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On Eulerian subgraphs and hamiltonian line graphs

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ABSTRACT

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A graph G is Hamilton-connected if for any pair of distinct vertices $u, v \in V(G)$, G has a spanning (u, v)-path; G is 1-hamiltonian if for any vertex subset $S \subseteq V(G)$ with $|S| \leq 1$, $G - S$ has a spanning cycle. Let $\delta(G)$, $\alpha'(G)$ and $L(G)$ denote the minimum degree, the matching number and the line graph of a graph G, respectively. The following result is obtained. Let G be a simple graph with $|E(G)| \geq 3$. If $\delta(G) \geq \alpha'(G)$, then each of the following holds.
(i) $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.
(ii) $L(G)$ is 1-hamiltonian if and only if $\kappa(L(G)) \geq 3$.

For a graph G, an integer $s \geq 0$ and distinct vertices $u, v \in V(G)$, an $(s; u, v)$-path-system of G is a subgraph H consisting of s internally disjoint (u, v)-paths. The spanning connectivity $\kappa^*(G)$ is the largest integer s such that for any k with $0 \leq k \leq s$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(k; u, v)$-path-system. It is known that $\kappa^*(G) \leq \kappa(G)$, and determining if $\kappa^*(G) > 0$ is an NP-complete problem. A graph G is maximally spanning connected if $\kappa^*(G) = \kappa(G)$. Let $msc(G)$ and $s_k(G)$ be the smallest integers m and m' such that $L^m(G)$ is maximally spanning connected and $\kappa^*(L^{m'}(G)) \geq k$, respectively. We show that every locally-connected line graph with connectivity at least 3 is maximally spanning connected, and that the spanning connectivity of a locally-connected line graph can be polynomially determined. As applications, we also determined best possible upper bounds for $msc(G)$ and $s_k(G)$, and characterized the extremal graphs reaching the upper bounds.

For integers $s \geq 0$ and $t \geq 0$, a graph G is (s, t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. Pulleyblank in [J. Graph Theory, 3 (1979) 309-310] showed that determining whether a graph is $(0, 0)$-supereulerian, even when restricted to planar graphs, is NP-complete. Settling an open problem of Bauer, Catlin in [J. Graph Theory, 12 (1988) 29-45] showed that every simple graph G on n vertices with $\delta(G) \geq \frac{n}{5} - 1$, when n is sufficiently large, is $(0, 0)$-supereulerian or is contractible to $K_{2,3}$. We prove the following for any nonnegative integers s and t.
(i) For any real numbers a and b with $0 < a < 1$, there exists a family of finitely many graphs $\mathcal{F}(a, b; s, t)$ such that if G is a simple graph on n vertices with $\kappa'(G) \geq t + 2$ and $\delta(G) \geq an + b$, then either G is (s, t)-supereulerian, or G is contractible to a member in $\mathcal{F}(a, b; s, t)$.
(ii) Let ℓK_2 denote the connected loopless graph with two vertices and ℓ parallel edges. If G is a simple graph on n vertices with $\kappa'(G) \geq t+2$ and $\delta(G) \geq \frac{n}{2} - 1$,

then when n is sufficiently large, either G is (s,t)-supereulerian, or for some integer j with $t+2 \leq j \leq s+t$, G is contractible to a jK_2.

For a hamiltonian property $\mathcal{P}$, Clark and Wormold introduced the problem of investigating the value $\mathcal{P}(a,b) = \max\{\min\{n : L^n(G) \text{ has property } \mathcal{P}\}$: $\kappa'(G) \geq a$ and $\delta(G) \geq b\}$, and proposed a few problems to determine $\mathcal{P}(a,b)$ with $b \geq a \geq 4$ when $\mathcal{P}$ is being hamiltonian, edge-hamiltonian and hamiltonian-connected. Zhan in 1986 proved that the line graph of a 4-edge-connected graph is Hamilton-connected, which implies a solution to the unsettled cases of above-mentioned problem. We consider an extended version of the problem. Let $ess'(G)$ denote the essential edge-connectivity of a graph G, and define $\mathcal{P}'(a,b) = \max\{\min\{n : L^n(G) \text{ has property } \mathcal{P}\}$: $ess'(G) \geq a$ and $\delta(G) \geq b\}$. We investigate the values of $\mathcal{P}'(a,b)$ when $\mathcal{P}$ is one of these hamiltonian properties. In particular, we show that for any values of $b \geq 1$, $\mathcal{P}'(4,b) \leq 2$ and $\mathcal{P}'(4,b) = 1$ if and only if Thomassen's conjecture that every 4-connected line graph is hamiltonian is valid.

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DEDICATION

To

everyone who wonders if I'm writing about them.

I am.

Contents

Chapter 1

Introduction

1.1 Notations and Terminology

This research project focusses on hamiltonian line graph problems and supereulerian problems. All graphs considered in this dissertation are undirected, finite and loopless. As in [87], for a graph G, let $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$ denote the stability number (also called the independence number), matching number, connectivity and edge connectivity of G, respectively.

Let G_1 and G_2 be two graphs. The **intersection** of G_1 and G_2, denote by $G_1 \cap G_2$, has the vertex set $V(G_1 \cap G_2)$ and edge set $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$; and the **union** of G_1 and G_2, denoted by $G_1 \cup G_2$, has the vertex set $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. If $G_2 \cong K_2$ with $E(G_2) = \{e\}$, then we simply write as $G_1 \cup e$.

The **line graph** $L(G)$ of a graph G is a simple graph obtained by taking $E(G)$ to be the vertex set $V(L(G))$, and any two vertex of $V(L(G))$ are adjacent if and only if they are adjacent in G as edges.

Hassler Whitney (1932) [148] proved that with one exceptional case the structure of a connected graph G can be recovered completely from its line graph. Many other properties of line graphs follow by translating the properties of the underlying graph from vertices into edges, and by Whitney's theorem the same translation can also be done in the other direction. Line graphs have been recognized as a class of graphs with many interesting properties. For example, line graphs are $_{1,3}$-free [148], and the line graphs of bipartite graphs are perfect [149].

Walk is a sequence of vertices and edges, where the edges connect the adjacent vertices in the sequence. **Tour** is a walk with no repeated edges. **Path** is a walk with no repeated vertices.

A graph is **hamiltonian** if it contains a spanning cycle, which is a cycle that passes through every vertex exactly once. In other words, a Hamiltonian graph is a graph in which it is possible to traverse every vertex in the graph

exactly once and return to the starting vertex by following the edges of the graph.

The hamiltonian cycle is named after the mathematician William Rowan Hamilton, who studied such cycles in the 19th century. In graph theory, the problem of determining whether a given graph contains a Hamiltonian cycle is known as the Hamiltonian cycle problem. This problem is NP-complete[150], meaning that it is computationally difficult to determine whether a given graph has a Hamiltonian cycle and it is believed that there is no efficient algorithm for solving this problem in general. However, there are efficient algorithms for finding Hamiltonian cycles in certain special classes of graphs.

Thomassen posed one of the leading conjectures on the hamiltonian line graph problem that every 4-connected line graph is hamiltonian. There are several known result listed below.

Theorem 1.1.1 *Let G be a graph.*
(i) (Zhan, Theorem 3 in [84]) If $\kappa(L(G)) \geq 7$, then $L(G)$ is hamiltonian-connected.
(ii) (Kaiser and Vrána [68]) If $\kappa(L(G)) \geq 5$ and $\delta(L(G)) \geq 6$, then $L(G)$ is hamiltonian.

We call a closed trail in a graph an Euler tour if it is traverses every edge of the graph exactly once. A graph is **Eulerian** if it admits an euler tour. In 1736, Euler proved a connected graph is Eulerian if and only if every vertex has even degree. This is known as Euler's Theorem.

The Chinese postman problem is a closely related problem, which seeks the shortest closed walk in a connected graph such that each edge traversed at least once. If the graph is eulerian, then the eulerian closed trail is an optimal solution to the Chinese postman problem. If not, then the optimization problem is to the minimum number of edges to be duplicate to result in an Eulerian graph". As a dual problem, Boesch, Suffey and Tindel [7] proposed the supereulerian problem, which seeks to determine whether a graph contains a spanning eulerian subgraphs. If a graph G contains a spanning Eulerian subgarph, then it is called **supereulerian**.

The supereulerian graph problem is also motivated by the study of Hamiltonian problems of graphs. A graph G is Hamiltonian if G has a spanning cycle. For integers $a, b > 0$, an $[a, b]$-factor F of G is a spanning subgraph of G such that for any $v \in V(F)$, $a \leq d_F(v) \leq b$. Thus a graph G is Hamiltonian if and only if G has a connected $[2, 2]$-factor; and is supereulerian if and only if G has a connected even $[2, \Delta(G)]$-factor. For a non-Hamiltonian graph G, and an even number k with $2 \leq k \leq \Delta(G)$, if G has a connected even $[2, k]$-factor, then the smaller k is, the closer G is to being Hamiltonian.

In particular, if G has an Eulerian subgraph H such that $E(G-V(H)) = \emptyset$, we say graph G is **dominating** by H. Harary and Nash-Williams find the

connection supereulerian graph and hamiltonian graph with line graph.

Theorem 1.1.2 (Harary and Nash-Williams, [125]) *Let G be a connected graph with at least 3 edges. Then $L(G)$ is hamiltonian if and only if G has an Eulerian subgraph H such that $E(G - V(H)) = \emptyset$.*

To achieve Thomassen's conjecture, by Harary and Nash-Williams Theorem, we need to study supereulerian graphs.

1.2 Background on Hamilton-Connected Line Graphs

This research is motivated by the following well-known theorem of Chvátal and Erdős on hamiltonian graphs.

Theorem 1.2.1 *(Chvátal and Erdős [92]) Let G be a simple graph with at least three vertices.*
(i) If $\kappa(G) \geq \alpha(G)$, then G has a Hamilton cycle.
(ii) If $\kappa(G) \geq \alpha(G) - 1$, then G has a Hamilton path.
(iii) If $\kappa(G) \geq \alpha(G) + 1$, then G is Hamilton-connected.

As shown in the survey of Saito in [112], there have been many extensions and variations of Theorem 1.2.1.

There are quite a few investigations using similar conditions involving edge connectivity, stability number or matching number to study supereulerian graphs, as seen in [93, 103, 104, 114, 117], among others.

Another motivation of this research comes from Thomassen's conjecture [143] that every 4-connected line graph is hamiltonian. A number most fascinating conjectures in this area are presented below. By an ingenious argument of Ryjáček [137], Conjecture 1(i) below is equivalent to a seeming stronger conjecture of Conjecture 1(ii). In [138], it is shown that all conjectures stated in Conjecture 1 below are equivalent to each other.

Conjecture 1 *(i) (Thomassen [143]) Every 4-connected line graph is hamiltonian.*
(ii) (Matthews and Sumner [135]) Every 4-connected claw-free graph is hamiltonian.
(iii) (Kužel and Xiong [128]) Every 4-connected line graph is Hamilton-connected.
(iv) (Ryjáček and Vrána [138]) Every 4-connected claw-free graph is Hamilton-connected.

Many researches have been conducted towards these conjectures, as can be found in the surveys in [88, 121, 122], among others. The best result by far is obtained by Kaiser, Ryjáček and Vrána in [97]. Algefari et al. (Corollary 1.1

of [85]) proved that every connected simple graph G with $|E(G)| \geq 3$ and with $\delta(G) \geq \alpha'(G)$ has a hamiltonian line graph. For an integer $s \geq 0$, a graph G is **s-hamiltonian** if for any vertex subset $X \subseteq V(G)$ with $|X| \leq s$, $G - X$ has a Hamilton cycle. The current research is to investigate similar relationship between the minimum degree and the matching number of a graph that would warrant Hamilton-connected line graphs and 1-hamiltonian line graphs. As Hamilton-connected graphs and 1-hamiltonian graphs must be 3-connected, it is natural to conduct the investigation within 3-connected line graphs.

1.3 Background on spanning connectivity of graphs

In [137], Ryjáček uses an ingenious argument to show that Conjecture 1(i) below is equivalent to a seeming stronger statement in Conjecture 1(ii). Later, Ryjáček and Vrána in [138] indicated that all the statements in Conjecture 1 are mutually equivalent.

There has been an effort to associate the study of the hamiltonicity and the connectivity of a graph. For any integer $s > 0$ and for $u, v \in V(G)$ with $u \neq v$, an $(s; u, v)$-path-system of G is a subgraph H consisting of s internally disjoint (u, v)-paths, and we say H is a **spanning $(s; u, v)$-path-system** if $V(H) = V(G)$. The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer k such that for any integer s with $0 \leq s \leq k$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(s; u, v)$-path-system. A graph G is **spanning k-connected** if for any $u, v \in V(G)$ with $u \neq v$, $\kappa^*(G) \geq k$. By define spanning connectivity. Thomassen conjecture is equivalent to that every 4 connected line graph is 2-spanning connected.

Suppose G is a graph. We define the maximal number of disjoint union of trees contained in G as packing number, denote as $\tau(G)$.

Theorem 1.3.1 (Catlin and Lai [90]) *Let G be a graph with $\tau(G) \geq 2$. Then $\kappa^*(L(G)) \geq 2$ if and only if $\kappa(L(G)) \geq 3$.*

Theorem 1.3.2 (Zhan) *If $\kappa'(G) \geq 4$, then $\kappa^*(L(G)) \geq 2$.*

Theorem 1.3.3 (Huang and Hsu[67]) *For any integer $k \geq 2$, if $\kappa'(G) \geq 2k \geq 4$, then $\kappa^*(L(G)) \geq k$.*

Chen et al in [**?**] improved Theorem 1.3.3 by using the core G_0 of a graph G to characterize all spanning 3-connected line graph $L(G)$ when $\tau(G_0) \geq 2$.

For an integer $m > 0$, define $L^0(G) = G$, and the **iterated line graph** $L^m(G) = L(L^{m-1}(G))$. A path P of G is a **divalent path** of G if every internal vertex of P has degree 2 in G. Following [129], we define a proper divalent path to be one that is not of length 2 and in a K_3, and

$$\ell(G) = \max\{m : G \text{ has a length } m \text{ proper divalent path} \qquad (1.1)$$

Define $\mathcal{G}$ to be the family of all connected nontrivial graphs that are not isomorphic to a path, a cycle or a $K_{1,3}$. To study iterated line graphs, we only consider graphs in $\mathcal{G}$. The iterated line graph index problem is also an intensively studied topic in graph theory. By the definition of line graphs, the iterated line graphs of a path will eventually becoming a K_1; the iterated line graphs of a cycle remains as unchanged. Therefore, in discussing iterated line graph problems, it is common to exclude paths, cycles and the graph $K_{1,3}$, whose line graph is a 3-cycle. Chartrand and Wall in [11] initiated the study the smallest integer $k \geq 0$, called the hamiltonian index of a graph G, such that the iterated line graph $L^k(G)$ becomes hamiltonian. we have the following definition.

Definition 1 *([130]) Let $\mathcal{P}$ denote a graphical property and G be a connected graph $G \in \mathcal{G}$. Then $\mathcal{P}(G)$, the* **$\mathcal{P}$-index** *of G, is defined by*

$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \text{ has property } \mathcal{P}\} \\ \qquad\qquad \text{if for some integer } j > 0,\ L^j(G) \text{ has property } \mathcal{P}, \\ \infty \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Clark and Wormald in [20] studied the existence of the indices for the properties of being edge-hamiltonian, pancyclic, vertex-pancyclic, edge-pancyclic, hamiltonian-connected, respectively. Additional studies of these indices can also be found in [130]. In [139], Ryjáček, Woeginger and Xiong indicated that determining the value of $h(G)$ is a difficult problem.

For an integer $k \geq 2$, and a graph $G \in \mathcal{G}$, let $s_k(G)$ be the smallest integer m such that $\kappa^*(L^m(G)) \geq k$. When k is small, upper bounds for $s_k(G)$ have been investigated.

Theorem 1.3.4 *Let $G \in \mathcal{G}$ be a connected graph with maximum degree $\Delta(G)$.*
(i) (Chen et al. Theorem 22 of [10]) $s_2(G) \leq |V(G)| - \Delta(G) + 1$.
(ii) (Xiong et al. Theorem 1.3 of [144]) $s_3(G) \leq \ell(G) + 6$.

As every Hamilton-connected graph must also be hamiltonian, we conclude that a graph G is Hamilton-connected if and only if $\kappa^*(G) > 0$. Thus determining if $\kappa^*(G) > 0$ in general is an NP-complete problem. One of the motivation of this research is to seek nontrivial common families of graphs in which spanning connectivity can be polynomially determined.

As it is known that the connectivity of a graph can be polynomially determined, for example, [119, 123]), the problem whether high connectivity could imply positive spanning connectivity was considered. While the complete bipartite graphs indicate that in general, high connectivity of a graph G does not warrant $\kappa^*(G) > 0$, researchers have been investigating graph families in which high connectivity of a graph G in these family would imply that $\kappa^*(G) > 0$.

Thomassen in [143] first conjectured that every 4-connected line graph is hamiltonian. This most fascinating conjecture has attracted many researchers.

Thomassen's above-mentioned conjecture is shown to be equivalent to each of the following.

Conjecture 2 *Let G be a graph and let Γ be a claw-free graph.*
(i) (Thomassen [143] and, Kučzel and Xiong [128]) Every 4-connected line graph has spanning connectivity at least 2.
(ii) (Matthews and Sumner [135], and Ryjáček and Vrána [138]) Every 4-connected claw-free graph has spanning connectivity at least 2.

As of today, little is known on maximally spanning connected graph families other than the complete graphs and a few others. This motivates the current study. For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The vertex v is locally connected if the induced subgraph $G[N_G(v)]$ is connected. A graph G is **locally connected** if very vertex v of G is locally connected. Asratian [1] and Y. Sheng, F. Tian and B. Wei [141] studied connectivity conditions for a locally connected claw-free graph G to have spanning connectivity at least 2. As line graphs are claw-free, their result is also valid for line graphs. A class of maximally spanning connected line graphs has also been studied in [127] and [18].

Theorem 1.3.5 *Let G be a connected graph.*
(i) (Asratian [1] and Y. Sheng, F. Tian and B. Wei [141]) If G is an locally connected claw-free graph with $\kappa(G) \geq 3$, then $\kappa^*(G) \geq 2$.
(ii) (Huang and Hsu [127], and Chen et al. [18]) Let $k \geq 3$ be an integer. If a graph G has k-edge-disjoint spanning trees, then $L(G)$ is maximally spanning connected.

Theorem 3.2.2(i) below, one of our main results, has identified a new family of graphs whose line graphs are maximally spanning connected, which extends Theorem 3.2.1(i). As connectivity of a graph can be polynomially determined.

In this research, we consider some indices related to spanning connectivity of graphs. For an integer $k \geq 2$, and a graph $G \in \mathcal{G}$, let $s_k(G)$ be the smallest integer m such that $\kappa^*(L^m(G)) \geq k$. When k is small, upper bounds for $s_k(G)$ have been investigated.

1.4 Background on (s,t)-supereulerian graphs

The supereulerian problem is introduced by Boesch, Suffel, and Tindell in [7], which seeks to characterize graphs with spanning closed trails. Catlin [9] proved the following theorems and start to use a reduction method to find spanning Eulerian subgraphs.

Lei et al. in [74] introduced (s,t)-supereulerian graphs, as a generalization of supereulerian graphs.

Definition 2 *For integers $s \geq 0$ and $t \geq 0$, a graph G is (s,t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y.*

By Definition 2, it is known that supereulerian graphs are precisely $(0,0)$-supereulerian graphs, and Catlin's reduction method is still a useful tool as (s,t)-supereulerian graphs. The notion of (s,t)-supereulerian was formally introduced in [73, 74], as a generalization of supereulerian graphs. For integers $s \geq 0$ and $t \geq 0$, a graph G is (s,t)-supereulerian if for any disjoint edge sets $X, Y \subseteq E(G)$ with $|X| \leq s$ and $|Y| \leq t$, G has a spanning closed trail that contains X and avoids Y. Thus supereulerian graphs are precisely $(0,0)$-supereulerian graphs. The (s,t)-supereulerian graph problem aims to determine graphs that are (s,t)-supereulerian. In particular, it is of interests to extend former results in (0,0)-supereulerian to (s,t)-supereulerian, for generic values of s and t. A number of research results on the (s,t)-supereulerian problem and similar topics have been obtained, as seen in [22, 23, 26, 73, 74, 30, 35], among others. Settling an open problem of Bauer posed in [2, 3], Catlin [9] proved the following theorem.

Theorem 1.4.1 *(Catlin, Theorem 9 of [9]) Let G be a simple graph on n vertices with $\kappa'(G) \geq 2$. If $\delta(G) \geq \frac{n}{5} - 1$, then when n is sufficiently large, G is $(0,0)$-supereulerian, or G can be contracted to a $K_{2,3}$.*

It is natural to consider whether Theorem 3.2.1 can be extended to (s,t)-supereulerian graphs for all possible values of s and t. By definition, if a graph G is (s,t)-supereulerian, then $\kappa'(G) \geq t+2$.

Theorem 1.4.2 (Lei et al.[73]) *Let $s \leq 2$ and $t \geq 0$ be two integers. G is a $(t+2)$-edge-connected locally connected graph on n vertices. Then exact one of the following holds: (i) G is $(2,t)$-supereulerian. (ii) The reduction of $(G-Y)_x$ is member of $\{K_1, K_2, K_2, t(t \geq 1)\}$*

Theorem 1.4.3 *(*(Lei et al.[74]) *Let $k \geq 1$ be an integer. If G is a connected, locally k- edge connected graph, then G is (s,t)-supereulerian for all pairs of nonnegative integers s and t with $s+t \leq k-1$.*

Theorem 1.4.4 (Lei et al.[74]) *Let G be a connected, locally k-edge-connected graph. Let s and t be nonnegative integers such that $s+k \leq k$. (i) If $t < k$ and $k \geq 3$, then G is (s,t)- supereulerian. (ii) If $\kappa'(G) \geq k+2$ and $k \geq 3$, then G is (s,t)-supereulerian.*

We define a relation "$\sim$" on $E(G)$ such that $e_1 \sim e_2$ if $e_1 = e_2$, or if e_1 and e_2 form a cycle in G. It is routine to check that $\sim$ is an equivalence relation and edges in the same equivalence class are parallel edges with the same end vertices. We use $[uv]$ to denote the set of all edges between u and v in a graph, and shorten $|[uv]|$ to $|uv|$. For a graph G, $\mu(G) = \max\{|uv| : uv \in E(G)\}$ is the **multiplicity** of G. Let ℓK_2 denote the connected loopless graph with two vertices and ℓ parallel edges. Thus for each edge $e \in E(G)$, the edges parallel to e in G induces a subgraph isomorphic to $|e|K_2$.

1.5 Background on Index Problem

For a hamiltonian property $\mathcal{P}$, Clark and Wormold introduced the problem of investigating the value $\mathcal{P}(a, b) = \max\{\min\{n : L^n(G) \text{ has property } \mathcal{P}\}$: $\kappa'(G) \geq a$ and $\delta(G) \geq b\}$, and proposed a few problems to determine $\mathcal{P}(a, b)$ with $b \geq a \geq 4$ when $\mathcal{P}$ is being hamiltonian, edge-hamiltonian and hamiltonian-connected. Zhan in 1986 proved that the line graph of a 4-edge-connected graph is Hamilton-connected, which implies a solution to the unsettled cases of above-mentioned problem. We consider an extended version of the problem. Let $ess'(G)$ denote the essential edge-connectivity of a graph G, and define $\mathcal{P}'(a, b) = \max\{\min\{n : L^n(G) \text{ has property } \mathcal{P}\}$: $ess'(G) \geq a$ and $\delta(G) \geq b\}$. We investigate the values of $\mathcal{P}'(a, b)$ when $\mathcal{P}$ is one of these hamiltonian properties. In particular, we show that for any values of $b \geq 1$, $\mathcal{P}'(4, b) \leq 2$ and $\mathcal{P}'(4, b) = 1$ if and only if Thomassen's conjecture that every 4-connected line graph is hamiltonian is valid.

For discussional convenience, we in this paper denote $\mathcal{G}$ to be the family of all connected nontrivial graphs that are not isomorphic to a path, a cycle or a $K_{1,3}$. To study iterated line graphs, we only consider graphs in $\mathcal{G}$. The iterated line graph index problem is an intensively studied topic in graph theory. Chartrand and Wall in [11] initiated the study of the smallest integer $k \geq 0$, called the hamiltonian index of a graph G, such that the iterated line graph $L^k(G)$ becomes hamiltonian. Other hamiltonian like indices were defined and studied by Clark and Wormald in [20].

Clark and Wormald in [20] initiated the study of the indices for the properties of being hamiltonian, being edge-hamiltonian and being Hamilton-connected, together with several other hamiltonian properties. They proved the existences of the indices of the properties listed above. Additional studies of these indices can also be found in [130], which showed that the above-mentioned hamiltonian-like properties are closed under taking iterated line graphs. In [139], Ryjáček, Woeginger and Xiong indicated that determining the value of the hamiltonian index is a difficult problem. The index problem for graphical properties has

been intensively studied, as seen in [11, 19, 10, 20, 120, 129, 130, 132, 56, 57, 140, 139, 144, 146], among others. Define

$$\begin{aligned}\mathcal{H} &= \{G \in \mathcal{G} : G \text{ is hamiltonian}\},\\ \mathcal{E}_h &= \{G \in \mathcal{G} : G \text{ is edge-hamiltonian}\},\\ \mathcal{H}_c &= \{G \in \mathcal{G} : G \text{ is Hamilton-connected}\}.\end{aligned}$$

For a hamiltonian property $\mathcal{P}$ and integers $a > 0$ and $b > 0$, Clark and Wormald in [20] define

$$\mathcal{P}(a,b) = \begin{cases} \max\{\min\{n : L^n(G) \in \mathcal{P}\} : G \in \mathcal{G} \text{with } \kappa'(G) \geq a,\ \delta(G) \geq b\}, \\ \qquad\qquad \text{if such max exists,} \\ \infty \\ \qquad\qquad \text{otherwise.} \end{cases} \tag{1.2}$$

and investigate the values of $\mathcal{P}(a,b)$ when $\mathcal{P}$ represents the properties of being hamiltonian, edge-hamiltoning, pancyclic, edge-pancyclic, vertex-pancyclic, Hamilton-connected and pan-connected, among others. Clark and Wormald in [20] showed that for all the above mentioned properties $\mathcal{P}$,

$$\mathcal{P}(1,1) = \mathcal{P}(1,2) = \mathcal{P}(2,2) = \infty. \tag{1.3}$$

Clark and Wormald in [20] also proved that for other cases with $b \geq a \geq 3$, $1 \leq \mathcal{P}(a,b) \leq 3$ except when $b \geq a \geq 4$ and $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$. The paper [20] ends with the following question: if $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, what is the value of $\mathcal{P}(a,b)$ when $b \geq a \geq 4$?

Zhan in [63] is the first addressing this question. He proved in [63] that the line graph of every 4-edge-connected graph is in $\mathcal{H}_c$. This result implies that if $b \geq a \geq 4$, then $\mathcal{H}(a,b) = \mathcal{H}_c(a,b) = 1$. For an Hamilton-connected graph G and an arbitrary edge $e = uv \in E(G)$, as G has a spanning (u,v)-path P, $E(P) \cup \{e\}$ induces a Hamilton cycle that contains e. Therefore by definition, we have

$$\mathcal{H}_c \subseteq \mathcal{E}_h \subseteq \mathcal{H}, \text{ We have, for any positive integers a and b,}$$
$$\mathcal{H}_c(a,b) \geq \mathcal{E}_h(a,b) \geq \mathcal{H}(a,b).$$

Hence Zhan's result gives rise to a complete answer to the question raised in [20], as follows.

Theorem 1.5.1 *(Zhan [63]) If $b \geq a \geq 4$, then $\mathcal{H}(a,b) = \mathcal{H}_c(a,b) = \mathcal{E}_h(a,b) = 1$.*

We consider an extension of the problem. Let $U, W \subseteq V(G)$ be vertex subsets. Define

$$(U,W)_G = \{uw \in E(G) : u \in U \text{ and } w \in W\}.$$

When $Y = V(G) - X$, then we define $\partial_G(U) = (U, V(G) - U)_G$.

For a hamiltonian property $\mathcal{P}$ and positive integers a, b, define

$$\mathcal{P}'(a,b) = \begin{cases} \max\{\min\{n : L^n(G) \in \mathcal{P}\} : G \in \mathcal{G} \text{ with } ess'(G) \geq a,\ \delta(G) \geq b\}, \\ \qquad\qquad \text{if such max exists,} \\ \infty \\ \qquad\qquad \text{otherwise.} \end{cases} \tag{1.4}$$

By (1.2) and (1.4) and as $ess'(G) \geq \kappa'(G)$, it is known that $\mathcal{P}'(a,b) \geq \mathcal{P}(a,b)$ for any property $\mathcal{P}$. By definition, if a graph G satisfies both $\delta(G) \geq k$ and $ess'(G) \geq k$, then G does not have an edge cut whose size is less than $ess'(G)$, and so we must have $ess'(G) = \kappa'(G)$ in this case. Thus

$$\text{for all } b \geq a \geq 1,\ \mathcal{P}'(a,b) = \mathcal{P}(a,b). \tag{1.5}$$

As $\delta(G) \geq \kappa'(G)$ for any graph G, we observe that when $a > b$, $\mathcal{P}(a,b)$ does not exist. However, it is meaningful to discuss $\mathcal{P}'(a,b)$ even when $a > b$. Unlike the behavior of $\mathcal{P}(a,b)$, the study of $\mathcal{P}'(a,b)$ is related to the following fascinating conjecture of Thomassen 1(i)

In the research of this part, we shall investigate the values of $\mathcal{P}'(a,b)$ when $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$. As (1.5) has suggested some relationship between $\mathcal{P}'(a,b)$ and $\mathcal{P}(a,b)$ when $b \geq a \geq 1$, we reformulate the results in [20] together with Theorem 3.2.3 as follows.

Theorem 1.5.2 *(Clark and Wormald [20], Zhan [63]) For $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, we have the following.*

$$P(a,b) = \begin{cases} \infty & \text{if } 1 \leq a \leq b \leq 2, \\ 3 & \text{if } \mathcal{P} \in \{\mathcal{E}_h, \mathcal{H}_c\} \text{ with } a = 1 \text{ and } b = 3, \\ 2 & \text{if } \mathcal{P} = \mathcal{H} \text{ and both } a = 1 \text{ and } b = 3, \\ 2 & \text{if } 2 \leq a \leq b \leq 3, \text{ or if } 1 \leq a \leq 3 < 4 \leq b, \\ 1 & \text{if } b \geq a \geq 4. \end{cases} \tag{1.6}$$

Chapter 2

A Condition on Hamilton-Connected Line Graphs

2.1 Main result

The following is our main result.

Theorem 2.1.1 *Let G be a simple graph with $|E(G)| \geq 3$ and $\delta(G) \geq \alpha'(G)$. Then each of the following holds.*
(i) $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.
(ii) $L(G)$ is 1-hamiltonian if and only if $\kappa(L(G)) \geq 3$.

2.2 Preliminaries

A cycle on n vertices is often called an **n-cycle**. For a subset $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ is the subgraph of G induced by X. A path from a vertex u to a vertex v is referred to as a (u, v)-path. An edge subset X of G is an **essential cut** if $G - X$ has at least two nontrivial components or if $|X| = |E(G)| - 1$. For an integer $k \geq 0$, a connected graph G is **essentially k-edge-connected** if G does not have an essential edge cut X with $|X| < k$. For a connected graph G, let $ess'(G)$ be the largest integer k such that G is essentially k-edge-connected. By the definition of a line graph, we have the following observation for a graph G and its line graph $L(G)$:

$$\kappa(L(G)) = ess'(G). \tag{2.1}$$

2.2.1 Maximum matching of a graph

Let M be a matching in G. We use $V(M)$ to denote the set $V(G[M])$. A path P in G is an **M-augmenting path** if the edges of P are alternately in M and

in $E(G) - M$, and if both end vertices of P are not in $V(M)$. We start with a fundamental theorem of Berge.

Theorem 2.2.1 *(Berge [86]) A matching M in G is a maximum matching if and only if G does not have M-augmenting paths.*

Applying Theorem 2.2.1, the following results are proved in [85], which will be utilized in our arguments in the proof of Theorem 3.2.2.

Lemma 2.2.2 *(Lemma 2.1 of [85]) Let $k > 0$ be an integer and G be a graph with a matching M such that $|M| = k$. Suppose that $V(G) - V(M)$ has a subset X with $|X| \geq 2$ such that for any $v \in X$, $d(v) \geq k$. If X has at least one vertex u such that $d(u) \geq k + 1$, then M is not a maximum matching of G.*

Theorem 2.2.3 *(Theorem 2.2 of [85]) Let G be a connected simple graph with $n = |V(G)| \geq 2$ and $k = \alpha'(G)$. If $\delta(G) \geq k$, then $\kappa'(G) \geq k$.*

2.2.2 Collapsible graphs and strongly spanning trailable graphs

We use a definition of collapsible graphs [101] that is equivalent to Catlin's original definition in [9]. For a graph G, we use $O(G)$ to denote the set of all vertices of odd degree in G. A graph G is **collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph H such that $O(H) = R$. If G is collapsible, then by definition with $R = \emptyset$, G is supereulerian and so $\kappa'(G) \geq 2$. As examples, Catlin [9] observed that cycles of length at most 3 are collapsible. In [9], Catlin showed that for any graph G, every vertex of G lies in a unique maximal collapsible subgraph of G. The **reduction** of G, denoted by G', is obtained from G by contracting all nontrivial maximal collapsible subgraphs of G. A graph is **reduced** if it is the reduction of some graph. As shown in [9], a reduced graph is simple.

Theorem 2.2.4 *Let G be a graph.*
(i) (Catlin, Theorem 3 of [9]) Suppose that H is a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible.
(ii) (Catlin, Lemma 3 of [9]) If G is collapsible, then any contraction of G is also collapsible.
(iii) (Catlin, Theorem 5 of [9]) A graph G is reduced if and only if G does not contain a nontrivial collapsible subgraph.
(iv) If G has a spanning connected subgraph Q, such that for any edge $e \in E(Q)$, G has a collapsible subgraph J_e with $e \in E(J_e)$, then G is collapsible.

Proof. We argue by induction on $n = |V(G)|$ to prove (iv). As (iv) holds for $n = 1$, we assume that $n \geq 2$. For any $e \in E(Q)$, let J_e denote a collapsible subgraph of G with $e \in E(J_e)$. We fix an edge $e_0 \in E(Q)$ and let $J = J_{e_0}$

be a collapsible subgraph of G that contains e_0. Define $G_1 = G/J$. As Q is a spanning subgraph in G, $Q_1 = Q/(Q \cap J)$ is a spanning subgraph of G_1. For any edge $e \in E(Q_1) \subseteq E(Q)$, there exists a collapsible subgraph J_e of G with $e \in E(J_e)$. By Theorem 2.2.4(ii), $J'_e = J_e/(J \cap J_e)$ is a collapsible subgraph of G_1 with $e \in E(J'_e)$. It follows by induction that G_1 is collapsible. By Theorem 2.2.4(i), G is collapsible. ∎

For $u, v \in V(G)$, a (u, v)-**trail** is a trail of G from u to v. For $e, e' \in E(G)$, an (e, e')-trail is a trail of G having end-edges e and e'. An (e, e')-trail T is **dominating** if each edge of G is incident with at least one internal vertex of T, and T is **spanning** if T is a dominating trail with $V(T) = V(G)$. A graph G is **spanning trailable** if for each pair of edges e_1 and e_2, G has a spanning (e_1, e_2)-trail. Suppose that $e = u_1v_1$ and $e' = u_2v_2$ are two edges of G. If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ with a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$, also denoted by $G(e)$, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$. For the recovering operation, we let $c_e(G(e, e'))$ be the graph obtained from $G(e, e')$ by replacing the path $u_1v_ev_1$ with the edge $e = u_1v_1$. Thus, $c_{e'}(c_e(G(e, e'))) = G$.

By the definition of $G(e', e'')$, we have the following observation.

If $G(e', e'')$ is collapsible, then $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail. (2.2)

In fact, if $G(e', e'')$ is collapsible, then $G(e', e'')$ has a spanning connected subgraph J with $O(J) = \{v_{e'}, v_{e''}\}$. Hence J is a spanning $(v_{e'}, v_{e''})$-trail.

As defined in [106], a graph G is **strongly spanning trailable** if for any $e, e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$-trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. Since $e = e'$ is possible, strongly spanning trailable graphs are both spanning trailable and supereulerian.

Theorem 2.2.5 (Luo et al. [107], see also Theorem 4 of [90]) *If $\kappa'(G) \geq 4$, then G is strongly spanning trailable.*

Let G be a graph with $|V(G)| \geq 3$. For each integer $i \geq 0$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$. Suppose that $ess'(G) \geq 3$. The **core** of this graph G, denoted by G_0, is obtained from $G - D_1(G)$ by contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$. By the definition of $D_i(G)$, $G - D_1(G)$ is connected if G is connected. As contraction does not decrease the edge connectivity, G_0 is connected if G is connected. Lemma 2.2.6 (iii) below is proved by using a similar argument in the proof of Theorem 5.2.1.

Lemma 2.2.6 *(Shao [140]) Let G be a connected nontrivial graph such that $\kappa(L(G)) \geq 3$, and let G_0 denote the core of G.*
(i) G_0 is uniquely determined by G with $\kappa'(G_0) \geq 3$.

(ii) (see also Lemma 2.9 of [99]) If G_0 is strongly spanning trailable, then $L(G)$ is Hamilton-connected.
(iii) (see also Proposition 2.2 of [99]) $L(G)$ is Hamilton-connected if and only if for any pair of edges $e', e'' \in E(G)$, G has a dominating (e', e'')-trail.

2.3 Proof of the main results

Theorem 2.1.1 will be proved in this section. As every Hamilton-connected graph must be 3-connected, and every 1-hamiltonian graph must be 3-connected, it suffices to prove that if G is a graph satisfying $\delta(G) \geq \alpha'(G)$ and $\kappa(L(G)) \geq 3$, then $L(G)$ is Hamilton-connected for Theorem 2.1.1(i) and $L(G)$ is 1-hamiltonian for Theorem 2.1.1(ii).

2.3.1 Proof of Theorem 2.1.1(i).

As $\kappa(L(G)) \geq 3$, we have $ess'(G) \geq 3$, and so by Lemma 2.2.6(i), the core G_0 of G is well-defined with $\kappa'(G_0) \geq 3$. We shall prove a slightly stronger Theorem 3.2.6 below, which implies the sufficiency of Theorem 2.1.1(i).

Theorem 2.3.1 *Let G be a connected simple graph with $|E(G)| \geq 3$ and $ess'(G) \geq 3$, and let G_0 denote the core of G.*
(i) If $\delta(G_0) \geq \alpha'(G_0)$, then G_0 is strongly spanning trailable.
(ii) Suppose that $\delta(G) \geq \alpha'(G)$. Then $L(G)$ is Hamilton-connected if and only if $\kappa(L(G)) \geq 3$.

To prove Theorem 3.2.6, we begin with some tools that would be used in the arguments. For a graph G, let $circ(G)$ denote the length of a longest cycle of G.

Proposition 2.3.2 *Let G be a connected simple graph with $|E(G)| \geq 3$ and $\delta(G) \geq \alpha'(G) = k$.*
(i) If $k \geq 4$, then $\kappa'(G) \geq k \geq 4$ and G is strongly spanning trailable.
(ii) (Lemma 3.1 of [85]) If $k = 1$, then $G \in \{K_3, K_{1,n-1}\}$.
(iii) If $k \geq 4$ or $k = 1$, then $L(G)$ is Hamilton-connected.

Proof. To prove Proposition 2.3.2(i), we apply Theorem 2.2.3 to conclude that $\kappa'(G) \geq k \geq 4$. Hence by Theorem 2.2.5, G is strongly spanning trailable. It remains to justify Proposition 2.3.2(iii). If $k \geq 4$, then as G is strongly spanning trailable, by Lemma 2.2.6(iii), $L(G)$ is Hamilton-connected. If $k = 1$, then $L(G)$ is a complete graph and so it is also Hamilton-connected. ∎

We define $P^-(10), P(10), P(11), K_{1,3}(1,1,1), K_{2,3}, T(1,2)$ to be the graphs as respectively depicted in Figure 1.

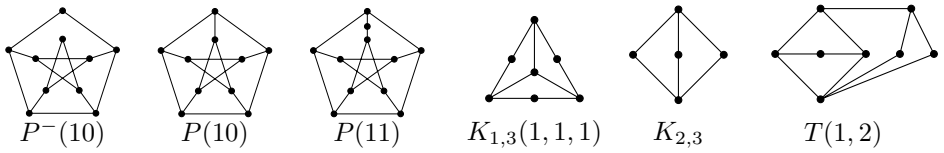

Figure 1. nontrivial reduced graphs in Theorem 2.3.3(ii).

Theorem 2.3.3 *Let G be a connected graph with $n = |V(G)|$, and let G' denote the reduction of G.*
(i) (Ma et al., Theorem 3.2 of [108], See also Theorem 4.5.4 of [116]) If $G = G'$, and G satisfies $\kappa'(G) \geq 2$, $circ(G) \leq 8$, $|D_2(G)| \leq 2$ and $ess'(G) \geq 3$, then G is collapsible.
(ii) (Theorem 1.7 of [102]) If $ess'(G) \geq 3$, $n \leq 11$, $|D_1(G)| = 0$ and $|D_2(G)| \leq 2$, then $G' \in \{K_1, K_{2,3}, K_{1,3}(1,1,1), T(1,2), P^-(10), P(10), P(11)\}$.

Corollary 2.3.4 *Each of the following holds.*
(i) Every graph G with $\kappa'(G) \geq 2$, $circ(G) \leq 8$, $|D_2(G)| \leq 2$ and $ess'(G) \geq 3$ is collapsible.
(ii) Every graph G with $\kappa'(G) \geq 3$ and $circ(G) \leq 6$ is strongly spanning trailable.
(iii) Let G be a graph with $ess'(G) \geq 3$ and $circ(G) \leq 6$, and let G_0 be the core of G. Then G_0 is strongly spanning trailable.

Proof. Let G be a graph with $\kappa'(G) \geq 2$, $circ(G) \leq 8$, $|D_2(G)| \leq 2$ and $ess'(G) \geq 3$, and let G' be the reduction of G. By the definition of contraction, we have $\kappa'(G') \geq \kappa'(G) \geq 2$, $circ(G') \leq circ(G) \leq 8$ and $ess'(G') \geq ess'(G) \geq 3$. Let $v \in D_2(G')$ be a vertex. Since $ess'(G) \geq 3$, v must be a trivial vertex and so $v \in D_2(G)$. This implies that $|D_2(G')| \leq |D_2(G)| \leq 2$. It follows by Theorem 2.3.3, G' is collapsible which implies that $G' = K_1$ and so G is collapsible. This proves (i).

To prove (ii), we assume that G with $\kappa'(G) \geq 3$ and $circ(G) \leq 6$. Let $e', e'' \in E(G)$ be two edges and let $H = G(e', e'')$. Then as $\kappa'(G) \geq 3$ and $circ(G) \leq 6$, we conclude that $\kappa'(H) \geq 2$, $circ(H) \leq 8$, $|D_2(H)| \leq 2$ and $ess'(H) \geq 3$. It follows by (i) that H is collapsible. Let $v_{e'}$ and $v_{e''}$ denote the only vertices in $D_2(H)$. As H is collapsible, H has a spanning connected subgraph T with $O(T) = \{v_{e'}, v_{e''}\}$. Thus T is a spanning $(v_{e'}, v_{e''})$-trail of H, and so by the randomness of e', e'', G is strongly spanning trailable. This proves (ii).

Now we assume that G is a graph with $ess'(G) \geq 3$ and $circ(G) \leq 6$. Let G_0 denote the core of G. By Lemma 2.2.6(i), $\kappa'(G_0) \geq 3$. As G_0 is a contraction of G, we have $circ(G_0) \leq circ(G) \leq 6$. By (ii), G_0 is strongly spanning trailable.
■

2.3.2 Proof of Theorem 3.2.6(i)

We assume that $\delta(G_0) \geq \alpha'(G_0)$. Let $n = |V(G_0)|$ and $k = \alpha'(G_0)$. As G is connected, by the definition of G_0, G_0 is also connected. Thus if $k = 0$, then $n = 1$, and so by definition, G_0 is strongly spanning trailable. Hence we assume that $k > 0$. Then $|V(G)| \geq n = |V(G_0)| \geq 2\alpha'(G_0) = 2k \geq 2$. Thus G is a connected nontrivial graph. As $ess'(G) \geq 3$, by (2.1) and Lemma 2.2.6(i), $\kappa'(G_0) \geq 3$. Thus $|E(G_0)| \geq 3$. If $k = 1$, then applying Proposition 2.3.2(ii) to G_0, G_0 is spanned either by a K_3 or by a $K_{1,n-1}$ with $\kappa'(G_0) \geq 3$. If G_0 is spanned by a K_3, then this K_3 must have at least two edges each of which lies in a 2-cycle. For any $e', e'' \in G_0$, if there exists a 2-cycle C in $G_0(e', e'')$, then after contracting this 2-cycle C in $G_0(e', e'')$, every edge of $G_0(e', e'')/C$ lies in a cycle of length at most 3. As C is collapsible, by Theorem 2.2.4(i) that $G_0(e', e'')$ is collapsible. If there does not exist a 2-cycle in $G_0(e', e'')$, every edge of $G_0(e', e'')$ lies in a cycle of length at most 3 in $G_0(e', e'')$. It follows by Theorem 2.2.4(iv) that $G_0(e', e'')$ is collapsible. When G_0 is spanned by a $K_{1,n-1}$, since $\kappa'(G_0) \geq 3$, every edge must be in a parallel class of at least three edges. In this case, every edge of $G_0(e', e'')$ lies in a cycle of length at most 3. It follows by Theorem 2.2.4(iv) that $G_0(e', e'')$ is collapsible. By (2.2), G_0 is strongly spanning trailable. If $k \geq 4$, then by Proposition 2.3.2(i), G_0 is strongly spanning trailable. Therefore, we assume that $k \in \{2, 3\}$. Suppose that $k = 2$. Then G_0 does not have a cycle of length longer than 5, and so by Corollary 2.3.4(iii), G_0 is strongly spanning trailable.

Hence we assume that $k = 3$, and so $circ(G_0) \leq 7$. If $circ(G_0) \leq 6$, then by Corollary 2.3.4(iii), G_0 is strongly spanning trailable, and we are done. Therefore, we assume that $circ(G_0) = 7$. Let C be a cycle of G_0 with $|V(C)| = 7$. If $V(G_0) - V(C) \neq \emptyset$, then as G_0 is connected, there must be a vertex $v \in V(G_0) - V(C)$ such that v is adjacent to a vertex on C, implying that $3 = \alpha'(G_0) \geq 4$, a contradiction. Thus $V(G_0) = V(C)$ and so $|V(G_0)| = 7$ and C is a Hamilton cycle of G_0.

For any $e', e'' \in E(G_0)$, let $H = G_0(e', e'')$ and let $v_{e'}, v_{e''}$ denote the new vertices newly added in the process of subdividing e' and e'', respectively. Then $|V(H)| = 9$. As $\kappa'(G_0) \geq 3$, we have $|D_1(H)| = 0$ and $|D_2(H)| = 2$. Let H' be the reduction of H. We claim that $H' = K_1$ and so H is collapsible. By contradiction, we assume that $1 < |V(H')| \leq |V(H)| = 9$. By Theorem 2.3.3 (ii), $H' \in \{K_{2,3}, K_{1,3}(1,1,1), T(1,2)\}$. Since $\kappa'(G_0) \geq 3$, $|D_2(H')| \leq |D_2(H)| \leq 2$. It follows that $H' \notin \{K_{2,3}, K_{1,3}(1,1,1), T(1,2)\}$, as any of these graphs have at least 3 vertices of degree 2. This contradiction implies that $H' = K_1$ and so H is collapsible. By (2.2), G_0 is strongly spanning trailable. This completes the proof of Theorem 3.2.6(i). ∎

2.3.3 Proof of Theorem 3.2.6(ii)

In this subsection, we assume Theorem 3.2.6(i) to prove Theorem 3.2.6(ii). It suffices to show that if $ess'(G) \geq 3$ and $\delta(G) \geq \alpha'(G)$, then $L(G)$ is Hamilton-connected. Let G_0 denote the core of G, $k = \alpha'(G)$ and $n = |V(G_0)|$.

By Proposition 2.3.2(i), if $\alpha'(G) \geq 4$, then G is strongly spanning trailable. By definition, any spanning $(v_{e'}, v_{e''})$-trail induces a spanning (e', e'')-trail in G. It follows by Lemma 2.2.6 that $L(G)$ is Hamilton-connected.

Hence we assume that $k \leq 3$. As G_0 is a contraction of G, we have $\alpha'(G_0) \leq \alpha'(G) \leq 3 \leq \kappa'(G_0) \leq \delta(G_0)$. By Theorem 3.2.6(i), G_0 is strongly spanning trailable. By Lemma 2.2.6(ii), $L(G)$ is Hamilton-connected. This completes the proof. ∎

2.3.4 Proof of Theorem 2.1.1(ii)

For a vertex $u \in V(G)$, define $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ to be the set of neighbors of u in G. The main purpose of this subsection is to prove Theorem 3.2.2(ii). As remarked at the beginning of this section, it suffices to assume that G is a graph satisfying $\delta(G) \geq \alpha'(G)$ and $\kappa(L(G)) \geq 3$ to show that $L(G)$ is 1-hamiltonian. In the proof, we will need the following former results.

Lemma 2.3.5 *Let G be a connected graph, and let $K_{3,3}^-$ denote the graph obtained from $K_{3,3}$ by deleting an edge. Each of the following holds.*
(i) (Catlin et al., Theorem 1.1 of [91]) If $\kappa'(G) \geq 4$, then for any edge subset $X \subseteq E(G)$ with $|X| \leq 2$, $G - X$ has two edge-disjoint spanning trees and is collapsible.
(ii) (Catlin [89]) $K_{3,3}^-$ is collapsible, and so $K_{3,3}$ is collapsible.
(iii) (Li et al., Lemma 2.1 of [105]) If $|V(G)| \leq 8$ with $|D_1(G)| = 0$ and $|D_2(G)| \leq 2$. Then the reduction of G is in $\{K_1, K_2, K_{2,3}\}$.

By the definition of the core, and imitating the arguments in [125, 140] and in Theorem 2.7 of [100], we have the following observation.

Observation 2.3.6 *Let $s \geq 0$ be an integer, G be a connected graph with $|E(G)| \geq s + 3$ and $ess'(G) \geq 3$, and G_0 be the core of G.*
(i) (Theorem 2.7 of [100]) The line graph $L(G)$ is s-hamiltonian if and only if for any $S \subseteq E(G)$ with $|S| \leq s$, $G - S$ has a dominating eulerian subgraph.
(ii) If for any $S \subseteq E(G_0)$ with $|S| \leq s$, $G_0 - S$ is supereulerian, then $L(G)$ is s-hamiltonian.

Proof. It suffices to justify Observation 2.3.6(ii). By Observation 2.3.6(i), we need to prove that for any $X \subseteq E(G)$ with $|X| \leq s$, $G - X$ has a dominating

eulerian subgraph. Let G_0 denote the core of G. Define

$$\begin{aligned} S_1 &= \{e \in E(G) : e \text{ is incident with a vertex in } D_1(G)\}, \\ S_2' &= \{e \in X : e \text{ is incident with a vertex in } D_2(G)\}. \end{aligned}$$

We shall adopt the following convention in our arguments. If $e', e'' \in S_2'$ are incident with a vertex $v \in D_2(G)$, then we may always assume that e'' is being contracted in the construction of G_0 and e' remains in $E(G_0)$. With this convention, for any $v \in D_2(G)$, we may denote $E_G(v) = \{e_v', e_v''\}$, and define $X_2 = \{e_v'' : v \in D_2(G)\}$. Hence by the definition of G_0, we may assume that $G_0 = G/(S_1 \cup X_2)$.

Let $S_2 = S_2' - X_2$, $S_3 = X - (S_1 \cup S_2')$. Then $S = S_2 \cup S_3 \subseteq E(G_0)$. As $S \subseteq X$, we have $|S| \leq |X| \leq s$. By the assumption of Observation 2.3.6(ii), $G_0 - S$ has a spanning eulerian subgraph H'. Let $S_2'' = \cup_{v \in D_2(G), e_v' \in E(H')} \{e_v', e_v''\}$. Define $H = G[E(H') \cup S_2'']$. Since H' is an eulerian subgraph of G_0, by the definition of G_0, every vertex in H not incident with an edge in $X_2 \cap S_2''$ has the same (even) degree as in H'. As H is obtained from $G[E(H')]$ by adding the edges in $X_2 \cap S_2''$, which amounts to subdividing the edges in $(\cup_{v \in D_2(G)} E_G(v)) \cap E(H')$ to form H, it follows that H is an eulerian subgraph of G. For any edge $e \in E(G)$, if $e \in E(G) - (S_1 \cup X_2) = E(G_0)$, then since H' is a spanning eulerian subgraph of G_0, e is incident with a vertex in $V(H)$. If $e \in S_1$, then by $ess'(G) \geq 3$, e is also incident with a vertex of degree at least 4 in G. Hence e is incident with a vertex in $V(H)$ as well. Finally, we assume that $e \in X_2$. As $X_2 = \{e_v'' : v \in D_2(G)\}$, there exists a vertex $v \in D_2(G)$ with $e = e_v''$. Let u, w be the neighbors of v in G, and so uvw is a path of length 2 in G. As $ess'(G) \geq 3$, it follows that $d_G(u) \geq 3$ and $d_G(w) \geq 3$. By the definition that $G_0 = G/(S_1 \cup X_2)$ and since H' spans G_0, we have $u, w \in V(G_0) = V(H')$. As $H = G[E(H') \cup S_2'']$, this implies that $u, w \in V(H)$, and so e must be incident with a vertex in $V(H)$. It follows by definition that H is a dominating eulerian subgraph of $G - X$, and so by Observation 2.3.6(i), $L(G) - X$ is hamiltonian. This proves Observation 2.3.6(ii). ∎

To prove Theorem 2.1.1(ii), we let $k = \alpha'(G)$ and G_0 denote the core of G. Then we will justify the following claim.

Claim 1 *If $k = 1$ or $k \geq 4$, then $L(G)$ is 1-hamiltonian.*

Suppose first that $k = 1$. By Proposition 2.3.2(ii), $G \in \{K_3, K_{1,n-1}\}$. As $\kappa(L(G)) \geq 3$, $G \in \{K_{1,n-1}\}$ where $n \geq 5$. By the definition of a line graph, $L(G) = K_{n-1}$ is 1-hamiltonian. Next we assume that $k = \alpha'(G) \geq 4$. By Theorem 2.2.3, $\kappa'(G) \geq 4$. By Lemma 2.3.5(i), for any $e \in E(G)$, $G - e$ is collapsible, and so is supereulerian. Thus by Observation 2.3.6, $L(G)$ is 1-hamiltonian. This proves Claim 1.

By Claim 1, it remains to discuss the cases when $k \in \{2, 3\}$. Suppose that $k = 2$. Let M be a maximum matching of G_0 and X be the set of vertices in G_0 not incident with any edges in M. As $\delta(G_0) \geq \kappa'(G_0) \geq 3$, it follows by Lemma 2.2.2 that $|X| \leq 1$ and so $|V(G_0)| \leq 5$. Thus for any edge $e \in E(G_0)$, we have $|V(G_0 - e)| \leq |V(G_0)| \leq 5$. As $\kappa'(G_0) \geq 3$, we have $|D_1(G_0 - e)| = 0$ and $|D_2(G_0 - e)| \leq 2$. By Lemma 2.3.5(iii), the reduction of $G_0 - e$ is in $\{K_1, K_2, K_{2,3}\}$. Again by $\kappa'(G_0) \geq 3$, $\kappa'(G_0 - e) \geq 2$ and $G_0 - e$ has at most two edge cuts of size 2. Thus the reduction of $G_0 - e$ is 2-edge-connected and has at most two edge cuts of size 2. Then the reduction of $G_0 - e$ is K_1 and so $G_0 - e$ is collapsible. By Observation 2.3.6, $L(G)$ is 1-hamiltonian.

Hence we assume that $k = 3$. We shall show that

$$\text{for any } e \in E(G_0),\ G_0 - e \text{ is collapsible.} \tag{2.3}$$

We prove (2.3) by contradiction, and assume that for some $e_0 = z_1z_2 \in E(G_0)$, $G_0 - e_0$ is not collapsible. Let G_0' denote the reduction of $G_0 - e_0$. Since $G_0 - e_0$ is not collapsible, $|V(G_0')| \geq 2$.

Let w_1, w_2 be the vertices in $V(G_0')$, each of whose preimages in $G_0 - e_0$ contains an end vertex of e_0. We claim that $w_1 \neq w_2$. By contradiction, we assume that $w_1 = w_2$. As G_0' is the reduction of $G_0 - e_0$, there exists a collapsible subgraph H in $G_0 - e_0$ with $V(e_0) \subseteq V(H)$, and so $(G_0 - e_0)/H = G_0/H$. Since G_0/H is a contraction of G_0, we have $\kappa'(G_0/H) \geq \kappa'(G_0) \geq 3$ and $\alpha'(G_0/H) \leq \alpha'(G) = k \leq 3$. It follows that $\kappa'(G_0/H) \geq 3$ and $circ(G_0/H) \leq 7$. By Corollary 2.3.4(i), G_0/H is collapsible which implies that $(G_0 - e_0)/H$ is collapsible. Thus by Theorem 2.2.4(i), $G_0 - e_0$ is collapsible, which is contrary to the assumption that $G_0 - e_0$ is not collapsible. This proves $w_1 \neq w_2$.

Define G_0^+ to be the graph obtained from G_0' by adding a new edge linking w_1 and w_2. Thus G_0^+ is a contraction of G_0, and $G_0' = G_0^+ - e_0$. As G_0 is a contraction of G and G_0' is a contraction of $G_0 - e_0$, it follows that $\alpha'(G_0') \leq \alpha'(G_0) \leq \alpha'(G) = k \leq 3$. Since $\kappa'(G_0) \geq 3$ and G_0^+ is a contraction of G_0, we have $\kappa'(G_0^+) \geq 3$. As $G_0' = G_0^+ - e_0$, we conclude that $\kappa'(G_0') \geq 2$, $|D_2(G_0')| \leq 2$ and $ess'(G_0') \geq 3$. Thus by Corollary 2.3.4(i), G_0' is collapsible. As G_0' is the reduction of $G_0 - e_0$, we have $G_0' = K_1$ and so $G_0 - e_0$ is collapsible. This leads to a contradiction to the assumption that $G_0 - e_0$ is not collapsible, and completes the proof of Theorem 3.2.2(ii).

■

Chapter 3

Polynomially determining spanning connectivity of locally connected line graphs

3.1 Main result

Theorem 3.1.1 *Each of the following holds.*
(i) Every 3-connected, locally connected line graph $L(G)$ is maximally spanning connected.
(ii) The spanning connectivity of a locally connected line graph can be polynomially determined.

Theorem 3.1.2 *Let $G \in \mathcal{G}$ be a connected graph with maximum degree $\Delta(G)$.*
(i) (Chen et al. Theorem 22 of [10]) $s_2(G) \leq |V(G)| - \Delta(G) + 1$.
(ii) (Xiong et al. Theorem 1.3 of [144]) $s_3(G) \leq \ell(G) + 6$.

The results in Theorem 3.2.3 also motivate our current study. A divalent path P of G is a **bridge divalent path** if every edge of P is a cut edge of G; and is a **divalent (s,t)-path** if the two end vertices of P are of degree s and t, respectively. The next main result studies best possible bounds for $s_k(G)$. When $k = 2$, Theorem 3.1.3(iv) improves Theorem 3.2.3(i) and when $k = 3$, Theorem 3.1.3(iii) sharpens Theorem 3.2.3(ii).

Theorem 3.1.3 *Let $G \in \mathcal{G}$ be a graph and let $k \geq 3$ be an integer.*
(i) $s_2(G) \leq \ell(G) + 2$.
(ii) $s_k(G) \leq \ell(G) + k - 1$. Furthermore, $s_k(G) = \ell(G) + k - 1$ only if for some integer $t \geq 3$, G has a bridge divalent $(3,t)$-path of length $\ell(G)$.
(iii) $s_3(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3,t)$-path of length $\ell(G)$.
(iv) $s_k(G) \leq |V(G)| - \Delta(G) + k - 2$.

For a graph $G \in \mathcal{G}$, define $msc(G)$ to be the smallest integer m such that $L^m(G)$ is maximally spanning connected. A best possible upper bound for $msc(G)$ is also obtained.

Theorem 3.1.4 *Let $G \in \mathcal{G}$ be a graph.*
(i) $msc(G) \leq \ell(G) + 2$, and for any integer $m \geq \ell(G) + 2$, $\kappa(L^m(G)) = \kappa^*(L^m(G))$. Moreover, $msc(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$.
(ii) $msc(G) \leq |V(G)| - \Delta(G) + 2$, and for any integer $m \geq |V(G)| - \Delta(G) + 2$, $\kappa(L^m(G)) = \kappa^*(L^m(G))$.

The tools to assist our arguments to prove the main results are summarized and developed in the next section. In Section 3, we will prove the main results. Related open problems will be discussed in the last section.

3.2 Mechanisms

By the well-known Menger's Theorems (Theorems 9.1 and 9.7 of [87]), we define a graph G to be **k-connected** (or k-edge-connected, respectively) if for any pair of distinct vertices u and v, G contains a $(k; u, v)$-path-system (or a $(k; u, v)$-trail-system, respectively). Therefore, the connectivity $\kappa(G)$ of a graph G (or the edge-connectivity $\kappa'(G)$ of G, respectively) equals the maximum number k such that for every pair of distinct vertices u and v, G has a $(k; u, v)$-path-system (or a $(k; u, v)$-trail-system, respectively). In [126], an $(s; u, v)$-path-system and a spanning $(s; u, v)$-path-system are also called a k-container and a k^*-container, respectively. The **spanning connectivity** $\kappa^*(G)$ of a graph G is the largest integer s such that for any integer k with $0 \leq k \leq s$ and for any $u, v \in V(G)$ with $u \neq v$, G has a spanning $(k; u, v)$-path-system. A graph G is **s-spanning connected** if $\kappa^*(G) \geq s$. There have been quite a few studies on spanning connectivity and an edge counterpart of it, as seen in [18, 131, 133, 134, 142, 144, 145], among others. As shown in [126], many former studies on spanning conductivities have been focused on results involving degree conditions to assure a simple graph to have spanning connectivity at least a given integer s; as well as investigations of spanning connectivity of certain family of graphs such as Harary graphs, hypercubes and hypercube-like graphs.

By definition, a graph G is hamiltonian if and only if for any distinct vertices $u, v \in V(G)$, G has a spanning $(2; u, v)$-path-system. Thus as remarked in [126], spanning connectivity of graphs can be viewed as a hybrid concept of Hamiltonicity and connectivity. Following [87], a graph G is **Hamilton-connected** if for any $u, v \in V(G)$ with $u \neq v$, G has a spanning (u, v)-path P. Thus $\kappa^*(G) \geq 1$ implies that G is Hamilton-connected. It is well known that every Hamilton-connected graph with at leats 4 vertices must be 3-connected.

Hence the following fact (3.1) is observed.

$$\text{If } G \text{ is a graph with } |V(G)| \geq 4 \text{ and } \kappa^*(G) > 0, \text{ then } \kappa(G) \geq 3. \tag{3.1}$$

As every Hamilton-connected graph must also be hamiltonian, we conclude that a graph G is Hamilton-connected if and only if $\kappa^*(G) > 0$. Thus determining if $\kappa^*(G) > 0$ in general is an NP-complete problem. One of the motivation of this research is to seek nontrivial common families of graphs in which spanning connectivity can be polynomially determined.

As it is known that the connectivity of a graph can be polynomially determined, (see, for example, [119, 123]), the problem whether high connectivity could imply positive spanning connectivity was considered. While the complete bipartite graphs indicate that in general, high connectivity of a graph G does not warrant $\kappa^*(G) > 0$, researchers have been investigating graph families in which high connectivity of a graph G in these family would imply that $\kappa^*(G) > 0$. Thomassen in [143] first conjectured that every 4-connected line graph is hamiltonian. This most fascinating conjecture has attracted many researchers.

Let $L(G)$ denote the **line graph** of a graph G, which is a simple graph with vertex set $E(G)$, and with edge set $E(L(G)) = \{e'e'' : e', e'' \in E(G)$ and e', e'' are adjacent in $G\}$. A graph that does not have an induced subgraph isomorphic to $K_{1,3}$ is a **claw-free** graph. Beineke [4] and Robertson (Page 74 of [125]) showed that line graphs are claw-free graphs. By several ingenious closure concepts developed by Ryjáček [137] and by Ryjáček and Vrána [138], Thomassen's above-mentioned conjecture is shown to be equivalent to each of the following.

Conjecture 3 *Let G be a graph and let Γ be a claw-free graph.*
(i) (Thomassen [143] and, Kučzel and Xiong [128]) Every 4-connected line graph has spanning connectivity at least 2.
(ii) (Matthews and Sumner [135], and Ryjáček and Vrána [138]) Every 4-connected claw-free graph has spanning connectivity at least 2.

There have been intensive studies towards Conjecture 3, as shown in the surveys [118, 121, 122]. By Menger's Theorem ([136], see also Theorem 9.1 of [87]), for any graph G, we always have $\kappa(G) \geq \kappa^*(G)$. Thus graphs G with $\kappa(G) = \kappa^*(G)$ are of particular interests. In view of (3.1), we define a connected graph G to be **maximally spanning connected** if both $\kappa(G) \geq 3$ and $\kappa(G) = \kappa^*(G)$. A similar concept of super spanning connected graph is formerly defined in [126], which implies that K_2 is super spanning connected. By the definition in this paper, K_2 is not maximally spanning connected. As examples, complete graphs of order at least 4 are maximally spanning connected, but complete bipartite graphs of any orders are not maximally spanning connected.

As of today, little is known on maximally spanning connected graph families other than the complete graphs and a few others. This motivates the current study. For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The vertex v is locally connected if the induced subgraph $G[N_G(v)]$ is connected. A graph G is **locally connected** if very vertex v of G is locally connected. Asratian [1] and Y. Sheng, F. Tian and B. Wei [141] studied connectivity conditions for a locally connected claw-free graph G to have spanning connectivity at least 2. As line graphs are claw-free, their result is also valid for line graphs. A class of maximally spanning connected line graphs has also been studied in [127] and [18].

Theorem 3.2.1 *Let G be a connected graph.*
(i) (Asratian [1] and Y. Sheng, F. Tian and B. Wei [141]) If G is an locally connected claw-free graph with $\kappa(G) \geq 3$, then $\kappa^*(G) \geq 2$.
(ii) (Huang and Hsu [127], and Chen et al. [18]) Let $k \geq 3$ be an integer. If a graph G has k-edge-disjoint spanning trees, then $L(G)$ is maximally spanning connected.

Theorem 3.2.2(i) below, one of our main results, has identified a new family of graphs whose line graphs are maximally spanning connected, which extends Theorem 3.2.1(i). As connectivity of a graph can be polynomially determined, Theorem 3.2.2(ii) follows from Theorem 3.2.2(i).

Theorem 3.2.2 *Each of the following holds.*
(i) Every 3-connected, locally connected line graph $L(G)$ is maximally spanning connected.
(ii) The spanning connectivity of a locally connected line graph can be polynomially determined.

For an integer $m > 0$, define $L^0(G) = G$, and the iterated line graph $L^m(G) = L(L^{m-1}(G))$. A path P of G is a **divalent path** of G if every internal vertex of P has degree 2 in G. Following [129, 132, 144, 146], define a divalent path that is not of length 2 and in a K_3 as proper divalent path,

$$\ell(G) = \max\{m : G \text{ has a length } m \text{ proper divalent path}\}. \tag{3.2}$$

For discussional convenience, we in this paper denote $\mathcal{G}$ to be the family of all connected nontrivial graphs that are not isomorphic to a path, a cycle or a $K_{1,3}$. To study iterated line graphs, we only consider graphs in $\mathcal{G}$. The iterated line graph index problem is also an intensively studied topic in graph theory. Chartrand and Wall in [11] initiated the study the smallest integer $k \geq 0$, called the hamiltonian index of a graph G, such that the iterated line graph $L^k(G)$ becomes hamiltonian. Other hamiltonian like indices were defined and studied by Clark and Wormald in [20]. More generally, we have the following definition.

Definition 3 *([130]) Let $\mathcal{P}$ denote a graphical property and G be a connected graph $G \in \mathcal{G}$. Then $\mathcal{P}(G)$, the* **$\mathcal{P}$-index** *of G, is defined by*

$$\mathcal{P}(G) = \begin{cases} \min\{k : L^k(G) \text{ has property } \mathcal{P}\} \\ \qquad\qquad \text{if for some integer } j > 0,\ L^j(G) \text{ has property } \mathcal{P}, \\ \infty \\ \qquad\qquad \text{otherwise.} \end{cases}$$

Clark and Wormald in [20] studied the existence of the indices for the properties of being edge-hamiltonian, pancyclic, vertex-pancyclic, edge-pancyclic, hamiltonian-connected, respectively. Additional studies of these indices can also be found in [130]. In [139], Ryjáček, Woeginger and Xiong indicated that determining the value of $h(G)$ is a difficult problem. The index problem for graphical properties has been intensively studied, as seen in [11, 19, 10, 20, 120, 129, 130, 132, 140, 139, 144, 146], among others.

In this research, we consider some indices related to spanning connectivity of graphs. For an integer $k \geq 2$, and a graph $G \in \mathcal{G}$, let $s_k(G)$ be the smallest integer m such that $\kappa^*(L^m(G)) \geq k$. When k is small, upper bounds for $s_k(G)$ have been investigated.

Theorem 3.2.3 *Let $G \in \mathcal{G}$ be a connected graph with maximum degree $\Delta(G)$.*
(i) (Chen et al. Theorem 22 of [10]) $s_2(G) \leq |V(G)| - \Delta(G) + 1$.
(ii) (Xiong et al. Theorem 1.3 of [144]) $s_3(G) \leq \ell(G) + 6$.

The results in Theorem 3.2.3 also motivate our current study. A divalent path P of G is a **bridge divalent path** if every edge of P is a cut edge of G; and is a **divalent (s,t)-path** if the two end vertices of P are of degree s and t, respectively. The next main result studies best possible bounds for $s_k(G)$. When $k = 2$, Theorem 3.2.4(iv) improves Theorem 3.2.3(i) and when $k = 3$, Theorem 3.2.4(iii) sharpens Theorem 3.2.3(ii).

Theorem 3.2.4 *Let $G \in \mathcal{G}$ be a graph and let $k \geq 3$ be an integer.*
(i) $s_2(G) \leq \ell(G) + 2$.
(ii) $s_k(G) \leq \ell(G) + k - 1$. Furthermore, $s_k(G) = \ell(G) + k - 1$ only if for some integer $t \geq 3$, G has a bridge divalent $(3,t)$-path of length $\ell(G)$.
(iii) $s_3(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3,t)$-path of length $\ell(G)$.
(iv) $s_k(G) \leq |V(G)| - \Delta(G) + k - 2$.

For a graph $G \in \mathcal{G}$, define $msc(G)$ to be the smallest integer m such that $L^m(G)$ is maximally spanning connected. A best possible upper bound for $msc(G)$ is also obtained.

Theorem 3.2.5 *Let $G \in \mathcal{G}$ be a graph.*
(i) $msc(G) \leq \ell(G) + 2$, and for any integer $m \geq \ell(G) + 2$, $\kappa(L^m(G)) = \kappa^*(L^m(G))$. Moreover, $msc(G) = \ell(G) + 2$ if and only if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path of length $\ell(G)$.
(ii) $msc(G) \leq |V(G)| - \Delta(G) + 2$, and for any integer $m \geq |V(G)| - \Delta(G) + 2$, $\kappa(L^m(G)) = \kappa^*(L^m(G))$.

The tools to assist our arguments to prove the main results are summarized and developed in the next section. In Section 3, we will prove the main results. Related open problems will be discussed in the last section.

To facilitate our proofs of the main results, a number of tools will be displayed and developed in this section. Given a graph G and an integer $i \geq 0$, let $D_i(G)$ be the set of all vertices of degree i in G and $O(G) = \cup_{j \geq 0} D_{2j+1}(G)$ be the set of all odd degree vertices in G. By an **n-cycle** we mean a cycle C with $|V(C)| = n$; and C is a **short cycle** if $2 \leq |E(C)| \leq 3$. Extending the definition in [5], a graph G is **triangular** if every edge $e \in E(G)$ lies in a short cycle C_e of G.

By definition, a spanning $(2, u, v)$-path system is a Hamilton cycle and a $(2; u, v)$-trail system is a spanning eulerian subgraph in a graph G.

Chen et al in [18] extended Theorem [25] by displaying a relationship between spanning connectivity in $L(G)$ and certain type of dominating trail systems in G. This will be a key tool in our arguments. As in [87], a trail in a graph G can be expressed as a sequence

$$T = v_0, e_1, v_1, e_2, \cdots, e_k, v_k \tag{3.3}$$

such that for each i with $1 \leq i \leq k$, the edge e_i is incident with the two vertices v_{i-1} and v_i, and such that if $1 \leq i < j \leq k$, then $e_i \neq e_k$. A trial T (with the notation in (3.3)) is **open** (or **closed**, respectively) if $v_0 \neq v_k$ (or $v_0 = v_k$, respectively). We define the **internal vertices** of the trail in (3.3) to be the set $\{v_1, v_2, \cdots, v_{k-1}\}$, if T is open, and to be $V(T)$ if T is closed. As in an open trail, vertices may occur more than once, it is also possible for the end vertices v_0 or v_k in (3.3) to be internal. A trail T of G is **dominating** if every edge of G is incident with an internal vertex of T, and is **spanning** if it is dominating with $V(T) = V(G)$.

Let $e', e'' \in E(G)$ be two edges of G. A trail T of G is an **(e', e'')-trail** of G if the two end edges of T are e' and e'', respectively. As an example, the trail in (3.3) is an (e_1, e_k)-trail. Two (e', e'')-trails T_1 and T_2 are **internally edge-disjoint** if $E(T_1) \cap E(T_2) = \{e', e''\}$. For a given integer $s \geq 0$, an $(s; e', e'')$-trail system in G is a subgraph J consisting of s internally edge-disjoint (e', e'')-trails $(T_1, T_2, \cdots, T_s)$. A vertex v is an **internal vertex of** J if for some i with $1 \leq i \leq s$, v is an internal vertex of T_i. For an $(s; e', e'')$-trail system J, define

$$\partial_G(J) = \{e \in E(G) - E(J) : \ e \text{ is incident with an internal vertex of } J\}.$$

An $(s; e', e'')$-trail system J is **dominating** if $E(G) - E(J) = \partial_G(J)$, and is **spanning** if it is dominating with $V(G) = V(J)$.

Theorem 3.2.6 *(Chen et al., Theorem 2.1 of [18]) Let G a graph with $|E(G)| \geq 3$ and let $s \geq 3$ be an integer. Then $\kappa^*(L(G)) \geq s$ if and only if for any edge e', $e'' \in E(G)$, and for each integer k with $1 \leq k \leq s$, G has a dominating $(k; e', e'')$-trail-system.*

Recall that $\ell(G)$ is defined in (3.2), the connectivity of iterated line graphs have been investigated. The following former results will be useful in our arguments.

Theorem 3.2.7 *Let $k > 0$ be an integer and a graph $G \in \mathcal{G}$ be a graph.*
(i) (Zhang et al, Lemma 3.2 [146]) If $G \in \mathcal{G}$, then $L^{\ell(G)}(G)$ is triangular.
(ii) (Zhang et al, Proposition 2.3 [147]) If G a connected triangular simple graph, then $L(G)$ is triangular. If, in addition, G is k-connected, then $L(G)$ is $(k+1)$-connected.
(iii) For any integer $m \geq \ell(G) + k - 1$, $\kappa(L^m(G)) \geq m - \ell(G) + 1 \geq k$.

Proof. It suffices to justify (iii). Let $\ell = \ell(G)$. By Theorem 3.2.7 (i), $L^\ell(G)$ is 1-connected and triangular. By repeated application of Theorem 3.2.7 (ii), $\kappa(L^m(G)) \geq m - \ell(G) + 1 \geq k$. ∎

In the following of this section, we always assume that $G \in \mathcal{G}$ is a connected graph. We shall show certain relationship between the subgraphs of a graph G and the subgraphs of its line graph $L(G)$. Let $\mathcal{H}(G)$ denote the collection of all edge-induced subgraphs of G and let $\mathcal{L}(G)$ denote the collection of all induced subgraphs of $L(G)$. Thus for any subgraph $H \in \mathcal{H}(G)$, we have $L(H) = L(G[E(H)]) \in \mathcal{L}(G)$. If $J \in \mathcal{L}(G)$ then $V(J) \subseteq E(G)$ and so the edge-induced subgraph $G[V(J)] \in \mathcal{H}(G)$ satisfying $L(G[V(J)]) = J$. Thus we may view $L : \mathcal{H}(G) \to \mathcal{L}(G)$ as a bijective mapping and let L^{-1} denote the inverse mapping of L. By the definition of iterated line graphs, if $s \geq 1$ is an integer, then we denote L^s to be the mapping that maps subgraphs in $\mathcal{H}(G)$ into subgraphs in $L^s(G)$, and we use L^{-s} to denote the pull back mapping that sends induced subgraphs in $L^s(G)$ to back to subgraphs in $\mathcal{H}(G)$. For notational convenience, If j and k are nonnegative integers, then we also use L^j to denote the corresponding mapping from $\mathcal{H}(L^k(G))$ to $\mathcal{L}(L^{k+j}(G))$, and and L^{-j} its corresponding pull back mapping. Using the notation thus defined, we summarize some observations from the definition of line graphs in the following proposition.

Proposition 3.2.8 *Let $G \in \mathcal{G}$ be a connected graph and let $L : \mathcal{H}(G) \to \mathcal{L}(G)$ denote the bijection mapping defined above. For each edge $e \in E(G)$, define*

$v(e) = L(e)$. Each of the following holds.
(i) For each edge $e \in E(G)$, the vertex v_e is a cut vertex of $L(G)$ if and only if $\{e\}$ is an essential edge-cut of G.
(ii) Let $e_1, e_2 \in E(G)$. Then if $v_{e_1}v_{e_2}$ is an edge in $E(L(G))$ not lying in a complete graph of order at least 3 in $L(G)$, then $G[\{e_1, e_2\}]$ is a divalent path of G.
(iii) Let P be a divalent path in G with $|E(P)| = h > 0$. For any integer k with $0 \leq k < h$, $L^k(P)$ is a divalent path in $L^k(G)$ with $|E(L^k(P))| = h - k$, and $L^h(P)$ is a vertex of $L^h(G)$. Furthermore, if P is a bridge divalent path of G, then $L^k(P)$ is also a bridge divalent path in $L^k(G)$, and $L^h(P)$ is a cut vertex of $L^h(G)$.
(iv) Let s and t be integers with $s \geq t \geq 2$. If v is a cut vertex of $L^s(G)$, then $L^{-t}(v)$ is a bridge divalent path of length t in $L^{s-t}(G)$ in which every edge is an essential cut edge; Likewise, if e is an edge which is not in a complete subgraph of order at least 3 in $L^s(G)$, then $L^{-t}(e)$ is a divalent path of length $t+1$ in $L^{s-t}(G)$.
(v) Let $e', e'' \in E(G)$ be distinct edges and let $s \geq 1$ be an integer. If $L(G)$ has an $(s;, e', e'')$-path-system, then G has an $(s; e', e'')$-trail-system.

Proof. Proposition 3.2.8(i), (ii) and (iii) follow from the definitions of line graphs, of divalent paths and of bridge divalent paths. To prove (iv), let $v \in V(L^s(G))$ be a vertex. Then there must be an edge $e \in E(L^{s-1}(G))$ such that $v = v_e := L(e)$, or $e = L^{-1}(v)$. Since $G \in \mathcal{G}$, it follows by the definition of line graphs that $L^{s-1}(G) \in \mathcal{G}$, and so $L^s(G) - v$ has at least two components, which implies that e is a bridge divalent path of length 1 in $L^{s-1}(G)$ and $\{e\}$ is an essential edge-cut of $L^{s-1}(G)$; and $L^{-2}(v)$ is a bridge divalent path of length 2 in $L^{s-2}(G)$ in which every edge is an essential cut edge of $L^{s-2}(G)$. Inductively, for $t \geq 2$, $L^{-t}(v)$ is a bridge divalent path of length t in $L^{s-t}(G)$ in which every edge is an essential cut edge of $L^{s-t}(G)$. The proof for the edge part is similar and so it is omitted.

We are to prove (v). Let H be an $(s;, e', e'')$-path-system consisting of s internally disjoint (e', e'')-paths $P_1, P_2, ..., P_s$. Choose such an $(s; e', e'')$-path-system H. For each i with $1 \leq i \leq s$, as $V(P_i) \subseteq E(G)$, $G[V(P_i)])$ is an edge-induced connected subgraph in G containing both edges e' and e'', and so $G[V(P_i)])$ contains an (e', e'')-trail T_i. Since $P_1, P_2, ..., P_s$ are internally disjoint in $L(G)$, we conclude that $P_1, P_2, ..., P_s$ are internally edge-disjoint in G, and so G has an $(s; e', e'')$-trail-system. ∎

3.3 Proofs of the main results

The **symmetric difference** of two sets X and Y, is

$$X \triangle Y = X \cup Y - (X \cap Y).$$

Let G be a connected graph and $k > 0$ be an integer. An edge-cut X of G is an **essential k-edge-cut** of G if $|X| = k$ and each side of $G - X$ has an edge. The **essential edge-connectivity** of a connected graph G, denoted by $ess'(G)$, is the smallest integer k such that G has an essential k-edge-cut, if G has at least one essential edge cut; or $ess'(G) = |E(G)| - 1$, if G does not have an essential edge cut. We say that G is essentially k-edge-connected if $ess'(G) \geq k$. By the definition of a line graph, we observe that

$$\kappa(L(G)) \geq k \text{ if and only if } ess'(G) \geq k. \tag{3.4}$$

3.3.1 Maximally spanning connectedness in locally connected line graphs

We start with some preliminary results to understand the impact of local connectedness of $L(G)$ on the graph G. For a vertex $v \in V(G)$, define $E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}$.

Lemma 3.3.1 *Let G be a connected graph with $|E(G)| \geq 3$. The following are equivalent.*
(i) $L(G)$ is locally connected.
(ii) Every edge $e = uv \in E(G)$ with $\min\{d_G(u), d_G(v)\} \geq 2$ lies in a short cycle C_e of G.

Proof. Assume (i). Let $e = uv \in E(G)$ be an edge with $\min\{d_G(u), d_G(v)\} \geq 2$, which is not lying in a cycle of length 2. By the definition of a line graph, $N_{L(G)}(e) = (E_G(u) \cup E_G(v)) - \{e\}$. Since $\min\{d_G(u), d_G(v)\} \geq 2$, each of $E_G(u)$ and $E_G(v)$ is not empty. Since $L(G)[N_{L(G)}(e)]$ is connected, there must be an edge $e_u \in E_G(u)$ and $e_v \in E_G(v)$ such that $e_u e_v \in E(L(G))$. It follows that e_u and e_v would share a common vertex in G, and so $C_e = G[\{e, e_u, e_v\}]$ is a 3-cycle in G that contains e. Thus (ii) must hold.

Conversely, we assume that (ii) holds. Let $e \in V(L(G))$ be given. We shall show that e is a locally connected vertex in $L(G)$. By symmetry, we assume that $e = uv \in E(G)$ with $|N_G(u)| \geq |N_G(v)|$. If $|N_G(v)| = 1$, then $N_{L(G)}(e) = E_G(u) - \{e\}$, and so $L(G)[N_{L(G)}(e)]$ is a complete graph. Assume that $|N_G(v)| \geq 2$. By definition, $N_{L(G)}(e) = E_G(u) \cup E(G(v) - \{e\}$, and so $N_{L(G)}(e)$ is spanned by two complete subgraphs $L(G)[E_G(u) - \{e\}]$ and $L(G)[E_G(v) - \{e\}]$. By (ii), e lies in a short cycle C_e of G. If $E(C_e) = \{e, e_1\}$, then $e_1 \in (E_G(u) \cap E(G(v)) - \{e\}$, and so $L(G)[N_{L(G)}(e)]$ is connected. Now assume that $E(C_e - e) = \{e_1, e_2\}$. We may assume that $e_1 \in E_G(u)$ and $e_2 \in E_G(v)$. Since C_e is a 3-cycle, e_1 and e_2 are incident with a common vertex in G, and so in $L(G)$, $e_1 e_2 \in E(L(G))$. This implies that in any case, (i) must hold. ∎

In view of Lemma 3.3.1, we define a graph G to be **almost triangular** if every edge $e = uv \in E(G)$ with $\min\{d_G(u), d_G(v)\} \geq 2$ lies in a short cycle in G. A subgraph H is **near spanning** in G if $V(G) - D_1(G) = V(H)$. The next lemma is useful.

Lemma 3.3.2 *Let $s \geq 1$ be an integer and G be a connected almost triangular graph with $ess'(G) \geq 3$. For any $e', e'' \in E(G)$, if G has an $(s;, e', e'')$-trail system, then G has a near spanning and dominating $(s; e', e'')$-trail system.*

Proof. Suppose that G has an $(s;, e', e'')$-trail system. Choose an $(s;, e', e'')$-trail system J of G such that

$$|V(J)| + |E(J)| \text{ is maximized, among all } (s;, e', e'')\text{-trail systems of } G, \tag{3.5}$$

and subject to (3.5),

$$|\partial_G(J)| \text{ is as large as possible.} \tag{3.6}$$

Let $J = (T_1, T_2, ..., T_s)$, where each T_i is an (e', e'')-trail, $1 \leq i \leq s$.

Claim 2 $V(J) = V(G) - D_1(G)$.

By contradiction, and as G is connected, we assume that there must be a vertex $v \in V(G) - (V(J) \cup D_1(G))$ such that for some $w_v \in V(J)$, $vw_v \in E(G)$. Assume further that there exists a vertex $v \in V(G) - V(J)$ such that for some i with $1 \leq i \leq s$, w_v is an internal vertex T_i. As $v \notin D_1(G)$ and w_v is an internal vertex of J, we have $\min\{d_G(v), d_G(w_v)\} \geq 2$. Since G is almost triangular, there must be a short cycle C_{vw} with $vw_v \in E(C_{vw})$. Since $v \notin V(J)$, both edges incident with v in C_{vw} are not in J. If $|E(C_{vw})| = 2$, then T_i can be extended to $G[E(T_i) \cup E(C_{vw})]$, which is also an (e', e'')-trail, internally edge-disjoint from the other (e', e'')-trail in J, contrary to (3.5). Hence we assume that $|E(C_{vw})| = 3$.

Let e_v denote the edge in C_{vw} that is not incident with v. We assume that if $e_v \in E(J)$, (including the case when $e_v \in \{e', e''\}$), then $e_v \in E(T_i)$. Define

$$T_i' = \begin{cases} G[E(T_i) \triangle E(C_{vw})] & \text{if } e_v \notin \{e', e''\} \\ G[E(T_i) \cup E(C_{vw})] & \text{if } e_v \in \{e', e''\} \end{cases}.$$

Then T_i' is also an (e', e'')-trail. As $v \notin V(J)$, T_i' is also internally edge-disjoint from T_j, where $1 \leq j \leq s$ and $j \neq i$. It follows that $J' = (T_1, ..., T_{i-1}, T_i', T_{i+1}, ..., T_s)$ is an $(s; e', e'')$-trail system with $|V(J)| + |E(J)| < |V(J')| + |E(J')|$, contrary (3.5). Hence we assume that no vertex in $V(G) - (D_1(G) \cup V(J))$ is incident with an internal vertex of any T_i. This implies that any path connecting a vertex in $V(J)$ and a vertex in $V(G) - V(J)$ must use at least one of the edges

$\{e', e''\}$. This implies that $\{e', e''\}$ is an essential edge cut of G, contrary to the assumption that $ess'(G) \geq 3$. This justifies Claim 2.

By Claim 1, J is near spanning. If J is also dominating, then done. Hence we assume that

there exists an edge $e_0 \in E(G) - E(J)$ not incident with any internal vertex of J. (3.7)

Suppose that $e_0 = u_0v_0$ is incident with $v_0 \in D_1(G)$. By $ess'(G) \geq 3$, $d_G(u_0) \geq 4$. By (3.7), u_0 cannot be an internal vertex of J. Hence we may assume that $e' = u'v'$ with $u_0 = u'$ not being an internal vertex of J. This implies that v' must be an internal vertex of J. It follows that we have $\min\{d_G(u_0), d_G(v')\} \geq 2$. Since G is almost triangular, G has a short cycle C_0 containing $e' = u_0v$. If $E(C_0) - E_G(u_0)$ has an edge in $E(J)$, then we may assume that this edge is in T_1. Define $T_1'' = G[(E(T_1) \triangle E(C_0)) \cup \{e'\}]$. Thus $J'' = (T_1'', T_2, ..., T_s)$ is also an $(s; e', e'')$-trail system of G with $|V(J)| + |E(J)| \leq |V(J'')| + |E(J'')|$ and $\partial_G(J) \cup \{e_0\} \subseteq \partial_G(J'')$, contrary to (3.6).

Hence e_0 is not incident with a vertex of degree 1 in G. By (3.7), we may assume that $e' = u'v'$, $e'' = u''v''$, $e_0 = u'u''$ and for any i with $1 \leq i \leq s$, T_i is an (u', u'')-trial with the first edge being e' and the last edge being e''. Since G is triangular, $e' = u'v'$ lies in a short cycle $C_{e'}$ of G. Let e_1' denote the edge in $C_{e'} - \{e'\}$ that is incident with u'. If $e_1' \in E(J)$, then u' is an internal vertex of J, contrary to (3.7). Hence $e_1' \notin E(J)$. By definition, $|E(C_{e'})| \in \{2, 3\}$. When $E(C_{e'}) = \{e', e_1', e_2'\}$, we assume by symmetry that, if $e_2' \in E(J)$, then $e_2' \in E(T_1)$. With this assumption, define $T_1''' = G[(E(T_1) \triangle E(C_{e'})) \cup \{e'\}]$. Thus $J''' = (T_1''', T_2, ..., T_s)$ is also an $(s; e', e'')$-trail system of G with $|V(J)| + |E(J)| \leq |V(J''')| + |E(J''')|$ and $\partial_G(J) \cup \{e_0\} \subseteq \partial_G(J'')$, contrary to (3.6). Thus every possibility of the assumption (3.7) always leads to a contradiction, and so J must be dominating. This completes the proof of the lemma. ∎

Lemma 3.3.3 *Let $k \geq 1$ be an integer and G be a graph.*
(i) Let $e', e'' \in E(G)$. If $L(G)$ has a $(k; e', e'')$-trial system, then G has a $(k; e', e'')$-trail system.
(ii) Suppose that G is a connected almost triangular graph with $ess'(G) \geq 3$. Then $L(G)$ is maximally spanning connected.

Proof. For any $e', e'' \in E(G)$, assume that $L(G)$ has a $(k; e', e'')$-trial system H consisting of internally edge-disjoint (e', e'')-trails $P_1, P_2, ..., P_k$ with $|E(H)|$ minimized. Then by the minimality of $|E(H)|$, each P_i in H must be a path. By Proposition 3.2.8(v), G has an $(k; e', e'')$-trail system. This proves (i).

Let $\kappa(L(G)) = k$. As $ess'(G) \geq 3$, by (3.4), $k \geq 3$. Thus for every integer s with $1 \leq s \leq k$, G is s-connected. By Menger theorem, for any $e', e'' \in E(G)$,

$L(G)$ has an $(s; e', e'')$-path system. By Lemma 3.3.3(i), G has an $(s; e', e'')$-trail system.

Fix an integer s with $1 \leq s \leq k$, and let e', e'' be arbitrarily chosen edges in G. Since G is a connected almost triangular graph with $ess'(G) \geq 3$, and since G has an $(s; e', e'')$-trail system, it follows by Lemma 3.3.2 that G must also have a dominating $(s; e', e'')$-trail system. By Theorem 3.2.6, $L(G)$ has a spanning $(s; e', e'')$-path system. As e', e'' are arbitrarily chosen, it follows from the definition of spanning connectivity that $k = \kappa(L(G)) \geq \kappa^*(L(G)) \geq k$, completing the proof for (ii). ∎

Proof of Theorem 3.2.2(i). Since $L(G)$ is locally connected, by Lemma 3.3.1, G is almost triangular. Since $\kappa(L(G)) \geq 3$, it follows from (3.4) that G must be essentially 3-edge-connected. By Lemma 3.3.3(ii), $L(G)$ is maximally connected. This proves Theorem 3.2.2(i). ∎

3.3.2 The spanning connected indices of a graphs

The main purpose of this subsection is to prove Theorems 3.2.4 and 3.2.5. Before proving these theorems, we present the following examples, which are useful to illustrate the process determining the graphs in Theorems 3.2.4(i) and 3.2.5(i) that reach the upper bounds. (See Figure 1 for an illustration of the iterated line graphs of a bridge divalent $(3, t)$-path with $\ell = 2$ in Example 3.3.4.)

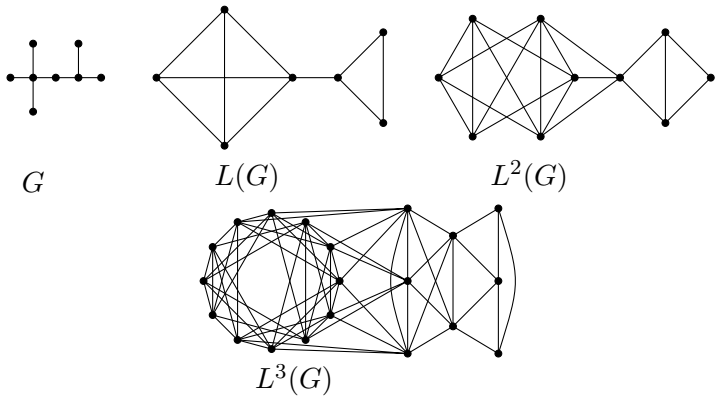

Figure 1.

Example 3.3.4 *Let d', d'' and ℓ be positive integers with $d'' \neq 2$, $d' \geq 3$ and $\ell \geq 2$; and $G \in \mathcal{G}$ be a graph with $\ell(G) = \ell$ that contains a bridge divalent (d', d'')-path $P = v_0e_1v_1 \cdots v_{\ell-1}e_\ell v_\ell$ with $d_G(v_0) = d'$ and $d_G(v_\ell = d''$. If $d' = 3$, then let $e'_1, e'_2, e_1 \in E(G)$ be the three edges incident with v_0 in G. It is routine to apply Proposition 3.2.8 to verify the following.*
(i) For any j with $0 \leq j \leq \ell - 1$, $L^j(G)$ has a bridge divalent (d', d'')-path.
(ii) If $d' = 3$ and $d'' \geq 3$, then $L^\ell(G)$ has a cut vertex which is incident with an essential edge cut of size 2.
(iii) If $d' = 3$ and $d'' \geq 3$, then $L^{\ell+1}(G)$ is triangular and has a vertex 2-cut.

(iv) If $d' = 3$ and $d'' \geq 3$, then $msc(G) \geq \ell + 2$.
(v) If G does not have a bridge divalent $(3,t)$-path for some integer $t \geq 3$, then $L^\ell(G)$ is essentially 3-edge-connected.

Proof. By the definition of line graphs, an edge e incident with a vertex of degree d lies in a maximal clique of order d in the line graph. Hence the edge incident with the vertex of degree s in a bridge divalent (s,t)-path of length at least 2 becomes a vertex in a bridge divalent path of degree s. By Proposition 3.2.8(iii), $L^j(G)$ has a bridge divalent (d', d'')-path. This justifies (i).

Assume that $d' = 3$ and $d'' \geq 3$. By (i), $L^{\ell-1}(G)$ has a bridge divalent $(3, d'')$-path of length 1, which is a cut edge $f_0 = w_1w_2$ in $L^{\ell-1}(G)$. Assume that the edges incident with w_1 in $L^{\ell-1}(G)$ are f_0, f_1, f_2. Since $d'' \geq 3$, f_0 is an essential cut edge. By Proposition 3.2.8(ii), in $L^\ell(G)$, f_0 is a cut vertex, and so $\{f_0f_1, f_0f_2\}$ is an essential edge cut in $L^\ell(G)$. This proves (ii).

As (iii) implies that $L^{\ell+1}(G)$ is not 3-connected, (iv) follows from (iii), and so it suffices to justify (iii). By Theorem 3.2.7(i) and (ii), $L^{\ell+1}(G)$ is triangular. By (ii), the essential edge cut of size 2 in $L^\ell(G)$ becomes a vertex 2-cut in $L^{\ell+1}(G)$. Hence (iii) must hold.

Now assume that G does not have a bridge divalent $(3,t)$-path for some integer $t \geq 3$. Let X be an essential edge cut of $L^\ell(G)$. By Theorem 3.2.7(i), $L^\ell(G)$ is triangular, and so $|X| \geq 2$. By contradiction, we assume that $X = \{f_1, f_2\}$ is an edge cut of $L^\ell(G)$. As $L^\ell(G)$ is triangular, f_1, f_2 must be incident with a common vertex w_0 in $L^\ell(G)$. Since $\{f_1, f_2\}$ is an essential edge cut of $L^\ell(G)$, w_0 must be a cut vertex of $L^\ell(G)$. By Proposition 3.2.8, for some integer $t \geq 3$, $L^{-\ell}(w_0)$ is a bridge divalent $(3,t)$-path of length ℓ in G, contrary to the assumption of (v). ∎

Lemma 3.3.5 *Let $G \in \mathcal{G}$ with $\ell = \ell(G)$. Then*

$$L^{\ell(G)+1}(G) \text{ is triangular and } \kappa'(L^{\ell(G)+1}(G)) \geq \kappa(L^{\ell(G)+2}(G)) \geq 3. \quad (3.8)$$

Proof. By Theorem 3.2.7 (i), $L^{\ell(G)}(G)$ is triangular. By definition, a connected triangular graph must also be 2-edge-connected. It follows by Theorem 3.2.7 (ii) that $L^{\ell(G)+1}(G)$ is triangular and $\kappa'(L^{\ell(G)+1}(G)) \geq \kappa(L^{\ell(G)+1}(G)) \geq 3$. ∎

For graphs that does not have bridge divalent $(3,t)$-path of length ℓ, a slightly stronger assertion can be stated.

Lemma 3.3.6 *Let $G \in \mathcal{G}$ be a connected graph and let $\ell = \ell(G) \geq 2$. Suppose that for any integer $t \geq 3$, G does not have a bridge divalent $(3,t)$-path of length ℓ. Each of the following holds.*
(i) $L^\ell(G)$ is triangular with $ess'(L^\ell(G)) \geq 3$.
(ii) For any integer $j \geq 1$, $L^{\ell+j}(G)$ is triangular with

$$\kappa^*(L^{\ell+j}(G)) = \kappa(L^{\ell+j}(G)) \geq j + 2.$$

Proof. (i) follows from Theorem 3.2.7(i) and Example 3.3.4(v). By Lemma 3.3.6(i), (3.4), and by Lemma 3.3.3(ii). Lemma 3.3.6(ii) holds when $j = 1$. Inductively, assume that (ii) holds for smaller values of j and $j \geq 2$. Then by induction, $L^{\ell+j-1}(G)$ is triangular with $\kappa^*(L^{\ell+j-1}(G)) = \kappa(L^{\ell+j-1}(G)) \geq (j-1)+2$. By Theorem 3.2.7(ii) and Lemma 3.3.3(ii), we conclude that $L^{\ell+j}(G)$ is triangular with $\kappa^*(L^{\ell+j}(G)) = \kappa(L^{\ell+j}(G)) \geq j+2$, and so (ii) follows by induction. This proves the lemma. ∎

Lemma 3.3.7 *Let $G \in \mathcal{G}$ with $\ell = \ell(G) \geq 2$. Each of the following holds.*
(i) $\ell \leq |V(G)| - \Delta$.
(ii) If G has a bridge divalent path P of length ℓ, then $\ell = |V(G)| - \Delta$ only if G has a unique bridge divalent $(\Delta, 1)$-path of length ℓ.
(iii) If for some integers $d', d'' \geq 3$, G has a bridge divalent (d', d'')-path P of length ℓ, then $\ell(G) \leq |V(G)| - \Delta - 2$.

Proof. Let $G \in \mathcal{G}$ be a graph with $\Delta = \Delta(G)$ and $\ell = \ell(G)$. Since $G \in \mathcal{G}$, we have $\Delta > 2$. Pick a vertex $w_0 \in V(G)$ with $d_G(w_0) = \Delta$ with $N_G(w_0) = \{u_1, u_2, ..., u_\Delta\}$. Let P be a longest divalent path in G. Thus $|E(P)| = \ell(G)$. Since P is a divalent path, $|E(P) \cap \{u_1w_0, u_2w_0, ..., u_\Delta w_0\}| \leq 2$. Hence at least $\Delta - 1$ vertices in $N_G(w_0) \cup \{w_0\}$ cannot be the internal vertices of P, which implies that $\ell = |E(P)| \leq |V(G)| - (\Delta - 1) + 1 = |V(G)| - \Delta$. This justifies Lemma 3.3.7(i).

To prove (ii) and (iii), we assume that

$$P = v_0v_1...v_{\ell-1}v_\ell \tag{3.9}$$

is a bridge divalent path of length ℓ in G. By symmetry, we assume that $d' = d_G(v_0) \geq d_G(v_\ell) = d''$. (We allow that in the proof for (ii), $d'' = 1$.) By definition of $\ell(G)$, $v_0, v_\ell \notin D_2(G)$. Assume that $d_G(v_0) < \Delta$ and w_0 is a vertex in $V(G)$ with $d_G(w_0) = \Delta$. Then since P is a divalent path, we observe that $w_0 \notin V(P)$, and so $N_G(w_0) \cap V(P) = \emptyset$. Hence $V(P) \subseteq V(G) - (N_G(w_0) \cup \{w_0\})$. It follows that

$$\begin{aligned} \ell &= |E(P)| \leq |V(G) - (N_G(w_0) \cup \{w_0\})| - 1 \qquad (3.10) \\ &= |V(G)| - (\Delta + 1) - 1 = |V(G)| - \Delta - 2. \end{aligned}$$

To complete the proof for (ii), we assume that $\ell = |V(G)| - \Delta$. By (3.10) we may assume that $d_G(v_0) = \Delta$. If $d_G(v_\ell) \neq 1$, then $d_G(v_\ell) \geq 3$. Since P is a bridge divalent path and $\ell \geq 2$, $|N_G(v_0) \cap N_G(v_\ell)| \leq 1$. Hence the vertices in $(N_G(v_0) \cup N_G(v_\ell)) - (N_G(v_0) \cap N_G(v_\ell))$ cannot be internal vertices of P. It follows that

$$\begin{aligned} \ell &= |E(P)| \leq |V(G)| - |(N_G(v_0) \cup N_G(v_\ell)) - (N_G(v_0) \cap N_G(v_\ell))| - 1 \qquad (3.11) \\ &\leq |V(G)| - (\Delta + 3 - 1) - 1 = |V(G)| - \Delta - 3, \qquad (3.12) \end{aligned}$$

a contradiction. This forces that $d_G(v_\ell) = 1$, and so P must be a divalent $(\Delta, 1)$-path.

To show that uniqueness in (ii), we assume that G has two bridge divalent $(\Delta, 1)$-paths P and P', each of length ℓ. As shown above, $|V(P) \cap V(P')| \leq 1$ and there is at most one vertex of degree Δ in $V(P) \cap V(P')$. Thus there are at most two internal vertices of P and P' incident with a vertex of a vertex of degree Δ in G, and so there are at least $\Delta - 1$ vertices in $N_G(w_0) \cup \{w_0\}$ that cannot be internal vertices of P or P'. It follows that the total number of internal vertices of P and P' is at most $|V(G)| - (\Delta - 1) - 2 = |V(G)| - \Delta - 1$. This implies that $2\ell = |E(P)| + |E(P')| \leq |V(G)| - \Delta - 1 + 2$. As $\ell = |V(G)| - \Delta$, this forces that $|V(G)| = \Delta + 1$, implying that G is spanned by a $K_{1,|V(G)|-1}$, contrary to the assumption that $\ell \geq 2$. Thus (ii) must hold.

To prove (iii), we assume that the path P in (3.9) is a bridge divalent (d', d'')-path P of length ℓ with $d \geq 3$. By contradiction assume that $\ell(G) > |V(G)| - \Delta - 2$. By (3.10), we must have $d_G(v_0) = \Delta$, and so $V(G) = N_G(v_0) \cup V(P) \cup N_G(v_\ell)$. It follows by $d'' \geq 3$ that $\ell = |E(P)| = |V(G) - (N_G(v_0) \cup N_G(v_\ell))| + 1 = |V(G)| - \Delta - 2$, contrary to the assumption that $\ell(G) > |V(G)| - \Delta - 2$. ∎

We are now ready to complete the proofs of the main results. For some technical reason, we first prove Theorem 3.2.5.

Proof of Theorem 3.2.5. By (3.8) and by Lemmas 3.3.3 and 3.3.2, we must have $\kappa(L^{\ell(G)+2}(G)) = \kappa^*((L^{\ell(G)+2}(G)))$. Thus $L^{\ell(G)+2}(G)$ is triangular and maximally spanning connected. For any integer $m > \ell(G) + 2$, assuming that $L^{m-1}(G)$ is 3-connected, triangular and maximally spanning connected. By Theorem 3.2.7(ii), $L^m(G)$ is also 3-connected and triangular; and by Lemma 3.3.2, $L^m(G)$ is maximally spanning connected. It follows by induction that for any $m \geq \ell(G) + 2$, $L^m(G)$ is also maximally spanning connected.

By Example 3.3.4 (iv), if for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path, then $msc(G) \geq \ell(G) + 2$. This, together with the conclusions above, forces that $msc(G) = \ell(G) + 2$. Conversely, we assume that $G \in \mathcal{G}$ satisfies $msc(G) = \ell(G) + 2$. If G does not have a bridge divalent $(3, t)$-path for some integer $t \geq 3$, then by Example 3.3.4(v) and Theorem 3.2.7(i), $L^\ell(G)$ is essentially 3-edge-connected and triangular. Hence by Lemma 3.3.2, $L^{\ell+1}(G)$ is maximally spanning connected, contrary to the assumption of $msc(G) = \ell(G) + 2$. This completes the proof of Theorem 3.2.5(i).

If $\ell(G) = 1$, then $|V(G)| - \Delta(G) + 2 \geq 3$. By Theorem 3.2.5(i), for any $m \geq 3$, $L^m(G)$ is maximally spanning connected. Assume that $\ell(G) \geq 2$. By Lemma 3.3.7(i), $\ell(G) \leq |V(G| - \Delta(G) + 1$, and so by Theorem 3.2.5(i), we have

$$msc(G) \leq \ell(G) + 2 \leq |V(G)| - \Delta + 3. \tag{3.13}$$

If $G \in \mathcal{G}$ satisfying $msc(G) = |V(G)| - \Delta + 3$, then by (3.13), we must have $msc(G) = \ell(G) + 2$. It follows by Theorem 3.2.5(i) that G must have bridge

divalent path of length ℓ. By Lemma 3.3.7(ii), G has a unique bridge divalent $(\Delta, 1)$-path of length ℓ. By Theorem 3.2.7(i), $L^{\ell}(G)$ is triangular. By Example 3.3.4(v), $L^{\ell}(G)$ is essentially 3-edge-connected. It follows from Lemma 3.3.2 and Theorem 3.2.6 that $L^{\ell+1}(G)$ is maximally spanning connected. This contradicts to the assumption of $msc(G) = |V(G)| - \Delta + 3$. Hence, for any $G \in \mathcal{G}$, we must have $msc(G) \leq |V(G)| - \Delta + 2$. By (3.8), Lemmas 3.3.3 and 3.3.2, it is routine to show that for any $m \geq |V(G)| - \Delta + 2$, $L^m(G)$ is maximally spanning connected. This completes the proof of Theorem 3.2.5. ∎

Proof of Theorem 3.2.4. Let $\ell = \ell(G)$. Assume first that $k \in \{2, 3\}$. By Theorem 3.2.5(i), $L^{\ell+2}(G)$ is maximally spanning connected. By Lemma 3.3.5, $\kappa^*(L^{\ell+2}(G)) \geq 3$. Thus $s_2(G) \leq s_3(G) \leq \ell + 2$. This proves Theorem 3.2.4(i).

Let $k \geq 3$ be an integer and let $m(k) = \ell + k - 1 \geq \ell + 2$. By Theorem 3.2.5(i), $L^{m(k)}(G)$ is maximally spanning connected. This, together with Theorem 3.2.7(iii), implies $\kappa^*(L^{m(k)}(G)) = \kappa(L^{m(k)}(G)) \geq k$. This shows that $s_k(G) \leq m(k) = \ell + k - 1$. Suppose that for any integer $t \geq 3$, G does not have a bridge divalent $(3, t)$-path of length ℓ. By Lemma 3.3.6(ii) with $j = k - 2$, we conclude that if for any integer $t \geq 3$, G does not have a bridge divalent $(3, t)$-path of length ℓ, then $s_k(G) \leq \ell + k - 2$. This completes the proof of Theorem 3.2.4(ii).

To prove (iii), by Theorem 3.2.4(ii) with $k = 3$, we assume that for some integer $t \geq 3$, G has a bridge divalent $(3, t)$-path. By Example 3.3.4(iii), $\kappa(L^{\ell+1}(G)) < 3$, and so $s_3(G) \geq \ell(G) + 2$. This implies that in this case, we must have $s_3(G) = \ell(G) + 2$. This completes the proof of Theorem 3.2.4(iii).

Let $k \geq 3$ be an integer. If G has a bridge divalent $(3, t)$-path P of length ℓ for some integer $t \geq 3$, then by Lemma 3.3.7(iii), $\ell(G) \leq |V(G)| - \Delta - 2$. By Theorem 3.2.4(ii), $s_k(G) \leq |V(G)| - \Delta + k - 3$. Theorem 3.2.4(iv) follows in this case.

Suppose that for any integer $t \geq 3$, G does not have a bridge divalent $(3, t)$-path. If every bridge divalent (d', d'')-path P of length ℓ satisfies $\min\{d', d''\} = 1$, then as the degree 1 vertex cannot be an internal vertex of P, there are at least $\Delta + 2$ vertices in G that are not internal vertices of P. It follows that $\ell \leq |V(G)| - \Delta - 1$. If every divalent (d', d'')-path P of length ℓ satisfies $\min\{d', d''\} \geq 4$, then either P is a bridge divalent path, whence $\ell = |E(P)| \leq |V(G)| - \Delta$

Also by Lemma 3.3.6(ii) with $j = k-2$, we conclude that $s_k(G) \leq \ell + k - 2 \leq |V(G)| - \Delta - 1 + k - 2 = |V(G)| - \Delta + k - 3$. Thus Theorem 3.2.4(iv) follows in this case also.

Hence we assume that G has a bridge divalent path of length ℓ, and every bridge divalent path P of length ℓ is a (d', d'')-path with $\min\{d', d''\} \geq 4$. By Example 3.3.4(iv) and Theorem 3.2.7(i), $L^{\ell}(G)$ is triangular with $ess'(G) \geq 3$.

∎

3.4 Concluding remarks

The research has found a new family of maximally spanning connected line graphs and the tools developed in this research have also improved some of the former results. The existence of other natural and commonly studied graph families that are also maximally spanning connected would be of interests. Motivated by Conjecture 3, we present the following problems for future researches.

Open Problem 1 *Let G be a connected graph and $s \geq 2$ be an integer.*
(i) Determine the existence of, and if exists, the smallest value of an integer $f(s)$, such that every $f(s)$-connected line graph is s-spanning connected.
(i) Determine the existence of, and if exists, the smallest value of an integer $h(s)$, such that every $h(s)$-connected claw-free graph is s-spanning connected.

As every line graph is a claw-free graph, we have $h(s) \geq f(s)$ if they exist. As stated in Conjecture 3, Thomassen [143] and, Kučzel and Xiong [128] conjecture that $f(2) = 4$, and Matthews and Sumner [135], and Ryjáček and Vrána [138] conjectured that $h(2) = 4$ also. Furthermore, Ryjáček and Vrána [138] proved that $f(2) = 4$ is equivalent to $h(2) = 4$. We conjecture that these values $f(s)$ and $h(s)$ exist for all $s \geq 2$, and Theorem 3.2.2 supports the conjecture that $f(s)$ exists.

Chapter 4

On (s,t)-supereulerian graphs with linear degree bounds

4.1 Main result

Theorem 4.1.1 *For any nonnegative integers s and t, and any real numbers a and b with $0 < a < 1$, there exists a family of finitely many graphs $\mathcal{F}(a,b;s,t)$ such that if G is a simple graph on n vertices with $\kappa'(G) \geq t+2$ and $\delta(G) \geq an+b$, then one of the following must hold.*
(i) G is (s,t)-supereulerian.
(ii) G is contractible to a member in $\mathcal{F}(a,b;s,t)$.

Let n, s, t and m be positive integers with $n = 2m \geq s+t$. Define G to be the graph from a disjoint union of two graphs G_1 and G_2, with $G_1 \cong G_2 \cong K_m$, and by adding a set W of $s+t-1$ new edges linking vertices in G_1 to vertices in G_2. Then $\delta(G) = \frac{n}{2} - 1$. Choose a subset $X \subset W$ satisfying $1 < |X| \leq s$, $|W - X| \leq t$ and $|X| \equiv 1 \pmod 2$. As $|X| \equiv 1 \pmod 2$, $G - (W - X)$ cannot have a spanning closed trail containing X. This example indicates that the bound in the next result is best possible in some sense.

Theorem 4.1.2 *Let s and t be two nonnegative integers. If G is a simple graph on n vertices with $\kappa'(G) \geq t+2$ and $\delta(G) \geq \frac{n}{2} - 1$, then when n is sufficiently large, one of the following must hold.*
(i) G is (s,t)-supereulerian.
(ii) For some integer j with $t+2 \leq j \leq s+t$, G is contractible to a jK_2.

In the next section, we summarize former results and develop needed tools in our arguments to prove the main results. The main results will be validated in the last section.

4.2 Mechanisms

For a graph G, let $\tau(G)$ be the maximum number of edge-disjoint spanning trees in G, and $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with two edge-disjoint spanning tree. Thus $\tau(G) \geq 2$ if and only if $F(G) = 0$. Theorem 4.2.1 below presents useful properties related to collapsible graphs.

Theorem 4.2.1 *Let G be a graph and let H be a collapsible subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*
(i) (Catlin, Theorem 3 of [9]) G is collapsible (or supereulerian, respectively) if and only if G/H is collapsible (or supereulerian, respectively). In particular, G is collapsible if and only if the reduction of G is K_1.
(ii) (Catlin, Theorem 5 of [9]) A graph is reduced if and only if it does not have a nontrivial collapsible subgraph.
(iii) (Catlin [9]) Cycles of length at most 3 are collapsible.
(iv) (Catlin [9]) The contraction of a collapsible graph is collapsible.
(v) Let $X \subseteq E(G)$ be an edge subset of G. If $G - X$ is collapsible, then G has a spanning eulerian subgraph H with $X \subseteq E(H)$.

Proof. It remains to prove (v). Let $R = O(G[X])$. Then $R \subseteq V(G)$, and $|R| \equiv 0 \pmod 2$. Since $G - X$ is collapsible, $G - X$ has a spanning connected subgraph H_R with $O(H_R) = R$. It follows that $H = G[E(H_R \cup X)]$ is a spanning eulerian subgraph of G with $X \subseteq E(H)$. ■Theorem 4.2.2(iii) below can be obtained by applying Theorem 1.4 of [17] to maximal 2-connected subgraph of G.

Theorem 4.2.2 *Let G be a connected graph. Each of the following holds.*
(i) (Catlin, Theorem 7 of [9]) If $F(G) \leq 1$, then G is collapsible if and only if $\kappa'(G) \geq 2$. In particular, every graph G with $\tau(G) \geq 2$ is collapsible.
(ii) (Catlin et al, Theorem 1.3 of [17]) If $F(G) \leq 2$, then either G is collapsible or its reduction is a member in $\{K_2, K_{2,t} : t \geq 1\}$.
(iii) (Catlin et al, Theorem 1.4 of [17]) If $F(G) \leq 2$ and $\kappa'(G) \geq 3$, then G is collapsible.
(iv) (Catlin et al, Lemma 2.3 of [17]) If G is a reduced graph with $|V(G)| \geq 2$, then $F(G) = 2|V(G)| - |E(G)| - 2$.

As $F(G) = 0$ amounts to $\tau(G) \geq 2$, utilizing the spanning tree packing theorem of Nash-Williams [32] and Tutte [36], the following is obtained.

Theorem 4.2.3 *(Catlin et al, Theorems 1.1 and 1.3 of [91]). Let G be a graph, $\epsilon \in \{0, 1\}$ and $\ell \geq 1$ be integers. The following are equivalent:*

(i) G is $(2\ell+\epsilon)$-edge-connected;
(ii) For any $X \subseteq E(G)$ with $|X| \leq \ell+\epsilon$, $\tau(G-X) \geq \ell$.

Theorem 4.2.3 has a seemingly more general corollary, as stated below.

Corollary 4.2.4 *(Xiong et al. [34]) Let G be a connected graph, and ϵ, k, ℓ be integers with $\epsilon \in \{0,1\}$, $\ell \geq 2$ and $2 \leq k \leq \ell$. The following are equivalent.*
(i) $\kappa'(G) \geqslant 2\ell+\epsilon$.
(ii) For any $X \subseteq E(G)$ with $|X| \leq 2\ell-k+\epsilon$, $\tau(G-X) \geq k$.

An **elementary subdivision** of an $e=uv \in E(G)$ is the operation to form a new graph $G(e)$ from $G-e$ by adding a path uv_ev with v_e being a new vertex in $G(e)$. If $X \subseteq E(G)$ is an edge subset, then $G(X)$ denotes the resulting graph formed by elementarily subdividing each edge in X. Observation 5.3.1 below follows immediately from the definition.

Observation 4.2.5 *For an edge subset $X \subseteq E(G)$, let $V_X = \{v_e : e \in X\}$, $E_X = \{uv_e, v_ev : e = uv \in X\}$ and $E'_X = \{v_ev : e = uv \in X\}$. Each of the following holds.*
(i) $V_X = V(J) - V(G)$ and $E_X = E(G(X)) - E(G)$.
(ii) There exists a bijection between X and $\{v_eu : e \in X\}$ and so $G(X)/E'_X \cong G$.
(iii) For any 2-edge-connected subgraph H' of $G(X)$, and for any $e=uv \in X$, if $v_e \in V(H')$, then both $v_eu, v_ev \in E(H')$; and if $\{uv_e, vv_e\} \cap E(H') \neq \emptyset$, then $\{uv_e, vv_e\} \subset E(H')$. Thus in view of Observation 5.3.1(ii), $H = H'/(E'_X \cap E(H'))$ is a subgraph of G, called the **restoration** *of H' in G.*
(iv) G has a spanning eulerian subgraph H with $X \subseteq E(H)$ and $Y \cap E(H) = \emptyset$ if and only if $(G-Y)(X)$ is supereulerian.

Chen, Chen and Luo (Theorem 4.1 of [23]) prove that if $\kappa'(G) \geq 4$, $t \leq \frac{\kappa'(G)}{2}$ and $s+t+1 \leq \kappa'(G)$, then G is (s,t)-supereulerian. Proposition 4.2.6(ii) below extends this result when $\kappa'(G) \geq 5$.

Proposition 4.2.6 *Let s, t be nonnegative integers and let G be a graph. Each of the following holds.*
(i) If G is a (s,t)-supereulerian graph, then any contraction of G is also (s,t)-supereulerian graph.
(ii) Suppose that H is a graph with $\kappa'(H) \geq \max\{s+t+1, t+2, 5\}$. Then H is a (s,t)-supereulerian graph.
(iii) If $H = \ell K_2$ with $\ell \geq \max\{s+t+1, t+2, 4\}$, then G is (s,t)-supereulerian if and only if G/H is (s,t)-supereulerian.

Proof. Suppose that G is (s,t)-supereulerian and $e_0 \in E(G)$. Let $\Gamma = G/e_0$. To prove (i), it suffices to show that Γ is also (s,t)-supereulerian. Let $X, Y \subseteq$

$E(\Gamma)$ be arbitrary edge subsets with $X \cap Y = \emptyset$, $|X| \leq s$ and $|Y| \leq t$. As $E(\Gamma) \subseteq E(G)$, and since G is (s,t)-supereulerian, it follows from Observation 5.3.1(iv) that $(G-Y)(X)$ has a spanning eulerian subgraph J. As $e_0 \in E(G-(X \cup Y))$, let $J + e_0$ denote the subgraph of $(G-Y)(X)$ induced by $E(J) \cup \{e_0\}$. Since J is eulerian, it follows that $J' = (J+e_0)/e_0$ is also a connected graph without a vertex of odd degree, and so J' is a spanning eulerian subgraph of Γ. Hence (i) holds.

Assume that $\kappa'(H) \geq \max\{s+t+1, t+2, 5\}$. Let X, Y be disjoint edge subsets of H with $|X| \leq s$ and $|Y| \leq t$. By adding edges to X if needed, we assume that $|X| = s$. If $s + t \leq \kappa'(H) - 2$, then by Corollary 4.2.4 (with $k = 2$), $H - (X \cup Y)$ has two edge-disjoint spanning trees, and so by Theorem 4.2.1(i), $H - (X \cup Y)$ is collapsible. It follows from Theorem 4.2.1(iii) that $H - Y$ has a spanning eulerian subgraph containing X. Hence we assume that $s + t = \kappa'(H) - 1$, and so $s = \kappa'(H) - t - 1 \geq 1$. Let $W \subseteq X \cup Y$ with $|W| = 2$ and $|W \cap X| > 0$ such that if $s \geq 2$, then $W \subseteq X$; and let $Z = (X \cup Y) - W$. Hence $|Z| \leq s + t - 2$, and so $\kappa'(H - Z) \geq 3$. By Corollary 4.2.4, $\tau(H - Z) \geq 2$. It follows that $F((H-Z)(W)) \leq 2$. As $\kappa'(H-Z) \geq 3$, then only edge cuts of size 2 in $(H-Z)(W)$ are those of the form $\partial_{(H-Z)(W)}(v_e)$ for some $e \in W$. By Theorem 4.2.2(ii), either $(H-Z)(W)$ is collapsible or the reduction of $(H-Z)(W)$ is a $K_{2,|W|} = K_{2,2}$. As the latter case contradicts to the fact that $\kappa'(H-Z) \geq 3$, we conclude that $(H-Z)(W)$ is collapsible. By Theorem 4.2.1(v), $(H-Y)(W)$ has a spanning eulerian subgraph that contains $X - W$, and so $H - Y$ has a spanning eulerian subgraph that contains X. This proves (ii).

By (i), to prove (iii), it remains to show that G/H is (s,t)-supereulerian. Let $G' = G/H$ and let v_H denote the vertex in G' onto which H is contracted. By (ii), we may assume that H is not a spanning subgraph of G, and so G' is nontrivial. Let X, Y be disjoint edge subsets of G with $|X| \leq s$ and $|Y| \leq t$. Define $X' = X - E(H)$, $X'' = X \cap E(H)$, $Y' = Y - E(H)$, and $Y'' = Y \cap E(H)$. Then $|X'| \leq s$ and $|Y'| \leq t$. Since G' is a nontrivial (s,t)-supereulerian graph, it follows by Observation 5.3.1(iv) that $(G'-Y')(X')$ contains a spanning eulerian subgraph L'.

We need to extend L' to a spanning eulerian subgraph of $(G-Y)(X)$. Let $G'' = (G-Y)(X)$ and $H'' = (H-Y'')(X'')$. Then as $E(L') \cap Y'' = \emptyset$, by their definitions, both $E(L') \subseteq E((G'-Y')(X')) \subseteq E(G'')$ and H'' is a subgraph of G''. It follows that

$$\begin{aligned}(G'-Y')(X') = (G/H - Y')(X') &= (G-Y')(X')/H \\ &= (G-Y)(X)/[(H-Y'')(X'')] = G''/H''.\end{aligned}$$

Since $H = \ell K_2$ with $\ell \geq \max\{s+t+1, t+2, 4\}$, and since $|X''| \leq s$ and $|Y''| \leq t$, H'' is a graph in which every edge lies in a cycle of length at most 3,

and so by Theorem 4.2.1(i) and (iii), H'' is collapsible. Let $R = O(G''[E(L')])$. Then $|R| \equiv 0 \pmod 2$. As L' is an eulerian subgraph of $(G' - Y')(X') = (G - Y)(X)/H = G''/H''$, we have $R \subseteq V(H'')$. Since H'' is collapsible, H'' has a spanning connected subgraph L'' with $O(L'') = R$. It follows that $G''[E(L') \cup E(L'')]$ is a spanning eulerian subgraph of $G'' = (G - Y)(X)$. By definition, G is (s,t)-supereulerian. ∎

For given non negative integers s and t, let $\mathcal{L}_{s,t}$ denote the family of all (s,t)-supereulerian graphs. By definition, $K_1 \in \mathcal{L}_{s,t}$. A graph H is a **contractible configuration** of $\mathcal{L}_{s,t}$ (or (s,t)**-contractible**, in short), if for any graph G containing H as a subgraph, the following always holds:

$$G \in \mathcal{L}_{s,t} \text{ if and only if } G/H \in \mathcal{L}_{s,t}.$$

Proposition 4.2.6 indicates that $\mathcal{L}_{s,t}$ is closed under taking contraction, and, if $\ell \geq \max\{s+t+1, t+2, 4\}$, then ℓK_2 is a contractible configuration of $\mathcal{L}_{s,t}$. A a graph Γ is (s,t)**-reduced** if Γ does not contain any nontrivial subgraph that is a contractible configuration of $\mathcal{L}_{s,t}$. For a graph G, the (s,t)**-reduction** of G, is the graph Γ formed from G by contracting all maximal (s,t)-contractible subgraphs of G. By definition, if Γ is the (s,t)-reduction of G, then

$$G \in \mathcal{L}_{s,t} \text{ if and only if } \Gamma \in \mathcal{L}_{s,t}. \tag{4.1}$$

For a graph G, the (collapsible) reduction of G and the (s,t)-reduction of G may not be the same. To describe the relationship between the two, we need a few more terms.

Definition 4 *Let s and t be nonnegative integers, G be a graph, X and Y be disjoint edge subsets of G with $|X| \leq s$ and $|Y| \leq t$, and let $J = (G - Y)(X)$ and J' be the reduction of J. For any vertex $z \in V(J')$, let H'_z denote the preimages of z in J, and let H_z be the restorations of H'_z in $G - Y$. Define*

$$\begin{aligned} M &= G[\bigcup_{z \in V(J')} E(H_z)], \\ M' &= J[\bigcup_{z \in V(J')} E(H'_z)], \\ X' &= X \cap E(M') \text{ and } J'' = (G - Y)(X')/M'. \end{aligned}$$

Define $Y' = \{uv \in Y : \text{there exists a graph } L \in \{H_z : z \in V(J')\} \text{ such that } u, v \in V(L)\}$, and $Y'' = Y - Y'$.

The following lemma describes a relationship between the (collapsible) reduction of G and the (s,t)-reduction of G, and will be needed in our arguments.

Lemma 4.2.7 *We adopt the notation in Definition 4 and let $X'' = X - X'$. Each of the following holds.*

(i) $X'' \subseteq E(J'')$ *and* $J'' = (G-Y)(X')/M' = (G-Y'')(X')/M'$.
(ii) $J' = J''(X'') = ((G-Y'')/M)(X'')$.
(iii) If J *is not supereulerian, then* G *can be contracted to an* (s,t)*-reduced and non* (s,t)*-supereulerian graph with order at most* $|V(J')|$.

Proof. Let G, J and J' be graphs defined as in Definition 4, for given edge subsets $X, Y \subseteq E(G)$ with $X \cap Y = \emptyset$, $|X| \leq s$ and $|Y| \leq t$.

Since J' is the reduction of $J = (G-Y)(X)$, for any vertex $z \in V(J')$, let H'_z denote the preimage of z in J, and let H_z be the restoration of H'_z in $G-Y$. Thus $V(G) = V(G-Y) = \cup_{z \in V(J')} V(H_z)$.

By Definition 4, $J'' = (G-Y)(X')/M' = (G-Y'')(X')/M'$. As $X' = X \cap E(M')$, we have $X'' \subseteq E(J'')$, and so (i) follows.

Fix an arbitrary vertex $z \in V(J')$. Since H'_z is collapsible, $\kappa'(H_z) \geq 2$, and so for any vertex $v \in V(H_z) \cap V_X$, both edges incident with v in J must also be in $E(H'_z)$. It follows from Theorem 4.2.1(iv) that H_z is a collapsible subgraph of G. By definition, $J' = J/M'$. Then by their definitions, the edges in Y' will become loops and be deleted in the process of contracting M'. It follows that $J' = J/M' = [(G-Y)(X)]/M' = [(G-Y'')(X)]/M' = [(G-Y'')(X')]/M'(X'') = J''(X'')$. By Definition 4, $J'' = (G-Y)(X')/M' = (G-Y'')/M$, and so $J' = J''(X'') = ((G-Y'')/M)(X'')$. This justifies (ii).

Since J is not supereulerian, it follows by Theorem 4.2.1(i) that J' is not supereulerian. By Lemma 4.2.7(i) and (ii), the graph

$$\begin{aligned}[(G/M)-Y''](X'') &= ((G-Y'')/M)(X'') \\ &= [(G-Y'')(X')]/M'(X'') = J''(X'') = J'\end{aligned}$$

is not supereulerian. Since $|X''| \leq |X| \leq s$ and $|Y''| \leq |Y| \leq t$, G/M is not (s,t)-supereulerian. Let Γ be the (s,t)-reduction of G/M. It follows by (4.1) that Γ is not (s,t)-supereulerian. By the equality we showed above, the restoration of J' is $G/M - Y''$ and so $|V(\Gamma)| \leq |V(G/M - Y'')| = |V(G/M)| \leq |V(J')|$. This completes the proof of the lemma. ∎

In [34], an edge-connectivity necessary condition for (s,t)-supereulerian graph has been found.

Proposition 4.2.8 *(Xiong et al. [34]) Let* s, t *be nonnegative integers. Define*

$$j_0(s,t) = \begin{cases} s+t+\frac{1-(-1)^s}{2} & \text{if } s \geq 1 \text{ and } s+t \geq 3, \\ t+2 & \text{otherwise.} \end{cases} \tag{4.2}$$

If a graph G *is* (s,t)*-supereulerian, then* $\kappa'(G) \geq j_0(s,t)$.

The next lemma is also useful.

Lemma 4.2.9 *(Liu et al. Lemma 3.1 of [31]) Let* G *be a simple graph with* $\delta = \delta(G)$*, and* $X \subseteq V(G)$ *be a subset. If* $|\partial_G(X)| < \delta$*, then* $|X| \geq \delta + 1$.

4.3 Proof of Theorem 4.1.1

Let a, b, s, t be given as in the statement of Theorem 4.1.1, $\ell = \max\{s + t + 1, t + 2, 5\}$, and

$$c = \max\left\{\frac{10a}{1+a} + 1, 4\right\}. \tag{4.3}$$

Define $N = N(a, b, s, t)$ by

$$N = \max\left\{\frac{1}{a} + s + 3, \frac{4-b}{a}, \frac{|b+1| - a(b+1)}{a^2}, \frac{c+t-b+1}{a}, \frac{(1+a)(c+1) - 10a}{a(c-3)}\right\}, \tag{4.4}$$

and define $\mathcal{F} = \mathcal{F}(a, b; s, t)$ to be the family of all (s, t)-reduced non (s, t)-supereulerian graphs of order at most N. By Proposition 4.2.6(iii), every graph G in $\mathcal{F}$ has multiplicity at most $\ell - 1$. Thus $\mathcal{F}$ is a family of finitely many graphs. In particular, by Proposition 4.2.8,

$$\{jK_2 : 1 \le j \le j_0 - 1\} \subset \mathcal{F}. \tag{4.5}$$

To prove Theorem 4.1.1, we argue by contradiction, and assume that there exists a counterexample graph G such that $n = |V(G)|$ is smallest in those counterexample to the theorem for which we have the following observations, stated as Claim 3 below.

Claim 3 *The graph G satisfies the hypotheses of Theorem 4.1.1, as well as each of the following.*
(i) Among all counterexamples to Theorem 4.1.1, $n = |V(G)|$ is smallest.
(ii) G cannot be contracted to a member in $\mathcal{F}$, and so $n \ge N + 1$.
(iii) There exist disjoint edge subsets $X, Y \subseteq E(G)$ with $|X| = s$ and $|Y| = t$ such that $G - Y$ does not have a spanning closed trail that contains all edges in X.

Let X and Y be the edge subsets assured by Claim 3(iii), and define $J = (G - Y)(X)$. We adopt the notation in Observation 5.3.1 for the definition of V_X and E_X. As $\kappa'(G) \ge t + 2$ and by Observation 5.3.1(iv),

$$\kappa'(J) \ge 2 \text{ and } J \text{ is not supereulerian.} \tag{4.6}$$

Let J' denote the reduction of J, and define $h = |D_2(J')|$ and $h_1 = |D_2(J') \cap V_X|$. We have the following claim.

Claim 4 $F(J') \ge 3$.

Suppose that $F(J') \le 2$. By Theorem 4.2.2(ii), either J' is supereulerian, whence by Theorem 4.2.1(i), J is supereulerian; or $J' = K_{2,h}$ with $h \equiv 1 \pmod 2$ and $h \ge 3$. By (4.6), we must have $J' = K_{2,h}$. Let $D_h(J') = \{u_1, u_2\}$, and

let H_1', H_2' be the preimages of u_1 and u_2 in J, respectively; and let H_1 and H_2 be the restorations of H_1' and H_2' in $G - Y$, respectively. Thus $V(G) = V(G - Y) = V(H_1) \cup V(H_2)$.

If $h = h_1$, then $h \leq |X| \leq s \leq \max\{s+t, 1\}$, and so by (4.5), $G/(H_1 \cup H_2) = hK_2$ is a member in $\mathcal{F}$, contrary to Claim 3(ii). Thus we must have $h > h_1$. Then for each vertex $z \in D_2(J') - V_X$, let H_z' denote the preimage of z in J, and H_z be the restoration of H_z' in $G - Y$. Since H_z' is collapsible, we have $\kappa'(H_z) \geq 2$. Pick a vertex $v \in V(H_z)$. As $z \in D_2(J') - V_X$ and by $n > N \geq \frac{4-b}{a}$, we have $|V(H_z)| \geq |N_G[v]| - 2 \geq an + b - 1 \geq 3$, It follows that there must be a vertex $v' \in V(H_z)$ such that $N_G[z'] \subseteq V(H_z)$. Thus for each $z \in D_2(J') - V_X$, $|V(H_z)| \geq an + b + 1$. This implies, by $n > N \geq \frac{|b+1|-a(b+1)}{a^2}$ in (4.4), that

$$h - h_1 \leq \frac{n}{an+b+1} = \frac{an+b+1-b-1}{a(an+b+1)} = \frac{1}{a} - \frac{b+1}{a(an+b+1)} < \frac{1}{a} + 1.$$

It follows by $h_1 \leq s$ and (4.4) that $|V(J')| = 2 + h = 2 + h_1 + (h - h_1) < \frac{1}{a} + s + 3 \leq N$. By Lemma 4.2.7 and by (4.6), G can be contracted to an (s,t)-reduced graph with at most N vertices, which is in $\mathcal{F}$, contrary to Claim 3(ii). This proves the Claim 4.

For each integer i, let $d_i = |D_i(J')|$. By Claim 4, $F(J') \geq 3$ and so by Theorem 4.2.2(iv), we have $4|V(J')| - 2|E(J')| \geq 10$. As $|V(J')| = \sum_{\imath \geq 2} d_i$ and $2|E(J')| \geq \sum_{i \geq 2} i d_i$, we have

$$2d_2 + d_3 \geq 10 + \sum_{i \geq 5} (i-4) d_i. \tag{4.7}$$

For each vertex $v \in V(J') - V_X$, let H_v' be the maximal collapsible subgraph in J which is the contraction preimage of v, and let H_v be the restoration of H_v'. Thus H_v is a subgraph of G.

Claim 5 *Let $n' = |V(J')|$, and define $Z_c = \{v \in V(J') : d_{J'}(v) \leq c\}$. Each of the following holds.*
(i) For any $z \in Z_c$, $|V(H_z)| \geq an + b + 1$.
(ii) $|Z_c| \leq \frac{1}{a} + 1$.
(iii) $n' \leq N$.

Fix a vertex $z \in Z_c$. Then by (4.4), for any $v \in V(H_z)$, as $n > N \geq \frac{c+t-b+1}{a}$, we have $|\partial_G(V(H_z))| \leq c + t < an + b$. It follows by Lemma 4.2.9 that $|V(H_z)| \geq an + b + 1$. Thus (i) holds. By (i), we have

$$n = |V(G)| \geq \sum_{z \in Z_c} |V(H_z)| \geq |Z_c|(an+b+1), \text{ and so } |Z_c| \leq \frac{n}{an+b+1}.$$

By (4.4), $n \geq N \geq \frac{|b+1|-a(b+1)}{a^2}$, implying that $|Z_c| \leq \frac{1}{a} + 1$, and so (ii) follows as well.

To prove (iii), we observe that for any vertex $v \in V(J') - Z_c$, $d_{J'}(v) \geq c+1$, and so by $F(J') \geq 3$,

$$(c+1)|V(J') - Z_c| \leq \sum_{v \in V(J')} d_{J'}(v) = 2|E(J')| \leq 4n' - 10.$$

It follows that $|V(J') - Z_c| \leq \frac{4n'-10}{c+1}$, and so by Claim 5(ii),

$$\frac{1}{a}+1 \geq |Z_c| = n' - |V(J') - Z_c| \geq n' - \frac{4n'-10}{c+1} = n'\left(1 - \frac{4}{c+1}\right) + \frac{10}{c+1}. \quad (4.8)$$

By algebraic manipulations and by (4.8), (4.3) and (4.4), we have

$$n' \leq \frac{(1+a)(c+1) - 10a}{a(c-3)} \leq N.$$

Thus (iii) holds, and so the claim is justified.

By Claim 5(iii), and by Lemma 4.2.7, G can be contracted to a member in $\mathcal{F}$, contrary to Claim 3(ii). This completes the proof of Theorem 3.2.2.

4.4 Proof of Theorem 4.1.2

Let G be a graph satisfying the hypothesis of Theorem 4.1.2, and set

$$N = \max\{2t+9, 2(2s+t+2)\}. \quad (4.9)$$

We shall assume that $n \geq N$ and that Theorem 4.1.2(i) fails to show that Theorem 4.1.2(ii) must hold. As Theorem 4.1.2(i) fails, by Observation 5.3.1(iv), there exist edge disjoint subsets $X, Y \subseteq E(G)$ such that $|X| \leq s$, $|Y| \leq t$ and

$$(G - Y)(X) \text{ is not supereulerian.} \quad (4.10)$$

Let $J = (G - Y)(X)$ and J' be the reduction of J. Since $\kappa'(G) \geq t+2$, we have $\kappa'(J') \geq 2$. If $F(J') \leq 1$, then by Theorem 4.2.2(i), J' is collapsible, and so by Theorem 4.2.1(i), J is supereulerian, contrary to (4.10). Hence we must have $F(J') \geq 2$. For each integer i, we again let $d_i = |D_i(J')|$. By Theorem 4.2.2(iv), $2|V(J')| - |E(J')| - 2 = F(J') \geq 2$, and so $4\sum_{i\geq 2} d_i \geq 8 + \sum_{i\geq 2} i d_i$. It follows that

$$2d_2 + d_3 \geq 8 + \sum_{i \geq 5} (i-4)d_i. \quad (4.11)$$

We will validate the following claim.

Claim 6 *Each of the following holds.*
(i) $\Delta(J') \leq 2s$.
(ii) Every vertex in $(\bigcup_{i=3}^{2s} D_i(J')) \cup (D_2(J') - V_X)$ *is nontrivial.*
(iii) Let m *be the number of nontrivial vertices in* J'. *Then* $m \leq 2$.
(iv) Let $h = |D_2(J')|$. *Then* $h \equiv 1$ *(mod 2),* $h \geq 3$, $J' \cong K_{2,h}$ *and* $D_2(J') \subseteq V_X$.

By contradiction, we assume that $\Delta(J') \geq 2s+1$. Then for some $j \geq 2s+1$, $d_j > 1$, and so by (4.11), $2(d_2+d_3) \geq 8+(2s+1-4) = 2s+5$. As both sides of the inequality are integers, we have $d_2+d_3 \geq s+3$. Since $|V_X \cap D_2(J')| \leq |V_X| = s$, there must be at least three vertices $z_1, z_2, z_3 \in D_2(J') \cup D_3(J') - V_X$. For each $i \in \{1,2,3\}$, let H'_{z_i} denote the contraction preimage of z_i in J, and let H_{z_i} denote the restoration of H'_{z_i} in $G-Y$. By (4.9), $n \geq N \geq 2t+9$, and so $\delta(G) \geq \frac{n}{2} - 1 > 3 + t \geq |\partial_G(H_{z_i})|$. By Lemma 4.2.9, $|V(H_{z_i})| \geq \frac{n}{2}$. It follows that $n = |V(G)| \geq \sum_{i=1}^{3} |V(H_{z_i})| \geq \frac{3n}{2}$, contrary to the fact $n > 0$. This proves (i).

Let $z \in (\bigcup_{i=3}^{2s} D_i(J')) \cup (D_2(J') - V_X)$, H'_z be the contraction preimage of z in J, and H_z denote the restoration of H'_z in $G-Y$. By (4.9), $n \geq N \geq 2(2s+t+2) \geq 4$, and so $\delta(G) \geq \frac{n}{2} - 1 > 2s + t \geq |\partial_G(H_z)|$. By Lemma 4.2.9, $|V(H_z)| \geq \frac{n}{2} \geq 2$, and so (ii) follows.

By contradiction, we assume that J' has at least three nontrivial vertices, say w_1, w_2, w_3. For each $i \in \{1,2,3\}$, let H'_{w_i} denote the contraction preimage of w_i in J, and let H_{w_i} denote the restoration of H'_{w_i} in $G-Y$. By (4.9), $n \geq N \geq 2(2s+t+2)$, and so by Claim 6(i) that $\delta(G) \geq \frac{n}{2} - 1 > 2s + t \geq |\partial_G(H_{w_i})|$. By Lemma 4.2.9, $|V(H_{w_i})| \geq \frac{n}{2}$. It follows that $n = |V(G)| \geq \sum_{i=1}^{3} |V(H_{w_i})| \geq \frac{3n}{2}$, contrary to the fact $n > 0$. This proves (iii).

By Claim 6(i), $d_j = 0$ for any $j \geq 2s+1$, and so by Claim 6(ii), $|V(J')| - |D_2(J') \cap V_X| = \sum_{i \geq 2} d_i - |D_2(J') \cap V_X| \leq 2$. Thus $|V(J')| \leq |D_2(J')| + 2$. By Claim 6(iii), $m \leq 2$. Let $u_1, ..., u_m$ denote the nontrivial vertices of J'. If at least one of the w_i's is of even degree in J', then since the number of odd degree vertices of a graph must be even, it follows by $m \leq 2$ that J' is an eulerian graph, and so supereulerian. By Theorem 4.2.1(i), J is supereulerian, contrary to (4.10). Hence we must have $m = 2$ and both u_1 and u_2 are of odd degree in J'. Since J' is reduced, J' contains no cycles of length at most 3, and so we must have $N_{J'}(u_1) = N_{J'}(u_2) = D_2(J')$. By (4.10), J' cannot be eulerian, and so $h \equiv 1 \pmod 2$. Since $\kappa'(J') \geq 2$, we must have $h \geq 3$. Finally, since both u_1 and u_2 are not in $D_2(J')$, it follows by Claim 6(ii) and (iii) that $D_2(J') \subseteq V_X$. This proves (iv), as well as Claim 6.

By Claim 4(iv), $J' \cong K_{2,h}$ for some odd integer $h \geq 3$. We continue using u_1, u_2 to denote the two vertices of degree h in J', and define H'_{u_i} to be the preimage of u_i in J, and H_{u_i} the restoration of H'_{u_i} in $G-Y$. By Claim 4(iv), $D_2(J') \subseteq V_X$. Let $X'' = \{e \in X : v_e \in D_2(J')\}$. Since $J' \cong K_{2,h}$, we have $V(G) = V(H_{u_1}) \cup V(H_{u_2})$ and $X'' \subseteq E_G[V(H_{u_1}), V(H_{u_2})] \subseteq X'' \cup Y$. Let $j = |E_G[V(H_{u_1}), V(H_{u_2})]|$. Then by $\kappa'(G) \geq t+2$, we have $t+2 \leq j \leq |X''| + |Y| \leq s+t$ and $G/(H_{u_1} \cup H_{u_2}) = jK_2$. Thus Theorem 4.1.2(ii) must hold. This completes the proof of Theorem 4.1.2.

Chapter 5

On the Extended Clark-Wormold Hamiltonian-Like Index Problem

5.1 Main result

Theorem 5.1.1 *If $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, then each of the following holds.*
(i) If $b \geq a \geq 1$, then $\mathcal{P}'(a,b) = \mathcal{P}(a,b)$.
(ii) For $a = 2$, $\mathcal{P}'(2,1) = \infty$.
(iii) For $a = 3$, $\mathcal{P}'(3,1) = \mathcal{P}'(3,2) = 3$.
(iv) If $a \geq 4$ and $b \geq 1$, then $\mathcal{P}'(4,b) \leq 2$. Furthermore, $\mathcal{P}'(4,b) = 1$ if and only if Conjecture 1 is valid.
(v) If $a \geq 6$ and $b \geq 1$, or if $a = 5$ and $b \geq 4$, then $\mathcal{P}'(a,b) = 1$.

In the next section, we summarize and develop former results and needed tools in our arguments to prove the main results. The main results will be validated in the last section.

5.2 Preliminaries

Given a trail $T = v_0e_1v_1...e_{n-1}v_{n-1}e_nv_n$ in a graph G, we often refer this trail as a (v_0, v_n)-trail to emphasize the end vertices, or as an (e_1, e_n)-trail to emphasize the end edges. The vertices $v_1, v_2, ..., v_{n-1}$ are the **internal vertices** of T. As a vertex may occur more than once in a trail, when either v_0 or v_n occurs in the trail as a v_i with $0 < i < n$, it is also an internal vertex by definition. A trail T of G is **internally dominating** if every edge of G is incident with an internal vertex of T, is **spanning** if T is internally dominating

with $V(T) = V(G)$. A graph G is **spanning trailable** if for any pair of edges $e', e'' \in E(G)$, G has a spanning (e', e'')-trail. If H is an eulerian subgraph (a closed trail) of G, then every vertex of H is an internal vertex. Thus H is **dominating** if $E(G - V(H)) = \emptyset$. Harary and Nash-Williams discovered a close relationship between dominating eulerian subgraphs and hamiltonian line graphs. The **essential edge connectivity** of G, denoted $ess'(G)$, is the smallest size of an essential edge cut of G. A graph G is **essentially k-edge-connected** if G is connected and $ess'(G) \geq k$. By definition, it is observed in [140] that the following holds for a connected graph G with $|E(G)| \geq 3$.

$$\kappa(L(G)) = ess'(G). \tag{5.1}$$

Theorem 5.2.1 *(Harary and Nash-Williams, [39]) Let G be a connected graph with at least three edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Following the same idea of Theorem 5.2.1, the following have been observed.

Proposition 5.2.2 *Let G be a connected graph with at least three edges.*
(i) The line graph $L(G)$ has a Hamilton path if and only if G has an internally dominating trail.
(ii) (Shao [140], see also Theorem 1.5 of [105]) The line graph $L(G)$ is Hamilton-connected if and only if for any edges $e, e' \in E(G)$, G has an internally dominating (e, e')-trail. In particular, if G is spanning trailable, then $L(G)$ is Hamilton-connected.

Let G be a graph, and define $\tau(G)$ to be the maximum number of edge-disjoint spanning trees in G. For each integer $i \geq 0$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\}.$$

Thus $O(G) := \cup_{s \geq 0} D_{2s+1}(G)$ is the set of all odd degree vertices of G. A graph G is **eulerian** if G is connected with $O(G) = \emptyset$; and is **supereulerian** if it contains a spanning eulerian subgraph. For a subset $Y \subseteq E(G)$, the **contraction** G/Y is the graph obtained from G by identifying the two ends of each edge in Y and then by deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. For a vertex $v \in V(G/X)$, we define $PI_G(v)$ to be the contraction preimage of v in G.

A graph G is called **collapsible** if for any $R \subseteq V(G)$ with $|R|$ is even, G has a spanning subgraph S_R with $O(S_R) = R$. The following theorem shows the property of collapsible graphs.

Theorem 5.2.3 *Let $k \geq 1$ be an integer and G be a graph. Each of the following holds.*

(i) (Catlin [9]) (Corollary 1 of [9]) If G has a spanning tree of which every edge lies in a cycle of length 3 in G, then G is collapsible. In particular, cycles of length at most 3 are collapsible.
(ii) (Gusfield [38] and Kundu [45]) If $\kappa'(G) \geq 2k$, then $\tau(G) \geq k$.

Lemma 5.2.4 *(Li et al., Proposition 2.3 of [51]) Let $k \geq 1$ be an integer, and let $\mathcal{T}_k = \{G : \tau(G) \geq k\}$. Then $\mathcal{T}_k$ satisfies each of the following.*
(C1) $K_1 \in \mathcal{T}_k$.
(C2) If $G \in \mathcal{T}_k$ and $e \in E(G)$, then $G/e \in \mathcal{T}_k$.
(C3) Let H be a subgraph of G. If $H, G/H \in \mathcal{T}_k$, then $G \in \mathcal{T}_k$.

Definition 5 *Let $e = uv$ be an edge of G. Define $G(e)$ to be the graph obtained from G by replacing $e = uv$ with a path uv_ev, where v_e is a new vertex not in $V(G)$. We say that $G(e)$ is formed by performing an* **elementary subdivision** *on $e \in E(G)$. For an edge subset $X \subseteq E(G)$, we use $G(X)$ to denote the the graph formed by performing an elementary subdivision on each edge in X. When $X = \{e_1, e_2\}$, we also use $G(e_1, e_2)$ for $G(X)$.*

As defined in [53, 104], a graph G is **strongly spanning trailable** if for any e, $e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$-trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. By definition, every strongly spanning trailable graph is spanning trailable. As observed in [54] (also in Chapter 1 of [61]), there exist graphs that are spanning trailable but not strongly spanning trailable.

Lemma 5.2.5 *Let G be a connected graph. Then each of the following holds.*
(i) (Lei et al., Theorem 2.2(iv) of [104]) Suppose that $\tau(G) \geq 2$. For any $e', e'' \in E(G)$, $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail if and only if $\{e', e''\}$ is not an edge-cut of G. Moreover, if $\{e', e''\}$ is an edge-cut of G and G_1, G_2 are the two components of $G - \{e', e''\}$, then for any $i \in \{1, 2\}$, G has an (e', e'')-trail containing all vertices in $V(G_i)$.
(ii) (Proposition 1.1 of [105]) If $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail, then G has a spanning (e', e'')-trail.
(iii) If $\tau(G) \geq 2$ and $ess'(G) \geq 3$, then $L(G)$ is Hamilton-connected.

Proof. It remains to prove (iii). Suppose that $\tau(G) \geq 2$ and $ess'(G) \geq 3$. By Proposition 5.2.2(ii), we shall show that G is spanning trailable. Let $e', e'' \in E(G)$. If $\{e', e''\}$ is not an edge-cut of G, then Lemma 5.2.5(i) and (ii) imply that G has a spanning (e', e'')-trail. If $\{e', e''\}$ is an edge-cut of G, then as $ess'(G) \geq 3$, there exists a vertex v of degree 2 in G incident with both e' and e'', and so by Lemma 5.2.5(i), G has a spanning spanning (e', e'')-trail. ∎

5.3 Proof of Theorem 5.1.1

Theorem 5.1.1 will be justified in this section. The arguments will utilize the **symmetric difference** of two sets X and Y, which is defined as

$$X \triangle Y = X \cup Y - (X \cap Y).$$

We have the following observations.

Observation 5.3.1 *Let G be a graph and let $u, v \in V(G)$ be two distinct vertices.*
(i) If $\{u, v\}$ is a vertex cut of G, then G does not have a spanning (u, v)-path.
(ii) If $e = uv \in E(G)$ and $\{u, v\}$ is a vertex cut of G, then G does not have a Hamilton cycle containing e.
(iii) If G is Hamilton-connected, then $\kappa(G) \geq 3$.

Shao [59] proves some useful properties for essential edge-connectivity of line graphs.

Theorem 5.3.2 *(Shao, Theorem 1.3 of [59]) Let $G \in \mathcal{G}$ be a connected graph with $|E(G)| \geq 4$. If $D_2(G) = \emptyset$, then $ess'(L(G)) \geq 2ess'(G) - 2$.*

Lemma 5.3.3 *Let G be a connected graph with $|E(G)| \geq 4$, $ess'(G) \geq 1$ and $\delta(G) \geq 3$. Then $ess'(L(G)) \geq \min\{ess'(G) + 1, 4\}$.*

Proof. By Theorem 5.3.2, if $ess'(G) \geq 3$, then $ess'(L(G)) \geq 2ess'(G) - 2 \geq \min\{ess'(G) + 1, 4\}$. Hence we assume that $ess'(G) \in \{1, 2\}$. Since $\delta(G) \geq 3$, we have $\delta(L(G)) \geq 4$. As $|V(L(G))| = |E(G)| \geq 4$, we have $|E(L(G))| \geq 8$. Hence we may assume that $L(G)$ has two connected nontrivial components L_1 and L_2, with $V(L(G)) = V(L_1) \cup V(L_2)$ and $V(L_1) \cap V(L_2) = \emptyset$, such that $F = (V(L_1), V(L_2))_{L(G)}$ is a minimum essential edge-cut of $L(G)$.

Let $c = |F|$ and denote $F = \{f_1, f_2, ..., f_c\}$. Then $1 \leq c \leq 2$. For each $i \in \{1, 2, ..., c\}$, denote $f_i = e_i e_i'$ for edges $e_i, e_i' \in E(G)$, with $e_i \in V(L_1)$ and $e_i' \in V(L_2)$. Thus we may assume that there exist distinct vertices $u_i, v_i, w_i \in V(G)$ such that $e_i = u_i v_i$ and $e_i' = v_i w_i$. Since $\delta(G) \geq 3$, there must be an edge $e_i'' = z_i v_i \in E(G) - \{e_i, e_i'\}$. As F is an essential edge cut of $L(G)$, it follows by definition that $\{e_1, e_2, ..., e_c\}$ is an essential edge cut of G. Thus $ess'(G) \leq c$.

Suppose that $c = 1$. Then e_1 is an essential cut edge of G. As $e_1, e_1', e_1'' \in E(G)$, we by symmetry may assume that $e_i'' \in V(L_2)$, and so $e_1 e_1', e_1 e_1'' \in (V(L_1), V(L_2))_{L(G)} = F$. It follows that $ess'(L(G)) = c \geq 2$.

Suppose now that $c = ess'(G) = 2$. Then we may assume that $e_1 \neq e_2$ and $\{e_1, e_2\}$ is an essential edge cut of G. With the notation above, we have $e_1, e_1', e_1'', e_2, e_2', e_2'' \in E(G) = V(L(G))$ with e_1, e_2, e_1', e_1'' being mutually distinct edges in G. If $e_1' \neq e_2'$, then F contains three distinct edges $e_1 e_1', e_1 e_1'', e_2 e_2'$,

and so we have $2 = |F| \geq |\{e_1e_1', e_1e_1'', e_2e_2'\}| = 3$, a contradiction. Assume that $e_1' = e_2'$. If $e_2'' \in V(L_1)$, then $2 = |F| \geq |\{e_1e_1', e_1e_1'', e_1e_2'\}| = 3$, and if $e_2'' \in V(L_2)$, then $2 = |F| \geq |\{e_1e_1', e_1e_2, e_2e_2''\}| = 3$. A contradiction occurs in any case. Hence we must have $c \geq 3$. This proves the lemma. ∎

As $\mathcal{P}'(a,b) \geq \mathcal{P}(a,b)$, it follows from (1.3) that

$$\mathcal{P}'(1,1) = \mathcal{P}'(1,2) = \mathcal{P}'(2,2) = \infty. \tag{5.2}$$

The following lemma shows an upper bound of $\mathcal{P}'(a,b)$ when $a \geq 3$.

Lemma 5.3.4 *Let $G \in \mathcal{G}$ be a connected graph with $|E(G)| \geq 4$ and $ess'(G) \geq 3$. Then $L^3(G)$ is Hamilton-connected. Thus for any $a \geq 3$ and $b \geq 1$, $\mathcal{H}_c'(a,b) \leq 3$.*

Proof. As $G \in \mathcal{G}$ and by the definition of iterated line graphs, we have $|E(L^i(G))| \geq 4$ for $i \geq 1$. As $ess'(G) \geq 3$, we have $ess'(L(G)) \geq \kappa'(L(G)) \geq \kappa(L(G)) \geq 3$. Thus by Theorem 5.3.2 and $\delta(L(G)) \geq \kappa(L(G)) \geq 3$, we have $\kappa'(L^2(G)) = ess'(L^2(G)) \geq 4$, and so by Theorem 5.2.3(ii), $\tau(L^2(G)) \geq 2$. It follows from Lemma 5.2.5(iii) that $L^3(G)$ is Hamilton-connected. ∎

5.3.1 Justification of Theorem 5.1.1(i), (ii) and (iii)

It is straightforward that Theorem 5.1.1(i) is a consequence of (1.5). It suffices to prove Theorem 5.1.1 (ii) and (iii).

Proposition 5.3.5 *For any integer $k > 0$, there exists an infinite family $\mathcal{G}_1(k)$ of connected graphs such that every $G \in \mathcal{G}_1(k)$ satisfies $ess'(G) = 2$, $\delta(G) = 1$ and $L^k(G)$ is not hamiltonian. Thus $\mathcal{H}'(2,1)$ cannot be bounded above by a finite number.*

Proof. Let s_1, s_2 be nonnegative integers, w_1, w_2 be two distinct vertices, and for $i \in \{1,2\}$, $X_i = \{x_1^i, x_2^i, ..., x_{s_i}^i\}$ be a set of vertices, and for $j \in \{1,2,3\}$, $P_j = v_1^j ... v_{k+1}^j$ be a path of length k, such that the sets $\{w_1, w_2\}$, X_1, X_2 and $V(P_1), V(P_2)$ and $V(P_3)$ are mutually disjoint. Define $G = G(k, s_1, s_2)$ to be the graph with

$$\begin{aligned} V(G) &= \{w_1, w_2\} \cup X_1 \cup X_2 \cup V(P_1) \cup V(P_2) \cup V(P_3), \\ E(G) &= E(P_1) \cup E(P_2) \cup E(P_3) \cup \{w_1v_1^j : 1 \leq j \leq 3\} \cup \{w_2v_{k+1}^j : 1 \leq j \leq 3\} \\ &\quad \cup \{w_1x_s^1 : 1 \leq s \leq s_1\} \cup \{w_2x_s^2 : 1 \leq s \leq s_2\}. \end{aligned}$$

Hence $G - (X_1 \cup X_2)$ can be viewed as a subdivision of $K_{2,3}$. By the definition of iterated line graphs, we observe that $L^{k-1}(G)$ can be contracted to a $K_{2,3}$ in which every vertex in $D_2, K_{2,3}$ has a nontrivial contraction preimage. It follows by Theorem 5.2.1 that $L^k(G)$ is not hamiltonian. ∎

By Proposition 5.3.5, we conclude that for any $\mathcal{P} \in \{\mathcal{H}, \mathcal{E}_h, \mathcal{H}_c\}$, $\mathcal{P}'(2,1) = \infty$. This proves Theorem 5.1.1 (ii).

To prove Theorem 5.1.1 (iii), we start with a proposition showing the lower bounds.

Proposition 5.3.6 *For any integer $k > 0$, each of the following holds.*
(i) There exists an infinite family of connected graphs $\mathcal{F}_5$ such that for any $G \in \mathcal{F}_5$, $ess'(G) = 3$, $\delta(G) = 1$ and $L^2(G)$ is not hamiltonian. Thus $\mathcal{H}'_c(3,1) \geq \mathcal{E}'_h(3,1) \geq \mathcal{H}'(3,1) \geq 3$.
(ii) There exists an infinite family of connected graphs $\mathcal{F}_6$ such that for any $G \in \mathcal{F}_6$, $ess'(G) = 3$, $\delta(G) = 2$ and $L^2(G)$ is not hamiltonian. Thus $\mathcal{H}'_c(3,2) \geq \mathcal{E}'_h(3,2) \geq \mathcal{H}'(3,2) \geq 3$.

Proof. Let $P(10)$ denote the Petersen graph with $E = E(P(10))$ and $V(P(10)) = \{v_i : 1 \leq i \leq 10\}$. As in Definition 5, $P(10)(E)$ is the graph formed from $P(10)$ by performing an elementary subdivision on each edge in E.

(i) For each i with $1 \leq i \leq 10$, let $J_i \cong K_{1,d_i}$ be a star with $d_i \geq 2$ and with w_i being the only vertex of degree d_i in J_i. Obtain a graph $G = P(10)(d_i : 1 \leq i \leq 10)$ from $P(10)(E)$ by identifying v_i with w_i, for each i with $1 \leq i \leq 10$. Define $\mathcal{F}_5$ to be the graph family such that $G \in \mathcal{F}_5$ if and only if $G = P(10)(d_i : 1 \leq i \leq 10)$ for some integers $d_i \geq 2$. Thus for each $G \in \mathcal{F}_5$, $ess'(G) = 3$ and $\delta(G) = 1$. As $L(G)$ is contractible to the Petersen graph with every vertex in the contraction having a nontrivial contraction preimage, it follows by Theorem 5.2.1 that $L^2(G)$ is not hamiltonian. This, together with (1.5), implies that $\mathcal{H}'_c(3,1) \geq \mathcal{E}'_h(3,1) \geq \mathcal{H}'(3,1) \geq 3$.

(ii) For each i with $1 \leq i \leq 10$, let $J_i \cong K_{2,d_i}$ be a star with $d_i \geq 3$ and with w_i being one of the two vertex of degree d_i in J_i. Obtain a graph $G = P(10)'(d_i : 1 \leq i \leq 10)$ from $P(10)(E)$ by identifying v_i with w_i, for each i with $1 \leq i \leq 10$. Define $\mathcal{F}_6$ to be the graph family such that $G \in \mathcal{F}_6$ if and only if $G = P(10)'(d_i : 1 \leq i \leq 10)$ for some integers $d_i \geq 3$. Thus for each $G \in \mathcal{F}_6$, $ess'(G) = 3$ and $\delta(G) = 2$. As $L(G)$ is contractible to the Petersen graph with every vertex in the contraction having a nontrivial contraction preimage, it follows by Theorem 5.2.1 that $L^2(G)$ is not hamiltonian. This, together with (1.5), implies that $\mathcal{H}'_c(3,2) \geq \mathcal{E}'_h(3,2) \geq \mathcal{H}'(3,2) \geq 3$. ∎

By Lemma 5.3.4, for any positive integer b, we have $\mathcal{H}'(3,b) \leq \mathcal{H}'_c(3,b) \leq 3$. By Proposition 5.3.6(ii) and (iii), we conclude that $\mathcal{H}'_c(3,1) = \mathcal{E}'_h(3,1) = \mathcal{H}'(3,1) = 3$ and $\mathcal{H}'_c(3,2) = \mathcal{E}'_h(3,2) = \mathcal{H}('3,2) = 3$. This completes the proof for Theorem 5.1.1(ii).

5.3.2 Justification of Theorem 5.1.1(iv)

While Conjecture 1 remains open, there have been many researches done towards the conjecture. The following theorem summarizes some efforts on the

hamiltonian properties of 4-connected iterated line graphs.

Theorem 5.3.7 *Let G be a connected graph. Each of the following holds.*
(i) (Corollary 3.9 of [13]) If $L^2(G)$ is 4-connected, then $L^2(G)$ is hamiltonian.
(ii) (Kriesell, [43]) If $L^2(G)$ is 4-connected, then $L^2(G)$ is Hamilton-connected.
(iii) (Theorem 1.3 of [99]) Let G be a connected graph with $|E(G)| \geq 4$ and $ess'(G) \geq 3$. If every 3-edge-cut of G has at least one edge lying in a short cycle of G, then $L(G)$ is Hamilton-connected.

In fact, Kriesell in [43] proved that every 4-connected line graph of a graph without an induced $K_{1,3}$ is Hamilton-connected, which apparently implies Theorem 5.1.1-1(i). As shown in Corollary 1.5 of [99], Theorem 5.1.1(iii) is an extension of the above mentioned results in [13] and [43]. By Theorem 5.1.1 and (1.5), we observe that

$$\text{for any integer } b \geq 1,\ \mathcal{H}'(4,b) \leq \mathcal{E}'_h(4,b) \leq \mathcal{H}'_c(4,b) \leq 2. \tag{5.3}$$

By (2.3.5) and (5.3), we are led to the conclusion that $\mathcal{H}'(4,b) = 1$ if and only if Conjecture 1 holds. To complete the justification of Theorem 5.1.1(v), we need the following result of Kučzel and Xiong in [44].

Theorem 5.3.8 *(Kučel and Xiong [44]) The following are equivalent.*
(i) Every 4-connected line graph is hamiltonian.
(ii) Every 4-connected line graph is Hamilton-connected.

By (5.3), $\mathcal{H}'(4,b) \leq \mathcal{H}'_c(4,b) \leq 2$. This, together with Theorem 5.3.8, has led us to the following observation.

Observation 5.3.9 *The following statements are equivalent.*
(i) For any positive integer b, $\mathcal{H}'(4,b) = 1$.
(ii) Every 4-connected line graph is hamiltonian.
(iii) Every 4-connected line graph is Hamilton-connected.
(iv) For any positive integer b, $\mathcal{H}'_c(4,b) = 1$.

In fact, assume that Observation 5.3.9(i) holds, and so for any positive integer b, $\mathcal{H}'(4,b) = 1$. Then the definition of $\mathcal{H}'(4,b)$ implies that every connected, essentially 4-edge-connected graph has a hamiltonian line graph. By (5.1), this is equivalent to Observation 5.3.9(ii). Next we assume that Observation 5.3.9(ii) is valid. Then by Theorem 5.3.8, Observation 5.3.9(iii) follows. By the definition of $\mathcal{H}'_c(4,b)$, we conclude that Observation 5.3.9(iii) implies Observation 5.3.9(iv). Finally, by (5.3), we observe that Observation 5.3.9(iv) implies Observation 5.3.9(i). This completes the justification of Observation 5.3.9. By (5.3), (1.5) and Observation 5.3.9, Theorem 5.1.1(iv) is now validated.

5.3.3 Justification of Theorem 5.1.1(v)

The first result towards Conjecture 1 was done by Zhan in [64]. In 2012, Kaiser and Vrána made a breakthrough after Zhan's first result. Later in 2014, Kaiser, Ryjáček and Vrána gave a further improvement, as presented below.

Theorem 5.3.10 *A graph G is 1-Hamilton-connected if for any vertex subset $S \subseteq V(G)$ with $|S| \leq 1$, $G - S$ is Hamilton-connected.*
(i) (Zhan, Theorem 3 in [64]) If $\kappa(L(G)) \geq 7$, then $L(G)$ is Hamilton-connected.
(ii) (Kaiser and Vrána [42]) Every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian.
(iii) (Kaiser, Ryjáček and Vrána [97]) Every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.

Theorem 5.3.10 has an immediate corollary, which implies Theorem 3.2.2(v).

Corollary 5.3.11 *Each of the following holds.*
(i) For any $a \geq 6$ and $b \geq 1$, $\mathcal{H}'_c(a, b) = 1$.
(ii) For any $a \geq 5$ and $b \geq 4$, $\mathcal{H}'_c(a, b) = 1$.

5.3.4 Remark

The cases we cannot determine the values of $\mathcal{P}'(a, b)$ are those when $a = 4$ and $1 \leq b \leq 5$, or $a = 5$ and $1 \leq b \leq 3$. Using the same arguments as in Subsection 5.3.2, it is clear that if Conjecture 1 holds, then $\mathcal{P}'(a, b) = 1$ for all these above-mentioned unsettled values of a and b.

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