# Longest Path and Cycle Transversal and Gallai Families 

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# Longest Path and Cycle Transversals and Gallai Families 

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Dissertation submitted to the Eberly College of Arts and Sciences at West Virginia University in partial fulfillment of the requirements for the degree of<br>Doctorate of Philosophy<br>in<br>Mathematics

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## Abstract

Longest Path and Cycle Transversals and Gallai Families<br>James A. Long Jr.

A longest path transversal in a graph $G$ is a set of vertices $S$ of $G$ such that every longest path in $G$ has a vertex in $S$. The longest path transversal number of a graph $G$ is the size of a smallest longest path transversal in $G$ and is denoted $\operatorname{lpt}(G)$. Similarly, a longest cycle transversal is a set of vertices $S$ in a graph $G$ such that every longest cycle in $G$ has a vertex in $S$. The longest cycle transversal number of a graph $G$ is the size of a smallest longest cycle transversal in $G$ and is denoted $\operatorname{lct}(G)$. A Gallai family is a family of graphs whose connected members have longest path transversal number 1. In this paper we find several Gallai families and give upper bounds on $\operatorname{lpt}(G)$ and $\operatorname{lct}(G)$ for general graphs and chordal graphs in terms of $|V(G)|$.

## Dedication

To my parents, James and Teresa Long

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## Chapter 1

## Introduction

A graph $G$ is a pair consisting of a vertex set $V(G)$, and an edge set $E(G)$ with each edge consisting of a unordered pair of vertices called its endpoints. Two vertices are adjacent, and are called neighbors, if there is an edge between them. The neighborhood of a vertex $v$ in a graph $G$ is denoted $N_{G}(v)$ and is the set of all vertices adjacent to $v$ in $G$. We abbreviate $N_{G}(v)$ to $N(v)$ when $G$ is clear from context. Two edges are incident if they have a common endpoint. The degree of a vertex $v$ in a graph $G$ is the number of times that vertex appears as an endpoint of an edge of $G$ and is denoted $d_{G}(v)$. We write $d(v)$ instead when it is clear from context what graph we are considering. A graph $G$ is $k$-regular if $d(v)=k$ for each $v \in V(G)$. The maximum degree of a graph $G$ is denoted $\Delta(G)$ and is equal to $\max \{d(v): v \in V(G)\}$. Similarly, we define the minimum degree of a graph $G$ to be $\delta(G)$. A graph is finite if $|V(G)|$ and $|E(G)|$ are finite. Unless otherwise stated, all graphs should be assumed to be finite.

A directed graph or digraph $G$ is a pair consisting of a vertex set $V(G)$ and an edge set $E(G)$ such that each edge in $G$ is an ordered pair of vertices. If $(u, v)$ is an edge of a directed graph $G$, then the edge goes from the tail $u$ to the head $v$. To simplify notation we write $u v$ for $(u, v)$.

A multigraph $G$ is similar to a graph except that $E(G)$ is a multiset, allowing multiple parallel edges to join the same pair of vertices.

A graph $G$ is isomorphic to a graph $H$ if there is a bijective function $f: V(G) \rightarrow V(H)$ such that $u$ is adjacent to $v$ in $G$ if and only if $f(u)$ is adjacent to $f(v)$ in $H$.

A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say $G$ contains $H$ if $H$ is a isomorphic to a subgraph of $G$ and write $H \subseteq G$. An induced subgraph $H$ of a graph $G$ is a subgraph of $G$ such that $E(H)=\{u v \in E(G): u, v \in V(H)\}$. We say two subgraphs of a graph intersect if they have a vertex in common.

A path is a graph in which its vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the ordering. For any $n \in \mathbb{N}$, we define $P_{n}$ as the path on $n$ vertices. The size or order of a path is $|V(G)|$. The length of a path $P$ is $|E(P)|$. A longest path in a graph $G$ is a path of maximum length in $G$. The endpoints of a path are the first and last vertices of the path. For a graph $G$, and vertices $u, v \in V(G)$, a uv-path is a path with endpoints $u$ and $v$. The interior vertices of a path $P$ are the vertices of $P$ which are not endpoints of $P$. Two paths are internally disjoint if their interiors are
comprised of distinct vertices.
A graph $G$ is connected if there is a $u v$-path in $G$ for every pair of vertices $u, v$ in $G$. A graph is disconnected otherwise. A graph $G$ is $k$-connected if it has at more than $k$ vertices and the removal of fewer than $k$ vertices from $G$ does not disconnect the graph. The connectivity of a graph $G$ is the maximum $k$ such that $G$ is $k$-connected and is denoted $\kappa(G)$. The components of a graph $G$ are the maximal connected subgraphs of $G$. A (vertex) cut in a connected graph is a set of vertices whose removal increases the number of components.

A cycle is a 2-regular connected graph and note that all cycles on $n$ vertices are isomorphic. We define $C_{n}$ as the cycle on $n$ vertices. The length of a cycle $C$ in a graph $G$ is $|E(C)|$. A longest cycle in a graph $G$ is a cycle of maximum length. The graph $C_{3}$, the cycle on 3 vertices, is known as a triangle.

A tree is a connected graph with no cycles. A leaf in a tree is a vertex of degree 1 . If $T$ is a tree with $n$ vertices, then $T$ has $n-1$ edges.

A graph $G$ is complete if every pair of vertices is adjacent. We define $K_{n}$ as the complete graph on $n$ vertices. A graph is bipartite if its vertices can be partitioned into two parts such that no edge has both endpoints in the same part. The complete bipartite graph $K_{m, n}$ has one part $X$ containing $m$ vertices, one part $Y$ containing $n$ vertices, and has edge set $\{x y: x \in X$ and $y \in Y\}$. A clique is a set of pairwise adjacent vertices in a graph $G$. The clique number of a graph $G$, denoted $\omega(G)$, is the size of a largest clique in $G$.

With some of the basics of graph theory out of the way, we introduce the Helly property.
A family of sets $\mathcal{F}$ is a $k$-Helly family and has the $k$-Helly property if, in every subfamily of $\mathcal{F}$ with empty intersection, we can find a set consisting of at most $k$ sets whose intersection is empty. That is, if $\mathcal{F}^{\prime} \subseteq \mathcal{F}$ with $\bigcap \mathcal{F}^{\prime}=\varnothing$, then $\bigcap \mathcal{F}^{\prime \prime}=\varnothing$ for some $\mathcal{F}^{\prime \prime} \subseteq \mathcal{F}^{\prime}$ with $\left|\mathcal{F}^{\prime \prime}\right| \leq k$. A 1-Helly family is a family of sets whose non-empty members have a common element. If $\mathcal{F}$ is a 2 - Helly family then we say $\mathcal{F}$ is a Helly family and has the Helly property [1]. The Helly property is named after Eduard Helly who proved that every finite family of convex sets in a Euclidean space of dimension $n$ has the $(n+1)$-Helly property. For example, any finite family of intervals on the real line have the Helly property.

Of interest to us is the family of longest paths in a connected graph. It is well known that the family of longest paths in a connected graph are pairwise intersecting, as proved in Proposition 1 below.

Proposition 1. If $G$ is a connected graph, then the longest paths of $G$ are pairwise intersecting.

Proof. Let $P$ and $Q$ be longest paths in a connected graph $G$ and assume, for a contradiction, $V(P) \cap V(Q)=$ $\varnothing$. Since $G$ is connected, there is a path from any vertex of $P$ to any vertex of $Q$. Let $R$ be a shortest path with one endpoint in $P$ and one in $Q$. Note that, by extremal choice of $R$, its interior is disjoint from both $V(P)$ and $V(Q)$. Let $p$ be the endpoint of $R$ on $P$ and similarly define $q$. The vertex $p$ is at least half the length of $P$ from one endpoint of $P$. A similar observation can be made of $q$. Taking the longer segment from an endpoint of $P$ to $p$, traveling across $R$, then taking the longer segment to an endpoint of $Q$ from $q$, we make a longer path. This is impossible since $P$ and $Q$ are longest paths.

In 1966, Tibor Gallai [2] asked if there was a vertex common to all longest paths in a connected graph. Gallai was asking whether the family of longest paths in a connected graph has the Helly property. As it turns out the answer to Gallai's question is no. Walther found an example of a connected graph on 25
vertices for which the family of longest paths does not have a common vertex [3]. He and Voss went on to find a smaller example on 12 vertices [4]. Zamfirescu independently found the same example [5]. That example is derived from the Peterson graph, pictured in Figure 1.1, by splitting one vertex into three vertices of degree one. We refer to this graph as the Peterson fragment and it is pictured in Figure 1.2.

Figure 1.1: The Peterson graph


Figure 1.2: The Peterson fragment


A subgraph in a graph is spanning if it contains all the vertices of the graph. The Peterson fragment $G$ has no spanning paths as no path in $G$ can use all three vertices of degree one. Furthermore, no path in $G$ uses 11 vertices as this path would necessarily use all vertices of degree three and two vertices of degree one, which must act as endpoints. This path corresponds to a spanning cycle in the Peterson graph, which famously does not have such a cycle. Paths on 10 vertices in $G$ can be found by inspection, and are longest paths in $G$. Since any such path omits at most two vertices of degree one, there is no longest path avoiding two vertices of degree three in $G$.

The last statement of the previous paragraph is saying that, while we can't always find a single vertex common to all longest paths, in the case of the Peterson fragment, we can find a set of two vertices which intersect all longest paths. Such a set vertices is called a longest path transversal. The idea of finding a longest path transversal in general graphs is explored in Chapter 2. The majority of the content in Chapter 2 appears in a paper written by Long, Milans, and Munaro [6].

We could also ask for what families of graphs is there a vertex common to all longest paths. These would be families that have a positive answer to Gallai's question. Such ideas are explored in Chapter 3. The majority of the content for Chapter 3 appears in a paper written by Long, Milans, and Munaro [7].

Additionally, we may mix the concepts and look for small longest path transversal in a graph with a particular structure. Chapter 4 deals with finding a small longest path transversal in chordal graphs. Some of the results from this section will eventually appear in a paper with Dr. Milans and Michael Wigal, but, at the current time, is still being written.

Another logical question not explored in this paper is the following: if any two longest paths in a connected graph intersect, what about any three longest paths? The answer to this question remains unknown, but there are some partial results. For example, Axenovich shows that if the union of three longest paths is outerplanar, then those paths have a vertex in common [8]. It has been conjectured that any three longest paths in a connected graph share a common vertex as noted in [8]. If this could be proved, the next step would be to see if any four longest paths have a common intersection in connected graphs.

The Peterson fragment has a family of nine longest paths having empty intersection. Skupień constructed a connected graph having a set of seven longest paths with empty intersection [9]. It is unknown if there is a connected graph with six longest paths having no common vertex. One can try to find the maximum $k$ such that every set of $k$ longest paths in a connected graph has nonempty intersection. Proposition 1 shows $k \geq 2$ and Skupien's example shows $k<7$.

## Chapter 2

## Longest Path Transversals

### 2.1 Definitions and History

A longest path transversal in a graph $G$ is a set of vertices $S$ in $G$ such that every longest path in $G$ has at least one vertex in $S$. The longest path transversal number of a graph $G$, denoted $\operatorname{lpt}(G)$, is the size of a minimum longest path transversal in $G$.

Overall, it is non-trivial to construct a connected graph without a vertex common to all longest paths. Regardless, we have one in the Peterson fragment, which, as previously noted, has longest path transversal number equal to two. This is the smallest known example of a connected graph with $\operatorname{lpt}(G)>1$. That is, there is no known connected graph $G$ on 11 or fewer vertices with $\operatorname{lpt}(G)>1$. See Shabbir, C. Zamefirescu, T. Zamefirescu, where it is claimed to be the smallest [10].

Graünbaum constructed a 324 -vertex connected graph $G$ with $\operatorname{lpt}(G)=3$ [11]. Soon after Zamfirescu found a 270 vertex example with the same transversal number [5]. There is no known example of a graph $G$ with $\operatorname{lpt}(G) \geq 4$. Hence, we might expect that longest path transversal number is small and many have asked if it is bounded by a constant. Moreover, graphs with high connectivity seem to admit a vertex common to longest paths. Indeed, there is no known 4 -connected graph $G$ such that $\operatorname{lpt}(G)>1$. Graünbaum's example is 3 -connected, so we do know 3 -connected graphs need not have a vertex common to all longest paths.

As difficult as it is to find graphs with large longest path transversal number, it seems just as challenging to show upper bounds on longest path transversal number for general, connected graphs. Once longest paths start intersecting even a few times, their structures get quite complex.

The following proposition is more a set of observations about longest path transversals in connected graphs.

Proposition 2. Let $G$ be a connected graph and let $\mathcal{P}$ be the family of longest paths in $G$.

1. $V(G)$ is a longest path transversal.
2. $V(P)$ is a longest path transversal for any $P \in \mathcal{P}$.
3. If $|\mathcal{P}|=k$, then $\operatorname{lpt}(G) \leq k$.

Proof. (1) Every longest path in $G$ is made from the vertices of $G$, hence $V(G)$ is a transversal. Therefore, we may always assume $\operatorname{lpt}(G) \leq|V(G)|$. If $G$ is disconnected, this bound is best possible since the edgeless graph has $n$ longest paths of length zero.
(2) Since longest paths are pairwise intersecting in a connected graph, every other longest path in $\mathcal{P}$ must have at least one vertex in common with $P$.
(3) If there are $k$ longest paths in $G$, taking a vertex from each one yields a longest path transversal of size $k$.

Based off Proposition 2, to have large longest path transversal number, we need a lot of long paths. Indeed, if there are only 'a few' paths, or the longest paths are 'short', then we can find a 'small' transversal.

Proposition 3. If $G$ is an n-vertex connected graph, then $\operatorname{lpt}(G) \leq\left\lceil\frac{n}{2}\right\rceil$
Proof. Let $(X, Y)$ be an arbitrary bipartition of the vertices of $G$. If $X$ contains a longest path, then $X$ is a longest path transversal. Alternatively, if $X$ does not contain a longest path, there is at least one vertex from every longest path in $Y$. It follows that $X$ or $Y$ is a longest path transversal. Since this is true for an arbitrary bipartition, the result follows by balancing the bipartition as much as possible.

Proposition 4. If $G$ is an n-vertex, connected graph, then $\operatorname{lpt}(G) \leq\left\lceil\frac{n}{4}\right\rceil$

Proof. Let $P$ be a longest path in $G$ and let $k$ be a positive integer to be determined later. Let $S$ be the middle $k$ vertices of $P$, possibly being slightly off-center depending on the parity of the path length. Assume $S$ is not a longest path transversal and let $Q$ be a longest path avoiding $S$. Since $G$ is connected, $P$ and $Q$ have a vertex in common. Note that $Q$ cannot only share vertices with $P$ on one side of $S$ as we could follow $P$ from the opposite end, through $S$, until the first vertex $P$ and $Q$ share, then follow $Q$ to whichever endpoint is further away. This would result in a longer path, which is impossible. Let $q_{l}$ and $q_{r}$ be the vertices in $V(P) \cap V(Q)$ that are closest to $S$ on each side of $P$. Note that removing $q_{l}$ and $q_{r}$ from $Q$ splits $Q$ into three paths. If any of these paths have fewer than $k$ vertices we can make a longer path by replacing it with the interior of the $q_{l} q_{r}$-subpath of $P$. Hence, we get at least $4 k$ vertices in $G$. We get $k$ vertices from $S$ and $k$ vertices from each components of $Q-\left\{q_{l}, q_{r}\right\}$. It follows that $k \leq \frac{n}{4}$ as there are only $n$ vertices in $G$. Hence, if we choose $k \geq\left\lceil\frac{n}{4}\right\rceil$, then $S$ must be a longest path transversal.

A closely related problem to finding longest path transversals is finding longest cycle transversals. That is, given a graph $G$ a set of vertices $S$ in $G$ is a longest cycle transversal if every longest cycle in $G$ has at least one vertex in $S$. We define the longest cycle transversal number of a graph $G$, which we denote $\operatorname{lct}(G)$, as the smallest non-negative integer such that $G$ has a longest cycle transversal of that size. For connected graphs, Thomassen proved $\operatorname{lct}(G) \leq \frac{n}{3}$ and this is best possible [12]. An example achieving this bound is a family of $\frac{n}{3}$ disjoint triangles strung together by a path (see Figure 2.1).

Note that in the example in Figure 2.1, the longest cycles are not pairwise intersecting. Longest cycles do, however, pairwise intersect in 2-connected graphs, as is well-known and a consequence of Lemma 5 .

One useful tool for proving both those claims is Menger's Theorem, given below. Given sets of vertices $X$ and $Y$ of $G$, an $(X, Y)$-separator is a set of vertices $S$ such that no path in $G-S$ has one endpoint

in $X$ and the other endpoint in $Y$. We allow an $(X, Y)$-separator to contain vertices in $X$ and $Y$. An $(X, Y)$-connector is a collection of vertex-disjoint paths $\left\{P_{1}, \ldots, P_{k}\right\}$ such that each $P_{i}$ has one endpoint in $X$, the other endpoint in $Y$, and the interior vertices of $P_{i}$ are outside $X \cup Y$. A variant of Menger's Theorem asserts that the minimum size of an $(X, Y)$-separator equals the maximum size of an $(X, Y)$-connector (see, e.g., Theorem 3.3.1 in [13]).

Lemma 5. If $G$ is 2-connected and $C_{1}$ and $C_{2}$ are longest cycles in $G$, then $C_{1} \cup C_{2}$ is 2-connected. In particular, $\left|V\left(C_{1}\right) \cap V\left(C_{2}\right)\right| \geq 2$.

Proof. Note that $C_{1} \cup C_{2}$ is connected, or else using 2-connectivity of $G$ and Menger's theorem, we obtain two disjoint paths joining $C_{1}$ and $C_{2}$ and hence a longer cycle. Let $H=C_{1} \cup C_{2}$, and suppose for a contradiction that $z$ is a cut vertex in $H$. Note that $V\left(C_{1}\right)$ and $V\left(C_{2}\right)$ cannot both contain a vertex $y$ with $y \neq z$, since then $C_{1}-z$ and $C_{2}-z$ would be in the same component of $H-z$, contradicting that $H-z$ is disconnected. Since $V\left(C_{1}\right)$ and $V\left(C_{2}\right)$ intersect, it follows that $V\left(C_{1}\right) \cap V\left(C_{2}\right)=\{z\}$. Since $G-z$ is connected, there is a path $P$ in $G$ joining $V\left(C_{1}-z\right)$ and $V\left(C_{2}-z\right)$. Let $x$ be the endpoint of $P$ in $C_{1}-z$ and let $y$ be the endpoint of $P$ in $C_{2}-z$. We form a cycle $C$ by combining $P$ with the longer $x z$ subpath of $C_{1}$ and the longer $y z$ subpath of $C_{2}$. Since $C$ is longer than $C_{1}$ and $C_{2}$, we obtain a contradiction.

The previous lemma tells us longest cycles are pairwise intersecting in 2-connected graphs. In fact, it tells us they must intersect at least twice. With the stronger assumption of 2-connectivity, we can try to improve upon $\operatorname{lct}(G) \leq\left\lceil\frac{n}{3}\right\rceil$. Hence, when dealing with longest cycles, we will require graphs to be 2 connected. In most known cases, a technique to get a bound on longest paths in connected graphs can be applied to get a longest cycle transversal in 2-connected graphs and vice versa.

The first significant improvement to the $\frac{n}{4}$ and $\frac{n}{3}$ bounds for longest path and longest cycle transversal number was by Rautenbach and Sereni when they proved the following two theorems.
Theorem 6. If $G$ is a connected n-vertex graph, then $\operatorname{lpt}(G) \leq\left\lceil\frac{n}{4}-\frac{n^{\frac{2}{3}}}{90}\right\rceil$ [14]
Theorem 7. If $G$ is a 2 -connected n-vertex graph, then $\operatorname{lct}(G) \leq\left\lceil\frac{n}{3}-\frac{n^{\frac{2}{3}}}{36}\right\rceil$ [14]
We were able to improve the upper bounds on $\operatorname{lpt}(G)$ and $\operatorname{lct}(G)$ to sublinear bounds. To do this, we recognized longest path transversals and longest cycle transversals are special cases of a more general transversal problem. These improvements, detailed in the rest of this chapter, appeared in our article Sublinear longest path transversals[6].

### 2.2 Maximum Subdivision Transversals

Given a multigraph $F$ and an edge $e \in E(F)$ with endpoints $u$ and $v$, the subdivision operation produces a new multigraph $F^{\prime}$ in which $e$ is replaced by a path $u w v$ through a new vertex $w$ in $F^{\prime}$. A subdivision of $F$ is a graph obtained from a sequence of zero or more subdivision operations. For a multigraph $R$ and a graph $G$, an $R$-subdivision in $G$ is a subgraph of $G$ isomorphic to a subdivision of $R$. We ask for a small set of vertices in $G$ that intersect every $R$-subdivision in $G$ of maximum size. The cases of longest path transversals and longest cycle transversals arise as $R=P_{2}$ and $R=C_{2}$ (the multigraph 2-vertex cycle), respectively. We prove that for each connected multigraph $R$, if the family $\mathcal{F}$ of maximum $R$-subdivisions in $G$ is pairwise intersecting, then $\mathcal{F}$ admits a transversal of size at most $C n^{3 / 4}$, where $C$ is a constant depending on $R$.

Let $R$ be a multigraph. Recall that an $R$-subdivision in $G$ is a subgraph of $G$ isomorphic to a subdivision of $R$, and a maximum $R$-subdivision is an $R$-subdivision $F$ in $G$ that maximizes $|V(F)|$. An $R$-transversal of $G$ is a set of vertices intersecting each maximum $R$-subdivision. Let $\tau_{R}(G)$ be the minimum size of an $R$-transversal in $G$.

Our next result shows that when the maximum $R$-subdivisions in a graph $G$ pairwise intersect, $G$ has sublinear $R$-transversals. We make no attempt to optimize the multiplicative constant 8 or the dependence on $m$.

Theorem 8. Let $R$ be a connected m-edge multigraph with $m \geq 1$ and let $G$ be an $n$-vertex graph. If the maximum $R$-subdivisions in $G$ pairwise intersect, then $\tau_{R}(G) \leq 8 m^{5 / 4} n^{3 / 4}$.

Proof. Let $m=|E(R)|$ and let $\varepsilon=2(m / n)^{1 / 4}$. We may assume that $m \leq n$, since otherwise we may take $V(G)$ as our $R$-transversal. Let $\mathcal{F}$ be the family of maximum $R$-subdivisions in $G$. An $\varepsilon$-partial transversal is a triple $(H, X, Y)$ such that $H$ is a subgraph of $G, X=V(G)-V(H), Y \subseteq X$ with $|Y| \leq \varepsilon|X|$, and each $F \in \mathcal{F}$ is a subgraph of $H$ or contains a vertex in $Y$. Given an $\varepsilon$-partial transversal $(H, X, Y)$, we either obtain an $\varepsilon$-partial transversal $\left(H^{\prime}, X^{\prime}, Y^{\prime}\right)$ with $\left|V\left(H^{\prime}\right)\right|<|V(H)|$ or we produce an $R$-transversal with at most $8 m^{5 / 4} n^{3 / 4}$ vertices. Starting with $(H, X, Y)=(G, \varnothing, \varnothing)$ and iterating gives the result.

Let $(H, X, Y)$ be an $\varepsilon$-partial transversal, and let $\mathcal{F}_{0}$ be the set of $F \in \mathcal{F}$ such that $F$ is a subgraph of $H$. We may assume that $H$ contains vertex-disjoint paths $P_{1}$ and $P_{2}$ each of size $\lceil\varepsilon n\rceil$. Otherwise, every path in $H$ has size less than $2\lceil\varepsilon n\rceil$, and so each $F \in \mathcal{F}_{0}$ has at most $2 m\lceil\varepsilon n\rceil$ vertices. Since $\mathcal{F}_{0}$ is pairwise intersecting, we have that $V(F) \cup Y$ is an $R$-transversal for each $F \in \mathcal{F}_{0}$. It follows that $\tau_{R}(G) \leq|Y|+2 m\lceil\varepsilon n\rceil \leq \varepsilon n+2 m\lceil\varepsilon n\rceil \leq(2 m+1) \varepsilon n+2 m \leq(2 m+2) \varepsilon n \leq 4 m \varepsilon n=8 m^{5 / 4} n^{3 / 4}$.

Suppose that $H$ has a $\left(V\left(P_{1}\right), V\left(P_{2}\right)\right)$-separator $S$ of size at most $\varepsilon^{2} n$. Since graphs in $\mathcal{F}_{0}$ are connected, each $F \in \mathcal{F}_{0}$ has a vertex in $S$ or is contained in some component of $H-S$. Also, since $\mathcal{F}_{0}$ is pairwise intersecting, at most one component $H^{\prime}$ of $H-S$ contains graphs in $\mathcal{F}_{0}$. Since $S$ is a separator, $H^{\prime}$ is disjoint from at least one of $\left\{P_{1}, P_{2}\right\}$. With $X^{\prime}=V(G)-V\left(H^{\prime}\right)$ and $Y^{\prime}=Y \cup S$, we have $\left|X^{\prime}\right|-|X| \geq \varepsilon n$ and $\left|Y^{\prime}\right|=|Y|+|S| \leq \varepsilon|X|+\varepsilon^{2} n \leq \varepsilon|X|+\varepsilon\left(\left|X^{\prime}\right|-|X|\right) \leq \varepsilon\left|X^{\prime}\right|$. It follows that $\left(H^{\prime}, X^{\prime}, Y^{\prime}\right)$ is an $\varepsilon$-partial transversal. Also $\left|V\left(H^{\prime}\right)\right|<|V(H)|$ since $\left|X^{\prime}\right|>|X|$.

Otherwise, by Menger's Theorem, $H$ has a $\left(V\left(P_{1}\right), V\left(P_{2}\right)\right)$-connector $\mathcal{P}$ with $|\mathcal{P}| \geq \varepsilon^{2} n$. Let $\mathcal{P}^{\prime}$ be the set of paths in $\mathcal{P}$ of size at most $2 / \varepsilon^{2}$. Note that $\left|\mathcal{P}^{\prime}\right| \geq|\mathcal{P}| / 2$, or else $\mathcal{P}$ has at least $\left(\varepsilon^{2} n\right) / 2$ paths of size more than $2 / \varepsilon^{2}$, contradicting that the paths in $\mathcal{P}$ are disjoint. So we have $\left|\mathcal{P}^{\prime}\right| \geq|\mathcal{P}| / 2 \geq\left(\varepsilon^{2} / 2\right) n=2 m^{1 / 2} n^{1 / 2} \geq 2$. Combining $P_{1}$ with two paths in $\mathcal{P}^{\prime}$ whose endpoints in $V\left(P_{1}\right)$ are as far apart as possible and a segment of $P_{2}$


Figure 2.2: $(V(C), V(F))$-connector case. The subpath $W$ of the cycle $C$ is dashed, and the cycle $D$ is displayed in bold.
gives a cycle $C_{0}$ such that $\left(\varepsilon^{2} / 2\right) n \leq\left|V\left(C_{0}\right)\right| \leq 2\lceil\varepsilon n\rceil+4 / \varepsilon^{2}-4 \leq 2 \varepsilon n+4 / \varepsilon^{2}$, where the lower bound counts vertices in $V\left(P_{1}\right) \cap V\left(C_{0}\right)$ and the upper bound counts at most $2\lceil\varepsilon n\rceil$ vertices in $\left(V\left(P_{1}\right) \cup V\left(P_{2}\right)\right) \cap V\left(C_{0}\right)$, at most $4 / \varepsilon^{2}$ vertices on the paths in $\mathcal{P}^{\prime}$ linking $P_{1}$ and $P_{2}$, and observing that the 4 endpoints of the linking paths are counted twice.

Let $C$ be a longest cycle in $H$ subject to $|V(C)| \leq 2 \varepsilon n+4 / \varepsilon^{2}$, let $\ell=|V(C)|$, and note that $\ell \geq$ $\left|V\left(C_{0}\right)\right| \geq\left(\varepsilon^{2} / 2\right) n$. If $V(C)$ intersects each subgraph in $\mathcal{F}_{0}$, then $Y \cup V(C)$ witnesses $\tau_{R}(G) \leq|V(C)|+|Y| \leq$ $\left(2 \varepsilon n+4 / \varepsilon^{2}\right)+\varepsilon n=3 \varepsilon n+(n / m)^{1 / 2}<8 m^{5 / 4} n^{3 / 4}$. Otherwise, choose $F \in \mathcal{F}_{0}$ that is disjoint from $C$. We may assume $|V(F)| \geq \ell$, or else $Y \cup V(F)$ witnesses that $\tau_{R}(G) \leq|V(F)|+|Y|<\left(2 \varepsilon n+4 / \varepsilon^{2}\right)+\varepsilon n<8 m^{5 / 4} n^{3 / 4}$.

If $H$ has a $(V(C), V(F))$-separator $T$ of size at most $\varepsilon \ell$, then we obtain an $\varepsilon$-partial transversal as follows. At most one component $H^{\prime}$ of $H-T$ contains graphs in $\mathcal{F}_{0}$. Let $X^{\prime}=V(G)-V\left(H^{\prime}\right)$ and let $Y^{\prime}=Y \cup T$. Since $H^{\prime}$ is disjoint from one of $\{C, F\}$, it follows that $\left|X^{\prime}\right|-|X| \geq \ell$. We compute $\left|Y^{\prime}\right|=|Y|+|T| \leq \varepsilon|X|+\varepsilon \ell \leq \varepsilon|X|+\varepsilon\left(\left|X^{\prime}\right|-|X|\right) \leq \varepsilon\left|X^{\prime}\right|$. Hence $\left(H^{\prime}, X^{\prime}, Y^{\prime}\right)$ is an $\varepsilon$-partial transversal with $\left|V\left(H^{\prime}\right)\right|<|V(H)|$.

Otherwise, $H$ has a $(V(C), V(F))$-connector $\mathcal{Q}$ with $|\mathcal{Q}| \geq \varepsilon \ell$. We use $\mathcal{Q}$ to obtain a contradiction. For $e \in E(R)$, let $Q_{e}$ be the path in $F$ corresponding to $e$, and let $\mathcal{Q}_{e}$ be the set of paths in $\mathcal{Q}$ which have an endpoint in $Q_{e}$. Since $|E(R)|=m$, it follows that $\left|\mathcal{Q}_{e}\right| \geq|\mathcal{Q}| / m \geq \varepsilon \ell / m$ for some edge $e \in E(R)$. Let $\mathcal{Q}^{\prime}$ be the set of paths in $\mathcal{Q}_{e}$ of size at most $\frac{2 m n}{\varepsilon \ell}$, and note that $\left|\mathcal{Q}^{\prime}\right| \geq\left|\mathcal{Q}_{e}\right| / 2 \geq \frac{\varepsilon \ell}{2 m}$, or else $\mathcal{Q}_{e}$ has at least $\frac{\varepsilon \ell}{2 m}$ paths of size more than $\frac{2 m n}{\varepsilon \ell}$, a contradiction. The endpoints of paths in $\mathcal{Q}^{\prime}$ divide $Q_{e}$ into $\left|\mathcal{Q}^{\prime}\right|-1$ edge-disjoint subpaths. Choose $Q_{1}, Q_{2} \in \mathcal{Q}^{\prime}$ to minimize the length of such a subpath $Q_{0}$ of $Q_{e}$, and note that $Q_{0}$ has length at most $\frac{n-1}{\left|\mathcal{Q}^{\prime}\right|-1}$; see Figure 2.2. Since $m \leq n$, we have $2 m \leq 2 m^{3 / 4} n^{1 / 4}=\frac{\varepsilon^{3}}{4} n \leq \frac{\varepsilon \ell}{2}$, and hence $\frac{n-1}{\left|\mathcal{Q}^{\prime}\right|-1}<\frac{n}{\frac{\varepsilon \ell}{2 m}-1}=\frac{2 m n}{\varepsilon \ell-2 m} \leq \frac{4 m n}{\varepsilon \ell}$.

The endpoints of $Q_{1}$ and $Q_{2}$ on $C$ partition $C$ into two subpaths; let $W$ be the longer subpath. If $|E(W)| \geq\left|E\left(Q_{0}\right)\right|$, then we would obtain a larger $R$-subdivision by using $Q_{1}, W$, and $Q_{2}$ to bypass $Q_{0}$. Since $F$ is a maximum $R$-subdivision, we have $|E(W)|<\left|E\left(Q_{0}\right)\right|$. Therefore using $Q_{1}, Q_{0}$, and $Q_{2}$ to bypass $W$ gives a cycle $D$ with $|E(D)|>|E(C)|$. By the extremal choice of $C$, it follows that $|V(D)|>2 \varepsilon n+4 / \varepsilon^{2}$.

On the other hand, $|V(D)|=|E(D)| \leq \frac{\ell}{2}+\left|E\left(Q_{1}\right)\right|+\left|E\left(Q_{0}\right)\right|+\left|E\left(Q_{2}\right)\right| \leq \frac{\ell}{2}+\frac{2 m n}{\varepsilon \ell}+\frac{4 m n}{\varepsilon \ell}+\frac{2 m n}{\varepsilon \ell}=\frac{\ell}{2}+\frac{8 m n}{\varepsilon \ell}$.
Therefore $2 \varepsilon n+\frac{4}{\varepsilon^{2}}<|V(D)| \leq \frac{\ell}{2}+\frac{8 m n}{\varepsilon \ell} \leq \varepsilon n+\frac{2}{\varepsilon^{2}}+\frac{8 m n}{\varepsilon \ell} \leq \varepsilon n+\frac{2}{\varepsilon^{2}}+\frac{16 m}{\varepsilon^{3}}$, where the last inequality uses $\ell \geq\left(\varepsilon^{2} / 2\right) n$. Simplifying gives $\varepsilon n<\frac{16 m}{\varepsilon^{3}}-\frac{2}{\varepsilon^{2}}<\frac{16 m}{\varepsilon^{3}}$, and this inequality is violated when $\varepsilon \geq(16 m / n)^{1 / 4}$.

Applying Theorem 8, we obtain the following corollary.
Corollary 9. Let $G$ be an $n$-vertex graph. If $G$ is connected, then $\operatorname{lpt}(G) \leq 8 n^{3 / 4}$. If $G$ is 2 -connected, then $\operatorname{lct}(G) \leq 20 n^{3 / 4}$.

Proof. When $R=P_{2}$, an $R$-transversal is a longest path transversal. As we have seen in Proposition 1 longest paths in a connected graph are pairwise intersecting. By Theorem 8 , we have $\operatorname{lpt}(G)=\tau_{R}(G) \leq 8 n^{3 / 4}$.

Similarly, when $R=C_{2}$, an $R$-transversal is a longest cycle transversal. If $G$ is 2 -connected, then the longest cycles pairwise intersect, as we have seen in Lemma 5. By Theorem 8, we have $\operatorname{lct}(G)=\tau_{R}(G) \leq$ $8 \cdot 2^{5 / 4} \cdot n^{3 / 4} \leq 20 n^{3 / 4}$.

We do not know whether the assumption in Theorem 8 that $R$ is connected is necessary to obtain sublinear $R$-transversals. To obtain analogues of Corollary 9 for general $R$, we show that the maximum $R$-subdivisions pairwise intersect when the connectivity of $G$ is sufficiently large.

Lemma 10. Let $R$ be a connected $m$-edge multigraph with $m \geq 1$. If $\kappa(G)>m^{2}$, then the maximum $R$-subdivisions in $G$ are pairwise intersecting.

Proof. Suppose for a contradiction that $G$ has disjoint maximum $R$-subdivisions $F_{1}$ and $F_{2}$, and let $k=$ $\left|V\left(F_{1}\right)\right|=\left|V\left(F_{2}\right)\right|$. By Menger's Theorem, there is an $\left(V\left(F_{1}\right), V\left(F_{2}\right)\right)$-connector $\mathcal{P}$ with $|\mathcal{P}|=\min \left\{k, m^{2}+1\right\}$. If $|\mathcal{P}|=k$, then every vertex in $F_{1}$ is an endpoint of a path in $\mathcal{P}$, and we obtain an $R$-subdivision of size more than $k$ by replacing an edge $u v \in E\left(F_{1}\right)$ with a path in $\mathcal{P}$ having $u$ as an endpoint, a path in $\mathcal{P}$ having $v$ as an endpoint, and an appropriate path in the connected subgraph $F_{2}$.

So we may assume $|\mathcal{P}|=m^{2}+1$. For each $e \in E(R)$, let $F_{i}(e)$ be the path in $F_{i}$ corresponding to $e$. Since $R$ has no isolated vertices, we may associate each $P \in \mathcal{P}$ with an ordered pair of edges $\left(e_{1}, e_{2}\right) \in(E(R))^{2}$ such that $P$ has its endpoint in $F_{1}$ in $F_{1}\left(e_{1}\right)$ and its endpoint in $F_{2}$ in $F_{2}\left(e_{2}\right)$. Since $|\mathcal{P}|>m^{2}$, some pair $\left(e_{1}, e_{2}\right)$ is associated with distinct paths $P, Q \in \mathcal{P}$. Let $W_{i}$ be the subpath of $F_{i}\left(e_{i}\right)$ whose endpoints are in $V(P) \cup V(Q)$. If $\left|E\left(W_{1}\right)\right| \geq\left|E\left(W_{2}\right)\right|$, then we modify $F_{2}$ to obtain a larger $R$-subdivision by using $P, W_{1}$, and $Q$ to bypass $W_{2}$. Similarly, if $\left|E\left(W_{2}\right)\right| \geq\left|E\left(W_{1}\right)\right|$, then we modify $F_{1}$ to obtain a larger $R$-subdivision by using $P, W_{2}$, and $Q$ to bypass $W_{1}$.

Corollary 11. Let $R$ be a connected $m$-edge multigraph. If $G$ is an n-vertex graph with $\kappa(G)>m^{2}$, then $\tau_{R}(G) \leq 8 m^{5 / 4} n^{3 / 4}$.

Note, we make no real attempt to minimize the required connectivity to guarantee pairwise intersection in Lemma 10. This is because, in the two cases were care most about, paths and cycles, we already know we need connected and two connected graphs, respectively.

This result has since been improved by Kierstead and Ren (2023+) in the specific case of longest path and cycle transversals to $O\left(n^{2 / 3}\right)$ (personnel communication).

## Chapter 3

## Gallai Families

Instead of searching for a bound on longest path transversal number for general connected graphs, we could try to find all graphs $G$ such that $\operatorname{lpt}(G)=1$. That is, we can try to characterize graphs with a positive answer to Gallai's question; with a vertex common to all longest paths. We call these graphs Gallai graphs. If we are given a family of graphs $\mathcal{G}$ whose connected members are Gallai graphs, we say $\mathcal{G}$ is a Gallai family. Additionally, if we have a vertex $v$ in a graph $G$ that is common to all longest paths in $G$ we say $v$ is a Gallai vertex. When it is clear we are talking about a family of graphs, a specific graph, or a vertex, we simply say that that family, that graph, or that vertex is Gallai.

There is, as of yet, no known characterization for Gallai graphs. There are several families of graphs that are known to be Gallai, and there are a handful of graph constraints which also guarantee Gallai vertices. Conversely, there are a few families of graphs which are known not to be Gallai.

Given a graph $G$ and a graph $H$ we say $G$ is $H$-free if $G$ does not have $H$ as an induced subgraph. If $\mathcal{G}$ is a family of graphs that are $H$-free, then we say $\mathcal{G}$ is $H$-free. If $\mathcal{G}$ is the family of all $H$-free graphs, we call $H$ the forbidden subgraph for $\mathcal{G}$. As an extension, given a family of graphs $\mathcal{H}$, we can define $\mathcal{H}$-free graphs. We say that a graph $G$ is $\mathcal{H}$-free if $G$ is $H$-free for each $H \in \mathcal{H}$.

Several known Gallai families, including many of our own results are about families of graph characterized by forbidden subgraphs.

### 3.1 Known Gallai Families

Our next proposition shows that a family of subtrees of a tree have the Helly property.
Proposition 12. A family of pairwise intersecting subtrees of a tree have non-empty intersection.

Proof. Let $T$ be a tree and let $\mathcal{S}$ be a family of pairwise intersecting subtrees of $T$. Assume, by way of contradiction, no vertex of $T$ is common to each member of $\mathcal{S}$. For each vertex $v$ of $T$, there is exactly one component of $T-v$ containing a subtree of $\mathcal{S}$. There must be at least one since $v$ is not contained in every subtree of $\mathcal{S}$ and there cannot be two components containing members of $\mathcal{S}$ as those subtrees would not intersect. Let $u$ be the neighbor of $v$ in the component of $T-v$ containing a member of $\mathcal{S}$. We construct an auxiliary digraph $T^{\prime}$ with vertex set $V(T)$ by adding the directed edge $v u$. Adding such a directed edge
for each vertex $v \in V(T)$ results in a digraph $T^{\prime}$ with $|V(T)|$ edges. Since $T$ only has $|V(T)|-1$ edges, there is some edge of $u v$ in $T$ such that $u v$ and $v u$ both appear in $E\left(T^{\prime}\right)$. This means there is a subtree in $\mathcal{S}$ contained in the component of $T-v$ containing $u$ and a subtree of $\mathcal{S}$ contained in the component of $V(T)-u$ containing $v$. These two subtrees are disjoint, contradicting that $\mathcal{S}$ is pairwise intersecting.

Corollary 13. Trees form a Gallai family.

Proof. Note that the family of longest paths in any tree $T$ is a pairwise intersecting family of subtrees in $T$. The result follows from Proposition 12.

A set of vertices $S$ in a graph $G$ is independent if every pair of vertices in $S$ is nonadjacent. The size of a largest independent set in a graph $G$ is called the independence number of the graph and is denoted $\alpha(G)$.

Split graphs, which are graphs whose vertex set can be partitioned into an independent set and a clique, are a known Gallai family. Additionally, cacti graphs, which are graphs in which any two cycles share at most one vertex, is a known Gallai family. Both results are shown by Klavžar and Petkovšek [15].

An intersection graph is a graph whose vertices are sets such that two vertices are adjacent if and only if they intersect. An interval graph is an intersection graph where the vertices are intervals on the real line. A circular arc graph is an intersection graph whose vertices are arcs on a circle. It was claimed in [16] that circular arc graphs were a Gallai family. However, Joos found a missing case in the proof, which he solved [17]. The former paper does solve the problem for interval graphs while the latter gives a completion of the argument for circular arc graphs.

Let $G$ be a graph and $s$ and $t$ be two of its vertices. We say $G$ is series parallel with terminals $s$ and $t$ if it can be turned into the edge st be a sequence of the following operations: replacement of a pair of parallel edges with a single edge that connects their common endpoints or replacement of a pair of edges incident to a vertex of degree 2 other than $s$ or $t$ with a single edge. A graph $G$ is 2 -terminal series parallel if there are vertices $s$ and $t$ in $G$ such that $G$ is series parallel with terminals $s$ and $t$. A graph $G$ is series parallel if each of its 2-connected components is a 2 -terminal series parallel graph. Chen et al. showed that each series parallel graph has a Gallai vertex [18].

A matching in a graph is a set of edges having no common endpoints. The matching number of a graph $G$, denoted $\nu(G)$, is the size of a largest matching in $G$. Chen showed connected graphs with $\nu(G) \leq 3$ are Gallai [19].

A chordal graph is a graph having no induced cycle of length more than three. A vertex $u$ is a maximum neighbor of a vertex $v$ if $N(w) \subseteq N(u)$ for each $w \in N(v)$. Note that $u=v$ is allowed in this definition. A maximum neighborhood ordering of a graph $G$ is a linear ordering $v_{1}, \ldots, v_{n}$ of $V$ such that $v_{i}$ has a maximum neighbor in $G_{i}$ where $G_{i}$ is the subgraph of $G$ induced by $\left\{v_{i}, \ldots, v_{n}\right\}$ for all $i$ from 1 to $n$. A graph is dually chordal if it has a maximum neighborhood ordering. Jobson et al. showed dually chordal graphs are a Gallai family [20].

A permutation graph is a graph whose vertices represent the elements of a permutation and whose edges represent pairs of elements reversed by the permutation. A bipartite permutation graph is a permutation graph that is also bipartite. Cerioli et al. showed bipartite permutation graphs are Gallai [21].

Golan and Shan showed $2 K_{2}$-free graphs are Gallai [22]. A graph is $P_{4}$-sparse is any set of five vertices induce at most one chordless path. Cerioli and Lima showed both $P_{4}$-sparse and $\left\{P_{5}, K_{1,3}\right\}$-free graphs are Gallai [23].

The graph $B_{i, j}$ has a triangle consisting of the vertices $x, y$, and $z$ where $x$ is also an endpoint of a path of length $i$ and $y$ is an endpoint of a path of length $j$. The graph $Z_{i}$ is equivalent to $B_{i, 0}$. Gao and Shan showed that the $\left\{K_{1,3}, H\right\}$-free graphs are a Gallai family for every $H$ in $\left\{C_{3}, P_{4}, P_{5}, P_{6}, Z_{1}, Z_{2}, Z_{3}, B_{1,1}, B_{1,2}\right\}$ [24].

It is entirely reasonable to pick our favorite graph family $\mathcal{H}$ and see if forbidding $\mathcal{H}$ results in a Gallai family. A more structured approach though is to look at the Peterson fragment $G_{0}$ and see what is 'broken' and see what 'fixes' it. That is, we can look at induced subgraphs in $G_{0}$ and forbid them.

We denote the family of $H$-free graphs Free $(H)$. We say $H$ is a fixer if Free $(H)$ is a Gallai family. It is so named since forbidding $H$ 'fixes' Gallai's question.

A linear forest is forest whose components are paths.
Proposition 14. If $H$ is a fixer, then $H$ is a linear forest on at most 9 vertices.

Proof. Let $H$ be a fixer. By definition, if $G$ is a graph with $\operatorname{lpt}(G)>1$, then $H$ is an induced subgraph of $G$.
Recall the Peterson fragment $G_{0}$ obtained from the Peterson graph (Figure 1.2) and the fact every path in $G_{0}$ omits at least 2 vertices. Let $R$ be the three vertices of degree one in $G_{0}$. Since the Petersen graph is vertex-transitive [25] and has a 9-cycle, it follows that for each vertex $x \in V\left(G_{0}\right)-R$, there is a longest path in $G_{0}$ with both ends in $R$ that omits only $x$ and the other vertex in $R$.

Let $M$ be the set of 3 edges incident to the vertices in $R$. Let $G_{1}$ be the graph obtained from $G_{0}$ by replacing each edge in $M$ with a path of length $q$ and replacing each edge outside $M$ with a path of length $p$, where $p>|V(H)|$. Provided that $q>\left|E\left(G_{0}\right)\right| \cdot p$, the longest paths in $G_{1}$ are in bijective correspondence with the longest paths in $G_{0}$ that have both ends in $R$. Recalling that, for each $x \in V\left(G_{0}\right)-R$, there is a longest path in $G_{0}$ with both ends in $R$ that omits $x$, we have $\operatorname{lpt}\left(G_{1}\right)>1$. Since $G_{1}$ has girth larger than $|V(H)|$ and $H$ is an induced subgraph of $G_{1}$, it follows that $H$ is acyclic.

Let $S$ be the set of cubic vertices in $G_{1}$. We obtain $G_{2}$ from $G_{1}$ by replacing each vertex $w \in S$ with a triangle $T_{w}$ such that the three edges incident to $w$ in $G_{1}$ are incident to distinct vertices of $T_{w}$ in $G_{2}$. Clearly, $G_{2}$ is claw-free. Let $P$ be a longest path in $G_{2}$. Again, provided that $q$ is sufficiently large, $P$ has its ends in $R$. When $P$ visits a vertex in some $T_{w}$, it must visit all vertices in $T_{w}$ before leaving. It follows that the longest paths in $G_{2}$ are in bijective correspondence with the longest paths in $G_{1}$ and $\operatorname{lpt}\left(G_{2}\right)>1$.

Since $H$ is an induced subgraph of $G_{1}$ and $G_{2}$, it follows that $H$ is triangle-free and claw-free, and so $\Delta(H) \leq 2$. Recalling that $H$ is acyclic, we have that $H$ is a linear forest. But $H$ is also an induced subgraph of $G_{0}$ and to obtain an induced linear forest as a subgraph of $G_{0}$, a vertex must be deleted from the closed neighborhood of each cubic vertex of $G_{0}$. Let $R^{\prime}$ be the set of neighbors of vertices in $R$. Since the vertices in $R^{\prime}$ are cubic and have disjoint closed neighborhoods, each induced linear forest has at most $\left|V\left(G_{0}\right)\right|-\left|R^{\prime}\right|$ vertices, and so $|V(H)| \leq\left|V\left(G_{0}\right)\right|-\left|R^{\prime}\right|=12-3=9$.

Remark 15. Gao and Shan asked whether all longest paths in a connected claw-free graph have a non-empty intersection [24]. Proposition 14 answers this question in the negative.

For $|V(H)| \leq 4$, we show that $H$ is a fixer if and only if $H$ is a linear forest. Necessity follows from Proposition 14. For sufficiency, we show that every 4-vertex linear forest is a fixer. The linear forests of order 4 are $P_{4}, P_{3}+P_{1}, 2 P_{2}, P_{2}+2 P_{1}$, and $4 P_{1}$. Cerioli and Lima showed that $P_{4}$-sparse graphs, a superclass of $P_{4}$-free graphs, form a Gallai family [23], whereas Gao and Shan showed that $2 P_{2}$-free graphs form a Gallai family [22]. In other words, $P_{4}$ and $2 P_{2}$ are fixers. In the following, we address the cases: $P_{3}+P_{1}, P_{2}+2 P_{1}$, and $4 P_{1}$. These, along with the rest of the results in the chapter appear in our paper Non-empty Intersection of Longest Paths in $H$-free Graphs [7].

Note that if $H$ and $K$ are graphs such that $H$ is a subgraph of $K$, then $\operatorname{Free}(H) \subseteq \operatorname{Free}(K)$ as any graph that is $H$-free is also $K$-free, however a graph that is $K$-free may not be $H$-free. This means, for example, showing $4 P_{1}$ is a fixer also shows any subgraph of $4 P_{1}$ is a fixer. Since the cases mentioned above turn out to be fixers, they also show the linear forests on fewer than 4 vertices are also fixers.


Figure 3.1: The linear forests on 4 vertices. These are exactly the graphs $H$ on 4 vertices such that Free $(H)$ is a Gallai family.

We begin with some basic but useful observations. Given vertices $x, y \in V(G)$, an $x y$ - fiber is a longest path among all the $x y$-paths. Similarly, an $x$-fiber is a longest path among all the paths having $x$ as an endpoint, and a fiber is a longest path in $G$. Note that every fiber is an $x$-fiber for some vertex $x$, and every $x$-fiber is an $x y$-fiber for some vertex $y$.

The following two basic lemmas are used repeatedly, sometimes implicitly. Similar ideas are key to the results in [26]. The first basic lemma treats single neighbors of fibers.

Lemma 16. Let $P$ be an $x y$-path in a graph $G$, where $P=v_{0} \cdots v_{\ell}$ with $x=v_{0}$ and $y=v_{\ell}$. Let $H$ be $a$ component of $G-V(P)$ with a neighbor $v_{i}$ on $P$. If $P$ is an $x$-fiber, then $i<\ell$. Moreover, if $0<i$, then $v_{\ell} v_{i-1} \notin E(G)$. Similarly, if $P$ is a $y$-fiber, then $0<i$, and if $i<\ell$, then $v_{0} v_{i+1} \notin E(G)$.

Proof. Suppose $P$ is an $x$-fiber. No vertex in $H$ is adjacent to $y$, or else $P$ extends to a longer $x$-fiber, a contradiction. Therefore, $i<\ell$. Also, if $i>0$ and $v_{i-1} v_{\ell} \in E(G)$, then following $P$ from $v_{0}$ to $v_{i-1}$, traversing $v_{i-1} v_{\ell}$, following $P$ backward from $v_{\ell}$ to $v_{i}$, and traveling to $H$ produces a longer $x$-fiber. The case that $P$ is a $y$-fiber is symmetric.

In many of our arguments, we show that a path $P$ in $G$ has some desired property or else we obtain a longer path. We now formalize two common ways to obtain longer paths. Given two lists of objects $a$ and $b$, a splice of $a$ with $b$ is a sequence obtained from $a$ by (1) replacing a non-empty interval of $a$ with $b$, or (2) inserting $b$ between consecutive elements in $a$, or (3) prepending or appending $b$ to $a$. Given a host path $P$ and a patching path $Q$, a splice of $P$ with $Q$ is a path whose vertices are ordered according to a splice of the ordered list of vertices in $P$ with the ordered list of vertices in $Q$. A splice of $P$ that has the same endpoints as $P$ is an interior splice; otherwise, the splice is exterior.

A detour of an $x y$-path $P$ is a path obtained from $P$ by using two patching paths $Q_{1}$ and $Q_{2}$ as follows. Suppose that $Q_{i}$ is a $u_{i} w_{i}$-path for $i \in\{1,2\}$ and $u_{1}, u_{2}, w_{1}, w_{2}$ are distinct vertices appearing in order along
$P$. We follow $P$ from $x$ to $u_{1}$, traverse $Q_{1}$, follow $P$ backward from $w_{1}$ to $u_{2}$, traverse $Q_{2}$, and finally follow $P$ from $w_{2}$ to $y$.

Note that our definitions of a splice and detour require the resulting object to be a path and therefore implicitly impose certain disjointness conditions on segments of the host and the patching paths. Also, note that interior splices and detours of $P$ have the same endpoints as $P$. A splice or detour of $P$ is augmenting if it is longer than $P$.

Let $P$ be a path in $G$ and let $H$ be a component of $G-V(P)$. A vertex $s \in V(P)$ with a neighbor in $H$ is an attachment point of $H$. Our next lemma concerns pairs of attachment points.

Lemma 17. Let $P$ be an xy-path in a graph $G$ and let $H$ be a component of $G-V(P)$ with attachment points $s$ and $s^{\prime}$, where $s$ appears before $s^{\prime}$ when traversing $P$ from $x$ to $y$. The following hold.

1. If $s$ and $s^{\prime}$ are consecutive on $P$, then there is an augmenting interior splice of $P$.
2. If $s$ and $s^{\prime}$ are not consecutive along $P, w$ and $w^{\prime}$ immediately follow $s$ and $s^{\prime}$ respectively, and $w w^{\prime} \in$ $E(G)$, then there is an augmenting detour of $P$.
3. If $s$ and $s^{\prime}$ are not consecutive along $P, w$ and $w^{\prime}$ immediately precede $s$ and $s^{\prime}$ respectively, and $w w^{\prime} \in E(G)$, then there is an augmenting detour of $P$.

Proof. For (1), since $s$ and $s^{\prime}$ are consecutive attachment points on $P$, we obtain an augmenting interior splice by inserting an appropriate path in $H$ between $s$ and $s^{\prime}$. For (2), let $Q_{1}$ be an $s s^{\prime}$-path with interior vertices in $H$ and let $Q_{2}$ be the path $w w^{\prime}$. There is an augmenting detour of $P$ using patching paths $Q_{1}$ and $Q_{2}$. The case (3) is symmetric.

When $P$ is a kind of fiber and a component $H$ of $G-V(P)$ has many attachment points, our next lemma obtains a large independent set contained in $P$ consisting of non-attachment points.

Lemma 18. Let $P$ be an xy-path in $G$, let $H$ be a component of $G-V(P)$, let $k$ be the number of attachment points of $H$. There is an independent set $A$ of $G$ such that $A \subseteq V(P)$, no edge joins a vertex in $A$ and $a$ vertex in $V(H)$, and the following hold.

1. If $P$ is an $x y$-fiber, then $A \subseteq V(P)-\{x, y\}$ and $|A| \geq k-1$.
2. If $P$ is an $x$-fiber, then $A \subseteq V(P)-\{x\}$ and $|A| \geq k$.
3. If $P$ is a fiber, then $A \subseteq V(P)$ and $|A| \geq k+1$.

Proof. Let $s_{1}, \ldots, s_{k}$ be the attachment points of $H$, with indices increasing from $x$ to $y$ along $P$, and let $S=\left\{s_{1}, \ldots, s_{k}\right\}$ (see Figure 3.2).

For (1), let $A$ be the set of vertices in $P$ that immediately follow some $s_{i}$ with $1 \leq i<k$. Since $P$ is an $x y$-fiber, Lemma 17 implies that $s_{i}$ and $s_{i+1}$ are not consecutive along $P$. Therefore, $S$ and $A$ are disjoint and so no vertex in $A$ has a neighbor in $H$. By Lemma 17, it follows that $A$ is an independent set.

For (2), suppose in addition that $P$ is an $x$-fiber. By Lemma $16, s_{k} \neq y$, and we may take $A$ to be the set of vertices that immediately follow some $s_{i}$ with $1 \leq i \leq k$.

For (3), suppose in addition that $P$ is a fiber. By Lemma 16, we have $s_{1} \neq x$. Let $A$ be the set of vertices that immediately follow an attachment point together with $x$. Note that since $P$ is also a $y$-fiber, it follows from Lemma 16 that $x$ has no neighbor in $A$, and so $A$ is an independent set of size $k+1$.


Figure 3.2: Construction of $A$ in the proof of Lemma 18.
We can finally show in the following sections that $P_{3}+P_{1}, P_{2}+2 P_{1}$, and $4 P_{1}$ are all fixers.

## $3.2 \quad P_{3}+P_{1}$ is a fixer

Theorem 19. If $G$ is a connected $\left(P_{3}+P_{1}\right)$-free graph, then every vertex of degree at least $\Delta(G)-1$ is a Gallai vertex.

Proof. Let $P$ be a longest path in $G$, where $P=v_{0} \cdots v_{\ell}$ with $x=v_{0}$ and $y=v_{\ell}$. Suppose for a contradiction that there is a vertex $u$ with $d(u) \geq \Delta(G)-1$ but $u \notin V(P)$. Let $H$ be the component of $G-V(P)$ containing $u$. Let $T=V(H)$, let $S$ be the set of attachment points of $H$ on $P$, let $k=|S|$, and let $t=|T|$.

Note that $H$ is a complete graph, or else an induced copy of $P_{3}$ in $H$ together with an endpoint of $P$ would induce a copy of $P_{3}+P_{1}$ in $G$. We now claim that $x v_{i} \in E(G)$ for each $v_{i} \in S$. Otherwise, by Lemma 16, given a neighbor $z$ of $v_{i}$ in $H,\left\{z, v_{i}, v_{i+1}, x\right\}$ would induce a copy of $P_{3}+P_{1}$.

Next we claim that $v_{i-1} v_{i+1} \notin E(G)$ when $v_{i} \in S$. Otherwise, we obtain a longer path by starting with a neighbor $z$ of $v_{i}$ in $H$, walking along $z v_{i} x$, following $P$ from $x$ to $v_{i-1}$, traversing $v_{i-1} v_{i+1}$, and following $P$ from $v_{i+1}$ to $y$. Therefore $z v_{i} \in E(G)$ for each $z \in T$ and $v_{i} \in S$, otherwise $\left\{z, v_{i-1}, v_{i}, v_{i+1}\right\}$ would induce a copy of $P_{3}+P_{1}$. It follows that $N(z)=(T \backslash\{z\}) \cup S$ for each $z \in T$. In particular, $d(u)=(t-1)+k$.

Next we claim that, if $v_{i}, v_{j} \in S$ with $i \neq j$, then $v_{i} v_{j+1} \in E(G)$. Otherwise, given a neighbor $z$ of $v_{i}$ in $H$, the set $\left\{z, v_{i}, v_{i+1}, v_{j+1}\right\}$ would induce a copy of $P_{3}+P_{1}$ since $v_{i+1} v_{j+1} \notin E(G)$ by Lemma 17. This implies that, if $v_{i} \in S$, then the neighborhood of $v_{i}$ contains $x, T$, and $\left\{v_{j+1}: v_{j} \in S\right\}$, and so $d\left(v_{i}\right) \geq 1+t+k$. Therefore $\Delta(G) \geq d\left(v_{i}\right) \geq d(u)+2$, a contradiction.

The degree assumption in Theorem 19 is best possible. Indeed, the complete bipartite graph $K_{t, t+2}$ is $\left(P_{3}+P_{1}\right)$-free, has maximum degree $t+2$, and the vertices of degree $t$ are not Gallai.

## $3.3 \quad P_{2}+2 P_{1}$ is a fixer

Proposition 20. If $G$ is a connected $\left(P_{2}+2 P_{1}\right)$-free graph, then every vertex of maximum degree is a Gallai vertex.

Proof. Let $G$ be a connected $\left(P_{2}+2 P_{1}\right)$-free graph and let $P=v_{0} \cdots v_{\ell}$ be a longest path in $G$ with ends $x=v_{0}$ and $y=v_{\ell}$. Suppose for a contradiction that $u$ is a vertex of maximum degree and $u \notin V(P)$. Let $k=d(u)=\Delta(G)$, and let $H$ be the component of $G-V(P)$ containing $u$, and let $k=\Delta(G)$. Note that $x y \notin E(G)$, or else we obtain a longer path by starting at a vertex in $H$ with a neighbor on $P$ and traveling around the cycle $P+x y$. Also, $V(G)-V(P)$ is an independent set, or else, by Lemma 16, an adjacent pair of vertices in $V(G)-V(P)$ together with $x$ and $y$ would induce a copy of $P_{2}+2 P_{1}$.

Let $S$ be the set of attachment points of $H$. Since $H$ has one vertex, we have $|S|=k$. Applying Lemma 18 where $H$ is graph with the single vertex $u$, there is an independent set $A \subseteq V(P)$ such that $|A|=k+1$ and $A \cap S=\varnothing$.

If some vertex $s \in S$ has two non-neighbors $w_{1}, w_{2} \in A$, then $\left\{u, s, w_{1}, w_{2}\right\}$ induces a copy of $P_{2}+2 P_{1}$. Hence every vertex in $S$ has at least $k$ neighbors in $A$. Counting $u$, every vertex in $S$ has degree at least $k+1$, contradicting that $\Delta(G)=k$.

Vertices of degree $\Delta(G)-1$ in a $\left(P_{2}+2 P_{1}\right)$-free graph $G$ need not be Gallai. Indeed, consider the graph $G$ obtained from $K_{t, t+2}$ by removing a matching saturating the part of size $t . G$ is $\left(P_{2}+2 P_{1}\right)$-free and $\Delta(G)=t+1$. The longest paths in $G$ omit one vertex, and the Gallai vertices are those in the smaller part. Two of the non-Gallai vertices in the larger part have degree $t$, which equals $\Delta(G)-1$.

## 3.4 $4 P_{1}$ is a fixer

For a path $P$ in a graph $G$ containing the vertices $x$ and $y$, the closed subpath of $P$ with boundary points $x$ and $y$, denoted $P[x, y]$, is the subpath of $P$ with endpoints $x$ and $y$. The open subpath of $P$ with boundary points $x$ and $y$, denoted $P(x, y)$, is $P[x, y]-\{x, y\}$. Additionally, we define the semi-open subpaths $P[x, y)$ and $P(x, y]$ analogously.

Let $x, y \in V(G)$, let $P$ be an $x y$-path in $G$, and let $H$ be a component of $G-V(P)$. For each nonattachment point $w \in V(P)$, we define the rank of $w$, denoted $\operatorname{rank}(w)$, to be the maximum length of a subpath of $P[x, w]$ containing $w$ but no attachment points. Note that if $s_{1}, \ldots, s_{k}$ are the attachment points with indices increasing from $x$ to $y$, then the rank of a non-attachment point $w \in V\left(P\left(s_{i}, s_{i+1}\right)\right)$ is $\operatorname{dist}_{P}\left(s_{i}, w\right)-1$.

Lemma 21. Let $P$ be an $x y$-path in a graph $G$ and let $H$ be a complete component of $G-V(P)$. Let $S$ be the set of attachment points of $H$ on $P$, where $S=\left\{s_{1}, \ldots, s_{k}\right\}$, with indices increasing from $x$ to $y$, and suppose that the induced $\left(S, V(H)\right.$ )-bigraph has a matching saturating $S_{0}$ when $S_{0} \subseteq S$ and $\left|S_{0}\right| \leq|V(H)|$. The following hold.

1. If $s_{1}=x$, then $P$ has an augmenting splice with endpoint $y$. If $s_{k}=y$, then $P$ has an augmenting splice with endpoint $x$. If $s_{i}$ and $s_{i+1}$ are consecutive on $P$, then $P$ has an augmenting interior splice.
2. If some component $P_{0}$ of $P-S$ has fewer than $|V(H)|$ vertices, then $P$ has an augmenting splice replacing $P_{0}$.
3. If $w$ and $w^{\prime}$ are in distinct components of $P-S-V\left(P\left[x, s_{1}\right]\right)$, $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<|V(H)|$, and $w w^{\prime} \in E(G)$, then $P$ has an augmenting detour.
4. If $w$ and $w^{\prime}$ are in distinct components of $P-S$, $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<|V(H)|$, and $w w^{\prime} \in E(G)$, then $G$ has a path with endpoint $y$ that is longer than $P$.

Proof. For part 1, if $s_{1}=x$ or $s_{k}=y$, then we obtain an augmenting splice of $P$ by prepending or appending a Hamiltonian path of $H$. If $s_{i}$ and $s_{i+1}$ are consecutive along $P$, then it follows from Lemma 17 that $P$ has an augmenting interior splice.

For part 2, let $P_{0}$ be a component of $P-S$ with $1 \leq\left|V\left(P_{0}\right)\right|<|V(H)|$. Note that $P_{0}$ is $P\left[x, s_{1}\right)$, or $P\left(s_{k}, y\right]$, or $P\left(s_{i}, s_{i+1}\right)$ for some $i$. Suppose that $P_{0}=P\left(s_{i}, s_{i+1}\right)$. Hence there is a matching $\left\{s_{i} z, s_{i+1} z^{\prime}\right\}$ joining $s_{i}$ and $s_{i+1}$ to distinct vertices $z, z^{\prime} \in V(H)$. Since $H$ is complete, $H$ contains a spanning $z z^{\prime}$-path $Q$. Since $\left|V\left(P_{0}\right)\right|<|V(H)|$, we obtain an augmenting interior splice by replacing $P_{0}$ with $Q$. The cases $P_{0}=P\left[x, s_{1}\right)$ and $P_{0}=P\left(s_{k}, y\right]$ are similar, except that we obtain an augmenting external splice.

For part 3, we may assume that $w$ appears before $w^{\prime}$ when traversing $P$ from $x$ to $y$. Let $i$ and $j$ be indices such that $w \in V\left(P\left(s_{i}, s_{i+1}\right)\right)$ and $w^{\prime} \in V\left(P\left(s_{j}, s_{j+1}\right)\right)$ except that we set $j=k$ if $w^{\prime} \in V\left(P\left(s_{k}, y\right]\right)$. Since $w$ and $w^{\prime}$ are in distinct components of $P-S$, we have $i<j$. If $|V(H)|=1$, then $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<|V(H)|$ implies that $w$ immediately follows $s_{i}$ and $w^{\prime}$ immediately follows $s_{j}$. By Lemma 17 part (2), we have that $P$ has an augmenting detour. Otherwise, $|V(H)| \geq 2$ and there is a matching $\left\{s_{i} z, s_{j} z^{\prime}\right\}$ joining $s_{i}$ and $s_{j}$ to distinct vertices $z, z^{\prime} \in V(H)$. Let $Q_{1}$ be an $s_{i} s_{j}$-path whose interior vertices form a spanning $z z^{\prime}$-path in $H$, and let $Q_{2}$ be the path $w w^{\prime}$. The detour of $P$ with patching paths $Q_{1}$ and $Q_{2}$ adds the vertices in $V(H)$ but omits the $\operatorname{rank}(w)$ vertices in $P\left(s_{i}, w\right)$ and the $\operatorname{rank}\left(w^{\prime}\right)$ vertices in $P\left(s_{j}, w^{\prime}\right)$. Since $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<|V(H)|$, the detour is augmenting.


Figure 3.3: Part 3 in Lemma 21.

For part 4, we may apply the argument for (2) unless $w \in V\left(P\left[x, s_{1}\right]\right)$. As before, let $j$ be the index such that and $w^{\prime} \in V\left(P\left(s_{j}, s_{j+1}\right)\right)$ except that we set $j=k$ if $w^{\prime} \in V\left(P\left(s_{k}, y\right]\right)$. We obtain a new path $P^{\prime}$ by following $P$ backward from $y$ to $w^{\prime}$, traversing $w^{\prime} w$, following $P$ forward from $w$ to $s_{j}$, traversing an edge joining $s_{j}$ and a vertex in $H$, and finishing with a Hamiltonian path in $H$. The path $P^{\prime}$ includes all of $V(H)$ but omits the $\operatorname{rank}(w)$ vertices in $P[x, w)$ and the $\operatorname{rank}\left(w^{\prime}\right)$ vertices in $P\left(s_{j}, w^{\prime}\right)$. Since $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<$ $|V(H)|$, the path $P^{\prime}$ is longer than $P$.

Our next lemma provides additional structure when $G$ is $k$-connected and $\alpha(G) \leq k+2$.
Lemma 22. Let $P$ be a longest path in $G$ with endpoints $x$ and $y$, and let $H$ be a component of $G-V(P)$. Suppose that $G$ is $k$-connected and $\alpha(G) \leq k+2$. The following hold.

1. The set $S$ of attachment points of $H$ on $P$ has size $k$.
2. The subgraph $H$ is complete.
3. The graph $P-S$ has $k+1$ components, and each has at least $|V(H)|$ vertices.
4. If $w$ and $w^{\prime}$ are in distinct components of $P-S$ and $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<|V(H)|$, then $w w^{\prime} \notin E(G)$.
5. The vertices in each component of $P-S$ of rank less than $|V(H)|$ form a clique.

Proof. Let $S=\left\{s_{1}, \ldots, s_{r}\right\}$, with indices increasing from $x$ to $y$. Since $G$ is $k$-connected and $H$ is a component of $G-V(P)$, it follows that $r \geq k$, or else $S$ separates $V(H)$ from $x$ and $y$. Since $P$ is a fiber, it follows from Lemma 18 that $G$ contains an independent set $A$ with $|A|=r+1$ such that $A \subseteq V(P)$ and no edge joins $A$ and $V(H)$. Since $1+(k+1) \leq \alpha(H)+(r+1)=\alpha(H)+|A| \leq \alpha(G) \leq k+2$, it follows that $\alpha(H)=1$ and $r=k$. Hence, there are exactly $k$ attachment points and $H$ is complete.

Let $S_{0} \subseteq S$ with $\left|S_{0}\right| \leq|V(H)|$ and let $B$ be the induced $\left(S_{0}, V(H)\right.$ )-bigraph. If $B$ has no matching saturating $S_{0}$, then Hall's Theorem [25] implies that there exists $S_{1} \subseteq S_{0}$ such that $\left|N_{B}\left(S_{1}\right)\right|<\left|S_{1}\right|$. It follows that $N_{B}\left(S_{1}\right) \cup\left(S-S_{1}\right)$ is a cutset of size less than $k$, contradicting that $G$ is $k$-connected. Therefore Lemma 21 applies, and since $P$ is a longest path, parts 3 and 4 follow.

It remains to establish part 5. Suppose for a contradiction that $w$ and $w^{\prime}$ are distinct vertices in the same component $W$ of $P-S$ such that $\operatorname{rank}(w), \operatorname{rank}\left(w^{\prime}\right)<|V(H)|$ and $w w^{\prime} \notin E(G)$. Let $A$ be the set of non-attachment points in $P$ with rank 0 , and obtain $A^{\prime}$ from $A$ by deleting the vertex in $W \cap A$ and adding $w$ and $w^{\prime}$. Note that, with the possible exception of $\left\{w, w^{\prime}\right\}$, each pair of vertices in $A^{\prime}$ has rank sum less than $|V(H)|$ and intersects two components of $P-S$. It follows from (4) that $A^{\prime}$ is an independent set. Since $\left|A^{\prime}\right|=k+2$ and $A$ consists of non-attachment points, we may add any vertex in $H$ to obtain an independent set of size $k+3$, a contradiction.

Theorem 23. Let $k \in\{1,2\}$. If $G$ is $k$-connected and $\alpha(G) \leq k+2$, then every longest path in $G$ contains every vertex of degree at least $\Delta(G)-(2-k)$.

Proof. Let $P$ be a longest path in $G$ with endpoints $x$ and $y$, and suppose for a contradiction that there exists $u \notin V(P)$ with $d(u) \geq \Delta(G)-(2-k)$. Let $H$ be the component of $G-V(P)$ containing $u$, and let $t=|V(H)|$. Let $s_{1}, \ldots, s_{k}$ be the attachment points of $H$ on $P$, indexed in order from $x$ to $y$, and let $S=\left\{s_{1}, \ldots, s_{k}\right\}$. Note that $\Delta(G) \leq d(u)+(2-k) \leq((t-1)+k)+(2-k)=t+1$.

For each component $W$ of $P-S$, let $f(W)$ be the set of vertices $w$ in $W$ with $\operatorname{rank}(w)<t$. We claim that $N\left(s_{1}\right)$ either contains $V(H)$ or $f(W)$, for some component $W$ of $P-S$. If not, then let $A$ be the set of vertices consisting of the lowest-ranked non-neighbor of $s_{1}$ in each component of $P-S$. Note that if $\left\{w, w^{\prime}\right\}$ is a pair of vertices in $A$, then $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<t$, or else $s_{1}$ has a set $B$ of at least $t$ neighbors in the components of $P-S$ containing $w$ and $w^{\prime}$. Let $z$ be the vertex in $P\left[x, s_{1}\right]$ that precedes $s_{1}$. Note that $z \notin B$, since some non-neighbor of $s_{1}$ separates $z$ and the initial segment of $P\left[x, s_{1}\right)$ consisting of vertices belonging to $B$. Counting $B$ together with $z$, it follows that $d\left(s_{1}\right) \geq t+2$, contradicting that $\Delta(G) \leq t+1$.

Hence $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right)<t$ and it follows from Lemma 22 part (4) that $A$ is an independent set. But $A$ together with $s_{1}$ and a non-neighbor of $s_{1}$ in $H$ forms an independent set of size $k+3$, contradicting that $\alpha(G) \leq k+2$. Therefore $N\left(s_{1}\right)$ either contains $V(H)$ or $f(W)$ for some component $W$ of $P-S$.

Note that $|V(H)|=t$ and $|f(W)|=t$ for each component $W$ of $P-S$. Let $v$ and $v^{\prime}$ be the immediate neighbors of $s_{1}$ along $P$, and let $v^{\prime \prime}$ be a neighbor of $s_{1}$ in $H$. Noting that $V(H)$ and each $f(W)$ intersect $\left\{v, v^{\prime}, v^{\prime \prime}\right\}$ in at most one vertex, it follows that $d\left(s_{1}\right) \geq t+3-1$, contradicting that $\Delta(G) \leq t+1$.

We note two consequences.
Corollary 24. If $G$ is a connected graph with $\alpha(G) \leq 3$ and $\Delta(G)-\delta(G) \leq 1$, or if $G$ is a 2-connected regular graph with $\alpha(G) \leq 4$, then $G$ has a Hamiltonian path.

Corollary 25. The graph $4 P_{1}$ is a fixer.

### 3.5 A 5-vertex fixer

In this section, we show that $5 P_{1}$ is a fixer. Although $5 P_{1}$ is a fixer, there are connected $5 P_{1}$-free graphs in which no vertex of maximum degree is Gallai (see Example 33). By contrast, for each fixer $F$ of order at most 4, the vertices of maximum degree in a connected $F$-free graph are all Gallai. ([22] show this for $F=2 P_{2}$, our results in this chapter show this for $F \in\left\{P_{3}+P_{1}, P_{2}+2 P_{1}, 4 P_{1}\right\}$, and it is also true for $F=P_{4}$ ).

The statement that $5 P_{1}$ is a fixer is equivalent to the statement that if $G$ is a connected graph with $\alpha(G) \leq 4$, then $G$ has a Gallai vertex. In the case that $G$ is 2 -connected, the result already follows from Theorem 23. When $G$ has cut-vertices, we exploit the block-cutpoint structure of $G$. We need the following two variants of Theorem 23 in the case that $P$ is an $x$-fiber or an $x y$-fiber for distinguished vertices $x, y \in V(G)$.

Lemma 26. Let $G$ be a 2-connected graph with a distinguished vertex $x$. If $\alpha(G-x) \leq 3$, then every $x$-fiber contains every vertex in $G$ of maximum degree.

Proof. Let $P$ be an $x$-fiber with other endpoint $y$, and suppose for a contradiction that $u$ is a vertex of maximum degree not on $P$. Let $H$ be the component of $G-V(P)$ containing $u$, and let $r$ be the number of attachment points of $H$ on $P$. Note that $r \geq 2$, or else there is at most one attachment point separating $y$ and $H$, contradicting that $G$ is 2-connected. Moreover, by Lemma 18 part (2), we have that $r+\alpha(H) \leq \alpha(G-x) \leq 3$. Since $r \geq 2$ and $\alpha(H) \geq 1$, it follows that $r=2$ and $\alpha(H)=1$. Therefore $H$ is a complete graph. Let $\left\{s_{1}, s_{2}\right\}$ be the set of attachment points of $H$ on $P$, with indices increasing from $x$ to $y$, and let $S=\left\{s_{1}, s_{2}\right\}$.

Since $G$ is 2-connected, there is a matching in the induced $(S, V(H)$ )-bigraph saturating $S$ or $|V(H)|=1$. Let $t=|V(H)|$ and note that $d(u) \leq(t-1)+2=t+1$. Since $P$ is an $x$-fiber, it follows from Lemma 21 that both $P\left(s_{1}, s_{2}\right)$ and $P\left(s_{2}, y\right]$ are non-empty (part (1)) and have at least $t$ vertices (part (2)). If $s_{2}$ has at least $t$ neighbors in some set in $\left\{V(H), V\left(P\left(s_{1}, s_{2}\right)\right), V\left(P\left(s_{2}, y\right]\right)\right\}$, then $d\left(s_{2}\right) \geq t+2>d(u)$, contradicting that $u$ has maximum degree. Hence $s_{2}$ has fewer than $t$ neighbors in each of $V(H), V\left(P\left(s_{1}, s_{2}\right)\right)$, and $V\left(P\left(s_{2}, y\right]\right)$. Let $w_{1}$ and $w_{2}$ be the non-neighbors of $s_{2}$ of minimum rank in $P\left(s_{1}, s_{2}\right)$ and $P\left(s_{2}, y\right]$, respectively, and let $z$ be a non-neighbor of $s_{2}$ in $H$.

We claim that $\left\{s_{2}, z, w_{1}, w_{2}\right\}$ is an independent set, contradicting $\alpha(G-x) \leq 3$. By construction, $s_{2}$ has no neighbor in $\left\{z, w_{1}, w_{2}\right\}$. Since $w_{1}$ and $w_{2}$ are not attachment points, $z$ has no neighbor in $\left\{w_{1}, w_{2}\right\}$. If $w_{1} w_{2} \in E(G)$, then Lemma 21 part (3) and the fact that $P$ is an $x$-fiber imply that $\operatorname{rank}\left(w_{1}\right)+\operatorname{rank}\left(w_{2}\right) \geq t$. Hence $s_{2}$ is adjacent to all vertices in $P\left(s_{1}, w_{1}\right)$ and $P\left(s_{2}, w_{2}\right)$, and there are at least $t$ of them. Together with the vertex preceding $s_{2}$ in $P$ and a neighbor of $s_{2}$ in $H$, we have $d\left(s_{2}\right) \geq t+2$, contradicting that $u$ has maximum degree.

Lemma 27. Let $G$ be a 2 -connected graph and let $x$ and $y$ be distinct vertices of $G$. If $\alpha(G-\{x, y\}) \leq 2$, then every xy-fiber contains every vertex in $G$ of maximum degree or $G-\{x, y\}$ is the disjoint union of two complete graphs.

Proof. Let $P$ be an $x y$-fiber, let $u$ be a vertex of maximum degree not on $P$, and let $H$ be the component of $G-V(P)$ containing $u$. Let $\left\{s_{1}, \ldots, s_{r}\right\}$ be the set of attachment points of $H$, with indices increasing from $x$ to $y$, and let $S=\left\{s_{1}, \ldots, s_{r}\right\}$. Since $G$ is 2-connected, we have $r \geq 2$, or else deleting $S$ separates $H$ from $V(P)-S$ (which is non-empty since $x \neq y$ ). By Lemma 18, there is an independent set $A \subseteq V(P-\{x, y\})$ such that $|A|=r-1$ and there are no edges joining $A$ and $V(H)$. Therefore $1+1 \leq(r-1)+\alpha(H) \leq$ $\alpha(G-\{x, y\}) \leq 2$. It follows that $r=2$ and $\alpha(H)=1$.

Let $t=|V(H)|$. Note that $H$ is complete and, since $G$ is 2-connected, there is a matching in the induced $\left(S, V(H)\right.$ )-bigraph saturating $S$ or $|V(H)|=1$. By Lemma 21, we have $\left|V\left(P\left(s_{1}, s_{2}\right)\right)\right| \geq t$ or else there is an augmenting interior splice of $P$ replacing $P\left(s_{1}, s_{2}\right)$, contradicting that $P$ is an $x y$-fiber.

Let $W=V\left(P\left(s_{1}, s_{2}\right)\right)$. Note that $W$ is a clique, or else a non-adjacent pair of vertices in $W$ together with a vertex in $H$ gives an independent set of size 3 , contradicting $\alpha(G-\{x, y\}) \leq 2$.

If $(x, y)=\left(s_{1}, s_{2}\right)$, then $G-\{x, y\}$ is the disjoint union of the complete graph $H$ and the complete graph on $W$. Otherwise, if $x \neq s_{1}$, then $s_{1}$ has a non-neighbor in $H$ and a non-neighbor in $W$, or else $d\left(s_{1}\right) \geq t+2>d(u)$. So $s_{1}$ together with a non-neighbor in $W$ and a non-neighbor in $H$ form an independent set of size 3 in $G-\{x, y\}$, a contradiction. The case that $y \neq s_{2}$ is similar.

A block $B$ of $G$ is special if every longest path in $G$ contains an edge in $B$.
Lemma 28. If no cut-vertex in a connected graph $G$ is Gallai, then $G$ has a special block.
Proof. Let $G$ be a connected graph such that no cut-vertex is Gallai. Suppose for a contradiction that no block of $G$ is special. Let $T$ be the block-cutpoint tree of $G$ (see, e.g., [25]). We construct a digraph $D$ on $V(T)$ in which each vertex has out-degree 1 . Let $B$ be a block in $G$. We identify a particular cut-vertex $x \in V(B)$ and we include the directed edge $B x$ in $D$. Since $B$ is not special, some longest path of $G$ is contained in some component $H$ of $G-E(B)$. Note that $H$ and $B$ have exactly one vertex in common, and we take $x$ to be this cut-vertex.

Let $x$ be a cut-vertex in $G$. We specify a particular block $B$ that contains $x$ and we include the directed edge $x B$ in $D$. Since $x$ is not Gallai, some component $H$ of $G-x$ contains a longest path in $G$. Let $B$ be the block containing $x$ such that $B-x \subseteq H$. We add the directed edge $x B$ to $E(D)$.

Since $|E(D)|=|V(T)|>|E(T)|$, it follows that there is a block $B$ and a cut-vertex $x$ such that both $B x$ and $x B$ are edges in $D$. This implies that $G$ has vertex-disjoint longest paths, a contradiction.

Lemma 29. If $G$ is a connected graph, $\alpha(G) \leq 4$, and $G$ has a special block, then $G$ has a Gallai vertex.

Proof. Let $G$ be a connected graph with $\alpha(G) \leq 4$ and with a special block $B$. Let $S$ be the set of cut-vertices in $B$, with $S=\left\{x_{1}, \ldots, x_{k}\right\}$. Since $\alpha(G) \leq 4$, we have $k \leq 4$.

Case $k=0$. In this case, $G=B$ and so $G$ is 2-connected. It follows from Theorem 23 that $G$ has a Gallai vertex.

Case $k=1$. Let $u \in V(B)$ with $d_{B}(u)=\Delta(B)$. We claim that $u$ is a Gallai vertex in $G$. Let $P$ be a longest path in $G$. If $P$ is contained in $B$, then $u \in V(P)$ by Theorem 23. If $P$ leaves $B$ through the cut-vertex $x_{1}$, then $P \cap B$ is an $x_{1}$-fiber in $B$ and it follows that $u \in V(P)$ by Lemma 26 .

Case $k=2$. Suppose first that $B-S$ is not the disjoint union of two complete graphs. Let $u \in V(B)$ with $d_{B}(u)=\Delta(B)$. We claim that $u$ is a Gallai vertex. Let $P$ be a longest path in $G$. Since $B$ is special, it follows that $P \cap B$ is a nontrivial subpath of $P$. Note that, as a subgraph of $B$, the path $P \cap B$ is either a fiber, an $x_{1}$-fiber or an $x_{2}$-fiber, or an $x_{1} x_{2}$-fiber, depending on whether $P$ has two, one, or zero endpoints in $B$, respectively. It follows from Theorem 23 , Lemma 26 , or Lemma 27 that $u \in V(P \cap B)$, respectively.

Otherwise, suppose that $B-S$ is the disjoint union of two complete graphs $W_{1}$ and $W_{2}$ (see Figure 3.4). Since $B$ is 2-connected, for $i \in\{1,2\}$, there is a matching in the induced $\left(S, V\left(W_{i}\right)\right)$-bigraph saturating $S$ or $\left|V\left(W_{i}\right)\right|=1$. Also, since $S$ is a minimum cut in $B$, each vertex in $S$ has neighbors in $V\left(W_{1}\right)$ and $V\left(W_{2}\right)$. It follows that $B$ has a Hamiltonian cycle. We claim that $x_{2}$ is a Gallai vertex. Let $P$ be a longest path in $G$, and suppose for a contradiction that $x_{2} \notin V(P)$. Since $B$ is special, $P$ has at least one endpoint in $B$. Replacing the subpath of $P$ inside $B$ with an appropriate Hamiltonian path gives a longer path in $G$.


Figure 3.4: Case $k=2$ in the proof of Lemma 29.

Case $k=3$. Note that $B-S$ is a complete graph $W_{1}$ or else $\alpha(G)>4$. Suppose there is a pair of cut-vertices, say $\left\{x_{1}, x_{3}\right\}$, such that $B-\left\{x_{1}, x_{3}\right\}$ is the disjoint union of two complete graphs. These are necessarily $W_{1}$ and the 1-vertex subgraph consisting of $x_{2}$; let $W_{2}$ be this 1-vertex subgraph. As in the case $k=2$, it follows that $B$ has a Hamiltonian cycle containing $x_{1} x_{2} x_{3}$ as a subpath. We claim that $x_{3}$ is a Gallai vertex. Let $P$ be a longest path in $G$ and suppose for a contradiction that $x_{3} \notin V(P)$. Note that $P$ cannot have an endpoint in $B$, or else replacing $P \cap B$ with an appropriate Hamiltonian path gives a longer path in $G$. Therefore, as a subgraph of $B$, the path $P \cap B$ is an $x_{1} x_{2}$-fiber. But $B$ has a spanning $x_{1} x_{2}$-path, contradicting $x_{3} \notin V(P)$.

Otherwise, there is no pair of cut-vertices whose removal from $B$ results in the disjoint union of two complete graphs. Let $u \in V(B)$ with $d_{B}(u)=\Delta(B)$. We claim that $u$ is a Gallai vertex. Let $P$ be a longest path in $G$. It follows that, as a subgraph of $B$, the path $P \cap B$ is a fiber, an $x_{i}$-fiber for some $x_{i} \in S$, or an $x_{i} x_{j}$-fiber for some $x_{i}, x_{j} \in S$, depending on whether $P$ has two, one, or zero endpoints in $B$, respectively. It follows from Theorem 23, Lemma 26, or Lemma 27 that $u \in V(P \cap B)$, respectively.

Case $k=4$. The condition $\alpha(G) \leq 4$ requires that $|V(B)|=4$ and 2 -connectivity requires that $B$ contains a 4 -cycle $C$. Let $x_{i}$ be a cut-vertex in $B$ which maximizes the length of an $x_{i}$-fiber in $G-E(B)$. We claim that $x_{i}$ is a Gallai vertex. Let $P$ be a longest path in $G$, and suppose for a contradiction that $x_{i} \notin V(P)$. The path $P$ decomposes into three subpaths $P_{1}, P_{2}$, and $P_{3}$, where $P_{2}=P \cap B$. Let $x_{j}$ be the vertex in $V\left(P_{1}\right) \cap V\left(P_{2}\right)$, and let $x_{k}$ be the vertex in $V\left(P_{2}\right) \cap V\left(P_{3}\right)$. Since $|V(B)|=4$, it follows that $x_{j}$ or $x_{k}$ is a neighbor of $x_{i}$ in $C$. If $x_{k} x_{i} \in E(C)$, then we find a longer path in $G$ by keeping $P_{1}$, extending $P_{2}$ by the edge $x_{k} x_{i}$ to obtain $P_{2}^{\prime}$, and replacing $P_{3}$ with an $x_{i}$-fiber $P_{3}^{\prime}$ in $G-E(B)$. Since $P_{2}^{\prime}$ is longer than $P_{2}$ and $P_{3}^{\prime}$ is at least as long as $P_{3}$ by our choice of $x_{i}$, the path obtained by combining $P_{1}, P_{2}^{\prime}$, and $P_{3}^{\prime}$ is longer than $P$. The case $x_{j} x_{i} \in E(C)$ is symmetric.

Applying our lemmas gives the following.
Theorem 30. Let $G$ be a connected graph. If $\alpha(G) \leq 4$, then $G$ has a Gallai vertex. Equivalently, $5 P_{1}$ is a fixer.

Proof. If some cut-vertex in $G$ is Gallai, then the claim follows. Otherwise, we have that $G$ has a special block by Lemma 28, and hence $G$ has a Gallai vertex by Lemma 29 .

The graph $G_{0}$ from Figure 1.2 shows that there is a connected graph $G$ such that $G$ has no Gallai vertex and $\alpha(G)=6$. The case $\alpha(G) \leq 5$ remains open.

Conjecture 31. If $\alpha(G) \leq 5$ and $G$ is connected, then $G$ has a Gallai vertex.
When $G$ is 3 -connected, $\alpha(G) \leq 5$, and $G$ is sufficiently large, Theorem 32 shows that $G$ has a Gallai vertex. Outside of a finite number of cases when $\kappa(G) \geq 3$, resolving Conjecture 31 reduces to the cases that $\kappa(G)=1$ and $\kappa(G)=2$. Although it is reasonable to expect that the case $\kappa(G)=1$ may be treated by analyzing the block structure of $G$, it is less clear how to handle the case $\kappa(G)=2$.

### 3.6 A Chvátal-Erdős type result

A celebrated result of Chvátal and Erdős [26] states that if $\alpha(G) \leq \kappa(G)$, then $G$ has a Hamiltonian cycle, and the same technique shows that $G$ has a Hamiltonian path when $\alpha(G) \leq \kappa(G)+1$. Clearly, when $G$ has a Hamiltonian path, every vertex in $G$ is Gallai. We show that if $\alpha(G) \leq \kappa(G)+2$ and $G$ is sufficiently large in terms of $\kappa(G)$, then the maximum degree vertices in $G$ are Gallai.

Theorem 32. For each positive integer $k$, there exists an integer $n_{0}$ such that if $G$ is an $n$-vertex $k$-connected graph with $\alpha(G) \leq k+2$ and $n \geq n_{0}$, then each vertex of maximum degree is Gallai.

Proof. We take $n_{0}=k(k+2)(2 k+3)+1$. Let $P$ be a longest path in $G$ with endpoints $x$ and $y$, and suppose for a contradiction that $u \in V(G)-V(P)$ and $d(u)=\Delta(G)$. Let $H$ be the component of $G-V(P)$ containing $u$, and let $t=|V(H)|$. From Lemma 22, it follows that $H$ is complete and $H$ has a set $S$ of $k$ attachment
points on $P$. Let $S=\left\{s_{1}, \ldots, s_{k}\right\}$ with indices increasing from $x$ to $y$. For $1 \leq i<k$, let $W_{i}=V\left(P\left(s_{i}, s_{i+1}\right)\right)$; we also define $W_{0}=V\left(P\left[x, s_{1}\right)\right)$ and $W_{k}=V\left(P\left(s_{k}, y\right]\right)$. By Lemma 22, we have that $\left|W_{i}\right| \geq t$ for $0 \leq i \leq k$. Since $u \in V(H)$, we have that $N(u) \subseteq(V(H)-\{u\}) \cup S$ and therefore $\Delta(G)=d(u) \leq(t-1)+k$. If $t \leq 2 k(k+1)$, then $\Delta(G) \leq k(2 k+3)-1$ and so $\alpha(G) \geq n /(\Delta(G)+1) \geq n /[k(2 k+3)]>k+2$, since $n \geq n_{0}$. Therefore we may assume that $t>2 k(k+1)$.

We claim that $H$ is the only component of $G-V(P)$. If $G-V(P)$ contains a second component $H^{\prime}$, then let $S^{\prime}$ be the set of attachment points of $H^{\prime}$ on $P$. By Lemma 22, it follows that $\left|S^{\prime}\right|=k$. For each $i$, choose $a_{i} \in W_{i}$ among the vertices with ranks in $\{0, \ldots, k\}$ so that $a_{i} \notin S^{\prime}$. Let $A=\left\{a_{0}, \ldots, a_{k}\right\}$. Since $t>2 k(k+1)>2 k$, it follows from Lemma 22 that $A$ is an independent set of size $k+1$. Since $A$ is disjoint from $S \cup S^{\prime}$, we may extend $A$ to an independent set of size $k+3$ by adding a vertex in $H$ and a vertex in $H^{\prime}$. Since $\alpha(G) \leq k+2$, we obtain a contradiction, and so $H$ is the only component of $G-V(P)$.

Next, we claim that each vertex $w \in W_{i}$ has at most $k$ neighbors outside $W_{i}$. Let $A$ be the subset of $V(P)-S$ consisting of the vertices $w$ such that $\operatorname{rank}(w)=0$. By Lemma 22, we have that $A$ is an independent set with $|A|=k+1$. Note that each vertex $w \in V(P)-(S \cup A)$ has at least one neighbor in $A$, or else $w$ together with $A$ and a vertex in $H$ would give an independent set of size $k+3$. Since $|A|=k+1$ and $\Delta(G) \leq t+k-1$, it follows that $|V(P)-(S \cup A)| \leq(k+1)(t+k-1)$ and hence $|V(P)-S| \leq(k+1)(t+k)=t(k+1)+k(k+1)$. Since $V(P)-S=\bigcup_{i=0}^{k} W_{i}$ and $\left|W_{i}\right| \geq t$ for each $i$, it follows that $t \leq\left|W_{i}\right| \leq t+k(k+1)$. By Lemma 22, in each $W_{i}$, the $t$ vertices of smallest rank form a clique. By symmetry, in each $W_{i}$, the $t$ vertices of largest rank also form a clique. Since $\left|W_{i}\right| \leq t+k(k+1)<2 t$, it follows that each vertex in $W_{i}$ is among the $t$ vertices with smallest rank or the $t$ vertices with largest rank. In particular, each vertex in $W_{i}$ has at least $t-1$ neighbors in $W_{i}$ and hence at most $k$ neighbors outside $W_{i}$.

It now follows that each $W_{i}$ is a clique. Indeed, if $w_{i}, w_{i}^{\prime} \in W_{i}$ but $w_{i} w_{i}^{\prime} \notin E(G)$, then we obtain an independent set $A$ with $A \subseteq V(P)-S$ and $|A|=k+2$ as follows. Starting with $A=\left\{w_{i}, w_{i}^{\prime}\right\}$, we add a vertex to $A$ from each $W_{j}$ with $j \neq i$. Since $\left|W_{j}\right| \geq t>k(k+1)$ and each of the vertices already in $A$ have at most $k$ neighbors in $W_{j}$, some vertex in $W_{j}$ can be added to $A$. The set $A$ together with a vertex in $H$ gives an independent set of size $k+3$, a contradiction. Hence each $W_{i}$ is a clique.

A vertex $z$ dominates a set of vertices $B$ if $z$ is adjacent to each vertex in $B$. Next, we claim that each $s_{i} \in S$ dominates some set in $\left\{W_{0}, \ldots, W_{k}, V(H)\right\}$. If some attachment point $s_{i}$ has more than $k^{2}$ non-neighbors in each $W_{j}$ and a non-neighbor $v$ in $H$, then we may obtain an independent set of size $k+3$ by starting with $\left\{s_{i}, v\right\}$ and adding one vertex from each $W_{j}$. It follows that each $s_{i}$ has at least $t-k^{2}$ neighbors in some set in $\left\{W_{0}, \ldots, W_{k}, V(H)\right\}$. Let $W_{k+1}=V(H)$, let $s_{i}$ be an attachment vertex, and choose $j$ such that $0 \leq j \leq k+1$ and $s_{i}$ has at least $t-k^{2}$ neighbors in $W_{j}$. We claim that $s_{i}$ dominates $W_{j}$. Indeed, if $w \in W_{j}$ but $s_{i} w \notin E(G)$, then we obtain an independent set $A$ of size $k+3$ starting with $A=\left\{s_{i}, w\right\}$ and adding one vertex from each $W_{\ell}$ with $0 \leq \ell \leq k+1$ and $\ell \neq j$. Since $s_{i}$ has at most $(t+k-1)-\left(t-k^{2}\right)$ neighbors in $W_{\ell}$, each of the other vertices already in $A$ has at most $k$ neighbors in $W_{\ell}$, and $\left|W_{\ell}\right| \geq t>(k(k+1)-1)+(k+1) k$, it follows that $W_{\ell}$ contains a vertex that can be added to $A$. Since $\alpha(G) \leq k+2$, we obtain a contradiction, and so $s_{i}$ dominates $W_{j}$.

Let $1 \leq i<k$. Since $W_{i}$ is a clique and $W_{i}=V\left(P\left(s_{i}, s_{i+1}\right)\right)$, we obtain a path $P^{\prime}$ with $V(P)=V\left(P^{\prime}\right)$ and the same set of attachment points by reordering the vertices in $W_{i}$ arbitrarily, so long as the first vertex is adjacent to $s_{i}$ and the last vertex is adjacent to $s_{i+1}$. Similarly, we may reorder $W_{0}$ provided that the last
vertex in $W_{0}$ is adjacent to $s_{1}$ and we may reorder $W_{k}$ provided that the first vertex in $W_{k}$ is adjacent to $s_{k}$. Let $R$ be the set of neighbors of $S$ in $P$. Note that for each $w \in W_{i}-R$ and each $q$ with $1 \leq q \leq\left|W_{i}\right|-2$, we may obtain a path $P^{\prime}$ with $V(P)=V\left(P^{\prime}\right)$ and the same attachment points in which $\operatorname{rank}(w)=q$ by an appropriate reordering of $W_{i}$. It follows that if $w w^{\prime} \in E(G)$, for some $w \in W_{i}$ and $w^{\prime} \in W_{j}$, with $i$ and $j$ distinct in $\{0, \ldots, k\}$, then $w, w^{\prime} \in R$. Otherwise, we may reorder $W_{i}$ and $W_{j}$ to obtain a new path $P^{\prime}$ in which either $\operatorname{rank}(w) \leq 1$ and $\operatorname{rank}\left(w^{\prime}\right) \leq 1$, or $\operatorname{rank}(w) \geq\left|W_{i}\right|-2$ and $\operatorname{rank}\left(w^{\prime}\right) \geq\left|W_{j}\right|-2$. In the latter case, reversing $P^{\prime}$ gives a path $P^{\prime \prime}$ in which $\operatorname{rank}(w) \leq 1$ and $\operatorname{rank}\left(w^{\prime}\right) \leq 1$. This contradicts Lemma 22 with respect to $P^{\prime}$ or $P^{\prime \prime}$ since $\operatorname{rank}(w)+\operatorname{rank}\left(w^{\prime}\right) \leq 2$ but $|V(H)|=t>2 k(k+1) \geq 4$.

We obtain a final contradiction by showing that some attachment point has degree exceeding $\Delta(G)$. Let $D=\sum_{i=1}^{k} d\left(s_{i}\right)$ and note that $D \leq k(t+k-1)$. We give a lower bound on $D$ using three sets of edges. First, for each $s_{i}$, let $T_{i}$ be a set of 3 edges incident to $s_{i}$ consisting of the edges joining $s_{i}$ to its two neighbors in $R$ and a third edge joining $s_{i}$ and a vertex in $H$. Second, for $0 \leq i \leq k$, there is a matching $M_{i}$ of size $k$ joining vertices in $W_{i}$ and $V(G)-W_{i}$, or else Kőnig-Egerváry Theorem [25] implies that the induced $\left(W_{i}, V(G)-W_{i}\right)$-bigraph has a vertex cover of size less than $k$, which is also a vertex cut since $\left|W_{i}\right|,\left|V(G)-W_{i}\right| \geq t>k$. Obtain $M_{i}^{\prime}$ from $M_{i}$ by discarding edges incident to vertices in $W_{i} \cap R$. Note that $\left|M_{i}^{\prime}\right| \geq\left|M_{i}\right|-2 \geq k-2$ always, but for $i \in\{0, k\}$ we have $\left|M_{i}^{\prime}\right| \geq\left|M_{i}\right|-1 \geq k-1$. Suppose that $e \in M_{i}^{\prime}$, let $w$ be the endpoint of $e$ in $W_{i}$, and let $v$ be the other endpoint of $e$ in $V(G)-W_{i}$. Since $w$ is not an attachment point, we have $v \notin V(H)$, and since $H$ is the only component of $G-V(P)$, it follows that $v \in V(P)-W_{i}$. Since $w \notin R$, it follows that $v$ must be an attachment point. Hence each edge in $M_{i}^{\prime}$ joins a vertex in $W_{i}-R$ and a vertex in $S$. Moreover, $M_{i}^{\prime}$ and $T_{j}$ are disjoint, as each edge in $T_{j}$ has an endpoint in $R \cup V(H)$ and no edge in $M_{i}^{\prime}$ has such an endpoint. With $Z=\bigcup_{i=0}^{k} M_{i}^{\prime} \cup \bigcup_{j=1}^{k} T_{j}$, we have $|Z| \geq[(k-1)(k-2)+2(k-1)]+3 k=k(k+2)$. Third, for $1 \leq i \leq k$, let $F_{i}$ be the set of edges joining $s_{i}$ and a set in $\left\{W_{0}, \ldots, W_{k}, V(H)\right\}$ dominated by $s_{i}$. Note that $\left|F_{i} \cap Z\right| \leq 2$, since $F_{i}$ contains at most one edge in $\bigcup_{i=0}^{k} M_{i}^{\prime}$ and at most one edge in $\bigcup_{j=1}^{k} T_{j}$. Let $F=\bigcup_{j=1}^{k} F_{i}$, and note that $|F| \geq t k$ and $|F \cap Z| \leq 2 k$.

We compute $D \geq|F \cup Z|=|F|+|Z|-|F \cap Z| \geq t k+k(k+2)-2 k=t k+k^{2}=k(t+k)$. Since $D \leq k(t+k-1)$, it follows that $k(t+k) \leq D \leq k(t+k-1)$, contradicting that $k$ is positive.

Example 33. The assumption $\alpha(G) \leq \kappa(G)+2$ in Theorem 32 is best possible. Let $G$ be the graph obtained from the star $K_{1, k+2}$ with leaves $\left\{x_{1}, \ldots, x_{k+2}\right\}$ by replacing the center vertex with a $k$-clique $S$ and replacing each leaf vertex $x_{i}$ with a t-clique $X_{i}$ containing a set of $k$ distinguished vertices $Y_{i}$ that are joined to $S$. Since $V(G)$ can be covered by $k+3$ cliques, we have $\alpha(G) \leq k+3$. Also, we have $\kappa(G)=k$ since $S$ is a cutset of size $k$ and when $R \subseteq V(G)$ and $|R|<k$, the graph $G-R$ contains at least one vertex in each of $S, Y_{1}, \ldots, Y_{k+2}$, implying that $G-R$ is connected.

We claim that the set of Gallai vertices in $G$ is $S$. Since $|S|=k$ and $G-S$ is the disjoint union of $k+2$ copies of $K_{t}$, it follows that every path in $G$ has at most $|V(G)|-t$ vertices. Paths in $G$ that achieve this bound contain $S$ and all but one of $X_{1}, \ldots, X_{k+2}$, implying that $u \in V(G)$ is Gallai if and only if $u \in S$. By construction, each vertex in $S$ has degree $k(k+2)+(k-1)$. Hence, when $t$ is sufficiently large, the set of vertices in $G$ of maximum degree is $Y_{1} \cup \cdots \cup Y_{k+2}$, and none of these is Gallai.

Although maximum degree vertices are not Gallai, our construction still has Gallai vertices. It is natural to ask whether every graph with sufficiently high connectivity has a Gallai vertex [27, 28]. As noted at the beginning of Chapter 3, there are $k$-connected graphs having no Gallai vertices when $k \leq 3$. The question remains open for $k \geq 4$.

The complete bipartite graphs $K_{s, s+2}$ show that the condition $\alpha(G) \leq \kappa(G)+1$ cannot in general be relaxed to $\alpha(G) \leq \kappa(G)+2$ while still guaranteeing existence of Hamiltonian paths [26]. However, Theorem 32 immediately implies that this is possible for sufficiently large regular graphs.

Corollary 34. For each positive integer $k$, there exists $n_{0}$ such that every $k$-connected regular graph $G$ with $\alpha(G) \leq k+2$ and $n \geq n_{0}$ vertices has a Hamiltonian path.

We do not know whether the condition $\alpha(G) \leq k+2$ in Corollary 34 is best possible. The following construction from [29] shows that it cannot be relaxed to $\alpha(G) \leq k+5$.

Example 35. Let $k \geq 6$ be even. Let $G_{1}$ be $K_{k+1}$ minus an edge and let $G_{2}$ be $K_{k+1}$ minus a matching on $k-4$ vertices. Let $G$ be the graph obtained from two copies of $G_{1}$ and one copy of $G_{2}$ by adding a new vertex adjacent to all $k$ vertices of degree $k-1$. We have that $G$ is a 1 -connected regular graph with $\alpha(G)=6$ and no Hamiltonian path.

## Chapter 4

## Chordal Graphs

### 4.1 Motivation and Tree Representation

A graph $G$ is chordal if every induced cycle in $G$ of length 4 or more has a chord, which is an edge not on the cycle, but having both endpoints on the cycle. Several subfamilies of chordal graphs are known to be Gallai. For example, trees, series parallel graphs and interval graphs are Gallai families. Thus, it is natural to investigate transversals in chordal graphs as a progression of previous results.

Chordal graphs are interesting in their own right; indeed, chordal graphs have many nice properties and features. One such property is that minimal cut sets in chordal graphs are cliques. A simplicial vertex is a vertex whose neighborhood is a clique. Another useful property is that every chordal graph $G$ has a simplicial elimination ordering $v_{1}, \ldots, v_{n}$ such that $v_{j}$ is simplicial in the subgraph induced by $\left\{v_{1}, \ldots, v_{j}\right\}$ for each $j$. Many graph theory texts provide general information on chordal graphs (see [25], for example).

The most useful property for this work is a chordal graph's tree representation. A tree representation for a chordal graph $G$ is the intersection graph of subtrees of a tree $T$. It is well known [30] that a graph is chordal if and only if it has a tree representation. The representation associates a vertex $v$ in $G$ with a subtree of $T$, which we denote $S(v)$. If two vertices are adjacent in $G$, then their subtrees intersect in $T$. The converse is also true. That is, if $S(u)$ and $S(v)$ are the subtrees of $T$ corresponding to vertices $u$ and $v$ in $G$ and $S(u)$ intersects $S(v)$, then $u$ and $v$ are adjacent in $G$. This means if a vertex $v$ in $T$ is common to a family of subtrees corresponding to a set of vertices in $G$, then those vertices are pairwise adjacent. Hence, each vertex $x$ in $T$ corresponds to a clique in $G$, which we call a bag denoted by $B(x)$.

The Graph Minors Project [31] introduced the tree width graph parameter which is well-studied and exceedingly useful. The parameter measures how far a graph is from being a tree. It has received significant attention due to its applications in fixed parameter tractable algorithms. The tree width of a graph $G$, denoted $t w(G)$, is defined as:

$$
t w(G)=\min \{\omega(H)-1: G \subseteq H \text { and } H \text { is chordal }\}
$$

This is yet another reason to look at chordal graphs.
A tree representation $T$ for a chordal graph $G$ is minimal if there is no tree representation of $G$ with a
host tree on fewer than $|V(T)|$ vertices. A contraction of an edge $u v$ in a graph $G$ removes $u$ and $v$ from $G$ and creates a new vertex $w$ such that $N(w)=N(u) \cup N(v)$

Proposition 36. Given a connected chordal graph $G$ and a tree representation $T$ of $G$, we have $T$ is a minimal tree representation if and only if $B(x) \nsubseteq B(y)$ for each $x, y \in V(T)$. Moreover, if $T$ is a minimal tree representation of $G$, then the bags are exactly the maximal cliques in $G$.

Proof. Assume $T$ is a minimal tree representation but that there is a pair of vertices $x, y \in V(T)$ such that $B(x) \subseteq B(y)$. Let $P$ be the $x y$-subpath of $T$ and let $x^{\prime}$ be the vertex adjacent to $x$ in $P$. Note that $B(x) \subseteq B\left(x^{\prime}\right)$. We can obtain a smaller tree representation for $G$ by contacting the edge $x x^{\prime}$ in $T$ as well as any subtrees containing $x x^{\prime}$ to form a new tree representation $T^{\prime}$.

Conversely, suppose $B(x) \nsubseteq B(y)$ for each pair $x, y \in V(T)$ and that $T^{\prime}$ is another tree representation of $G$. Note that every clique in $G$ is contained in some bag $B(x)$ for some $x \in V(T)$ by the Helly property. Since each bag of a vertex in $T$ is a clique in $G$, and since no bag contains another, $G$ has at least $|V(T)|$ maximal cliques. Since the maximal cliques in $G$ are contained in distinct bags of vertices of $T^{\prime}$, we have $|V(T)| \leq\left|V\left(T^{\prime}\right)\right|$.

Note that minimal tree representations for a given chordal graph need not be unique since it may be possible to 'connect' maximal cliques in $G$ in different ways.

A bramble of a graph is a set of connected subgraphs that pairwise intersect or are joined by an edge. In particular, the set of longest paths of a connected graph and the set of longest cycles in a 2-connected graph form brambles. The order of a bramble is the size of a minimum vertex transversal. It is well known [32] that the tree width of a graph is equal to the maximum order of a bramble minus one. As observed by Rautenbach and Sereni [14], this proves $\operatorname{lpt}(G) \leq t w(G)+1$. In particular, for chordal graphs this gives $\operatorname{lpt}(G) \leq \omega(G)$. (This is also easy to see from the tree representation of a chordal graph, as shown in Proposition 49). Later, Harvey and Payne [33] improved this bound to most $4\left\lceil\frac{\omega(G)}{5}\right\rceil$ for chordal graphs. For the case of longest cycles, they obtained a transversal of size at most $2\left\lceil\frac{\omega(G)}{3}\right\rceil$ for 2-connected chordal graphs. We find upper bounds of $\operatorname{lpt}(G) \leq O\left(\log ^{2}(n)\right.$ and $\operatorname{lct}(G) \leq O(\log (n)$ for connected and 2-connected $n$-vertex chordal graphs, respectively.

### 4.2 Longest Cycle Transversals

A rooted tree is a tree with a distinguished vertex $r$, called the root of $T$. If $T$ is a rooted tree with root $r$ and $u \in V(T)$, then the subtree of $T$ rooted at $u$ is the tree induced by the set of all vertices $w \in V(T)$ such that the $r w$-path in $T$ contains $u$. A subtree of $T$ is a rooted subtree if it is the subtree of $T$ rooted at $u$ for some $u \in V(T)$. Our convention is to use $X$ to name a rooted subtree of $T$ with root vertex $x$. It is convenient to also allow empty rooted trees with no vertices and no edges. If $X$ is an empty rooted subtree of $T$, then $V(X)=\varnothing$.

For convenience, we assume that each host tree $T$ is equipped with a distinguished root vertex $r$. Given a subgraph $H$ of a chordal graph $G$ with tree representation $T$, the core of $H$ is the union, over all $u v \in E(H)$, of $V(S(u) \cap S(v))$. Note that when $H$ is connected, the core of $H$ induces a connected subgraph of $T$. A set $W \subseteq V(T)$ has the core capture property with respect to a family of subgraphs $\mathcal{H}$ of $G$ if each $H \in \mathcal{H}$ has a core which intersects $W$.

Lemma 37. Let $G$ be a graph with rooted tree representation $T$. Let $X$ be a rooted subtree of $T$ with root $x$, and let $H$ be a connected subgraph of $G$ such that $H$ has a core vertex in $X$ but $x$ is not a core vertex of $H$. There is a vertex $w \in V(H)$ such that $S(w) \subseteq X-x$.

Proof. Let $y \in X$ be a core vertex of $H$, and let $u v \in E(H)$ such that $y \in V(S(u)) \cap V(S(v))$. Note that for some $w \in\{u, v\}$, we have that $x \notin S(w)$, or else $x$ would also be a core vertex of $H$. Since $S(w)$ is a subtree of $T$, it must be that $S(w)$ is contained in a component of $T-x$. Since $y \in V(S(w))$ and $y$ is in a component of $X-x$, it follows that $S(w) \subseteq X-x$.

We make frequent use of the following lemma.
Lemma 38 (Jordan's Tree Separator). Let $T$ be a tree, then there exists a vertex $z \in V(T)$ such that each component of $T-z$ has at most $|V(T)| / 2$ vertices [34].

Let $G$ be a chordal graph with tree representation $T$, let $x, y \in V(T)$, and let $P$ be the $x y$-path in $T$. The components of $T-E(P)$ containing an endpoint of $P$ are exterior components, and the components containing an interior vertex of $P$ are interior components. (Note that if $P$ has no interior vertices, then $T-E(P)$ has no interior components and one or two exterior components according to $|V(P)|=1$ or $|V(P)|=2$.

Let $T$ be a rooted tree, let $X$ be the subtree rooted at a vertex $x \in V(T)$, and let $Q$ be a subpath of $X$ with endpoint $x$. For each $y \in V(Q)$, we define the descendants of $y$ in $X$ relative to $Q$, denoted $D(y ; Q, X)$, to be the component of $X-E(Q)$ containing $y$. For a subpath $Q_{0}$ of $Q$, we define $D\left(Q_{0} ; Q, X\right)$ to be the union, over $y \in V\left(Q_{0}\right)$ of $D(y ; Q, X)$.

Lemma 39. Let $G$ be a chordal graph with a rooted tree representation $T$, let $x$ and $y$ be distinct vertices in $T$, and let $P$ be the xy-path in $T$. If $G$ has a subgraph $H$ such that $H$ has a core vertex in each exterior component of $T-E(P)$ but no core vertex in any interior component, then $H$ contains $\kappa(H)$ vertices $v$ such that $S(v)$ contains $P$.

Proof. Let $T_{x}$ and $T_{y}$ be the exterior components of $T-E(P)$ containing $x$ and $y$ respectively. Let $e_{x}$ and $e_{y}$ be edges in $H$ having a core vertex in $T_{x}$ and $T_{y}$ respectively. Both endpoints of $e_{x}$ have subtrees intersecting $T_{x}$, and since $H$ has no core vertex in an interior component of $T-E(P)$, at least one of these endpoints $u_{x}$ has a subtree $S\left(u_{x}\right)$ that is contained in $T_{x}$. Similarly, let $u_{y}$ be an endpoint of $e_{y}$ with $S\left(u_{y}\right) \subseteq T_{y}$. With $k=\kappa(H)$, let $Q_{1}, \ldots, Q_{k}$ be internally disjoint $u_{x} u_{y}$-paths in $H$.

We claim that each $Q_{i}$ contains a vertex $v_{i}$ with $P \subseteq S\left(v_{i}\right)$. If not, then each vertex $v$ in $Q_{i}$ has a subtree $S(v)$ that is either disjoint from $T_{y}$ or disjoint from $T_{x}$. Since $Q_{i}$ has endpoints $u_{x}$ and $u_{y}$ which are disjoint from $T_{y}$ and $T_{x}$ respectively, it follows that $Q_{i}$ has adjacent vertices $w w^{\prime}$ such that $S(w)$ is disjoint from $T_{y}$ and $S\left(w^{\prime}\right)$ is disjoint from $T_{x}$. It follows that $S(w) \cap S\left(w^{\prime}\right)$ is disjoint from $T_{x} \cup T_{y}$, and so $S(w) \cap S\left(w^{\prime}\right)$ contains a vertex in an interior component of $T-E(P)$, contradicting that the core of $H$ is disjoint from those components.

Let $T$ be a rooted tree let $X$ be the subtree rooted at a vertex $x \in V(T)$, and let $Q$ be a subpath of $X$ with endpoint $x$. For each $y \in V(Q)$, we define $D(y ; Q, X)$ to be the component of $X-E(Q)$ containing $y$. For a subpath $Q_{0}$ of $Q$, we define $D\left(Q_{0} ; Q, X\right)$ to be the union, over $y \in V\left(Q_{0}\right)$ of $D(y ; Q, X)$.

Lemma 40. Let $G$ be a chordal graph with a rooted tree representation $T$, let $X$ be the subtree of $T$ with root $x$, and let $Q$ be a subpath of $X$ such that $x$ is an endpoint of $Q$. Let $\mathcal{H}$ be a nonempty family of subgraphs of $G$ such that $V(X)$ has the core capture property for $\mathcal{H}$ and let $k=\min \left\{\kappa\left(H_{1} \cup H_{2}\right): H_{1}, H_{2} \in \mathcal{H}\right\}$. Let $Q_{0}$ be a minimal subpath of $Q$ such that $V\left(D\left(Q_{0} ; Q, X\right)\right)$ has the core capture property for $\mathcal{H}$. If $\left|V\left(Q_{0}\right)\right| \geq 2$, then $G$ has $k$ vertices $v$ such that $S(v)$ contains $Q_{0}$.

Proof. Let $y_{1}$ and $y_{2}$ be the endpoints of $Q_{0}$. By minimality of $Q_{0}$, for $i \in\{1,2\}$, there exists $H_{i} \in \mathcal{H}$ such that $H_{i}$ has a core vertex in $D\left(y_{i} ; Q, X\right)$ but no core vertex in $D\left(Q_{0}-y_{i} ; Q, X\right)$. Let $H=H_{1} \cup H_{2}$ and note that $H$ has a core vertex in each exterior component of $T-E\left(Q_{0}\right)$ but no core vertex in an interior component of $T-E\left(Q_{0}\right)$. It follows from Lemma 39 that $G$ has $\kappa(H)$ vertices $v$ such that $S(v)$ contains $Q_{0}$, and $\kappa(H) \geq k$.

Lemma 41. Let $G$ be a graph, let $W$ be a pair of glue vertices, and let $\mathcal{H}$ be a family of subgraphs of $G$, where each $H \in \mathcal{H}$ is a maximum path or maximum cycle in $G$ and $V(H) \cap W=\varnothing$. An attachment point is a vertex $u \in V(G)$ such that $W \subseteq N(u)$. Let $\mathcal{R}$ be a nonempty family of paths in $G$ such that each $R \in \mathcal{R}$ is disjoint from $W$ and the endpoints of $R$ are distinct attachment points in $G$. If each $H \in \mathcal{H}$ contains a subpath in $\mathcal{R}$, then a longest path in $\mathcal{R}$ intersects every $H$ in $\mathcal{H}$.

Proof. Let $R$ be a longest path in $\mathcal{R}$, and suppose for a contradiction that $R$ is disjoint from some $H \in \mathcal{H}$. Let $R_{0}$ be a subpath of $H$ in $\mathcal{R}$, and let $x$ and $y$ be the endpoints of $R_{0}$. We have $\left|V\left(R_{0}\right)\right| \leq|V(R)|$. Let $W=\left\{w_{1}, w_{2}\right\}$. We modify $H$ to obtain a longer path or cycle in $G$ by replacing the subpath $R_{0}$ with the path $x w_{1} R w_{2} y$.

Lemma 42. Let $G$ be a connected chordal graph with minimal tree representation $T$. Let $\mathcal{C}$ be a family of longest cycles in $G$, and let $X$ be a subtree of $T$ with root $x$ having the core capture property for $\mathcal{C}$. There is a rooted subtree $X^{\prime}$ of $X$ with $\left|V\left(X^{\prime}\right)\right| \leq|V(X)| / 2$ and a set of vertices $A \subseteq V(G)$ with $|A| \leq 4$ such that for each $C \in \mathcal{C}$, we have that $V(C) \cap A \neq \varnothing$ or $C$ has a core vertex in $X^{\prime}$.

Proof. Suppose that $y$ is a vertex in $T$ with $|B(y)|=1$. Since $T$ is a minimal representation and $B(y)$ is not contained in any other bag, it follows that the vertex $u \in B(y)$ satisfies $V(S(u))=\{y\}$ and so $u$ has no neighbors in $G$. Since $G$ is connected, the lemma is satisfied with $A=V(G)=\{u\}$ and $X^{\prime}$ empty. Hence we may assume $|B(y)| \geq 2$ for each $y \in V(T)$.

Apply Lemma 38 to obtain a vertex $z \in V(X)$ such that each component of $X-z$ has at most $|V(X)| / 2$ vertices, and let $Q$ be the $x z$-path in $X$. By Lemma 5, we have that $C_{1} \cup C_{2}$ is a 2-connected subgraph of $G$. Let $Q_{0}$ be a minimal subpath of $Q$ such that $V\left(D\left(Q_{0} ; Q, X\right)\right)$ has the core capture property for $\mathcal{C}$. We claim that $G$ has a pair of vertices $\left\{w_{1}, w_{2}\right\}$ with each $S\left(w_{i}\right)$ containing $Q_{0}$. If $\left|V\left(Q_{0}\right)\right| \geq 2$, then this follows from Lemma 40. Otherwise, if $Q_{0}$ consists of a single vertex $y$, then we choose $\left\{w_{1}, w_{2}\right\}$ to be a pair of vertices from $B(y)$ arbitrarily. Let $W=\left\{w_{1}, w_{2}\right\}$.

Let $\mathcal{C}_{1}$ be the set of all cycles $C \in \mathcal{C}$ such that $C$ is disjoint from $\left\{w_{1}, w_{2}\right\}$. We may assume $\mathcal{C}_{1} \neq \varnothing$, or else the lemma is satisfied with $A=W$ and $X^{\prime}$ empty. Let $C \in \mathcal{C}_{1}$. We claim that $C$ contains a vertex whose subtree is contained in a component of $D\left(Q_{0} ; Q, X\right)-V\left(Q_{0}\right)$. Since $C$ has a core vertex in $V\left(D\left(Q_{0} ; Q, X\right)\right)$, it follows that $C$ contains adjacent vertices $u_{1}$ and $u_{2}$ such that $S\left(u_{1}\right) \cap S\left(u_{2}\right)$ intersects $D\left(Q_{0} ; Q, X\right)$. Since
$C$ is a longest cycle and $w_{1} \notin V(C)$, at least one of $\left\{S\left(u_{1}\right), S\left(u_{2}\right)\right\}$ is disjoint from $V\left(Q_{0}\right)$ and hence is contained in a component of $D\left(Q_{0} ; Q, X\right)-V\left(Q_{0}\right)$.

Suppose that there exists $C \in \mathcal{C}_{1}$ such that all but at most one vertex $v \in V(C)$ satisfies $S(v) \subseteq Y$ for some component $Y$ of $D\left(Q_{0} ; Q, X\right)-V\left(Q_{0}\right)$. It follows from Lemma 5 that each cycle in $\mathcal{C}_{1}$ intersects $C$ in at least one vertex whose subtree is contained in $Y$, implying that $V(Y)$ has the core capture property for $\mathcal{C}_{1}$. Hence the lemma is satisfied with $A=W$ and $X^{\prime}=Y$. So we may assume that each $C \in \mathcal{C}_{1}$ contains a vertex $u$ with $S(u) \subseteq Y$ for some component $Y$ of $D\left(Q_{0} ; Q, X\right)-V\left(Q_{0}\right)$ and a pair of distinct vertices $\left\{v_{1}, v_{2}\right\}$ with $S\left(v_{i}\right)$ intersecting $Q_{0}$.

Let $\mathcal{R}$ be the family of paths $R$ in $G-W$ such that the endpoints of $R$ are distinct and have subtrees intersecting $Q_{0}$ and each interior vertex $v$ of $R$ satisfies $S(v) \subseteq Y$ for some component $Y$ of $D\left(Q_{0} ; Q, X\right)$ $V\left(Q_{0}\right)$. Note that each $C \in \mathcal{C}_{1}$ contains a subpath in $\mathcal{R}$; since $\mathcal{C}_{1}$ is nonempty, so is $\mathcal{R}$. Let $R$ be a longest path in $\mathcal{R}$. It follows from Lemma 41 that each $C \in \mathcal{C}_{1}$ intersects $R$. Let $w_{3}$ and $w_{4}$ be the endpoints of $R$, and let $\mathcal{C}_{2}$ be the set of all $C \in \mathcal{C}_{1}$ that are disjoint from $\left\{w_{3}, w_{4}\right\}$. Let $Y$ be the component of $D\left(Q_{0} ; Q, X\right)-V\left(Q_{0}\right)$ such that each internal vertex $v$ of $R$ satisfies $S(v) \subseteq Y$. Note that each $C \in \mathcal{C}_{2}$ must intersect $R$ in an interior vertex of $R$ and hence each $C \in \mathcal{C}_{2}$ has a core vertex in $Y$. We set $A=\left\{w_{1}, \ldots, w_{4}\right\}$ and $X^{\prime}=Y$. Since each $C \in \mathcal{C}$ which is disjoint from $A$ is in $\mathcal{C}_{2}$, the lemma is satisfied.

Theorem 43. If $G$ is an $n$-vertex 2 -connected chordal graph, then $\operatorname{lct}(G) \leq 4(1+\lceil\lg n\rceil)$.

Proof. Let $T$ be a minimal tree representation for $G$. Since bags in $T$ correspond to maximal cliques in $G$, it follows from a simplicial elimination ordering on $G$ that $|V(T)| \leq n$. Choose a root vertex $x \in$ $V(T)$ arbitrarily. Let $\mathcal{C}$ be the family of longest cycles in $G$. We apply Lemma 42 iteratively to obtain $\left(\mathcal{C}_{0}, X_{0}\right), \ldots,\left(\mathcal{C}_{t}, X_{t}\right)$ and $A_{1}, \ldots, A_{t}$ as follows. We set $\left(\mathcal{C}_{0}, X_{0}\right)=(\mathcal{C}, T)$. Given $\left(\mathcal{C}_{i}, X_{i}\right)$ such that $\varnothing \subsetneq \mathcal{C}_{i} \subseteq \mathcal{C}$ and $V\left(X_{i}\right)$ has the core capture property for $\mathcal{C}_{i}$, we obtain a set of vertices $A_{i+1} \subseteq V(G)$ with $\left|A_{i+1}\right| \leq 4$ and a subtree $X_{i+1}$ with $\left|V\left(X_{i+1}\right)\right| \leq\left|V\left(X_{i}\right)\right| / 2$ such that each $C \in \mathcal{C}_{i}$ intersects $A_{i+1}$ or has a core vertex in $X_{i+1}$. We let $\mathcal{C}_{i+1}$ be the set of all $C \in \mathcal{C}_{i}$ that are disjoint from $A_{i+1}$, obtaining the next pair $\left(\mathcal{C}_{i+1}, X_{i+1}\right)$ in the iteration. The iteration ends with a pair $\left(\mathcal{C}_{t}, X_{t}\right)$ such that $X_{t}$ is empty (and necessarily $\mathcal{C}_{t}=\varnothing$ also, since $V\left(X_{t}\right)$ has the core capture property for $\left.\mathcal{C}_{t}\right)$. Since $\left|V\left(X_{i}\right)\right| \leq n / 2^{i}$, it follows that $t \leq 1+\lceil\lg n\rceil$. Let $A=\bigcup_{i=1}^{t} A_{i}$, and note that $A$ is a longest cycle transversal for $G$ since $C \in \mathcal{C}$ intersects $A_{i}$, where $C \in \mathcal{C}_{i-1}-\mathcal{C}_{i}$. Since each $A_{i}$ has size at most 4 , the bound follows.

### 4.3 Longest Path Transversals

In this section, we prove that each connected chordal graph $G$ with $n$ vertices has a longest path transversal of size $O\left(\log ^{2} n\right)$.

Note that if $X$ is a rooted subtree of a rooted tree $T$ and $Q$ is a subpath of $X$, then $V(D(Q ; Q, X))=$ $V(X)$. It follows that if $V(X)$ has the core capture property for a family of subgraphs $\mathcal{H}$ of $G$, then for each subpath $Q$, there is a minimal subpath $Q_{0}$ of $Q$ such that $V\left(D\left(Q_{0} ; Q, X\right)\right)$ has the core capture property for $\mathcal{H}$.

Lemma 44. Let $G$ be a connected chordal graph, and let $T$ be a minimal rooted tree representation of $G$. Let $X$ be a rooted subtree of $T$ and let $\mathcal{P}$ be a family of longest paths in $G$ such that $V(X)$ has the core capture property for $\mathcal{P}$. Moreover, suppose that for each path $P \in \mathcal{P}$, both $S(u)$ and $S(v)$ have a vertex outside $X$,
where $u$ and $v$ are the endpoints of $P$. There exists a set $A \subseteq V(G)$ with $|A| \leq 4$ and a subtree $X^{\prime}$ of $X$ with $\left|V\left(X^{\prime}\right)\right| \leq|V(X)| / 2$ such that for each $P \in \mathcal{P}:$

1. P has a vertex in A, or
2. $P$ has a core vertex in $X^{\prime}$.

Proof. Note that if $X$ is an empty rooted subtree, then $\mathcal{P}$ must be empty, since $X$ has the core capture property for $\mathcal{P}$. In this case, the lemma is satisfied with $A=\varnothing$ and $X^{\prime}$ empty. Hence we may assume that $X$ is a nonempty subtree of $T$ rooted at $x \in V(T)$.

Note that since $T$ is a minimal tree representation, we may assume that each bag in $T$ has size at least 2 , as a singleton bag would give a vertex in $G$ forming a maximal clique of size 1 . Since $G$ is connected, this would imply that $G$ is just a single vertex, in which case we may take $A=V(G)$ and $X$ empty.

Choose $z \in V(X)$ so that each component of $X-z$ has at most $|V(X)| / 2$ vertices. Let $Q$ be the $x z$-path in $T$. Note that $V(D(Q ; Q, X))=V(X)$, and so $V(D(Q ; Q, X))$ has the core capture property for $\mathcal{P}$. Let $Q_{1}$ be a minimal subpath of $Q$ such that $V\left(D\left(Q_{1} ; Q, X\right)\right)$ has the core capture property for $\mathcal{P}$. We claim that $G$ has a vertex $w_{1}$ such that $S\left(w_{1}\right)$ spans $Q_{1}$. If $\left|V\left(Q_{1}\right)\right|=1$ and $Q_{1}=y$, then minimality of $T$ gives that $B(y)$ is nonempty and we may choose $w_{1} \in B(y)$. Otherwise, if $\left|V\left(Q_{1}\right)\right| \geq 2$, then we apply Lemma 40 to obtain $w_{1}$.

Let $\mathcal{P}_{1}$ be the set of all paths $P \in \mathcal{P}$ that do not contain $w_{1}$; note that if $\mathcal{P}_{1}$ is empty, then the lemma is satisfied with $A=\left\{w_{1}\right\}$ and $X^{\prime}$ empty. So assume $\mathcal{P}_{1}$ is nonempty and let $Q_{2}$ be a minimal subpath of $Q_{1}$ such that $V\left(D\left(Q_{2} ; Q, X\right)\right)$ has the core capture property with respect to $\mathcal{P}_{1}$. Our next aim is to obtain $w_{2} \in V(G)$ such that $w_{2} \neq w_{1}$ and $S\left(w_{2}\right)$ spans $Q_{2}$. Indeed, if $\left|V\left(Q_{2}\right)\right|=1$ with $Q_{2}=y$, then we may use $|B(y)| \geq 2$ to choose $w_{2} \in B(y)$ distinct from $w_{1}$. Otherwise, if $\left|V\left(Q_{2}\right)\right| \geq 2$, then we obtain $w_{2}$ by applying Lemma 40 with $\mathcal{H}=\mathcal{P}_{1}$, and observing that $w_{2}$ is chosen from the union of two paths in $\mathcal{P}_{1}$, none of which contain $w_{1}$.

Let $\mathcal{P}_{2}$ be the set of all paths $P \in \mathcal{P}_{1}$ that do not contain $w_{2}$. Again, we may assume $\mathcal{P}_{2}$ is nonempty, or else the lemma is satisfied with $A=\left\{w_{1}, w_{2}\right\}$ and $X^{\prime}$ empty. Let $\mathcal{R}$ be the family of paths in $G-\left\{w_{1}, w_{2}\right\}$ of size at least 3 whose endpoints have subtrees intersecting $Q_{2}$ and whose interior vertices $u$ satisfy $S(u) \subseteq$ $D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$. We claim that each $P \in \mathcal{P}_{2}$ has a subpath in $\mathcal{R}$. Indeed, since $D\left(Q_{2} ; Q, X\right)$ has the core capture property for $\mathcal{P}_{1}$ and $P \in \mathcal{P}_{2} \subseteq \mathcal{P}_{1}$, it follows that $P$ has a core vertex in $V\left(D\left(Q_{2} ; Q, X\right)\right)$. Since $P$ is a longest path, $V\left(Q_{2}\right) \subseteq S\left(w_{2}\right)$, and $w_{2} \notin V(P)$, it follows that $P$ has no core vertex in $V\left(Q_{2}\right)$. Since $P$ has a core vertex in $D\left(Q_{2} ; Q, X\right)$ but no core vertex in $V\left(Q_{2}\right)$, it follows that there exists $u \in V(P)$ with $S(u) \subseteq D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$. Since the endpoints of $P$ have subtrees intersecting $T-V(X)$, it follows that $u$ is an interior vertex in a subpath of $P$ in $\mathcal{R}$.

Let $W=\left\{w_{1}, w_{2}\right\}$, and let $R$ be a path in $\mathcal{R}$ of maximum length. Applying Lemma 41 with $\mathcal{H}=\mathcal{P}_{2}$ to $W$ and $\mathcal{R}$, it follows that $V(R)$ intersects each path in $\mathcal{P}_{2}$. Let $w_{3}$ and $w_{4}$ be the endpoints of $R$. Note that $R-\left\{w_{3}, w_{4}\right\}$ is a (nonempty) connected subgraph of $G$, all of whose subtrees are contained in a single component of $D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$, which happens to be a rooted subtree $X^{\prime}$ of $X$.

Let $A=\left\{w_{1}, \ldots, w_{4}\right\}$. Let $P$ be a path in $\mathcal{P}$ disjoint from $A$, and note that $P \in \mathcal{P}_{2}$. Recall that $P$ and $R$ intersect, and since the endpoints of $R$ are contained in $A$, it follows that $P$ contains a vertex $u$ in the interior of $R$. Since $S(u) \subseteq X^{\prime}$ and since $|V(P)|>1$ follows that $P$ has a core vertex in $X^{\prime}$.

Corollary 45. Let $G$ be an n-vertex chordal graph with minimal rooted tree representation $T$, and let $X$ be $a$ rooted subtree of $T$. Let $\mathcal{P}$ be a family of longest paths in $G$ such that $V(X)$ has the core capture property for $\mathcal{P}$ and the endpoints of each $P \in \mathcal{P}$ have subtrees that contain a vertex outside $X$. There is a set $A \subseteq V(G)$ that intersects each path in $\mathcal{P}$ with $|A| \leq 4(1+\lfloor\lg |V(X)|\rfloor)$.

Proof. Iteratively apply Lemma 44 to obtain a sequence $\left(\mathcal{P}_{0}, X_{0}\right), \ldots,\left(\mathcal{P}_{t}, X_{t}\right)$ and sets $A_{1}, \ldots, A_{t}$ starting with $\left(\mathcal{P}_{0}, X_{0}\right)=(\mathcal{P}, X)$ and terminating when $X_{t}$ is empty, such that $V\left(X_{i}\right)$ has the core capture property for $\mathcal{P}_{i}$, the family $\mathcal{P}_{i+1}$ is the set of all paths in $\mathcal{P}_{i}$ that are disjoint from $A_{i+1}$, and $\left|V\left(X_{i+1}\right)\right| \leq\left|V\left(X_{i}\right)\right| / 2$. Since $\left|V\left(X_{i}\right)\right| \leq|V(X)| / 2^{i}$, it follows that $X_{i}$ is empty when $i>\lg |V(X)|$ and so $t \leq 1+\lfloor\lg |V(X)|\rfloor$. Also, since $X_{t}$ is empty but has the core capture property for $\mathcal{P}_{t}$, it follows that $\mathcal{P}_{t}=\varnothing$. Therefore $\bigcup_{i=1}^{t} A_{i}$ is a transversal for $\mathcal{P}$ of size at most $4 t$.

Lemma 41 considers a set of paths $\mathcal{R}$ such that each $R \in \mathcal{R}$ has endpoints that are adjacent to the glue vertices; these paths start in and return to the neighborhood of the glue vertices. Our next lemma is an analogue for "one-way" paths that start in the neighborhood of a glue vertex and need not return. A suffix of a path $P$ is a subpath of $P$ containing an endpoint of $P$.

Lemma 46. Let $G$ be a graph, let $w$ be a glue vertex, and let $\mathcal{P}$ be a family of longest paths in $G$ with each $P \in \mathcal{P}$ avoiding $w$. An attachment point is a vertex $u \in N(w)$. Let $\mathcal{R}$ be a nonempty family of paths in $G$ such that each $R \in \mathcal{R}$ avoids $w$ and has at least one attachment endpoint. If every path $P \in \mathcal{P}$ contains a suffix in $\mathcal{R}$, then a longest path in $\mathcal{R}$ intersects every path in $\mathcal{P}$.

Proof. Let $R$ be a longest path in $\mathcal{R}$ and suppose that some $P \in \mathcal{P}$ is disjoint from $R$. Let $R_{0}$ be a suffix of $P$ with $R_{0} \in \mathcal{R}$. Let $x$ and $y$ be the endpoints of $R_{0}$, with $y$ also serving as an endpoint of $P$. Note that $y$ is not an attachment point, or else we may extend $P$ by appending $w$ at $y$. Therefore $x$ is an attachment point. Since $|V(R)| \geq\left|V\left(R_{0}\right)\right|$, we obtain a longer path by replacing $R_{0}$ with the path $x w R$ (with $R$ oriented appropriately).

Lemma 47. Let $G$ be a connected chordal graph with minimal rooted tree representation $T$. Let $X$ be a rooted subtree of $T$ and let $\mathcal{P}$ be a family of longest paths in $G$ such that $V(X)$ has the core capture property for $\mathcal{P}$. There exists a set $A \subseteq V(G)$ with $|A| \leq 4\lfloor\lg |V(X)|\rfloor+5$ and a rooted subtree $X^{\prime}$ of $X$ with $\left|V\left(X^{\prime}\right)\right| \leq|V(X)| / 2$ such that for each $P \in \mathcal{P}$ :

1. P has a vertex in $A$, or
2. $P$ has a core vertex in $X^{\prime}$.

Proof. If $X$ is empty, then $\mathcal{P}=\varnothing$ since $X$ has the core capture property for $\mathcal{P}$. In this case, the lemma is satisfied with $A=\varnothing$ and $X^{\prime}$ empty. So we assume $X$ is nonempty. Let $x$ be the root of $X$, and choose $z \in V(X)$ so that each component of $X-z$ has at most $|V(X)| / 2$ vertices. Let $Q$ be the $x z$-path in $T$.

Let $Q_{1}$ be a minimal subpath of $Q$ such that $V\left(D\left(Q_{1} ; Q, X\right)\right)$ has the core capture property for $\mathcal{P}$. We claim some vertex $w_{1} \in V(G)$ satisfies $Q_{1} \subseteq S\left(w_{1}\right)$. If $Q_{1}$ is a single vertex $y$, then we may take $w_{1}$ to be any vertex in $B(y)$. Otherwise, we apply Lemma 40 with $\mathcal{H}=\mathcal{P}$ to obtain $w_{1}$.

Let $\mathcal{P}_{1}$ be the set of all $P \in \mathcal{P}$ such that $w_{1} \notin V(P)$. We may assume $\mathcal{P}_{1}$ is nonempty, or else the lemma is satisfied with $A=\left\{w_{1}\right\}$ and $X^{\prime}$ empty. Let $Q_{2}$ be a minimal subpath of $Q_{1}$ such that $V\left(D\left(Q_{2} ; Q, X\right)\right)$ has
the core capture property for $\mathcal{P}_{1}$. We claim there is a vertex $w_{2} \in V(G)$ such that $w_{2} \neq w_{1}$ and $Q_{2} \subseteq S\left(w_{2}\right)$. Indeed, if $Q_{2}$ is a single vertex $y$, then since $|B(y)| \geq 2$ and we may choose $w_{2} \in B(y)$ distinct from $w_{1}$. Otherwise, we apply Lemma 40 with $\mathcal{H}=\mathcal{P}_{1}$ to obtain $w_{2}$. Since $w_{2}$ is chosen from the union of two paths in $\mathcal{P}_{1}$, neither of which contains $w_{1}$, we have $w_{2} \neq w_{1}$. Let $\mathcal{P}_{2}$ be the set of paths in $\mathcal{P}_{1}$ that do not contain $w_{2}$. Since $S\left(w_{2}\right)$ contains $Q_{2}$ and all paths in $\mathcal{P}_{2}$ avoid $w_{2}$, it follows that the endpoints of each path in $\mathcal{P}_{2}$ have subtrees that are disjoint from $V\left(Q_{2}\right)$ (or else appending $w_{2}$ would extend the path).

Let $\mathcal{P}_{3}$ be the set of all paths $P \in \mathcal{P}_{2}$ that have an endpoint $v$ such that $S(v) \subseteq D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$ and let $\mathcal{P}_{4}=\mathcal{P}_{2}-\mathcal{P}_{3}$. Our goal is to apply Corollary 45 to obtain a small set of vertices $B$ such that every path in $\mathcal{P}_{4}$ intersects $B$. If $\mathcal{P}_{4}=\varnothing$, then we may simply take $B=\varnothing$. Suppose $\mathcal{P}_{4}$ is nonempty. Let $\mathcal{R}$ be the family of paths $R$ in $G$ of size at least 3 such that each endpoint of $R$ has a subtree intersecting $Q_{2}$ and each interior vertex $u$ of $R$ satisfies $S(u) \subseteq D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$. We claim that if $P \in \mathcal{P}_{4}$, then $P$ has a subpath in $\mathcal{R}$. Since $P$ has a core vertex in $V\left(D\left(Q_{2} ; Q, X\right)\right)$ but no core vertex in $V\left(Q_{2}\right)$, it follows from Lemma 37 that $P$ contains a vertex $u$ such that $S(u) \subseteq D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$. Since $P \notin \mathcal{P}_{3}$, it follows that $u$ is an interior vertex in a subpath of $P$ contained in $\mathcal{R}$. Let $R$ be a longest path in $\mathcal{R}$. Applying Lemma 41 to $\mathcal{P}_{4}$ with glue vertices $W=\left\{w_{1}, w_{2}\right\}$, it follows that each path in $\mathcal{P}_{4}$ intersects $R$. Choose $y \in V\left(Q_{2}\right)$ and $y^{\prime} \in V(D(y ; Q, X)) \cap N(y)$ such that each interior vertex $u$ of $R$ has a subtree $S(u)$ contained in the subtree $Y$ of $T$ rooted at $y^{\prime}$. Let $w_{3}$ and $w_{4}$ be the endpoints of $R$, and observe that each $P \in \mathcal{P}_{4}$ that avoids $w_{3}$ and $w_{4}$ must contain an interior vertex in $R$ and hence have a core vertex in $Y$. By Corollary 45, there exists $B \subseteq V(G)$ such that $|B| \leq 4(1+\lfloor\lg |Y|\rfloor) \leq 4\lfloor\lg |X|\rfloor$ and each path in $\mathcal{P}_{4}$ contains a vertex in $B \cup\left\{w_{3}, w_{4}\right\}$.

Our next goal is to process $\mathcal{P}_{3}$. If $\mathcal{P}_{3}=\varnothing$, then the lemma is satisfied with $X^{\prime}$ empty and $A=$ $\left\{w_{1}, w_{2}, w_{3}, w_{4}\right\} \cup B$. So we amy assume $\mathcal{P}_{3}$ is nonempty. If $\mathcal{P}_{3}$ contains a path $P$ such that $S(P) \subseteq$ $D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$, then we choose $y \in V\left(Q_{2}\right)$ and $y^{\prime} \in V\left(D\left(y ; Q_{2}, X\right)\right) \cap N(y)$ such that $S(P)$ is contained in the subtree $X^{\prime}$ of $T$ rooted at $y^{\prime}$. In this case, the lemma is satisfied with $X^{\prime}$ and $A=\left\{w_{1}, w_{2}, w_{3}, w_{4}\right\} \cup B$. So we may assume each path $P \in \mathcal{P}_{3}$ has an endpoint $u$ with $S(u) \subseteq D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$ and some other vertex $v$ with $S(v)$ intersecting $V\left(Q_{2}\right)$.

Let $\mathcal{R}$ be the set of paths $R$ in $G$ of size at least 2 such that some endpoint $u$ has a subtree $S(u)$ that intersects $V\left(Q_{2}\right)$ but all other vertices $v$ in $R$ satisfy $S(v) \subseteq D\left(Q_{2} ; Q, X\right)-V\left(Q_{2}\right)$. Note that each path in $\mathcal{P}_{3}$ has a subpath in $\mathcal{R}$, and it follows from Lemma 46 with $w=w_{2}$ that a longest path $R \in \mathcal{R}$ intersects each path in $\mathcal{P}_{3}$. As above, choose $y \in V\left(Q_{2}\right)$ and $y^{\prime} \in V\left(D\left(Q_{2} ; Q, X\right)\right) \cap N(y)$ such that all but one vertex in $R$ has a subtree contained in the subtree $X^{\prime}$ of $T$ rooted at $y^{\prime}$. Let $w_{5}$ be the endpoint of $R$ such that $S\left(w_{5}\right)$ intersects $V\left(Q_{2}\right)$, and note that each path in $\mathcal{P}_{3}$ that avoids $w_{5}$ contains a vertex from $R$ whose subtree is contained in $X^{\prime}$. Therefore $X^{\prime}$ has the core capture property for the family of paths in $\mathcal{P}_{3}$ that avoid $w_{5}$.

Let $A=\left\{w_{1}, \ldots, w_{5}\right\} \cup B$, and note that each path $P \in \mathcal{P}$ either intersects $A$ or has a core vertex in $X^{\prime}$.

Our goal is to show that each connected $n$-vertex chordal graph has a longest path transversal of size at most $O\left(\log ^{2} n\right)$. In our bound below, we make no attempt to optimize the multiplicative constant 4 on the leading $\lg ^{2} n$ term.

Theorem 48. If $G$ is a connected $n$-vertex chordal graph, then $\operatorname{lpt}(G) \leq(4\lfloor\lg n\rfloor+5)(\lfloor\lg n\rfloor+1)$.

Proof. Let $T$ be a minimal rooted tree representation for $G$. Since $T$ is a minimal tree representation, the vertices $x \in V(T)$ correspond to maximal cliques in $G$, and it is easy to see from a simplicial elimination
ordering of $V(G)$ that $G$ has at most $n$ maximal cliques. Hence $|V(T)| \leq n$.
Let $X=T$ and let $\mathcal{P}$ be the family of longest paths in $G$. We may assume that each $P \in \mathcal{P}$ has an edge, or else $G$ is a single vertex. It follows that $X$ has the core capture property for $\mathcal{P}$.

We apply Lemma 47 iteratively to obtain $\left(\mathcal{P}_{0}, X_{0}\right), \ldots,\left(\mathcal{P}_{t}, X_{t}\right)$ and $A_{1}, \ldots, A_{t}$ as follows. We set $\left(\mathcal{P}_{0}, X_{0}\right)=(\mathcal{P}, X)$. Given $\left(\mathcal{P}_{i}, X_{i}\right)$ such that $X_{i}$ is a rooted subtree of $T$ with $V\left(X_{i}\right)$ having the core capture property for $\mathcal{P}_{i}$, we obtain $\left(\mathcal{P}_{i+1}, X_{i+1}\right)$ and a set $A_{i+1} \subseteq V(G)$ with $\left|A_{i+1}\right| \leq 4\left\lfloor\lg \left|V\left(X_{i}\right)\right|\right\rfloor+5$ such that $\left|V\left(X_{i+1}\right)\right| \leq\left|V\left(X_{i}\right)\right| / 2$, each path in $\mathcal{P}_{i+1}-\mathcal{P}_{i}$ intersects $A_{i+1}$, and $X_{i+1}$ has the core capture property for $\mathcal{P}_{i+1}$. The iteration terminates when $\mathcal{P}_{t}=\varnothing$.

Note that since $\left|V\left(X_{i}\right)\right| \leq n / 2^{i}$, we have that $t \leq 1+\lfloor\lg n\rfloor$. Let $A=\bigcup_{i=1}^{t} A_{i}$, and note that $A$ is a longest path transversal for $G$. Since $\left|A_{i}\right| \leq 4(\lfloor\lg n\rfloor-(i-1))+5 \leq 4\lfloor\lg n\rfloor+5$, we have $|A| \leq t(4\lfloor\lg n\rfloor+5)$.

### 4.4 Leafage Bound

The leafage of a connected chordal graph $G$, denoted leaf $(G)$, is the minimum number of leaves in a tree representation of $G$. Previously introduced by Lin, McKee and West [35], leafage is a measure of how far a chordal graph is from being an interval graph. It is known to be computable in polynomial time [36].

For a tree $T$ on at least 2 vertices, let $f(T)$ be the maximum, over all $e \in E(T)$, of the minimum number of leaves of $T$ contained in a component of $T-e$. For a connected chordal graph $G$ with tree representation $T$, we show that $\operatorname{lpt}(G) \leq f(T)$.

Balister, Györi, Lehel, and Schelp [16] observed that tree representations of connected chordal graphs contain longest path transversal bags; we include the simple argument for completeness.

Proposition 49 (Balister-Györi-Lehel-Schelp [16]). Let $T$ be a tree representation of a connected chordal graph $G$. There is a vertex $x \in V(T)$ such that $B(x)$ is a longest path transversal of $G$.

Proof. Let $\mathcal{P}$ be the family of longest paths in $G$. For $P, Q \in \mathcal{P}$, we have that $P$ and $Q$ meet in a common vertex $w$, and it follows that $S(P)$ and $S(Q)$ both contain $S(w)$. Therefore $\{S(P): P \in \mathcal{P}\}$ is a pairwise intersecting family of subtrees of $T$, and since subtrees of a tree have the Helly property, there exists $x \in V(T)$ belong to each $S(P)$ for $P \in \mathcal{P}$. It follows that $B(x)$ is a longest path transversal for $G$.

Let $T$ be a tree with $k$ leaves. For $x, y \in V(T)$, we let $T[x, y]$ denote the path in $T$ with endpoints $x$ and $y$. A finger of $x$ is a path of the form $T[x, z]$, where $z$ is a leaf in $T$. Note that every vertex of a tree $T$ has $k$ fingers.

Let $G$ be a connected chordal graph and let $T$ be a minimal tree representation of $G$ with vertex $x$. A path $u_{1} \ldots u_{t}$ is handy with respect to $x$ if $x \in V\left(S\left(u_{1}\right)\right)$ but $x \notin V\left(S\left(u_{j}\right)\right)$ for $j>1$. With $Q=u_{1} \ldots u_{t}$, we have that $S\left(Q-u_{1}\right)$ is contained in a component of $T-x$.

Proposition 50. Let $T$ be a tree representation for a chordal graph $G$, and let $x$ be a vertex in $T$ such that $B(x)$ is a longest path transversal in $G$. If $Q$ is a handy path with respect to $x$ of maximum size, then $V(Q)$ is a longest path transversal in $G$.

Proof. By way of contradiction, let $P$ be a longest path in $G$ that is disjoint from $Q$. Note that $P$ contains at least one vertex in $B(x)$. Let $u$ and $v$ be the endpoints of $P$, let $w$ be the vertex in $B(x) \cap V(P)$ that is closest to $v$, and let $P_{0}=P[w, v]$. Note that $P_{0}$ is a handy path with respect to $x$, and by our selection of $Q$, we have $|V(Q)| \geq\left|V\left(P_{0}\right)\right|$. Since an endpoint of $Q$ has a subtree containing $x$, we obtain a longer path by attaching $P[u, w]$ to $Q$.

Let $T$ be a tree representation for a chordal graph $G$, let $x, y \in V(T)$. We say that $u \in V(G)$ is maximal from $x$ toward $y$ if, among all vertices in $B(x)$, the vertex $u$ maximizes $|V(S(u)) \cap V(T[x, y])|$.

Lemma 51. Let $G$ be a chordal graph with tree representation $T$, and let $w w^{\prime} \in E(G)$. If $w \in B(x)$ and $S\left(w^{\prime}\right)$ is contained in the component $X$ of $T-x$, then there is a leaf $y$ in $T$ belonging to $X$ such that every vertex in $G$ maximal from $x$ toward $y$ completes a triangle with $w w^{\prime}$.

Proof. Since $w w^{\prime} \in E(G)$, we have that $\varnothing \subsetneq V(S(w)) \cap V\left(S\left(w^{\prime}\right)\right) \subseteq V(X)$. It follows that there is a vertex $z \in V(T)$ that is common to all three subtrees in $\left\{S(w), S\left(w^{\prime}\right), X\right\}$. Let $y$ be a leaf in $T$ such that $z$ is on $T[x, y]$, and let $u \in V(G)$ be maximal from $x$ toward $y$. Since $x, z \in V(S(w))$, it follows from our choice of $u$ that $x, z \in V(S(u))$ also. Since $z$ is common to $S(w), S\left(w^{\prime}\right)$, and $S(u)$, it follows that $\left\{w, w^{\prime}, u\right\}$ is a triangle in $G$.

Lemma 52 (The Hand Lemma). Let $G$ be a connected chordal graph with tree representation $T$ and a vertex $x \in V(T)$ such that $B(x)$ is a longest path transversal of $G$. Let $Q$ be a handy path with respect to $x$ of maximum size, and suppose that $|V(Q)| \geq 2$. Let $X$ be the component of $T-x$ containing $S(Q)-x$. Let $y_{1}, \ldots, y_{k}$ be the leaf vertices of $T$ in $X$. For each $i$, let $u_{i}$ be a vertex in $B(x)$ that extends maximally towards $y_{i}$. The set $\left\{u_{1}, \ldots, u_{k}\right\}$ is a longest path transversal for $G$.

Proof. Let $A=\left\{u_{1}, \ldots, u_{k}\right\}$, and let $Q=v_{1} \ldots v_{t}$ with $v_{1} \in B(x)$. Since $v_{1} v_{2} \in E(G)$ with $S\left(v_{2}\right) \subseteq X$, it follows from Lemma 51 that some $u \in A$ completes a triangle with $v_{1} v_{2}$. Therefore $u v_{2} \ldots u_{k}$ is also handy with respect to $x$. Hence we may assume without loss of generality that $v_{1} \in A$.

Let $P$ be a longest path in $G$ and suppose for a contradiction that $V(P) \cap A=\varnothing$. By Proposition 50, we have $V(P) \cap V(Q) \neq \varnothing$, and so $P$ contains a vertex whose subtree is contained in $X$. Additionally, since $B(x)$ is a longest path transversal, it follows that $P$ contains adjacent vertices $w w^{\prime}$ such that $w \in B(x)$ and $S\left(w^{\prime}\right) \subseteq X$. By Lemma 51, some vertex in $A$ completes a triangle with $w$ and $w^{\prime}$, and we obtain a longer path by inserting this vertex in $P$ between $w$ and $w^{\prime}$.

Recall that $f(T)$ is the maximum, over all $e \in E(T)$, of the minimum number of leaves of $T$ contained in a component of $T-e$.

Theorem 53 (Leafage Bound). If $G$ is a connected chordal graph with tree representation $T$, then either $G$ is complete or $\operatorname{lpt}(G) \leq f(T)$.

Proof. Note that if $T^{\prime}$ is obtained from $T$ by contracting an edge, then $f\left(T^{\prime}\right) \leq f(T)$. Hence we may assume that $T$ is a minimal tree representation. If $|V(T)|=1$, then clearly $G$ is complete, so we assume $|V(T)| \geq 2$. Let $x \in V(T)$ and let $y$ be a neighbor of $x$. By minimality of $T$, we have that $v \in B(y)-B(x)$ for some vertex $v \in V(G)$. By connectivity of $G$, there exists a vertex $u \in B(x) \cap B(y)$. Note that $u v \in E(G)$ and $u v$
is a handy path with respect to $x$. It follows that for each $x \in V(T)$, a maximum handy path with respect to $x$ has size at least 2 .

We construct an auxiliary digraph $H$ on $V(T)$ as follows. For each $x \in V(T)$ such that $B(x)$ is a longest path transversal, we select a maximum handy path $Q$ with respect to $x$, and we direct an edge in $H$ from $x$ to $y$, where $y$ is the neighbor of $x$ belonging to the component of $T-x$ containing $S\left(Q-v_{1}\right)$, where $v_{1}$ is the endpoint of $Q$ in $B(x)$.

Note that in $H$, the outdegree of $x$ is 1 if $B(x)$ is a longest path transversal of $G$ and 0 otherwise. By Proposition 49, there exists an edge in $H$. Suppose $x y \in E(H)$. By Lemma 52, we obtain a longest path transversal contained in $B(x) \cap B(y)$. Hence $x y \in E(H)$ implies that $B(y)$ is a longest path transversal in $G$, and therefore $y$ also has outdegree 1. It follows that $H$ contains a directed cycle. Since the underlying graph of $H$ is acyclic, it follows that $x y \in E(H)$ and $y x \in E(H)$ for some vertices $x, y \in V(H)$.

Let $e$ be the edge $x y$ in $T$, let $T_{x}$ be the component of $T-e$ containing $x$, and let $T_{y}$ be the component of $T-e$ containing $y$. Since $x y \in E(H)$, it follows from Lemma 52 that maximal vertices from $x$ toward leaves in $T_{y}$ form a longest path transversal. Similarly, maximal vertices from $y$ toward leaves in $T_{x}$ also form a longest path transversal. Therefore $\operatorname{lpt}(G) \leq \min \left\{\ell\left(T_{x}\right), \ell\left(T_{y}\right)\right\} \leq f(T)$, where $\ell\left(T_{x}\right)$ and $\ell\left(T_{y}\right)$ are the number of leaves in $T$ belonging to $T_{x}$ and $T_{y}$, respectively.

Corollary 54. If $G$ is a connected chordal graph, then $\operatorname{lpt}(G) \leq \operatorname{leaf}(G) / 2$.

Proof. Let $T$ be a tree representation of $G$ with leaf $(G)$ leaves. By Theorem 53, we have $\operatorname{lpt}(G) \leq f(T) \leq$ $\operatorname{leaf}(G) / 2$.

A graph $G$ is a star if it is isomorphic to $K_{1, k}$ for some $k \in \mathbb{N}$. We call such graphs $k$-stars. A vertex of degree $k$ in a $k$-star is a center (vertex), and this vertex is unique except when $k=1$. A subdivided star is a star whose edges are subdivided some number of times (possibly zero). Two edges of the original star need not be subdivided the same number of times.

Corollary 55. The family of chordal graphs admitting a tree representation $T$ such that $T$ is a subdivided star tree is Gallai.

Proof. Let $G$ be a connected chordal graph with tree representation $T$ such that $T$ is a subdivision of a star. Note that $f(T)=1$. It follows from Theorem 53 that $G$ is complete (implying $\operatorname{lpt}(G)=1$ ) or $\operatorname{lpt}(G) \leq f(T)=1$.

## Chapter 5

## Conclusion

It has been said that one of the worst things to do in research is solve a problem completely. The idea being there is no more to be done. One of the greatest strengths of longest path transversals and all the work presented here is just how many more avenues there are to explore. Yes, there are some nice results here, and some good progress, but the problems are far from solved. And that is all the better.

In the second chapter, we give a sublinear upper bound on longest path transversal number for connected graphs. Like many others, though, we think the best possible bound should be constant. Obtaining such a bound appears to be difficult, though. Much of the resistance seems to come from the freedom longest paths enjoy and how complicated their intersections can be.

It was suggested to us to try to bound $\operatorname{lpt}(G)$ in terms of various graph parameters. For example, a fraction of the clique number added to another fraction of independence number. Perhaps there is progress to made here.

Instead of looking at upper bounds on $\operatorname{lpt}(G)$, we could try to raise the lower bound. We could search for a connected graph $G$ with $\operatorname{lpt}(G)=4$, which is deceptively hard. If we find one, we could search for a connected graph $G$ with $\operatorname{lpt}(G)=k$ for larger $k$. This task is challenging as it will likely require a reasonably large number of vertices (at least dozens).

Graünbaum's $\operatorname{lpt}(G)=3$ example comes from replacing degree three vertices in a graph with $\operatorname{lpt}(G)=2$ with the Peterson fragment. This trick, however, fails to produce a graph with $\operatorname{lpt}(G)=4$ as there (usually) is some Peterson fragment that is a transversal, so one can just select its three vertices of degree one as the transversal. Hence, we will need a new gadget to find a graph with $\operatorname{lpt}(G)=4$.

Additionally, we can look at how various graph parameters affect transversals. We have given a result giving a Gallai vertex in large graphs with $\alpha(G) \leq \kappa(G)+2$. It seems possible it can be extended to all graphs $G$ with $\alpha(G) \leq \kappa(G)+2$, not just large graphs. This does not seem like the whole truth is there, either. Recall that we know of no 4 -connected graph $G$ with $\operatorname{lpt}(G)>1$. It is possible that $\alpha(G) \leq \kappa(G)+3$ or even $\alpha(G) \leq \kappa(G)+4$ is sufficient to guarantee a Gallai vertex. The Peterson fragment shows that $\alpha(G) \leq \kappa(G)+5$ is not sufficient.

Another reasonable direction to continue is to forbid other linear forests in the Peterson fragment. All linear forests in $G_{0}$ on at most four vertices have been taken care of, and $5 P_{1}$ is also a fixer. It remains to
look at the other linear forests in $G_{0}$ having between 5 and 9 vertices. If we could show, for example, $3 P_{3}$ is a fixer then all of its subgraphs are also fixers. An easier task is to start with smaller subgraphs.

Next, as in the previous chapter and as many others have done, we could try to give upper bounds for longest path transversal number for other families of graphs. We conjecture that chordal graphs form a Gallai family and, at worst, should have logarithmic longest path transversals.

If we cannot show chordal graphs form a Gallai family, we could look at other host trees for the tree representation, or try to tighten the leafage bounds.

As things stand, our bound on the longest cycle transversal number of a 2-connected chordal graph has smaller order of magnitude than our bound on the longest path transversal number of a connected chordal graph. It seems possible that they should be of the same order of magnitude.

An asteroidal triple in a graph $G$ is a set of three distinct vertices such that each pair of vertices is connected by some path avoiding the neighborhood of the third vertex. Interval graphs are the chordal graphs without asteroidal triples [37], and notably interval graphs are Gallai. Hence, graphs with no asteroidal triple may form a Gallai family.

Overall, there is simply a lot to do here. The various problems regarding longest path transversals, and their longest cycle transversal variants, all could be studied. There are plenty of interesting graph families and graph parameters that impose structure that could help in solving these transversal problems.

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