# Natorp's Neo-Kantian Mathematical Philosophy of Science

*[A filosofia matemática da ciência no neokantismo de Natorp]*

**Thomas Mormann**[1]

(Tsukuba, Japan)

DOI: 10.5380/sk.v20i2.90396

## Abstract

This paper deals with Natorp's version of the Marburg mathematical philosophy of science characterized by the following three features: The core of Natorp's mathematical philosophy of science is contained in his "*knowledge equation*" that may be considered as a mathematical model of the "transcendental method" conceived by Natorp as the essence of the Marburg Neo-Kantianism. For Natorp, the object of knowledge was an infinite task. This can be elucidated in two different ways: Carnap, in the *Aufbau*, contended that this endeavor can be divided into two distinct parts, namely, a finite "constitution" of the object of knowledge and an infinite incompletable empirical description. In contrast, and more in the original spirit of Cohen and Natorp, the physicist and philosopher Margenau in *The Nature of Physical Reality* (Margenau, 1950) conceived the infinity of this "Aufgabe" as an infinite dialectical process, in which relative "data" and "conceptual constructs" determine each other. This dialectical process eliminates the dichotomy between Anschauung and Begriff that distinguished the Marburg Neo-Kantianism from Kantian orthodoxy, namely, the abandonment of the difference between intuition and concept. Finally, the paper deals with the non-Archimedean geometrical systems that played a central role in Natorp's defense of Cohen's "infinitesimal" metaphysics.
**Keywords:** Natorp's mathematical philosophy of science; Cohen; Cassirer; Natorp's knowledge equation; Non-archimedian geometry; infinitesimal metaphysics.

## Resumo

Este artigo trata da versão de Natorp da filosofia matemática da ciência de Marburgo caracterizada pelas seguintes três características: O núcleo da filosofia matemática da ciência de Natorp está contido em sua "equação do conhecimento" que pode ser considerada como um modelo matemático do "método transcendental", concebida por Natorp como a essência do neokantismo de Marburgo. Para Natorp, o objeto do conhecimento era uma tarefa infinita. Isso pode ser elucidado de duas maneiras diferentes: Carnap, no *Aufbau*, sustentou que esse esforço pode ser dividido em duas partes distintas, a saber, uma "constituição" finita do objeto de conhecimento e uma descrição empírica infinita incompletável. Em contraste, e mais no espírito original de Cohen e Natorp, o físico e filósofo Margenau em *The Nature of Physical Reality* (Margenau, 1950) concebeu a infinitude desse "Aufgabe" como um processo dialético infinito, no qual "dados" relativos e "construções conceituais" determinam-se mutuamente. Esse processo dialético elimina a dicotomia entre *Anschauung* e *Begriff* que distinguia o neokantismo de Marburgo da ortodoxia kantiana, ou seja, o abandono da diferença entre intuição e conceito. Finalmente, o artigo trata dos sistemas geométricos não arquimedianos que desempenharam um papel central na defesa de Natorp da metafísica "infinitesimal" de Cohen.
**Palavras-chave:** filosofia matemática da ciência de Natorp; Cohen; Cassirer; equação do conhecimento de Natorp; geometria não arquimediana; metafísica infinitesimal.

1 thomasarnold.morman@ehu.es

## 1. The Mathematical Philosophy of Science of the Marburg Neo-kantianism

According to classical wisdom, the most exact science of all sciences is mathematics. Thus, it is hardly surprising that in the Marburg account of philosophy of science mathematics played a preeminent role. It should be noted that from the beginning, however, the Marburg account of philosophy of science in general and philosophy of mathematics in particular had some rather specific features that distinguish the account from contemporary mainstream ideas about philosophy of mathematics. From the Marburg Neo-Kantian perspective, any philosophically acceptable "perspective on mathematics" must take into account not mathematics alone but mathematics and the exact sciences, particularly physics. This was succinctly expressed by Cassirer in his early programmatic paper *Kant und die moderne Mathematik* (1907):

> Der Blick der Philosophie darf—wenn man dieses Verhältnis einmal schroff und paradox ausdrücken will— weder auf die Mathematik noch auf die Physik gerichtet sein; er richtet sich einzig auf den Zusammenhang beider Gebiete (Cassirer, 1907, p. 48).

When dealing with the issue of what is to be understood by the "mathematical philosophy of science of the Marburg Neo-Kantianism", first of all, it is expedient to point out that the Marburg perspective is characterized by some specific features that distinguish it from the contemporary understanding of philosophy of science and from philosophy of mathematics in particular.

For the philosophers of the Marburg School, mathematics was the constitutive method of exact empirical sciences überhaupt. Mathematics was the guarantor and the expression of the scientific character of these disciplines. At the same time, for the Marburg philosophers, this insight secured the scientific status of philosophy (of science) itself. From a Neo-Kantian perspective, any philosophically acceptable "perspective on mathematics" had to take into account not mathematics alone but mathematics and the exact sciences, in particular physics. This "paradoxical" stance was not Cassirer's idiosyncrasy; all members of the Marburg school subscribed to a similar conception, in particular, Cohen and Natorp. In the following, I would like to show that this stance gave the Marburg philosophy of science a peculiar flavor that justifies its characterization as a "mathematical philosophy of science".

In this paper I would like to deal with Natorp's version of the Marburg mathematical philosophy of science, which has until now been a rather neglected topic in the (history of) philosophy of science.[2] First, a trivial clarification must be made: Mathematical philosophy of science is not the philosophy of mathematics. Indeed, a mathematical philosophy of science is to be understood in a more complex way:

(1) Philosophy of mathematics proper is only one part of a comprehensive mathematical philosophy of science in general.

(2) The application of mathematics in the empirical sciences is a central issue for the mathematical philosophy of science.

(3) The methods of mathematical philosophy of science are inspired by mathematics.

Although all members of the Marburg school subscribed to a mathematical philosophy of science in this sense, the Marburg school mathematical philosophy of science was not a monolithic doctrine that remained constant over the whole existence of the school and that was interpreted in the same way by all of its members.

2 Arguably, Cassirer's version of the Marburg school of mathematical philosophy of science is the best known, Cohen's is the most obscure, and Natorp's is the most ignored version of the Marburg school's mathematical philosophy of science. As I would like to show in the following, this does not entail that Natorp's version is the least interesting version.

True, the Marburg philosophers attempted to make this impression on nonmembers of the school. However, a closer look reveals that from early on, there existed more or less subtle differences and inconsistencies between the various versions of the Marburg mathematical philosophy, as put forward by Cohen, Cassirer, and Natorp. In this paper, I would like to concentrate on Natorp's version, but this clearly requires dealing with Cohen's and Cassirer's versions to some extent as well.

Somewhat paradoxically, from the very beginning, the Marburg mathematical philosophy of science had a rather tense relationship with contemporaneous mathematics. This situation may be compared with an unhappy love-affair — on the side of the Marburg school. While the Marburg philosophers never became tired of emphasizing the fundamental importance of mathematics for the sciences and for reason in general, mathematicians and logicians used to have a negative attitude toward the advances of the Marburg philosophers — often criticizing them as mathematically incompetent (cf. Frege (1885), Cantor (1884), Fraenkel (1967)). This ranged from the harsh criticisms that Cohen's *Das Prinzip der Infinitesimalmethode und seine Geschichte* (1883) received from most mathematicians to the rejection of Natorp's intended "synthese" of Cantor's and Veronese's accounts (cf. Natorp (1910)) by mathematicians such as Abraham Fraenkel and Abraham Robinson in the 1960s (cf. Fraenkel (1968), Robinson (1966)).

To some extent, these criticisms were justified; in other respects, they were unfair and overstated, ignoring the different intentions of mathematicians and philosophers. One reason for these sharp disagreements was that at the turn of the previous century, the concepts of the infinitely large and the infinitely small (infinitesimal) were hotly discussed within the field of mathematics itself, and the Marburg school in some sense clashed with the concepts that were to become mainstream in 20th century mathematics. In an oversimplified way, this mainstream conception can be characterized by the thesis that the concept of the infinitely small is, as Russell put it, an inconsistent pseudoconcept that had to be eliminated from the discourse of mathematics. In contrast, Hermann Cohen in *Das Prinzip* (1883) proposed the bold claim that the concept of the infinitesimal is the central notion in philosophy of science überhaupt.

For a long time, the generally accepted conviction of mathematicians and philosophers was that the adversaries of infinitesimals were the winners of this dispute. However, the situation has become more complicated: The concept of the infinitesimal has been rehabilitated in mathematics. It is a recognized topic in the disciplines of non-Archimedean mathematics, in particular in nonstandard analysis, and smooth analysis, where systems containing infinitesimals of different kinds are studied. Today, infinitesimals are recognized as mathematically respectable objects with the same ontological dignity as finite or infinitely large magnitudes.

In hindsight then, Natorp's *Logische Grundlagen* (1910) may be considered as having been ahead of its time, insofar as it is one of the very few treatises on philosophy of science that at least partially understood the relevance of non-Archimedean mathematics. This is not to deny that Natorp's account was deeply flawed insofar as it got the relation between infinitely small and infinitely large (Cantorian) numbers quite wrong.

Natorp's *Grundlagen* was published in the same year as Cassirer's *Substanzbegriff und Funktionsbegriff*, namely, in 1910. Since then, the work has always stood in the shadow of Cassirer's more brilliant opus. It would be unjustified, however, to consider the work as straightforwardly obsolete: In certain respects, Natorp's *Grundlagen* was more modern, more mathematically profound and certainly more faithful to the original spirit of the Marburg neo-Kantianism than Cassirer's *Substance and Function*. More precisely, Natorp intended to elucidate and make more precise Cohen's often obscure approach and, at the same time, he attempted to preserve the conceptual essence of Cohen's approach.[3]

3 Natorp was well aware of the fact that he did not fully succeed in this respect. In a letter to Görland (November 21, 1902) (Holzhey, 1986, p. 302), he characterized Cohen's way of philosophizing as follows: "Er ist u[nd] bleibt Poet in der *Art* seines Philosophierens, obwohl in sehr vielen Fällen die Ergebnisse sich nachher auch auf logischen Wegen

The outline of the paper is as follows: In section 2, *The Transcendental Method and Natorp's Knowledge*, we deal with Natorp's "*knowledge equation*", a mathematical model of the "transcendental method" that Natorp considered the core of the Marburg. Section 3, *The Object of Knowledge as an Infinite Task: Two Opposing Elucidations,* deals with two opposing elucidations of how science copes with this "infinite task": As Carnap contended in the *Aufbau* (Carnap, 1928), this endeavor can be divided into two quite distinct parts, namely, a finite "constitution" of the object and an infinite incompletable empirical description. In contrast, and more in the original spirit of Cohen and Natorp, the physicist and philosopher Margenau in *The Nature of Physical Reality* (Margenau, 1950) conceived the infinity of this "Aufgabe" as an infinite dialectical process, in which relative "data" and "conceptual constructs" determine each other. Section 4, *Eliminating the Dichotomy between Anschauung and Begriff*, deals with one of the most important features that distinguishes the Marburg Neo-Kantianism from Kantian orthodoxy, namely, the abandonment of the crucial difference between intuition and concept. Section 5 is concerned with the non-Archimedean geometrical systems that played a central role in Natorp's defense of Cohen's "infinitesimal" metaphysics. Section 6, *Infinitesimals: the Revolution of Rigor and Natorp's Failed Synthesis*, discusses Natorp's attempted synthesis of Carnap's and Veronese's account of infinitely large and infinitely small (infinitesimal) numbers.

## 2. The Transcendental Method and Natorp's Knowledge Metaphor

The Neo-Kantian approach to epistemology and philosophy aimed to be faithful to the spirit but not necessarily the letter of Kant's philosophy. For Natorp, this meant restituting the "transcendental method" as the true core of the Kantian approach and to give up all ingredients of Kant's system that did not sit well with that method. The transcendental method deals with the problem of the possibility of experience. The Neo-Kantians interpreted Kant as contending that the object of experience is determined by the laws and methods of the knowing subject. Therefore, the object is no longer something given ("gegeben") but something "posed" ("aufgegeben") (cf. Kinkel, 1923, p. 405). Conceiving Neo-Kantian philosophy as based on the transcendental method has two implications:

(i) Philosophy recognizes the historical, societal and scientific context in which it exists. It is aware that it is rooted in the specific theoretical and practical experiences of its time and refuses to build "high towers of metaphysical speculations" (cf. Natorp, 1912, p. 195, Kinkel, 1923, pp. 402/403).

(ii) Philosophy accepts the facts of science, morality, art and religion. The task of philosophy is to carry out a *deductio iuris* of these facts, i.e., to provide a kind of "logical analysis" that shows the reasons why these facts are possible, thereby revealing what is their "quid iuris". In other words, and going beyond the epistemological sphere, philosophy has to show the lawfulness and reasonableness of the cultural achievements of humankind.

Therefore, the critical idealism of the Marburg school (as the self-proclaimed true heir of Kant's philosophy) led to a "genetic" epistemology and theory of science that regarded the ongoing process of scientific and cultural creation as essential, and its temporary results were considered of secondary importance. As Natorp famously put it with respect to scientific knowledge, the fact of scientific knowledge is always a "becoming fact" ("Werdefaktum") and is never "closed" or "finished". There never is something "given" that is not transformed in the

darstellen lassen; in einigen aber vielleicht nicht, wenigstens reichen die *mir* zugänglichen Pfade der Logik nicht zu *allen* seinen Resultaten, obwohl zu recht vielen". Natorp's admitted inability to fully reformulate Cohen's "poetic way of philosophizing" in a more sober logical way had the consequence that he did not publish a fuller account of Cohen's *Logik* in his *Die logischen Grundlagen der exakten Wissenschaften* (Natorp, 1910)) as he had originally planned (cf. Holzheyl, 1986, Section II).

ongoing and, strictly speaking, infinite process of cognition.

The rejection of a nonconceptual given in any form brings the Marburg brand of Neo-Kantianism into open conflict with some of the cornerstones of Kant's epistemology: namely, the dualism of "scheme" and "intuition" and related dualisms such as that of the "spontaneity" and "receptivity" of thinking. Natorp was well aware of this fact: "Maintaining this dualism of epistemic factors (receptivity and spontaneity, T.M.) is virtually impossible if one takes seriously the core idea of the transcendental method" (Natorp, 1912, p. 9).

Subscribing to a "genetic" account of knowledge that emphasizes the process character of knowledge gives the relation of knowledge priority over its relata, namely, the knowing subject and the object of knowledge. Both are constituted in the ongoing process of knowledge. Taken in themselves, the subject and object are just abstractions from the more basic relation of knowledge. Although it may sometimes be expedient to treat the subject of knowledge and the object of knowledge separately, this separation is to be considered only a methodological device by which one may distinguish between two complementary accounts: an account in which the object occupies center stage and an account that emphasizes the role of the cognizing subject. In a Kantian framework, object-oriented accounts emphasize the role of the receptivity of cognition, in particular perception, while subject-oriented, epistemic accounts are inclined to stress the constructive aspects of cognition. According to the Neo-Kantian doctrine, both accounts are incomplete and therefore mistaken. For Neo-Kantianism, ontology and epistemology are two sides of the same coin. Ontology without epistemology would be some kind of magic, which would leave unexplained how knowledge accesses its object, while epistemology without ontology would be without content, since it would deny the objectual character of cognition. Expressed in Kantian language, object-oriented approaches tend to emphasize the receptivity of cognition. According to these approaches, cognition is essentially a passive and receptive behavior. The thinking mind is confronted with something that is outside and independent of the sphere of reason. Ignoring more subtle differences, this amounts to some kind of "copy theory" or "mirror theory" of knowledge. Subject-oriented approaches, on the other hand, emphasize the spontaneity of cognition. According to these approaches, cognizing is essentially a creative activity. Such a conception does not admit a "given" as a mind-independent presupposition in the cognizing process. Rather, the given ("das Gegebene") is to be conceived of as the product ("Ergebnis") of the immanent determination of thought. Therefore, subject-oriented approaches are in danger of underestimating the resistant power of the real world in favor of the unrestricted creative power of the knowing mind. According to Natorp, critical idealism, which employs the "transcendental method" as its fundamental guideline, avoids the shortcomings and deficits of both the subject-oriented and the object-oriented accounts.

Natorp famously (or notoriously) used to elucidate the transcendental method with the aid of a metaphor, namely, the metaphor of the "knowledge equation" (cf. also Kinkel, 1923, p. 405). The metaphor compared the evolution of science with the solution of numerical equations. More precisely, the "x" of a polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x + a_0 = 0$. According to this metaphor, coming to know an object—the "Erkenntnisobjekt"—is analogous to the process of solving such a numerical equation, i.e., determining the "unknown x". To be specific, consider a numerical polynomial in an equation such as

$$x^3 + x^2 + x + 1 = 0$$

The equation has the solutions x = (-1, +i, -i). In line with Natorp's didactic intentions, this equation sought to convey several ideas concerning knowledge and its objects.

1. The fact that the equation has several different solutions indicates that the process of research may not lead to unique results.

2. The fact that two of the knowledge solutions, namely, +i and –i, are imaginary reflects the fact that the research process may lead to an expansion of the original fundamental concepts

that one started with: The admission of complex numbers as solutions to an equation with only real parameters transcends the conceptual space in which the equation was originally formulated insofar as all these parameters are real numbers — even integers.

According to Natorp, the structure of the knowledge equation, namely, its peculiar interweaving of determined and yet to be determined ingredients, has important consequences for the concept of knowledge:

> If the object is to be the x of the equation of inquiry, then it must be possible to determine the meaning of this x by the nature of this equation (i.e., by the inquiry itself) in relation to its known factors (our fundamental concepts). From this, it must follow whether and in what sense the solution of this problem is possible for us. This is the very idea of the transcendental or critical method (Natorp, 1903, p. 10).

The transcendental method does not aim to extend our knowledge beyond the limits of the scientific method. Rather, the method seeks to clarify the limits of scientific knowledge. The method is called "transcendental" since it goes beyond the cognition that is immediately directed toward the objects but aims to obtain information about the general direction of the path to be taken. The method does not provide us with any specific knowledge about an object beyond experience. Hence, following the established Kantian terminology, the method is transcendental but not transcendent.

Natorp and his fellow philosophers of the Marburg school viewed the object of knowledge not as an unproblematic starting point of the ongoing process of scientific investigation but rather as its ideal limit. For the Marburg Neo-Kantians, the object was not given: the object was a problem to be solved. In various versions, this equational account of knowledge can be found in virtually all of Natorp's epistemological writings (cf. Natorp, 1902, 1910, 1912, 1927).

One might object that Natorp's equational model of scientific cognition is too simple, in the sense that empirical objects hardly ever show up as solutions to a finite equation such as the one considered above. Hence, what is still missing in this version of the metaphor is the "infinite character" of the knowledge equation. Natorp was aware of this shortcoming and tried to remedy it by emphasizing the knowledge equation as an infinite *Aufgabe*. Elaborating the equational model, he pointed out that the knowledge equation was not to be understood simply as a finite *Aufgabe* (problem) but as an infinite *Aufgabe* (problem) that could be solved only approximately by finite creatures such as us. Thereby, Natorp sought to escape from the trap of an overstated "Hegelian" rationalism in which all problems were merely conceptual problems.[4] Against Hegel's intellectualism, he wanted to characterize the Marburg approach as a more modest approach as follows:

> Although we conceive of the object of knowledge (= x), similarly as Hegel does, only in relation to the functions of knowledge itself, and consider it…as the x of the equation of knowledge, …we have understood that this equation is of such a kind that it leads to an infinite calculation. This means that the x is never fully determined by the parameters a, b, c, … of the equation. Moreover, the sequence of the parameters is not to be thought of as "closed" but rather as extendable further and further (Natorp, 1912, pp. 211-212).

According to Natorp, the transcendental method was the point of departure (from Kant) and the guideline for the scientific philosophy of the Marburg school. The method distinguished the transcendental approach of the Marburg philosophers from other rival approaches that Natorp characterized as using psychological, metaphysical, and logical methods (cf. Natorp, 1912, pp. 194ff).

4 For a contemporaneous critical discussion of the affinities between Hegelianism and the Marburg school from the point of view of Southwest Neo-Kantianism, see (Marck, 1913).

## 3. The object of knowledge as an infinite Task: Two opposing elucidations

The members of the Marburg school were not the only philosophers to consider the "knowledge equation" a characteristic and convenient metaphor for the epistemology of the Marburg school: philosophers who did not belong to the school also used the equation as an expedient means to separate their own approach from that of the Marburg philosophers. A good example is provided by Carnap's *The Logical Construction of the World* (*Aufbau*) (Carnap, 1928). One of Carnap's aims in this work was to separate the empiricist account of the *Aufbau* from the Marburg Neo-Kantianism by dismissing the knowledge equation as follows:

> According to the conception of the Marburg School (cf. Natorp's *Grundlagen*, pp. 18ff) the object is the eternal X, its determination is an incompleatable task. In opposition to this, it is to be noted that finitely many determinations suffice for the constitution of the object — and thus for its univocal description among the objects in general. Once such a description is set up, the object is no longer an X, but rather something univocally determined — whose complete description then certainly still remains an incompleteable task (Aufbau, § 179).

In *A Parting of the Ways* (2000), Michael Friedman argues that Carnap's attempt to dissociate himself from the Marburg school fails, since due to certain technical difficulties of the *Aufbau* (as already noted by Quine), it can be shown that Carnap did not succeed in constructing stable physical objects at a definite rank in the type-theoretical hierarchy of the quasi-analytical reconstruction of the world (cf. Friedman, 2000, p. 83). Thereby, Friedman concludes that:

> Carnap's construction of the physical world therefore appears never to close off at a definite rank in the hierarchy of types ... . And this means, of course, that the Marburg doctrine of the never completed "X" turns out to be correct, at least so far as physical (and hence all higher level) objects are concerned (Friedman, 2000, p. 84).

Whether this argument against Carnap's finite constructivism is thus definite, as Friedman claims, need not be discussed here. The more modest point I want to make is that in any case Carnap himself admitted that he did not escape the "infinite" character of the task of determining the "knowledge object" claimed by the Marburg school. Carnap simply divided the process of determining the object into two essentially different parts: a finite formal part, namely, that of constitution, and an infinite part, namely, empirical determination. How these two parts are related Carnap does not say. Rather, Carnap is content to invoke the metaphor of determining the geographical coordinates of the location of a physical object and the open-ended, infinite process of determining its empirical properties. Metaphorically, it may be obvious that the former can be determined without the latter, but on a second thought, this is revealed as being far from obvious.

In some sense then, the *Aufbau* falls back onto a Kantian position that conceives of cognition, i.e., the determination of the object of knowledge, as a bipartite process consisting of two separate ingredients, namely, empirical determination and conventional constitution. Arguably, the Marburg Neo-Kantian account of scientific knowledge, which denied any kind of Kantian dichotomy, was much farther away from traditional Kantianism and closer to what is truly occurring in science than Carnap's early logical empiricism. The *Aufbau's* unintended (and by its author vigorously denied) accordance with the original Marburg account concerning the infinite character of the determination of the knowledge object indicates that this feature is well entrenched in our understanding of the knowledge process. This does not mean, of course, that the Marburg description of this process is fully satisfying.

One of the very few attempts to improve and elaborate the original Marburg account was the approach taken by the physicist and philosopher Henry Margenau in his various works from the middle of the 1930s onwards (cf. Margenau, 1935, 1950). Margenau's work is particularly interesting, since he was one of the very few "working physicists" who was sympathetic to and knowledgeable about Neo-Kantian philosophy of science. As far as I know, he was the only philosopher-scientist who ever took notice of Cohen's *Die Logik der reinen Erkenntnis* (Cohen, 1902). Morgenau attempted to elucidate Natorp's "infinite task" of determining the knowledge object in an unending process in terms of a dialectics between the perceptual and the domain of conceptual constructs. In contrast to Kantian orthodoxy, according to Margenau, percepts (*Anschauungen*) and concepts (*Begriffe*) were to be distinguished, but they nevertheless belonged to the same connected realm:

> Sensation as part of the process of knowledge is not wholly *sui generis* and a passage from the qualities that signify an act of clear perception to those characterizing pure thought may well be gradual. ...
>
> On the one hand, many *concepts* have sensory-empirical aspects because of their reference to the immediately given...and...on the other hand, *sensory data* require concepts for their interpretation. Torn out of its context in experience, the immediately given becomes as grotesque as its counterpart, the rational, has often been when nourished in seclusion. Unless one is careful not to disturb the natural setting of data and thought, one's philosophy is artificial and certainly unrepresentative of science (Margenau 1950, p. 55).

Margenau explicitly and approvingly mentioned Cohen's basic "thesis of the origin" (*Die These des Ursprungs*): "Only thinking can generate what can be considered as being". Even more remarkable, Morgenau claimed that this thesis, usually considered evidence of Cohen's extravagant and overstated idealism, was fully compatible with Locke's empiricist *dictum* "nihil est in intellectu, quod prius non fuerit erat in sensu". Going beyond Nartorp, Margenau was not content to explain the infinite determination of the knowledge object by the metaphor of a mathematical equation; rather, he relied on a detailed "dialectical" description of the historical evolution of theories. In its most elementary form, the description went like this:

> We observe a falling body, or many different falling bodies, we then take the typical body into mental custody and endow it with the abstract properties expressed in the law of gravitation. It is no longer the body we originally perceived, for we have added properties which are neither immediately evident nor empirically necessary. If it be doubted that these properties are in a sense arbitrary, we need merely recall the fact that there is an alternate, equally or even more successful physical theory — that of general relativity — which ascribes to the typical bodies the power of influencing the metric of space, i.e., entirely different properties from those expressed in Newton's law of gravitation (Margenau 1935, p. 57).

Margenau continued by explaining the general process of the dialectical generation of knowledge in the empirical sciences as follows:

> The full course of physical explanation...begins in the range of perceptible awareness, swings over into what we shall now term the field of symbolic construction, and returns to perceptible awareness, or as we have said nature... The essential feature of physical explanation is evidently the transition from nature to the realm of construct, and the reverse (ibid., p. 59).

This kind of dialectic determination of the object of knowledge, in Margenau's terms, the ongoing process of the explanation of the physical object, can already be found in Natorp's knowledge equation, although in more abstract, mathematical terms. In Natorp's Neo-Kantianism, this dialectic between "percept" and "concept" also affected the Kantian dichotomy between intuition (*Anschauung*) and understanding (*Begriff*), leading the Marburg account to collapse the architectonic of the original Kantian system.

## 4. Eliminating the dichotomy between *Anschauung* and *Begriff*

For Natorp, one of the all-important consequences of the transcendental method was that the Kantian dualism between intuition and thought had to be given up. Natorp, like all members of the Marburg school, considered Kant's philosophy a promising starting point for modern epistemology and philosophy of science but not a doctrine that had to be followed literally. Like all Neo-Kantians, Natorp emphatically subscribed to the slogan of 'going with Kant beyond Kant'. The most important deviation from Kantian orthodoxy was to give up Kant's sharp separation between understanding and sensibility as two faculties of the mind. Beginning with Cohen, the Marburg philosophers replaced Kant's two faculties of the mind by a single comprehensive activity of 'pure thought' (*reines Denken*).[5] Pure thought primarily expressed itself in the progressive evolution of the mathematized empirical sciences. According to Cohen's well-known slogan, philosophy had to take 'the fact of science' as the starting point in its considerations. This attitude was but a consequence of the 'transcendental method' considered by the Neo-Kantians to be the core of Kantian philosophy. According to it, philosophy of science, such as philosophy in general, did not operate in empty space but had to rely on the historically established facts of science, ethics, art, and religion that provided it with its proper content (*cf*. Natorp, 1912, pp. 196–197). The task of philosophy was to 'justify' these facts by elucidating their reasonableness and thereby give a real sense to them. In other words, philosophy had to explicate the meaning of human culture and, in particular, the meaning of science.

In a nutshell, for Cohen, Natorp, and Cassirer the task of the philosophy of science was to make explicit the method of science as 'the method of an infinite and unending creative evolution of reason. Fulfilling this task was the indestructible core of Kant's philosophy' (Natorp, 1912, p. 200). In line with this dynamic concept of science, the Marburg school did not conceive the 'fact of science' as something static to be found "out there". Rather, science was to be conceived as a 'fact in becoming' (*Werdefaktum*). This led to a genetic epistemology that regarded the process of scientific evolution as essential and not so much its temporary results.[6] Thus, on the one hand, Natorp fully endorsed Cohen's elimination of the Kantian dualism between understanding and sensibility:

> So bleibt "Anschauung" nicht länger als denkfremder Faktor in der Erkenntnis dem Denken gegenüber — und entgegenstehend, sondern ist Denken, nur nicht blosses Gesetzesdenken, sondern volles Gegenstandsdenken; Anschauung verhält sich zum Denken des Begriffs, wie zum Gesetze der Funktion die Funktion selbst in ihrer Ausübung, im Vollzug (Natorp, 1912, p. 204).

However, the original Kantian distinction between understanding and sensibility reappears or is maintained in an attenuated form as the distinction between two aspects of "*Denken*", namely, between "*Gesetzesdenken*" and "*Gegenstands–denken*". Interestingly, Natorp attempted to elucidate this contrast by relying on the notion of function. According to Natorp, the relation between intuition and understanding can be considered analogous to the relation between two aspects of a function, namely, the aspect of a function as an object and the aspect of a function as a tool (for calculation). Whether this attempt to save some version of the

5 As has been observed by many scholars, giving up the Kantian dualism between a logical or conceptual faculty of pure understanding and an intuitive or nonconceptual faculty of pure sensibility amounts to an important difference between Kantian and Neo-Kantian philosophy (cf. Edel, 1991, pp. 60ff; Friedman, 2000, p. 27f).

6 Cf. Mormann & Katz, 2013.

traditional Kantian dichotomy is successful seems dubious.

## 5. Archimedean and Non-Archimedean Systems

Let us now consider in some more detail what may be considered one of the characteristic features of the Marburg mathematical philosophy of science, namely, the philosophical attention that it paid to certain non-Euclidean geometries. Today, virtually every philosopher agrees with Cassirer's thesis:

> In all the history of mathematics, there are few events of such immediate and decisive importance for the shaping and development of the problem of knowledge as the discovery of the various forms of non-Euclidean geometries (Cassirer 1950, p. 21).

This does not entail that all philosophers are seriously interested in the existing multiplicity of non-Euclidean geometries. Many acknowledge this multiplicity just as a fact, without paying much attention to its consequences. Actually, this is the attitude of most philosophers today, who take into account only Riemannian, i.e., locally Euclidean, geometry — due to its relevance to Einstein's general theory of relativity.

Most other types of non-Euclidean geometries are ignored or at least considered philosophically uninteresting. They are considered a topic of technical and formal interest for mathematicians only. One of the very few exceptions to this general philosophical attitude was Natorp. More precisely, in *Die logischen Grundlagen,* Natorp pointed out that the Marburg account of philosophy of science, which ascribed a central role to the concept of the infinitesimal, had to make philosophical sense of a "highly non-Euclidean" and even non-Riemannian geometries, namely, non-Archimedean geometries. Today, for philosophers, "non-Euclidean geometry" usually means "non-Euclidean Riemannian geometry" as it is used in Einstein's theory of general relativity. This amounts to a considerable restriction on what may be understood as a possibly empirically meaningful general concept of space. This view not only excludes many examples of non-Euclidean geometries but *a fortiori* dismisses modern general mathematical theories of spatial concepts such as topology as philosophically irrelevant.

In mathematics, systems of magnitudes that contain infinitely small magnitudes are called non-Archimedean systems, and systems of magnitudes with only finite elements are called Archimedean systems. A paradigmatic example of an Archimedean system is the system of natural numbers (N, <), with the relation < defined as $a < c := \exists b \in N(a + b = c)$. A non-Archimedean system would be a system that contains magnitudes that are — absurdly — infinitely close to 0 but nevertheless distinct from each other and 0, to borrow a definition from Quine.[7]

The "official" characterization of geometrical Archimedean magnitudes can be found in Hilbert's *Foundation of Geometry* (1899): "The Axiom of Archimedes: If AB and CD are any segments, then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B" (Hilbert, 1899, V.1, p. 26).

To those people who believe that we live in a Euclidean world, this axiom seems very natural. Infinitesimals, i.e., magnitudes that do not satisfy this axiom appear to be absurd. If CD were a magnitude that did not satisfy the axiom of Archimedes with respect to AB, then CD would be infinitesimally small with respect to AB, or, from the opposite perspective, AB

7 (Quine, 1976, §51, p. 428).

would be infinitely large with respect to CD, since no multiple of CD could ever be larger than AB. The nonsatisfaction of the Archimedean axiom asserts that there are infinitely small and infinitely large lines, i.e., exactly the kinds of magnitudes that Russell, Carnap, Quine and many others considered "absurd", probably because we have difficulties imagining them and because they allegedly contradict ordinary imagination.

Since 1880s, it was well known that non-Archimedean systems were mathematically possible. In §12 of the *Foundations of Geometry* Hilbert, there existed models of geometry that satisfied all axioms of Euclidean geometry except Archimedean geometry. As a mathematician, Hilbert was very open-minded with respect to non-Archimedean systems. He explicitly pointed out that these systems may be interesting not only for mathematical reasons but also from a philosophical point of view. The independence of the Archimedean axiom from the other geometrical axioms could also be of principal interest to physics:

> [The logical independence of the Archimedean axiom] leads to the following result: the fact that we reach by concatenating terrestrial distances the dimensions and the distances of celestial bodies, i.e., that we can determine lengths in space by terrestrial measures, as well as the fact that the distances in the interior of atoms can be expressed by the yardstick, is not at all a logical consequence of the theorems on triangle congruences and geometrical configurations, but only a result of empirical investigation. The validity of the Archimedean axiom in nature requires confirmation by experiment in the same familiar sense as the theorem on the sum of angles in the triangle (Hilbert, 1917, pp. 408-409).

Additionally, Poincaré, with a somewhat different focus, emphasized several times that non-Archimedean systems might be interesting for the empirical sciences (cf. Poincaré, 1906). Already in *Science and Hypothesis,* Poincaré pointed out that psycho-physical continua determined by sensations had quite a different structure from punctiform mathematical continua (Ibid., p. 22f.). Giovanelli has shown that Cohen himself obtained important impulses for his non-Archimedean "metaphysics of infinitesimals" from Fechner's experiments concerning very small differences in psycho-physical continua.[8] (cf. Fechner, 1860). This means that Cohen's approach did not emerge solely from mathematical and philosophical speculations but that from its very beginnings, the approach was not unrelated to considerations concerning the empirical sciences (cf. Giovanelli, 2016).

To ensure the mathematical meaningfulness of non-Archimedean structures, it is sufficient to consider general systems of magnitudes. Already the later logical empiricist Hans Hahn had done this in his trail-blazing paper Über *die nichtarchimedischen Grössensysteme* (Hahn, 1907). Hahn defined "systems of magnitudes" (*Grössensysteme*) as linearly ordered commutative groups, G = (G, +, <, 0), endowed with an associative and commutative addition "+" and a linear order "<" that satisfy the familiar axioms, as they hold, for example, for the system of integer numbers Z. If the neutral element of G is denoted by 0, the elements a of G that satisfy 0 < a are called positive, and the elements b that satisfy b < 0 are called negative. A system of magnitudes satisfies the Archimedean axiom if the following holds:

<u>The Archimedean axiom for algebraic systems of magnitudes.</u> Let a and b be positive elements of a system of magnitudes G and a < b. The system G is an Archimedean system iff there is a natural number n such that na > b (na: = a + a + … + a (n-times)).

Obviously, the system of integers (Z, +, <, 0) is an Archimedean system. It is quite elementary, however, to construct from this Archimedean system non-Archimedean systems: Let (Z x Z, +, 0) be the Cartesian product of pairs of integers {(a, b); a, b ∈ Z}, and this system may be endowed with a non-Archimedean order by the following recipe:

$$(a, b) < (a', b') := a < a' \text{ or } a = a' \text{ and } b < b'$$

8 Cf. Giovanelli, 2016, p. 10.

Then, for all n, the pairs one has (0, n) < (1, 1), i.e., relative to (1,1), the elements (0, n) are infinitesimally small; vice versa, (1,1) is infinitely large relative to (0, n). One may object that the system of magnitudes (Z x Z, +, <, 0) is rather "artificial". However, this is hardly a strong argument against the mathematical meaningfulness of non-Archimedean systems. It must be admitted, however, that these kinds of systems do not provide any better understanding of the infinitesimal magnitudes dx, dy, etc., on which traditional infinitesimal calculus is based. At least, systems such as (Z x Z, <, 0) and its type show that the concept of an infinitesimal magnitude is not openly "absurd", as many of its foes have contended. Rather, the system's "absurdity" is the result of an uncritical adherence to an obsolete criterion of "imaginability". Moreover, Hahn was completely clear about the possible empirical relevance of non-Archimedean systems.[9] In a popular lecture, Hahn explicitly pointed out:

> Spaces can be devised in which the Archimedean postulate is replaced by its opposite, that is, in which there are lengths that are greater than any multiple of a given length. Hence, in these spaces infinitely large and infinitely small lengths can exist… In a "non-Archimedean" space, lengths can be measured, and a system of analytical geometry can be developed. Of course, the real numbers of ordinary arithmetic are of no help in this geometry, but one uses "non-Archimedean" number systems, which can be interpreted and applied in calculations as well as the real numbers of ordinary arithmetic (Hahn, 1934, p. 99).

Hahn pointed out, however, that until now, there was no evidence that non-Archimedean systems could have applications outside mathematics. This is reminiscent of the situation at the end of the 19th century when the analogous question of the applicability of non-Euclidean but nevertheless Archimedean geometries was on the agenda. Similar to Hahn, who around 1930 did not see any possibility of a real, i.e., empirical, application of non-Archimedean systems, around 1900, Poincaré saw no options for applying non-Euclidean, Riemannian geometries, but did not exclude this possibility once and for all. As is well known, a few years after Poincaré's death, non-Euclidean (Archimedean) geometry turned out to be an essential conceptual tool for general relativity theory.

Although the mathematical meaningfulness of infinitesimal magnitudes could hardly be denied after Hilbert's *Foundations of Geometry* and Hahn's *Non-Archimedean Systems of Magnitudes*, nevertheless, there seemed to remain an unsurmountable weakness in non-Archimedean systems which had already been formulated by Felix Klein in 1908. Klein asked whether the non-Archimedean systems of magnitudes could be used to reformulate the traditional infinitesimal calculus in such a way that it would satisfy the modern standards of rigor put forward by Dedekind and Weierstraß. Surveying the results obtained so far, Klein resignedly concluded that this was not the case (cf. Klein, 1908, 2016, p. 236). Twenty years later, Hermann Weyl, in his *Philosophy of Mathematics and Natural Sciences*, arrived at the same negative conclusion (cf. Weyl, 1928).

This shortcoming of non-Archimedean theories was overcome only in the 1960s by Abraham Robinson (Robinson, 1966). Robinson constructed non-Archimedean systems of magnitudes for which the basic results of classical infinitesimal calculus could be proved in a rigorous manner.

In a sense then, one may say that Natorp was ahead of his time in *Grundlagen* (1910). Natorp was one of the very few philosophers who realized that non-Archimedean mathematics might be philosophically and scientifically relevant beyond the realm of pure mathematics. In *Grundlagen*, Natorp even (unsuccessfully) attempted to bring about something like a synthesis of

9 Hahn's recognition of the possible empirical (and philosophical) relevance of non-Archimedean systems remained a maverick position in logical empiricist philosophy of science. The logical empiricist mainstream, represented by Carnap, adhered to the verdict that the concept of infinitely small magnitudes is simply a meaningless pseudoconcept (cf. Carnap, 1928; Quine, 1976).

the various accounts of the infinitely large and the infinitely small being discussed at the turn of the 20th century.

## 6. The Infinitesimal, the Revolution of Rigor, and Natorp's Failed Synthesis

The origin of the Marburg school of mathematical philosophy of science was Cohen's *Das Prinzip der Infinitesimalmethode und seine Geschichte* (1883). The central claim of *Das Prinzip der Infinitesimalmethode* was that the infinitesimal was the central concept of philosophy of science überhaupt:

> The foundation of the concept of the infinitesimal is an issue of philosophy in a double sense. First, the conscience of traditional logic is not soothed until this basic notion of mathematized natural science has not been characterized and explained philosophically as far as possible. Further, there remains an irretrievable gap in the foundations of knowledge as long as this fundamental instrument as a presupposition of the mathematical and thus empirical knowledge has not been recognized and demarcated (Cohen, 1883, §1).

For Cohen, the philosophical elucidation of the concept of the infinitesimal was of utmost importance for any philosophy of science worth its salt. From *Das Prinzip* (1883) until the end of his philosophical career, Cohen unflinchingly held onto this thesis. Few followed him on this road. The logicians, mathematicians and philosophers who did not belong to the Marburg school harshly criticized *Das* Prinzip.

Recently, Giovanelli characterized *Das Prinzip* as a remarkably "unsuccessful book" (cf. Giovanelli, 2016). In a sense, this verdict is correct, particularly if one takes into account the book's negative reception on the part of mathematicians; however, in another sense, this verdict needs to be qualified. "Unsuccessful" books are usually forgotten and do not leave a trace. This, however, was not the fate of Cohen's book. In contrast, all members of the Marburg school defended it — admittedly, with more or less energy. Cassirer dissociated his account from Cohen's *Das Prinzip* rather early, without ever criticizing it explicitly[10]. In contrast, Natorp kept faith with Cohen and defended Cohen's infinitesimal-centered philosophy of science against all attacks from the outside, for instance, against Russell's fierce criticism in the *Principles of Mathematics* (Russell, 1903). Russell's violent attack on Cohen was not original; actually, it relied on arguments put forward by the mathematicians Cantor, Weierstraß and others. In general, the concept of the infinitesimal saw hard times in the last decades of the 19th century, when the "great revolution of rigor" took place.

The received historical narrative concerning infinitesimals in that period runs as follows. The idea of infinitesimals has been with us since antiquity. Mathematicians have used one or another variety of infinitesimals or indivisibles without truly understanding what they were doing. Eventually, infinitesimals fell into disrepute for logical and philosophical reasons, as enunciated by Berkeley and others. Despite Berkeley's allegedly devastating criticism, mathematicians continued to use infinitesimals in the 19th century with more or less good intellectual conscience. Finally, according to the traditional narrative, Cauchy, followed by Cantor, Dedekind, and Weierstraß, succeeded in formulating a rigorous foundation for calculus

10 In *Substanzbegriff und Funktionsbegriff* (Cassirer, 1910), the infinitesimal calculus is only mentioned in passing as one calculus among many others. This implicit betrayal of one of the central dogmas of the Marburg school did not go unnoticed by Cohen. In a letter to Cassirer, he praised Cassirer's *Substanzbegriff und Funktionsbegriff* in general but criticized that the concept of function instead of the infinitesimal occupied center stage in this work (cf. Mormann & Katz, 2013, p. 272f).

in terms of the epsilon-delta approach. Thereupon, infinitesimals were "officially" expelled from the realm of legitimate mathematics once and for all.

Natorp neither clung to the obsolete, intuitive, and logically dubious approach of infinitesimals that Cohen had proposed in *Das Prinzip* (Cohen, 1883) nor did he consider infinitesimals "inconsistent fictions", as Vaihinger proposed, nor did he whiggishly subscribe to the new orthodoxy of the "great triumvirate" (Cantor, Dedekind, Weierstrass) that insisted on the elimination of infinitesimals from any respectable mathematical discourse in favor of an approach based on Weierstrass's epsilontics. Instead, in *Grundlagen,* Natorp attempted to do justice to infinitely large numbers, infinitesimals and limit concepts. More precisely, Natorp attempted to reconcile the various doctrines of the infinitely large and the infinitely small put forward by Cantor, Dedekind, Weierstraß, Stolz, du Bois-Reymond, Veronese and others in one comprehensive synthesizing Neo-Kantian framework.

In an orthodox Neo-Kantian vein, Natorp argued as follows: If one takes the transcendental method seriously, the concept of the infinitesimal cannot be founded on any kind of intuition, as Cohen had claimed in *Das Prinzip*. Rather, the infinitesimal has to have its origin in "pure thought". Thus, Cohen had been right when he had changed his mind in *Die Logik der reinen Erkenntnis* (Cohen, 1902) and when he claimed that the infinitesimal is to be grounded in the principle of the origin. One of the aims of Natorp's *Grundlagen* was to elucidate Cohen's revised account based on a rather obscure "principle of the origin".

More precisely, Natorp undertook the bold attempt to reconcile Cantor's and Veronese's accounts (Veronese 1894), in the sense that he ascribed to Veronese the merit of having completed Cantor's work by extending his work on infinitely large numbers to the infinitely small. Thereby, Natorp presented Veronese as the perfector of the Cantorian revolution (cf. Natorp, 1910, p. 184, p. 200).[11]

Natorp's argument for his thesis was based on a vague and intuitive "Gedanken–experiment", according to which the infinitely large numbers and the infinitesimal are somehow "dual" with each other: Clearly, for any n Î N , the unit interval [0,1] can be divided into $2^n$ subintervals [k/$2^n$, k+1/$2^n$] of length 1/$2^n$. Without any further argument, Natorp assumed that this process could somehow be extended to infinity ("∞"), such that [0,1] could be divided into infinitely many subintervals of infinitesimal length 1/2∞. Thus, according to Natorp, 1/2∞ might be considered infinitesimally small with respect to 1. This process could be continued, starting with an infinitesimal interval [0, 1/2∞], to yield a magnitude even infinitesimally small with respect to 1/2∞, and so on. Thereby, one could obtain a series of infinitesimals directly corresponding to Cantor's ordinals (cf. Natorp, 1910, p. 195). Natorp's reconstruction of Cantor is, of course, a gross mathematical blunder. There is no direct correspondence between Cantor's infinite ordinals and the various non-Archimedean systems of infinitesimals of Veronese and others.[12]

Natorp's immature attempt at synthesis was severely criticized by mathematicians. Abraham Fraenkel is an example. In his *Lebenskreise. Aus den Erinnerungen eines jüdischen Mathematikers (1967)*, Fraenkel harshly criticized the mathematical philosophy of science of the Marburg school:

> In Cohen's reference to the "fact" of mathematics and the mathematical natural sciences, I missed any discussion of consistency ... In particular, I was deeply concerned with how the Marburg school treated the infinitely small,

11 This is, in every respect, quite an untenable interpretation. Cantor himself vigorously denied that Veronese's various kinds of infinitesimals had anything to do with his infinitely large (ordinal or cardinal) numbers. For a discussion of Veronese's philosophical background, see Cantú (2010).

12 For a modern account of the rise of non-Archimedean mathematics in general and systems of non-Archimedean magnitudes in particular, see Ehrlich (1994, 2006). Natorp's mathematically simplistic philosophical discussion of non-Archimedean systems in *Grundlagen* does not require a discussion of these issues in any detail here.

> beginning with Cohen's "Prinzip der Infinitesimalmethode (1883) up to Natorp's "Die Logischen Grundlagen der exakten Wissenschaften" (1910), where the infinitesimal was directly correlated with Cantor's transfinite numbers (Fraenkel, 1967, p. 107).

To be sure, Fraenkel did not militate (as Cantor did) against infinitesimals in general. Rather, he rightly rejected Natorp's mistaken "synthesis". Moreover, Fraenkel enthusiastically welcomed the quite unexpected achievement of his former student Abraham Robinson:

> A quite different, legitimate and surprising rehabilitation of the actual infinitely small has recently been achieved — from 1960 onwards — by my former student Abraham Robinson, now professor at the University of California in Los Angeles (ibid.).

## 7. Concluding Remarks

Natorp's mathematical philosophy of science may be considered the most radical version of the Marburg mathematical philosophy of science. Natorp understood mathematical philosophy of science as a philosophy in which mathematics occupied center stage in philosophy of science and epistemology, and even in wider sense, namely, as a representative of reason in general.

*Grundlagen* was the last serious defense of Cohen's "infinitesimal metaphysics" that was centered around the concept of the infinitesimal. For this purpose, in *Grundlagen*, Natorp used all logical and mathematical means available at that time, in particular, the various theories of contemporaneous theories of infinity. In this sense, Natorp's philosophy of science attempted to be a truly timely philosophy of science.[13] To be sure, mathematically, the philosophy was deeply flawed, but it seems to me this was a failure that does not deserve to be completely forgotten.

## Bibliographic References

CANTOR, G. 1884. 'Review of Cohen, Das Princip der Infinitesimal-Methode (Cohen 1884)'. In: *Deutsche Literaturzeitung, 5*, pp. 266–268, 1884.

CARNAP, R. *Der Logische Aufbau der Welt*. Hamburg, Meiner Verlag, 1961.

CARNAP, R. *Scheinprobleme in der Philosophie und andere metaphysikkritische Schriften*. herausgegeben, eingeleitet und mit Anmerkungen versehen von Thomas Mormann, Hamburg, Meiner Verlag, 2004.

CASSIRER, E. *Leibniz' System in seinen wissenschaftlichen Grundlagen*. Hamburg, Meiner, 1902.

CASSIRER, E. 'Kant und die moderne Mathematik'. In: *Kant-Studien 12*, pp. 1–49, 1907.

CASSIRER, E. *Substanzbegriff und Funktionsbegriff. Untersuchungen über die Grundfragen der*

13 For instance, Natorp's *Grundlagen* was the first Neo-Kantian work in philosophy of science that dealt with Einstein's (special) theory of relativity — Cassirer's *Substanzbegriff und Funktionsbegriff*, which was published in the same year of 1910, only dealt with classical physics.

*Erkenntniskritik*. Wissenschaftliche Buchgesellschaft, Darmstadt, 1980.

CASSIRER, E. 'Hermann Cohen und die Erneuerung der kantischen Philosophie'. In: *Kant-Studien 17*, pp. 252–273, 1912.

CASSIRER, E. *The Problem of Knowledge. Philosophy, Science, and History since Hegel*. New Haven and London, Yale University Press, 1950.

COHEN, H. *Das Prinzip der Infinitesimal-Methode und seine Geschichte: Ein Kapitel zur Grundlegung der Erkenntniskritik*. Berlin, Dümmler, 1883.

COHEN, H. *Einleitung mit kritischem Nachtrag zur „Geschichte des Materialismus" von F.A. Lange*. edited by H. Holzhey, Hildesheim, Olms, 1984.

CANTÚ, P. 'The role of epistemological models in Veronese's and Betazzi's theory of magnitudes'. In: M. D'Agostino, G. Giorello, F. Landisa, T. Pievani and C. Sinigaglia (eds.). *New Essays in Logic and Philosophy of Science*. College Publication, pp. 229–240, 2010.

EDEL, G. 'Kantianismus oder Platonismus?, Hypothesis als Grundbegriff der Philosophie Cohens'. In: *Il Cannochiale, Revista di studi filosofi*, 1-2, pp. 59–87, 1991.

EHRLICH, P. *Real Numbers, Generalizations of the Reals, and Theories of Continua.* Dordrecht, Kluwer Academic Publishers, 1994.

EHRLICH, P. 'The Rise of non-Archimedian Mathematics and the Roots of a Misconception I: The Emergence of non-Archimedian Systems of Magnitudes'. In: *Archive of the History of Exact Sciences*, 60, pp. 1–121, 2006.

FECHNER, G.T. *Elemente der Psychophysik*. Leipzig. Breitkopf und Härtel, 1860.

FRAENKEL, A.A. *Lebenskreise. Aus den Erinnerungen eines jüdischen Mathematikers*. Stuttgart, Deutsche Verlagsanstalt, 1967.

FREGE, G. 'Review of Cohen, Das Princip der Infinitesimal-Methode (Cohen 1884)'. In: *Zeitschrift für Philosophie und philosophische Kritik*, 87, pp. 324–229, 1885.

GIOVANELLI, M. *Reality and Negation – Kant's Principle of Anticipations of Perception: An Investigation of its Impact on the Post-Kantian Debate, Studies in German Idealism volume 11.* Berlin, Springer, 2011.

GIOVANELLI, M. 'Hermann Cohen's Das Princip der Infinitesimal-Methode: The History of an Unsuccessful Book'. In: *Studies in History and Philosophy of Science*, 58 A, pp. 9–23, 2016.

HAHN, H. 'Über die nichtarchimedischen Grössensysteme: Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften'. In: *Mathematisch-Naturwissenschaftliche Klasse*, Abteilung Iia, pp. 601–655, 1906.

HAHN, H. 'Gibt es Unendliches?' (1934). In: HAHN, H., pp. 115–140, 1988.

HAHN, H. 'Die Krise der Anschauung' (1933). In: HAHN, H., pp. 86–114, 1988.

HAHN, H. *Empiricism, Logic and Mathematics*. Philosophical Papers. smus, Logik, Mathematik, Frankfurt/Main, Suhrkamp Verlag, 1980.

HEIS, J. "Critical philosophy begins at the very point where logistic leaves off': Cassirer's response to Frege and Russell'. In: *Perspectives on Science*, 18, pp. 383–408, 2010.

HILBERT, D., *Grundlagen der Geometrie*. Leipzig, Teubner, 1899.

HILBERT, D. 'Axiomatisches Denken'. In: *Mathematische Annalen*, 78, pp. 405–415, 1917.

HOLZHEY, H. *Cohen und Natorp. Der Marburger Neukantianismus in Quellen*. Band 2, Basel, Schwabe, 1986.

KINKEL, W. 'Paul Natorp und der kritische Idealismus'. In: *Kant-Studien*, 28, pp. 398–418, 1923.

KLEIN, F. *Elementary mathematics from a Higher Standpoint. Arithmetic. Algebra, Analysis, vol. I*, Berlin, Springer, 2016.

MARCK, S. 'Die Lehre vom erkennenden Subjekt in der Marburger Schule'. In: *Logos IV*, pp. 364–386, 1913.

MARGENAU, H. 'Methodology of Modern Physics". In: *Philosophy of Science* 2(1), pp. 48–72, 1935.

MARGENAU, H. *The Nature of Physical Reality*. New York, McGraw Hill, 1950.

MORMANN, T., KATZ, M. 'Infinitesimals as an Issue of Neo-Kantian Philosophy of Science'. In: *HOPOS* 3 (2), pp. 236–280, 2013.

MORMANN, T. 'Zur mathematischen Wissenschaftsphilosophie des Marburger Neukantianismus'. In: DAMBÖCK, C. (Hrg.), *Philosophie und Wissenschaft bei Hermann Cohen*. Veröffentlichungen des Instituts Wiener Kreis, Bd. 28, Springer, pp. 101–132, 2018.

NATORP, P. 'Zu Cohens Logik, Manuskript'. In: HOLZHEY, H. *Cohen und Natorp*. Band 2, Der Marburger Neukantianismus in Quellen, pp. 41–78, 1986.

NATORP, P. 'Brief an Albert Görland, 21. November 1902'. In: HOLZHEY, H. *Cohen und Natorp*. Band 2, Der Marburger Neukantianismus in Quellen, pp. 299–303, 1902.

NATORP, P. 'Logik (Grundlegung und logischer Aufbau der Mathematik und mathematischen Naturwissenschaft)'. In: *Leitsätzen zu akademischen Vorlesungen*, 2. Auflage, N.G. Elwert'sche Verlagsbuchhandlung, Marburg, 1910.

NATORP, P. *Die logischen Grundlagen der exakten Wissenschaften*. Leipzig, Teubner, 1910.

NATORP, P. 'Kant und die Marburger Schule'. In: *Kant-Studien 17*, pp. 193–221, 1912.

NATORP, P. 'Philosophische Propädeutik (Allgemeine Einleitung in die Philosophie und Anfangsgründe der Logik, Ethik, und Psychologie)'. In: *Leitsätzen zu akademischen Vorlesungen*, 5. Auflage, N.G, Elwert'sche Verlagsbuchhandlung, Marburg, 1927.

POINCARÉ, H. *The Value of Science*. New York, Dover Publications, 1906.

QUINE, W.V.O. *Word and Object*. Cambridge/Massachusetts, MIT Press, 1980.

ROBINSON, A. *Non-Standard Analysis*. Amsterdam, London, North-Holland, 1966.

RUSSELL, B. *The Principles of Mathematics*. London, Routledge and Kegan Paul, 1903.

VERONESE, G. *Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten geradliniger Einheiten in elementarer Form entwickelt*, aus dem Italienischen übersetzt von Adolf Schepp. Leipzig, Teubner, 1894.

WEYL, H. *Philosophie der Mathematik und Naturwissenschaft*, München, Oldenbourg, 1928.