t[i])
    J = np.random.normal(muJ, sigmaJ, size=dN)
    S[i+1] = S[i] + mu*S[i]*dt
             + (sigma*S[i]+betaa)*dW
             + bb*S[i]*J.sum()
```

```
# Plot the stock price and jump times
fig, ax1 = plt.subplots()
ax1.plot(t, S, 'b-')
ax1.set_xlabel('Time')
ax1.set_ylabel('Stock Price
       with Jumps and crisis', color='b')

# Add vertical lines for jump times
for tjump in Jtimes:
    ax1.axvline(x=tjump,
    color='r', linestyle='--')

# Add a second y-axis for jump sizes
ax2 = ax1.twinx()
ax2.plot(Jtimes, np.zeros_like(Jtimes), 'r.')
ax2.set_ylabel('Jump Size', color='r')

plt.show()
```

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The main goal of this thesis is to merge high volatility and jumps in one model. This thesis suggests a jump model with high volatility for the underlying asset price. The suggested SDE for the underlying asset is solved, the PDE of the price of an Asian option under the suggested model is discussed and numerical simulations are presented to show the efficiency of the model.

Zeeshan Khalid received his Master of Science in Mathematics from the Department of Mathematics, College of Science and his Bachelor of Science in Electrical Engineering from the College of Engineering, University of Lahore, Pakistan.