Modular labelled calculi for relevant logics

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Abstract

In this article, we perform a detailed proof theoretic investigation of a wide number of relevant logics by employing the well-established methodology of labelled sequent calculi to build our intended systems. At the semantic level, we will characterise relevant logics by employing *reduced* Routley-Meyer models, namely, relational structures with a ternary relation between worlds along with a unique distinct element considered as the *real* (or *actual*) world. This paper realizes the idea of building a variety of modular labelled calculi by reflecting, at the syntactic level, semantic informations taken from reduced Routley-Meyer models. Central results include proofs of soundness and completeness, as well as a proof of CUT-admissibility.

1 Introduction

§1 Relevant logics are a well-known family of non-classical logics introduced to cope with so-called paradoxes of material and strict implication. According to relevantists, $\to$ is intended to express a more fine-grained and philosophically motivated notion of conditional. Part of the philosophical intuition of relevant logics, at least in the early development by Anderson and Belnap [1], was that the antecedent and consequent of a valid conditional must be relevant to each other, in the sense that, in expressions of the form $A \to B$, there must be a strong connection between antecedent and consequent.
Relevant logics have attracted a lot of attention among logicians and many formal structures were applied to offer detailed and systematic characterizations. For the purposes of this paper, however, we will introduce relevant logics in terms of *reduced* Routley-Meyer models, i.e., by means of relational structures employing

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a ternary relation between states (see, e.g., [33, 34]), along with a distinct element interpreted as the *real* (or *actual*) world. Intuitively, in Routley-Meyer models, a relevant implication $A \rightarrow B$ is true at world a just in case, for all worlds b, c, related to a, if A is true at b, then B is true at c. The aim of this article is to define modular proof systems for a variety of relevant logics on the basis of these models. More specifically, we will introduce a family of labelled sequent calculi for relevant logic **B** and its extensions, namely, **DW**, **DJ**, **TW**, **T**, **RW**, **R** and **RM**. The calculi are based on Routley-Meyer semantics, in the sense that, by following the well-established methodology proposed by [20], sequents internalize, by means of syntactic tools, semantic informations taken exactly from reduced Routley-Meyer models.

§2 Proof theoretic studies on relevant logics have a long and troubled history.[1] Gentzen-style sequents were proposed, among others, in [1, 28]. For what concerns generalizations of sequents, instead, there are different trends in the literature. To cite a few of them: *cognate* sequents ([16]), *hypersequents* ([2, 3]), Dunn-Mints calculi ([6]), *consecution* calculi ([5]), *display* sequents [31, 32, 5]. From the perspective of labelled proof systems, instead, there is a variety of approaches. Among the early significant contributions, one finds A. Urquhart and S. Giambrone's U- and G-systems for some positive fragments of a family of relevant logics (called *semilattice logics*) in [10]. Urquhart and Giambrone's systems correspond to a *weakly labelled calculus* in the sense of [13, 204], that is, labels are limited technical devices supporting proof construction. Indeed, no special rules operating on labels are introduced. More precisely, the behaviour of labels in derivations is subject only to some specific restrictions, established directly on the application of rules. Moreover, the labelling of formulas in the rules for $\rightarrow$ refers to a different treatment of the ternary relation $Rabc$ at the semantic level, that is, by putting $c = a \cup b$. In other words, rules of U- and G-systems employ the union of worlds a and b as element of the ternary relation, rather than a third distinct state c. An analogous work was conducted by R. Kashima in [15, 14] always in the context of semilattice relevant logics. L. Viganò [37] pursued a characterization of some relevant logics by using a calculus enriched with rules acting on labels and which restates the presence of the third element c, rather than $a \cup b$. Similarly in [17], H. Kurokawa and S. Negri introduced a wide range of labelled calculi constructed with reference to the *original* (or *non reduced*) ternary relational semantics proposed by Routley and Meyer.

Layout of the article In Section 2, relevant logics are introduced in terms of both, reduced Routley-Meyer models and axiomatic systems. Sections 3 and 4 present the rules of the labelled calculi for a variety of relevant logics and some related preliminary results, as well as a comparison with Kurokawa and Negri's article [17] mentioned above. Section 5 includes a proof of soundness, while

[1] The remarks that follows are not meant to be a comprehensive historical source, but just a sketchy introduction to the vast realm of sequent-based calculi for relevant logics.

Section 6 contains proofs of completeness. Finally, in Section 7, we will proceed towards the proof of CUT-admissibility.

2 Preliminaries

2.1 Semantics and axioms for relevant logic B

In this section, we will introduce Routley-Meyer relational semantics and an axiomatic system for relevant logic **B** (standing for *basic*). The former structures, employing a ternary relation between states, can be considered as generalizations of Kripke models for intuitionistic and modal logics. Notice that the interpretation of ternary relations is a controversial topic and there are different orientations in the literature.[2] Some possible readings are (notations adapted):

> "Well, to say that x determines $A \to B$ is to say that whenever we can conclude A on the basis of a piece of information y, we can conclude B on the basis of x and y jointly, that is, on the basis of $x \cup y$." [36, 160]

> "Consider a natural English rendering of Kripke's binary R. xRy 'says' that 'world' y is possible relative to world x. An interesting ternary generalization is to read $xRyz$ to say that 'worlds' y and z are compossible (better, maybe, compatible) relative to x. (The reading is suggested by Dunn.)" [33, 200]

> "$Rabc$ iff b and c are pairwise accessible from a, or, to take a more revealing modal analogue, iff a and b are compatible relative to c, or conversely iff c is compatible with a and b." [34, 299-300]

> "[...] we may read $Rxyz$ as meaning that z contains all the information obtainable by pooling the information x and y. [Alternatively,] $Rxyz$ is [...] interpreted as saying that the information in y is carried to z by x." [29, 207]

Let's turn to the formal details.

Definition 2.1. Let $\mathcal{L}$ be the language of **B**. We denote by At a set of atomic formulas $p, q, \ldots$. The set of **B** formulas, denoted Form, is defined recursively for all A as follows:

$$A ::= p \mid {\sim}A \mid A \wedge A \mid A \vee A \mid A \to A$$

Definition 2.2. A *reduced Routley-Meyer frame* for relevant logic **B**, denoted $\mathcal{F}$, is a quadruple $\langle W, 0, *, R \rangle$, where W is a set of points, with 0 denoting its

[2] A detailed overview can be found in [4].

base element, * is a unary function $W \mapsto W$. Finally, $R \subseteq W^3$ and satisfies the following conditions:

$$a^{**} = a \tag{F1}$$
$$R0aa \tag{F2}$$
$$R0ab \wedge R0bc \implies R0ac \tag{F3}$$
$$R0da \wedge Rabc \implies Rdbc \tag{F4}$$
$$R0ab \implies R0b^*a^* \tag{F5}$$

Notice that relations of the form $R0ab$ and $R0ab \wedge R0ba$ can be abbreviated by writing $a \leq b$ and $a = b$, respectively. However, given that both symbols, $\leq$ and $=$, are precisely defined in terms of the ternary accessibility relation, we can employ only R to characterize relevant logics.

Definition 2.3. A *reduced Routley-Meyer model* for **B**, denoted $\mathcal{M}$, is a pair $\langle \mathcal{F}, v \rangle$, where $\mathcal{F}$ is a reduced Routley-Meyer frame and $v : \mathsf{At} \mapsto \wp(W)$ is a valuation function on atomic formulas, such that, if $R0ab$ and $a \in v(p)$, then $b \in v(p)$, for all $p \in \mathsf{At}$. The valuation is then extended to the whole language in the following way:

$$\mathcal{M}, a \Vdash p \quad \text{iff} \quad a \in v(p) \tag{1}$$
$$\mathcal{M}, a \Vdash {\sim}A \quad \text{iff} \quad \mathcal{M}, a^* \nVdash A \tag{2}$$
$$\mathcal{M}, a \Vdash A \wedge B \quad \text{iff} \quad \mathcal{M}, a \Vdash A \text{ and } \mathcal{M}, a \Vdash B \tag{3}$$
$$\mathcal{M}, a \Vdash A \vee B \quad \text{iff} \quad \mathcal{M}, a \Vdash A \text{ or } \mathcal{M}, a \Vdash B \tag{4}$$
$$\mathcal{M}, a \Vdash A \to B \quad \text{iff} \quad \forall b, c \in W, \text{ if } Rabc \text{ and } \mathcal{M}, b \Vdash A, \text{ then } \mathcal{M}, c \Vdash B \tag{5}$$

Finally, we say that a formula A is *satisfied* in a model $\mathcal{M} = \langle \mathcal{F}, v \rangle$ iff $\mathcal{M}, 0 \Vdash A$ and that 'A *entails* B in $\mathcal{M}$' iff, for all $a \in W$, if $a \Vdash A$, then $a \Vdash B$. A formula A is *valid* in a frame $\mathcal{F} = \langle W, 0, *, R \rangle$ iff, for all valuations v, the formula A is satisfied in $\mathcal{M}$.

Observation 1. In the previous definitions we have introduced a so-called *reduced* model for relevant logics (see e.g., [35, 9]). These models were introduced as alternative structures to what might be called *non reduced* models, see e.g., [33, 34].[3] There are some main differences to consider. Let $\mathcal{F}'$ and $\mathcal{M}'$ be denoting non reduced frames and models, respectively. $\mathcal{F}'$ is the following structure $\langle W, 0, T, *, R \rangle$, where, 0 is taken to be a subset of W, rather than a singleton, and T is a distinct element $T \in 0$, called *designated situation*. The members of 0 are referred to as *regular situations*. A model $\mathcal{M}'$ is the structure $\langle \mathcal{F}', v \rangle$. Finally, satisfaction in a model is defined with respect to regular situations, i.e., A is *satisfied* in a model $\mathcal{M}'$ iff $\mathcal{M}', x \Vdash A$, for all $x \in 0$. Validity on $\mathcal{F}'$ is defined as before.

An important, standard lemma is that preservation of truth along the heredity ordering holds for arbitrary formulas:

[3] According to [9, 442], "reduced models are technically and practically important for the practicing logician. They are simpler and hence easier to use".

Lemma 2.1 ([30, 32, 7])**.** *If $R0ab$ and $\mathcal{M}, a \Vdash A$, then $\mathcal{M}, b \Vdash A$.*

Furthermore, we state a result showing the equivalence between the *satisfaction* of an implication in a model and the notion of *entailment* in that model. This results is often referred to as *verification lemma* (see [7]).

Lemma 2.2 ([30, 32, 7])**.** *A entails B in a given model $\mathcal{M}$ iff $A \to B$ is satisfied in that model, i.e., for all $a \in W$, ($\mathcal{M}, a \Vdash A \implies \mathcal{M}, a \Vdash B$) iff $\mathcal{M}, 0 \Vdash A \to B$.*

From the perspective of axiomatic systems, **B** is the least set of formulas containing all instances of the following axioms and closed under the following rules. (We employ $\Rrightarrow$ as a rule-forming operator, distinct from both, the meta-level symbol $\implies$ and the sequent arrow $\Rightarrow$.)

(A1)	$A \to A$	(R1)	$A, A \to B \Rrightarrow B$
(A2)	$A_1 \wedge A_2 \to A_i$	(R2)	$A, B \Rrightarrow A \wedge B$
(A3)	$(A \to B) \wedge (A \to C) \to (A \to (B \wedge C))$	(R3)	$A \to B \Rrightarrow (C \to A) \to (C \to B)$
(A4)	$A_i \to (A_1 \vee A_2)$	(R4)	$A \to B \Rrightarrow (B \to C) \to (A \to C)$
(A5)	$(A \to C) \wedge (B \to C) \to ((A \vee B) \to C)$	(R5)	$A \to B \Rrightarrow {\sim}B \to {\sim}A$
(A6)	$A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$		
(A7)	${\sim}{\sim}A \to A$		

2.2 Stronger relevant logics

In this subsection, we will present some Hilbert systems for some common stronger relevant logics, which can be obtained by the addition of axioms to the system for **B**. Likewise, frames for **B**, $\mathcal{F}_{\mathbf{B}}$, can be extended to capture stronger relevant logics by adding some further constraints on R. In what follows, we display a list of axioms and the frame conditions imposed on Routley-Meyer frames to validate them. Some of these conditions appeal to the standard definitions, $Rabcd ::= \exists x(Rabx \wedge Rxcd)$ and $Ra(bc)d ::= \exists x(Raxd \wedge Rbcx)$:

(A8)	$(A \to B) \to ({\sim}B \to {\sim}A)$	(F6)	$Rabc \implies Rac^*b^*$
(A9)	$(A \to B) \wedge (B \to C) \to (A \to C)$	(F7)	$Rabc \implies Ra(ab)c$
(A10)	$(A \to B) \to ((B \to C) \to (A \to C))$	(F8)	$Rabcd \implies Rb(ac)d$
(A11)	$(A \to B) \to ((C \to A) \to (C \to B))$	(F9)	$Rabcd \implies Ra(bc)d$
(A12)	$(A \to (A \to B)) \to (A \to B)$	(F10)	$Rabc \implies Rabbc$
(A13)	$(A \wedge (A \to B)) \to B$	(F11)	$Raaa$
(A14)	$(A \to {\sim}A) \to {\sim}A$	(F12)	Raa^*a
(A15)	$(A \to (B \to C)) \to (B \to (A \to C))$	(F13)	$Rabcd \implies Racbd$
(A16)	$A \to ((A \to B) \to B)$	(F14)	$Rabc \implies Rbac$
(A17)	$A \vee {\sim}A$	(F15)	$R00^*0$
(A18)	$((A \to A) \to B) \to B$	(F16)	$Ra0a$
(A19)	$A \to (A \to A)$	(F17)	$Rabc \implies (R0ac \vee R0bc)$

The following well known relevant logics can be obtained by combinations of the indicated axioms and frame conditions.

$\mathbf{B} = (\mathsf{A1}) - (\mathsf{A7}) + (\mathsf{R1}) - (\mathsf{R5})$	$\mathcal{F}_{\mathbf{B}} = (\mathsf{F1}) - (\mathsf{F5})$
$\mathbf{DW} = \mathbf{B} + (\mathsf{A8})$	$\mathcal{F}_{\mathbf{DW}} = \mathcal{F}_{\mathbf{B}} + (\mathsf{F6})$
$\mathbf{DJ} = \mathbf{DW} + (\mathsf{A9})$	$\mathcal{F}_{\mathbf{DJ}} = \mathcal{F}_{\mathbf{DW}} + (\mathsf{F7})$
$\mathbf{TW} = \mathbf{DJ} + (\mathsf{A10}) + (\mathsf{A11})$	$\mathcal{F}_{\mathbf{TW}} = \mathcal{F}_{\mathbf{DJ}} + (\mathsf{F8}) + (\mathsf{F9})$
$\mathbf{T} = \mathbf{TW} + (\mathsf{A12}) + (\mathsf{A13}) + (\mathsf{A14}) + (\mathsf{A17})$	$\mathcal{F}_{\mathbf{T}} = \mathcal{F}_{\mathbf{TW}} + (\mathsf{F10}) + (\mathsf{F11}) + (\mathsf{F12}) + (\mathsf{F15})$
$\mathbf{RW} = \mathbf{TW} + (\mathsf{A15}) + (\mathsf{A16})$	$\mathcal{F}_{\mathbf{RW}} = \mathcal{F}_{\mathbf{TW}} + (\mathsf{F13}) + (\mathsf{F14})$
$\mathbf{R} = \mathbf{B} + (\mathsf{A8}) - (\mathsf{A18})$	$\mathcal{F}_{\mathbf{R}} = \mathcal{F}_{\mathbf{B}} + (\mathsf{F6}) - (\mathsf{F16})$
$\mathbf{RM} = \mathbf{R} + (\mathsf{A19})$	$\mathcal{F}_{\mathbf{RM}} = \mathcal{F}_{\mathbf{R}} + (\mathsf{F17})$

Let $\mathbf{X} = \{\mathbf{B}, \mathbf{DW}, \mathbf{DJ}, \mathbf{TW}, \mathbf{T}, \mathbf{RW}, \mathbf{R}, \mathbf{RM}\}$.

Theorem 2.3 ([30, 32, 7])**.** *A formula A is a theorem of $\mathbf{X}$ if and only if A is valid in all Routley-Meyer frames, $\mathcal{F}_{\mathbf{X}}$.*

Let us now proceed towards the construction of our intended labelled calculi.

3 Proof System

In this section, we shall define a family of modular calculi for relevant logics. First of all, we enrich our language with labels $(a, b, c, \dots, x, y, z, \dots)$ denoting states in Routley-Meyer models and an expression to formalize the forcing relation. Formally:

Definition 3.1. Let W be a set of labels, including a distinguished label denoted 0, and $\mathcal{L}$ be the language of $\mathbf{B}$. To express the forcing relation $a \Vdash A$ via sequents we use the notation $a : A$, for $A \in \mathsf{Form}$ and $a \in W$. The set of well-formed formulas consists of (1) labelled formulas $a : A$ and (2) relational atoms $Rabc$, for all $A \in \mathsf{Form}$ and $a, b, c \in W$. Finally, given two multisets Γ, Δ of labelled formulas and relational atoms, a *labelled sequent* is an object of the following form: $\Gamma \Rightarrow \Delta$.

Furthermore, the labelled rules of our sequent system are subject to the following closure condition. Consider a rule $\mathcal{R}$ of the following form:

$$\frac{A, B_1, \dots, B_n, B_{n+1}, B_{n+1}, \Gamma \Rightarrow \Delta}{B_1, \dots, B_n, \Gamma \Rightarrow \Delta}\ \mathcal{R}$$

Applying the *closure condition* on $\mathcal{R}$ means to substitute the multiple occurrences B_{n+1}, B_{n+1} with a single one to obtain a rule $\mathcal{R}^*$ of the following shape:

$$\frac{A, B_1, \dots, B_n, B_{n+1}, \Gamma \Rightarrow \Delta}{B_1, \dots, B_n, \Gamma \Rightarrow \Delta}\ \mathcal{R}^*$$

We remark that the rules of **G3rB** are defined by analysing the semantic conditions of Definition 2.3 of the corresponding operators. More precisely, the sequent

Axioms For p atomic:

$$R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$$

(possibly, a^*, b^*)

Logical rules

$$\frac{\Gamma \Rightarrow \Delta, a^* : A}{a : {\sim}A, \Gamma \Rightarrow \Delta}\ L{\sim} \qquad \frac{a^* : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : {\sim}A}\ R{\sim}$$

$$\frac{a : A, a : B, \Gamma \Rightarrow \Delta}{a : A \wedge B, \Gamma \Rightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, a : A \quad \Gamma \Rightarrow \Delta, a : B}{\Gamma \Rightarrow \Delta, a : A \wedge B}\ R\wedge$$

$$\frac{a : A, \Gamma \Rightarrow \Delta \quad a : B, \Gamma \Rightarrow \Delta}{a : A \vee B, \Gamma \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow \Delta, a : A, a : B}{\Gamma \Rightarrow \Delta, a : A \vee B}\ R\vee$$

$$\frac{Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \quad Rabc, a : A \to B, c : B, \Gamma \Rightarrow \Delta}{Rabc, a : A \to B, \Gamma \Rightarrow \Delta}\ L \to$$

$$(b, c \text{ fresh})\ \frac{Rabc, b : A, \Gamma \Rightarrow \Delta, c : B}{\Gamma \Rightarrow \Delta, a : A \to B}\ R \to$$

Relational rules for R

$$\frac{R0a^{**}a, R0aa^{**}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R1}$$

$$\frac{R0aa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{R2} \qquad \frac{R0ac, R0ab, R0bc, \Gamma \Rightarrow \Delta}{R0ab, R0bc, \Gamma \Rightarrow \Delta}\ \text{R3}$$

$$\frac{Rdbc, R0da, Rabc, \Gamma \Rightarrow \Delta}{R0da, Rabc, \Gamma \Rightarrow \Delta}\ \text{R4} \qquad \frac{R0b^*a^*, R0ab, \Gamma \Rightarrow \Delta}{R0ab, \Gamma \Rightarrow \Delta}\ \text{R5}$$

Figure 1: **G3rB**

$$\frac{Rac^*b^*, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \mathrm{R6} \qquad (x \text{ fresh})\ \frac{Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \mathrm{R7}$$

$$(y \text{ fresh})\ \frac{Rbyd, Racy, Rabx, Rxcd, \Gamma \Rightarrow \Delta}{Rabx, Rxcd, \Gamma \Rightarrow \Delta}\ \mathrm{R8}$$

$$(y \text{ fresh})\ \frac{Rayd, Rbcy, Rabx, Rxcd, \Gamma \Rightarrow \Delta}{Rabx, Rxcd, \Gamma \Rightarrow \Delta}\ \mathrm{R9}$$

$$(x \text{ fresh})\ \frac{Rabx, Rxbc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \mathrm{R10}$$

$$\frac{Raaa, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{R11} \qquad \frac{Raa^*a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{R12}$$

$$(y \text{ fresh})\ \frac{Racy, Rybd, Rabx, Rxcd, \Gamma \Rightarrow \Delta}{Rabx, Rxcd, \Gamma \Rightarrow \Delta}\ \mathrm{R13} \qquad \frac{Rbac, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \mathrm{R14}$$

$$\frac{R00^*0, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{R15} \qquad \frac{Ra0a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{R16}$$

$$\frac{R0ac, Rabc, \Gamma \Rightarrow \Delta \qquad R0bc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ \mathrm{R17}$$

Figure 2: Further mathematical rules for R

system is obtained by formulating the rules according to the methodology introduced for modal and intermediate logics in [20, 21]. We remark that axiomatic sequents are stated in their weakening-absorbing version, while the premises of $L \to$ are contraction-absorbing. Importantly, $R \to$ has the eigenvariable condition, that is, each root-first application of the rule requires the introduction of *fresh* (i.e., not previously used) labels.
In addition to the newly introduced rules for $\to$, there are also rules for R constructed through the method of conversion of frame conditions into sequent calculus rules. More precisely, we first have observed that all frame conditions are formulated either as *universal axioms* or *geometric implications* and, then, by following the methodology described in [20] (but previously also in [19, 23, 27]), we have transformed them into well-constructed sequent-style rules. Universal axioms are first turned into conjunctive normal form, namely, $P_1 \wedge \cdots \wedge P_i \to Q_1 \vee \cdots \vee Q_j$ and, then, into suitably formulated rules. Geometric implications, instead, are formulas of the following shape $\forall \overline{z}(A \to B)$, where A and B are *geometric formulas*, i.e., they do not contain neither $\forall$ nor $\to$. As before, we first turn them into conjunctive normal form, namely, $\forall \overline{x}(P_1 \wedge \cdots \wedge P_i \to \exists y_1 M_1 \vee \cdots \vee \exists y_j M_j)$ and, then, convert them into the corresponding rule-schemes. Notice that according to this strategy, we are allowed to obtain modular extensions of **G3rB** (Figure 1) by transforming further frame conditions (see list on p. 51) into sequent-style rules (Figure 2). Such extensions can be characterised as follows:

$\mathbf{G3rDW} = \mathbf{G3rB} + \mathrm{R6}$	$\mathbf{G3rDJ} = \mathbf{G3rDW} + \mathrm{R7}$
$\mathbf{G3rTW} = \mathbf{G3rDJ} + \mathrm{R8} + \mathrm{R9}$	$\mathbf{G3rT} = \mathbf{G3rTW} + \mathrm{R10} + \mathrm{R11} + \mathrm{R12} + \mathrm{R15}$
$\mathbf{G3rRW} = \mathbf{G3rTW} + \mathrm{R14} + \mathrm{R13}$	$\mathbf{G3rR} = \mathbf{G3rRW} + \mathbf{G3rT} + \mathrm{R16}$
$\mathbf{G3rRM} = \mathbf{G3rR} + \mathrm{R17}$	

Observation 2. Kurokawa and Negri [17] developed a family of labelled calculi for a wide range of relevant logics by using *non reduced* Routley-Meyer models as starting point. We recall that in these latter (i) 0 is taken to be a subset of W, rather than a singleton, and (ii) there is an element $T \in 0$. Although we followed the same methodology to obtain our intended systems, there are some substantial differences.

1. The notion of validity is not defined at the base element T, but it refers to all regular situations (see Observation 1 and [17, §3.2]) and this is reflected at the calculus level as follows: for all $x \in 0$, if $x \Vdash A$, then $0x \Rightarrow x : A$ (see [17, §6]).
2. The formulations of the rules for relevant implication involves an auxiliary unary operator, i.e., the indexed modality $\Box_a$. The index a gives a ternary relation, denoted bR_ac, which is taken as an assignment of a binary relation to an index, rather than expressing a compossibility relation between situations. However, as the authors themselves remark, this "choice is not mandatory, i.e., the ternary relation for implication could be directly handled without using the indexed modality. But via the indexed modality

we can obtain a uniformity with [...] works on conditional logics" [17, §1], i.e., with labelled systems proposed for conditional logics, for example, in [24, 11].

3. The semantic condition for indexed modalities is $b \Vdash \Box_a A$ iff $\forall c(bR_a c \Longrightarrow a \Vdash A)$ and it is used to to formulated the clause for $\rightarrow$ as follows $a \Vdash A \rightarrow B$ iff $\forall b(b \Vdash A \Longrightarrow b \Vdash \Box_a B)$. Accordingly, the rules for both, $\Box_a$ and $\rightarrow$, are formulated as follows:

$$\dfrac{a : A \rightarrow B, \Gamma \Rightarrow \Delta, b : A \qquad b : \Box_a B, a : A \rightarrow B, \Gamma \Rightarrow \Delta}{a : A \rightarrow B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L_2 \rightarrow}$$

$$\text{(}b\text{ fresh)}\ \dfrac{b : A, \Gamma \Rightarrow \Delta, b : \Box_a B}{\Gamma \Rightarrow \Delta, a : A \rightarrow B}\ {\scriptstyle R_2 \rightarrow}$$

$$\dfrac{c : A, bR_a c, b : \Box_a A, \Gamma \Rightarrow \Delta, b : \Box_a B}{bR_a c, b : \Box_a A, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\Box_a} \qquad \text{(}c\text{ fresh)}\ \dfrac{bR_a c, \Gamma \Rightarrow \Delta, c : A}{\Gamma \Rightarrow \Delta, b : \Box_a A}\ {\scriptstyle R\Box_a}$$

4. Axiomatic sequents are only of the form $a : p, \Gamma \Rightarrow \Delta, a : p$ and, in order to preserve the heredity property at the calculus level, the following rule is included:

$$\dfrac{b : p, a \leq b, a : p, \Gamma \Rightarrow \Delta}{a \leq b, a : p, \Gamma \Rightarrow \Delta}\ \textsc{AtHer}$$

   Since this rule is a form of contraction, it is preferable to have a system in which this rule is height-preserving admissible (proved in Lemma 4.3). This is the reason why we have heredity incorporated in axioms. Moreover, in the presence of Proposition 4.2 (below) the generalized version of AtHer can be derived using (admissible) cut and contraction (see Proposition 4.4).

Although the non reduced Routley-Meyer semantics allows for a characterization of a wider range of relevant logics, the labelled systems constructed out of it can be shown to be semantically complete only indirectly (at least for the moment), and this is mainly due to the definition of validity on regular situations (elements of 0), see [17, §6]. Nonetheless, Kurokawa and Negri observe that the lack of a direct proof seems to be far from being an insurmountable problem and argue that such "a proof of completeness by proof-search must be possible, since labelled sequent calculi are in general suitable for proof-search and invertible rules preserve countermodels" [17, §8]. Instead, notice that if validity is defined w.r.t. the distinct element $0 \in W$ (considered as a singleton), we can lay out a direct completeness proof without encountering the difficulties connected to the presence of regular situations. Indeed, in [26, 276], it was noticed:

> "The labelled approach allows for a fine distinction between various notions of logical consequence that can be adopted: *actualistic* logical consequence is logical consequence relative to the actual world,

whereas *universal* (or *strong*) consequence is relative to an arbitrary world."

By keeping this distinction in mind, we will provide an *actualistic* completeness proof, i.e., we will show that if a formula A is valid at the actual world 0, then the sequent $\Rightarrow 0 : A$ is derivable (see Section 6).
Before going ahead, let us summarize the central results contained in the following sections.

1. A is a theorem of $\mathbf{X}$.
2. A is provable in $\mathbf{G3rX} + \textsc{cut}$, and $\textsc{cut}$ has the following shape:

$$\frac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

3. A is provable in $\mathbf{G3rX}$.
4. A is valid in every Routley-Meyer frame for $\mathbf{X}$.

The equivalence between 1 and 4 is stated in Theorem 2.3. $1 \Longrightarrow 2$ and $4 \Longrightarrow 3$ are both proved in Section 6 (Theorems 6.2 and 6.3); $2 \Longrightarrow 3$ is proved in Section 7 (Theorem 7.4), $3 \Longrightarrow 4$ is proved in Section 5 (Theorem 5.1).

4 Preliminary results

In this section, we show some preliminary results. Let us start by introducing the notions of *weight* of formulas and *height* of derivations in the standard way. (Let $\mathbf{X} = \{\mathbf{B}, \mathbf{DW}, \mathbf{DJ}, \mathbf{TW}, \mathbf{T}, \mathbf{RW}, \mathbf{R}, \mathbf{RM}\}$.)

Definition 4.1. Let $\mathcal{A}$ be any labelled formula of the form $a : A$. We denote by $l(\mathcal{A})$ the label of a formula $\mathcal{A}$, and by $p(\mathcal{A})$ the pure part of the formula, that is, the part of the formula without the label. The *weight* (or *complexity*) of a labelled formula is defined as a lexicographically ordered pair: $\langle \mathsf{w}(p(\mathcal{A})), \mathsf{w}(l(\mathcal{A}))\rangle$, where:

1. for all state labels $a \in W$, $\mathsf{w}(a) = 1$;
2. for all $p \in \mathsf{At}$, $\mathsf{w}(p) = 1$;
3. $\mathsf{w}({\sim}A) = \mathsf{w}(A) + 1$;
4. $\mathsf{w}(A \circ B) = \mathsf{w}(A) + \mathsf{w}(B) + 1$, for $\circ \in \{\wedge, \vee, \rightarrow\}$.

Definition 4.2. We denote by $h(\delta)$ the natural number indicating the height of a derivation. We associate the height with the longest branch in a proof-tree $\delta - 1$. The height of a derivation $h(\delta)$ is defined by induction on the construction of δ:

$$\delta \equiv \{\ \Gamma \Rightarrow \Delta \qquad\qquad\qquad\qquad h(\delta) = 0$$

$$\delta \equiv \left\{ \begin{array}{c} \vdots \}\delta_1 \\ \dfrac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta} \mathcal{R} \end{array} \right. \qquad h(\delta) = h(\delta_1) + 1$$

$$\delta \equiv \left\{ \begin{array}{c} \dfrac{\dfrac{\vdots \}\delta_1}{\Gamma' \Rightarrow \Delta'} \mathcal{R}' \qquad \dfrac{\vdots \}\delta_2}{\Gamma'' \Rightarrow \Delta''} \mathcal{R}''}{\Gamma \Rightarrow \Delta} \mathcal{R} \end{array} \right. \qquad h(\delta) = h(\delta_1) + 1,\ h(\delta_2) + 1$$

Finally, let $\mathcal{S}$ be a sequent. The notation '$\delta \vdash \mathcal{S}$' stands for '$\delta$ is a proof/derivation of $\mathcal{S}$' with $h(\delta) \leq n$ and '$\vdash^n \mathcal{S}$' stands for '$\mathcal{S}$ has a derivation δ of height n'.

Definition 4.3. A rule $\mathcal{R}$ is *height-preserving admissible* just in case: if there is a derivation of the premise(s) of $\mathcal{R}$, then there is a derivation of the conclusion of $\mathcal{R}$ that contains no application of $\mathcal{R}$ (with the height at most n, where n is the maximal height of the derivation of the premise(s)).

Definition 4.4. We define substitution as follows:

- $Rabc(d/e) \equiv Rabc$, if $e \neq a$, $e \neq b$ and $e \neq c$.
- $Rabc(d/a) \equiv Rdbc$, if $a \neq b$ and $a \neq c$.
- $Rabc(d/b) \equiv Radc$, if $b \neq a$ and $b \neq c$.
- $Rabc(d/c) \equiv Rabd$, if $c \neq a$ and $c \neq b$.
- $Raac(d/a) \equiv Rddc$, if $a = b$ and $a \neq c$.
- $Rabb(d/b) \equiv Radd$, if $b = c$ and $b \neq a$.
- $Rcbc(d/c) \equiv Rdbd$, if $c = a$ and $c \neq b$.
- $Raaa(d/a) \equiv Rddd$, if $a = b$ and $a = c$.
- $a : A(d/b) \equiv a : A$, if $b \neq a$.
- $a : A(d/a) \equiv d : A$.

Next we extend this definition to multisets. Similar proofs for labelled calculi for logics characterised by ternary relations are included, e.g., in [24, 12, 17].

Lemma 4.1. *Let the variable e stand for either a, b or c. If* $\mathbf{G3rX} \vdash^n \Gamma \Rightarrow \Delta$ *and, provided d is free for e in Γ, Δ, then* $\mathbf{G3rX} \vdash^n \Gamma(d/e) \Rightarrow \Delta(d/e)$ *(allowing $*$-variables to be substituted to variables as well).*

Proof. Let $n = 0$. If $\Gamma \Rightarrow \Delta$ is an axiom and (d/e) is not a vacuous substitution, then the substitution $\Gamma(d/e) \Rightarrow \Delta(d/e)$ is also an axiom. Let $n > 0$. If we are considering a propositional rule, we apply the inductive hypothesis to the premise(s) of the rule, and then the rule again. For example, let $\Gamma = e : {\sim}A, \Gamma'$ and $e = a, b, c$:

$$\frac{\vdash^n \Gamma' \Rightarrow \Delta, e^* : A}{\vdash^{n+1} e : {\sim}A, \Gamma' \Rightarrow \Delta}\ L{\sim}$$

In this case, in order to apply $L{\sim}$, we substitute d^*/e^* by the inductive hypothesis, and get the following derivation of the same height:

$$\frac{\vdash^n \Gamma'(d^*/e^*) \Rightarrow \Delta(d^*/e^*), d^* : A}{\vdash^{n+1} d : {\sim}A, \Gamma'(d^*/e^*) \Rightarrow \Delta(d^*/e^*)}\ L{\sim}$$

We proceed similarly if the last rule is $L\to$ (without the variable condition). Finally, let's consider the only rule with the eigenvariable condition, namely $R\to$.
(1) If the substitution is vacuous ($e \neq a, b, c$), then there's nothing to do.
(2) Assume the substitution d/e is not vacuous and d is not a fresh variable. We have to consider the case where d is substituted for a. Let $\Delta = a : A \to B, \Delta'$:

$$\frac{\vdash^n Razx, z : A, \Gamma \Rightarrow \Delta', x : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \to B}\ R\to$$

By the application of the inductive hypothesis (d/a) we obtain the following application of $R\to$ with the same derivation height:

$$\frac{\vdash^n Rdzx, z : A, \Gamma(d/a) \Rightarrow \Delta'(d/a), x : B}{\vdash^{n+1} \Gamma(d/a) \Rightarrow \Delta'(d/a), d : A \to B}\ R\to$$

(3) The substitution is non-vacuous, and d is an eigenvariable. So, our derivation ends as follows:

$$\frac{\vdash^n Radc, d : A, \Gamma \Rightarrow \Delta', c : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \to B}\ R\to \qquad \frac{\vdash^n Rabd, b : A, \Gamma \Rightarrow \Delta', d : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \to B}\ R\to$$

First, we rename the fresh variables d, c and b, d with z, x and x, z, respectively. By the variable condition the substitution does not affect Γ, Δ'. Indeed, we get the following premise of height n:

$$Raxz, x : A, \Gamma \Rightarrow \Delta', z : B \text{ and } Razx, z : A, \Gamma \Rightarrow \Delta', x : B$$

. So, by applying inductive hypothesis, we substitute the labels d/b and d/c, respectively, to conclude:

$$\frac{\vdash^n Razx, z : A, \Gamma(d/b) \Rightarrow \Delta'(d/b), x : B}{\vdash^{n+1} \Gamma(d/b) \Rightarrow \Delta'(d/b), a : A \to B}\ R\to \qquad \frac{\vdash^n Raxz, x : A, \Gamma(d/c) \Rightarrow \Delta'(d/c), z : B}{\vdash^{n+1} \Gamma(d/c) \Rightarrow \Delta'(d/c), a : A \to B}\ R\to$$

Analogous results follow also for relational rules. Some of them subject to the eigenvariable condition and, as usual, more care is needed. Roughly, the cases for such relational rules follow the pattern of case 3 above: to avoid clashes of variables, we apply height-preserving substitution before the inductive hypothesis and conclude the argument by finally applying the rule. □

As in the case of other labelled calculi for intermediate logics (e.g., [8, 18]), the heredity property of the forcing relation (Lemma 2.1) can be expressed by means of formal derivations in the calculus:

Proposition 4.2. *Sequents of the following form are derivable in* **G3rX**: $R0ab, a : A, \Gamma \Rightarrow \Delta, b : A$.

Proof. By induction on A. Let $A = {\sim}B$ and consider the following derivation:

$$\dfrac{\dfrac{\dfrac{R0b^*a^*, R0ab, b^* : B, \Gamma \Rightarrow \Delta, a^* : B}{R0ab, b^* : B, \Gamma \Rightarrow \Delta, a^* : B}\ \text{R5}}{R0ab, b^* : B, a : {\sim}B, \Gamma \Rightarrow \Delta}\ L{\sim}}{R0ab, a : {\sim}B, \Gamma \Rightarrow \Delta, b : {\sim}B}\ R{\sim}$$

where the premises are derivable by inductive hypothesis.
If $A = B \to C$, then we obtain the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Racd, \mathcal{S}, c : B, \Gamma \Rightarrow \Delta, d : C, c : B}{Racd, \mathcal{S}, c : B, \Gamma \Rightarrow \Delta, d : C, c : B}\ \text{R2} \quad \dfrac{R0dd, Racd, \mathcal{S}, d : C, c : B, \Gamma \Rightarrow \Delta, d : C}{Racd, \mathcal{S}, d : C, c : B, \Gamma \Rightarrow \Delta, d : C}\ \text{R2}}{Racd, R0ab, Rbcd, a : B \to C, c : B, \Gamma \Rightarrow \Delta, d : C}\ L\to}{R0ab, Rbcd, a : B \to C, c : B, \Gamma \Rightarrow \Delta, d : C}\ \text{R4}}{R0ab, a : B \to C, \Gamma \Rightarrow \Delta, b : B \to C}\ R\to\ (c, d \text{ fresh})$$

where $\mathcal{S}$ abbreviates $R0ab, Rbcd, a : B \to C$. The cases for A being $B \wedge C$ or $B \vee C$ are straightforward. □

Lemma 4.3. *The following rules:*

$$\dfrac{b : p, R0ab, a : p, \Gamma \Rightarrow \Delta}{R0ab, a : p, \Gamma \Rightarrow \Delta}\ \textsc{AtHer-l} \qquad \dfrac{R0ab, \Gamma \Rightarrow \Delta, b : p, a : p}{R0ab, \Gamma \Rightarrow \Delta, b : p}\ \textsc{AtHer-r}$$

are height-preserving admissible.

Proof. We display the details for AtHer-l, but the argument is the same for AtHer-r. By induction on the height of δ, we prove that for any proof of $b : p, R0ab, a : p, \Gamma \Rightarrow \Delta$, there exists a proof of $R0ab, a : p, \Gamma \Rightarrow \Delta$ of the same (or smaller) height. The base cases are obtained as follows:

$$\vdash^n b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p \quad \overset{i.h.}{\rightsquigarrow} \quad \vdash^n R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$$

$$\dfrac{\vdash^n R0aa^{**}, R0a^{**}a, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n+1} b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\ \text{R1} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n R0aa^{**}, R0a^{**}a, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n+1} R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\ \text{R1}$$

$$\dfrac{\vdash^n R0aa, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n+1} b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\ \text{R2} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n R0aa, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n+1} R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\ \text{R2}$$

$$\dfrac{\vdash^n R0ac, R0bc, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n+1} R0bc, b : p, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\ \text{R3} \quad \overset{i.h.}{\rightsquigarrow} \quad \dfrac{\vdash^n R0ac, R0bc, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}{\vdash^{n+1} R0bc, R0ab, a : p, \Gamma \Rightarrow \Delta, b : p}\ \text{R3}$$

$$\frac{\vdash^n Racd, Rbcd, b:p, R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}{\vdash^{n+1} Rbcd, b:p, R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}\text{ R4} \quad \overset{i.h.}{\rightsquigarrow} \quad \frac{\vdash^n Racd, Rbcd, R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}{\vdash^{n+1} Rbcd, R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}\text{ R4}$$

$$\frac{\vdash^n R0b^*a^*, b:p, R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}{\vdash^{n+1} b:p, R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}\text{ R5} \quad \overset{i.h.}{\rightsquigarrow} \quad \frac{\vdash^n R0b^*a^*, R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}{\vdash^{n+1} R0ab, a:p, \Gamma \Rightarrow \Delta, b:p}\text{ R5}$$

The remaining cases are dealt with analogously. The inductive step is completed by permutation of the rules. □

Proposition 4.4. *The following rules:*

$$\frac{b:A, R0ab, a:A, \Gamma \Rightarrow \Delta}{R0ab, a:A, \Gamma \Rightarrow \Delta}\textsc{ GenHer-l} \qquad \frac{R0ab, \Gamma \Rightarrow \Delta, b:A, a:A}{R0ab, \Gamma \Rightarrow \Delta, b:A}\textsc{ GenHer-r}$$

corresponding to the heredity rules for compound formulas, are admissible.

Proof. GenHer-l can be derived as follows:

$$\frac{R0ab, a:A, \Gamma \Rightarrow \Delta, b:A \qquad b:A, R0ab, a:A, \Gamma \Rightarrow \Delta}{R0ab, a:A, \Gamma \Rightarrow \Delta}\textsc{ cut+lc+rc+lc}_L$$

For GenHer-r we have the following derivation:

$$\frac{R0ab, \Gamma \Rightarrow \Delta, b:A, a:A \qquad R0ab, a:A, \Gamma \Rightarrow \Delta, b:A}{R0ab, \Gamma \Rightarrow \Delta, b:A}\textsc{ cut+lc+rc+lc}_L$$

where the leftmost (resp., rightmost) premise is derivable by Proposition 4.2, while the applications of contraction and cut are admissible by Lemma 7.3 and Theorem 7.4.[4] □

5 Soundness

This section is devoted to the proof of the soundness theorem for our systems ($3 \implies 4$, p. 57). We will show that the rules of each labelled calculus **G3rX** preserve validity over Routley-Meyer frames obeying the conditions appropriate for each relevant logic **X**. In order to do that, we start by extending semantic notions to sequents as follows:

Definition 5.1. Let $\mathcal{M} = \langle W, 0, *, R_{\mathcal{M}}, v\rangle$ be a model and let $\mathcal{S}$ be the sequent $\Gamma \Rightarrow \Delta$. We define a *$\mathcal{S}$-interpretation in $\mathcal{M}$* is a mapping $[\![\cdot]\!]$ from the labels in $\mathcal{S}$ to the set W of states in $\mathcal{M}$, such that (i) $0 = [\![0]\!]$ and (ii) if $Rabc$ is in Γ, then $R_{\mathcal{M}}[\![a]\!][\![b]\!][\![c]\!]$. Now we can define:

[4] We observe that this proposition can be proved in the same way as we proved admissibility of AtHer-l and AtHer-r, i.e., by induction on the height of the derivation. This, in fact would provide us with a stronger result, namely that GenHer-l and GenHer-r are height-preserving admissible in **G3rX**. However, here we omit the details of such a proof as we do not need this result throughout the paper.

$\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{S}$ iff if for all $a : A \in \Gamma$, we have $\mathcal{M}, [\![a]\!] \Vdash A$, then there exists $b : B \in \Delta$, such that $\mathcal{M}, [\![b]\!] \Vdash B$.

Definition 5.2. A sequent $\mathcal{S}$ is *satisfied* in $\mathcal{M} = \langle W, 0, *, R, v\rangle$ if for all $\mathcal{S}$-interpretations $[\![\cdot]\!]$ we have $\mathcal{M}, [\![\cdot]\!] \Vdash \mathcal{S}$. A sequent $\mathcal{S}$ is valid in a frame $\mathcal{F} = \langle W, 0, *, R\rangle$, if for all valuations v, the sequent $\mathcal{S}$ is satisfied in $\mathcal{M} = \langle W, 0, *, R, v\rangle$.

Finally, we can prove the soundness theorem:

Theorem 5.1. *If a sequent $\mathcal{S}$ is provable in* **G3rX**, *then it is valid in every Routley-Meyer frame for* **X**.

Proof. We proceed by induction on the height of the derivation of $\mathcal{S}$. We show that for each rule $\mathcal{R}$ of the form $\mathcal{P}_1, \ldots, \mathcal{P}_n/\mathcal{C}$, if the premises $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are valid in all Routley-Meyer frames, then so is $\mathcal{C}$. It follows from a case analysis on $\mathcal{R}$:

- Ax. By way of contradiction, assume that $R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$ is not valid in all Routley-Meyer frames. This means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash R0ab, a : p, \Gamma \Rightarrow \Delta, b : p$, i.e., $R_{\mathcal{M}}[\![0]\!][\![a]\!][\![b]\!]$ and $\mathcal{M}, a \Vdash p$, but $\mathcal{M}, b \nVdash p$. However, this is not possible given heredity (lemma 2.1).

- $L{\sim}$. By way of contradiction, assume that $\Gamma \Rightarrow \Delta, a^* : A$ is valid in all Routley-Meyer frames, but $a : {\sim}A, \Gamma \Rightarrow \Delta$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash a : {\sim}A, \Gamma \Rightarrow \Delta$, i.e., $\mathcal{M}, a \Vdash {\sim}A$, but $\mathcal{M}, d \nVdash C$ for all $d : C \in \Delta$. However, by the forcing clause (2), we also have $\mathcal{M}, a^* \nVdash A$. Consequently, $\mathcal{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, a^* : A$. Contradiction.

- $R{\sim}$. By way of contradiction, assume that $a^* : A, \Gamma \Rightarrow \Delta$ is valid in all Routley-Meyer frames, but $\Gamma \Rightarrow \Delta, a : {\sim}A$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, a : {\sim}A$, i.e., $\mathcal{M}, d \Vdash C$, for all $d : C \in \Gamma$ but $\mathcal{M}, a \nVdash {\sim}A$. However, by the forcing clause (2), we also have $\mathcal{M}, a^* \Vdash A$. Then, $\mathcal{M}, [\![\cdot]\!] \nVdash a^* : A, \Gamma \Rightarrow \Delta$. Contradiction.

- $L\rightarrow$. By way of contradiction, assume that $Rabc, \Gamma \Rightarrow \Delta, b : A$ and $Rabc, c : B, \Gamma \Rightarrow \Delta$ are valid in all Routley-Meyer frames, but $Rabc, a : A \rightarrow B, \Gamma \Rightarrow \Delta$ is not. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash Rabc, a : A \rightarrow B, \Gamma \Rightarrow \Delta$, i.e., $R_{\mathcal{M}}[\![a]\!][\![b]\!][\![c]\!]$ and $\mathcal{M}, a \Vdash A \rightarrow B$, but $\mathcal{M}, d \nVdash C$ for all $d : C \in \Delta$. However, by the forcing clause (5), we also have $\mathcal{M}, b \nVdash A$ or $\mathcal{M}, c \Vdash B$. Consequently, $\mathcal{M}, [\![\cdot]\!] \nVdash Rabc, \Gamma \Rightarrow \Delta, b : A$ or $\mathcal{M}, [\![\cdot]\!] \nVdash Rabc, c : B, \Gamma \Rightarrow \Delta$. Contradiction.

- $R\rightarrow$. By way of contradiction, assume that $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$ is valid in all Routley-Meyer frames, but $\Gamma \Rightarrow \Delta, a : A \rightarrow B$ is not, where $b, c \notin \Gamma, \Delta$. The latter means that there is a model $\mathcal{M}$ and an interpretation $[\![\cdot]\!]$, such that $\mathcal{M}, [\![\cdot]\!] \nVdash \Gamma \Rightarrow \Delta, a : A \rightarrow B$. In particular, we know that there are worlds b' and c' such that $R_{\mathcal{M}}[\![a]\!]b'c'$ and $\mathcal{M}, b' \Vdash A$, but $\mathcal{M}, c' \nVdash B$. Now we define an extension $[\![\cdot]\!]'$ of $[\![\cdot]\!]$ such that $[\![b]\!]' = b'$, $[\![c]\!]' = c'$ and $[\![\cdot]\!]' = [\![\cdot]\!]$. Then, $\mathcal{M}, [\![\cdot]\!]' \nVdash Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$. Contradiction.

The other cases are similar and simpler. In particular, note that the cases for the mathematical rules are trivial, as all Routley-Meyer frames have to obey the corresponding conditions. □

6 Completeness

In this section, we will show the completeness of **G3rB**, and its extensions, by deriving the axioms of the corresponding logics ($1 \implies 2$, p. 57).
Let $\mathbf{X} = \{\mathbf{B}, \mathbf{DW}, \mathbf{DJ}, \mathbf{TW}, \mathbf{T}, \mathbf{RW}, \mathbf{R}, \mathbf{RM}\}$.
Before turning to the proof the theorem, we show a syntactic version of Lemma 2.2 within our labelled calculi:

Lemma 6.1. $\mathbf{G3rX} + \textit{CUT} \vdash a : A \Rightarrow a : B$ *iff* $\mathbf{G3rX} + \textit{CUT} \vdash \Rightarrow 0 : A \to B$.

Proof. ($\Longrightarrow$)

$$\dfrac{\dfrac{a : A \Rightarrow a : B \qquad R0ab, a : B \Rightarrow b : B}{R0ab, a : A \Rightarrow b : B}\ \textsc{cut}}{\Rightarrow 0 : A \to B}\ R\to$$

($\Longleftarrow$)

$$\dfrac{\dfrac{\Rightarrow 0 : A \to B \qquad \dfrac{R0aa, 0 : A \to B, a : A \Rightarrow a : A, a : B \qquad R0aa, 0 : A \to B, a : A, a : B \Rightarrow a : B}{R0aa, 0 : A \to B, a : A \Rightarrow a : B}\ L\to}{R0aa, a : A \Rightarrow a : B}\ \textsc{cut}}{a : A \Rightarrow a : B}\ \text{R2}$$

where, in both derivations, the rightmost premise(s) is (are) derivable by Proposition 4.2. □

Theorem 6.2. *If a formula A is provable in an axiomatic system* $\mathbf{X}$, *then the sequent* $\Rightarrow 0 : A$ *is derivable in the corresponding labelled system* $\mathbf{G3rX} + \textit{CUT}$.

Proof. The proof proceeds by deriving root-first the axioms of each relevant logic $\mathbf{X}$ in the corresponding labelled system $\mathbf{G3rX} + \textsc{cut}$. As the derivations occupy much space, we display them in Appendix A. □

Alternatively, one might be interested in proving a theorem of semantic completeness, that is, for every sequent $\mathcal{S}$, the proof search either terminates in a proof or fails, and the failed proof tree is used to obtain a countermodel for $\mathcal{S}$. Intuitively, to see whether A is derivable, we check if it is valid at the actual world $0 \in W$, i.e., $0 \Vdash A$. This, indeed, will amount to have the sequent $\Rightarrow 0 : A$ in our calculus. As said above, this correspond to reflect, at the calculus level, the *actualistic* notion of validity employed in reduced Routley-Meyer models. Finally, notice that the countermodel construction argument, allows us to show completeness directly (although non-constructively, as the proof relies on König's lemma), for any labelled sequent and not only specifically for formulas.

Theorem 6.3. *Let* $\Gamma \Rightarrow \Delta$ *be a sequent of* **G3rX**. *Then either the sequent is derivable in* **G3rX** *or it has a countermodel with the frame properties peculiar for* **X**.

Proof. We follow the pattern of the completeness proof in [22, 25]. We proceed with the construction of a derivation tree for $\Gamma \Rightarrow \Delta$ by applying the rules of **G3rX** root-first (see Appendix B). If the reduction tree is finite, i.e., all leaves are axiomatic sequents, we have a proof in **G3rX**. Assume that the derivation tree is infinite. By König's lemma, it has an infinite branch that is used to build the needed counterexample. Suppose that $\Gamma \Rightarrow \Delta \equiv \Gamma_0 \Rightarrow \Delta_0, \Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_i \Rightarrow \Delta_i \dots$ is one of such branches. Consider the sets $\mathbf{\Gamma} \equiv \bigcup \Gamma_i$ and $\mathbf{\Delta} \equiv \bigcup \Delta_i$, for $i \geq 0$. We now construct a countermodel, i.e. a model that makes all labelled formulas and relational atoms in $\mathbf{\Gamma}$ true and all labelled formulas in $\mathbf{\Delta}$ false. Let $\mathcal{F}_{\mathbf{X}}$ be a frame, whose elements are all the labels occurring in $\mathbf{\Gamma}$. $\mathcal{F}_{\mathbf{X}}$ is defined as follows:

- for all $a : p$ in $\mathbf{\Gamma}$ it holds that $a \Vdash p$ in $\mathcal{F}_{\mathbf{X}}$.
- for all $Rabc$ in $\mathbf{\Gamma}$ it holds that $R_{\mathcal{M}}abc$ in $\mathcal{F}_{\mathbf{X}}$.
- for all $a : p$ in $\mathbf{\Delta}$ it holds that $a \nVdash p$ in $\mathcal{F}_{\mathbf{X}}$.

It can then be shown that A is forced in the model at 0 if $0 : A$ is in $\mathbf{\Gamma}$ and A is not forced at 0 if $0 : A$ is in $\mathbf{\Delta}$. We will end up with a countermodel to the endsequent.

1. If p is atomic, the claim holds by definition of the model.
2. If $0 : {\sim}A$ is in $\mathbf{\Gamma}$, then $0^* : A$ is in $\mathbf{\Delta}$. By the inductive hypothesis $0^* \nVdash A$, i.e., $0 \Vdash {\sim}A$.
3. If $0 : {\sim}A$ is in $\mathbf{\Delta}$, then $0^* : A$ is in $\mathbf{\Gamma}$. By the inductive hypothesis $0^* \Vdash A$, i.e., $0 \nVdash {\sim}A$.
4. If $0 : A \wedge B$ is in $\mathbf{\Gamma}$, then there exists i such that $0 : A \wedge B$ appears first in Γ_i, and, therefore, for some $j \geq 0$, we have $0 : A$ and $0 : B$ in Γ_{i+j}. By the inductive hypothesis $0 \Vdash A$ and $0 \Vdash B$ and, consequently, $0 \Vdash A \wedge B$. (The case for $0 : A \vee B$ in $\mathbf{\Delta}$ is analogous.)
5. If $0 : A \wedge B$ is in $\mathbf{\Delta}$, then either $0 : A$ or $0 : B$ in $\mathbf{\Delta}$. By the inductive hypothesis either $0 \nVdash A$ or $0 \nVdash B$ and, therefore, $0 \nVdash A \wedge B$. (The case for $0 : A \vee B$ in $\mathbf{\Gamma}$ is analogous.)
6. If $0 : A \to B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms $R0ab$ that occur in $\mathbf{\Gamma}$. If there's no relational atom, the accessibility condition is vacuously satisfied and, therefore, $0 \Vdash A \to B$ is in the model. For any occurrence of $R0ab$ in $\mathbf{\Gamma}$, by construction of the tree $a : A$ is in $\mathbf{\Delta}$ or $b : A$ is in $\mathbf{\Gamma}$. By the inductive hypothesis $a \nVdash A$ or $b \Vdash B$, and since $R_{\mathcal{M}}0ab$, we obtain $0 \Vdash A \to B$ in the model.
7. If $0 : A \to B$ is in $\mathbf{\Delta}$, at the successive step of the reduction tree we find that $R0ab$ and $a : A$ in $\mathbf{\Gamma}$, whereas $b : B$ is in $\mathbf{\Delta}$. By the inductive hypothesis we obtain $R_{\mathcal{M}}0ab$ and $a \Vdash A$ but $b \nVdash B$, that is, $0 \nVdash A \to B$ in the model.

□

This result directly implies the implication $4 \implies 3$ stated on p. 57.

Corollary 6.3.1. *If a sequent* $\Gamma \Rightarrow \Delta$ *is valid in every Routley-Meyer frame for* **X**, *then it is derivable in the system* **G3rX**.

7 Proof analysis and Cut-admissibility

In this section we prove the CUT-admissibility theorem for our labelled sequent calculi. The general proof presented here is similar to the proof for labelled systems for modal and intermediate logics (see, e.g., [20, 25, 8, 12, 17, 18]). More precisely, we proceed with the proofs of weakening and contraction admissibility. In conclusion, we show the central theorem of the section, i.e., CUT-admissibility. As there are many cases to be analysed in these proofs, we only outline the important parts here.

Lemma 7.1. *The rules of weakening:*

$$\frac{\Gamma \Rightarrow \Delta}{d : C, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, d : C}\ RW \qquad \frac{\Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ LW_L$$

are height-preserving admissible in **G3rX**.

Proof. By induction on the height of the derivation. (1) For $n = 0$, the case is trivial. For $n > 0$, we simultaneously display the transformed derivations for LW and RW. (Analogous results hold for LW_L)
(2) For rules without variable condition, the lower sequent of the transformed derivation is the same as the lower one of the original derivation, obtained by applying several times weakening. This is also the case for $L \rightarrow$.
(3) Consider the rules with the variable condition, e.g., $R \rightarrow$. The derivations end as follows:

$$\frac{\vdash^n Radc, d : A, \Gamma \Rightarrow \Delta', c : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \rightarrow B}\ R\rightarrow \qquad \frac{\vdash^n Rabd, b : A, \Gamma \Rightarrow \Delta', d : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \rightarrow B}\ R\rightarrow$$

To avoid clashes of variables we apply height-preserving substitution (x/d) to obtain:

$$Raxc, x : A, \Gamma(x/d) \Rightarrow \Delta'(x/d), c : B \text{ and } Rabx, b : A, \Gamma(x/d) \Rightarrow \Delta'(x/d), x : B$$

Finally, by applying the inductive hypothesis (on the left and on the right) to the premise and, finally, also the rule, we obtain the requested derivations:

$$\frac{\vdash^n d : C, Raxc, x : A, \Gamma \Rightarrow \Delta', c : B}{\vdash^{n+1} d : C, \Gamma \Rightarrow \Delta', a : A \rightarrow B}\ R\rightarrow \qquad \frac{\vdash^n Raxc, x : A, \Gamma \Rightarrow \Delta', c : B, d : C}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a : A \rightarrow B, d : C}\ R\rightarrow$$

and

$$\frac{\vdash^n d:C, Rabx, b:A, \Gamma \Rightarrow \Delta', x:B}{\vdash^{n+1} d:C, \Gamma \Rightarrow \Delta', a:A \to B} R\to \qquad \frac{\vdash^n Rabx, b:A, \Gamma \Rightarrow \Delta', x:B, d:C}{\vdash^{n+1} \Gamma \Rightarrow \Delta', a:A \to B, d:C} R\to$$

where, in all cases, the lower derivations are the result of applying weakening (on the left and on the right) to the premises of the derivations displayed above. If we consider relational rules without variable condition, the proof follows straightforwardly by applications of the inductive hypothesis.

For relational rules with eigenvariable conditions, we always are in need to consider possible clashes of variables. As an example, suppose that the rule applied is R7:

$$\frac{\vdash^n Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta} \text{R7}$$

If $d \neq x$, that is, the variable condition is not violated, then desired derivations follow by the inductive hypothesis and an application of the rule:

$$\frac{\vdash^n d:C, Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} d:C, Rabc, \Gamma \Rightarrow \Delta} \text{R7} \qquad \frac{\vdash^n Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta, d:C}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta, d:C} \text{R7}$$

If the fresh variable condition is violated, we substitute the clashing variable with a fresh one, apply the inductive hypothesis and then the rule. If the application of the rule looks like:

$$\frac{\vdash^n Rabd, Radc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta} \text{R7}$$

we substitute d with a fresh one, say y, to obtain the following premise

$$\vdash^n Raby, Rayc, Rabc, \Gamma(y/d) \Rightarrow \Delta(y/d)$$

By applying the inductive hypothesis and the rule, we obtain the desired derivations:

$$\frac{\vdash^n d:C, Raby, Rayc, Rabc, \Gamma \Rightarrow \Delta}{\vdash^{n+1} d:C, Rabc, \Gamma \Rightarrow \Delta} \text{R7} \qquad \frac{\vdash^n Raby, Rayc, Rabc, \Gamma \Rightarrow \Delta, d:C}{\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta, d:C} \text{R7}$$

where, as before, the lower derivations are the results of applying weakening (on the left and on the right) to the premise of the rule displayed above. □

Definition 7.1. A rule $\mathcal{R}$ is *height-preserving invertible* just in case: if there is a derivation of the conclusion of $\mathcal{R}$, then there is a dedrivation of premise(s) of $\mathcal{R}$ (with the height at most n, where n is the maximal height of the derivation of the conclusion).

Lemma 7.2. *All rules of* **G3rX** *are height-preserving invertible.*

Proof. For each rule $\mathcal{R}$, we have to show that if there is a derivation δ of the conclusion, then there is a derivation δ' of the premise(s), of the same height. For $L\!\sim$, $R\!\sim$, $L\vee$, $R\vee$, $R\wedge$, $L\wedge$ and $L\rightarrow$ we use a standard induction on the height of δ. For $R\rightarrow$ as well, but we need to be sure that in the transformed derivation we make use of a fresh label by applying the substitution lemma inside δ', if needed. The same procedures apply to all relational rules (R1-R17).
As an interesting example, we show height-preserving invertibility of $R\rightarrow$. It is proved by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta, a : A \rightarrow B$. We distinguish three main cases. (1) If $n = 0$, $\Gamma \Rightarrow \Delta, a : A \rightarrow B$ is an axiom, and then also $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$ is an axiom. Let $n > 0$. (2) If $\vdash^{n+1}$ $\Gamma \Rightarrow \Delta, a : A \rightarrow B$ is concluded by any rule $\mathcal{R}$ other than $R\rightarrow$, we apply the inductive hypothesis to the premise(s) $\Gamma' \Rightarrow \Delta', a : A \rightarrow B$ ($\Gamma'' \Rightarrow \Delta'', a : A \rightarrow B$) to obtain derivation(s) of height n of $Rabc, b : A, \Gamma' \Rightarrow \Delta', c : B$ ($Rabc, b : A, \Gamma'' \Rightarrow \Delta'', c : B$). By applying $\mathcal{R}$ we obtain a derivation of height $n+1$ of $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$, as desired. (3) If $\vdash^{n+1} \Gamma \Rightarrow \Delta, a : A \rightarrow B$ is concluded by $R\rightarrow$, then $Rabc, b : A, \Gamma \Rightarrow \Delta, c : B$ is the requested conclusion of height n, possibly with different eigenvariables, but the desired ones can be obtained by height-preserving substitutions (Lemma 4.1). As an example for relational rules, we only deal with R7, i.e., a rule with eigenvariable. (1) If $n = 0$, $Rabc, \Gamma \Rightarrow \Delta$ is an axiom, and then also $Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta$ is an axiom. If $\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta$ is concluded by any rule $\mathcal{R}$ other than R7, we apply the inductive hypothesis to the premise(s) $Rabc, \Gamma' \Rightarrow \Delta'$ ($Rabc, \Gamma'' \Rightarrow \Delta''$) to obtain derivation(s) of height n of $Rabx, Raxc, Rabc, \Gamma' \Rightarrow \Delta'$ ($Rabx, Raxc, Rabc, \Gamma'' \Rightarrow \Delta''$). By applying $\mathcal{R}$ we obtain a derivation of height $n+1$ of $Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta$, as desired. (3) If $\vdash^{n+1} Rabc, \Gamma \Rightarrow \Delta$ is concluded by R7, then $Rabx, Raxc, Rabc, \Gamma \Rightarrow \Delta$ is the requested conclusion of height n (possibly by applying Lemma 4.1). □

Lemma 7.3. *The rules of contraction:*

$$\frac{a : C, a : C, \Gamma \Rightarrow \Delta}{a : C, \Gamma \Rightarrow \Delta}\ {\scriptstyle LC} \qquad \frac{\Gamma \Rightarrow \Delta, a : C, a : C}{\Gamma \Rightarrow \Delta, a : C}\ {\scriptstyle RC} \qquad \frac{Rabc, Rabc, \Gamma \Rightarrow \Delta}{Rabc, \Gamma \Rightarrow \Delta}\ {\scriptstyle LC_L}$$

are height-preserving admissible in **G3rX**.

Proof. By induction on the height of derivation. As usual, if $n = 0$, then the premise is an axiomatic sequent and so also the contracted sequent is an axiomatic one. If $n > 0$, we consider the last rule applied to the premise of contraction. If the contraction formula is not principal in the premise of some $\mathcal{R}$, then both occurrences are found in the premises of the rule and they have a smaller derivation height. By applying the induction hypothesis, we contract them and apply $\mathcal{R}$ to obtain a derivation of the conclusion with the same derivation height. If the contraction formula is principal, we distinguish three cases: (1) $\mathcal{R}$ is a rule where active formulas are proper subformulas of the principal formula (all rules for $\sim, \wedge, \vee$); (2) $\mathcal{R}$ is a rule where both, labels $Rabc$ and proper subformulas of the principal formula, are active formulas ($R\rightarrow$); (3) $\mathcal{R}$ is a rule in which the principal formula is repeated also in the premises of the rule ($L\rightarrow$).
(1) In the cases for $\sim, \wedge, \vee$ the contraction is reduced to contraction on formulas

of smaller complexity (as in the cases for modal and intermediate logics, see, e.g., [20, 21, 8]).
(2) We consider a rule where the principal formula and relational atoms are both active, for instance:

$$\dfrac{\vdash^n Rabc, b : A, \Gamma \Rightarrow \Delta, a : A \to B, c : B}{\vdash^{n+1} \Gamma \Rightarrow \Delta, a : A \to B, a : A \to B}\ R\to$$

By height-preserving invertibility (Lemma 7.2) applied to the premise, we obtain the following derivation:

$$\dfrac{\dfrac{\vdash^n Rabc, b : A, Rabc, b : A, \Gamma \Rightarrow \Delta, c : B, c : B}{\vdash^n Rabc, b : A, \Gamma \Rightarrow \Delta, c : B}\ i.h.}{\vdash^{n+1} \Gamma \Rightarrow \Delta, a : A \to B}\ R\to$$

as requested. Notice that if both contraction formulas are principal in $R\to$, we apply the closure condition.
(3) Finally, we consider a rule in which only the labelled formula is principal, namely $L\to$:

$$\dfrac{\vdash^n Rabc, a : A \to B, a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad \vdash^n Rabc, c : B, a : A \to B, a : A \to B, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, a : A \to B, a : A \to B, \Gamma \Rightarrow \Delta}\ L\to$$

Again, by applying the inductive hypothesis to the premises, we obtain the desired derivation:

$$\dfrac{\vdash^n Rabc, a : A \to B, \Gamma \Rightarrow \Delta, b : A \qquad \vdash^n Rabc, c : B, a : A \to B, \Gamma \Rightarrow \Delta}{\vdash^{n+1} Rabc, a : A \to B, \Gamma \Rightarrow \Delta}\ L\to$$

□

Finally, we can prove that CUT is an admissible rule. This theorem directly entails the implication 2 $\implies$ 3 stated on p. 57:

Theorem 7.4. *The rule of* CUT*:*

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

is admissible in **G3rX**.

Proof. The proof is by a lexicographic induction on the complexity of the CUT-formula $a : A$ and the sum of the heights $h(\delta_1) + h(\delta_2)$. We perform a case analysis on the last rule used in the derivation above the CUT and whether it applies to the CUT-formula or not. We show that each application of CUT can either be eliminated, or be replaced by one or more applications of CUT of smaller complexity. The proof proceeds similarly to the CUT-elimination proofs for several logics, e.g., [20, 25, 12, 18]. Intuitively, we eliminate the left- and topmost CUT first, and proceed by repeating the procedure until we reach a

CUT-free derivation. We start by showing that CUT can be eliminated if one of the CUT premises is an axiom (**case 1**). Then we show that the CUT-height can be reduced in all cases in which the CUT-formula is not principal in at least one of the CUT-premises (**case 2**). Finally, we show that if the CUT-formula is principal in both CUT-premises, then the CUT is reduced to one or more CUTs on less complex formulas or on shorter derivations (**case 3**). The complete case analysis is performed in Appendix C.
Here, we present two interesting cases where the CUT-formula A is principal in both premises. We start by considering a derivation where the last rules applied to obtain the CUT-premises are $R{\sim}$ and $L{\sim}$, respectively. Let $A = {\sim}B$:

$$\dfrac{\dfrac{a^* : B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, a : {\sim}B}\, R{\sim} \qquad \dfrac{\Gamma' \Rightarrow \Delta', a^* : B}{a : {\sim}B, \Gamma' \Rightarrow \Delta'}\, L{\sim}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \text{CUT}$$

It is transformed into the following derivation:

$$\dfrac{\Gamma' \Rightarrow \Delta', a^* : B \qquad a^* : B, \Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \text{CUT}$$

where CUT is applied on a formula of smaller complexity.
Assume that the premises of CUT are derived by $R \rightarrow$ and $L \rightarrow$, respectively. Let $A = B \rightarrow C$:

$$\dfrac{(b, c \text{ fresh})\ \dfrac{Rabc, b : B, \Gamma \Rightarrow \Delta, c : C}{\Gamma \Rightarrow \Delta, a : B \rightarrow C}\, R\rightarrow \qquad \dfrac{Rade, a : B \rightarrow C, \Gamma' \Rightarrow \Delta', d : B \qquad Rade, e : C, a : B \rightarrow C, \Gamma' \Rightarrow \Delta'}{Rade, a : B \rightarrow C, \Gamma' \Rightarrow \Delta'}\, L\rightarrow}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \text{CUT}$$

It is transformed into the following derivation:

$$\dfrac{\dfrac{\dfrac{\vdots\, \delta_1}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', e : C} \qquad \dfrac{\vdots\, \delta_2}{Rade, e : C, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{Rade, Rade, \Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta'}\, \text{CUT}}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, (\text{Lemma 7.3})\ \text{LC+RC+LC}_L$$

where the conclusion of δ_1 is derived by:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : B \rightarrow C \qquad Rade, a : B \rightarrow C, \Gamma' \Rightarrow \Delta', d : B}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' d : B}\, \text{CUT} \qquad \dfrac{\dfrac{Rabc, b : B, \Gamma \Rightarrow \Delta, c : C}{Rabe, b : B, \Gamma \Rightarrow \Delta, e : C}\, (\text{Lemma 4.1})\ \text{SUB}(e/c)}{Rade, d : B, \Gamma \Rightarrow \Delta, e : C}\, (\text{Lemma 4.1})\ \text{SUB}(d/b)}{Rade, Rade, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta', e : C}\, \text{CUT}}{Rade, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', e : C}\, (\text{Lemma 7.3})\ \text{LC+RC+LC}_L$$

while the conclusion of δ_2 is derived by:

$$\dfrac{\Gamma \Rightarrow \Delta, a : B \rightarrow C \qquad Rade, e : C, a : B \rightarrow C, \Gamma' \Rightarrow \Delta'}{Rade, e : C, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \text{CUT}$$

Notice that the two topmost CUTs, those on $a : B \to C$, are derived with a shorter derivation height, while the other two are applied on formulas of smaller complexity, i.e., $d : B$ and $e : C$. □

8 Conclusions

In this paper, we have presented labelled sequent calculi for a wide range of relevant logics by reflecting at the syntactic level semantic informations taken from reduced Routley-Meyer models, and have proved soundness and (syntactic and semantic) completeness. Least but not last, we have shown height-preserving invertibility of the rules, height-preserving admissibility of structural rules, and CUT-admissibility.

To conclude, we would like to point out some further topics of research, directly connected to the work developed so far:

- Along with labelled calculi, many generalizations of sequent systems have been proposed over the years. This flourishing of systems has also paved the way to investigations concerning the relations between them. In this context, an interesting task for future work is represented by establishing correspondences between the calculi presented in this work with other characterizations obtained by application of different proof-theoretic structures, e.g., hypersequents and display sequents.

- Notice that relevant logics face some troubles when it comes to establish decidability results and, indeed, many of them are undecidable. Given the subtleties that such a discussion might involve, we leave (un)decidability issues out from this investigation and we limit ourselves to some observations. One of the main consequences that can be drawn from CUT-elimination proofs is a fundamental trait of sequent systems, namely the so-called *subformula property*. This ensures that all formulas in a derivation are subformulas of formulas in the endsequent. Unfortunately, labelled sequent calculi, given the presence of geometrical rules in which relational atoms disappear from premise to conclusion, do not have a *full* subformula property. Nonetheless, by following the considerations expressed in [20], we observe that all of our calculi enjoy a *weak* version of the property, namely: *All formulas in a derivation are either subformulas of formulas in the endsequent or formulas of the form Rabc.* This property alone, however, is not enough to prove syntactic decidability. Firstly, in order to provide such a proof, one needs to find a bound on the number of eigenvariables (fresh labels) in a derivation of a given sequent. Secondly, since the repetition of the principal formula in the premises of $L\to$ is another source of potentially non-terminating proof search, there's also the need of finding a bound on applications of $L\to$. This amounts to binding the number of applications of $L\to$ with principal formula $a : A \to B$ to the number of relational atoms of the form *Rabc* that appear on the left-hand side of sequents in the derivation. This number, in turn, will be bounded by the number of existing relational atoms of that form and relational atoms that can be introduced by applications of $R\rightarrow$ with principal formula $a : A \rightarrow B$.

- Throughout our paper, we have considered labelled rules for the following connectives $\sim$, $\wedge$, $\vee$ and $\rightarrow$. However, occasionally, relevant logics are presented also with further connectives, such as for example, 'fusion' (also known as 'intensional conjunction') and 'fission' (also known as 'intensional disjunction'). Some other relevantists would also welcome the addition of the so-called 'Ackerman truth constant' (often denote as $\mathbf{t}$). Nonetheless, given our intentions in this paper, we have preferred to omit the consideration of wider sets of connectives and have decided to leave this topic for further research. We only notice that all connectives mentioned above can be, in line of principle, treated according to the methodology we have adopted so far.

A Appendix

Proof of Theorem 6.2 (Syntactic Completeness). We show that (Ax1)-(Ax16) can be derived in the calculi **G3rX**:
G3rB $\vdash \Rightarrow 0 : A \to A$

$$(a, b \text{ fresh}) \; \dfrac{R0ab, a : A \Rightarrow b : A}{\Rightarrow 0 : A \to A} \; R\to$$

G3rB $\vdash \Rightarrow 0 : A \wedge B \to A$ and **G3rB** $\vdash \Rightarrow 0 : A \wedge B \to B$.

$$(a, b \text{ fresh}) \; \dfrac{\dfrac{R0ab, a : A, a : B \Rightarrow b : A}{R0ab, a : A \wedge B \Rightarrow b : A} \; L\wedge}{\Rightarrow 0 : A \wedge B \to A} \; R\to \qquad (a, b \text{ fresh}) \; \dfrac{\dfrac{R0ab, a : A, a : B \Rightarrow b : B}{R0ab, a : A \wedge B \Rightarrow b : B} \; L\wedge}{\Rightarrow 0 : A \wedge B \to B} \; R\to$$

G3rB $\vdash \Rightarrow 0 : (A \to B) \wedge (A \to C) \to (A \to (B \wedge C))$. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix} \vdots\, \delta_1 \\ Racd, \mathcal{S}, c : A, a : A \to B \Rightarrow d : B \end{matrix} \qquad \begin{matrix} \vdots\, \delta_2 \\ Racd, \mathcal{S}, c : A, a : A \to C \Rightarrow d : C \end{matrix}}{Racd, Rbcd, R0ab, c : A, a : A \to B, a : A \to C \Rightarrow d : B \wedge C} \; R\wedge}{Rbcd, R0ab, c : A, a : A \to B, a : A \to C \Rightarrow d : B \wedge C} \; \mathrm{R4}}{R0ab, a : A \to B, a : A \to C \Rightarrow b : A \to (B \wedge C)} \; R\to \; (c, d \text{ fresh})}{R0ab, a : (A \to B) \wedge (A \to C) \Rightarrow b : A \to (B \wedge C)} \; L\wedge}{\Rightarrow 0 : (A \to B) \wedge (A \to C) \to (A \to (B \wedge C))} \; R\to \; (a, b \text{ fresh})$$

where the conclusion of δ_1 is obtained by:

$$\dfrac{\dfrac{R0cc, Racd, \mathcal{S}', c : A \Rightarrow d : B, c : A}{Racd, \mathcal{S}', c : A \Rightarrow d : B, c : A} \; \mathrm{R2} \qquad \dfrac{R0dd, Racd, \mathcal{S}', d : B, c : A \Rightarrow d : B}{Racd, \mathcal{S}', d : B, c : A \Rightarrow d : B} \; \mathrm{R2}}{Racd, \mathcal{S}, c : A, a : A \to B \Rightarrow d : B} \; L\to$$

while the conclusion of δ_2 is derived by:

$$\dfrac{\dfrac{R0cc, Racd, \mathcal{S}'', c : A \Rightarrow d : C, c : A}{Racd, \mathcal{S}'', c : A \Rightarrow d : C, c : A} \; \mathrm{R2} \qquad \dfrac{R0dd, Racd, \mathcal{S}'', d : C, c : A \Rightarrow d : C}{Racd, \mathcal{S}'', d : C, c : A \Rightarrow d : C} \; \mathrm{R2}}{Racd, \mathcal{S}, c : A, a : A \to C \Rightarrow d : C} \; L\to$$

where $\mathcal{S} = Rbcd, R0ab$, $\mathcal{S}' = Rbcd, R0ab, a : A \to B$ and $\mathcal{S}' = Rbcd, R0ab, a : A \to C$.
G3rB $\vdash \Rightarrow 0 : A \to (A \vee B)$ and **G3rB** $\vdash \Rightarrow 0 : B \to (A \vee B)$.

$$(a, b \text{ fresh}) \; \dfrac{\dfrac{R0ab, a : A \Rightarrow b : A, b : B}{R0ab, a : A \Rightarrow b : A \vee B} \; R\vee}{\Rightarrow 0 : A \to (A \vee B)} \; R\to \qquad (a, b \text{ fresh}) \; \dfrac{\dfrac{R0ab, a : B \Rightarrow b : A, b : B}{R0ab, a : B \Rightarrow b : A \vee B} \; R\vee}{\Rightarrow 0 : B \to (A \vee B)} \; R\to$$

G3rB $\vdash \Rightarrow 0 : (A \to C) \wedge (B \to C) \to ((A \vee B) \to C)$.
The derivation is as follows:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\ \delta_1}{Racd, \mathcal{S}, c : A \vee B, a : B \to C \Rightarrow d : C, c : A} \qquad Racd, \mathcal{S}, d : C, c : A \vee B, a : B \to C \Rightarrow d : C}{Racd, Rbcd, R0ab, c : A \vee B, a : A \to C, a : B \to C \Rightarrow d : C}}{Rbcd, R0ab, c : A \vee B, a : A \to C, a : B \to C \Rightarrow d : C}\ \mathrm{R4}}{Rbcd, R0ab, c : A \vee B, a : (A \to C) \wedge (B \to C) \Rightarrow d : C}\ L\wedge}{R0ab, a : (A \to C) \wedge (B \to C) \Rightarrow b : (A \vee B) \to C}\ R\!\to\ (c, d \text{ fresh})}{\Rightarrow 0 : (A \to C) \wedge (B \to C) \to ((A \vee B) \to C)}\ R\!\to\ (a, b \text{ fresh})
$$

where the conclusion of δ_1 is derived by:

$$
\dfrac{\dfrac{R0cc, Racd, \mathcal{S}, c : A, a : B \to C \Rightarrow d : C, c : A}{Racd, \mathcal{S}, c : A, a : B \to C \Rightarrow d : C, c : A}\ \mathrm{R2} \qquad \overset{\vdots\ \delta_1'}{Racd, \mathcal{S}, c : B, a : B \to C \Rightarrow d : C, c : A}}{Racd, \mathcal{S}, c : A \vee B, a : B \to C \Rightarrow d : C, c : A}\ L\vee
$$

while δ_1' is derived by:

$$
\dfrac{\dfrac{R0cc, Racd, \mathcal{S}', c : B \Rightarrow d : C, c : A, c : B}{Racd, \mathcal{S}', c : B \Rightarrow d : C, c : A, c : B}\ \mathrm{R2} \qquad \dfrac{R0dd, Racd, \mathcal{S}', d : C, c : B \Rightarrow d : C, c : A}{Racd, \mathcal{S}', d : C, c : B \Rightarrow d : C, c : A}\ \mathrm{R2}}{Racd, \mathcal{S}, c : B, a : B \to C \Rightarrow d : C, c : A}\ L\!\to
$$

with $\mathcal{S} = Rbcd, R0ab, a : A \to C$ and $\mathcal{S}' = Rbcd, R0ab, a : A \to C, a : B \to C$.
G3rB $\vdash \Rightarrow 0 : A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$. We obtain the following derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\ \delta_1}{R0ab, a : A, a : B \Rightarrow b : A \wedge B, b : A \wedge C} \qquad \overset{\vdots\ \delta_2}{R0ab, a : A, a : C \Rightarrow b : A \wedge B, b : A \wedge C}}{R0ab, a : A, a : B \vee C \Rightarrow b : A \wedge B, b : A \wedge C}\ L\vee}{R0ab, a : A \wedge (B \vee C) \Rightarrow b : A \wedge B, b : A \wedge C}\ L\wedge}{R0ab, a : A \wedge (B \vee C) \Rightarrow b : (A \wedge B) \vee (A \wedge C)}\ R\vee}{\Rightarrow 0 : A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)}\ R\!\to\ (a, b \text{ fresh})
$$

where the conclusion of δ_1 is derived by:

$$
\dfrac{R0ab, a : A, a : B \Rightarrow b : A, b : A \wedge C \qquad R0ab, a : A, a : B \Rightarrow b : B, b : A \wedge C}{R0ab, a : A, a : B \Rightarrow b : A \wedge B, b : A \wedge C}\ R\wedge
$$

while the conclusion of δ_2 is obtained by:

$$
\dfrac{R0ab, a : A, a : C \Rightarrow b : A, b : A \wedge B \qquad R0ab, a : A, a : C \Rightarrow b : C, b : A \wedge B}{R0ab, a : A, a : C \Rightarrow b : A \wedge B, b : A \wedge C}\ R\wedge
$$

G3rB $\vdash \Rightarrow 0 : {\sim}{\sim} A \to A$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{R0a^{**}b, R0a^{**}a, R0aa^{**}, R0ab, a^{**}:A \Rightarrow b:A}{R0a^{**}a, R0aa^{**}, R0ab, a^{**}:A \Rightarrow b:A}\ \text{R3}}{R0ab, a^{**}:A \Rightarrow b:A}\ \text{R1}}{R0ab \Rightarrow b:A, a^{*}:{\sim}A}\ R{\sim}}{R0ab, a:{\sim}{\sim}A \Rightarrow b:A}\ L{\sim}}{\Rightarrow 0:{\sim}{\sim}A \to A}\ R{\to}\quad (a, b \text{ fresh})$$

If **G3rB** $\vdash \Rightarrow 0:A$, **G3rB** $\vdash \Rightarrow 0:A \to B$, then **G3rB** $\vdash \Rightarrow 0:B$.

$$\cfrac{\Rightarrow 0:A \qquad \cfrac{\cfrac{\Rightarrow 0:A \to B}{a:A \Rightarrow a:B}\ (\text{Lemma 6.1})}{0:A \Rightarrow 0:B}\ (\text{Lemma 4.1})\ \text{SUB}(0/a)}{\Rightarrow 0:B}\ \text{CUT}$$

If **G3rB** $\vdash \Rightarrow 0:A$, **G3rB** $\vdash \Rightarrow 0:B$, then **G3rB** $\vdash \Rightarrow 0:A \wedge B$.

$$\cfrac{\Rightarrow 0:B \qquad \Rightarrow 0:A}{\Rightarrow 0:A \wedge B}\ R\wedge$$

If **G3rB** $\vdash \Rightarrow 0:A \to B$, then **G3rB** $\vdash \Rightarrow 0:(C \to A) \to (C \to B)$. We have the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{R0bb, Rabc, a:C \to A, b:C \Rightarrow c:B, b:C}{Rabc, a:C \to A, b:C \Rightarrow c:B, b:C}\ \text{R2} \qquad \cfrac{\cfrac{\Rightarrow 0:A \to B}{c:A \Rightarrow c:B}\ (\text{Lemma 6.1})}{Rabc, c:A, a:C \to A, b:C \Rightarrow c:B}\ (\text{Lemma 7.1})\ \text{LW+LW}_L}{Rabc, a:C \to A, b:C \Rightarrow c:B}\ L{\to}}{a:C \to A \Rightarrow a:C \to B}\ R{\to}\ (b, c \text{ fresh})}{\Rightarrow 0:(C \to A) \to (C \to B)}\ (\text{Lemma 6.1})$$

If **G3rB** $\vdash \Rightarrow 0:A \to B$, then **G3rB** $\vdash \Rightarrow 0:(B \to C) \to (A \to C)$. We have the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Rightarrow 0:A \to B}{b:A \Rightarrow b:B}\ (\text{Lemma 6.1})}{Rabc, a:B \to C, b:A \Rightarrow c:C, b:B}\ (\text{Lemma 7.1})\ \text{LW+RW+LW}_L \qquad \cfrac{R0cc, Rabc, c:C, a:B \to C \Rightarrow c:C}{Rabc, c:C, a:B \to C \Rightarrow c:C}\ \text{R2}}{Rabc, a:B \to C, b:A \Rightarrow c:C}\ L{\to}}{a:B \to C \Rightarrow a:A \to C}\ R{\to}\ (b, c \text{ fresh})}{\Rightarrow 0:(B \to C) \to (A \to C)}\ (\text{Lemma 6.1})$$

If **G3rB** $\vdash \Rightarrow 0:A \to B$, then **G3rB** $\vdash \Rightarrow 0:{\sim}B \to {\sim}A$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Rightarrow 0:A \to B}{a:A \Rightarrow a:B}\ (\text{Lemma 6.1})}{a^{*}:A \Rightarrow a^{*}:B}\ (\text{Lemma 4.1})\ \text{SUB}(a^{*}/a)}{a^{*}:A, a:{\sim}B \Rightarrow}\ L{\sim}}{a:{\sim}B \Rightarrow a:{\sim}A}\ R{\sim}}{\Rightarrow 0:{\sim}B \to {\sim}A}\ (\text{Lemma 6.1})$$

where $\mathcal{S} = R0ab, 0 : A \to B$. This completes the completeness proof for **G3rB**.
G3rDW $\vdash \Rightarrow 0 : (A \to B) \to ({\sim}B \to {\sim}A)$.

$$
\begin{prooftree}
\AxiomC{$R0d^*d^*, Rad^*c^*, \mathcal{S}, d^* : A \Rightarrow c^* : B, d^* : A$}
\RightLabel{R2}
\UnaryInfC{$Rad^*c^*, \mathcal{S}, d^* : A \Rightarrow c^* : B, d^* : A$}
\AxiomC{$R0c^*c^*, Rad^*c^*, \mathcal{S}, c^* : B, d^* : A \Rightarrow c^* : B$}
\RightLabel{R2}
\UnaryInfC{$Rad^*c^*, \mathcal{S}, c^* : B, d^* : A \Rightarrow c^* : B$}
\RightLabel{$L\to$}
\BinaryInfC{$Rad^*c^*, Rbd^*c^*, Rbcd, R0ab, d^* : A, a : A \to B \Rightarrow c^* : B$}
\RightLabel{R4}
\UnaryInfC{$Rbd^*c^*, Rbcd, R0ab, d^* : A, a : A \to B \Rightarrow c^* : B$}
\RightLabel{R6}
\UnaryInfC{$Rbcd, R0ab, d^* : A, a : A \to B \Rightarrow c^* : B$}
\RightLabel{$L{\sim}$}
\UnaryInfC{$Rbcd, R0ab, d^* : A, a : A \to B, c : {\sim}B \Rightarrow$}
\RightLabel{$R{\sim}$}
\UnaryInfC{$Rbcd, R0ab, a : A \to B, c : {\sim}B \Rightarrow d : {\sim}A$}
\LeftLabel{$(c, d \text{ fresh})$}
\RightLabel{$R\to$}
\UnaryInfC{$R0ab, a : A \to B \Rightarrow b : {\sim}B \to {\sim}A$}
\LeftLabel{$(a, b \text{ fresh})$}
\RightLabel{$R\to$}
\UnaryInfC{$\Rightarrow 0 : (A \to B) \to ({\sim}B \to {\sim}A)$}
\end{prooftree}
$$

where $\mathcal{S} = Rbd^*c^*, Rbcd, R0ab, a : A \to B$.
G3rDJ $\vdash \Rightarrow 0 : (A \to B) \wedge (B \to C) \to (A \to C)$.

$$
\begin{prooftree}
\AxiomC{$R0cc, Racx, Raxd, \mathcal{S}', c : A \Rightarrow d : C, c : A$}
\RightLabel{R2}
\UnaryInfC{$Racx, Raxd, \mathcal{S}', c : A \Rightarrow d : C, c : A$}
\AxiomC{$\vdots\ \delta_1$}
\noLine
\UnaryInfC{$Racx, Raxd, \mathcal{S}, x : B, a : B \to C, \Rightarrow d : C$}
\RightLabel{$L\to$}
\BinaryInfC{$Racx, Raxd, Racd, Rbcd, R0ab, c : A, a : A \to B, a : B \to C \Rightarrow d : C$}
\LeftLabel{$(x \text{ fresh})$}
\RightLabel{R7}
\UnaryInfC{$Racd, Rbcd, R0ab, c : A, a : A \to B, a : B \to C \Rightarrow d : C$}
\RightLabel{R4}
\UnaryInfC{$Rbcd, R0ab, c : A, a : A \to B, a : B \to C \Rightarrow d : C$}
\LeftLabel{$(c, d \text{ fresh})$}
\RightLabel{$R\to$}
\UnaryInfC{$R0ab, a : A \to B, a : B \to C \Rightarrow b : A \to C$}
\RightLabel{$L\wedge$}
\UnaryInfC{$R0ab, a : (A \to B) \wedge (B \to C) \Rightarrow b : A \to C$}
\LeftLabel{$(a, b \text{ fresh})$}
\RightLabel{$R\to$}
\UnaryInfC{$\Rightarrow 0 : (A \to B) \wedge (B \to C) \to (A \to C)$}
\end{prooftree}
$$

and δ_1 is derived by:

$$
\begin{prooftree}
\AxiomC{$R0xx, Racx, Raxd, \mathcal{S}', x : B \Rightarrow d : C, x : B$}
\RightLabel{R2}
\UnaryInfC{$Racx, Raxd, \mathcal{S}', x : B \Rightarrow d : C, x : B$}
\AxiomC{$R0dd, Racx, Raxd, \mathcal{S}, d : C, x : B \Rightarrow d : C$}
\RightLabel{R2}
\UnaryInfC{$Racx, Raxd, \mathcal{S}, d : C, x : B \Rightarrow d : C$}
\RightLabel{$L\to$}
\BinaryInfC{$Racx, Raxd, \mathcal{S}, x : B, a : B \to C, \Rightarrow d : C$}
\end{prooftree}
$$

where $\mathcal{S} = Racd, Rbcd, R0ab, a : A \to B$ and $\mathcal{S}' = Racd, Rbcd, R0ab, a : A \to B, a : B \to C$.
G3rTW $\vdash \Rightarrow 0 : (A \to B) \to ((B \to C) \to (A \to C))$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{R0ee, \mathcal{S}, a : A \to B, e : A \Rightarrow e : A, f : C}{\mathcal{S}, a : A \to B, e : A \Rightarrow e : A, f : C}\;\text{R2} \qquad \begin{array}{c}\vdots\,\delta_1\\ Raex, Rcxf, \mathcal{S}', x : B, c : B \to C \Rightarrow f : C\end{array}}{Raex, Rcxf, Rbex, Rdef, Rbcd, R0ab, a : A \to B, c : B \to C, e : A \Rightarrow f : C}\;L\to}{Rcxf, Rbex, Rdef, Rbcd, R0ab, a : A \to B, c : B \to C, e : A \Rightarrow f : C}\;\text{R4}}{\text{(}x\text{ fresh)}\quad Rdef, Rbcd, R0ab, a : A \to B, c : B \to C, e : A \Rightarrow f : C}\;\text{R8}}{\text{(}e, f\text{ fresh)}\quad Rbcd, R0ab, a : A \to B, c : B \to C \Rightarrow d : A \to C}\;R\to}{\text{(}c, d\text{ fresh)}\quad R0ab, a : A \to B \Rightarrow b : (B \to C) \to (A \to C)}\;R\to}{\text{(}a, b\text{ fresh)}\quad \Rightarrow 0 : (A \to B) \to ((B \to C) \to (A \to C))}\;R\to
$$

and δ_1 is derived by:

$$
\cfrac{\cfrac{R0xx, Raex, Rcxf, \mathcal{S}'', x : B \Rightarrow f : C, x : B}{Raex, Rcxf, \mathcal{S}'', x : B \Rightarrow f : C, x : B}\;\text{R2} \qquad \cfrac{R0ff, Raex, Rcxf, \mathcal{S}'', f : C, x : B \Rightarrow f : C}{Raex, Rcxf, \mathcal{S}'', f : C, x : B \Rightarrow f : C}\;\text{R2}}{Raex, Rcxf, \mathcal{S}', x : B, c : B \to C \Rightarrow f : C}\;L\to
$$

where $\mathcal{S} = Raex, Rcxf, Rbex, Rdef, Rbcd, R0ab, c : B \to C$,
$\mathcal{S}' = Rbex, Rdef, Rbcd, R0ab, e : A, a : A \to B$ and $\mathcal{S}'' = Rbex, Rdef, Rbcd, R0ab, e : A, a : A \to B, c : B \to C$.

G3rTW $\vdash \Rightarrow 0 : (A \to B) \to ((C \to A) \to (C \to B))$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{R0ee, Rcex, \mathcal{S}, e : C, \Rightarrow f : B, e : C}{Rcex, \mathcal{S}, e : C, \Rightarrow f : B, e : C}\;\text{R2} \qquad \begin{array}{c}\vdots\,\delta_1\\ Raxf, Rcex, \mathcal{S}', x : A, e : C, a : A \to B \Rightarrow f : B\end{array}}{\text{(}x\text{ fresh)}\quad Raxf, Rcex, Racd, Rdef, Rbcd, R0ab, e : C, a : A \to B, c : C \to A \Rightarrow f : B}\;L\to}{Racd, Rdef, Rbcd, R0ab, e : C, a : A \to B, c : C \to A \Rightarrow f : B}\;\text{R9}}{Rdef, Rbcd, R0ab, e : C, a : A \to B, c : C \to A \Rightarrow f : B}\;\text{R4}}{\text{(}e, f\text{ fresh)}\quad Rbcd, R0ab, a : A \to B, c : C \to A \Rightarrow d : C \to B}\;R\to}{\text{(}c, d\text{ fresh)}\quad R0ab, a : A \to B \Rightarrow b : (C \to A) \to (C \to B)}\;R\to}{\text{(}a, b\text{ fresh)}\quad \Rightarrow 0 : (A \to B) \to ((C \to A) \to (C \to B))}\;R\to
$$

where the conclusion of δ_1 is derived by:

$$
\cfrac{\cfrac{R0xx, Raxf, Rcex, \mathcal{S}', x : A, e : C \Rightarrow f : B, x : A}{Raxf, Rcex, \mathcal{S}'', x : A, e : C \Rightarrow f : B, x : A}\;\text{R2} \qquad \cfrac{R0ff, Raxf, Rcex, \mathcal{S}', f : B, x : A, e : C \Rightarrow f : B}{Raxf, Rcex, \mathcal{S}'', f : B, x : A, e : C \Rightarrow f : B}\;\text{R2}}{Raxf, Rcex, \mathcal{S}', e : C, a : A \to B \Rightarrow f : B}\;L\to
$$

with $\mathcal{S} = Raxf, Racd, Rdef, Rbcd, R0ab, a : A \to B, c : C \to A$,
$\mathcal{S}' = Racd, Rdef, Rbcd, R0ab, c : C \to A$ and $\mathcal{S}'' = Racd, Rdef, Rbcd, R0ab, c : C \to A, a : A \to B$.

G3rT $\vdash \Rightarrow 0 : (A \to (A \to B)) \to (A \to B)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0cc, Racx, Rxcd, \mathcal{S}, c : A \Rightarrow d : B, c : A}{Racx, Rxcd, \mathcal{S}, c : A \Rightarrow d : B, c : A}\ \text{R2} \qquad \begin{matrix}\vdots\ \delta_1 \\ Racx, Rxcd, \mathcal{S}, x : A \to B, c : A \Rightarrow d : B\end{matrix}}{Racx, Rxcd, Racd, Rbcd, R0ab, a : A \to (A \to B), c : A \Rightarrow d : B}\ L\to}{Racd, Rbcd, R0ab, a : A \to (A \to B), c : A \Rightarrow d : B}\ \text{R10}\ (x \text{ fresh})}{Rbcd, R0ab, a : A \to (A \to B), c : A \Rightarrow d : B}\ \text{R4}}{R0ab, a : A \to (A \to B) \Rightarrow b : A \to B}\ R\to\ (c, d \text{ fresh})}{\Rightarrow 0 : (A \to (A \to B)) \to (A \to B)}\ R\to\ (a, b \text{ fresh})$$

and δ_1 is derived by:

$$\dfrac{\dfrac{R0cc, Racx, Rxcd, \mathcal{S}', c : A \Rightarrow d : B, c : A}{Racx, Rxcd, \mathcal{S}', c : A \Rightarrow d : B, c : A}\ \text{R2} \qquad \dfrac{R0dd, Racx, Rxcd, \mathcal{S}', d : B, c : A \Rightarrow d : B}{Racx, Rxcd, \mathcal{S}', d : B, c : A \Rightarrow d : B}\ \text{R2}}{Racx, Rxcd, \mathcal{S}, x : A \to B, c : A \Rightarrow d : B}\ L\to$$

where $\mathcal{S} = Racd, Rbcd, R0ab, a : A \to (A \to B)$ and $\mathcal{S}' = Racd, Rbcd, R0ab, a : A \to (A \to B), x : A \to B$.

G3rT $\vdash \Rightarrow 0 : (A \wedge (A \to B)) \to B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0aa, Raaa, \mathcal{S}, a : A \Rightarrow b : B, a : A}{Raaa, \mathcal{S}, a : A \Rightarrow b : B, a : A}\ \text{R2} \qquad Raaa, R0ab, a : B, a : A, a : A \to B \Rightarrow b : B}{Raaa, R0ab, a : A, a : A \to B \Rightarrow b : B}\ L\to}{R0ab, a : A, a : A \to B \Rightarrow b : B}\ \text{R11}}{R0ab, a : A \wedge (A \to B) \Rightarrow b : B}\ L\wedge}{\Rightarrow 0 : (A \wedge (A \to B)) \to B}\ R\to\ (a, b \text{ fresh})$$

where $\mathcal{S} = R0ab, a : A \to B$.

G3rT $\vdash \Rightarrow 0 : (A \to {\sim}A) \to {\sim}A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0b^*a^*, R0ab, \mathcal{S}, b^* : A, \Rightarrow a^* : A}{R0ab, \mathcal{S}, b^* : A, \Rightarrow a^* : A}\ \text{R5}}{R0ab, \mathcal{S} \Rightarrow a^* : A, b : {\sim}A}\ R{\sim} \qquad \dfrac{\dfrac{\dfrac{R0b^*a^*, R0ab, \mathcal{S}, b^* : A \Rightarrow a^* : A}{R0ab, \mathcal{S}, b^* : A \Rightarrow a^* : A}\ \text{R5}}{R0ab, \mathcal{S} \Rightarrow b : {\sim}A, a^* : A}\ R{\sim}}{R0ab, \mathcal{S}, a : {\sim}A \Rightarrow b : {\sim}A}\ L{\sim}}{Raa^*a, R0ab, a : A \to {\sim}A \Rightarrow b : {\sim}A}\ L\to}{R0ab, a : A \to {\sim}A \Rightarrow b : {\sim}A}\ \text{R12}}{\Rightarrow 0 : (A \to {\sim}A) \to {\sim}A}\ R\to\ (a, b \text{ fresh})$$

where $\mathcal{S} = Raa^*a, a : A \to {\sim}A$.

G3rT $\vdash \Rightarrow 0 : A \vee {\sim}A$.

$$\dfrac{\dfrac{\dfrac{R00^*0, 0^* : A \Rightarrow 0 : A}{0^* : A \Rightarrow 0 : A}\ \text{R15}}{\Rightarrow 0 : A, 0 : {\sim}A}\ R{\sim}}{\Rightarrow 0 : A \vee {\sim}A}\ R\vee$$

G3rRW $\vdash \Rightarrow 0 : (A \to (B \to C) \to (B \to (A \to C)).$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0ee, Raey, Rycf, \mathcal{S}, e : A, c : B \Rightarrow f : C, e : A}{Raey, Rycf, \mathcal{S}, e : A, c : B \Rightarrow f : C, e : A}\,\text{R2} \qquad \begin{matrix} \vdots\, \delta_1 \\ Raey, Rycf, \mathcal{S}, y : B \to C, e : A, c : B \Rightarrow f : C \end{matrix}}{Raey, Rycf, Racd, Rdef, Rbcd, R0ab, e : A, c : B, a : A \to (B \to C) \Rightarrow f : C}\,L\to\ (y\text{ fresh})}{Racd, Rdef, Rbcd, R0ab, e : A, c : B, a : A \to (B \to C) \Rightarrow f : C}\,\text{R13}}{Rdef, Rbcd, R0ab, e : A, c : B, a : A \to (B \to C) \Rightarrow f : C}\,\text{R4}}{Rbcd, R0ab, c : B, a : A \to (B \to C) \Rightarrow d : A \to C}\,R\to\ (e, f\text{ fresh})}{R0ab, a : A \to (B \to C) \Rightarrow b : B \to (A \to C)}\,R\to\ (c, d\text{ fresh})}{\Rightarrow 0 : (A \to (B \to C)) \to (B \to (A \to C))}\,R\to\ (a, b\text{ fresh})
$$

and δ_1 is derived by:

$$
\dfrac{\dfrac{R0cc, Raey, Rycf, \mathcal{S}', e : A, c : B \Rightarrow f : C, c : B}{Raey, Rycf, \mathcal{S}', e : A, c : B \Rightarrow f : C, c : B}\,\text{R2} \qquad \dfrac{R0ff, Raey, Rycf, \mathcal{S}', f : C, e : A, c : B \Rightarrow f : C}{Raey, Rycf, \mathcal{S}', f : C, e : A, c : B \Rightarrow f : C}\,\text{R2}}{Raey, Rycf, \mathcal{S}, y : B \to C, e : A, c : B \Rightarrow f : C}\,L\to
$$

where $\mathcal{S} = Racd, Rdef, Rbcd, R0ab, a : A \to (B \to C)$ and $\mathcal{S}' = Racd, Rdef, Rbcd, R0ab, a : A \to (B \to C), y : B \to C$.
G3rRW $\vdash \Rightarrow 0 : A \to ((A \to B) \to B).$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{R0aa, Rcad, \mathcal{S}, a : A \Rightarrow d : B, a : A}{Rcad, \mathcal{S}, a : A \Rightarrow d : B, a : A}\,\text{R2} \qquad \dfrac{R0dd, Rcad, \mathcal{S}, d : B, a : A \Rightarrow d : B}{Rcad, \mathcal{S}, d : B, a : A \Rightarrow d : B}\,\text{R2}}{Rcad, Racd, Rbcd, R0ab, a : A, c : A \to B \Rightarrow d : B}\,L\to}{Racd, Rbcd, R0ab, a : A, c : A \to B \Rightarrow d : B}\,\text{R14}}{Rbcd, R0ab, a : A, c : A \to B \Rightarrow d : B}\,\text{R4}}{R0ab, a : A \Rightarrow b : (A \to B) \to B}\,R\to\ (c, d\text{ fresh})}{\Rightarrow 0 : A \to ((A \to B) \to B)}\,R\to\ (a, b\text{ fresh})
$$

where $\mathcal{S} = Racd, Rbcd, R0ab, c : A \to B$.
G3rR $\vdash \Rightarrow 0 : ((A \to A) \to B) \to B.$

$$
\dfrac{\dfrac{\dfrac{\dfrac{R0cd, Ra0a, R0ab, c : A \Rightarrow b : A, d : A}{Ra0a, R0ab \Rightarrow b : B, 0 : A \to A}\,R\to\ (c, d\text{ fresh}) \qquad Ra0a, R0ab, a : B \Rightarrow b : B}{Ra0a, R0ab, a : (A \to A) \to B \Rightarrow b : B}\,L\to}{R0ab, a : (A \to A) \to B \Rightarrow b : B}\,\text{R16}}{\Rightarrow 0 : ((A \to A) \to B) \to B}\,R\to\ (a, b\text{ fresh})
$$

G3rRM $\vdash \Rightarrow 0 : A \to (A \to A).$

$$\cfrac{\cfrac{\cfrac{\cfrac{R0ad, Racd, \mathcal{S}, a : A, c : A \Rightarrow d : A \qquad R0cd, Racd, \mathcal{S}, a : A, c : A \Rightarrow d : A}{Racd, Rbcd, R0ab, a : A, c : A \Rightarrow d : A}\ \text{R17}}{Rbcd, R0ab, a : A, c : A \Rightarrow d : A}\ \text{R4}}{R0ab, a : A \Rightarrow b : A \to A}\ R\to\ (c, d \text{ fresh})}{\Rightarrow 0 : A \to (A \to A)}\ R\to\ (a, b \text{ fresh})$$

where $\mathcal{S} = Rbcd, R0ab$. □

B Appendix

Proof of Theorem 6.3 (Semantic Completeness cont.) In this appendix we construct a reduction tree for an arbitrary sequent $\mathcal{S}$, by applying, root-first, all rules for **G3rX** according to a specific order. This construction is used to define a countermodel to $\mathcal{S}$ (displayed above). Importantly, recall that, to reflect the notion of validity at the actual world, we will consider derivability at 0.
The reduction tree is defined inductively in stages as follows: (1) If $n = 0$, then $\Gamma \Rightarrow \Delta$ stands at the root of the tree. (2) If $n > 0$, we distinguish two subcases. (2.1) If every topmost sequent is an axiomatic sequent reduction the tree terminates; (2.2) If no axiomatic sequent is reached, the construction of the reduction tree does not terminate and we continue applying, root-first, all rules of **G3rX** according to a specific order. There are $8 + j$ different stages: 8 for the rules for the propositional connectives and j for the mathematical rules. We start, for $n = 1$, with $L\sim$ and consider topmost sequents of the following form:

$$0 : \sim B_1, \ldots, 0 : \sim B_k, \Gamma' \Rightarrow \Delta$$

where $0 : \sim B_1, \ldots, 0 : \sim B_k$, are all formulas in Γ with $\sim$ as outermost connective. By applying, root-first, k times, $L\sim$ we obtain the following sequent:

$$\Gamma' \Rightarrow \Delta, 0^* : B_1, \ldots, 0^* : B_k$$

placed on top of the former.
For $n = 2$, we consider sequents of the form:

$$\Gamma \Rightarrow \Delta', 0 : \sim B_1, \ldots, 0 : \sim B_k$$

By applying, root-first, k times, $R\sim$ we obtain the following sequent:

$$0^* : B_1, \ldots, 0^* : B_k, \Gamma \Rightarrow \Delta'$$

placed on top of the former.
For $n = 3$, we consider sequents of the form:

$$0 : B_1 \wedge C_1, \ldots, 0 : B_k \wedge C_k, \Gamma' \Rightarrow \Delta$$

By applying, root-first, k times, $L\wedge$ we obtain the following sequent:

$$0 : B_1, 0 : C_1, \ldots, 0 : B_k, 0 : C_k, \Gamma' \Rightarrow \Delta$$

The case for $n=6$, with $R\vee$ is symmetric.
For $n=4$, we consider sequents of the form:

$$\Gamma \Rightarrow \Delta', 0 : B_1 \wedge C_1, \ldots, 0 : B_k \wedge C_k$$

By applying, root-first, k times, $R\wedge$ we obtain the following sequents:

$$\Gamma \Rightarrow \Delta', 0 : B_1, \ldots, 0 : B_k \qquad \text{and} \qquad \Gamma \Rightarrow \Delta', 0 : C_1, \ldots, 0 : C_k$$

placed on top of the former as its premises. The case for $n=5$, with $L\vee$ is symmetric.
For $n=7$, we consider topmost sequents of the following form:

$$R0a_1b_1, \ldots, R0a_kb_k,\ 0 : B_1 \to C_1, \ldots, 0 : B_k, \to C_k,\ \Gamma' \Rightarrow \Delta$$

where labels and principal formulas are in Γ'. By applying, root-first, k times, $L\to$ (with $R0a_1b_1, \ldots, R0a_kb_k, 0 : B_1 \to C_1, \ldots, 0 : B_k, \to C_k$ principal) we obtain the following sequent:

$$R0a_1b_1, \ldots, R0a_kb_k,\ b_{m_1} : C_{m_1}, \ldots, b_{m_l} : C_{m_l},\ \Gamma' \Rightarrow \Delta,\ a_{j_{l+1}} : B, \ldots, a_{j_k} : B$$

where $\{m_1, \ldots, m_l\} \subseteq \{1, \ldots, k\}$ and $j_{l+1}, \ldots, j_k \in \{1, \ldots, k\} - \{m_1, \ldots, m_l\}$, and placed on top of the former as its premises.
For $n=8$, we consider all the labelled sequents that have implications in the succedent. We consider topmost sequents of the following form:

$$\Gamma \Rightarrow \Delta', 0 : B_1 \to C_1, \ldots, 0 : B_k, \to C_k$$

Let $a_1, ..., a_k$ and $b_1, \ldots, b_k$ be fresh variables, not yet used in the reduction tree and apply, root-first, k times, $R\to$ to obtain the following sequent:

$$R0a_1b_1, \ldots, R0a_kb_k,\ a_1 : B, \ldots, a_k : B,\ \Gamma \Rightarrow \Delta',\ b_1 : C_1, \ldots, b_k : C_k$$

placed on top of the former as its premise.
Finally, we consider relational rules. If it is a rule without eigenvariable condition, we write on top of the lower sequent the result of applying the relational rule under consideration. For relational rules with eigenvariable condition, the proof proceeds analogously to the proof at stage $n=8$. As an example, consider R7 and a topmost sequent of the following form:

$$Ra_1b_1c_1, \ldots, Ra_kb_kc_k, \Gamma' \Rightarrow \Delta$$

Let $x_1, \ldots, x_k$ be variables not yet used in the reduction tree. By applying k times, root-first, R7, we obtain the following sequent, placed on top of the former:

$$Ra_1b_1x_1, Ra_1x_1c_1, Ra_1b_1c_1, \ldots, Ra_kb_kx_k, Ra_kx_kc_k, Ra_kb_kc_k, \Gamma' \Rightarrow \Delta$$

This construction is then used in the development of the second part of the proof displayed in Section 6.

C Appendix

Proof of Theorem 7.4 (CUT-admissibility cont.). We finish the proof of CUT-admissibility by displaying some other salient examples.We distinguish three main cases.

Case 1: If at least one of the premises of CUT is an axiom, we distinguish 4 subcases:

Case 1.1: The left premise of CUT is an axiom and the CUT-formula is not principal. If the derivation has the following shape:

$$\dfrac{R0bc, b : B, \Gamma \Rightarrow \Delta, c : B, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B}\ \textsc{cut}$$

It is transformed into:

$$R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B$$

without applications of CUT.

Case 1.2: The left premise of CUT is an axiom and the CUT-formula is principal. The derivation:

$$\dfrac{R0ba, b : A, \Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{R0ba, b : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

is transformed into:

$$\text{(Lemma 7.1)}\ \dfrac{\text{(Lemma 4.1)}\ \dfrac{a : A, \Gamma' \Rightarrow \Delta'}{b : A, \Gamma' \Rightarrow \Delta'}\ \textsc{sub}(b/a)}{R0ba, b : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{lw+rw+lw}_L$$

Case 1.3: The right premise of CUT is an axiom and the CUT-formula is not principal. The derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, R0bc, b : B, \Gamma' \Rightarrow \Delta', c : B}{R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B}\ \textsc{cut}$$

It is transformed into:

$$R0bc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', c : B$$

without applications of CUT.

Case 1.4: The right premise of CUT is an axiom and the CUT-formula is principal. The derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad R0ab, a : A, \Gamma' \Rightarrow \Delta', b : A}{R0ab, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b : A}\ \textsc{cut}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A}{\Gamma \Rightarrow \Delta, b : A}\ \textsc{sub}(b/a)\ (\text{Lemma 4.1})}{R0ab, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b : A}\ \textsc{lw+rw+lw}_L\ (\text{Lemma 7.1})$$

Case 2: The CUT-formula A is not principal in at least one premise. The proof proceeds by permuting the application of CUT with the rule under consideration, to move the CUT upwards in the transformed derivation.

Case 2.1: A is not principal in the left premise. We distinguish two subcases.
Subcase 2.1.1: Let $\Gamma = x : {\sim}B, \Gamma''$:

$$\dfrac{\dfrac{\Gamma'' \Rightarrow \Delta, a : A, x^* : B}{x : {\sim}B, \Gamma'' \Rightarrow \Delta, a : A}\ L{\sim} \qquad a : A, \Gamma' \Rightarrow \Delta'}{x : {\sim}B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

and transform it into the following one:

$$\dfrac{\dfrac{\Gamma'' \Rightarrow \Delta, a : A, x^* : B \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma'', \Gamma' \Rightarrow \Delta, \Delta', x^* : B}\ \textsc{cut}}{x : {\sim}B, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ L{\sim}$$

where the CUT-height is reduced.
Let $\Gamma = Rxbc, x : B \to C, \Gamma''$ and consider as an example $L \to$. We have the following derivation:

$$\dfrac{\dfrac{Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta, a : A, b : B \qquad Rxbc, c : C, x : B \to C, \Gamma'' \Rightarrow \Delta, a : A}{Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta, a : A}\ L{\to} \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, x : B \to C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

and transform it into the following one:

$$\dfrac{\dfrac{Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta, a : A, b : B \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, x : B \to C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta', b : B}\ \textsc{cut} \qquad \dfrac{Rxbc, c : C, x : B \to C, \Gamma'' \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, c : C, x : B \to C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}}{Rxbc, x : B \to C, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ L{\to}$$

with two CUTs of lower height.
Subcase 2.1.2: Let $\Delta = \Delta'', x : {\sim}B$:

$$\dfrac{\dfrac{x^* : B, \Gamma \Rightarrow \Delta'', a : A}{\Gamma \Rightarrow \Delta'', x : {\sim}B, a : A}\ R{\sim} \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : {\sim}B}\ \textsc{cut}$$

it is transformed into the following application of CUT with a shorter derivation height:

$$\dfrac{\dfrac{x^* : B, \Gamma \Rightarrow \Delta'', a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{x^* : B, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta'}\ \textsc{cut}}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : {\sim}B}\ R{\sim}$$

Let $\Delta = \Delta'', x : A \to B$:

$$\dfrac{(b, c \text{ fresh}) \quad \dfrac{Rxbc, b : B, \Gamma \Rightarrow \Delta'', c : C, a : A}{\Gamma \Rightarrow \Delta'', x : B \to C, a : A}\ R\!\to \qquad a : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : B \to C}\ \textsc{cut}$$

it is transformed into the following application of CUT with a shorter derivation height:

$$(b, c \text{ fresh}) \quad \dfrac{\dfrac{Rxbc, b : B, \Gamma \Rightarrow \Delta'', c : C, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rxbc, b : B, \Gamma, \Gamma' \Rightarrow \Delta'', \Delta', c : C}\ \textsc{cut}}{\Gamma, \Gamma' \Rightarrow \Delta'', \Delta', x : B \to C}\ R\!\to$$

As an example for the relational rules, we deal with R7 (with variable condition). Let $\Gamma = Rabc, \Gamma''$:

$$\dfrac{\dfrac{Rabx, Raxc, Rabc, \Gamma'' \Rightarrow \Delta, a : A}{Rabc, \Gamma'' \Rightarrow \Delta, a : A}\ \text{R7} \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rabc, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

(x is a fresh variable) It is transformed in the following one:

$$\dfrac{\dfrac{Rabx, Raxc, Rabc, \Gamma'' \Rightarrow \Delta, a : A \qquad a : A, \Gamma' \Rightarrow \Delta'}{Rabx, Raxc, Rabc, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \textsc{cut}}{Rabc, \Gamma'', \Gamma' \Rightarrow \Delta, \Delta'}\ \text{R7}$$

The other cases for relational rules are dealt with analogously.
Case 2.2: A is principal in the left premise only. We distinguish two subcases.
Subcase 2.2.1: Similarly to the preceding subcase. Let $\Gamma' = x : {\sim}B, \Gamma''$:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad \dfrac{a : A, \Gamma'' \Rightarrow \Delta', x^* : B}{a : A, x : {\sim}B, \Gamma'' \Rightarrow \Delta'}\ L{\sim}}{x : {\sim}B, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, \Gamma'' \Rightarrow \Delta', x^* : B}{\Gamma, \Gamma'' \Rightarrow \Delta, \Delta', x^* : B}\ \textsc{cut}}{x : {\sim}B, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ L{\sim}$$

with a shorter derivation height.
Let $\Gamma' = Rxbc, x : B \to C, \Gamma''$ and consider $L\!\to$. We have the following derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad \dfrac{a : A, Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta', b : B \qquad a : A, Rxbc, c : C, x : B \to C, \Gamma'' \Rightarrow \Delta'}{a : A, Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta'}\ L\!\to}{Rxbc, x : B \to C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ \textsc{cut}$$

is reduced to the following one:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, Rxbc, x : B \to C, \Gamma'' \Rightarrow \Delta', b : B}{Rxbc, x : B \to C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta', b : B}\text{ CUT} \qquad \dfrac{\Gamma \Rightarrow \Delta, a : A \qquad a : A, Rxbc, c : C, x : B \to C, \Gamma'' \Rightarrow \Delta'}{Rxbc, c : C, x : B \to C, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\text{ CUT}}{Rxbc, x : A \to B, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\ L\to$$

with two CUTs of lower height.
Subcase 2.2.2: Let $\Delta' = \Delta'', x : \sim B$:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad \dfrac{x^* : B, a : A, \Gamma' \Rightarrow \Delta''}{a : A, \Gamma' \Rightarrow \Delta'', x : \sim B}\ R\sim}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : \sim B}\text{ CUT}$$

it is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad x^* : B, a : A, \Gamma' \Rightarrow \Delta''}{x^* : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta''}\text{ CUT}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : \sim B}\ R\sim$$

with a shorter derivation height.
Let $\Delta' = \Delta'', a : B \to C$ and the derivation:

$$\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad {\scriptstyle (b, c \text{ fresh})}\ \dfrac{Rxbc, a : A, b : B, \Gamma' \Rightarrow \Delta'', c : C}{a : A, \Gamma' \Rightarrow \Delta'', x : B \to C}\ R\to}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : B \to C}\text{ CUT}$$

It is reduced to the following one:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, a : A \qquad Rxbc, a : A, b : B, \Gamma' \Rightarrow \Delta'', c : C}{Rxbc, b : B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'', c : C}\text{ CUT}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'', x : B \to C}\ R\to \ {\scriptstyle (b, c \text{ fresh})}$$

with a shorter derivation height.
Case 3: The procedure for A being $\sim B$ or $B \to C$, can be found on p. 68.

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