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Some coupled fixed point theorems in complete $E-$ fuzzy metric spaces

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Abstract

In this paper some coupled coincidence fixed point theorems are established for the mappings under ϕ- contraction in complete generalized E- fuzzy metric spaces. Also give an example to validate the theorem and present an application of this theorem to integral equations.

Keywords:Fuzzy metric space, G-metric space, E-fuzzy metric space, Complete E-fuzzy metric space, coupled fixed point, coupled coincidence point, ϕ map.

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1 Introduction

L.A.Zadeh [1965] introduced fuzzy set theory. This concept was used in almost all branches of mathematics. George.A and P.Veeramani [1994] modified the notion of fuzzy metric space proposed by O.Kramosil and J.Michalek [1975]. Many authors investigated the idea of contractive mappings and constructed fixed point theorems on the basis of this fuzzy metric space.
The notion of Generalized metric space was investigated by Mustafa.Z and B.Sims [2006]. Many researchers proved existence and uniqueness of fixed point of mappings under some contractions in G-metric space. The concept of coupled fixed point for a function $F : X^2 \to X$ was introduced by Bhaskar.T.G and Lakshmikantham.G.V [2006] and proved some coupled fixed point theorems in partially ordered metric space. Also they presented some applications of this in the existence and uniqueness of a solution of periodic boundary value problems. Lakshmikantham.V and Ciric.Lj. [2009] gave the concept of coupled coincidence and coupled common fixed point for the mappings $F : X^2 \to X$ and $g : X \to X$ and established some results. Sedghi et al. [2012] discussed a coupled fixed point theorem for contractions in fuzzy metric space and as a generalization of this result, Xin-Qi-Hu [2011] established a common fixed point theorem for mapping under $\phi-$ contraction in fuzzy metric space. Many authors have studied coupled fixed point theorems in metric space, fuzzy metric space and G- metric space. Using the concept of G-metric space, Sukanya.K.P and Jose [2017] proposed E-fuzzy metric space and proved some fixed point theorems in this space.
The purpose of this paper is to establish coupled coincidence fixed point theorems for the mappings under ϕ- contraction in complete E-fuzzy metric space. Also an example to illustrate the theorem is given and an application of this theorem to integral equations is presented.

2 Preliminaries

Definition 2.1. *B.Schweizer and A.Sklar [1983]: A binary operation* $*$ *:* $[0, 1]^2 \to [0, 1]$ *is a continuous t-norm if it satisfies the following:*

1. *commutative and associative,*

2. *continuous,*

3. $t * 1 = t$ *for all* $t \in [0, 1]$*,*

4. $t * s \le t' * s'$ *whenever* $t \le t'$ *and* $s \le s'$ *for all* $t, s, t', s' \in [0, 1]$.

Definition 2.2. *Sukanya.K.P and Jose [2017] Let X is an arbitrary set and * is a continuous t-norm, a 3-tuple $(X, E, *)$ is called E- fuzzy metric space if E is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following:*

1. *$E(\alpha, \beta, \gamma, t) > 0$ and $E(\alpha, \alpha, \beta, t) \leq E(\alpha, \beta, \gamma, t)$ for all $\alpha, \beta, \gamma \in X$ with $\gamma \neq \beta$,*
2. *$E(\alpha, \beta, \gamma, t) = 1$, for all $t > 0$ if and only if $\alpha = \beta = \gamma$,*
3. *$E(\alpha, \beta, \gamma, t) = E(p(\alpha, \beta, \gamma), t)$ (symmetry), where p is a permutation function,*
4. *$E(\alpha, a, \gamma, t) * E(a, \beta, \gamma, s) \leq E(\alpha, \beta, \gamma, t + s)$,*
5. *E is continuous, $\forall \alpha, \beta, \gamma, a \in X$ and $t, s > 0$.*

Definition 2.3. *Sukanya.K.P and Jose [2022] A sequence $\{\alpha_n\}$ in $(X, E, *)$ is said to be converges to a point $\alpha \in X$ if $E(\alpha_n, \alpha, \alpha, t) \to 1$ as $n \to \infty$.*

Remark 2.1. *Shahana.A.R and Jose [in press] A sequence $\{\alpha_n\}$ in $(X, E, *)$ is said to be a cauchy sequence if for each $0 < r < 1$ and $t > 0$ there exists $k \in N$ such that $E(\alpha_m, \alpha_m, \alpha_n, t) > 1 - r$ for each $m, n \geq k$.*

Definition 2.4. *Sukanya.K.P and Jose [2022] A complete E-fuzzy metric space is the E- fuzzy metric space in which every cauchy sequence is convergent.*

Remark 2.2. *Shahana.A.R and Jose [in press] Let $(X_1, E_1, *)$ and $(X_2, E_2, *)$ be E-fuzzy metric spaces. A function $f : X_1 \to X_2$ is continuous at a point $\alpha \in X_1$ if and only if every convergent sequence $\{\alpha_n\} \to \alpha$ in X_1, $\{f(\alpha_n)\} \to f(\alpha)$ in X_2.*

Definition 2.5. *Bhaskar.T.G and Lakshmikantham.G.V [2006] Let X be a nonempty set, an element $(\alpha, \beta) \in X^2$ is called a coupled fixed point of a mapping $F : X^2 \to X$ if $F(\alpha, \beta) = \alpha$ and $F(\beta, \alpha) = \beta$.*

Definition 2.6. *Lakshmikantham.V and Ciric.Lj. [2009] Let X be a non empty set, the element $(\alpha, \beta) \in X^2$ is called a coupled coincidence point of the mappings $F : X^2 \to X$ and $g : X \to X$ if $F(\alpha, \beta) = g\alpha$ and $F(\beta, \alpha) = g\beta$.*

Definition 2.7. *J.X.Fang [2009] An element $\alpha \in X$ is called common fixed point of the mappings $F : X^2 \to X$ and $g : X \to X$ if $F(\alpha, \alpha) = g\alpha = \alpha$, then the F and g.*

Definition 2.8. *Lakshmikantham.V and Ciric.Lj. [2009] The mappings $F : X^2 \to X$ and $g : X \to X$ are said to be commutative if $gF(\alpha, \beta) = F(g\alpha, g\beta)$.*

3 Coupled fixed point theorems in E-fuzzy metric spaces

Let ϕ be the function , $\phi : (0, \infty) \to (0, \infty)$ such that $\phi(t) < t$ and for any $t > 0$, $lim_{n\to\infty} \phi^n(t) = 0$ for any $t > 0$. Some coupled fixed point theorems for ϕ contraction in complete $E-$ fuzzy metric space are established in this section.
In this paper we denote,
$[E(\alpha,\beta,\gamma,t)]^n = \underbrace{E(\alpha,\beta,\gamma,t) * E(\alpha,\beta,\gamma,t) * ... * E(\alpha,\beta,\gamma,t)}_{n}$ for all $n \in N$.

Theorem 3.1. *Let $(X, E, *)$ be a complete E-Fuzzy metric space. Let $F : X^2 \to X$ and $g : X \to X$ be mappings such that $E(F(\alpha,\beta), F(\mu,\eta), F(\gamma,\omega), \phi(t))$ $\geq E(g(\alpha), g(\mu), g(\gamma), t) * E(g(\beta), g(\eta), g(\omega), t)$, $\forall \alpha,\beta,\mu,\eta,\gamma,\omega \in X$. Suppose that F and g satisfy the following:*

- *$F(X^2) \subseteq g(X)$*
- *g is continuous and commutes with F*

Then there exists a unique α in X such that $\alpha = g\alpha = F(\alpha,\alpha)$.

Proof. Since $F(X^2) \subseteq g(X)$, we can construct two sequences $\{g\alpha_n\}$ and $\{g\beta_n\}$ in X such that $g(\alpha_{n+1}) = F(\alpha_n, \beta_n)$, $g(\beta_{n+1}) = F(\beta_n, \alpha_n)$ $\forall n \geq 0$. For any $0 < r < 1$ there exists a $0 < s < 1$ such that $\underbrace{(1-s) * (1-s) * ... * (1-s)}_{k} >$ $(1-r)$ $\forall k \in N$.
We have

$$\lim_{t\to\infty} E(\alpha,\beta,\gamma,t) = 1 \ \forall \alpha,\beta,\gamma \in X \tag{1}$$

Thus,
for any $t > 0$ and $s > 0$ there exists $t_0 > 0$ such that $E(g\alpha_0, g\alpha_1, g\alpha_2, t_0) > 1-s$, $E(g\beta_0, g\beta_1, g\beta_2, t_0) > 1-s$ and $E(g\alpha_2, g\alpha_1, g\alpha_0, t_0) > 1-s$, $E(g\beta_2, g\beta_1, g\beta_0, t_0) > 1-s$. For a ϕ function there exists $k \in N$ such that $\sum_{n=k_0}^{\infty} \phi^n(t_0) < \frac{t}{2}$.

$$\begin{aligned} E(g\alpha_1, g\alpha_2, g\alpha_3, \phi(t_0)) &= E(F(\alpha_0,\beta_0), F(\alpha_1,\beta_1), F(\alpha_2,\beta_2), \phi(t_0)) \\ &\geq E(g\alpha_0, g\alpha_1, g\alpha_2, t_0) * E(g\beta_0, g\beta_1, g\beta_2, t_0). \end{aligned}$$

Similarly,

$$E(g\beta_1, g\beta_2, g\beta_3, \phi(t_0)) \geq E(g\beta_0, g\beta_1, g\beta_2, t_0) * E(g\alpha_0, g\alpha_1, g\alpha_2, t_0).$$

Now,

$$\begin{aligned} E(g\alpha_2, g\alpha_3, g\alpha_4, \phi^2(t_0)) &= E(F(\alpha_1,\beta_1), F(\alpha_2,\beta_2), F(\alpha_3,\beta_3), \phi^2(t_0)) \\ &\geq E(g\alpha_1, g\alpha_2, g\alpha_3, \phi(t_0)) * E(g\beta_1, g\beta_2, g\beta_3, \phi(t_0)) \\ &\geq [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^2 * [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^2. \end{aligned}$$

Similarly,

$$\begin{aligned} E(g\beta_2, g\beta_3, g\beta_4, \phi^2(t_0)) &\geq [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^2 \\ &* [E(g(\alpha_0, g\alpha_1, g\alpha_2, t_0))]^2. \end{aligned}$$

Continuing this, we get

$$E(g\alpha_n, g\alpha_{n+1}, g\alpha_{n+2}, \phi^n(t_0)) \geq [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{n-1}} * [E(g\beta_0, g\beta_1, g\beta_2, t_0))]^{2^{n-1}}$$ and

$$E(g\beta_n, g\beta_{n+1}, g\beta_{n+2}, \phi^n(t_0)) \geq [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^{n-1}} * [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{n-1}}.$$

In the same way,

$$E(g\alpha_{n+2}, g\alpha_{n+1}, g\alpha_n, \phi^n(t_0)) \geq [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{n-1}} * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{n-1}}$$ and

$$E(g\beta_{n+2}, g\beta_{n+1}, g\beta_n, \phi^n(t_0)) \geq [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{n-1}} * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{n-1}}.$$

For $m, n \in N$ with $m > n \geq k_0$

$$\begin{aligned}
&E(g\alpha_m, g\alpha_m, g\alpha_n, t) \geq E(g\alpha_{n+1}, g\alpha_m, g\alpha_n, \tfrac{t}{2}) * E(g\alpha_m, g\alpha_{n+1}, g\alpha_n, \tfrac{t}{2}) \\
&\geq E(g\alpha_{n+1}, g\alpha_m, g\alpha_n, \textstyle\sum_{k=n}^{m-2} \phi^k(t_0)) * E(g\alpha_m, g\alpha_{n+1}, g\alpha_n, \textstyle\sum_{k=n}^{m-2} \phi^k(t_0)) \\
&\geq E(g\alpha_n, g\alpha_{n+1}, g\alpha_{n+2}, \phi^n(t_0)) * E(g\alpha_{n+1}, g\alpha_{n+2}, g\alpha_{n+3}, \phi^{n+1}(t_0)) * ... \\
&\quad * E(g\alpha_{m-2}, g\alpha_{m-1}, g\alpha_m, \phi^{m-2}(t_0)) * E(g\alpha_{n+2}, g\alpha_{n+1}, g\alpha_n, \phi^n(t_0)) \\
&\quad * E(g\alpha_{n+3}, g\alpha_{n+2}, g\alpha_{n+1}, \phi^{n+1}(t_0)) * ... \\
&* E(g\alpha_m, g\alpha_{m-1}, g\alpha_{m-2}, \phi^{m-2}(t_0)) \\
&\geq [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{n-1}} [E(g\beta_0, g\beta_1, g\beta_2, t_0))]^{2^{n-1}} \\
&* [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^n} * [E(g\beta_0, g\beta_1, gy_2, t_0))]^{2^n} \\
&* ... * [E(g\alpha_0, g\alpha_1, g\alpha_2, t_0)]^{2^{m-3}} \\
&* [E(g\beta_0, g\beta_1, g\beta_2, t_0))]^{2^{m-3}} * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{n-1}} \\
&* [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{n-1}} * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^n} \\
&\quad * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^n} * ... * [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{m-3}} \\
&* [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{m-3}} \\
&\geq [E(g\alpha_0, g\alpha_1, gx_2, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} * [E(g\beta_0, g\beta_1, g\beta_2, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} \\
&\quad [E(g\alpha_2, g\alpha_1, g\alpha_0, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} * [E(g\beta_2, g\beta_1, g\beta_0, t_0)]^{2^{\frac{m-n-1}{2}(m+n-4)}} \\
&> \underbrace{(1-s) * (1-s) * ...(1-s)}_{2^{2(m-n-1)(m+n-4)}} \\
&> (1-r).
\end{aligned}$$

Thus $\{g\alpha_n\}$ is a cauchy sequence. Similarly we can show that $\{g\beta_n\}$ is a cauchy sequence. Since E is a complete E- fuzzy metric space, we get $\{g\alpha_n\}$ converges to some $\alpha \in X$ and $\{g\beta_n\}$ converges to some $\beta \in X$. Thus $\lim_{n\to\infty} F(\alpha_n, \beta_n) = \lim_{n\to\infty} g(\alpha_n) = \alpha$ and $\lim_{n\to\infty} F(\beta_n, \alpha_n) = \lim_{n\to\infty} g(\beta_n) = \beta$.

As g is continuous, we have $\{gg\alpha_n\}$ converges to $g\alpha$ and $\{gg\beta_n\}$ converges to $g\beta$.

Also since g and F commute, we have

$$\begin{aligned} E(gg\alpha_{n+1}, F(\alpha,\beta), F(\alpha,\beta), t) &= E(F(g\alpha_n, g\beta_n, F(\alpha,\beta), F(\alpha,\beta), t) \\ &\geq E(F(g\alpha_n, g\beta_n), F(\alpha,\beta), F(\alpha,\beta), \phi(t)) \\ &\geq E(gg\alpha_n, g\alpha, g\alpha, t) * E(gg\beta_n, g\beta, g\beta, t). \end{aligned}$$

Since E is continuous on its variable and letting $n \to \infty$, we get
$E(g\alpha, F(\alpha,\beta), F(\alpha,\beta), t) \geq E(g\alpha, g\alpha, g\alpha, t) * E(g\beta, g\beta, g\beta, t)$.

Hence $E(g\alpha, F(\alpha,\beta), F(\alpha,\beta), t) = 1$. That is $F(\alpha,\beta) = g\alpha$.
Similarly we can show $F(\beta,\alpha) = g\beta$. Thus (α,β) is a coupled coincidence point of F and g.
Now, for any $r > 0$ there exists $s > 0$ such that
$\underbrace{(1-s) * (1-s) * ... * (1-s)}_{k} > (1-r)\ \forall k \in N$.

Using (1), for any $t > 0$ and $s > 0$ there exists $t_0 > 0$ such that $E(g\alpha, \beta, \beta, t_0) > 1 - s$ and $E(g\beta, \alpha, \alpha, t_0) > 1 - s$. Since $\phi^n(t_0) \to 0$ there exists $k \in N$ such that $\phi^n(t_0) < t\ \forall n \geq k$.

$$\begin{aligned} E(g\alpha, g\beta_{n+1}, g\beta_{n+1}, \phi(t_0)) &= E(F(\alpha,\beta), F(\beta_n,\alpha_n), F(\beta_n,\alpha_n), \phi(t_0)) \\ &\geq E(g\alpha, g\beta_n, g\beta_n, t_0) * E(g\beta, g\alpha_n, g\alpha_n, t_0). \end{aligned}$$

As $n \to \infty$
$E(g\alpha, \beta, \beta, \phi(t_0)) \geq E(g\alpha, \beta, \beta, t_0) * E(g\beta, \alpha, \alpha, t_0)$.
Similarly, $E(g\beta, \alpha, \alpha, \phi(t_0)) \geq E(g\beta, \alpha, \alpha, t_0) * E(g\alpha, \beta, \beta, t_0)$.
$E(g\alpha, \beta, \beta, \phi(t_0)) * E(g\beta, \alpha, \alpha, \phi(t_0)) \geq [E(g\alpha, \beta, \beta, t_0)]^2 * [E(g\beta, \alpha, \alpha, t_0)]^2$.

Continuing this process, we get, $\forall n \in N$

$$\begin{aligned} E(g\alpha, \beta, \beta, \phi^n(t_0)) * E(g\beta, \alpha, \alpha, \phi^n(t_0)) &\geq [E(g\alpha, \beta, \beta, t_0)]^{2^n} \\ &* [E(g\beta, \alpha, \alpha, t_0)]^{2^n} \end{aligned}$$

Therefore

$$\begin{aligned} E(g\alpha, \beta, \beta, t) * E(g\beta, \alpha, \alpha, t) &\geq E(g\alpha, \beta, \beta, \phi^k(t_0)) * E(g\beta, \alpha, \alpha, \phi^k(t_0)) \\ &\geq [E(g\alpha, \beta, \beta, t_0)]^{2^k} * [E(g\beta, \alpha, \alpha, t_0)]^{2^k} \\ &> \underbrace{(1-s) * (1-s) * ... * (1-s)}_{2^{2k}} \\ &> 1 - r. \end{aligned}$$

Thus for any $r > 0$, $E(g\alpha, \beta, \beta, t) * E(g\beta, \alpha, \alpha, t) > 1 - r\ \forall t > 0$.
Therefor $g\alpha = \beta$ and $g\beta = \alpha$. Now for any $r > 0$ there exists a $s > 0$ such that
$\underbrace{(1-s) * (1-s) * ... * (1-s)}_{k} > (1-r)\ \forall k \in N$.

Using (1), for any $t > 0$ and $s > 0$ there exists $t_0 > 0$ such that $E(\alpha, \beta, \gamma, t_0) > 1 - s$. Since $\phi^n(t_0) \to 0$ there exists $k \in N$ such that $\phi^n(t_0) < t\ \forall n \geq K$.

$$\begin{aligned} E(g\alpha_{n+1}, g\beta_{n+1}, g\beta_{n+1}, \phi(t_0)) &= E(F(\alpha_n,\beta_n), F(\beta_n,\alpha_n), F(\beta_n,\alpha_n), \phi(t_0)) \\ &\geq E(g\alpha_n, g\beta_n, g\beta_n, t_0) * E(g\beta_n, g\alpha_n, x_n, t_0) \end{aligned}$$

As $n \to \infty$, $E(\alpha, \beta, \beta, \phi(t_0)) \geq E(\alpha, \beta, \beta, t_0) * E(\beta, \alpha, \alpha, t_0)$
Continuing this process, we get
$E(\alpha, \beta, \beta, \phi^n(t_0)) \geq [E(\alpha, \beta, \beta, t_0)]^n * [E(\beta, \alpha, \alpha, t_0)]^n$.
Thus

$$\begin{aligned} E(\alpha, \beta, \beta, t) &\geq E(\alpha, \beta, \beta, \phi^n(t_0)) \\ &\geq [E(\alpha, \beta, \beta, t_0)]^n * [E(\beta, \alpha, \alpha, t_0)]^n \\ &> \underbrace{(1-s) * (1-s) * ... * (1-s)}_{2k} \\ &> 1 - r. \end{aligned}$$

Hence for any $r > 0, E(\alpha, \beta, \beta, t) > 1 - r$ $\forall t$, implies $\alpha = \beta$, hence $g(\alpha) = F(\alpha, \alpha) = \alpha$. Thus (α, α) is the unique common coupled fixed point of F and g. That is F and g have a unique common fixed point in X.□

Corolary 3.1. *Let $(X, E, *)$ be a complete E-Fuzzy metric space and $F : X^2 \to X$ be a mapping such that $E(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega), \phi(t)) \geq E(\alpha, \mu, \gamma, t) * E(\beta, \eta, \omega, t)$ $\forall \alpha, \beta, \gamma, \mu, \eta, \omega \in X$ and $t > 0$. Then there exists a unique $\alpha \in X$ such that $\alpha = F(\alpha, \alpha)$.*

Proof. Take g as the identity map in Theorem(3.1), we get the result. □

Corolary 3.2. *Let $(X, E, *)$ be a complete E-Fuzzy metric space and $F : X^2 \to X$ be a mapping such that $E(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega), kt) \geq E(\alpha, \mu, \gamma, t) * E(\beta, \eta, \omega, t)$ $\alpha, \beta, \gamma, \mu, \eta, \omega \in X$, $0 < k < 1$ and $t > 0$. Then there exists a unique $\alpha \in X$ such that $\alpha = F(\alpha, \alpha)$.*

Proof. Define $\phi(t) = kt$. Then by Corollary(3.1) we get the result. □

Example 3.1. *Let $X = [0, 1]$. Define $t * s = ts$ $\forall t, s \in X$ and $E : X^3 \to [0, 1]$ as $E(\alpha, \beta, \gamma, t) = [exp(\frac{G(\alpha,\beta,\gamma)}{t})]^{-1}$ where $G(\alpha, \beta, \gamma) = |\alpha - \beta| + |\beta - \gamma| + |\gamma - \alpha|$. Let $\phi(t) = \frac{t}{6}$. Define $g : X \to X$ as $g(\alpha) = \alpha$ and $F : X^2 \to X$ as $F(\alpha, \beta) = \frac{\alpha\beta}{6}$ $\forall \alpha, \beta \in X$.*
Then there exists unique common fixed point of F and g, by Theorem3.1.

$$\begin{aligned} G(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega) &= |\tfrac{\alpha\beta}{6} - \tfrac{\mu\eta}{6}| + |\tfrac{\mu\eta}{6} - \tfrac{\gamma\omega}{6}| + |\tfrac{\gamma\omega}{6} - \tfrac{\alpha\beta}{6}| \\ &\leq \tfrac{1}{6}\{(|\alpha - \mu| + |\mu - \gamma| + |\gamma - \alpha|) + (|\beta - \eta| \\ &\qquad + |\eta - \omega| + |\omega - \beta|)\} \\ &\leq \tfrac{1}{6}\{G(g\alpha, g\mu, g\gamma) + G(g\beta, g\eta, g\omega)\} \end{aligned}$$

$$\begin{aligned} E(F(\alpha, \beta), F(\mu, \eta), F(\gamma, \omega), \phi(t)) &= [exp(\tfrac{G(F(\alpha,\beta),F(\mu,\eta),F(\gamma,\omega)}{\phi(t)})]^{-1} \\ &\geq [exp\tfrac{(\frac{1}{6}G(g\alpha,g\mu,g\gamma)+\frac{1}{6}G(g\beta,g\eta,g\omega)}{\frac{1}{6}}]^{-1} \\ &\geq [exp\tfrac{G(g\alpha,g\mu,g\gamma)}{t}]^{-1} * [exp\tfrac{G(g\beta,g\eta,g\omega)}{t}]^{-1} \\ &\geq E(g\alpha, g\mu, g\gamma, t) * E(g\beta, g\eta, g\omega, t). \end{aligned}$$

It satisfies Theorem(3.1).

Theorem 3.2. *Let $(X, E, *)$ be a complete $E-$ fuzzy metric space and let $F : X^2 \to X$ and $g : X \to X$ be mappings such that $\forall\ \alpha, \beta, \gamma, \mu, \eta, \omega \in X$, $E(F(\alpha, \beta), F(\mu, \eta), F(\omega, \gamma), \phi(t)) \geq min\{E(g\alpha, g\mu, g\omega, t), E(g\beta, g\eta, g\gamma, t)\}$. Assume that $F(X^2) \subseteq g(X)$ and g is continuous and commutes with F. Then there exists unique $\alpha \in X$ such that $F(\alpha, \alpha) = g(\alpha) = \alpha$.*

Proof. The procedure of proof of this theorem is similar to that of theorem(3.1), except for the implementation of the condition given in this statement. □

4 Application

This section presents the existence and uniqueness for the solution of the following integral equation by using the theoretical results discussed in the previous section.

Let $X = C([0, 1])$. Define $E : X^3 \times (0, \infty) \to (0, 1]$ as $E(\alpha, \beta, \gamma, t') = [exp(\frac{G(\alpha,\beta,\gamma)}{t'})]^{-1}$ where $G : X^3 \to R$ is $G(\alpha, \beta, \gamma) = sup_{\epsilon\in[0,1]}|\alpha(\epsilon) - \beta(\epsilon)| + sup_{\epsilon\in[0,1]}|\beta(\epsilon) - \gamma(\epsilon)| + sup_{\epsilon\in[0,1]}|\gamma(\epsilon) - \alpha(\epsilon)|$ and the $t-$norm $*$ is $t * s = ts$. Then $(X, E, *)$ is a complete $E-$ fuzzy metric space. Let $\phi : (0, \infty) \to (0, \infty)$ be defined as $\phi(t') = rt'$, $0 < r < 1$ and $g : X \to X$ as $g(\alpha) = \alpha$. Consider the integral equation,

$F(\alpha, \beta)(\epsilon) = f(\epsilon) + \int_0^1 k(\epsilon, \delta)h(\delta, \alpha(\delta), \beta(\delta))d\delta$, $\epsilon \in [0, 1]$

with the following conditions

1. $h : [0, 1] \times (0, \infty)^2 \to (0, \infty)$ and $f : [0, 1] \to (0, \infty)$ are continuous.

2. there exists $0 < r < 1$ such that $|h(\delta, \alpha, \beta)-h(\delta, \mu, \eta) \leq r(|\alpha-\mu|+|\beta-\eta|)$ $\forall \alpha, \beta, \mu, \eta \in (0, \infty)$ and $\forall \delta \in [0, 1]$.

3. $\forall \epsilon, \delta \in [0, 1]$, $sup_{\epsilon\in[0,1]} \int_0^1 k(\epsilon, \delta)d\delta < 1$.

Then by Theorem(3.1) there exists an unique solution of the integral equation. For

$$G(F(\alpha, \beta)(\epsilon), F(\mu, \eta)(\epsilon), F(\gamma, \omega)(\epsilon))$$
$$= sup_{\epsilon\in[0,1]}|F(\alpha, \beta)(\epsilon) - F(\mu, \eta)(\epsilon)| + sup_{\epsilon\in[0,1]}|F(\mu, \eta)(\epsilon) - F(\gamma, \omega)(\epsilon)|$$
$$+ sup_{\epsilon\in[0,1]}|F(\gamma, \omega)(\epsilon) - F(\alpha, \beta)(\epsilon)|$$

$$\leq sup_{\epsilon\in[0,1]}| \int_0^1 k(\epsilon, \delta)h(\delta, \alpha(\delta), \beta(\delta))d\delta - \int_0^1 k(\epsilon, \delta)h(\delta, \mu(\delta), \eta(\delta))d\delta|$$
$$+ sup_{\epsilon\in[0,1]}| \int_0^1 k(\epsilon, \delta)h(\delta, \mu(\delta), \eta(\delta))d\delta - \int_0^1 k(\epsilon, \delta)h(\delta, \gamma(\delta), \omega(\delta))d\delta|$$
$$+ sup_{\epsilon\in[0,1]}| \int_0^1 k(\epsilon, \delta)h(\delta, \gamma(\delta), \omega(\delta))d\delta - \int_0^1 k(\epsilon, \delta)h(\delta, \alpha(\delta), \beta(\delta))d\delta|$$

$$\leq sup_{\epsilon\in[0,1]}| \int_0^1 k(\epsilon, \delta)\{|h(\delta, \alpha(\delta), \beta(\delta) - h(\delta, \mu(\delta), \eta(\delta))\}d\delta|$$
$$+ sup_{\epsilon\in[0,1]}| \int_0^1 k(\epsilon, \delta)\{|h(\delta, \mu(\delta), \eta(\delta)) - h(\delta, \gamma(\delta), \omega(\delta))\}d\delta|$$

$$+ sup_{\epsilon\in[0,1]}|\int_0^1 k(\epsilon,\delta)\{h(\delta,\gamma(\delta),\omega(\delta)) - h(\delta,\alpha(\delta),\beta(\delta))\}d\delta|$$

$$\leq sup_{\epsilon\in[0,1]}\int_0^1 k(\epsilon,\delta)d\delta\ \{sup|\alpha(\epsilon)-\mu(\epsilon)| + sup|\beta(\epsilon)-\eta(\epsilon)| + sup|\mu(\epsilon)-\gamma(\epsilon)| \\ +sup|\eta(\epsilon)-\omega(\epsilon)| + sup|\gamma(\epsilon)-\alpha(\epsilon)| + sup|\omega(\epsilon)-\beta(\epsilon)|\}$$

$\leq r\{G(\alpha,\mu,\gamma)+G(\beta,\eta,\omega)\}$. Then

$$\begin{aligned} E(F(\alpha,\beta)(t),F(\mu,\eta)(t),F(\gamma,\omega)(t),\phi(t')) &= [exp\tfrac{G(F(\alpha,\beta)(t),F(\mu,\eta(t),F(\gamma,\omega)(t))}{\phi(t')}]^{-1} \\ &\geq [exp\tfrac{r\{G(\alpha,\mu,\gamma)+G(\beta,\eta,\omega)\}}{rt'}]^{-1} \\ &\geq [exp\tfrac{G(\alpha,\mu,\gamma)}{t'}]^{-1}[exp\tfrac{G(\alpha,\mu,\gamma)}{t'}]^{-1} \\ &\geq E(\alpha,\mu,\gamma,t') * E(\beta,\eta,\omega,t'). \end{aligned}$$

Hence it satisfies Theorem(3.1) Thus there exists unique $\alpha \in C[0,1]$ as the solution of the given integral equation.

5 Conclusions

In this work, coupled coincidence fixed point theorems for the mappings under ϕ-contraction in complete E-fuzzy metric space are established and applied these results to show the existence and uniqueness for the solution of the integral equations. These are the generalization of existing results in the literature of fuzzy metric spaces. Using different types of contractive conditions, we can develop coupled fixed point theorems in complete E-fuzzy metric spaces.

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