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# QUASITRIANGULAR COIDEAL SUBALGEBRAS OF $U_q(\mathfrak{g})$ IN TERMS OF GENERALIZED SATAKE DIAGRAMS

VIDAS REGELSKIS AND BART VLAAR

Abstract. Let $\mathfrak{g}$ be a finite-dimensional semisimple complex Lie algebra and $\theta$ an involutive automorphism of $\mathfrak{g}$. It is well-known from works of Letzter, Kolb and Balagović that the fixed-point subalgebra $\mathfrak{k} = \mathfrak{g}^\theta$ has a quantum counterpart $B$, a coideal subalgebra of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ possessing a cylinder-twisted universal $K$-matrix $\mathcal{K}$. The objects $\theta$, $\mathfrak{k}$, $B$ and $\mathcal{K}$ can all be described in terms of a combinatorial datum, a Satake diagram. In the present work we extend this construction to generalized Satake diagrams, objects first considered by Heck. A generalized Satake diagram defines a semisimple automorphism of $\mathfrak{g}$ restricting to the standard Cartan subalgebra $\mathfrak{h}$ as an involution. We show that it naturally leads to a subalgebra $\mathfrak{k} \subset \mathfrak{g}$, not necessarily a fixed-point subalgebra, but still satisfying $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$. Such a subalgebra $\mathfrak{k}$ can be quantized to a coideal subalgebra of $U_q(\mathfrak{g})$ endowed with a cylinder-twisted universal $K$-matrix. We conjecture that all such coideal subalgebras of $U_q(\mathfrak{g})$ arise from generalized Satake diagrams in this way.

Contents



## 1. Introduction

Given a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ and an involutive Lie algebra automorphism $\theta \in \mathrm{Aut}(\mathfrak{g})$, a *symmetric pair* is a pair $(\mathfrak{g}, \mathfrak{k})$ where $\mathfrak{k} = \mathfrak{g}^\theta$ is the $\theta$-fixed subalgebra of $\mathfrak{g}$, see [Ara62, Sat71]. *Quantum symmetric pairs* are their quantum analogons. That is to say, the enveloping algebra $U(\mathfrak{g})$ can be quantized to a quasitriangular Hopf algebra, the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ endowed with the universal $R$-matrix $\mathcal{R}$, see [Ji85, Dr87]. Similarly, the $\theta$-fixed subalgebra $\mathfrak{k}$ can be quantized to a coideal subalgebra $B \subseteq U_q(\mathfrak{g})$ [Let99, Let02, Ko14] having a compatible quasitriangular structure, the cylinder-twisted universal $K$-matrix $\mathcal{K}$ [BK16, Ko17].

The involution $\theta$, the corresponding fixed-point subalgebra $\mathfrak{k}$, the coideal subalgebra $B$ and the universal object $\mathcal{K}$ are all defined in terms of a combinatorial data, the so-called Satake diagram $(X, \tau)$. Here $X$ is a subdiagram of the Dynkin diagram of $\mathfrak{g}$ and $\tau$ is an involutive diagram automorphism stabilizing $X$ and satisfying certain compatibility conditions, see [Let02, Ko14].

It is the aim of this paper to extend some of the above work to a more general setting than (quantizations of) fixed-point subalgebras. A direct motivation for this is the fact that the correct quantum group analogue of the fixed-point subalgebra in the Letzter-Kolb approach is not a fixed-point subalgebra itself, but merely tends to one as $q \to 1$, see [Ko14, Ch. 10]. This suggests that there may be a generalization of this approach that does not require a fixed-point subalgebra as input.

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A careful analysis of [Ko14, BK15, BK16] indeed indicates that the compatibility conditions for $X$ and $\tau$ can be weakened, leading to the notion of a *generalized Satake diagram*, see Definition 2.2, and the whole theory survives in this setting with very minor adjustments. The resulting Lie subalgebra $\mathfrak{k} = \mathfrak{k}(X, \tau)$ is given in Definition 3.1 and the corresponding coideal subalgebra $B = B(X, \tau)$ in Definition 4.1. Indeed, in [BK15, Rmks. 2.6, 3.14] it is explicitly suggested that some key passages of the theory are amenable for generalizations.

Our proposed generalization of Satake diagrams can be traced back to the work of A. Heck [He84]. In this work Heck provides a classification of involutions of finite root systems such that the corresponding restricted Weyl group is the Weyl group of the restricted root system. We will review this point-of-view and make a connection with a theorem of Lusztig stating that the restricted Weyl group is in fact a Coxeter group.

The characterization in terms of the restricted Weyl group is relevant in the context of the universal $R$- and $K$-matrices for quantum symmetric pairs. The universal $R$-matrix $\mathcal{R}$ has a distinguished factor called quasi $R$-matrix playing an important role in the theory of canonical bases for $U_q(\mathfrak{g})$ developed by Kashiwara and Lusztig, see [Ka90] and [Lu94, Part IV]. This object possesses a remarkable factorization property expressed in terms of the braid group action on $U_q(\mathfrak{g})$ of the Weyl group associated to $\mathfrak{g}$, see e.g. [KR90, LS90]. Recently it has become clear that many of these properties extend to the cylinder-twisted universal $K$-matrix $\mathcal{K}$. It has a distinguished factor called quasi $K$-matrix introduced in [BW13] for certain coideal subalgebras of $U_q(\mathfrak{sl}_N)$ and in a more general setting in [BK15], and featuring prominently in the theory of canonical bases for quantum symmetric pairs [BW16]. In [DK18] a factorization property is established for the quasi $K$-matrix using a braid group action of the aforementioned restricted Weyl group. In the present work we argue that the factorization property extends to quasi $K$-matrices defined in terms of the generalized Satake diagrams.

A generalization of this approach to the Kac-Moody setting will be addressed in a future work. Another outstanding issue is a Lie-theoretic motivation of the subalgebra $\mathfrak{k}$, which we define in a rather *ad hoc* manner directly in terms of the combinatorial data $(X, \tau)$, see Definition 3.1.

Therefore let us end the introduction with an additional motivation for the study of the subalgebra $\mathfrak{k}$ and its quantization $B$ by making some observations related to the representation theory of the pair $(U_q(\mathfrak{g}), B)$. Following [BK16, Ko17], there exists a suitable completion $\mathcal{U}$ of $U_q(\mathfrak{g})$ such that the objects $\mathcal{R} \in (\mathcal{U} \otimes \mathcal{U})^\times$ and $\mathcal{K} \in \mathcal{U}^\times$ have well-defined images under any finite-dimensional representation $\rho : U_q(\mathfrak{g}) \to \mathrm{End}(V)$. Furthermore, there exists an involutive Hopf algebra automorphism $\phi$ of $\mathcal{U}$ such that $(\phi \otimes \phi)(\mathcal{R}) = \mathcal{R}$ and the following quartic relation is satisfied, known as the *(universal) $\phi$-twisted reflection equation* (see [Ko17, Eqs. (3.22-3.23)]):

$$\mathcal{R}_{21}\,\mathcal{K}_2\,(\phi \otimes \mathrm{id})(\mathcal{R})\,\mathcal{K}_1 = \mathcal{K}_1\,(\phi \otimes \mathrm{id})(\mathcal{R}_{21})\,\mathcal{K}_2\,\mathcal{R} \qquad \in \mathcal{U} \otimes \mathcal{U} \tag{1.1}$$

where $\mathcal{K}_1 = \mathcal{K} \otimes 1$, $\mathcal{K}_2 = 1 \otimes \mathcal{K}$, $\mathcal{R}_{21} = \sigma(\mathcal{R})$ and $\sigma \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{U} \otimes \mathcal{U})$ is the flip map. Let $R \in \mathrm{GL}(V \otimes V)$ be proportional to $(\rho \otimes \rho)(\mathcal{R})$ and $K \in \mathrm{GL}(V)$ proportional to $\rho(\mathcal{K})$. In the case $\phi = \mathrm{id}$, applying $\rho \otimes \rho$ to (1.1) one obtains the matrix reflection equation

$$R_{21}\,K_2\,R\,K_1 = K_1\,R_{21}\,K_2\,R \qquad \in \mathrm{End}(V \otimes V) \tag{1.2}$$

where $K_1 = K \otimes \mathrm{Id}$, $K_2 = \mathrm{Id} \otimes K$ and $R_{21} = PRP$ with $P : V \otimes V \to V \otimes V$ the permutation operator. When $\phi \neq \mathrm{id}$ one naturally obtains the so-called twisted matrix reflection equation which we omit for simplicity, but this does not significantly affect any of the following remarks. In particular, starting with a Satake diagram, one will recover the solutions of (1.2) used in [NDS95, NS95] to define quantum symmetric pairs.

Treating the matrix $R$ as given, one can of course solve (1.2) for $K \in \mathrm{GL}(V)$. For $U_q(\mathfrak{sl}_N)$ and $V = \mathbb{C}^N$ this was done by A. Mudrov [Mu02]. Based on this result and computations for $U_q(\mathfrak{g})$ with $\mathfrak{g}$ of types $B_n$, $C_n$, $D_n$ ($n \leq 4$) and $G_2$, and $V$ the vector representation, we formulate the following conjecture.

**Conjecture 1.1.** *Let $\rho : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ be the vector representation of $U_q(\mathfrak{g})$. If $K \in \mathrm{GL}(V)$ is a solution of (1.2) then there exists a generalized Satake diagram $(X, \tau)$ such that $K$ is proportional to $\rho(\mathcal{K})$ where $\mathcal{K}$ is the universal $K$-matrix for the coideal subalgebra $B(X, \tau)$, i.e. the quantization of $U(\mathfrak{k}(X, \tau))$.*

Based on the available evidence in terms of solutions to (1.2) known to intertwine restrictions of $\rho$ to coideal subalgebras, we also make the following claim.

**Conjecture 1.2.** *Let $\rho : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ be the vector representation of $U_q(\mathfrak{g})$. Then $\rho$ can be used to identify coideal subalgebras, i.e. if the distinct coideal subalgebras $B, B' \subseteq U_q(\mathfrak{g})$ possess the universal $K$-matrices $\mathcal{K}$ and $\mathcal{K}'$, respectively, then $\rho(\mathcal{K})$ and $\rho(\mathcal{K}')$ are not scalar multiples of each other.*

If these two conjectures are true, the only coideal subalgebras of $U_q(\mathfrak{g})$ which possess a universal $K$-matrix in the sense of [Ko17] are those which are quantizations of $U(\mathfrak{k}(X,\tau))$ with $(X,\tau)$ a generalized Satake diagram.

We should remark that coideal subalgebras $B$ in the Letzter-Kolb approach carry additional parameters. The generators associated to the nodes $i \in I\backslash X$ depend on scalars $\gamma_i \neq 0$ and $s_i$, see Definition 4.1. We can thus sharpen Conjecture 1.1. Any invertible matrix solution $K$ of (1.2) is proportional to $\rho(\mathcal{K})$ for some $B(X,\tau)$ with the additional parameters satisfying certain constraints. Most of these constraints were found in [Let03, Ko14] given in terms of the sets $\Gamma_q$ and $\mathcal{S}_q$, see (4.3). Always, we must have $(\gamma_i)_{i\in I\backslash X} \in \Gamma_q$. For the conditions on $s_i$ it is helpful to consider the set $I_{\mathrm{ns}} = \{i \in I\backslash X \,|\, i \text{ does not neighbour } X, \tau(i) = i\}$, see (3.16). The constraints on the $s_i$ are as follows. If $i \notin I_{\mathrm{ns}}$ then $s_i = 0$. For all $(i,j) \in I_{\mathrm{ns}} \times I_{\mathrm{ns}}$ such that $i \neq j$ conjecturally one of three conditions must hold: the Cartan integer $a_{ij}$ is even, $s_j = 0$, or $s_i^2/\gamma_i$ lies in a particular finite subset of a quadratic completion of $\mathbb{C}(q)$. The defining condition of the set $\mathcal{S}_q$ does not cover the third possibility, which appeared in [BB10].

The paper is organized as follows. Section 2 contains the preliminaries and basic definitions. We define the necessary Lie-theoretic objects surrounding a finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$. We introduce the notion of a generalized Satake diagram as a decoration of the Dynkin diagram of $\mathfrak{g}$. We explain how the generalized Satake diagrams emerge in the work of A. Heck.

In Section 3 we define the main object of this paper, the subalgebra $\mathfrak{k} = \mathfrak{k}(X,\tau) \subseteq \mathfrak{g}$. Theorem 3.2 is the main result of this section. We show that $\mathfrak{k}$ satisfies the intersection condition $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta$ (which trivially holds when $\mathfrak{k} = \mathfrak{g}^\theta$ with $\theta^2 = \mathrm{id}_{\mathfrak{g}}$) precisely if $(X,\tau)$ is a generalized Satake diagram. We then study the derived subalgebra of $\mathfrak{k}$. When $\mathfrak{k}$ is not a reductive Lie algebra, Propositions 3.5 and 3.6 establish a semidirect product decomposition for $\mathfrak{k}$ in terms of a reductive subalgebra and a nilpotent ideal of class 2. We end this section with some results about the universal enveloping algebra $U(\mathfrak{k})$. (Appendix A contains three technical lemmas in aid of Section 3.)

In Section 4 we briefly review the quasitriangular structure behind the quantum symmetric pairs. We indicate the necessary modifications to the theory of Balagović-Kolb so that it would be applicable to the quantum pair algebras associated to the generalized Satake diagrams.

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## 2. Finite-dimensional semisimple Lie algebras and root system involutions

Let $I$ be a finite set and $A = (a_{ij})_{i,j\in I}$ a Cartan matrix. In particular, there exist positive rationals $d_i$ ($i \in I$) such that $d_i a_{ij} = d_j a_{ji}$. Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding finite-dimensional semisimple Lie algebra over $\mathbb{C}$. More precisely, $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i\in I}$ subject to

$$[h_i,h_j]=0, \qquad [h_i,e_j]=a_{ij}e_j, \qquad [h_i,f_j]=-a_{ij}f_j, \qquad [e_i,f_j]=\delta_{ij}h_i \tag{2.1}$$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0 \qquad \text{if } i \neq j, \tag{2.2}$$

for all $i,j \in I$. We denote the standard Cartan subalgebra by $\mathfrak{h} = \langle h_i \,|\, i \in I\rangle$ and also consider the corresponding nilpotent subalgebras $\mathfrak{n}^+ = \langle e_i \,|\, i \in I\rangle$, $\mathfrak{n}^- = \langle f_i \,|\, i \in I\rangle$.

The simple roots $\alpha_i \in \mathfrak{h}^*$ ($i \in I$) satisfy $\alpha_j(h_i) = a_{ij}$ for $i,j \in I$. Let $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$ denote the root lattice. In terms of the root spaces $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \forall h \in \mathfrak{h}, [h,x] = \alpha(h)x\}$ ($\alpha \in Q$), $\mathfrak{g}$ is a $Q$-graded Lie algebra and we have the following identities for $\mathfrak{h}$-modules:

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-, \qquad \mathfrak{n}^\pm = \bigoplus_{\alpha \in Q^+} \mathfrak{g}_{\pm\alpha}, \qquad \mathfrak{h} = \mathfrak{g}_0. \tag{2.3}$$

Hence the root system $\Phi := \{\alpha \in Q \,|\, \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0\}$ satisfies $\Phi = \Phi^+ \cup \Phi^-$ where $\Phi^\pm = \pm(\Phi \cap Q^+)$ and $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$.

The Weyl group $W$ is a finite subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by the simple reflections $r_i$ ($i \in I$) acting via $r_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$ for all $i \in I$, $\alpha \in \mathfrak{h}^*$. More precisely, $W$ is a normal subgroup of

$$\mathrm{Aut}(\Phi) := \{g \in \mathrm{GL}(\mathfrak{h}^*) \,|\, g(\Phi) = \Phi\}.$$

Since $W$ induces a simple transitive action on the set of bases of $\Phi$, one readily obtains that $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$, where

$$\mathrm{Aut}(A) = \{\sigma : I \to I \text{ invertible} \,|\, a_{\sigma(i)\,\sigma(j)} = a_{ij} \text{ for all } i, j \in I\}$$

is the group of diagram automorphisms (acting by relabelling).

The following subgroup of $\mathrm{Aut}(\mathfrak{g})$ will be important in what follows:

$$\mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) = \{\sigma \in \mathrm{Aut}(\mathfrak{g}) \,|\, \sigma(\mathfrak{h}) = \mathfrak{h}\} < \mathrm{Aut}(\mathfrak{g}).$$

We briefly review some important subgroups of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$. A braid group action on $\mathfrak{g}$ which extends the dual action of $W$ on $\mathfrak{h}$ is defined by $\mathrm{Ad}(r_i) = \exp(\mathrm{ad}(e_i)) \exp(\mathrm{ad}(-f_i)) \exp(\mathrm{ad}(e_i)) \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ for $i \in I$, yielding $\mathrm{Ad}(W) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$. We also have $\mathrm{Aut}(A) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ (acting by relabelling). The Chevalley involution $\omega \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ is defined by swapping $e_i$ and $-f_i$ for all $i \in I$; it commutes with $\mathrm{Ad}(W)$ and with $\mathrm{Aut}(A)$. Finally, the group $\widetilde{H} := \mathrm{Hom}(Q, \mathbb{C}^\times)$ naturally induces a subgroup $\mathrm{Ad}(\widetilde{H}) < \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ via $\mathrm{Ad}(\chi)|_{\mathfrak{g}_\alpha} = \chi(\alpha)\,\mathrm{id}_{\mathfrak{g}_\alpha}$ for all $\chi \in \widetilde{H}$, $\alpha \in Q$.

The elements of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ can be dualized to elements of $\mathrm{Aut}(\Phi)$. Conversely, given $g \in \mathrm{Aut}(\Phi)$ there are $\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ whose restriction to $\mathfrak{h}$ dualizes to $g$. Indeed, from $-\mathrm{id}_{\mathfrak{h}^*} \in \mathrm{Aut}(\Phi)$ and the direct product decomposition $\mathrm{Aut}(\Phi) = W \rtimes \mathrm{Aut}(A)$, there exist unique $(w, \tau) \in W \times \mathrm{Aut}(A)$ such that $g = -w\tau$. Then one easily checks that $\psi = \mathrm{Ad}(w)\omega\tau \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ satisfies $(\psi|_{\mathfrak{h}})^* = g$.

2.1. **Compatible decorations and involutions of $\Phi$.** Given a subset $X \subseteq I$ denote the corresponding Cartan submatrix by $A_X = (a_{ij})_{i,j \in X}$ and consider the corresponding semisimple Lie algebra $\mathfrak{g}_X := \langle e_i, f_i, h_i \,|\, i \in X \rangle \subseteq \mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_X = \mathfrak{h} \cap \mathfrak{g}_X$, dual Weyl vector $\rho_X^\vee \in \mathfrak{h}_X$ and Weyl group $W_X := \langle r_i \,|\, i \in X \rangle \leq W$. The unique longest element $w_X \in W_X$ is an involution and there exists $\tau_{0,X} \in \mathrm{Aut}(A_X)$ which satisfies

$$-\,w_X(\alpha_i) = \alpha_{\tau_{0,X}(i)} \qquad \text{for all } i \in X. \tag{2.4}$$

Note that $\mathrm{Ad}(w_X)|_{\mathfrak{g}_X} = \tau_{0,X}\,\omega|_{\mathfrak{g}_X}$ and $\mathrm{Ad}(w_X)^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha)\,\mathrm{id}_{\mathfrak{g}_\alpha}$ for all $\alpha \in \Phi$, where $\zeta = \zeta(X) \in \widetilde{H}$ is defined by

$$\zeta(\alpha_i) := (-1)^{2\alpha_i(\rho_X^\vee)} \qquad \text{for } i \in I.$$

We will study

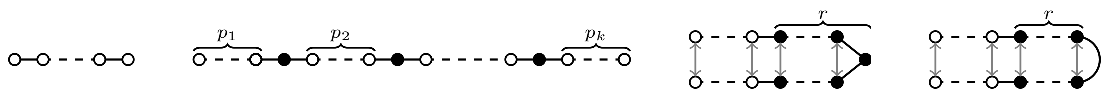
$$\mathrm{Aut}^{\mathrm{inv}}(\mathfrak{g}, \mathfrak{h}) := \{\psi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) \,|\, \psi^2|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\},$$
$$\mathrm{Aut}^{\mathrm{inv}}(\Phi) := \{g \in \mathrm{Aut}(\Phi) \,|\, g^2 = \mathrm{id}_{\mathfrak{h}^*}\}$$

by means of combinatorial data: we define

$$\mathrm{CDec}(A) = \{(X, \tau) \,|\, X \subseteq I, \tau \in \mathrm{Aut}(A), \tau^2 = \mathrm{id}_I, \tau(X) = X, \tau|_X = \tau_{0,X}\} \tag{2.5}$$

and call its elements *compatible decorations* (of $A$). In the Dynkin diagram associated to $\mathfrak{g}$ one marks this decoration by filling the nodes corresponding to $X$ and drawing two-sided arrows for the nontrivial orbits of $\tau$.

**Example 2.1.** Let $A$ be of type $\mathrm{A}_n$, $n \geq 2$. The compatible decorations $\mathrm{CDec}(A)$ are




where $p_1, p_k \in \mathbb{Z}_{\geq 0}$, $p_2, \ldots, p_{k-1} \in \mathbb{Z}_{\geq 1}$ for any $k \in \mathbb{Z}_{\geq 2}$ and $0 \leq r \leq \lceil n/2 \rceil$.

Given $(X, \tau) \in \mathrm{CDec}(A)$, we define

$$\theta = \theta(X, \tau) = -w_X \tau \in \mathrm{Aut}^{\mathrm{inv}}(\Phi). \tag{2.6}$$

As explained above, the map dual to $\theta$ can be extended to an element of $\mathrm{Aut}^{\mathrm{inv}}(\mathfrak{g}, \mathfrak{h})$ which we shall also call $\theta$. It is given by $\theta = \mathrm{Ad}(w_X)\tau\omega$ so that $\theta|_{\mathfrak{h}} = -w_X\tau$. Note that, as a consequence of properties of $\mathrm{Ad}(w_X)$ mentioned earlier, we have

$$\theta|_{\mathfrak{g}_X} = \mathrm{id}_{\mathfrak{g}_X}, \tag{2.7}$$

$$\theta^2|_{\mathfrak{g}_\alpha} = \zeta(\alpha)\,\mathrm{id}_{\mathfrak{g}_\alpha} \qquad \text{for all } \alpha \in \Phi. \tag{2.8}$$

2.2. **Generalized Satake diagrams and the restricted Weyl group.** We choose a subset $I^* \subseteq I \backslash X$ such that it contains precisely one element from each $\tau$-orbit in $I \backslash X$. For $i \in I^*$ denote by $\check{X}(i) \subseteq X$ the union of connected components of $X$ neighbouring $\{i, \tau(i)\}$ and $\check{X}[i] := \check{X}(i) \cup \{i, \tau(i)\}$. By a *minimal subdiagram* of $(X, \tau) \in \mathrm{CDec}(A)$ we mean any subdiagram of the form $\check{X}[i]$ for some $i \in I^*$. By definition $\check{X}[i]$ is a compatible decoration of $A_{\check{X}[i]}$; it is also known as a Satake diagram of (restricted) rank 1.

**Definition 2.2.** *Generalized Satake diagrams* are elements of the set

$\mathrm{GSat}(A) := \{(X, \tau) \in \mathrm{CDec}(A) \,|\, (X, \tau)$ contains no minimal subdiagram of the form ○—●$\}$. ⌀

The compatible decorations in Example 2.1 are generalized Satake diagrams when $p_1 = p_k = 0$ and $p_2 = \ldots = p_{k-1} = 1$.

*Remark* 2.3. Generalized Satake diagrams were first considered by Heck in [He84] where they are shown to classify involutions of root systems such that the restricted Weyl group is the Weyl group of the restricted root system. Heck uses the symbol $\sigma$ to denote the negative of our map $\theta$. He also uses the term Satake diagram for any $(X, \tau)$ such that $X \subseteq I$, $\tau \in \mathrm{Aut}(A)$, $\tau^2 = \mathrm{id}_I$ and $\tau(X) = X$ (this properly contains the set $\mathrm{CDec}(A)$) and the elements of $\mathrm{GSat}(A)$ are called admissible Satake diagrams. However, the term Satake diagram has become reserved for those combinatorial data which classify involutions of $\mathfrak{g}$ up to conjugacy (and their fixed-point subalgebras), which is the reason for our nomenclature "compatible decoration" and "generalized Satake diagram". ⌀

Note that $(X, \tau)$ is a generalized Satake diagrams precisely if

$$\forall (i,j) \in I \backslash X \times X \,:\, \tau(i) = i,\, w_X(\alpha_i) = \alpha_i + \alpha_j \implies a_{ij} \neq -1, \tag{2.9}$$

which is precisely the condition needed in [Ko14, Proof of Lemma 5.11, Step 1] and [BK16, Proof of Lemma 6.4]. One can show that (2.9) is equivalent to either of the following more compact conditions:

$$\forall i, j \in I : \theta(\alpha_i) = -(\alpha_i + \alpha_j) \implies a_{ij} \neq -1,$$
$$\forall i \in I : (\theta(\alpha_i))(h_i) \neq -1.$$

Satake diagrams can be defined as the following subset of compatible decorations of $A$:

$$\mathrm{Sat}(A) = \{(X, \tau) \in \mathrm{CDec}(A) \,|\, \forall i \in I \backslash X : i = \tau(i) \implies \zeta(\alpha_i) = 1\}. \tag{2.10}$$

It is well-known that Satake diagrams classify involutive Lie algebra automorphisms up to conjugacy, see e.g. [Ara62]. More precisely, in the current setup, for $(X, \tau) \in \mathrm{Sat}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I^*}$ define $s_{\boldsymbol{\gamma}} \in \widetilde{H}$ by means of

$$s_{\boldsymbol{\gamma}}(\alpha_i) = \begin{cases} 1 & \text{if } i \in X, \\ \gamma_i & \text{if } i \in I^*, \\ \gamma_{\tau(i)}\zeta(\alpha_i) & \text{if } i \in (I\backslash X)\backslash I^*, \end{cases}$$

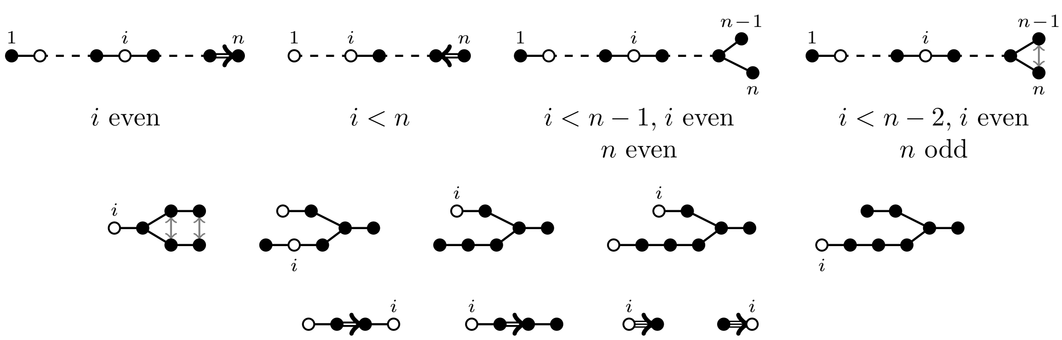
cf. [BK16, Eqs. (5.1-5.2)]. Then it follows from (2.8) that

$$\theta_{\boldsymbol{\gamma}} := \mathrm{Ad}(s_{\boldsymbol{\gamma}})\theta \tag{2.11}$$

satisfies $(\theta_{\boldsymbol{\gamma}})^2 = \mathrm{id}_{\mathfrak{g}}$.

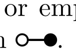
If $(X,\tau) \in \mathrm{CDec}(A)\backslash\mathrm{GSat}(A)$ then there exists a pair $(i,j) \in I\backslash X \times X$ such that the union of connected components of $X$ neighbouring $i$ is simply $\{j\}$ and $a_{ji} = -1$. Hence $\rho_X^\vee = \frac{1}{2}h_j$ so that $\zeta(\alpha_i) = (-1)^{a_{ji}} = -1$ implying $(X,\tau) \notin \mathrm{Sat}(A)$. Consequently $\mathrm{Sat}(A) \subseteq \mathrm{GSat}(A)$. The complement $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ is empty if and only if $A$ is of type $\mathrm{A}_n$. We refer the reader to the classification in [He84, Table I], which does not explicitly distinguish between elements of $\mathrm{Sat}(A)$ and $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$. It is convenient for our purposes to list the elements of $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$, which we do in Table 1.

Table 1. All elements of $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ for indecomposable Cartan matrices $A$. By a case-by-case analysis there is a unique $i \in I$ such that $\zeta(\alpha_i) = -1$; we have indicated the corresponding node in the diagrams. The classical diagrams are labelled in the usual way. For types $\mathrm{C}_n$ and $\mathrm{D}_n$ upper bounds on $i$ are imposed to avoid the cases when $\theta$ is an involution whose fixed-point subalgebra is isomorphic to $\mathfrak{gl}_n$.




Consider the real vector space $V = \mathbb{R}\Phi$. For a fixed $\theta \in \mathrm{Aut}^{\mathrm{inv}}(\Phi)$ we can decompose $V$ into the positive and negative $\theta$-eigenspaces, $V = V^\theta \oplus V^{-\theta}$. Denote by $\bar{\ }: V \to V$ the corresponding projection onto $V^{-\theta}$. The *restricted roots* are the elements of

$$\overline{\Phi} = \{\overline{\alpha} \,|\, \alpha \in \Phi\}\backslash\{0\}.$$

Given an arbitrary $\theta \in \mathrm{Aut}^{\mathrm{inv}}(\Phi)$, $\overline{\Phi}$ is not necessarily a root system in its own right. According to [He84, Thm. 6.1], $\overline{\Phi}$ is a (possibly non-reduced or empty) root system precisely if $\theta = \theta(X,\tau) = -w_X\tau$, where $(X,\tau) \in \mathrm{GSat}(A)$ or $(X,\tau)$ is the diagram .

Now consider the following groups:

$$W^\theta = \{w \in W \,|\, w = \theta w \theta\} = \{w \in W \,|\, w = w_X\tau(w)w_X\},$$
$$\overline{W} = \{w|_{V^{-\theta}} \,|\, w \in W,\, w(V^{-\theta}) \subseteq V^{-\theta}\}.$$

If $\theta = \theta(X,\tau)$ it follows straightforwardly that $W_X$ is a subgroup of $W^\theta$. Moreover, [He84, Prop. 3.1] implies that $\overline{W}$ is isomorphic to $W^\theta/W_X$. For $i \in I^*$ we define $\widetilde{r}_i := w_X w_{X[i]} \in W$ where $X[i] = X \cup \{i,\tau(i)\}$ and set $s_i \in \mathrm{GL}(V^{-\theta})$ to be the unique element satisfying $s_i(\overline{\alpha}_i) = -\overline{\alpha}_i$ and $s_i(\beta) = \beta$ for all $\beta \in V^{-\theta}$ such that $\beta(h_i) = 0$. In [He84, Lemma 3.2, Thm. 3.3, Thm. 4.4] the following result is proved.

**Theorem 2.4.** *Let $(X,\tau) \in \mathrm{CDec}(A)$. The following conditions are equivalent:*

*(i)* $(X,\tau) \in \mathrm{GSat}(A)$.
*(ii)* *For all $i \in I^*$, $s_i \in \overline{W}$.*
*(iii)* *For all $i \in I^*$, $\widetilde{r}_i$ lies in $W^\theta$ and satisfies $\widetilde{r}_i|_{V^{-\theta}} = s_i$.*
*(iv)* *For all $i \in I^*$, $\tau_{0,X[i]}$ preserves $X$.*
*(v)* $\overline{W} = W(\overline{\Phi})$.

In [Lu76, 5.9 (i)] it is shown that $(\widetilde{W}, \{\widetilde{r}_i\}_{i\in I^*})$ with $\widetilde{W} = \langle \widetilde{r}_i \rangle_{i \in I^*}$ is a Coxeter system if condition (iv) in Theorem 2.4 holds (also see [Lu02, 25.1]). If condition (iv) fails then for some $i \in I^*$, $w_{X[i]}$ and $w_X$ do not commute so that $\widetilde{r}_i^2 \neq \mathrm{id}_V$. Hence we obtain the following result.

**Corollary 2.5.** *Let $(X,\tau) \in \mathrm{CDec}(A)$. Then $(\widetilde{W}, \{\widetilde{r}_i\}_{i\in I^*})$ is a Coxeter system if and only if $(X,\tau) \in \mathrm{GSat}(A)$.*

## 3. The subalgebra $\mathfrak{k}$

For $(X,\tau) \in \mathrm{Sat}(A)$ and a suitable choice of $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I^*}$ the $\theta_{\boldsymbol{\gamma}}$-fixed subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ can be presented in terms of generators, see e.g. [Ko14, Lemma 2.8]. This motivates the following seemingly *ad hoc* definition, where we permit a more general $\boldsymbol{\gamma}$.

**Definition 3.1.** For $(X,\tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\backslash X}$ define $\mathfrak{k}_{\boldsymbol{\gamma}} = \mathfrak{k}_{\boldsymbol{\gamma}}(X,\tau)$ to be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_X$, $\mathfrak{h}^\theta$ and

$$b_{i,\boldsymbol{\gamma}} = f_i + \gamma_i \theta(f_i) \qquad \text{for all } i \in I\backslash X. \tag{3.1}$$

$\varnothing$

It is convenient to suppress the dependence on $\boldsymbol{\gamma}$ and simply write $b_i$ and $\mathfrak{k}$ if there is no cause for confusion. We denote $b_i = f_i$ if $i \in X$. Since $\mathfrak{h}_X \subseteq \mathfrak{h}^\theta$ it follows that $\mathfrak{k}$ is generated by $\mathfrak{n}_X^+ := \{e_i | i \in X\}$, $\mathfrak{h}^\theta$ and $b_i$ for $i \in I$. Owing to (2.1-2.2), these satisfy

$$[e_i, b_j] = \delta_{ij} h_i \in \mathfrak{h}^\theta \qquad \text{for all } i \in X,\, j \in I, \tag{3.2}$$

$$[h, b_j] = -\alpha_j(h) b_j \qquad \text{for all } h \in \mathfrak{h}^\theta,\, j \in I, \tag{3.3}$$

$$[h, e_j] = \alpha_j(h) e_j \qquad \text{for all } h \in \mathfrak{h}^\theta,\, j \in X, \tag{3.4}$$

$$[h, h'] = 0 \qquad \text{for all } h, h' \in \mathfrak{h}^\theta, \tag{3.5}$$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0 \qquad \text{for all } i, j \in X,\, i \neq j. \tag{3.6}$$

By setting $m = 1 - a_{ij}$ in Lemmas (A.1-A.3) one also obtains analogues of Serre relations among the generators $b_i$. Namely, for $i, j \in I$ such that $i \neq j$,

$$\mathrm{ad}(b_i)^{1-a_{ij}}(b_j) = \begin{cases} (1+\zeta(\alpha_i))\gamma_i [\theta(f_i),[f_i,f_j]] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i)+\alpha_i+\alpha_j \in \Phi^-,\, a_{ij}=-1, \\ -18\gamma_i^2 e_j & \text{if } \theta(\alpha_i)+\alpha_i+\alpha_j = 0,\, a_{ij}=-3, \\ -\gamma_i(2h_i+h_j) & \text{if } \theta(\alpha_i)+\alpha_i+\alpha_j = 0,\, a_{ij}=-1, \\ \big(\gamma_i+\zeta(\alpha_i)\gamma_j\big)[\theta(f_i),f_j] \in \mathfrak{n}_X^+ & \text{if } \theta(\alpha_i)+\alpha_j \in \Phi^-,\, a_{ij}=0, \\ \gamma_j h_i - \gamma_i h_j & \text{if } \theta(\alpha_i)+\alpha_j = 0,\, a_{ij}=0, \\ 2(\gamma_i+\gamma_j) b_i & \text{if } \theta(\alpha_i)+\alpha_j = 0,\, a_{ij}=-1, \\ -\gamma_i b_j & \text{if } \theta(\alpha_i)+\alpha_i = 0,\, j \in I\backslash X,\, a_{ij}=-1, \\ -3\gamma_i [b_i,b_j] & \text{if } \theta(\alpha_i)+\alpha_i = 0,\, j \in I\backslash X,\, a_{ij}=-2, \\ -6\gamma_i^2 b_j - 3\gamma_i [b_i,[b_i,b_j]] & \text{if } \theta(\alpha_i)+\alpha_i = 0,\, j \in I\backslash X,\, a_{ij}=-3, \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

In order to state the main result of this section, we need some more notation. Consider the subsets

$$I_{\mathrm{diff}} = \{i \in I^* \mid i \neq \tau(i) \text{ and } (\theta(\alpha_i))(h_i) \neq 0\} = \{i \in I^* \mid i \neq \tau(i) \text{ and } \exists j \in X[i] \text{ s.t. } a_{ij} < 0\}$$

and

$$\Gamma = \Gamma(X,\tau) = \{\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\backslash X} \,|\, \forall i \in I^* : \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\rm diff}\}.$$

For $\boldsymbol{i} \in I^\ell$ with $\ell \in \mathbb{Z}_{>0}$ we write $\alpha_{\boldsymbol{i}} = \sum_{r=1}^{\ell} \alpha_{i_r}$ and

$$b_{\boldsymbol{i}} = \mathrm{ad}(b_{i_1})\cdots\mathrm{ad}(b_{i_{\ell-1}})(b_{i_\ell}), \quad e_{\boldsymbol{i}} = \mathrm{ad}(e_{i_1})\cdots\mathrm{ad}(e_{i_{\ell-1}})(e_{i_\ell}), \quad f_{\boldsymbol{i}} = \mathrm{ad}(f_{i_1})\cdots\mathrm{ad}(f_{i_{\ell-1}})(f_{i_\ell}).$$

Observe that $\mathfrak{n}^- = \mathrm{Sp}\bigcup_{\ell>0}\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in I^\ell}$. Hence for all $\ell \in \mathbb{Z}_{>0}$ we can choose $\mathcal{J}_\ell \subseteq I^\ell$ such that $\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{J}_\ell}$ is a basis for $\mathrm{Sp}\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in I^\ell}$. Then $\{f_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\mathcal{J}}$ with $\mathcal{J} := \bigcup_{\ell\in\mathbb{Z}_{>0}}\mathcal{J}_\ell$ is a basis of $\mathfrak{n}^-$.

**Theorem 3.2.** *Let $(X,\tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I\backslash X}$. The following statements are equivalent:*

*(i)* $(X,\tau) \in \mathrm{GSat}(A)$ *and* $\boldsymbol{\gamma} \in \Gamma$.

*(ii) For all $i,j \in I$ such that $i \neq j$ we have the following bounded Serre relations:*

$$\mathrm{ad}(b_i)^{1-a_{ij}}(b_j) \in \mathfrak{n}^+_X \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\substack{\boldsymbol{k}\in I^\ell \\ \alpha_{\boldsymbol{k}}<\lambda_{ij}}} \mathbb{C}b_{\boldsymbol{k}} \tag{3.8}$$

*where* $\lambda_{ij} := (1-a_{ij})\alpha_i + \alpha_j \in Q^+\backslash\Phi^+$.

*(iii) We have the following identity for $\mathfrak{h}^\theta$-modules:*

$$\mathfrak{k} = \mathfrak{n}^+_X \oplus \mathfrak{h}^\theta \oplus \bigoplus_{\boldsymbol{i}\in\mathcal{J}} \mathbb{C}b_{\boldsymbol{i}}. \tag{3.9}$$

*(iv) We have*

$$\mathfrak{k} \cap \mathfrak{h} = \mathfrak{h}^\theta. \tag{3.10}$$

*Remark* 3.3. In the fixed-point case $\mathfrak{k} = \mathfrak{g}^{\theta_{\boldsymbol{\gamma}}}$ (3.10) is trivially satisfied (note that $\mathfrak{h}^\theta = \mathfrak{h}^{\theta_{\boldsymbol{\gamma}}}$). ⌀

*Proof of Theorem 3.2.*

*(i) $\iff$ (ii):* This is a direct consequence of (3.7).

*(ii) $\implies$ (iii):* Owing to (3.3-3.5) it is sufficient to prove (3.9) as an identity for vector spaces. First we prove that $\mathfrak{k} = \mathfrak{n}^+_X + \mathfrak{h}^\theta + \mathrm{Sp}\{b_{\boldsymbol{i}} | \boldsymbol{i} \in \mathcal{J}\}$. From (3.2-3.3) it follows that, as vector spaces,

$$\mathfrak{k} = \mathfrak{n}^+_X + \mathfrak{h}^\theta + \langle b_j\rangle_{j\in I} = \mathfrak{n}^+_X + \mathfrak{h}^\theta + \sum_{\ell\in\mathbb{Z}_{>0}}\sum_{\boldsymbol{i}\in I^\ell} \mathbb{C}b_{\boldsymbol{i}}. \tag{3.11}$$

As a consequence of this, we see that it suffices to prove that for all $\boldsymbol{j} \in \cup_\ell I^\ell$ we have

$$b_{\boldsymbol{j}} \in \mathfrak{n}^+_X + \mathfrak{h}^\theta + \mathrm{Sp}\{b_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathcal{J}\}. \tag{3.12}$$

We will prove this by induction with respect to the height $\ell$. Since for all $j \in I$ we have $\dim(\mathfrak{g}_{-\alpha_j}) = 1$ and hence $(j) \in \mathcal{J}$, the case $\ell = 1$ is trivial. Now fix $\ell \in \mathbb{Z}_{>1}$ and assume that (3.12) holds true for all smaller positive integers. Fix $\boldsymbol{j} \in I^\ell$ and repeatedly apply the Serre relations (2.2) to obtain that for all $\boldsymbol{i} \in \mathcal{J}_\ell$ there exist $a_{\boldsymbol{i}} \in \mathbb{C}$ such that

$$f_{\boldsymbol{j}} = \sum_{\boldsymbol{i}\in\mathcal{J}_\ell} a_{\boldsymbol{i}} f_{\boldsymbol{i}}.$$

Hence, by virtue of (ii) and equations (3.2-3.3) it follows that

$$b_{\boldsymbol{j}} - \sum_{\boldsymbol{i}\in\mathcal{J}_\ell} a_{\boldsymbol{i}} b_{\boldsymbol{i}} \in \mathfrak{n}^+_X + \mathfrak{h}^\theta + \mathrm{Sp}\left\{ b_{\boldsymbol{i}} \,\middle|\, \boldsymbol{i} \in \bigcup_{m=1}^{\ell-1} I^m \right\}.$$

Using the induction hypothesis for the elements $b_{\boldsymbol{i}}$ in the last summation one obtains (3.12).

It remains to show that the sum in (3.12) is direct. Let $\boldsymbol{j} \in \mathcal{J}$. Then $f_{\boldsymbol{j}}$ is nonzero. Because of the explicit formula (3.1) we have

$$b_{\boldsymbol{j}} - f_{\boldsymbol{j}} \in \mathfrak{n}^+_X + \mathfrak{h}^\theta + \mathbb{C}\theta(f_{\boldsymbol{j}}) + \mathrm{Sp}\{b_{\boldsymbol{i}} \,|\, \boldsymbol{i} \in \mathcal{J}, \alpha_{\boldsymbol{i}} < \alpha_{\boldsymbol{j}}\}. \tag{3.13}$$

Hence $f_{\boldsymbol{j}} = \pi_{-\alpha_{\boldsymbol{j}}}(b_{\boldsymbol{j}})$ for all $\boldsymbol{j} \in \mathcal{J}$, where $\pi_\alpha$ is the projection on $\mathfrak{g}_\alpha$ for $\alpha \in \Phi$, see (2.3). Thus the linear independence of $\{f_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathcal{J}}$ together with (2.3) implies that the sum is direct.

*(iii) $\implies$ (iv):* By definition, $\mathfrak{h}^\theta \subseteq \mathfrak{k} \cap \mathfrak{h}$ so it suffices to show that $\mathfrak{k} \cap \mathfrak{h} \subseteq \mathfrak{h}^\theta$. Suppose $h \in \mathfrak{k} \cap \mathfrak{h}^\theta$. Since $\pi_{-\alpha_{\boldsymbol{j}}}(b_{\boldsymbol{j}}) = f_{\boldsymbol{j}}$ and the triangular decomposition (2.3), part (iii) implies $h \in \mathfrak{n}_X^+ \oplus \mathfrak{h}^\theta \subseteq \mathfrak{g}^\theta$ so $h \in \mathfrak{h}^\theta$.

*(iv) $\implies$ (ii):* We prove the contrapositive. If (3.8) fails then (3.14) and (3.7) imply that either $\gamma_j h_i - \gamma_i h_j \in \mathfrak{k} \cap (\mathfrak{h}\backslash\mathfrak{h}^\theta)$ with $\gamma_i \neq \gamma_j$ or $2h_i + h_j \in \mathfrak{k} \cap (\mathfrak{h}\backslash\mathfrak{h}^\theta)$. In either case (3.10) fails. ☐

It is convenient to have an explicit description of $\mathfrak{h}^\theta$. Given $i \in I$, by applying $\theta$ to $\theta(h_i) - h_i - \theta(h_{\tau(i)}) + h_{\tau(i)} \in \mathfrak{g}_X \cap \mathfrak{h}$ one obtains $\theta(h_i - h_{\tau(i)}) = h_i - h_{\tau(i)}$. From this we straightforwardly deduce

$$\mathfrak{h}^\theta = \bigoplus_{i\in X} \mathbb{C}h_i \oplus \bigoplus_{\substack{i\in I^* \\ i\neq\tau(i)}} \mathbb{C}(h_i - h_{\tau(i)}) \tag{3.14}$$

We denote $\Phi_X = \Phi \cap Q_X$ and note that $|\mathcal{J}| = |\Phi|/2$; from (3.14) we also obtain $\dim(\mathfrak{h}^\theta) = |I| - |I^*|$. Hence, given $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$, Theorem 3.2 (iii) implies

$$\dim(\mathfrak{k}) = |\Phi_X|/2 + |I| - |I^*| + |\Phi|/2. \tag{3.15}$$

**Corollary 3.4.** *Let $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. The generating set*

$$\{h_i, e_i\}_{i\in X} \cup \{h_i - h_{\tau(i)}\}_{i\in I^*, i\neq\tau(i)} \cup \{b_i\}_{i\in I},$$

*and the relations (3.2-3.6) provide a presentation of $\mathfrak{k}$.*

*Proof.* There are no relations for the $b_i$ other than (3.2), (3.3) and (3.7): otherwise applying $\pi_{-\alpha}$ with $\alpha \in \Phi^+$ maximal produces a relation for the $f_i$ inequivalent to a relation (2.1), (2.2). ☐

3.1. **Ideal structure of $\mathfrak{k}$.** In this section we assume that $A$ is indecomposable, so that $\mathfrak{g}$ is simple. In order to describe the derived subalgebra of $\mathfrak{k}$ recall the set $I_{\mathrm{diff}} \in I^*$ and define

$$\begin{aligned} I_{\mathrm{ns}} &= \{i \in I \,|\, (\theta(\alpha_i))(h_i) = -2\} = \{i \in I \,|\, i = \tau(i),\, \check{X}(i) = \emptyset\}, \\ I_{\mathrm{nsf}} &= \{j \in I_{\mathrm{ns}} \,|\, \forall i \in I_{\mathrm{ns}}\, a_{ij} \in 2\mathbb{Z}\}. \end{aligned} \tag{3.16}$$

**Proposition 3.5.** *Let $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. As vector spaces we have*

$$\mathfrak{k} = \mathfrak{k}' \oplus \bigoplus_{i\in I_{\mathrm{diff}}} \mathbb{C}(h_i - h_{\tau(i)}) \oplus \bigoplus_{i\in I_{\mathrm{nsf}}} \mathbb{C}b_i.$$

*Proof.* Fix $(X,\tau) \in \mathrm{GSat}(A)$. Note that neither $h_i - h_{\tau(i)}$ ($i \in I_{\mathrm{diff}}$) nor $b_j$ ($j \in I_{\mathrm{nsf}}$) is a linear combination of Lie brackets in $\mathfrak{k}$. This follows from Corollary 3.4 and (3.2-3.7): these elements do not appear as in the expressions for Lie brackets in the defining relations of $\mathfrak{k}$.

It now suffices to show that the remaining basis elements specified in (3.9) are linear combinations of Lie brackets in $\mathfrak{k}$, for which we argue as follows.

*$b_{\boldsymbol{i}}$ with $\boldsymbol{i} \in \mathcal{J}_\ell$, $\ell > 1$:* This holds by definition.

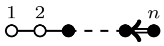
*$e_i$, $f_i$, $h_i$ with $i \in X$:* This follows from (3.2-3.4).

*$h_i - h_{\tau(i)}$ with $i \in I^*\backslash I_{\mathrm{diff}}$ and $i \neq \tau(i)$:* The given condition is equivalent to $w_X(\alpha_i) = \alpha_i$ and $a_{i\,\tau(i)} = 0$. Hence (3.7) implies that $h_i - h_{\tau(i)} = \gamma_i^{-1}[b_i, b_{\tau(i)}]$.

*$b_j$ with $\check{X}(j) \neq \emptyset$:* There exists $i \in X$ such that $a_{ij} \neq 0$. By (3.3) we have $b_j = -a_{ij}^{-1}[h_i, b_j]$.

*$b_j$ with $j \neq \tau(j)$:* Note that $a_{\tau(j)\,j} \leq 0$. By (3.3) we have $b_j = (a_{\tau(j)\,j} - 2)^{-1}[h_j - h_{\tau(j)}, b_j]$.

*$b_j$ with $j \in I_{\mathrm{ns}}\backslash I_{\mathrm{nsf}}$:* By definition of $I_{\mathrm{nsf}}$ there exists $i \in I_{\mathrm{ns}}$ such that $a_{ij} \in \{-1,-3\}$. According to (3.7), $b_j = -\gamma_i^{-1}\mathrm{ad}(b_i)^2(b_j)$ if $a_{ij} = -1$ and $b_j = -(2\gamma_i)^{-1}\mathrm{ad}(b_i)^2(b_j) - (6\gamma_i^2)^{-1}\mathrm{ad}(b_i)^4(b_j)$ if $a_{ij} = -3$; in either case $b_j \in \mathfrak{k}'$. ☐

It follows that the codimension of $\mathfrak{k}'$ in $\mathfrak{k}$ equals $|I_{\rm diff}| + |I_{\rm nsf}|$. For $(X,\tau) \in \mathrm{Sat}(A)$, in [Let02, Sec. 7, Variation 1] it was noted that $|I_{\rm diff}| \le 1$ if $A$ is of finite type. In light of the above it is natural to generalize this in two directions: also involve the set $I_{\rm nsf}$ and allow $(X,\tau) \in \mathrm{GSat}(A)$. It turns out the same upper bound holds true and there are generalized Satake diagrams with $|I_{\rm diff}| + |I_{\rm nsf}| = 1$ unless $A$ is of type $\mathrm{E}_8$, $\mathrm{F}_4$ or $\mathrm{G}_2$. From Table 1 it follows that the only elements of $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ for which $|I_{\rm diff}| + |I_{\rm nsf}| = 1$ are of the form  with $n > 2$ in which case $I_{\rm nsf} = \{1\}$ and $\zeta(\alpha_2) = -1$.

For the reasons that will become clear a bit later we introduce a further refinement of generalized Satake diagrams. In particular, we define the set of *weak Satake diagrams* by

$$\mathrm{WSat}(A) = \{(X,\tau) \in \mathrm{GSat}(A)\backslash\mathrm{Sat}(A) \,|\, (X,\tau) \text{ contains no minimal subdiagram of the form } \}.$$

As mentioned in Table 1, for elements of $\mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ a case-by-case analysis yields that there can be at most one $i \in I\backslash X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. For $(X,\tau) \in \mathrm{WSat}(A)$ we will obtain a semidirect product decomposition in terms of a reductive Lie subalgebra and a nilpotent ideal in which this unique $i \in i\backslash X$ plays an important role.

For any $r \in \mathbb{Z}_{\ge 0}$ and any $i \in I$ denote by $\mathfrak{k}(i)_r$ the span of all $b_{\boldsymbol{j}}$ such that the coefficient of $\alpha_i$ in $\alpha_{\boldsymbol{j}}$ is precisely $r$. We then have the following decomposition

$$\langle b_i \rangle_{i \in I} = \bigoplus_{r=0}^{\infty} \mathfrak{k}(i)_r.$$

Consider the subspace

$$\mathfrak{k}(i) := \bigoplus_{r=1}^{\infty} \mathfrak{k}(i)_r$$

and the subalgebras

$$\mathfrak{k}_{\hat{i}} := \langle \mathfrak{n}_X^+, \mathfrak{h}^\theta, \{b_j\}_{j \in I\backslash\{i\}} \rangle \subseteq \mathfrak{k}, \qquad \mathfrak{g}_{\hat{i}} := \langle \{e_j, f_j, h_j\}_{j \in I\backslash\{i\}} \rangle \subset \mathfrak{g}.$$

Note that $\mathfrak{k} = \mathfrak{k}_{\hat{i}} + \mathfrak{k}(i)$ (not necessarily a direct sum, since e.g. $b_i$ may lie in $\mathfrak{k}_{\hat{i}}$).

**Proposition 3.6.** *Let $(X,\tau) \in \mathrm{WSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. Denote by $i$ the unique element of $I\backslash X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. Then $\mathfrak{k}(i)_r = \{0\}$ if $r > 2$ and we have the lower central series*

$$\mathfrak{k}(i) = \mathfrak{k}(i)_1 \oplus \mathfrak{k}(i)_2 \supset \mathfrak{k}(i)_2 \supset \{0\}$$

*so that $\mathfrak{k}(i)$ is nilpotent of class 2. Furthermore, both $\mathfrak{k}(i)_1$ and $\mathfrak{k}(i)_2$ are $\mathfrak{k}_{\hat{i}}$-modules under the adjoint action, $\mathfrak{k}(i)$ is an ideal of $\mathfrak{k}$, $\mathfrak{k}_{\hat{i}}$ is the fixed-point subalgebra of $\theta|_{\mathfrak{g}_{\hat{i}}}$ and we have $\mathfrak{k} = \mathfrak{k}_{\hat{i}} \ltimes \mathfrak{k}(i)$.*

*Proof.* Note that (3.7) implies, for all $j \in I\backslash\{i\}$, that

$$\mathrm{ad}(b_i)^{1-a_{ij}}(b_j) = 0 \tag{3.17}$$

$$\mathrm{ad}(b_j)^{1-a_{ji}}(b_i) \in \sum_{r=1}^{-a_{ij}} \mathbb{F}\mathrm{ad}(b_j)^r(b_i) \subseteq \mathfrak{k}(i)_1. \tag{3.18}$$

Since (3.3) and (3.18) are the only relations in $\mathfrak{k}$ with $b_i$ appearing on the right-hand side, it follows that $\mathfrak{k}_{\hat{i}} = \langle \mathfrak{n}_+^X, \mathfrak{h}^\theta, \mathfrak{k}(i)_0 \rangle$ and $\mathfrak{k} = \mathfrak{k}_{\hat{i}} \oplus \mathfrak{k}(i)$ (as vector spaces). Deleting the node $i$ from any diagram in Table 1 one obtains a (possibly disconnected) Satake diagram such that $\theta|_{\mathfrak{g}_{\hat{i}}}$ by virtue of (2.8) is an involution. From Table 1 it also follows that $I^* = I\backslash X$ so that $\mathfrak{k}_{\hat{i}}$ is the fixed-point subalgebra of $\mathfrak{g}_{\hat{i}}$ for the involution $\theta_{\boldsymbol{\gamma}}$, see (2.11).

Combined with (3.2-3.3), (3.18) implies that each summand $\mathfrak{k}(i)_r$ is a $\mathfrak{k}_{\hat{i}}$-module. Hence $\mathfrak{k}(i)$ is a $\mathfrak{k}_{\hat{i}}$-module and by virtue of (3.17) it is a subalgebra of $\mathfrak{k}$. It follows that $\mathfrak{k}(i)$ is an ideal. Automatically we have that $[\oplus_{r=1}^s \mathfrak{k}(i)_r, \mathfrak{k}(i)_1] \subseteq \oplus_{r=1}^{s+1} \mathfrak{k}(i)_r$ for all $s \in \mathbb{Z}_{\ge 1}$. A case-by-case analysis using Table 1 yields that the coefficient in front of $\alpha_i$ in the highest root of $\Phi$ is always 2. This implies $\mathfrak{k}(i)_3 = 0$ so that $\mathfrak{k}(i)_2$ is the centre of $\mathfrak{k}(i)$ and we obtain the indicated lower central series. $\square$

Regarding the centre $\mathfrak{z}$ of $\mathfrak{k}$ for $(X,\tau) \in \mathrm{WSat}(A)$, recall the notation $i$ for the unique element of $I\backslash X$ such that $i = \tau(i)$ and $\zeta(\alpha_i) = -1$. Since the centre of the ideal $\mathfrak{k}(i)$ is $\mathfrak{k}(i)_2$, we must have $\mathfrak{z} \subseteq \mathfrak{k}(i)_2$. Define

$$\mathcal{J}_{\mathrm{even}} := \{\boldsymbol{j} \in \mathcal{J} \,|\, \forall k \in I\backslash X \text{ the coefficient of } \alpha_{\boldsymbol{j}} \text{ in front of } \alpha_k \text{ is even}\}$$

so that

$$\mathfrak{k}(i)_{2,\mathrm{even}} := \bigoplus_{\boldsymbol{j} \in \mathcal{J}_{\mathrm{even}}} \mathbb{C} b_{\boldsymbol{j}} \subset \mathfrak{k}(i)_2.$$

We claim without proof that $\mathfrak{z}$ is generated by a single element of $\mathfrak{k}(i)_{2,\mathrm{even}}$.

Let us now explain the motivation behind the definition of the set $\mathrm{WSat}(A)$. Consider the excluded generalized Satake diagram ●⇛○. By definition, $\mathfrak{k}$ is the subalgebra of $\mathfrak{g} = \mathrm{Lie}(G_2)$ generated by $e_1, h_1, b_1 = f_1$ and $b_2 = f_2 + \gamma_2 \theta(f_2)$ for some $\gamma_2 \in \mathbb{C}^\times$. The relations (3.2-3.7) specialize to

$$[e_1, b_1] = h_1, \qquad [e_1, b_2] = 0, \qquad [h_1, b_1] = -2b_1, \qquad [h_1, b_2] = b_2, \qquad [h_1, e_1] = 2e_1,$$
$$[b_1, [b_1, b_2]] = 0, \qquad [b_2, [b_2, [b_2, [b_2, b_1]]]] = -18\gamma_2^2 e_1.$$

According to (3.15) we have $\dim(\mathfrak{k}) = 8$. A natural basis is given by

$$e_1, \qquad b_1, \qquad h_1, \qquad b_2, \qquad b_{(2,1)}, \qquad b_{(2,2,1)}, \qquad b_{(2,2,2,1)}, \qquad b_{(1,2,2,2,1)}.$$

Using the adjoint action of $e_1$, $b_1$ and $b_2$ on $\mathfrak{k}$ it is easy to verify that an ideal of $\mathfrak{k}$ equals $\mathfrak{k}$ if it contains any of the generators listed above. This together with some straightforward computations shows that $\mathfrak{k}$ is in fact a simple Lie algebra. Since $\dim(\mathfrak{k}) = 8$, it is must be isomorphic to $\mathfrak{sl}_3$. On the other hand, if $(X,\tau) \in \mathrm{WSat}(A)$, since $\mathfrak{k}$ has a nonzero nilpotent ideal by Proposition 3.6, $\mathfrak{k}$ is not a reductive Lie algebra.

**Proposition 3.7.** *Let $(X,\tau) \in \mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. Then $\mathfrak{k}$ is not the fixed-point subalgebra of any automorphism of $\mathfrak{g}$.*

*Proof.* We first show this for the case when $(X,\tau)$ is ●⇛○. Suppose there exists $\phi \in \mathrm{Aut}(\mathfrak{g})$ such that $\mathfrak{k} = \mathfrak{g}^\phi$. From $[h_2, b_1] = 3b_1$ and $[h_2, e_1] = -3e_1$ one establishes straightforwardly that $\phi(h_2) \in \mathfrak{h}$ and hence that $\phi(h_2) = \frac{3}{2}(m-1)h_1 + mh_2$ for some $m \in \mathbb{C}$. Next, from $\theta(f_2) = e_{(2,1)}$ it follows that $[h_2, b_2] = -f_2 - b_2$; hence $\phi(f_2) = mf_2 + \frac{1}{2}(1-m)b_2$. Combining this with $[f_2, b_2] = 3e_1$ one obtains $m = 1$. But this means that $h_2$ and $f_2$ are also fixed points of $\phi$, contrary to assumption. Hence such $\phi$ does not exist. Now let $(X,\tau) \in \mathrm{WSat}(A)$. In this case $\mathfrak{k}$ is not a reductive Lie algebra and [Ja62, Thm. 1] implies that $\mathfrak{k}$ cannot be the fixed-point subalgebra of any automorphism of $\mathfrak{g}$. □

Nevertheless, in Section 4 we will show that for all $(X,\tau) \in \mathrm{GSat}(A)$, the subalgebra $\mathfrak{k}$ can be quantized resulting in a coideal subalgebra possessing a universal $K$-matrix.

## 3.2. **The universal enveloping algebra $U(\mathfrak{k})$.**

Let $(X,\tau) \in \mathrm{GSat}(A)$ and $\boldsymbol{\gamma} \in \Gamma$. We identify $\mathfrak{k}$ with its image in $U(\mathfrak{k})$ under the canonical Lie algebra embedding. The generators of $U(\mathfrak{k})$ corresponding to $b_i$ ($i \in I\backslash X$) can be modified by scalar terms, which is a straightforward generalization of [Ko14, Cor. 2.9].

**Proposition 3.8.** *For $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in \Gamma$ and $\boldsymbol{s} \in \mathbb{C}^{I\backslash X}$, the universal enveloping algebra $U(\mathfrak{k}_{\boldsymbol{\gamma}})_{\boldsymbol{s}}$ is generated by $e_i, f_i$ ($i \in X$), $h \in \mathfrak{h}^\theta$ and*

$$b_{i;\boldsymbol{\gamma},\boldsymbol{s}} = f_i + \gamma_i \theta(f_i) + s_i \qquad \text{for all } i \in I\backslash X. \tag{3.19}$$

Again, if there is no cause for confusion, we will suppress $\boldsymbol{\gamma}$ and $\boldsymbol{s}$ from the notation. Because of Corollary 3.4 we immediately obtain the following result, which addresses [Ko14, Rmk. 2.10].

**Proposition 3.9.** *For $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in \Gamma$ and $\boldsymbol{s} \in \mathbb{C}^{I\backslash X}$, the defining relations of the universal enveloping algebra $U(\mathfrak{k})$ are given by (3.2-3.6), with the Lie bracket interpreted as commutator.*

We may view $U(\mathfrak{k})$ as a Hopf subalgebra of $U(\mathfrak{g})$ so that Lie algebra automorphisms of $\mathfrak{g}$ lift to Hopf algebra automorphisms of $U(\mathfrak{g})$. Call two Hopf subalgebras $B, B'$ of $U(\mathfrak{g})$ equivalent if there exists $\phi \in \mathrm{Aut}_{\mathrm{Hopf}}(U(\mathfrak{g}))$ such that $B' = \phi(B)$. Define

$$\widetilde{\Gamma} := \{\boldsymbol{\gamma} \in \Gamma \mid \gamma_i = 1 \text{ unless } i \in I_{\mathrm{diff}}\}, \\ \mathcal{S} := \{\boldsymbol{s} \in \mathbb{C}^{I\backslash X} \mid s_i = 0 \text{ unless } i \in I_{\mathrm{nsf}}\}. \tag{3.20}$$

**Proposition 3.10.** *Let $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in \Gamma$ and $\boldsymbol{s} \in \mathbb{C}^{I\backslash X}$. There exist $\widetilde{\boldsymbol{\gamma}} \in \widetilde{\Gamma}$ and $\boldsymbol{s}' \in \mathcal{S}$ such that $U(\mathfrak{k}_{\boldsymbol{\gamma}})_{\boldsymbol{s}}$ is equivalent to $U(\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})_{\boldsymbol{s}'}$.*

*Proof.* The existence of $\widetilde{\boldsymbol{\gamma}}$ can be proven in an argument entirely analogous to the proof of [Ko14, Prop. 9.2 (i)]. It follows that $U(\mathfrak{k}_{\boldsymbol{\gamma}})_{\boldsymbol{s}}$ is equivalent to $U(\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})_{\widetilde{\boldsymbol{s}}}$ for some $\widetilde{\boldsymbol{s}} \in \mathbb{C}^{I\backslash X}$.

Regarding the existence of $\boldsymbol{s}' \in \mathcal{S}$, note that $b_{i,\widetilde{\boldsymbol{\gamma}}} \in (\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})'$ unless $i \in I_{\mathrm{nsf}}$ owing to Prop. 3.5. Hence $U(\mathfrak{k}_{\widetilde{\boldsymbol{\gamma}}})_{\widetilde{\boldsymbol{s}}}$ is already generated by $e_i, f_i$ $(i \in X)$, $h \in \mathfrak{h}^\theta$, $b_{i;\widetilde{\boldsymbol{\gamma}},0}$ for $i \in (I\backslash X)\backslash I_{\mathrm{nsf}}$ and $b_{i;\widetilde{\boldsymbol{\gamma}},\widetilde{\boldsymbol{s}}}$ for $i \in I_{\mathrm{nsf}}$. Hence we may take $s'_i = \widetilde{s}_i$ if $i \in I_{\mathrm{nsf}}$ and $s'_i = 0$ otherwise. □

## 4. The universal $K$-matrix revisited

Assume the $d_i$ are dyadic rationals and let $\mathbb{K}$ be a quadratic closure of $\mathbb{C}(q)$ where $q$ is an indeterminate, so that $q_i := q^{d_i} \in \mathbb{K}$ for all $i \in I$. The Drinfeld-Jimbo quantum group $U_q = U_q(\mathfrak{g})$ is an associative unital algebra over $\mathbb{K}$ which quantizes the universal enveloping algebra $U(\mathfrak{g})$. It is generated by $\{E_i, F_i, t_i^{\pm 1}\}$ where $i \in I$, satisfying the relations given in e.g. [Lu94, 3.1.1]. It is a Hopf algebra whose structure is defined by the choice of the coproduct:

$$\Delta(E_i) = E_i \otimes 1 + t_i \otimes E_i, \qquad \Delta(F_i) = F_i \otimes t_i^{-1} + 1 \otimes F_i, \qquad \Delta(t_i) = t_i \otimes t_i.$$

For $\alpha = \sum_i n_i \alpha_i \in Q$ with $n_i \in \mathbb{Z}$ we write $t_\alpha = \prod_{i \in I} t_i^{n_i}$. The Hopf subalgebra $U_q^0 = U_q(\mathfrak{h})$ is the subalgebra generated by $t_i^{\pm 1}$ for $i \in I$ and spanned by $\{t_\alpha\}_{\alpha \in Q}$. In terms of the quantum root spaces

$$(U_q)_\alpha = \{u \in U_q \,|\, \forall i \in I \; t_i u t_i^{-1} = q_i^{\alpha(h_i)} u\}$$

where $\alpha \in Q$, we have the $Q$-grading

$$U_q = \bigoplus_{\alpha \in Q} (U_q)_\alpha, \qquad (U_q)_\alpha (U_q)_\beta \subseteq (U_q)_{\alpha+\beta}. \tag{4.1}$$

According to [Tw92, Thm. 2.1] we have $\mathrm{Aut}_{\mathrm{Hopf}}(U_q) = \mathrm{Ad}(\widetilde{H}) \rtimes \mathrm{Aut}(A)$ with $\mathrm{Ad}(\chi)$ for $\chi \in \widetilde{H}$ acting on the root space $(U_q)_\alpha$ for $\alpha \in Q$ by multiplication by $\chi(\alpha)$, and $\mathrm{Aut}(A)$ acting by relabelling. Other relevant algebra automorphisms are Lusztig's automorphisms $T_i$ for $i \in I$ given as $T''_{i,1}$ in [Lu94, 37.1.3] which define a braid group action on $U_q$ restricting to the Weyl group action on $U_q^0$: $T_i(t_\alpha) = t_{r_i(\alpha)}$ for $i \in I$ and $\alpha \in Q$. For $X \subseteq I$ with $w_X = r_{i_1} \cdots r_{i_\ell}$ a reduced decomposition we define $T_X = T_{i_1} \cdots T_{i_\ell}$. Also, we define a quantum analogue of the Chevalley involution by

$$\omega_q(E_i) = -t_i^{-1} F_i, \qquad \omega_q(F_i) = -E_i t_i, \qquad \omega_q(t_i^{\pm 1}) = t_i^{\mp 1} \tag{4.2}$$

for $i \in I$. Then $\omega_q$ commutes with $\mathrm{Aut}(A)$ and with $T_i$ for $i \in I$, see [BK16, Lemma 7.1]. Assuming $\tau(X) = X$, one straightforwardly checks that $\tau$ commutes with $T_X$.

4.1. **Quantum pair algebras.** We will follow the approach of the papers [Ko14, BK15, BK16] and simply highlight where a definition or formula needs to be changed. The quantum analogon of the map $\theta = \mathrm{Ad}(w_X)\tau\omega$ is the map

$$\theta_q = \theta_q(X,\tau) = T_X \tau \omega_q \in \mathrm{Aut}_{\mathrm{alg}}(U_q).$$

Note the absence of the factor $\mathrm{Ad}(s)$, cf. [Ko14, Def. 4.3] or [BK16, Def. 5.4 and Eqn. (5.4)], which was present in *ibid.* to guarantee that $\theta_q$ specializes to the appropriate Lie algebra involution, see [Ko14, Prop. 10.2]. Similar to (3.14) it follows that $U_q(\mathfrak{h})^{\theta_q}$ consists of polynomials in $t_i^{\pm 1}$ $(i \in X)$ and $(t_i t_{\tau(i)}^{-1})^{\pm 1}$ $(i \in I^*, i \neq \tau(i))$. It is equal to the subalgebra denoted ${U_\Theta^0}'$ in [Ko14].

The quantization of the fixed-point subalgebra in the formalism by [Ko14] relies on the presentation of $\mathfrak{g}^{\theta_\gamma}$ in terms of generators given in [Ko14, Lemma 2.8]. Our $\mathfrak{k}(X,\tau)$ with $(X,\tau) \in \mathrm{GSat}(A)$ by definition can be quantized to a right coideal subalgebra in the same way.

**Definition 4.1.** Let $(X,\tau) \in \mathrm{GSat}(A)$, $\boldsymbol{\gamma} \in (\mathbb{K}^\times)^{I\backslash X}$ and $\boldsymbol{s} \in \mathbb{K}^{I\backslash X}$. Then $B = B_{\boldsymbol{\gamma},\boldsymbol{s}}(X,\tau)$ is the coideal subalgebra generated by $U_q(\mathfrak{g}_X)$, $U_q(\mathfrak{h})^{\theta_q}$ and the elements

$$B_i = B_{i;\boldsymbol{\gamma},\boldsymbol{s}} = F_i + \gamma_i \theta_q(F_i t_i) t_i^{-1} + s_i t_i^{-1} \qquad \text{for all } i \in I\backslash X. \qquad \varnothing$$

To make a direct match between the Kolb-Balagović formalism based on fixed-point subalgebras and our more general approach one should set, for all $i \in I\backslash X$,

$$\gamma_i = s(\alpha_{\tau(i)}) c_i,$$

see also [BK16, Eqn. (7.7)]. If the tuples $\boldsymbol{\gamma}$, $\boldsymbol{s}$ lie in the sets

$$\begin{aligned} \Gamma_q &= \{\boldsymbol{\gamma} \in (\mathbb{K}^\times)^{I\backslash X} \mid \forall i \in I^* \, \gamma_i \neq \gamma_{\tau(i)} \implies i \in I_{\mathrm{diff}}\}, \\ \mathcal{S}_q &= \{\boldsymbol{s} \in \mathbb{K}^{I\backslash X} \mid s_i = 0 \text{ unless } i \in I_{\mathrm{nsf}}\} \end{aligned} \tag{4.3}$$

respectively, then according to [Ko14, Sec. 5.3 and Sec. 6] one obtains decompositions of $B$ yielding the quantum analogue of (3.10), namely $B \cap U_q(\mathfrak{h}) = U_q(\mathfrak{h})^{\theta_q}$. The key condition for Satake diagrams, see (2.10), is only used in [Ko14, Proof of Lemma 5.11, Step 1], but it is clear that what is needed is precisely the weaker condition appearing in the definition of a generalized Satake diagram, see Definition 2.2. The rest of [Ko14] is applicable without change in the setting of generalized Satake diagrams; in particular in the specialization $(q \to 1)$ one recovers $U(\mathfrak{k})$, see [Ko14, Sec. 10].

In [BK15] the bar involutions for $U_q$ and $B$ are studied, following earlier work by [ES13] and [BW13] in the case of quantum symmetric pairs of $\mathfrak{gl}_N$ type. The proof of [BK15, Prop. 2.3] relies on a case-by-case analysis of Satake diagrams of finite type from Araki's work [Ara62]. We claim here without proof that a similar analysis using Table 1 yields the same result for all generalized Satake diagrams, in other words that [BK15, Prop. 2.5] holds with $\nu_i = 1$ for all $i \in I\backslash X$ (otherwise $\nu_i = -1$). In the remainder of [BK15] the defining condition of Satake diagrams or a case-by-case analysis is not used so that these results remain valid.

The universal $K$-matrix for the algebra $B$ is constructed in [BK16] in the case $(X,\tau) \in \mathrm{Sat}(A)$. We restate some key conditions in terms of the parameters $\boldsymbol{\gamma}$. Assuming $\nu_i = 1$ for all $i \in I\backslash X$, condition [BK16, Eqn. (5.17)] is equivalent to

$$\gamma_{\tau(i)} = \zeta(\alpha_i) q_i^{(\theta(\alpha_i) - 2\rho_X)(h_i)} \overline{\gamma_i},$$

where $\rho_X$ is the Weyl vector of $\mathfrak{g}_X$ and $\overline{\cdot}$ denotes the bar involution of $U_q$, which by definition fixes $E_i, F_i$ and inverts $t_i^{\pm 1}$ and $q$. In [BK16, Proof of Lemma 6.4] the defining condition of Satake diagrams is used, but as before the defining condition of generalized Satake diagrams is what is needed. Then [BK16, Eqn. (7.14)] needs to be replaced by

$$\overline{T_{w_X}(E_{\tau(i)})} = \zeta(\alpha_i) q_i^{-2\rho_X(h_i)} T_{w_X}^{-1}(E_{\tau(i)})$$

so that the scalar $\rho_i$ appearing in [BK16, Lemma 9.3] equals $q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)}$ since [BK16, Eqn. (9.8)] is equivalent to

$$\overline{\gamma_i T_{w_X}(E_{\tau(i)})} = q_i^{-\theta(\alpha_i)(h_i)} \gamma_{\tau(i)} T_{w_X}^{-1}(E_{\tau(i)}).$$

Finally, we highlight the paper [DK18] which establishes an elegant factorization property of the quasi $K$-matrix in terms of the restricted Weyl group of $\mathfrak{g}$. Sections 2.2 and 2.3 in *ibid.* entail an analysis of the restricted Weyl group and restricted root system following [Lu76]. For completeness, in reference to a comment in [DK18, between Eqs. (2.9) and (2.10)] we remark that also for all $(X,\tau) \in \mathrm{GSat}(A)\backslash\mathrm{Sat}(A)$ the set $X$ is invariant under the diagram automorphism $\tau_0 = \tau_{I,0}$ corresponding to the longest element of $W$; this follows from Table 1. The upshot of this in [DK18] is that $\tau_{0,X[i]}$ stabilizes $X$ (for all $i \in I^*$). This is used to derive that the $\widetilde{r}_i = w_X w_{X[i]}$ form a Coxeter system for the group they generate. Alternatively, this result follows from Corollary 2.5 for all generalized Satake diagrams.

## A. Deriving modified Serre relations for $\mathfrak{k}$

The following three technical lemmas are used to derive the key equation (3.7). It is convenient to introduce the notation $Q_X = \sum_{i \in X} \mathbb{Z}\alpha_i$ and $Q_X^+ := Q^+ \cap Q_X$.

**Lemma A.1.** *Let $(X, \tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I \backslash X}$. For all $i \in X$, $j \in I$ and $m \in \mathbb{Z}_{\geq 1}$ we have*

$$\mathrm{ad}(b_i)^m(b_j) = \begin{cases} \mathrm{ad}(f_i)^m(f_j) + \gamma_j\, \theta\left(\mathrm{ad}(f_i)^m(f_j)\right) & \text{if } j \in I\backslash X, \\ \mathrm{ad}(f_i)^m(f_j) & \text{if } j \in X. \end{cases}$$

*Proof.* This follows immediately from (2.7) and the fact that $\theta$ is a Lie algebra automorphism. □

**Lemma A.2.** *Let $(X, \tau) \in \mathrm{CDec}(A)$ and $\boldsymbol{\gamma} \in (\mathbb{C}^\times)^{I \backslash X}$. For all $i \in I\backslash X$, $j \in X$ and $m \in \mathbb{Z}_{\geq 1}$ we have*

$$\mathrm{ad}(b_i)^m(b_j) = \mathrm{ad}(f_i)^m(f_j) + \gamma_i^m\, \theta\left(\mathrm{ad}(f_i)^m(f_j)\right) + \mathrm{LO}_{ij}(m)$$

*where*

$$\mathrm{LO}_{ij}(m) = \begin{cases} (1+\zeta(\alpha_i))\gamma_i\,[\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+ & \text{if } \tau(i) = i,\, w_X(\alpha_i) - \alpha_i - \alpha_j \in \Phi^+,\, m = 2, \\ -\gamma_i\,(2h_i - a_{ij}h_j) & \text{if } \tau(i) = i,\, w_X(\alpha_i) = \alpha_i + \alpha_j,\, m = 2, \\ -3\,(2 + a_{ij})\,\gamma_i\,(f_i - \theta(f_i)) & \text{if } \tau(i) = i,\, w_X(\alpha_i) = \alpha_i + \alpha_j,\, m = 3, \\ -6\,a_{ij}(2 + a_{ij})\,\gamma_i^2\, e_j & \text{if } \tau(i) = i,\, w_X(\alpha_i) = \alpha_i + \alpha_j,\, m = 4, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By induction with respect to $m$. For $m = 1$, (2.7) implies

$$\mathrm{ad}(b_i)^1(b_j) = [f_i + \gamma_i\,\theta(f_i), f_j] = \mathrm{ad}(f_i)^1(f_j) + \gamma_i^1\,\theta\left(\mathrm{ad}(f_i)(f_j)\right) + \mathrm{LO}_{ij}(1)$$

with $\mathrm{LO}_{ij}(1) = 0$ as required. Now assume $m \in \mathbb{Z}_{>1}$ and suppose the statement holds for all smaller values. Then, by virtue of the induction hypothesis, the fact that $\theta$ is a Lie algebra automorphism and (2.7), we find

$$\begin{aligned} \mathrm{ad}(b_i)^m(b_j) &= \left[b_i, \mathrm{ad}(b_i)^{m-1}(b_j)\right] \\ &= \left[f_i + \gamma_i\,\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j) + \gamma_i^{m-1}\theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right) + \mathrm{LO}_{ij}(m-1)\right] \\ &= \mathrm{ad}(f_i)^m(f_j) + \gamma_i^m\theta\left(\mathrm{ad}(f_i)^m(f_j)\right) \\ &\quad + \gamma_i\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] + \gamma_i^{m-1}\left[f_i, \theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right)\right] + \left[b_i, \mathrm{LO}_{ij}(m-1)\right]. \end{aligned}$$

Using (2.8) we have $\theta^2(f_i) = \zeta(\alpha_i)f_i$ so that

$$\mathrm{LO}_{ij}(m) = \gamma_i\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] + \zeta(\alpha_i)\gamma_i^{m-1}\theta\left(\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right]\right) + [b_i, \mathrm{LO}_{ij}(m-1)]. \tag{A.1}$$

Suppose that $\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] \neq 0$. Then $w_X(\alpha_{\tau(i)}) - (m-1)\,\alpha_i - \alpha_j \in \Phi \cup \{0\}$. Now $\Phi = \Phi^+ \cup \Phi^-$ implies that $\tau(i) = i$ and $j \in \check{X}(i)$.

If $w_X(\alpha_{\tau(i)}) - (m-1)\,\alpha_i - \alpha_j \in \Phi^+$ we must have $\tau(i) = i$ and $m = 2$; because $w_X(\alpha_{\tau(i)}) - \alpha_i - \alpha_j \in Q_X^+$ it follows that $[\theta(f_i), [f_i, f_j]] \in \mathfrak{n}_X^+$. The claimed expression for $\mathrm{LO}_{ij}(2)$ follows immediately from (A.1); those for $\mathrm{LO}_{ij}(m)$ with $m > 2$ from (3.2).

Now suppose $w_X(\alpha_{\tau(i)}) - (m-1)\,\alpha_i - \alpha_j \in \Phi^- \cup \{0\}$. Then $\tau(i) = i$ and $w_X(\alpha_i) \leq (m-1)\,\alpha_i + \alpha_j$ so that $\check{X}(i) = \{j\}$ and hence $a_{ji} < 0$. In this case we readily obtain

$$w_X(\alpha_{\tau(i)}) - (m-1)\,\alpha_i - \alpha_j = (2-m)\,\alpha_i - (1 + a_{ji})\,\alpha_j.$$

From $\Phi = \Phi^+ \cup \Phi^-$ it follows that $a_{ji} = -1$. Now $\mathbb{Z}\alpha_i \cap \Phi = \{\pm\alpha_i\}$ implies that $m \in \{2, 3\}$. We straightforwardly compute

$$\left[\theta(f_i), \mathrm{ad}(f_i)^{m-1}(f_j)\right] = \begin{cases} a_{ij}h_j - h_i & \text{if } m = 2, \\ -2\,(1 + a_{ij})f_i & \text{if } m = 3, \end{cases}$$

and the claimed expressions for $\mathrm{LO}_{ij}(m)$ readily follow. □

For $i,j\in I$ and $m,r\in\mathbb{Z}$ such that $0\le r\le\lfloor m/2\rfloor$ define $p_{ij}^{(r,m)}\in\mathbb{Z}$ by

$$p_{ij}^{(0,m)}=-1,\qquad p_{ij}^{(\frac{m+1}{2},m)}=0,\qquad p_{ij}^{(r,m+2)}=p_{ij}^{(r,m+1)}-(m+1)(m+a_{ij})p_{ij}^{(r-1,m)}. \tag{A.2}$$

**Lemma A.3.** *Let $(X,\tau)\in\mathrm{CDec}(A)$ and $\boldsymbol{\gamma}\in(\mathbb{C}^\times)^{I\backslash X}$. For all $i,j\in I\backslash X$ such that $i\neq j$ and $m\in\mathbb{Z}_{\ge0}$ we have*

$$\mathrm{ad}(b_i)^m(b_j)=\mathrm{ad}(f_i)^m(f_j)+\theta\left(\mathrm{ad}(f_i)^m(f_j)\right)+\mathrm{LO}_{ij}(m)$$

*where*

$$\mathrm{LO}_{ij}(m)=\begin{cases}(\gamma_i+\zeta(\alpha_i)\gamma_j)\left[\theta(f_i),f_j\right]\in\mathfrak{n}_X^+ & \text{if } \tau(i)=j,\, w_X(\alpha_i)-\alpha_i\in\Phi^+,\, m=1,\\ \gamma_jh_i-\gamma_ih_j & \text{if } \tau(i)=j,\, w_X(\alpha_i)=\alpha_i,\, m=1,\\ 2\left((\gamma_j-a_{ij}\gamma_i)f_i-\gamma_i(\gamma_i-a_{ij}\gamma_j)e_j\right) & \text{if } \tau(i)=j,\, w_X(\alpha_i)=\alpha_i,\, m=2,\\ \displaystyle\sum_{r=1}^{\lfloor m/2\rfloor}p_{ij}^{(r,m)}\gamma_i^r\mathrm{ad}(b_i)^{m-2r}(b_j) & \text{if } \tau(i)=i,\, w_X(\alpha_i)=\alpha_i,\\ 0 & \text{otherwise.}\end{cases}$$

*Proof.* As in the proof of Lemma A.2 we apply induction with respect to $m$. For $m=0$ we have

$$\mathrm{ad}(b_i)^0(b_j)=b_j=f_j+\gamma_j\theta(f_j)=\mathrm{ad}(f_i)^0(f_j)+\gamma_i^0\gamma_j\theta\left(\mathrm{ad}(f_i)^0(f_j)\right)+\mathrm{LO}_{ij}(0)$$

with $\mathrm{LO}_{ij}(0)=0$ as required. Now assume $m\in\mathbb{Z}_{>0}$ and suppose the statement holds for all smaller values. Then, by the induction hypothesis,

$$\begin{aligned}\mathrm{ad}(b_i)^m(b_j)&=[b_i,\mathrm{ad}(b_i)^{m-1}(b_j)]\\&=[f_i+\gamma_i\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)+\gamma_i^{m-1}\gamma_j\theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right)+\mathrm{LO}_{ij}(m-1)].\end{aligned}$$

Rearranging terms and using that $\theta$ is a Lie algebra automorphism we obtain

$$\begin{aligned}\mathrm{ad}(b_i)^m(b_j)&=\mathrm{ad}(f_i)^m(f_j)+\gamma_i^{m-1}\gamma_j\theta\left(\mathrm{ad}(f_i)^m(f_j)\right)+\mathrm{LO}_{ij}(m)\qquad\text{where}\\ \mathrm{LO}_{ij}(m)&=\gamma_i\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]+\gamma_i^{m-1}\gamma_j\left[f_i,\theta\left(\mathrm{ad}(f_i)^{m-1}(f_j)\right)\right]+[b_i,\mathrm{LO}_{ij}(m-1)].\end{aligned}$$

Using (2.8) we obtain

$$\mathrm{LO}_{ij}(m)=\gamma_i\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]+\zeta(\alpha_i)\gamma_i^{m-1}\gamma_j\theta\left(\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]\right)+[b_i,\mathrm{LO}_{ij}(m-1)]. \tag{A.3}$$

If $\left[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)\right]\neq0$ then $w_X(\alpha_{\tau(i)})-(m-1)\alpha_i-\alpha_j\in\Phi\cup\{0\}$.

If $w_X(\alpha_{\tau(i)})-(m-1)\alpha_i-\alpha_j\in\Phi^+$ we must have $j=\tau(i)$, $\check{X}(i)\neq\emptyset$, $m=1$; since $w_X(\alpha_{\tau(i)})-\alpha_j\in Q_X^+$ it follows that $[\theta(f_i),f_j]\in\mathfrak{n}_X^+$. The expression for $\mathrm{LO}_{ij}(1)$ follows from (A.3); $\mathrm{LO}_{ij}(m)=0$ with $m>1$ is a consequence of (3.2).

Now suppose $w_X(\alpha_{\tau(i)})-(m-1)\alpha_i-\alpha_j\in\Phi^-\cup\{0\}$. It follows that $\check{X}(i)=\emptyset$, so $\zeta(\alpha_i)=1$, and $\tau(i)\in\{i,j\}$. If $\tau(i)=j$ then $\mathbb{Z}\alpha_i\cap\Phi=\{\pm\alpha_i\}$ implies that $m\in\{1,2\}$. Furthermore, $\theta(f_i)=-e_j$ and $a_{ij}=a_{ji}$. Now (A.3) implies, as required, $\mathrm{LO}_{ij}(1)=\gamma_jh_i-\gamma_ih_j$,

$$\begin{aligned}\mathrm{LO}_{ij}(2)&=\gamma_i\gamma_j\theta\left([-e_j,[f_i,f_j]]\right)+\gamma_i\left[-e_j,[f_i,f_j]\right]+[b_i,\mathrm{LO}_{ij}(1)]\\&=\gamma_i\gamma_j\theta\left([h_j,f_i]\right)+\gamma_i[h_j,f_i]+[\gamma_ih_j-\gamma_jh_i,f_i-\gamma_ie_j]\\&=2\left((\gamma_j-a_{ij}\gamma_i)f_i-\gamma_i(\gamma_i-a_{ij}\gamma_j)e_j\right)\end{aligned}$$

and $\mathrm{LO}_{ij}(m)=0$ if $m>2$.

It remains to deal with the case $\check{X}(i)=\emptyset$ and $\tau(i)=i$, in which case $\theta(f_i)=-e_i$. A straightforward computation gives

$$[\theta(f_i),\mathrm{ad}(f_i)^{m-1}(f_j)]=(m-1)(m-2+a_{ij})\mathrm{ad}(f_i)^{m-2}(f_j).$$

By virtue of the induction hypothesis, (A.3) simplifies to

$$\mathrm{LO}_{ij}(m)=(m-1)(m-2+a_{ij})\gamma_i\left(\mathrm{ad}(b_i)^{m-2}(b_j)-\mathrm{LO}_{ij}(m-2)\right)+[b_i,\mathrm{LO}_{ij}(m-1)],$$

from which the recursion (A.2) follows straightforwardly. □

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