




Strategies of Pre-Service Mathematics Teachers when Articulating Representations of a Function

Tulio Amaya De Armas ^a
 Eduardo Carrasco Henríquez ^b
 Humberto Álvarez Sepúlveda ^c

^a Institución educativa Madre Amalia de Sincelejo, Colombia

^b Departamento de Educación Básica de la Universidad Metropolitana de Ciencias de la Educación, Chile

^c Facultad de Educación de la Universidad Católica de la Santísima Concepción, Chile

Received for publication 24 Oct. 2020. Accepted after review 12 Aug. 2021

Designated editor: Claudia Lisete Oliveira Groenwald

ABSTRACT

Background: Several investigations have shown the difficulties that pre-service mathematics teachers present in articulating the elements of a function, in particular, its different representations. **Objective:** To analyse pre-service mathematics teachers' articulation of semiotic representations and partial meanings of the notion of function. **Design:** This research uses qualitative design. **Setting and participants:** The respondents' sample comprised 37 pre-service mathematics teachers from a Chilean university. **Data collection and analysis:** To collect the information, interviews were applied based on the answers given by the pre-service teachers when solving problem situations that involve functional relationships, of which they had to produce and connect their representations, both with conceptual elements and with the sociocultural context. The information was processed using the content analysis technique. **Results:** The results show a good state of development of the mathematical dimension of the participants' didactic-mathematical knowledge. Regarding didactic knowledge, while some managed to articulate the partial meanings of function and the representations produced adequately, others show limitations, by not putting at the service of mathematical knowledge the knowledge with which they can potentially carry out teaching situations, establishing few connections between the elements of the representations produced, with corresponding conceptual and sociocultural elements. **Conclusions:** There is a need for implementing strategies that allow pre-service teachers to articulate partial meanings and representations of a function with conceptual and sociocultural elements that leads them to envision potential teaching activities of this concept.

Keywords: Pre-service teachers; Articulation; Function; Semiotic representations; Partial meanings of the function notion.

Corresponding author: Tulio Amaya De Armas. Email: tuama1@hotmail.com

Estrategias de futuros profesores de matemáticas al articular representaciones de una función

RESUMEN

Contexto: Diferentes investigaciones han evidenciado las dificultades que presentan los futuros profesores de matemáticas para articular los elementos de una función, en particular sus diferentes representaciones, lo que ha motivado este estudio. **Objetivo:** Analizar la articulación que futuros profesores de matemáticas, hacen de las representaciones semióticas y significados parciales de la noción función. **Diseño:** Se utilizó un diseño cualitativo. **Entorno y participantes:** La muestra de informantes estuvo conformada por 37 futuros profesores de matemática de una universidad chilena. **Recolección y análisis de datos:** Para recoger la información se aplicaron entrevistas basadas en las respuestas dadas por los futuros profesores al resolver situaciones problema que involucran relaciones funcionales, de las que debían producir y conectar sus representaciones, tanto con elementos conceptuales, como del contexto sociocultural. La información se procesó utilizando la técnica análisis de contenido. **Resultados:** Los resultados evidencian buen estado de desarrollo de la dimensión matemática del conocimiento didáctico-matemático de los participantes. Respecto al conocimiento didáctico, mientras unos lograron articular adecuadamente los significados parciales de función, y las representaciones producidas, otros muestran limitaciones, al no poner al servicio del conocimiento matemático, el conocimiento con el que potencialmente pueden llevar a cabo situaciones de enseñanza, estableciendo pocas conexiones entre los elementos de las representaciones producidas, con elementos conceptuales y socioculturales correspondientes. **Conclusiones:** Se concluye sobre la necesidad de implementar estrategias que les permitan a los futuros profesores articular los significados parciales y las representaciones de una función con elementos conceptuales y socioculturales que los lleve a visionar potenciales actividades de enseñanza de este concepto.

Palabras claves: Futuros Profesores; Articulación; Función; Representaciones Semióticas; Significados Parciales de la Noción Función.

Estratégias de futuros professores de matemática ao articular representações de uma função

RESUMO

Contexto: Diferentes investigações têm mostrado as dificuldades que os futuros professores de matemática apresentam para articular os elementos de uma função, em particular suas diferentes representações, o que motivou este estudo. **Objetivo:** analisar a articulação que os futuros professores de matemática fazem das representações semióticas e significados parciais da noção de função. **Design:** Um design qualitativo foi usado. **Ambiente e participantes:** A amostra de informantes consistiu de 37 futuros professores de matemática de uma universidade chilena.

Recolha e análise dos dados: Para recolher as informações, foram aplicadas entrevistas a partir das respostas dadas pelos futuros professores na resolução de situações-problema envolvendo relações funcionais, das quais estes deviam produzir e ligar as suas representações, tanto com os elementos conceituais como com o contexto sociocultural. As informações foram processadas por meio da técnica de análise de conteúdo. **Resultados:** Os resultados mostram um bom estado de desenvolvimento da dimensão matemática do conhecimento didático-matemático dos participantes. No que diz respeito aos conhecimentos didáticos, enquanto uns conseguiram articular adequadamente os sentidos parciais da função e as representações produzidas, outros apresentam limitações por não colocarem a serviço do conhecimento matemático o saber com o qual potencialmente podem realizar situações de ensino, estabelecendo poucas conexões entre os elementos das representações produzidas, com os correspondentes elementos conceituais e socioculturais. **Conclusões:** Há a necessidade de implementação de estratégias que permitam aos futuros professores articular os significados e representações parciais de uma função com os correspondentes elementos conceituais e socioculturais que os levem a vislumbrar potencialidades pedagógicas desse conceito.

Palavras-chave: Futuros Professores; Articulação; Função; Representações Semióticas; Significados Parciais da Noção de Função.

INTRODUCTION

The design and analysis of activities for teaching purposes, which often include complex cognitive tasks, is not easy since, in mathematics, the access to the object of study is exclusively semiotic, and all mathematical activity consists in the transformation of a representation into another representation (Duval, 2017). Therefore, guiding a teaching and learning process of the mathematical object function requires the person to have well-developed functional thinking for didactic purposes and to master different meanings of this notion (Biehler, 2005). They must also have an excellent command of situations of variation and change to help generalise patterns and laws linking visualisation, exploration, manipulation, and use of multiple forms of representation (National Council of Teachers of Mathematics [NCTM], 2000), to help find a socially shared solution of situations that involve this concept (Dolores, 2013). According to Dolores (2013) and Kieran et al. (2016), studying situations of variation and change is essential to structuring and developing algebraic reasoning and indispensable to accessing calculus. Likewise, studying the functions through their multiple forms of representation helps the teacher foster an integral construction of this concept in the learner and minimise their difficulties in that construction process (Even, 1998; Gagatsis & Shiakalli, 2004). This is because each representation provides different

information about the object, making it easier for them to show different aspects. Although they may differ in appearance, they lead to an integral meaning, which is why the coordination and functioning in the synergy of two or more representations are required as a basis for conceptualisation (Duval, 2017; Siegler et al., 2013).

Despite the importance of function as a curriculum articulator in mathematics study plans (Steele et al., 2013) and as a modelling tool, students and pre-service teachers present gaps and conceptual limitations in their learning (Amaya, 2020). Various studies (Even, 1998; Amaya et al., 2016; Bueno & Pérez, 2018) have reported difficulties in understanding associated with their teaching and learning. Some of those difficulties are related to the ignorance, production, or articulation of the registers and representations of the notion of function (Panaoura et al., 2016) at all academic levels, besides its conscious use in sociocultural contexts (Amaya, 2020). The procedural is prioritised in its teaching and learning, leaving aside its conceptual development (Dolores et al., 2018).

The above could be problematic if one considers that the misunderstanding of the representations of a mathematical object by those who guide the teaching and learning process jeopardises the assignment of specific meanings to the objects they guide. In the case of function, it could lead to different conceptions of its meaning, allowing it to be associated with the several representations that can be reproduced from it (Adu-Gyamfi & Bossé, 2014; Rojas, 2015), leading students to confuse the object represented with its representative (Duval, 2017). According to Amaya (2020), such confusion is because the function is not studied integrally, only isolated representations disconnected from it are approached.

As can be seen, the approach to functions through their multiple representations plays a significant role in understanding the concept. If the person who guides the teaching process does not carry out an adequate study that facilitates the apprentice's visions of different meanings and makes the integration of their registers and representations, obstacles that limit their learning and that of other more advanced concepts that require prior understanding may arouse.

The question that guided the research process was: How do pre-service mathematics teachers articulate semiotic representations and partial meanings of function? The work was motivated by the authors' concern about obtaining results that would facilitate understanding how pre-service mathematics teachers produce and articulate representations of functional relationships, and

the analysis they make of the produced emergent semiotic systems, considered fundamental elements for teaching purposes, among which, the function. Said analysis emphasises the need to implement didactic strategies to help pre-service teachers approach functions through their registers and representations, articulating their elements so that they can project potential teaching activities that facilitate the reconstruction and articulation of different meanings of function in their future students. This, among other things, because the explicit education of pre-service teachers on different dimensions of the meanings of a function lets them base their teaching on multiple meanings of function in such a way that they simplify its global learning by restructuring and enriching the partial meanings that they associate with the concept (Biehler, 2005). Likewise, according to Zarhouti et al. (2014), registers and representations facilitate the comprehensive understanding of the concept and facilitate the analysis of student productions, so their understanding constitutes a contribution to the assessments in mathematics.

THEORETICAL APPROACHES

On didactic-mathematical knowledge

Besides mastering the dimension that accredits their mathematical knowledge (common and extended knowledge), the teacher must know how to teach (didactic knowledge); that is, they must simultaneously develop their didactic dimension (Schoenfeld & Kilpatrick, 2008). The mathematical dimension allows the teacher to choose appropriate tasks and solve them using different procedures, adapting them to the educational level where they teach and producing several representations of the mathematical object they teach, linking it with other mathematical objects of that level or previous and later levels (Ball et al., 2008; Pino-Fan & Godino, 2015).

The didactic dimension makes it easier for the teacher to know how to teach: make appropriate didactic transpositions for those who teach, and articulate the elements of the different representations that an object can produce (Amaya et al., 2016). Likewise, it allows the teacher to understand and mobilise the diversity of partial meanings for the same mathematical object, provide various arguments, and identify the knowledge put into play while solving a mathematical task. It also allows them to project and anticipate the possible difficulties that arise in the development of the teaching and learning process (Pino-Fan et al., 2015). Finally, they should be able to connect all this knowledge with learning theories and curricular elements (Steele et al., 2013)

that allow them to understand the learners' cognitive and ontogenetic needs and make decisions based on their students' achievements.

The tasks that the teacher proposes to their students must be appropriate to their abilities and allow them to manage their discourse (NCTM, 2000) and a permanent interaction in the classroom so that they gain confidence and autonomy. In this way, for these tasks to engage the student in a solution process, they must be of interest to him/her and closely associated with the context where the learning occurs (Dolores, 2013). A process with these characteristics could be the key to understanding the knowledge involved in a mathematics class (Rojas, 2015).

To achieve the above, it is necessary to introduce the necessary didactic transformations so that the student dedicates the necessary time to solving the proposed activities until they get to put in parallel the elements of the different representations of the studied mathematical object. This can be a strong determinant of the success of the process since, according to Pino-Fan et al. (2017), the teacher's ability to produce semiotic representations of the studied mathematical objects and the possibility of establishing connections between them could be the key for the student to assign meanings and senses to such objects, which may lead them to adequate understanding processes. But the teacher him/herself must master such knowledge, so, in their training process, they must gain the representational versatility that allows them to make connections between representations of the same object, making the corresponding conceptual and procedural interactions (Thomas, 2008), and, in their interaction with peers, validate the didactic activities that they design, their solutions, and project potential ways to develop them.

Semiotic registers and representations

Semiotics is the discipline that studies sign systems with communicative intentions (Rojas, 2015), and noesis is any cognitive act that facilitates the understanding of a concept (Peirce, 1974). Duval (2017) considers that noesis is not possible without semiosis. Furthermore, according to Peirce (1974), every person's experience is already semiotised, so it is natural that if a study is undertaken with a communicative intention, it is associated with a system of signs, as mathematical objects are abstract ideas that cannot be directly manipulated, while their representations can. That is why studying semiotic registers and representations is essential as the only means of accessing mathematical knowledge (Duval, 2017).

Semiotic representations are constructions through signs with which people externalise their mental representations (Duval, 2017), and registers are the containers that allow those representations to be configured (Amaya, 2020). They have their own rules and sign systems, although some rules and signs are shared with other registers. The representations of an object admit two types of transformations: conversion type and treatment type. A transformation is a conversion type done by changing the register, i.e., the transformations are done by semantic elaboration. When changing the register, the symbols and rules are also changed without changing the meaning of the represented object. A transformation is of a treatment type when it is done within a register, with its symbols and rules; however, it can be done both by semantic elaboration when the symbols are changed during the transformation, and by syntactic modification when it is maintained the same system of symbols and these are only manipulated when indicated operations are carried out or a previous procedure is continued.

According to the above, all mathematical activity consists of transforming a representation into another representation, either by conversion or by treatment. However, the mathematical idea or concept is hidden, encoded in its representations, there is no one-to-one correspondence between a mathematical object and any of its representations. We find particular characteristics of the object in each of them. Thus, all the representations do not contain the same information, so none represents the object in its entirety, but some elements may coincide in different representations (Duval, 2017). This makes it necessary to study the highest number of representations and match the identifiable elements in several of them to articulate the global characteristics of the object under study in the most comprehensive way possible.

When the elements of two representations coincide and have the same order of apprehension, they are congruent; otherwise, they are said to be incongruent (Duval, 2017). In some cases, the lack of congruence prevents the learner from articulating the elements of mathematical objects, causing differences in meaning for the same object. Thus, the learner associates it with the register or representation where he/she produces it (Rojas, 2015). This conditions the relationship between mathematical objects and their various registers and representations and the way of accessing them since, on the part of the person who guides the teaching and learning processes, it will depend to a great extent on students' conceptual awareness (Pino-Fan et al., 2017).

The functions and their epistemic configurations

The concept of function has been fundamental in the development of humanity (Hitt, 1998). According to Rey et al. (2008), its genesis is based on the development of the concept of numbers. The fundamental elements that constitute a function are the sets of values that its variables can acquire (Dolores, 2013) and the parameters that compose it, which can be evident in some of the representations that model it. Some of the epistemic configurations reported by Amaya (2016, p.33) are:

- As a correspondence relationship between variables: a relationship in which each value in the input variable corresponds to one and only one value of the output variable.
- As a correspondence between elements of two sets: a rule in which each element of the starting set must be related to a unique element of the target set.
- As an assignment rule: each value of the independent variable is matched to a single value of the dependent variable.
- As a set of ordered pairs: a function f is a set of ordered pairs $(a, b) \in f$, with the condition that the first component is not repeated in any pair of the set, and that for every element a in the domain of f , there exists an element b of the codomain of f such that $(a, b) \in f$.
- As a relationship between domain and image: passage from an initial state to a final or transformed state: the relationship that associates a unique numerical result among the images $f(x)$ to each number belonging to domain D .
- As a criterion of the vertical line: if a vertical line is drawn through any part of the plane, if the line cuts the graph, it cuts it in only one part, otherwise the graph does not represent a function.

Very hand in hand with the epistemic configurations of the notion of function in specific historical periods and mobilised to solve problem situations that involve them are their partial meanings (Pino-Fan et al., 2011), since both have been present in the evolution of this concept. Pino-Fan et al. (2019) report that in its historical development, function has had at least six partial meanings: i) function as correspondence; ii) function as a relationship between variable magnitudes; iii) function as a graphical representation; iv) function as analytical

expression; v) the function as arbitrary correspondence; and vi) function from set theory.

Associated with the epistemic configurations and the partial meanings of function are the functional relationships, which are identifiable functions in a particular concrete context that facilitate the development of functional thinking skills, understood as the mental process related to how a person thinks when connecting variable quantities to solve an everyday problem.

Due to its variational nature, in this study, function as one of the fundamental elements of calculus requires situations of variation and change, which refers to the extent to which a magnitude varies in relation to another or to itself, while variation is related to the quantification of the change or to the description or measurement of a changing feature in a process. Dolores (2013) and Rey et al. (2008) consider it essential to understand how the functions change and how their elements are related, and these with other curricular and sociocultural elements, as well as the relationship between their different registers and representations. Therefore, in the teaching and learning processes of the functions, we must resort to sociocultural contexts as containers of functional relationships. The use of functional relationships to study functions facilitates a covariational analysis between its variables (Hitt & Morasse, 2009) because it allows the study to focus on the interdependent covariation of two quantities and to analyse the effect of a change in the values of the dominance over those of the range or vice versa. Thompson and Carlson (2017) also state that understanding function as covariation is essential for students' mathematical development of students.

Just as the development of the concept of function has been costly (Wills et al., 2014), so have its teaching and learning processes (Font, 2011). The person who teaches functions must analyse the difficulties in teaching and learning them because it allows for minimising the imbalances in the learning process and could avoid epistemological obstacles in the learner. In this regard, Bueno and Pérez (2018) consider that it is possible to minimise the obstacles that have historically appeared with the learning of functions through appropriate didactic transpositions.

METHODOLOGICAL ASPECTS

Study type

It is a qualitative and exploratory work, where we analysed the articulation between representations and partial meanings of function notion done by students of a mathematics teacher education programme while solving

problem situations involving functions and projecting how to conduct a class in said situations. The pre-service teachers were expected to reflect on how to connect the mathematics they learn in mathematics courses in their formative programme to project potential teaching activities using functions that link their acquired knowledge with their future students' learning needs (Feikes & Schwingendorf, 2008). To this end, they worked with functional relationships that helped them connect the elements of the representations of the functions involved, both with conceptual elements and with elements of the sociocultural context. With this, we intended to extract information about pre-service teachers' understanding of the function notion. Interacting with their classmates and approaching knowledge socially shared and accepted by the community of mathematics educators (Arcavi, 2020), they reflected on how to teach it.

Sample of informants

The informant sample involved 37 students enrolled in a pedagogy programme in secondary education in mathematics from a Chilean university. The sample was selected intentionally. The inclusion criteria were that each participant should be attending the Didactics of Calculus subject and be supervised by one of the researchers. This subject is offered in the programme in the second semester of the third year of the degree and has as prerequisites Introduction to Algebra, Introduction to Analysis, Discrete Analysis, Continuous Analysis, and Integrals and Applications. When collecting the information, pre-service teachers had already taken the subjects Introduction to Didactics, Didactics of Arithmetic, Didactics of Geometry, Didactics of Algebra, and two of the four pedagogical practices offered in the programme. Therefore, they only needed to carry out professional practice. The programme lasts for nine semesters.

During the Didactics of Calculus subject, the pre-service teachers interacted while solving situations involving functions, preparing, and/or simulating classes for high school students. During the course, the supervisor focused on the knowledge and skills that a mathematics teacher might need to achieve instruction that best facilitates student learning (Schoenfeld & Kilpatrick, 2008; Ball et al., 2008; Pino-Fan & Godino, 2015). It was an interactive and dynamic peer collaboration process that highlighted and strengthened the progress, corrected, and tried to minimise errors in solving situations and designing classes. According to Schoenfeld (2011), this type of diagnostic teaching is traditionally not done with pre-service teachers; thus, after they complete their degree, they have to empower themselves

independently. In this way, he considers the above fundamental in pre-service teachers' formative process.

Collecting information

Given the above, we collected information for this analysis during the second semester of 2019, based on task-based semi-structured interviews applied (Goldin, 2000) to freely-formed groups of three or four pre-service teachers. The meetings were held in the Didactics of Calculus class, which lasted two hours each. Three professors participated as interviewers: the head of the subject and two guests, who rotated in the workgroups, conducting the interviews and recording them on video. In each meeting, each subgroup of pre-service teachers should choose a situation to work with, and while they solved it –or after solving it– they were asked questions based on their answers. Finally, we asked them about the elements produced, their relationships, and how to use them to design a class for high school students using the situation chosen.

Instruments used to collect information

In the process of selecting and designing the activities, together with the pre-service teachers, we created a bank of problem situations investigating reports or research articles (Arce et al., 2004), books (Farfán, 2012; Dolores, 2013) and online graded material (Godino et al., 2003; Secretaría de Educación Pública, 2004). We sought activities that could make it easier for the pre-service teachers to contextualise the functions for teaching purposes (Biehler, 2005). In the search process, we chose the activities that best fit didactic work with functions, as they facilitate the production of many representations and establish connections with conceptual elements and the sociocultural context (Amaya, 2016).

Below, Figures 1 and 2 present a couple of situations the pre-service teachers selected for the work reported here:

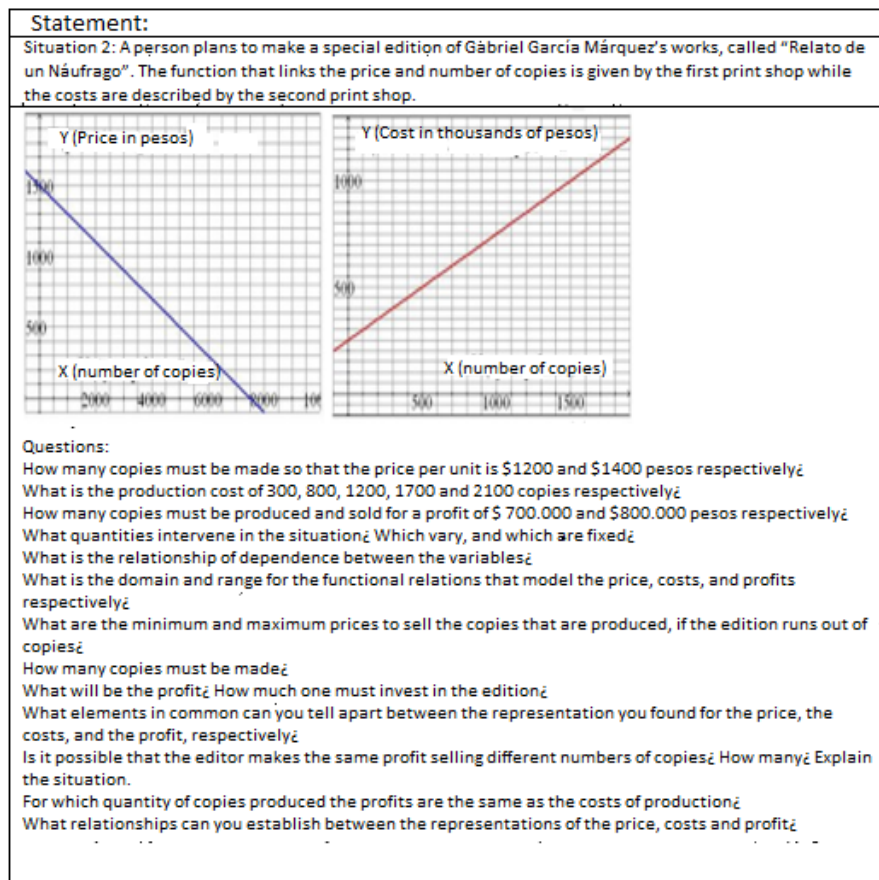
Figure 1

The first example of the situations that emerged and the questions asked.

Statement:
<p>Situation 1: If you want to build a box without a cover using a letter-size paper sheet (21,8 cm x 28,7 cm), when you cut at the corners squares with side ℓ, folding the crease upwards until making a box as illustrated below.</p>
<p>Questions:</p> <p>What are the dimensions of the box, if you want it to have the maximum volume?</p> <p>How do you know those dimensions give you the maximum volume?</p> <p>Which are the common elements between the representations found for the volume? Interpret them.</p> <p>What is the domain and range for the functional relation that models the volume of the box? What quantities intervene in the situation? Which vary and which remain fixed (are constant)?</p> <p>What is the relationship of dependence between the variables?</p> <p>What is the lateral area of the box?</p> <p>What relationships can you establish between the representations of the lateral area?</p> <p>What is the domain and range of the functional relation that models the lateral area of the box?</p> <p>Is there any type of dependence between the volume and the lateral area of the box?</p>

Figure 2

The first example of the situations that emerged and the questions asked.



The statements of both situations correspond to a functional relationship. Situation 1 is raised in a combination of colloquial and figural registers as the main register. Although this type of box as packaging is already in disuse, displaced by new computer-generated models, its usefulness for didactic purposes continues, as it facilitates producing a large number of its representations (Amaya, 2020), which makes it easier to connect them with each other and with conceptual and sociocultural elements. Situation 2 is raised in a combination of colloquial and graphic registers.

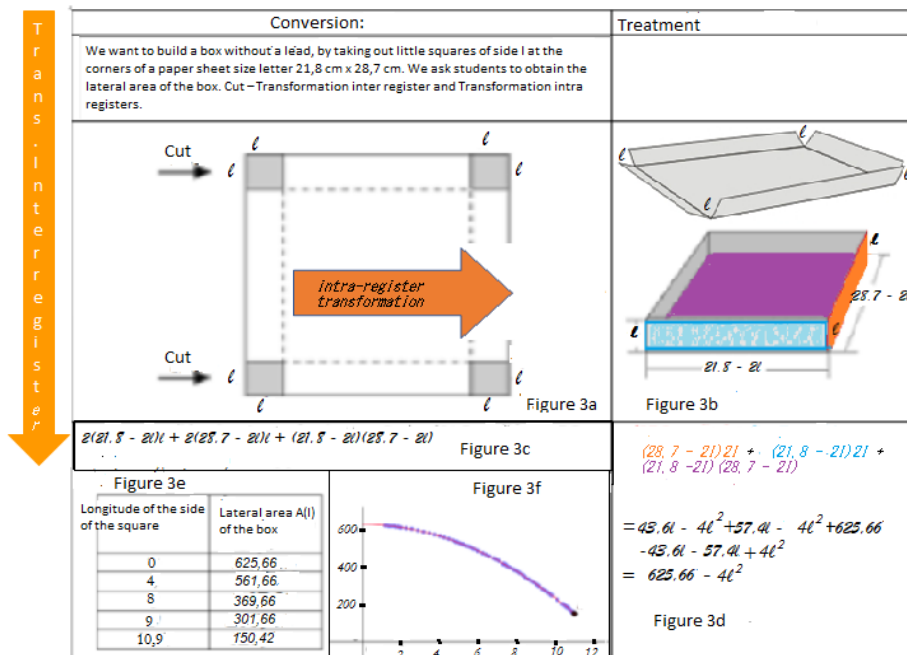
Next, we describe some transformations and show several representations of the functional relationship of Situation 1, in the figural, tabular, graphic, and algebraic registers, as auxiliary registers. We point out some common elements in several of them. The production order of the representations is decided by who produces them, i.e., anyone can do it.

When the participants start to manipulate the sheet and remove little squares from the corners, there is a phenomenological representation as the first auxiliary register, since this corresponds to the event generated by the manipulation of the sheet, until they obtain the box as requested (Amaya & Medina, 2013). When they fold the sides upward to form the box, a treatment-type transformation occurs, since we have not left the phenomenological register, going from a box of height zero (0) to another of height l . When trying to figuratively illustrate the situation described above, one passes from the phenomenological to the figural register through a conversion-type transformation, obtaining a drawing like the one in Figure 3a. By manipulating the sheet until obtaining a box like the one shown in Figure 3b, the transformations carried out are treatment-type, since they are done without abandoning the figural register.

If the box constructed in the phenomenological register or the figural representation shown in Figure 1 is taken as the main register and we use the definition of the area of a rectangle, by conversion, we obtain an algebraic representation of the lateral area of the box, such as the one shown in Figure 3c. As the transformations at this stage are of the conversion type, they are done by semantic elaboration, implying a change of register and symbols. By performing successive treatment-type transformations, such as those shown in Figure 3d, one finds the expression $A(l) = 625,66 - 4l^2$, which is a simplified algebraic representation of the lateral area.

Figure 3

Conversion-type and treatment-type transformations of the side area of a box without a lid.



Besides producing the representations of the object under study, it is possible to establish congruences (Duval, 2017) between the ostensive elements in some of them, i.e., it is possible to make a comparative analysis between some of the elements of two or more representations in different registers or the same register (Pino-Fan et al., 2017). In this case, when comparing the elements of the representations produced, it can be seen that in Figures 3b and 3d, the parts of the same colour in both are equivalent; therefore, the representations shown in these figures are quite homogeneous. In the tabular (Figure 3e), graphical (Figure 3f) and analytical algebraic (Figure 3d) representations, elements such as A (0) and A (10.9) of the analytic-arithmetical register can be evidenced, which in the Cartesian register would be A (0, 625.66) and A (10.9, 150.42), respectively. However, the concavity of the function is only evident in the graphic representation, so that, with respect to

this characteristic, the graphic representation (Figure 3f) is heterogeneous with the other representations.

On the other hand, there are very few ostensive elements in the colloquial register, which suggests that the representations produced in it are quite heterogeneous with the other representations. This lack of homogeneity is only apparent since elements common to various representations appear in the step by step of transformations. The colloquial register allows us to establish articulations between registers and representations. In other words, once a particular perceptual fluency has been developed by the person guiding the process (Rau et al., 2017), they must also develop some verbal fluency and argue about the common elements visible in various representations produced of the studied object. In this way, they can connect them with conceptual and sociocultural elements that facilitate their articulation and assign meanings and senses that positively affect the cognitive architecture of the learner.

The relationship between mathematical objects, their various representations, and sociocultural contexts as containers of functional relationships determines their connection, since transformations in mathematics cannot be done without a semiotic representation system (Pino-Fan et al., 2017). Also because the processes of assigning meaning and sense to those objects require establishing congruences between the different representations, with elements of the phenomenological representation associated with the studied object.

Information treatment and analysis

Trying to gain reliability in the information collected, a triadic interaction was done between the answers of the pre-service teachers, the questions that the researchers investigated, and the knowledge put into play (Koichu & Harel, 2007). In the pre-service teachers' answers, we focused on the partial meanings of function mobilised when they were solving the task (Pino-Fan et al., 2019) and on the conversion-type or treatment-type transformations that they could produce from the representations of a function and the connections that they could establish (Duval, 2017), both with conceptual and sociocultural elements (Amaya et al., 2016). Based on the identification of these elements, we analysed the participants' progress and their conceptual errors. Table 1 shows the previous categories of analysis used to review the work reported here.

Table 1*Categories of analysis and indicators or criteria used for the analysis*

Categories of analysis	Guidelines or criteria for analysis
Collaborative work in task solution and class simulation	a) Valuation of their own work and colleagues' b) Involvement in solving tasks c) Recognition of oneself and of the other as social educators
Characteristics of the solutions given to the tasks	a) Identification and interpretation of the elements of the function b) Meanings of the function mobilised in the solution of the task
Characteristics of the emergent semiotic systems produced	a) Production of different registers and representations. b) Relationship between the starting register used and the meaning of function mobilised in solving the task. c) Connections between representations and their connections with corresponding conceptual and sociocultural elements

The processing of oral and written information was done through the content analysis technique, according to the grouping of basic ideas by thematic or temporal criteria or by unit segmentation (Bernárdez, 1995). Subsequently, the characteristics of the findings were qualitatively described and analysed against the theoretical referents consulted. Labels G_i are written in the presentation of the results, where $i = 1, 2, \dots, 10$ corresponds to the i -th work group into which the pre-service teachers participating in this study were subdivided.

RESULTS AND ANALYSIS OF THE INFORMATION

Regarding collaborative work, the pre-service teachers first tried to solve the situation individually and then discussed their solutions. In this process of individual solutions to the situations, similar to that reported by Gonzato et al. (2011), all pre-service teachers managed to partially solve the

task. However, most were unsure of their solutions, which they had to validate with their classmates with the best academic averages in their career development or with the educators. We noticed that they valued the work of some colleagues better than that of others and even their own. As groups, they shared the fragments elaborated by each participant, and each one tried to put together a more comprehensive solution to defend their position before the educators' questions. However, only from the third meeting onwards did the G_3 and G_6 have all members solving the tasks jointly. This type of teamwork is essential in the formative development of a mathematics teacher since, according to Kaur et al. (2020), learning mathematics is a social activity that involves extensive peer interaction, working collaboratively around a problem.

Regarding the characteristics of the solutions given to the tasks, the results show some aspects common to all the groups of pre-service teachers: when starting the work with each situation, all had difficulties identifying some elements of the function, such as domain, range, intercepts, maximum or minimum points, and characteristics such as growth, decrease, and concavity. In addition, they made inferences about growth (decrease) or maximums (minimums), with two or three values showing that trend. However, the difficulties reduced over time, and they could easily identify those elements, being more careful when making inferences.

Regarding the partial meanings of function mobilised in the solutions given by the subgroups of pre-service teachers, the main register in which each functional relationship was presented determined the partial meaning of function used. Thus, one of the most widely used meanings of function at the beginning was that of function as a relationship between variable magnitudes, perhaps because the step-by-step construction process of functional relationships favoured it. In the solution given to each situation, they combined various partial function meanings, maintaining similar patterns of the combinations of said meanings in each solution. For example, the members of G_3 , G_6 , G_9 and G_{10} subgroups combined the partial meanings of function as a relation between variable magnitudes, as an arbitrary correspondence and as a graphical representation. Thus, working with the functional relationships that model the lateral area and the volume of the box without a lid, they built their boxes and compared with their partners which box had the largest volume. After building a couple of boxes each, by comparing and discussing with their classmates, they soon concluded that making new boxes and comparing their volumes was not the most efficient process to find the maximum volume. So, with a pattern of the formula that allows finding the volume of a parallelepiped, the G_6 members found an algebraic expression, started to give it values, and

made a table and a graph. When they analysed it, they found a maximum. Using this pattern, the members of G_3 began to try with several values and thus obtained the maximum volume sought. Then, with the pattern used, they found an algebraic expression that allowed them to find the volume of a box of edges of any dimension. As in Amaya (2020), the G_9 members converted from the phenomenological register to the analytic-algebraic one, and the latter was used as the central register. From the representations obtained in this register, they obtained the other representations they managed to produce. Below is a fragment of the interview with G_9 members when solving the problem of building a lidless box.

Interviewer: Explain: how did you find all those data?

Student 5: We only really made two boxes, and with the box, I applied the formula and then, I would substitute the values and write them down.

Interviewer: How did you find the formula?

Student 5: As in the picture, we see that on the four sides of the sheet, x is being removed; we see that in width (21,8), x must be removed on each side, i.e. $2x$, and also in length, and since the volume of a box is length times width times depth, then it would be $V(x) = (28.7 - 2x)(21.8 - 2x)x$. With this equation, we replaced x here –pointing to the formula–, found the volume and wrote it down.

Interviewer: How do you know that those data correspond to a function?

Student 4: Because the graph has only one line, and if we pass a vertical line, it only cuts it in part.

Interviewer: What is the domain?

Student 4: The domain goes from zero to 10.9 because since the function has three factors, when $x = 0$, the product gives zero and when $x = 10.9$, it also gives zero.

Interviewer: But when $x = 14.35$, it also gives zero

Student 5: Yes, but at that point, it's not this function –and he pointed to the graph– because I can't cut squares larger than 10.9 centimetres on this sheet.

Interviewer: How do you relate the points you have written down with those on the graph and with the boxes you are building?

Student 3: That is the volume; each point contains the height of the box, which is the size of the square, and the volume of the box.

Interviewer: Could you point to an example of what you are saying?

Student 3: Yes, I am showing here point (1, 528), corresponding to $x = 1\text{cm}$ and a volume of 528.66, corresponding to a one-centimetre high and 528-volume box.

Members of G_3 , G_6 , and G_{11} subgroups also reported $D_v = [0, 10.9]$ as domain, and $R_v = [0, 1143.08]$ as range, saying that: *The generalised variable function is not equivalent to the studied functional relationship because they do not have the same domain or range. As the associated generalised variable function has no restriction whatsoever, neither in its domain nor range, both sets coincide with the real numbers. In the functional relationship, in turn, its domain is determined by the dimensions of the sides of the cut squares, bounded by the dimensions of the width of the sheet, so it has a maximum in its domain. It also has two relative minima corresponding to the endpoints of the interval of variation that represents the range.* In those pre-service teachers' (G_3 , G_6 , G_9 , and G_{11}) production, significant advances are evidenced in the link of their didactic-mathematical knowledge about functions because they articulately used primary mathematical elements, justifying arguments from which we glimpse potential teaching activities (Ball et al., 2008; Zarhouti et al., 2014). This becomes evident when we analyse the primary mathematical elements (Godino et al., 2006) in their solutions, since the G_6 members could solve the task and used different solution strategies, making various representations of the functions involved, relating elements of multiple representations, and giving quite good arguments, as can be seen in Figure 4. We can also see that G_9 members who pointed out common elements between the representations produced related the dimensions of the cut squares with their associated boxes as corresponding elements, both to the sequence of points and to the points that made up the graph, establishing clear congruences between the representations produced.

Figure 4

G₆ members' written account when modelling the functional relationship between the volume and the lateral area of the lidless box.

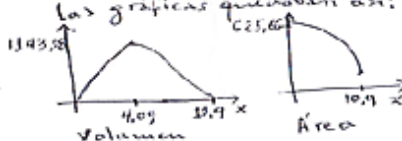
Luego de construir la caja utilizando hojas de dimensiones 21,8 cm x 28,7 cm, el reto estaba en elaborar la caja con mayor volumen posible. En el proceso nos dimos cuenta que al cortar un cuadradito pequeño, el volumen descendía y si era muy grande, el volumen también descendía, así que nos fuimos acercando en las medidas y llegamos a que el mayor volumen lo obteníamos si tomábamos como referencia de corte de las esquinas 4,07 cm.

Luego en conjunto establecimos el dominio de la función volumen al igual que el rango y la representación algebraica que resultó ser $V(x) = (21,8 - 2x)(28,7 - 2x) \cdot x$. Luego buscamos el área lateral de la caja construida e hicimos

lo mismo que con la función volumen, y nos dimos cuenta que son totalmente independientes: la función del volumen y la del área. Es decir, son importante lo grande del volumen, el área no será más grande, ya que la función del área será siempre el área de la hoja tomada, menos la medida de los cuatro cuadraditos que se recortan, es decir:

$$A(x) = 625,66 - 4x^2$$

Además, graficamos las dos funciones y entendimos por qué en el contexto de la situación las gráficas quedaban así:



In the situation where they had to find the lateral area, in order to find an algebraic expression and then its graph, a significant group of pre-service teachers (G₁, G₃, G₄, G₆, G₇, G₉, G₁₀ y G₁₂) used the information they found when looking for the volume of the box, combining the meaning of function as a relationship between variable magnitudes, that of function as arbitrary correspondence and that of function as a graphical representation. As in Chinnappan and Thomas (2001), they changed the starting register, establishing the analytic register as the main register and from this, they obtained the other representations they managed to produce.

In the analysis of the dependence relationship between the volume of the box and its lateral area, various groups of pre-service teachers (G₁, G₃, G₅, G₆, G₁₀ y G₁₂) soon realised that the volume of a box is independent of its lateral area, as stated by G₆ in the account shown in Figure 4, where they easily describe their processes. Moreover, in their description, we see a solid didactic-mathematical knowledge, in the sense of Pino-Fan and Godino (2015), since

they articulate various elements of the representations produced with conceptual and sociocultural elements.

Likewise, besides producing adequate representations, groups G_7 and G_{12} established congruences between the equivalent elements in several of them, as shown in G_7 's account (Figure 5). In their description, G_7 shows their procedures, obtained representations, and the congruences established between the elements of the representations, which is evidenced when they say, “625.66 gives me the total area of the sheet, $4x^2$ it gives me the four little squares that were removed from the sheet. Then, with this, the domain and range were found: $D_f = [0, 10.9]$ y $R_f = [0, 625.66]$ was deduced from the area formula”. Although the G_7 members make a mistake in reporting the range of the lateral area function, the fact that they report the domain and range of the functional relation instead of the domain and range corresponding to the generalised variable function is a real advance, since this generally constitutes an epistemic conflict, in the sense of Font (2011) and Godino et al. (2006), that seems to have originated in the lack of expanded knowledge at the service of common knowledge (Pino-Fan & Godino, 2015).

Figure 5

Part of G_7 's written account when describing the construction process of the side area of the boxes.

$$A(x) = 2x(21,8 - 2x) + 2x(28,7 - 2x) + (21,8 - 2x)(28,7 - 2x)$$

$$= 43,6x - 4x^2 + 57,4x - 4x^2 + 625,66 - 43,6x - 57,4x + 4x^2$$

me indica el área total de la hoja

$$A(x) = 625,66 - 4x^2$$

me indica los 4 cuadritos que se le quitó a la hoja. luego con esto se halló el dominio y el rango
 $D_f: [0, 10,9]$
 $R_f: [0, 625,66]$
 se dedujo de la formula del area

The fact that the G_7 members could produce several semiotic representations of the studied mathematical object, establish connections between them, and communicate them as they did reveals, in the sense of Duval

(2017) and Panaoura et al. (2016), that the function as a mathematical is well founded in them.

On the other hand, in their solutions, the G₂, G₅, G₈ y G₁₁ subgroups combined the meaning of *function as a relationship between magnitudes as a graphical representation* and *as an arbitrary correspondence*. In the case of the problem of the facsimile edition of the book *Relato de un naufrago*, based on a visual analysis of the graphs, they made a table of values and a Cartesian sequence with points, both for prices and costs and for profits and, from the manipulation of changing values, they established the notion of variable quantities and the relationship of dependence between them (Ruiz-Higueras, 1994). Although building such representations did not make it easier for them to see the mathematical object function in them, they returned to the graphs. From there, they could explain that the data correspond to a function using the epistemic configuration of the vertical line. In other words, the quantitative covariational analysis alone was insufficient for them to accept the values produced as representatives of a function. However, doing the qualitative covariational analysis and correlating them allowed them to conceive the function concept more comprehensively (Rolfes, 2018) and articulate their meanings with elements of the colloquial representation. Below is a fragment of the interview with the G₂ members when solving the problem of the facsimile edition of the book *Relato de un naufrago*:

Interviewer: How did you find the domain and range?

Student 2: It was very difficult to find the domain and range if the function does not have x or y .

Interviewer: But tell me, how did you do it?

Student 2: Well, we began to analyse the graphs and realised that we had an X axis and a Y axis, so, I immediately said, the domain is the specimens, and the range is the price, and in this other one –the woman pointed out the cost graph– the domain and also the copies and the range of the costs.

Interviewer: What was the domain and range that you found?

Student 3: The domain in both goes from negative infinity to positive infinity, and the range, too.

Interviewer: How did you find those values?

Student 1: It is that both the price graph and the cost graph are linear functions, and with a couple of points from each one, we can draw the equations.

Interviewer: But how did you get the domain and range of both functions from the equations?

Student 3: Because we know that, in a linear function, the domain and range are the real numbers, because there are no restrictions on x .

Interviewer: Tell me then, what does it mean that fewer than two copies are produced?

Student 3: Hmmm, it doesn't make sense, wait, and we'll check again,... Look, according to the graph, the domain in both goes from zero to 8,000, and the range, for Prices from zero to 1,500, and in costs, I tell you, ..., from zero to 4,002,941.

Interviewer: How did you get those values?

Student 3: Using the plot tool of the Derive program.

Interviewer: And how do you know that the data correspond to functions?

Student 1: We reviewed the graphs, and we see that each value of x has a value in y , and only one.

Interviewer: Where can you see that in the information you have?

Student: Because if y had two different values for the same x , one would be on top of the other, and that cannot be done in a function, i.e., we see that y only takes one value for each x in each graph. In other words, for each number of items, there is only one price and one cost, and that is the definition of the function.

Interviewer: What are the max and min prices at which the produced copies must be sold for a profit?

Student 2: The minimum price is when 270 copies are produced, and the maximum is when 4,700 are produced.

Interviewer: What is the maximum profit that will be obtained?
And how many copies should be made?

Student 2: The maximum profit is \$1,000,000, and it occurs when 2,517 copies are produced and sold.

Interviewer: How did you get those values?

Student 3: As I already told you, professor, using the plot tool of the Derive program.

Student 2: Tell me, how did you do it?

Student 3: We made the cost, income, and profit graphs on the same plane in Derive, and to answer those questions, we followed up each one with graphs, and the answers are in the points that are showing.

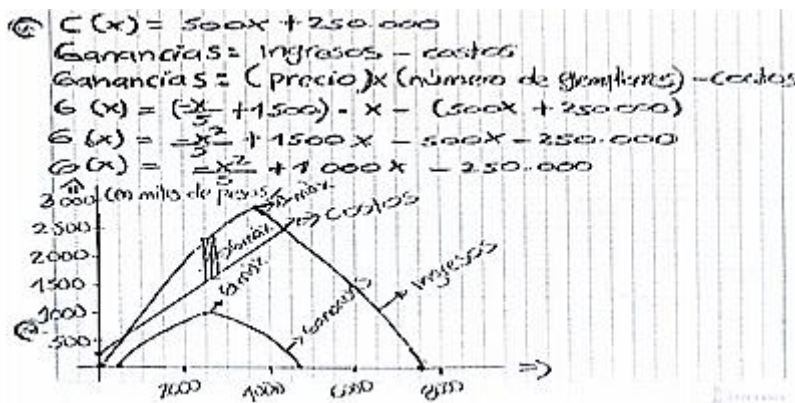
Interviewer: And for the maximums and minimums, why didn't you use derivatives?

Student 1: That's how we started to do it, but since the class is for intermediate-level students, they won't understand us if we use derivatives. Illustrating it graphically will.

We observed that the G_2 members based their initial answers on previously learned criteria, sometimes leading them to make mistakes. However, those errors were reduced during the interview (Koichu & Harel, 2007). They produced and explored different representations of functional relationships (Figure 6), making a quantitative and qualitative covariational analysis (Rolfes et al., 2018). They also made numerical calculations, analysed the changes that occurred, and visually explored the graphs, basing their answers on the visual inspection without including precise values of the functions. This allowed them to conduct a static analysis of punctual relationships and a dynamic analysis of the elements of the functions involved. According to Rolfes et al. (2020), dynamic representations help students better understand functions. Thompson and Carlson (2017) consider that the combination of static and dynamic representations plays a fundamental role in teaching the concept of function at school. In this case, it led them to view potential didactic activities (Zarhouti et al., 2014), using computational math tools, such as Derive. Solutions similar to that of G_2 were also given by G_5 , G_8 y G_{11} .

Figure 6

G_2 's written account when giving a solution to the facsimile edition of *Relato de un naufrago*.



These pre-service teachers (G_2 , G_5 , G_8 y G_{11}) found it easier to coordinate the elements of the representations they had produced, i.e., the establishment of congruences between the elements of the representations produced up to then was evidenced (Duval, 2017; Siegler et al., 2013). Likewise, in their solution processes, they gave meaning and adequately reasoned about the mathematical knowledge that will be taught, and contrary to what was reported by Dreher and Kuntze (2015), they paid attention to the significance of this knowledge to teach. In Schoenfeld's (2011) words, they put at the service of mathematical knowledge the knowledge with which they could potentially carry out teaching situations, establishing rich connections between the elements of the representations produced (Panaoura et al., 2016). Based on their answers, we noticed that they have an excellent command of their mathematical knowledge because they solved the situations easily, producing and articulating various representations (Figure 6) correctly, including analytical-arithmetic, analytical-algebraic, graphic, figural, and Cartesian representations, linking the theme with conceptual and sociocultural elements.

CONCLUSIONS

Regarding the relevance of the theoretical framework, we can say that the solution processes developed by the pre-service teachers were favoured by how they integrated the registers and representations and the partial meanings

of the function used when solving the problem situations (Kaur et al., 2020). When the functions are learned for teaching purposes, the panorama is different (Biehler, 2005). One is not only understanding for oneself, i.e., one must envision how to guide others so that they understand the functions.

In their solutions, the participating pre-service teachers resorted to several registers. However, all used two: the analytical-algebraic to produce other representations and the graphic, to define the function (Steele et al., 2013) and as a basis for analysing emerging semiotic systems (Amaya, 2020) because, in these registers, they could show relevant characteristics of the representations (Rolfes et al., 2018), making it easier to identify their elements and articulate them with the corresponding conceptual and sociocultural elements. The covariational analysis also depended on the relevant ostensive characteristics in each register and representation, facilitating more the figural, graphic, and Cartesian registers for the qualitative covariational analysis and the tabular and analytical ones for the quantitative one, while the colloquial register functioned as an element of link.

The use of functional relationships awakened in pre-service teachers an intuitive component closely linked to the sociocultural context, which gave consistency and meaning to the work carried out in the situations used (Arcavi, 2020). This fact led pre-service teachers to integrate the partial meanings of the function notion and establish strong connections between the concrete and the abstract by identifying, in elements of the sociocultural context, corresponding conceptual elements, as happened with the length of the sides of the cut squares (height of the box), which they related to the variable x , and the domain of the function.

The state of development of the didactic-mathematical knowledge of this group of pre-service teachers on functions evidences a good state of development of the mathematical dimension and limitations in some, in their didactic knowledge. Regarding the mathematical dimension, they solved the tasks adequately, produced various representations of the functional relationships studied, and related concepts of various levels. Regarding the didactic component, while some articulated correctly the partial meanings of function and the representations produced, even using dynamic representations, others produced the representations but could not articulate them.

The results invite us to implement some type of strategies in the instructional processes that can help pre-service teachers achieve the best coordination of the elements of the resulting emerging semiotic systems in the study processes since, according to Wilhelmi et al. (2015), it is not enough for

teachers to make operational use of the functions. They must also master the specialised knowledge for teaching, which can help them understand potential learning conflicts of the students they guide and design situations that allow those conflicts to be resolved.

DECLARATION OF AUTHORSHIP CONTRIBUTIONS

The total contribution percentage for conceptualisation, preparation, analysis of the information, and correction of this article was TAD 40%, HAS 30%, and ECH 30%.

DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the corresponding author [TAD].

REFERENCES

- Adu-Gyamfi, K. & Bossé, M. (2014). Processes and reasoning in representations of linear functions. *International Journal of Science and Mathematics Education*, 12, 167-192. <https://doi.org/10.1007/s10763-013-9416-x> .
- Amaya, T. (2020). Evaluación de la faceta epistémica del conocimiento didáctico- matemático de futuros profesores de matemáticas en el desarrollo de una clase utilizando funciones. *Revista Bolema*, 34(66), 110-131. <https://doi.org/10.1590/1980-4415v34n66a06> .
- Amaya, T. (2016). *Evaluación de los conocimientos didáctico-matemáticos de futuros profesores de matemáticas al hacer transformaciones de las representaciones de una función*. [Tesis de doctorado]. <https://creativecommons.org/licenses/by-nc-nd/4.0>
- Amaya, T. & Medina, A. (2013). Dificultades de los estudiantes de grado once al hacer transformaciones de representaciones de una función con el registro figural como registro principal. *Revista Educación Matemática*, 25(2), 119-140.
- Amaya, T., Pino-Fan, L., & Medina, A. (2016). Evaluación del conocimiento de futuros profesores de matemáticas sobre las transformaciones de

las representaciones de una función. *Revista Educación Matemática*, 28(3), 111-144. <https://doi.org/10.24844/EM2803.05> .

- Arcavi, A. (2020). Learning to Look at the World Through Mathematical Spectacles—A Personal Tribute to Realistic Mathematics Education. In: M. Heuvel-Panhuizen (Ed.) *International Reflections on the Netherlands Didactics of Mathematics Visions on and Experiences with Realistic Mathematics Education* (pp. 83–95). Springer.
- Arce, J., Torres, L., Ramírez, M., Valoyes, L., Malagón, M., & Arboleda, L. (2004). *Iniciación al álgebra escolar: situaciones funcionales, de generalización y modelación*. Universidad del Valle.
- Ball, D., Thames, M., & Phelps, G. (2008). Content Knowledge for Teaching. What Makes It Special? *Journal of Teacher Education*, 59(5), 389-407. <https://doi.org/10.1177/0022487108324554>
- Bernárdez, E. (1995). *El papel del léxico en la organización textual*. Universidad Complutense de Madrid.
- Biehler, R. (2005). Reconstruction of Meaning as a Didactical Task: The Concept of Function as an Example. In J. Kilpatrick, C. Hoyles, O. Skovsmose & P. Valero (Eds.) *Meaning in mathematics education* (pp. 61-81). Springer. https://doi.org/10.1007/0-387-24040-3_5 .
- Bueno, S. y Pérez, O. (2018). Prácticas actuales de la idoneidad epistémica y cognitiva del concepto función real de una variable real en carreras de ingeniería. *Revista Educación Matemática*, 30(2), 202-231. <https://doi.org/10.24844/em3002.08> .
- Chinnappan, M., & Thomas, M. (2001). Prospective teachers' perspectives on function representation. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.) *Numeracy and beyond (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 155–162). Merga.
- Dolores, C. (2013). *La variación y la derivada (2da ed.)*. Díaz de Santos.
- Dolores, C., Rivera, M. & García, J. (2018). Exploring mathematical connections of pre-university students through tasks involving rates of change. *International Journal of Mathematical Education in Science and Technology*, 50(3), 369-389. <https://doi.org/10.1080/0020739X.2018.1507050> .

- Dreher, A. & Kuntze, S. (2015). Teachers' professional knowledge and noticing: The case of multiple representations in the mathematics classroom. *Educational Studies in Mathematics*, 88(1), 89–114. <https://doi.org/10.1007/s10649-014-9577-8> .
- Duval, R. (2017). *Semiosis y pensamiento humano (2da ed.)*. Universidad del Valle.
- Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior*, 17(1), 105-121. [https://doi.org/10.1016/S0732-3123\(99\)80063-7](https://doi.org/10.1016/S0732-3123(99)80063-7) .
- Farfán, R. (2012). *El desarrollo del pensamiento matemático y la actividad docente*. Gedisa.
- Feikes, D. & Schwingendorf, K. (2008). The importance of compression in children's learning of mathematics and teachers' learning to teach mathematics. *Mediterranean Journal for Research in Mathematics Education*, 7(2), 1-11.
- Font, V. (2011). Las funciones y la competencia disciplinar en la formación docente matemática. *UNO. Revista de Didáctica de las Matemáticas*, 56, 86-94.
- Gagatsis, A. & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology*, 24(5), 645-657. <https://doi.org/10.1080/0144341042000262953> .
- Godino, J., Batanero, C., & Font, V. (2003). *Fundamentos de la enseñanza y el aprendizaje de las matemáticas para maestros*. Universidad de Granada.
- Goldin, G. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In: Kelly, A.; Lesh, R. (Ed.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517-545). Routledge.
- Gonzato, M., Godino, J., & Neto, T. (2011). Evaluación de conocimientos didáctico-matemático sobre la visualización de objetos tridimensionales. *Revista Educación Matemática*, 23(3), 5-37. <http://doi.org/10.17583/redimat.2016.1984> .
- Hitt, F. (1998). Difficulties in the Articulation of Different Representations Linked to the Concept of Function. *Journal of mathematical*

behavior, 17(1), 123-134. [https://doi.org/10.1016/S0732-3123\(99\)80064-9](https://doi.org/10.1016/S0732-3123(99)80064-9) .

- Hitt, F. & Morasse, C. (2009). Pensamiento numérico-algebraico avanzado: construyendo el concepto de covariación como preludio al concepto de función. *Electronic Journal of research in educational psychology*, 7(17) 243-260.
- Kieran, C., Pang, J., Schifter, D., & Ng, S.F. (2016). *Early Algebra*. 1st ed. Springer.
- Kaur, B., Wong, L., & Govindani, S. (2020). Graphing Linear Equations-A Comparison of the Opportunity-to-Learn in Textbooks Using the Singapore and the Dutch Approaches to Teaching Equations. In M. Heuvel-Panhuizen (Ed.) *International Reflections on the Netherlands Didactics of Mathematics Visions on and Experiences with Realistic Mathematics Education* (pp. 97–111). Springer. <https://doi.org/10.1007/978-3-030-20223-1> .
- Koichu, B. & Harel, G. (2007). Triadic interaction in clinical task-based interviews with mathematics teachers. *Educational Studies in Mathematics*, 65(3), 349-365.
- National Council of Teachers of Mathematics (2000). *Principios y Estándares para la Educación Matemática*. Traducción al español, Sociedad Andaluza de Educación Matemática Thales. Proyecto Sur.
- Panaoura, A., Michael-Chrysanthou, P., Gagatsis, A., Elia, I., & Philippou, A. (2016). A Structural Model Related to the Understanding of the Concept of Function: Definition and Problem Solving. *International Journal of Science and Mathematics Education*, 15(4), 723-740. <https://doi.org/10.1007/s10763-016-9714-1> .
- Peirce, Ch. (1974). *La ciencia de la semiótica*. Nueva Visión.
- Pino-Fan, L., Assis, A. & Castro, W. (2015). Towards a Methodology for the Characterization of Teachers' Didactic-Mathematical Knowledge. *Eurasia Journal of Mathematics, Science & Technology Education*, 11(6), 1429-1456.
- Pino-Fan, L. & Godino, J. (2015). Perspectiva ampliada del conocimiento didáctico – matemático del profesor. *Paradigma*, 36(1), 87-109.

- Pino-Fan, L., Godino, J., & Font, V. (2011). Faceta epistémica del conocimiento didáctico-matemático sobre la derivada. *Educação Matemática Pesquisa*, 13(1), 141-178.
- Pino-Fan, L., Guzmán, I., Font, V., & Duval, R. (2017). Analysis of the Underlying Cognitive Activity in the Resolution of a Task on Derivability of the Absolute-Value Function: Two Theoretical Perspectives. *PNA*, 11(2), 97-124.
- Pino-Fan, L., Parra, Y., & Castro, W. (2019). Significados de la función pretendidos por el currículo de matemáticas chileno. *Magis - Revista Internacional de Investigación en Educación*, 11(23), 201-220. <https://doi.org/10.11144/Javeriana.m11-23.sfpc> .
- Rey, G., Boubée, C., Sastre, P., & Cañibano, A. (2008). Ideas para Enseñar. Aportes didácticos para abordar el concepto de función. *Revista iberoamericana de educación matemática*, (20), 153-162.
- Rojas, P. (2015). Objetos matemáticos, representaciones semióticas y sentidos. *Revista Enseñanza de las ciencias*, 33(1), 151-165. <http://doi.org/10.5565/rev/ensciencias> .
- Rolfes, T. (2018). *Funktionales Denken: Empirische Ergebnisse zum Einfluss von Statischen Und Dynamischen Repräsentationen*. Springer, <https://doi.org/10.1007/978-3-658-22536-0> .
- Rolfes, T., Roth, J., & Schnotz, W. (2020). Learning the concept of function with dynamic visualizations. *Frontiers in Psychology*, 11, 1-16. <https://doi.org/10.3389/fpsyg.2020.00693> .
- Rolfes, T., Roth, J., & Schnotz, W. (2018). Effects of Tables, Bar Charts, and Graphs on Solving Function Tasks. *Journal für Mathematik Didaktik*, 39, 97–125. <https://doi.org/10.1007/s13138-017-0124-x>
- Ruiz-Higueras, L. (1994). *Concepciones de los alumnos de secundaria sobre la noción de función. Análisis epistemológico y didáctico*. [Tesis de doctorado]. <http://www.atd-tad.org/documentos/concepciones-de-los-alumnos-de-secundaria-sobre-la-nocion-de-funcion-analisis-epistemologico-y-didactico/> .
- Schoenfeld, A. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM Mathematics Education*, 43, 457–469. <https://doi.org/10.1007/s11858-011-0307-8> .

- Schoenfeld, A. & Kilpatrick, J. (2008). Towards a theory of proficiency in teaching mathematics. In D. Tirosh & T. Wood (eds.), *Tools and Processes in Mathematics Teacher Education* (pp. 321-354). Sense.
- Secretaría de Educación Pública. (2004). *El Libro para el maestro de Matemáticas. Educación secundaria*. <https://www.uv.mx/personal/grihernandez/files/2011/04/libromaestro.pdf>
- Siegler, R., Fazio, L., Bailey, D., & Zhou, X. (2013). Fractions: the new frontier for theories of numerical development. *Trends in Cognitive Sciences*, 17(1), 13-19. <http://doi.org/10.1016/j.tics.2012.11.004> .
- Steele, M., Hillen, A., & Smith, M. (2013). Developing mathematical knowledge for teaching in a methods course: the case of function. *Journal of Mathematics Teacher Education*, 16, 451-482. <https://doi.org/10.1007/s10857-013-9243-6> .
- Thomas, M. (2008). Developing versatility in mathematical thinking. *Mediterranean Journal for Research in Mathematics Education*, 7(2), 67-87.
- Thompson, P. & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). National Council of Teachers of Mathematics.
- Wilhelmi, M., Godino, G., & Lasa, A. (2015). *Significados conflictivos de ecuación y función en estudiantes de profesorado de secundaria*. In M. González; M. Codes; D. Arnau; T. Ortega, (Eds.), *Investigación en educación matemática* (pp. 573-582). Sociedad Española de Investigación en Educación Matemática.
- Wills, T., Shipley, T., Chang, B., Cromley, J., & Booth, J. (2014). What Gaze Data Reveal About Coordinating Multiple Mathematical Representation. *Proceedings of the Annual Meeting of the Cognitive Science Society*, 36(36), 312-3118.
- Zarhouti, M., Mouradi, M., & El Marouf, A. (2014). The teaching of the function at high school: The graphic representation of a function at first year, section experimental sciences. *Journal of Research & Method in Education*, 4(3), 56-65. <https://doi.org/10.9790/7388-04315665> .

