RATIONAL CURVES ON SMOOTH CUBIC HYPERSURFACES OVER FINITE FIELDS

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ABSTRACT. Let k be a finite field with characteristic exceeding 3. We prove that the space of rational curves of fixed degree on any smooth cubic hypersurface over k with dimension at least 11 is irreducible and of the expected dimension.

1. INTRODUCTION

The geometry of a variety is intimately linked to the geometry of the space of rational curves on it. Given a field k and a projective variety X defined over k, a natural object to study is the moduli space of k-rational curves on X. There are many results in the literature establishing the irreducibility of such mapping spaces, but most such statements are only proved for generic X, there being relatively few results which are valid for *all* X in a family. The aim of this paper is to prove such a result for all smooth cubic hypersurfaces of large enough dimension which are defined over a finite field of characteristic exceeding 3.

Suppose that $k = \mathbb{C}$ and $X \subset \mathbb{P}^{n-1}_{\mathbb{C}}$ is a smooth cubic hypersurface with $n \geqslant 6$. Let $\overline{\mathrm{Mor}_d}(\mathbb{P}^1_{\mathbb{C}}, X)$ be the Kontsevich moduli space of rational curves of degree d on X. Then it has been shown by Coskun and Starr [2] that $\overline{\mathrm{Mor}_d}(\mathbb{P}^1_{\mathbb{C}}, X)$ is irreducible and of the expected dimension $d(n-3)+n-5$. We would like to prove a similar result when $k = \mathbb{F}_q$ is a finite field with q elements and $X \subset \mathbb{P}^{n-1}_{\mathbb{F}_q}$ is a smooth cubic hypersurface defined over it. Rather than working with $\overline{\mathrm{Mor}_d}(\mathbb{P}^1_{\mathbb{F}_q}, X)$, which corresponds to "unparametrized" maps, we will study the moduli space $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$ of actual maps (see §2 for its construction). The expected dimension of $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$ is

$$D(d,n) = d(n-3) + n - 2, \tag{1.1}$$

since $\mathbb{P}^1_{\mathbb{F}_q}$ has automorphism group of dimension 3.

For a smooth cubic hypersurface $X \subset \mathbb{P}^{n-1}_{\mathbb{F}_q}$, the Lang–Tsen theorem (see [3, Thm. 3.6]) ensures that $X(\mathbb{F}_q(t)) \neq \emptyset$ as soon as $n \geqslant 10$, in which case X

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contains a rational curve defined over $\mathbb{F}_q$. One can go further if one enlarges the size of the finite field. Let $n \geqslant 4$. Then, according to Kollár [6, Example 7.6], there exists a constant c_n depending only on n such that for any $q > c_n$ and any point $x \in X(\mathbb{F}_q)$, the cubic hypersurface X contains a rational curve (of degree at most 216) which is defined over $\mathbb{F}_q$ and passes through x.

Following a suggestion of Ellenberg and Venkatesh, Pugin developed an "algebraic circle method" in his 2011 Ph.D. thesis [7] to study the spaces $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$. Thus, when $n \geqslant 13$ and $X \subset \mathbb{P}^{n-1}_{\mathbb{F}_q}$ is the diagonal cubic hypersurface

$$a_1x_1^3 + \cdots + a_nx_n^3 = 0, \quad (\text{for } a_1, \dots, a_n \in \mathbb{F}_q^*),$$

he succeeds in showing that the associated moduli space $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$ is irreducible and of the expected dimension $D(d,n)$, provided that $\mathrm{char}(\mathbb{F}_q) \neq 3$. Our main result extends Pugin's result to non-diagonal hypersurfaces, as follows.

Theorem 1.1. *Let* $\mathrm{char}(\mathbb{F}_q) > 3$ *and let* $X \subset \mathbb{P}^{n-1}_{\mathbb{F}_q}$ *be a smooth cubic hypersurface defined over* $\mathbb{F}_q$*, with* $n \geqslant 13$*. Then for each* $d \geqslant 1$ *the moduli space* $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$ *is irreducible and of dimension* $D(d,n)$*.*

Inspired by Pugin's approach, our proof of this result rests on an estimate for $\#\,\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)(\mathbb{F}_q)$, as $q \to \infty$. The cardinality of $\mathbb{F}_q$-points on $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$ is roughly equal to the number of $\mathbb{F}_q(t)$-points on X of degree d. We shall access the latter quantity through a function field version of the Hardy–Littlewood circle method. The traditional setting for this is a fixed finite field $\mathbb{F}_q$, with the goal being to understand the $\mathbb{F}_q(t)$-points on X of degree d, as $d \to \infty$. In contrast to this, Theorem 1.1 requires us to handle any fixed $d \geqslant 1$, as $q \to \infty$. The key ingredients will be drawn from work of Lee [4] on a $\mathbb{F}_q(t)$ version of Birch's work on systems of forms in many variables and our own recent contribution to the subject [1], which is specific to cubic forms. Perhaps the chief interest of Theorem 1.1 lies in the fact that a result in algebraic geometry can be proved using methods of analytic number theory.

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2. From moduli spaces to counting

Let k be a field and let $X \subset \mathbb{P}^{n-1}_k$ be a hypersurface cut out by an equation $F = 0$, where $F \in k[x_1, \dots, x_n]$ is a homogeneous cubic polynomial. Let $G_d(k)$ be the set of all homogeneous polynomials in u, v of degree $d \geqslant 1$, with coefficients in k. A *rational curve* on X is a non-constant morphism $f:\mathbb{P}^1_k \to X$. A morphism of degree d is given by

$$f=(f_1(u,v),\dots,f_n(u,v)),$$

with $f_1,\dots,f_n \in G_d(k)$, with no non-constant common factor in $k[u,v]$, such that $F(f_1(u,v),\dots,f_n(u,v))$ is identically zero. Using the coefficients of $f_1,\dots,f_n$ we can regard f as a point in $\mathbb{P}_k^{n(d+1)-1}$. The morphisms of degree d on X are parameterised by $\mathrm{Mor}_d(\mathbb{P}^1_k,X)$, which is an open subvariety of $\mathbb{P}_k^{n(d+1)-1}$ cut out by a system of $3d+1$ equations of degree 3. This directly leads to the naive expectation that $\mathrm{Mor}_d(\mathbb{P}^1_k,X)$ should have dimension

$$n(d+1)-1-(3d+1)=D(d,n),$$

in the notation of (1.1). The complement to $\mathrm{Mor}_d(\mathbb{P}^1_k,X)$ in its closure is the set of $(f_1,\dots,f_n)$ with a common zero. We can obtain explicit equations by noting that $f_1,\dots,f_n$ have a common zero if and only if the resultant $\mathrm{Res}(\sum_i \lambda_i f_i, \sum_j \mu_j f_j)$ is identically zero as a polynomial in λ_i,μ_j. This gives a system of equations of degree $2d$ in the coefficients of $f_1,\dots,f_n$.

Now let $k=\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q)>3$ in the above discussion. Assuming that $d\geqslant 1$ and $n\geqslant 13$ we need to show that $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q},X)$ is irreducible and of dimension $D(d,n)$. We note that $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q},X)$ is also defined over any finite extension $\mathbb{F}_{q^\ell}$ of $\mathbb{F}_q$. Following Pugin's approach [7], our proof of Theorem 1.1 relies on estimating $\#\,\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q},X)(\mathbb{F}_{q^\ell})$, as $\ell\to\infty$. According to Kollár [5, Thm. II.1.2/3], all irreducible components of $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q},X)$ have dimension at least $D(d,n)$. Hence, in view of the Lang–Weil estimate, Theorem 1.1 is a direct consequence of the following result.

Theorem 2.1. *Let* $\mathrm{char}(\mathbb{F}_q)>3$ *and let* $X\subset\mathbb{P}^{n-1}_{\mathbb{F}_q}$ *be a smooth cubic hypersurface defined over* $\mathbb{F}_q$*, with* $n\geqslant 13$*. Then for each* $d\geqslant 1$ *we have*

$$\lim_{\ell\to\infty} q^{-\ell D(d,n)}\#\,\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q},X)(\mathbb{F}_{q^\ell})\leqslant 1.$$

We henceforth redefine q^ℓ to be q. Our proof of Theorem 2.1 is based on the Hardy–Littlewood circle method over the function field $\mathbb{F}_q(t)$, always under the assumption that $\mathrm{char}(\mathbb{F}_q)>3$. The main input comes from our previous work [1] and a straightforward adaptation of work due to Lee [4]. We will adhere to the notation described in [1, §2.1 and §2.2] without further comment.

Assume that $F(\mathbf{x})=\sum_{\mathbf{i}} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, with variables $\mathbf{x}=(x_1,\dots,x_n)$ and coefficients $a_{\mathbf{i}}\in\mathbb{F}_q$. In particular the height H_F and discriminant Δ_F of F satisfy

$$H_F=\max_{\mathbf{i}}|a_{\mathbf{i}}|=1 \quad\text{and}\quad |\Delta_F|=1.$$

We will make frequent use of these facts in what follows. To establish Theorem 2.1 we work with the naive space

$$M_d = \{\mathbf{x} = (x_1, \dots, x_n) \in G_d(\mathbb{F}_q)^n \setminus \{\mathbf{0}\} : F(\mathbf{x}) = 0\},$$

which corresponds to the $\mathbb{F}_q$-points on the affine cone of $\mathrm{Mor}_d(\mathbb{P}^1_{\mathbb{F}_q}, X)$. Let us set

$$E(d,n) = D(d,n) + 1 = (n-3)(d+1) + 2.$$

It will clearly suffice to show that

$$\lim_{q\to\infty} q^{-E(d,n)} \# M_d \leqslant 1, \tag{2.1}$$

for $n \geqslant 13$. We proceed by relating the counting function $\# M_d$ to the counting function that lies at the heart of our earlier investigation [1].

Let $w : K_\infty^n \to \{0,1\}$ be given by $w(\mathbf{x}) = \prod_{1\leqslant i\leqslant n} w_\infty(x_i)$, where

$$w_\infty(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Putting $P = t^{d+1}$, we then have $\# M_d \leqslant N(P)$, where

$$N(P) = \sum_{\substack{\mathbf{x}\in\mathscr{O}^n \\ F(\mathbf{x})=0}} w(\mathbf{x}/P). \tag{2.2}$$

It follows from [1, Eq. (4.1)] that for any $Q \geqslant 1$ we have

$$N(P) = \sum_{\substack{r\in\mathscr{O} \\ |r|\leqslant \widehat{Q} \\ r \text{ monic}}} \ \sideset{}{^*}\sum_{|a|<|r|} \int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}} S\left(\frac{a}{r} + \theta\right) \mathrm{d}\theta, \tag{2.3}$$

where $\sum^*$ means that the sum is taken over residue classes $|a| < |r|$ for which $(a,r) = 1$, and where

$$S(\alpha) = \sum_{\mathbf{x}\in\mathscr{O}^n} \psi(\alpha F(\mathbf{x})) w(\mathbf{x}/P), \tag{2.4}$$

for any $\alpha \in \mathbb{T}$. We will work with the choice $Q = 3(d+1)/2$, so that $\widehat{Q} = |P|^{3/2}$.

We henceforth set

$$\delta = \frac{3}{d+1}.$$

Let $A(P)$ denote the contribution to $N(P)$ in (2.3) from values of r, θ such that either $|\theta| < \widehat{Q}^{-4}$, or else $r = 1$ and $|\theta| < |P|^{-3+\delta}$.

Lemma 2.2. *We have* $\lim_{q\to\infty} q^{-E(d,n)} A(P) = 1$.

Proof. Let us put $A_1(P)$ for the contribution from $r = 1$ and $|\theta| < |P|^{-3+\delta}$, and $A_2(P)$ for the remaining contribution. Taking the trivial bound $|S(\alpha)| \leqslant |P|^n$, it is easy to check that $\lim_{q\to\infty} q^{-E(d,n)} A_2(P) = 0$ and so our attention shifts to $A_1(P)$. For this we invoke [1, Lemma 2.2], which gives

$$\begin{aligned} A_1(P) &= \int_{|\theta|<|P|^{-3+\delta}} S(\theta)\mathrm{d}\theta \\ &= |P|^{-3+\delta} \# \left\{ \mathbf{x} \in \mathscr{O}^n : |\mathbf{x}| < |P|,\ |F(\mathbf{x})| < |P|^{3-\delta} \right\}. \end{aligned}$$

Note that our choice of δ implies that $|P|^{3-\delta} = q^{3(d+1)-3} = q^{3d}$ and so this result is applicable since $3d$ is an integer. Any $\mathbf{x}$ to be counted is an n-tuple of polynomials with jth component $x_j = a_{0,j}t^d + \dots + a_{d,j}$ for coefficients $a_{i,j} \in \mathbb{F}_q$. The condition $|F(\mathbf{x})| < |P|^{3-\delta}$ is therefore equivalent to the condition $F(a_{0,1}, \dots, a_{0,n}) = 0$. Since F is non-singular it is certainly absolutely irreducible over $\mathbb{F}_q$. Thus the Lang–Weil estimate implies that the total number of available $\mathbf{x}$ is $q^{dn+n-1}(1+O_n(q^{-1/2}))$, where the implied constant depends only on n. Thus

$$A_1(P) = q^{-3d+dn+n-1}(1 + O_n(q^{-1/2})),$$

from which the statement of the lemma follows. $\square$

Let us put $B(P)$ for the contribution to $N(P)$ in (2.3) from values of r, θ with $|\theta| \geqslant \widehat{Q}^{-4}$, such that either $|r| > 1$, or else $r = 1$ and $|\theta| \geqslant |P|^{-3+\delta}$. The remainder of this paper is devoted to a proof of the following result.

Lemma 2.3. *We have* $\lim_{q\to\infty} q^{-E(d,n)} B(P) = 0$ *for* $n \geqslant 13$.

Recalling that $\#M_d \leqslant A(P) + B(P)$, we see that (2.1) follows from Lemmas 2.2 and 2.3. Thus it remains to prove Lemma 2.3 in order to complete the proof of Theorem 2.1.

In our analysis of $B(P)$ it will be convenient to sort the sum according to the size of $|r|$ and $|\theta|$. Consequently, we let $S(d)$ denote the set of $(Y, \Theta) \in \mathbb{Z}^2$ such that

$$0 \leqslant Y \leqslant Q \quad \text{and} \quad -4Q \leqslant \Theta < -(Y + Q),$$

with either $Y \geqslant 1$, or else $Y = 0$ and $\widehat{\Theta} \geqslant |P|^{-3+\delta}$. In particular it is clear that $\#S(d) \leqslant 7(d + 1) = c_d$, say. We then have

$$B(P) \leqslant \sum_{(Y,\Theta)\in S(d)} |N(P, Y, \Theta)| \leqslant c_d \max_{(Y,\Theta)\in S(d)} |N(P, Y, \Theta)|,$$

where

$$N(P, Y, \Theta) = \sum_{\substack{r\in\mathscr{O} \\ |r|=\widehat{Y} \\ r \text{ monic}}} \sideset{}{^*}\sum_{|a|<|r|} \int_{|\theta|=\widehat{\Theta}} S\left(\frac{a}{r} + \theta\right) \mathrm{d}\theta. \tag{2.5}$$

We will use two basic methods for analysing $N(P, Y, \Theta)$.

Let

$$S_1(d) = \{(Y, \Theta) \in S(d) : Y \geqslant 1 \text{ and } \Theta \leqslant (n/6 - 4/3)Y - 2Q\}.$$

For (Y, Θ) belonging to this set we will apply our previous work [1], which is founded on Poisson summation. This is the object of §3. Alternatively, in §4, we will use a function field version of Weyl differencing to handle (Y, Θ) belonging to the set

$$S_2(d) = \{(Y, \Theta) \in S(d) : \text{If } Y \geqslant 1 \text{ then } \Theta > (n/6 - 4/3)Y - 2Q\}.$$

This part of the argument is essentially due to Lee [4]. It will be convenient to set

$$B_i(P) = \max_{(Y,\Theta) \in S_i(d)} |N(P, Y, \Theta)|, \quad \text{for } i = 1, 2,$$

so that $B(P) \leqslant c_d\{B_1(P) + B_2(P)\}$. Assuming that $n \geqslant 13$, it now suffices to show that $\lim_{q\to\infty} q^{-E(d,n)} B_i(P) = 0$ for $i = 1, 2$.

3. Poisson summation

The counting function (2.2) is equal to the counting function $N(P)$ considered in [1, §4] with $M = 1$ and $\mathbf{b} = \mathbf{0}$. (Equivalently this is [1, Eq. (7.4)] with $M = 1$, $\mathbf{b} = \mathbf{0}$, $L = 0$ and $\mathbf{x}_0 = \mathbf{0}$.) Throughout this section we shall assume that the cubic form F has $n \geqslant 13$ variables. The main part of [1] is actually concerned with non-singular cubic forms in only $n \geqslant 8$ variables. Intrinsic to the success of this endeavour is the choice of counting function, in which $\mathbb{F}_q(t)$-solutions are singled out for consideration if they are sufficiently close to a conveniently chosen solution over K_∞. The fact that we must consider all $\mathbb{F}_q(t)$-solutions in (2.2) directly accounts for this loss of precision.

Let $J(\Theta) = \max\{1, \widehat{\Theta}|P|^3\}$. Appealing to [1, Lemma 7.2], we find that

$$N(P, Y, \Theta) = |P|^n \sum_{\substack{r \in \mathscr{O} \\ |r| = \widehat{Y} \\ r \text{ monic}}} |r|^{-n} \int_{|\theta| = \widehat{\Theta}} \sum_{\substack{\mathbf{c} \in \mathscr{O}^n \\ |\mathbf{c}| \leqslant \widehat{C}}} S_r(\mathbf{c}) I_r(\theta; \mathbf{c}) \mathrm{d}\theta,$$

where $\widehat{C} = \widehat{Y}|P|^{-1}J(\Theta)$ and

$$S_r(\mathbf{c}) = \sideset{}{^*}\sum_{|a| < |r|} \sum_{\substack{\mathbf{y} \in \mathscr{O}^n \\ |\mathbf{y}| < |r|}} \psi\left(\frac{aF(\mathbf{y}) - \mathbf{c}.\mathbf{y}}{r}\right),$$

$$I_r(\theta; \mathbf{c}) = \int_{K_\infty^n} w(\mathbf{x}) \psi\left(\theta P^3 F(\mathbf{x}) + \frac{P\mathbf{c}.\mathbf{x}}{r}\right) \mathrm{d}\mathbf{x}.$$

It will be convenient to put $\gamma = \theta P^3$ in $I_r(\theta; \mathbf{c})$. The definition of w implies that the integral is over $\mathbb{T}^n$, whence an application of [1, Lemma 2.7] shows that

$$|I_r(\theta; \mathbf{c})| \leqslant \operatorname{meas}\{\mathbf{x} \in \mathbb{T}^n : |\gamma \nabla F(\mathbf{x}) + r^{-1} P\mathbf{c}| \leqslant \max\{1, |\gamma|^{1/2}\} = J(\Theta)^{1/2}.$$

The exponential sum $S_r(\mathbf{c})$ is a multiplicative function of r by [1, Lemma 4.5]. We will adopt the notation conceived in [1, Definition 4.6], so that associated to any $r \in \mathscr{O}$ and $i \in \mathbb{Z}_{>0}$ are the elements

$$b_i = \prod_{\varpi^i \| r} \varpi^i \quad \text{and} \quad r_i = \prod_{\substack{\varpi^e \| r \\ e \geqslant i}} \varpi^e.$$

Applying [1, Lemma 5.1], we therefore find that there exists a constant $A_n > 0$ depending only on n such that

$$\sum_{\substack{\mathbf{c} \in \mathscr{O}^n \\ |\mathbf{c}| \leqslant \widehat{C}}} |S_r(\mathbf{c}) I_r(\theta; \mathbf{c})| \leqslant A_n^{\omega(b_1 b_2)} |b_1 b_2|^{n/2+1} \int_{\mathbb{T}^n} \sum_{\mathbf{c} \in \mathscr{C}(\mathbf{x})} |S_{r_3}(\mathbf{c})| \mathrm{d}\mathbf{x},$$

where

$$\mathscr{C}(\mathbf{x}) = \left\{ \mathbf{c} \in \mathscr{O}^n : |\mathbf{c} + r\theta P^2 \nabla F(\mathbf{x})| \leqslant |P|^{-1} \widehat{Y} J(\Theta)^{1/2} \right\}.$$

It now follows from [1, Lemma 6.4] that for any $\varepsilon > 0$ there is a constant $c_{n,\varepsilon} > 0$, depending only on n and ε, such that

$$\sum_{\mathbf{c} \in \mathscr{C}(\mathbf{x})} |S_{r_3}(\mathbf{c})| \leqslant c_{n,\varepsilon} |r_3|^{n/2+1+\varepsilon} \left(|r_3|^{n/3} + \frac{\widehat{Y}^n J(\Theta)^{n/2}}{|P|^n} \right).$$

According to [1, Lemma 2.2] we have

$$\int_{|\theta| = \widehat{\Theta}} \mathrm{d}\theta = \widehat{\Theta + 1} - \widehat{\Theta} \leqslant \widehat{\Theta + 1}.$$

Hence, on integrating trivially over $\mathbf{x}$ and then over θ, we deduce the existence of a constant $c_{n,\varepsilon} > 0$ such that

$$\begin{aligned} \frac{|P|^n}{|r|^n} \int_{|\theta| = \widehat{\Theta}} \sum_{\substack{\mathbf{c} \in \mathscr{O}^n \\ |\mathbf{c}| \leqslant \widehat{C}}} |S_r(\mathbf{c}) I_r(\theta; \mathbf{c})| \mathrm{d}\theta & \\ \leqslant c_{n,\varepsilon} \widehat{Y}^{n/2+1+\varepsilon} \widehat{\Theta + 1} & \left(\frac{|r_3|^{n/3} |P|^n}{\widehat{Y}^n} + J(\Theta)^{n/2} \right). \end{aligned}$$

It remains to sum this over all monic $r \in \mathscr{O}$ such that $|r| = \widehat{Y}$, of which there are precisely $\widehat{Y}$. For this we note that

$$\sum_{\substack{r\in\mathscr{O}\\ |r|=\widehat{Y}\\ r \text{ monic}}} |r_3|^{n/3} \leqslant \widehat{Y}^{n/3} \sum_{\substack{r=b_1b_2r_3\in\mathscr{O}\\ |r|=\widehat{Y}\\ r \text{ monic}}} \frac{1}{|b_1b_2|^{n/3}} \leqslant c_n \widehat{Y}^{n/3+1/3},$$

for an appropriate constant $c_n > 0$ such that there are at most $c_n\widehat{Y}^{1/3}$ values of $|r_3| \leqslant \widehat{Y}$. Recalling that $Y \leqslant Q$ and $\Theta < -(Y+Q)$, we easily deduce that

$$J(\Theta)^{n/2} \leqslant \max\left\{1, \frac{|P|^3}{\widehat{Y}\widehat{Q}}\right\}^{n/2} = \frac{\widehat{Q}^{n/2}}{\widehat{Y}^{n/2}}.$$

Hence there is a constant $c_{n,\varepsilon} > 0$ such that

$$|N(P,Y,\Theta)| \leqslant c_{n,\varepsilon}\widehat{Y}^{n/2+1+\varepsilon}\widehat{\Theta+1}\left(\frac{\widehat{Y}^{n/3+1/3}|P|^n}{\widehat{Y}^n} + \frac{\widehat{Q}^{n/2}}{\widehat{Y}^{n/2-1}}\right),$$

whence in fact

$$|N(P,Y,\Theta)| \leqslant c_{n,\varepsilon}\widehat{\Theta+1}\left\{\frac{|P|^n}{\widehat{Y}^{n/6-4/3-\varepsilon}} + \widehat{Y}^2\widehat{Q}^{n/2+\varepsilon}\right\}.$$

Taking $\widehat{\Theta+1} \leqslant \widehat{Y}^{-1}\widehat{Q}^{-1}$ we see that the second term is at most

$$c_{n,\varepsilon}\widehat{Y}\widehat{Q}^{n/2-1+\varepsilon} \leqslant c_{n,\varepsilon}\widehat{Q}^{n/2+\varepsilon} \leqslant c_{n,\varepsilon}|P|^{3n/4+2\varepsilon}.$$

But we also have $\widehat{\Theta+1} \leqslant q\widehat{Y}^{n/6-4/3}/\widehat{Q}^2$ for any $(Y,\Theta) \in S_1(d)$, whence

$$B_1(P) \leqslant c_{n,\varepsilon}\left\{q|P|^{n-3+2\varepsilon} + |P|^{3n/4+2\varepsilon}\right\}.$$

Assuming that $\varepsilon > 0$ is taken to be sufficiently small in term of d, it easily follows that $\lim_{q\to\infty} q^{-E(d,n)}B_1(P) = 0$ for $n \geqslant 13$.

4. Weyl differencing

The goal of this section is to show that $\lim_{q\to\infty} q^{-E(d,n)}B_2(P) = 0$ for $n \geqslant 13$. Our starting point is an analysis of the exponential sum (2.4), for which we will use the function field version of Birch's Weyl differencing that was worked out by Lee [4]. Our task is to make the dependence on q completely explicit, but the argument is very standard and so we shall be brief where possible. Since we are only concerned with cubic forms one needs to take $R = 1$ and $d = 3$ in Lee's work [4, §3]. As usual we will assume that $\mathrm{char}(\mathbb{F}_q) > 3$.

Define the Hessian matrix

$$\mathbf{H}(\mathbf{x}) = \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_{1\leqslant i,j\leqslant n}$$

that is associated to our cubic form F. For any $\beta = \sum_{-\infty < i \leqslant N} b_i t^i \in K_\infty$, we let $\|\beta\| = |\sum_{-\infty<i<0} b_i t^i|$. Beginning with an application of [4, Cor. 3.3], it follows that

$$|S(\alpha)|^4 \leqslant |P|^{2n} \# \left\{ \mathbf{u}, \mathbf{v} \in \mathscr{O}^n : |\mathbf{u}|, |\mathbf{v}| < |P|, \ \|\alpha \mathbf{H}(\mathbf{u})\mathbf{v}\| < |P|^{-1} \right\}.$$

for any $\alpha \in \mathbb{T}$. We are only interested in values of α with rational approximation $\alpha = a/r + \theta$, where $|r| = \widehat{Y}$ and $|\theta| = \widehat{\Theta}$ for $(Y, \Theta) \in S_2(d)$. We recall here, for the sake of convenience, that this means

$$1 \leqslant \widehat{Y} \leqslant \widehat{Q} \quad \text{and} \quad \widehat{\Theta} < \frac{1}{\widehat{Y}\widehat{Q}},$$

with either $\widehat{Y} \geqslant q$ and $\widehat{\Theta} > \widehat{Y}^{n/6-4/3}/\widehat{Q}^2$, or else $\widehat{Y} = 1$ and $\widehat{\Theta} \geqslant |P|^{-3+\delta}$. In either case we therefore have $\widehat{\Theta} > \widehat{Y}^{n/6-4/3}/\widehat{Q}^2$. We note that $S_2(d)$ is non-empty only when $\widehat{Y} < |P|^{9/(n-2)}$, which we now assume.

The next stage in the analysis of $S(\alpha)$ is a double application of the function field analogue of Davenport's "shrinking lemma", as proved in [4, Lemma 3.4]. Let $\Gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix with entries in K_∞. For $1 \leqslant i \leqslant n$ we introduce the linear forms

$$L_i(u_1, \ldots, u_n) = \sum_{j=1}^n \gamma_{ij} u_j. \tag{4.1}$$

Next, for given real numbers a, Z, we let $N(a, Z)$ denote the number of vectors $(u_1, \ldots, u_{2n}) \in \mathscr{O}^{2n}$ such that

$$|u_j| < \widehat{a}\widehat{Z} \quad \text{and} \quad |L_j(u_1, \ldots, u_n) + u_{j+n}| < \frac{\widehat{Z}}{\widehat{a}} \quad \text{for } 1 \leqslant j \leqslant n.$$

In due course we will adapt the argument of [4, Lemma 3.4] to show that for any $a, Z_1, Z_2 \in \mathbb{R}$ with $Z_1 \leqslant Z_2 \leqslant 0$, we have

$$\frac{N(a, Z_1)}{N(a, Z_2)} \geqslant \widehat{K}^n, \tag{4.2}$$

where $K = \lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil$ and $\{a\}$ denotes the fractional part of a.

Taking this on faith for the moment, let Z be such that

$$\widehat{Z} = \sqrt{\widehat{Y}\widehat{\Theta}|P|}.$$

Our assumptions on Y, Θ easily imply that $\widehat{Z} \leqslant 1$ and $Z \in \frac{1}{2}\mathbb{Z}$. We may therefore apply the shrinking lemma first with $(\widehat{a}, \widehat{Z}_1, \widehat{Z}_2) = (|P|, \widehat{Z}, 1)$. This allows us to take $K \geqslant Z_1$ in (4.2). Next we apply the lemma a second time with $(\widehat{a}, \widehat{Z}_1, \widehat{Z}_2) = (\widehat{Z}^{-1/2}|P|, \widehat{Z}^{3/2}, \widehat{Z}^{1/2})$. We may write $Z/2 = N + k/4$ for some integer N and $k \in \{0, 1, 2, 3\}$. Thus

$$\lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil = (3N + k) - N = 2N + k \geqslant Z_1 - Z_2.$$

This therefore implies

$$|S(\alpha)|^4 \leqslant \frac{|P|^{2n}}{\widehat{Z}^{2n}} \# \left\{ \mathbf{u}, \mathbf{v} \in \mathscr{O}^n : |\mathbf{u}|, |\mathbf{v}| < \widehat{Z}|P|, \ \|\alpha \mathbf{H}(\mathbf{u})\mathbf{v}\| < \widehat{Z}^2 |P|^{-1} \right\}.$$

The next step is an application of the function field analogue of Heath-Brown's Diophantine approximation lemma, as worked out in [4, Lemma 3.6]. Noting that $|\mathbf{H}(\mathbf{u})\mathbf{v}| \leqslant |\mathbf{u}||\mathbf{v}|$, we shall apply this with $\widehat{M} = (\widehat{Z}|P|)^2$ and $\widehat{Y_0} = \widehat{Z}^{-2}|P|$. (In order to avoid a clash of notation we let Y_0 denote the parameter Y that features in [4, Lemma 3.6].) This result allows us to conclude that $\mathbf{H}(\mathbf{u})\mathbf{v} = \mathbf{0}$ provided that $\widehat{Y_0} > |r|$ and $\widehat{M}^{-1} > |r\theta| \geqslant \widehat{Y}_0^{-1}$. Since $|r| = \widehat{Y}$ and $|\theta| = \widehat{\Theta}$ for $(Y, \Theta) \in S_2(d)$ it is easy to check that our choice of Z ensures that all of these inequalities are satisfied. Hence

$$|S(\alpha)|^4 \leqslant \frac{|P|^{2n}}{Z^{2n}} \# \left\{ \mathbf{u}, \mathbf{v} \in \mathscr{O}^n : |\mathbf{u}|, |\mathbf{v}| < \widehat{Z}|P|, \ \mathbf{H}(\mathbf{u})\mathbf{v} = \mathbf{0} \right\}.$$

The proof of [1, Lemma 6.5] directly yields the existence of a constant $c_n > 0$ such that the remaining cardinality is bounded by $c_n(\widehat{Z}|P|)^n$. In conclusion we have shown that

$$|S(\alpha)| \leqslant \frac{c_n |P|^n}{(\widehat{Y}\widehat{\Theta}|P|^3)^{n/8}}.$$

Turning now to the estimation of $N(P, Y, \Theta)$, it follows from (2.5) that

$$\begin{aligned} B_2(P) &\leqslant c_n \max_{(Y,\Theta) \in S_2(d)} \frac{\widehat{Y}^2 \widehat{\Theta + 1} |P|^n}{(\widehat{Y}\widehat{\Theta}|P|^3)^{n/8}} \\ &= c_n q \max_{(Y,\Theta) \in S_2(d)} \widehat{Y}^{2-n/8} \widehat{\Theta}^{1-n/8} |P|^{5n/8}. \end{aligned}$$

Note that the exponent of $\widehat{\Theta}$ is negative for $n \geqslant 13$. Let $(Y, \Theta) \in S_2(d)$. Taking $\widehat{\Theta} > \widehat{Y}^{n/6-4/3}/\widehat{Q}^2$, we get

$$\widehat{Y}^{2-n/8} \widehat{\Theta}^{1-n/8} |P|^{5n/8} < \frac{\widehat{Y}^{2-n/8} |P|^{n-3}}{\widehat{Y}^{(n/8-1)(n/6-4/3)}} \leqslant |P|^{n-3},$$

since $\widehat{Y} \geqslant 1$ and $(2 - n/8) - (n/8 - 1)(n/6 - 4/3) \leqslant 0$ for $n \geqslant 13$. Hence $\lim_{q \to \infty} q^{-E(d,n)} B_2(P) = 0$ for $n \geqslant 13$.

Our final task is to show that (4.2) holds with $K = \lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil$. The argument is based on the geometry of numbers. Every matrix corresponds to an $\mathscr{O}$-lattice spanned by its columns. We will abuse notation and identify a matrix with its corresponding lattice. Given a lattice M, the adjoint lattice Λ is defined to satisfy $\Lambda^T \mathsf{M} = I$. Let $\Gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix with entries in K_∞. Given any integer m, we define the special lattice

$$\mathsf{M}_m = \begin{pmatrix} t^{-m} I_n & 0 \\ t^m \Gamma & t^m I_n \end{pmatrix},$$

with corresponding adjoint lattice

$$\mathsf{\Lambda}_m = \begin{pmatrix} t^m I_n & -t^m \Gamma \\ 0 & t^{-m} I_n \end{pmatrix}.$$

Let $\widehat{R}_1, ..., \widehat{R}_{2n}$ denote the successive minima of the lattice corresponding to M_m and note that the lattices M_m and $\mathsf{\Lambda}_m$ can be identified with one another. It follows from [4, Lemma B.6] that $R_\nu + R_{2n-\nu+1} = 0$ for each $1 \leqslant \nu \leqslant 2n$. Let $L_i(u_1, \dots, u_n)$ be the linear forms (4.1) for $1 \leqslant i \leqslant n$. Then for any real number Z, it is easy to see that

$$N(m, Z) = \{\mathbf{x} \in \mathsf{M}_m : |\mathbf{x}| < \widehat{Z}\},$$

in the notation of (4.2). We denote the right hand side by $M_m(Z)$ and proceed to establish the following inequality.

Lemma 4.1. *Let $m, Z_1, Z_2 \in \mathbb{Z}$ such that $Z_1 \leqslant Z_2 \leqslant 0$. Then we have*

$$\frac{M_m(Z_1)}{M_m(Z_2)} \geqslant \left(\frac{\widehat{Z}_1}{\widehat{Z}_2} \right)^n.$$

Proof. Let $1 \leqslant \mu, \nu \leqslant 2n$ be such that $R_\mu < Z_1 \leqslant R_{\mu+1}$ and $R_\nu < Z_2 \leqslant R_{\nu+1}$. Since R_j is a non-decreasing sequence which satisfies $R_j + R_{2n-j+1} = 0$, we must have $0 \leqslant R_{n+1}$, whence in fact $\mu \leqslant \nu \leqslant n$. It follows from [4, Lemma B.5] that

$$\frac{M_m(Z_1)}{M_m(Z_2)} = \begin{cases} 1 & \text{if } Z_1, Z_2 < R_1, \\ \left(\prod_{j=1}^{\nu} \widehat{R}_j / \widehat{Z_1} \right) (\widehat{Z_1}/\widehat{Z_2})^\nu & \text{if } Z_1 < R_1 \leqslant Z_2, \\ \left(\prod_{j=\mu+1}^{\nu} \widehat{R}_j / \widehat{Z_1} \right) (\widehat{Z_1}/\widehat{Z_2})^\nu & \text{if } R_1 \leqslant Z_1 \leqslant Z_2, \end{cases}$$

The statement of the lemma is now obvious. □

Now let $a \in \mathbb{R}$ and put $m = \lfloor a \rfloor$. For any real number Z it is clear that

$$M_m(Z - \{a\}) \leqslant N(a, Z) \leqslant M_m(Z + \{a\}).$$

Lemma 4.1 therefore yields (4.2) with $K = \lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil$, as required.

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