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Research article

On some generalized q-difference sequence spaces

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Abstract: In this study, we construct the spaces of q-difference sequences of order m. We obtain some inclusion relations, topological properties, Schauder basis and alpha, beta and gamma duals of the newly defined spaces. We characterize certain matrix classes from the newly defined spaces to any one of the spaces c_0, c, ℓ_∞ and ℓ_p.

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1. Introduction

A sequence space is defined as a vector subspace of ω, where ω is the set of all $\mathbb{K}$-valued sequences, where $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$. Some of the well-known examples of classical sequence space are the set of all bounded sequences ℓ_∞, null sequences c_0, convergent sequences c and p-absolutely summable sequences ℓ_p, where $1 \leq p < \infty$. A BK-space is a Banach sequence space with continuous coordinates. The space ℓ_p is BK-space accompanied by the norm $\|u\|_{\ell_p} = (\sum_k |u_k|^p)^{1/p}$.

Set $\mathcal{A} = (a_{nk})_{\mathbb{N}_0\times\mathbb{N}_0}$ be an infinite matrix with real or complex elements. We will denote by $\mathcal{A}_n = (a_{nk})$ the sequence in the n^{th} row of $\mathcal{A}$ for every $n \in \mathbb{N}\cup\{0\}$. For $x = (x_k) \in \omega$, the $\mathcal{A}$-transform of x is defined as the sequence $\mathcal{A}x = ((\mathcal{A}x)_n)_{n=0}^{\infty}$, where

$$(\mathcal{A}x)_n = \sum_k a_{nk}x_k$$

provided the series on the right side converges for each $n \in \mathbb{N}$. Furthermore, the sequence x is called $\mathcal{A}-$summable to the number l if $(\mathcal{A}x)_n \to l$, as $n \to \infty$. In that case, we write $x \to l(\mathcal{A})$ where l is called the $\mathcal{A}-$limit of x.

Define X, Y be two sequence spaces and $\mathcal{A}$ be an infinite matrix. Then, we call $\mathcal{A}$ a matrix mapping from X into Y, if $\mathcal{A}x$ exists and is in Y for every sequence $x = (x_k) \in X$. The class of all infinite matrices that map X into Y will be denoted by (X, Y).

For a sequence space E, we call $E_{\mathcal{A}}$ the *matrix domain* of an infinite matrix $\mathcal{A}$ if

$$E_{\mathcal{A}} = \{x = (x_k) \in \omega : \mathcal{A}x \in E\}. \tag{1.1}$$

Here $E_{\mathcal{A}}$ is a sequence space. A matrix $\mathcal{A}$ is called conservative if $\mathcal{A}x \in c$ for all $x \in c$. If in addition $\mathcal{A} - \lim x = \lim x$ for all $x \in c$, Then $\mathcal{A}$ is called a regular matrix.

Several authors in the literature have constructed sequence spaces using the domain of some special matrices. For instance, one may refer to these nice papers and summability books [1–7].

2. q-sequence spaces

This paper focuses on the q-analogue of difference matrices and seeks to obtain new results related to the q-analogue.

We start with q-integer definition generated by [8]. A q-integer is defined by

$$[u]_q = \begin{cases} \sum_{k=0}^{u-1} q^k, & (u = 1, 2, 3, ...), \\ 0, & (u = 0). \end{cases}$$

It is to be expected that, when $q \to 1^-$ then $[u]_q \to u$. We denote $[u]_q$ briefly by $[u]$. The q-binomial coefficient is defined by

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{cases} \frac{[u]!}{[v]![u-v]!}, & 0 \le v \le u, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where q-factorial $[u]!$ of u is given by

$$[u]! = \begin{cases} \prod_{k=1}^{u} [k], & (u = 1, 2, 3, ...), \\ 1, & (u = 0). \end{cases}$$

From the definition of q-binomial coefficients, we have

$$(x + r)_q^i = \sum_{j=0}^{i} \begin{bmatrix} i \\ j \end{bmatrix} q^{\binom{j}{2}} x^{i-j} r^j. \tag{2.2}$$

The last formula is called Gauss's q-binomial formula.

Let us now introduce the q-difference matrix definition, following [9]. We first define q-operator by

$$\Delta_q u = (u_0 - u_1, q(u_1 - u_2), q^2(u_2 - u_3), q^3(u_3 - u_4)...) \tag{2.3}$$

where $u = (u_0, u_1, u_2, ...)$. This operator leads directly to the q-binomial coefficients via iteration, $\Delta_q^m = \Delta_q(\Delta_q^{m-1})$:

$$(\Delta_q^m u)_{i,j} = q^{mj} \sum_{i=j}^{m} (-1)^{i-j} q^{\binom{i-j}{2}} \begin{bmatrix} m \\ i-j \end{bmatrix} u_i = q^{mj} \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m \\ i \end{bmatrix} u_{i+j}. \tag{2.4}$$

The q-difference matrix $\Delta_q^m = \left(\left(\delta_q^m\right)_{ij}\right)_{\mathbb{N}_0 \times \mathbb{N}_0}$ is given by

$$(\delta_q^m)_{i,j} = (-1)^{i-j} q^{\binom{i-j}{2}} \begin{bmatrix} m \\ i-j \end{bmatrix}$$

and this matrix can be explicitly represented as

$$\Delta_q^m = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -[m] & 1 & 0 & 0 & 0 & \cdots \\ q\begin{bmatrix} m \\ 2 \end{bmatrix} & -[m] & 1 & 0 & 0 & \cdots \\ -q^3\begin{bmatrix} m \\ 3 \end{bmatrix} & q\begin{bmatrix} m \\ 2 \end{bmatrix} & -[m] & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then, the inverse $(\Delta_q^m)^{-1} = (\delta_q^m)_{i,j}^{-1}$ of Δ_q^m is obviously given by

$$(\delta_q^m)_{i,j}^{-1} = (\delta_{\frac{1}{q}}^m)_{i,j} = (-1)^{i-j} q^{(i-j)(i-j-m)-\binom{i-j}{2}} \begin{bmatrix} m \\ i-j \end{bmatrix} = \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix}.$$

The primary purpose of this paper is to define a new sequence space using the Δ_q^m operator and to examine this sequence space.

3. q-difference sequence spaces

The q-analogue of Cesàro sequence spaces were defined by Demiriz and Şahin [10] and Yaying et al. [11]. Then, Yaying et al. [12, 13] studied over (p', q)-analogue of Euler sequence spaces and q-analogue of Catalan sequence spaces. Recently, Alotaibi et al. [14] and Yaying et al. [15,16] introduced q-difference sequence spaces of the second order. For other studies on q-analogue of sequence spaces, you can refer to the references [17, 18].

Let $D = \ell_\infty, c_0, c$. The concept of a difference sequence space was introduced by Kizmaz [19], who studied the difference sequence spaces $D(\Delta)$, where

$$D(\Delta) = \{u = (u_k) \in w : \Delta u \in D\}.$$

In the past, several authors studied matrix transformations on sequence spaces that are the matrix domain of the difference operator, or of the matrices of some classical methods of summability in different sequence spaces, for instance we refer to [20–24] and references therein.

Recently, Altay [24] introduced the spaces $D(\Delta^m)$ as follows:

$$D(\Delta^m) = \{u = (u_k) \in \omega : \{(\Delta^m u)_k\}_{k=0}^\infty \in D\},$$

where $m \in \mathbb{N}$ and $(\Delta^m x)_k = \left\{(\Delta \circ \Delta^{m-1})x\right\}_{k=0}^\infty$ for all $k \in \mathbb{N} \cup \{0\}$.

In this section, we introduce the spaces $\ell_p(\Delta_q^m)$ as a generalization spaces $\ell_p(\Delta^m)$. Now let's give the sequence space $\ell_p(\Delta_q^m)$ as the set of all sequences such that Δ_q^m-transforms of them are in the space ℓ_p, that is

$$\ell_p(\Delta_q^m) = \left\{ u = (u_j) \in \omega : \sum_j \left| q^{mj} \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m \\ i \end{bmatrix} u_{i+j} \right|^p < \infty \right\},$$

where $m \in \mathbb{N}$.

It is easy to check that when $q = 1$, the sequence space $\ell_p(\Delta_q^m)$ reduces to the ordinary difference sequence space $\ell_p(\Delta^m)$ as studied by Altay [24].

In the notation of (1.1), we can redefine the space $\ell_p(\Delta_q^m)$ by

$$\ell_p(\Delta_q^m) = \{\ell_p\}_{\Delta_q^m}. \tag{3.1}$$

Define the sequence $v = \left(v_j\right)_{j=0}^{\infty}$, which will be frequently used, as the Δ_q^m-transform of a sequence $u = (u_j)$, i.e.,

$$v_j = q^{mj} \sum_{i=0}^{j} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} m \\ i \end{bmatrix} u_{i+j}; \qquad (j \in \mathbb{N} \cup \{0\}). \tag{3.2}$$

Also the sequence $u = \left(u_j\right)_{j=0}^{\infty}$,

$$u_j = \sum_{i=0}^{j} \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix} v_i; \qquad (j \in \mathbb{N} \cup \{0\}). \tag{3.3}$$

Wilansky's Theorem 4.3.12 of [25, p.63] states that X is a BK-space and Λ is a triangle, then X_Λ is also a BK-space endowed with the norm $\|x\|_{X_\Lambda} = \|\Lambda x\|_X$. It is easily seen that the $\ell_p(\Delta_q^m)$ set becomes a linear space with the coordinatewise addition and scalar multiplication, which is a BK-space with the norm $\|u\|_{\ell_p(\Delta_q^m)} = \left\|\Delta_q^m u\right\|_{\ell_p}$.

Theorem 3.1. *The $\ell_p(\Delta_q^m)$ sequence space is linearly isomorphic to ℓ_p.*

Proof. The transformation Ψ can be defined with (3.2) notation from $\ell_p(\Delta_q^m)$ to ℓ_p by $u \mapsto v = \Psi u$. Clearly, Ψ is a linear bijection and norm preserving. □

Theorem 3.2. *The space $\ell_p(\Delta_q^m)$ is non-absolute type.*

Proof. Taking $v = (v_k) = (-1)^k$, it is clear from (2.3) that

$$\begin{aligned} \Delta_q v &= \left\{2, -2q, 2q^2, -2q^3, ...\right\} = 2\left\{1, -q, q^2, -q^3, ...\right\}, \\ \Delta_q^2 v &= 2\left\{1+q, -q^2(1+q), q^4(1+q), ...\right\} = 2(1+q)\left\{1, -q^2, q^4, ...\right\}, \\ \Delta_q^3 v &= 2(1+q)\left\{1+q^2, -q^3(1+q^2), q^6(1+q^2), ...\right\} = 2(1+q)(1+q^2)\left\{1, -q^3, q^6, ...\right\}, \end{aligned}$$

and finally

$$\Delta_q^m v = 2(1+q)(1+q^2)(1+q^3)...(1+q^{m-1})\left\{1, -q^m, q^{2m}, ...\right\}.$$

Therefore

$$\left(\Delta_q^m v\right)_k = 2\,(1+q)_q^{m-1}\,(-1)^k q^{mk}.$$

But

$$\Delta_q|v| \;=\; \left\{0, 0q, 0q^2, 0q^3, ...\right\} = \{0, 0, 0, 0, ...\}$$

then

$$\Delta_q^m|v| \;=\; \{0, 0, 0, ...\}\,.$$

So

$$\|v\|_{\ell_p(\Delta_q^m)} \neq \||v|\|_{\ell_p(\Delta_q^m)}$$

where $|v| = (|v_k|)_{k=0}^{\infty}$. □

Now, we discuss some inclusion relations concerning with the space $\ell_p(\Delta_q^m)$.

Theorem 3.3. *Let* $0 < q < 1$. *The inclusion* $\ell_p \subsetneq \ell_p(\Delta_q^m)$ *holds.*

Proof. It is fairly easy to see that the space $\ell_p \subset \ell_p(\Delta_q^m)$. To prove the strictness part, we consider the sequence $(r_k) = (k)$ for all $k \in \mathbb{N} \cup \{0\}$. Then, r is not a sequence in ℓ_p. On the other hand from Eq (2.3)

$$\Delta_q^m r \;=\; \left(-(1-q)(1-q^2)(1-q^3)\cdots(1-q^{m-1})q^{mk}\right)_{k=0}^{\infty} = \left(-(1-q)_q^{m-1}q^{mk}\right)_{k=0}^{\infty},$$

then

$$\sum_k \left|(1-q)_q^{m-1}q^{mk}\right|^p < \infty.$$

Since it is convergent, this means that $\Delta_q^m r \in \ell_p$ and as a result $r \in \ell_p(\Delta_q^m)$. Hence, $\ell_p \subsetneq \ell_p(\Delta_q^m)$ holds. □

A sequence Schauder basis for a linear metric space $\mathfrak{X}$ is a sequence $(u_k) \subset \mathfrak{X}$ with the property that for every $u \in \mathfrak{X}$, there exists a unique sequence (α_k) of scalars such that

$$\left\|u - \sum_{k=1}^{n} \alpha_k u_k\right\| \to 0, \;\; (n \to \infty).$$

If we take into consideration the fact that the matrix domain $\mathfrak{X}_{\mathcal{A}}$ of a normed sequence space $\mathfrak{X}$ has a basis if and only if $\mathfrak{X}$ has a basis whenever $\mathcal{A} = (a_{nk})_{\mathbb{N}_0\times\mathbb{N}_0}$ is a triangle. Then, we have:

Corollary 3.4. *Let* $1 \le p < \infty$ *and* $\beta_j(q) = (\Delta_q^m a)_j$ *for all* $j \in \mathbb{N} \cup \{0\}$. *Define the sequence* $b^{(j)}(q) = \left(b_i^{(j)}(q)\right)_{i=0}^{\infty}$ *of the elements of the space* $\ell_p(\Delta_q^m)$ *for all fixed* $j \in \mathbb{N} \cup \{0\}$ *by*

$$b^{(j)}(q) = \begin{cases} \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix}, & 0 \le j < i, \\ 0, & j \ge i. \end{cases}$$

Then, the sequence $\left(b_i^{(j)}(q)\right)_{j=0}^{\infty}$ *is a Schauder basis for the space* $\ell_p(\Delta_q^m)$.

It is well-known that a space which has a Schauder basis is separable, then we can give following corollary:

Corollary 3.5. *The sequence space* $\ell_p(\Delta_q^m)$ *for* $1 \le p < \infty$ *is separable.*

4. α-, β- and γ-duals

In this section, our next goal is to state and prove the theorems determining the α-, β- and γ-duals of our new sequence spaces. The following will assume that p^* is the conjugate of p, that is, $p^{-1}+p^{*-1} = 1$.

The α-, β- and γ-duals of a sequence space $\mathcal{U}$ are denoted by U^α, $\mathcal{U}^\beta$ and $\mathcal{U}^\gamma$, respectively, and are defined by

$$\begin{aligned}\mathcal{U}^\alpha &= \{u = (u_j) \in w : ua = (u_ja_j) \in \ell_1 \text{ for all } a = (a_j) \in \mathcal{U}\},\\ \mathcal{U}^\beta &= \{u = (u_j) \in w : ua = (u_ja_j) \in cs \text{ for all } a = (a_j) \in \mathcal{U}\},\\ \mathcal{U}^\gamma &= \{u = (u_j) \in w : ua = (u_ja_j) \in bs \text{ for all } a = (a_j) \in \mathcal{U}\}.\end{aligned}$$

First, let's give the Lemmas used in the proof of the Theorems we will give in this section:

Lemma 4.1. *[26] $\mathcal{U} \in (\ell_p : \ell_1)$ if and only if*

$$\sup_{N\in\mathcal{F}} \sum_{k=0}^{\infty} \left| \sum_{n\in N} u_{nk} \right|^{p^*} < \infty \ \ (1 < p \leq \infty). \tag{4.1}$$

Lemma 4.2. *[26] $\mathcal{U} \in (\ell_p : c)$ if and only if*

$$\forall k, \lim_{n\to\infty} u_{nk} \text{ exists} \tag{4.2}$$

and

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} |u_{nk}|^{p^*} < \infty. \tag{4.3}$$

Theorem 4.3. *Let the set $\mathfrak{c}_{\mathfrak{q}}$ be as follows:*

$$\mathfrak{c}_{\mathfrak{q}} = \left\{ u = (u_k) \in w : \sup_{N\in\mathcal{F}} \sum_{i=0}^{\infty} \left| \sum_{j\in N} \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix} \alpha_j \right|^{p^*} < \infty \right\}.$$

Then, $\left\{\ell_p(\Delta_q^m)\right\}^\alpha = \mathfrak{c}_{\mathfrak{q}}$.

Proof. Let $u = \left(u_j\right)_{j=0}^{\infty} \in \omega$. By (3.3), define the sequence $u = \left(u_j\right)_{j=0}^{\infty}$,

$$u_j = \sum_{i=0}^{j} \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix} v_i$$

for every $j \in \mathbb{N} \cup \{0\}$. Thus,

$$\alpha_j u_j = \sum_{i=0}^{j} \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix} \alpha_j v_i = (Av)_j, \quad (j \in \mathbb{N} \cup \{0\}); \tag{4.4}$$

where the sequence $A = (a_{ij})_{\mathbb{N}_0\times\mathbb{N}_0}$ is defined by

$$a_{ij} = \begin{cases} \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix} \alpha_j, & (0 \leq i \leq j) \\ 0, & (i > j) \end{cases}$$

$ux = (u_nx_n) \in \ell_1$ for $x \in \ell_p(\Delta_q^m)$ if and only if $\mathcal{A}v \in \ell_1$ for $v \in \ell_p$. That is $\mathcal{A} \in (\ell_p : \ell_1)$. Hence, by Lemma 4.1 from (4.1), it is concluded that $\left\{\ell_p(\Delta_q^m)\right\}^\alpha = \mathfrak{c}_{\mathfrak{q}}$. □

Theorem 4.4. *Let us consider $\mathcal{D} = (d_{nj})$ defined via a sequence $a = (a_j)$ by*

$$d_{nj} = \begin{cases} \sum_{i=j}^{n} \begin{bmatrix} m+i-j-1 \\ i-j \end{bmatrix} a_i, & (0 \leq j \leq n), \\ 0, & (j > n). \end{cases} \tag{4.5}$$

Define the sets $\mathfrak{d}_1$ and $\mathfrak{d}_2$ as follows

$$\begin{aligned} \mathfrak{d}_1 &= \left\{ u = (u_j) \in w : \lim_{n \to \infty} d_{nj} = \alpha_j \right\}, \\ \mathfrak{d}_2 &= \left\{ u = (u_j) \in w : \sup_{n \in \mathbb{N}} \sum_{j=0}^{\infty} |d_{nj}|^{p^*} < \infty \right\}. \end{aligned}$$

Then,

(i) $\left\{ \ell_p(\Delta_q^m) \right\}^{\beta} = \mathfrak{d}_1 \cap \mathfrak{d}_2$,

(ii) $\left\{ \ell_p(\Delta_q^m) \right\}^{\gamma} = \mathfrak{d}_2$.

Proof. We give the proof only for the case β-dual. Consider the equation

$$\sum_{j=0}^{n} a_j u_j = \sum_{j=0}^{n} \left[\sum_{i=0}^{j} \begin{bmatrix} m+j-i-1 \\ j-i \end{bmatrix} v_i \right] a_j = \sum_{j=0}^{n} \left[\sum_{i=j}^{n} \begin{bmatrix} m+i-j-1 \\ i-j \end{bmatrix} a_i \right] v_j = (\mathcal{D}v)_n$$

for any $n \in \mathbb{N}$. Then $au = (a_j u_j) \in cs$ for $u \in \ell_p(\Delta_q^m)$ if and only if $\mathcal{D}v \in c$ for $v \in \ell_p$. That is $\mathcal{D} \in (\ell_p : c)$. Hence, by Lemma 4.2 from (4.2) and (4.3), it is deduced that

$$\left\{ \ell_p(\Delta_q^m) \right\}^{\beta} = \mathfrak{d}_1 \cap \mathfrak{d}_2.$$

□

5. Matrix transformations

Let $\mu \in \{c_0, c, \ell_\infty, \ell_p\}$. In this section we will characterize the spaces let $\left(\ell_p(\Delta_q^m) : \mu\right)$ and $\left(\mu : \ell_p(\Delta_q^m)\right)$.

Theorem 5.1. *Define, for all $k, n \in \mathbb{N}$, elements of infinity matrices $\mathcal{U} = (u_{nk})$ and $\mathcal{V} = (v_{nk})$*

$$v_{nk} := \sum_{i=k}^{n} \begin{bmatrix} m+i-k-1 \\ i-k \end{bmatrix} u_{ni}. \tag{5.1}$$

In this case $\mathcal{U} \in (\ell_p(\Delta_q^m) : \mu)$ if and only if for all $n \in \mathbb{N}$, $(u_{nk})_{k=0}^{\infty} \in \{\ell_p(\Delta_q^m)\}^{\beta}$ and $\mathcal{V} \in (\ell_p : \mu)$.

Proof. Let μ be a sequence space. Then, $\mathcal{U} = (u_{nk})$ and $\mathcal{V} = (v_{nk})$ satisfy the condition in (5.1). Also, the spaces $\ell_p(\Delta_q^m)$ and ℓ_p are linearly isomorphic, as shown in Theorem 3.1.

Let $\mathcal{U} \in (\ell_p(\Delta_q^m) : \mu)$ and $y = (y_k) \in \ell_p$. Since $(u_{nk})_{k=0}^{\infty} \in \mathfrak{d}_1 \cap \mathfrak{d}_2$, we have $\{\delta_{nk}\}_{k=0}^{\infty} \in \ell_p$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, Δy exists and we have

$$\sum_k v_{nk} y_k = \sum_k u_{nk} x_k$$

for all $n \in \mathbb{N} \cup \{0\}$.

Hence, $\mathcal{V}y = \mathcal{U}x$. Thus, we deduce that $\mathcal{V} \in (\ell_p : \mu)$.

Conversely, suppose that $(u_{nk})_{k=0}^{\infty} \in \{\ell_p(\Delta_q^m)\}^{\beta}$ for all $n \in \mathbb{N} \cup \{0\}$, $\mathcal{V} \in (\ell_p : \mu)$ and $x = (x_k) \in \ell_p(\Delta_q^m)$. Then, $\mathcal{U}x$ exists. Therefore, we have

$$\sum_{k=0}^{\infty} u_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{i=k}^{n} \begin{bmatrix} m+i-k-1 \\ i-k \end{bmatrix} u_{ni} \right] y_k \quad (n \in \mathbb{N}).$$

Hence, $\mathcal{V}y = \mathcal{U}x$. This leads us to the result $\mathcal{U} \in (\ell_p(\Delta_q^m) : \mu)$. □

Theorem 5.2. *Let $\mathcal{U} = (u_{ij})$ be an infinite matrix and define the matrix $\mathcal{V} = (v_{ij})$ by*

$$v_{ij} := q^{mk} \sum_{i=k}^{m} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} m \\ i-k \end{bmatrix} u_{ij} \tag{5.2}$$

for all $i, j \in \mathbb{N} \cup \{0\}$ and μ be a sequence space. Then, $\mathcal{U} \in (\mu : \ell_p(\Delta_q^m))$ if and only if $\mathcal{V} \in (\mu : \ell_p)$.

Proof. Let $z = (z_k) \in \mu$. Then,

$$\begin{aligned} \sum_{j=0}^{r} v_{ij} z_j &= \sum_{j=0}^{r} \left(q^{mk} \sum_{i=k}^{m} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} m \\ i-k \end{bmatrix} u_{ij} \right) z_j \\ &= q^{mk} \sum_{i=k}^{m} (-1)^{i-k} q^{\binom{i-k}{2}} \begin{bmatrix} m \\ i-k \end{bmatrix} \left(\sum_{j=0}^{r} u_{ij} z_j \right) \end{aligned}$$

for all $i, r \in \mathbb{N} \cup \{0\}$ holds. Since $r \to \infty$, $(\mathcal{V}z)_r = \left(\Delta_q(\mathcal{U}z)\right)_r$ for all $r \in \mathbb{N} \cup \{0\}$. Thus, $z \in \mu$ holds. Hence, $\mathcal{U}z \in \ell_p(\Delta_q^m)$ if and only if $\mathcal{V}z \in \ell_p$. □

By Stieglitz and Tietz [26]

$$\lim_{k} u_{nk} = 0, \text{ for all } n, \tag{5.3}$$

$$\lim_{n\to\infty} u_{nk} = 0, \text{ for all } k, \tag{5.4}$$

$$\sup_{m} \sum_{k} \left| \sum_{n=0}^{m} u_{nk} \right|^{p^*} < \infty, \tag{5.5}$$

$$\sum_{n} u_{nk} \text{ converges, for all } k, \tag{5.6}$$

$$\sup_{K} \sum_{n} \left| \sum_{k\in K} u_{nk} \right|^{p} < \infty, \tag{5.7}$$

$$\sup_{K} \sum_{n} \left| \sum_{k\in K} (u_{nk} - u_{n,k+1}) \right|^{p} < \infty, \tag{5.8}$$

$$\sup_{K} \sum_{n} \left| \sum_{k\in K} (u_{nk} - u_{n,k-1}) \right|^{p} < \infty. \tag{5.9}$$

In this case, the Lemma below is obtained from these conditions.

Lemma 5.3. *Let $\mathcal{U} = (u_{nk})$ be an infinite matrix. Then,*

(1) $\mathcal{U} \in (\ell_p : c) \Leftrightarrow$ (4.2) and (4.3).

(2) $\mathcal{U} \in (\ell_p : c_0) \Leftrightarrow$ (4.3) and (5.4).

(3) $\mathcal{U} \in (\ell_p : bs) \Leftrightarrow$ (5.5).

(4) $\mathcal{U} \in (\ell_p : cs) \Leftrightarrow$ (5.5) and (5.6).

(5) $\mathcal{U} \in (c_0 : \ell_p) = (c : \ell_p) = (\ell_\infty : \ell_p) \Leftrightarrow$ (5.7).

(6) $\mathcal{U} \in (bs : \ell_p) \Leftrightarrow$ (5.3) and (5.8).

(7) $\mathcal{U} \in (cs : \ell_p) \Leftrightarrow$ (5.9).

Corollary 5.4. *Let $\mathcal{U} = (u_{nk})$ be an infinite matrix. Then, by Theorem 5.1, the following conditions hold:*

(i) *$\mathcal{U} \in (\ell_p(\Delta_q^m) : c_0)$ if and only if $(u_{nk})_{k=0}^{\infty} \in \{\ell_p(\Delta_q^m)\}^\beta$ for all $n \in \mathbb{N} \cup \{0\}$ and (4.3) and (5.4) hold with $\tilde{u}_{nk}$ instead of u_{nk}.*

(ii) *$\mathcal{U} \in (\ell_p(\Delta_q^m) : c)$ if and only if $(u_{nk})_{k=0}^{\infty} \in \{\ell_p(\Delta_q^m)\}^\beta$ for all $n \in \mathbb{N} \cup \{0\}$ and (4.2) and (4.3) hold with $\tilde{u}_{nk}$ instead of u_{nk}.*

(iii) *$\mathcal{U} \in (\ell_p(\Delta_q^m) : \ell_\infty)$ if and only if $(u_{nk})_{k=0}^{\infty} \in \{\ell_p(\Delta_q^m)\}^\beta$ for all $n \in \mathbb{N} \cup \{0\}$ and (4.3) holds with $\tilde{u}_{nk}$ instead of u_{nk}.*

(iv) *$\mathcal{U} \in (\ell_p(\Delta_q^m) : bs)$ if and only if $(u_{nk})_{k=0}^{\infty} \in \{\ell_p(\Delta_q^m)\}^\beta$ for all $n \in \mathbb{N} \cup \{0\}$ and (5.5) holds with $\tilde{u}_{nk}$ instead of u_{nk}.*

(v) *$\mathcal{U} \in (\ell_p(\Delta_q^m) : cs)$ if and only if $(u_{nk})_{k=0}^{\infty} \in \{\ell_p(\Delta_q^m)\}^\beta$ for all $n \in \mathbb{N} \cup \{0\}$ and (5.5) and (5.6) hold with $\tilde{u}_{nk}$ instead of u_{nk}.*

Corollary 5.5. *Let $\mathcal{U} = (u_{nk})$ be an infinite matrix. Then, by Theorem 5.2, the following conditions hold:*

(i) *$\mathcal{U} = (u_{nk}) \in (c_0 : \ell_p(\Delta_q^m)) = (c : \ell_p(\Delta_q^m)) = (\ell_\infty : \ell_p(\Delta_q^m))$ if and only if (5.7) holds with b_{nk} instead of u_{nk}.*

(ii) *$\mathcal{U} = (u_{nk}) \in (bs : \ell_p(\Delta_q^m))$ if and only if (5.3) and (5.8) hold with b_{nk} instead of u_{nk}.*

(iii) *$\mathcal{U} = (u_{nk}) \in (cs : \ell_p(\Delta_q^m))$ if and only if (5.9) holds with b_{nk} instead of u_{nk}.*

6. Conclusions

The theory of the q-analogue plays a significant role in various fields of mathematical, physical, and engineering sciences. Due to its vast applications in diverse fields of mathematics, several studies related to q-calculus can be found in the literature.

Recently, the construction of sequence spaces using q-calculus has been realized. The difference matrix is the most commonly used matrix in summability theory. In this study, we use the q-analogue version of the difference matrix of order m, thus providing new results.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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