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An improved optimal elemental method for updating finite element models

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Abstract: The optimal matrix method and optimal elemental method used to update finite element models may not provide accurate results. This situation occurs when the test modal model is incomplete, as is often the case in practice. An improved optimal elemental method is presented that defines a new objective function, and as a byproduct, circumvents the need for mass normalized modal shapes, which are also not readily available in practice. To solve the group of nonlinear equations created by the improved optimal method, the Lagrange multiplier method and Matlab function *fmincon* are employed. To deal with actual complex structures, the float-encoding genetic algorithm (FGA) is introduced to enhance the capability of the improved method. Two examples, a 7-degree of freedom (DOF) mass-spring system and a 53-DOF planar frame, respectively, are updated using the improved method. The example results demonstrate the advantages of the improved method over existing optimal methods, and show that the genetic algorithm is an effective way to update the models used for actual complex structures.

Keywords: model updating; optimal elemental method; Lagrange multiplier method; genetic algorithm

1 Introduction

Discrepancies between structural analytical models, which usually are finite element (FE) models, and test results are common in practice. Updating the structural models has evolved as a method to reconcile these differences. Generally, the existing methods used to update models can be classified as the optimization based method (Baruck,1982; Berman and Nagy,1983; Kabe,1985), sensitivity-based method (Ricles and Kosmatka, 1992; Farhat and Hemez,1993) and eigenstructure assignment method (Zimmerman and Widengren, 1990). Research studies by Baruch (1982) and Berman and Nagy (1983) defined a framework for the optimal matrix method (OMM). By nature, optimization-based methods are used to minimize the differences between analytical and updated models under the constraints provided by the test modal model. OMM provides a closed-form solution, which makes this method very appealing. However, updated models generated by OMM often lose their sparsity. The optimal elemental method (OEM) enforces the connectivity of updated model by preventing the error-free parts from updating. The sensitivity-based method (SBM) makes use of the derivatives of measured parameters, typically eigenmodes, to a set of physical parameters to calculate the changes in them. However, it can only be used to update models with small discrepancies to test models. While the eigenstructure assignment method for model updating is similar to the poles assignment method in structural control, it uses a fictitious controller to force the analytical model to respond like a test model. The control gains are used to calculate the perturbation to the analytical model. Loss of physical meaning of the updated model is the major limitation of this method. There are other methods that update the frequency response function directly (Friswell and Mottershead, 1995). An excellent survey on finite model updating methods was made by Mottershead and Friswell (1993).

In this paper, the optimization-based methods for updating finite element models are addressed, and an improved optimal elemental method is presented. The remainder of this paper is organized as follows. First, the problem of updating the model is outlined, and the OMM and OEM are briefly introduced. An improved model updating method is then presented by defining a more strict and consistent objective function. Next, an iterative solving strategy using the Matlab function *fmincon* and float-encoding genetic algorithm are adopted to solve the nonlinear optimal problem caused by the improved method. Finally, the proposed method

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is used to update two structures to demonstrate its effectiveness.

2 Optimization based model updating method

The governing equations for updating finite element models using test modal parameters are the orthogonality of updated mass and stiffness matrices to test modal shapes and the eigenequation, which are given as

$$\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\Phi}=\bar{\boldsymbol{M}} \tag{1}$$

$$\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\Phi}=\bar{\boldsymbol{K}} \tag{2}$$

$$\bar{\boldsymbol{K}}=\bar{\boldsymbol{M}}\boldsymbol{\Lambda} \tag{3}$$

where $\boldsymbol{M}$ and $\boldsymbol{K}$ are the updated mass and stiffness matrices, respectively; $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{K}}$ are the modal mass and stiffness matrices, $\boldsymbol{\Phi}$ is the measured modal shape matrix and $\boldsymbol{\Lambda}$ is the measured diagonal eigenvalue matrix. Superscript T denotes transpose.

In the procedure of OMM developed by Baruch (1982), Berman and Nagy (1983) and Zhang and Wei (1999), the mass and stiffness matrices are updated separately in the same way. For the stiffness matrix, the updated stiffness is found to be closest to the analytical stiffness matrix $\boldsymbol{K}_{\mathrm{A}}$ by minimizing the following objective function

$$\varepsilon=\frac{1}{2}\left\|\boldsymbol{M}^{-1/2}(\boldsymbol{K}-\boldsymbol{K}_{\mathrm{A}})\boldsymbol{M}^{-1/2}\right\| \tag{4}$$

subject to the eigenequation and the symmetry constraints given by

$$\boldsymbol{K}\boldsymbol{\Phi}=\boldsymbol{M}\boldsymbol{\Phi}\boldsymbol{\Lambda} \tag{5}$$

$$\boldsymbol{K}^{\mathrm{T}}=\boldsymbol{K} \tag{6}$$

where $\|\ \|$ is the Euclidean norm. A closed-form solution for this problem is available. The OEM for the model updating method is similar to the OMM, but excludes some error-free elements in the matrix from being updated, and thus preserves the sparsity of updated matrices.

As seen from Eqs. (4)-(6), the updated matrices should be those which are closest to the analytical matrices and satisfy the constraints. However, this argument may not be true when the test modal data are incomplete, which is often the case in practice. The matrices, which are the closest to the analytical matrices while satisfying the constraints, may not necessarily represent the actual model, which could be closer to the test model in practice. On the other hand, the matrices of the actual model, which are what the updating procedure actually searches for, may not be the closest to the analytical matrices. In other words, the matrices of the actual model are not on the minimum point of the objective function of Eq. (4). This idea can be illustrated by example.

Consider a three DOF structure. The mass and stiffness matrices, which represent the analytical model of the structure, are

$$\boldsymbol{M}_{\mathrm{A}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad \boldsymbol{K}_{\mathrm{A}}=\begin{bmatrix}40&-20&0\\-20&40&-20\\0&-20&20\end{bmatrix} \tag{7, 8}$$

Next, alter the element in the first row and the first column of $\boldsymbol{K}_{\mathrm{A}}$ from 40 to 30, and keep all other elements in $\boldsymbol{K}_{\mathrm{A}}$ and mass matrix $\boldsymbol{M}_{\mathrm{A}}$ unchanged. The distance between the analytical and altered models, in the form of a Euclidean norm of two stiffness matrices, is 10.

Then, the altered stiffness and mass matrices are used to produce the eigenvalues and eigenvectors, which could be the measured modal parameters in practice. Using the first two of the three modes in implementing the OMM and denoting the updated matrices as $\boldsymbol{M}_{\mathrm{U}}$ and $\boldsymbol{K}_{\mathrm{U}}$, we obtain that $\boldsymbol{M}_{\mathrm{U}}$ equates to $\boldsymbol{M}_{\mathrm{A}}$ and

$$\boldsymbol{K}_{\mathrm{U}}=\begin{bmatrix}30.58&-20.93&0.44\\-20.93&41.49&-20.71\\0.44&-20.71&20.34\end{bmatrix} \tag{9}$$

The Euclidean norm of the difference matrix $\boldsymbol{K}_{\Delta}=\boldsymbol{K}_{\mathrm{U}}-\boldsymbol{K}_{\mathrm{A}}$ is 9.70, which is less than 10. This implies that the matrix of the actual model is not at the minimum point of the objective function defined by Eq. (4). Therefore, the objective functions of OMM and OEM may mislead the updated model to deviate from the actual model. This is especially true when fewer measured eigenmodes are used for the updating. In this condition, however powerful the optimization technique is, the optimization result will never approximate to the real solution as expected.

In addition, for the existing optimal matrix and elemental methods, the measured modal shapes should be mass normalized. However, this requirement can seldom be met in modal test and analysis, especially for ambient vibrations, in which the inputs are unknown.

3 Improved optimal elemental method

Let's revisit the governing Eqs. (1)-(3). All the off-diagonal elements of $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{K}}$ are zeros due to the orthogonality of updated mass and stiffness matrices to modal shapes. The diagonals of the two matrices are non-zeros, but they are unknown because of the arbitrarily-scaled modal shapes. However, their relationships exist in the form of eigenequations of Eq. (3).

At this point, an improved model updating algorithm can be formulated by defining a new object function. The updated matrices are found to minimize the sum of square of distances of off-diagonal elements in $\bar{\boldsymbol{M}}$ and $\bar{\boldsymbol{K}}$ to zeros. Then, the new object function is

$$\varepsilon=\sum_{p=1}^{N}\sum_{\substack{q=1\\q\neq p}}^{N}[(\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ip}m_{ij}\varphi_{jq}-0)^2+(\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ip}k_{ij}\varphi_{jq}-0)^2] \tag{10}$$

The eigenequation and symmetry constraints for the optimal problem are given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ip}k_{ij}\varphi_{jp}-\lambda_p\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ip}m_{ij}\varphi_{jp}=0 \qquad p=1,2,3,\dots,N \tag{11}$$

$$m_{ij}=m_{ji},\quad k_{ij}=k_{ji},\quad i,j=1,2,3,\dots,n,\quad i\neq j \tag{12}$$

where φ_{ip} is the modal displacement at the i th degree of pth eigenvector, and λ_p is the pth eigenvalue. m_{ij} and k_{ij} are elements of the mass and stiffness matrices, and N and n are the numbers of the measured modes and degrees, respectively.

The matrices of the actual model, which satisfy the constraints of Eqs. (11)-(12), are always at the minimum point of the objective function defined in Eq. (10). That is to say that the actual model can consistently be achieved by minimizing this objective function. For the new objective function, the mass normalized modal shapes are not needed for the updating procedure.

However, the improved method creates a more complex optimization problem when compared with those used in Baruck and Berman's approaches. However, by employing a more efficient optimization technique and adding constraints to narrow the search domain, the optimization problem can be solved more efficiently.

Generally, the existing optimal methods are not capable of resolving large discrepancies between the analytical and test models. Additional constraints, i.e., that the diagonal elements of the mass and stiffness matrices be positive, as shown in Eq. (13), should be considered when dealing with two models with a large gap.

$$m_{ii}>0,\quad k_{ii}>0,\quad i=1,2,3,\dots,n \tag{13}$$

However, the constraints given in Eq. (13) result in a group of nonlinear equations, which makes solving the optimal problem more challenging.

4 Solving strategy for the optimization problem

Employing the Lagrange multiplier method, and considering the symmetry of the mass and stiffness matrices implicity, the Lagrange function is obtained as Eq. (14).

$$\begin{aligned}L=&\frac{1}{2}\sum_{p=1}^{N-1}\sum_{q=p+1}^{N}\{[\sum_{i=1}^{n}m_{ii}\varphi_{ip}\varphi_{iq}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}m_{ij}(\varphi_{ip}\varphi_{jq}+\varphi_{jp}\varphi_{iq})]^2+\\&[\sum_{i=1}^{n}k_{ii}\varphi_{ip}\varphi_{iq}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}k_{ij}(\varphi_{ip}\varphi_{jq}+\varphi_{jp}\varphi_{iq})]^2\}+\\&\sum_{p=1}^{N}\alpha_p[(\sum_{i=1}^{n}k_{ii}\varphi_{ip}^2+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}k_{ij}\varphi_{ip}\varphi_{jp})-\lambda_p(\sum_{i=1}^{n}m_{ii}\varphi_{ip}^2+2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}m_{ij}\varphi_{ip}\varphi_{jp})]+\\&\sum_{i=1}^{n}\xi_i(m_{ii}-g_i^2)+\sum_{i=1}^{n}\zeta_i(k_{ii}-h_i^2)\end{aligned} \tag{14}$$

where α_p, ξ_i, ζ_i, g_i and h_i are Lagrange multipliers. This is a nonlinear optimization problem, in which nonlinearity comes from the constraints represented by Eq. (13).

The Matlab function *fmincon* in the optimization toolbox is used to search for the solution. To take advantage of the large-scale algorithm in *fmincon*, which is more powerful and efficient, the optimization problem in Eqs. (10)-(13) is formulated with the objective function given as

$$
\begin{aligned}
\varepsilon = {} & \frac{1}{2}\sum_{p=1}^{N-1}\sum_{q=p+1}^{N}\{[\sum_{i=1}^{n} m_{ii}\varphi_{ip}\varphi_{iq} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} m_{ij}(\varphi_{ip}\varphi_{jq} + \varphi_{jp}\varphi_{iq})]^2 + \\
& [\sum_{i=1}^{n} k_{ii}\varphi_{ip}\varphi_{iq} + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} k_{ij}(\varphi_{ip}\varphi_{jq} + \varphi_{jp}\varphi_{iq})]^2\} + \\
& \frac{1}{2}\sum_{p=1}^{N}[\sum_{i=1}^{n} k_{ii}\varphi_{ip}^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} k_{ij}\varphi_{ip}\varphi_{jp} - \lambda_p(\sum_{i=1}^{n} m_{ii}\varphi_{ip}^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} m_{ij}\varphi_{ip}\varphi_{jp})]^2
\end{aligned}
\tag{15}
$$

The constraints are given by Eqs. (12) and (13). The eigenequation stated as constraints in Eq. (11) is included as part of the objective function in Eq. (15). After the reformulation, the large-scale algorithm of *fmincon* can be implemented.

The solving technique based on the gradients of the objective function, such as the one employed by the Matlab function *fmincon*, usually can not avoid possible local entrapments when the shape of the object functions become complex. The genetic algorithm (GA) provides a potential way to avoid these local entrapments. A float-encoding genetic algorithm (FGA) is proposed to solve this optimization problem. The FGA is characterized by its ranking-like selection process. The most superior individuals in each generation are directly moved to the next generation in addition to being involved in the genetic process. Penalty functions are adopted to deal with the equation and inequation constraints. According to the characteristics of the objective function, some simple and excellent genetic operators, such as crossover and mutation, are mixed. For example, arithmetic crossover and one-point crossover are combined when implementing the crossover operation, and boundary mutation and non-uniform mutation are combined when mutations are operated.

Reducing the size of the optimization problem and keeping the number of unknowns small facilitate the process of searching for the results. For optimization in model updating, using the sparsity of the matrices is an effective way to achieve this reduction. In addition, some parts of actual structures are error-free. By considering these error-free elements, the number of unknowns is further reduced.

The objective function is made up of contributions from the modal mass and modal stiffness matrices. Weighing on mass or stiffness matrices to ensure that their contributions are balanced is an efficient way to make the convergence faster and the solution more precise. Finally, setting the initial values of unknowns to be the analytical model matrices is a good technique to achieve convergence.

5 Examples

5.1 Example 1

Consider a 7-DOF mass-spring system (Heylen *et al.*,1997). The charateristic parameters for the actual or "test" model are taken as m_1 = 2kg, m_2 = 5kg, m_3 = 4kg, m_4 = 4kg, m_5 = 3kg, m_6 = 2kg, and m_7 = 1kg. All the spring stiffnesses are 10000 N/m. The analytical model is the same as the actual model except that m_3 = 3.5kg, m_4 = 4.5kg, $k_{1,2}$ = 8000N/m, $k_{2,4}$ = 7000N/m, $k_{4,5}$ = 9000N/m, $k_{6,7}$ = 6000N/m, and $k_{6,0}$ = 8500N/m. Measured noise and damping behavior are not present in this example. The natural frequencies of the test model and the analytical model are shown in Table 1.

Denoting BB as the OMM presented by Baruch (1982) and Berman (1983), and IM(GT) and IM(GA) as the improved method (IM) employing the gradient optimization technique and Genetic Algorithm, GT and GA, respectively, a different model updating method is implemented for comparison. In the BB method, the mass normalized modal shapes are used, and in the IM

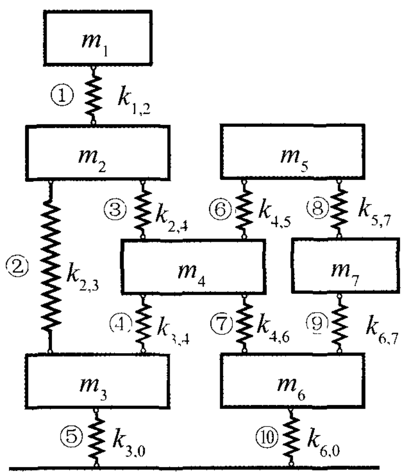

Fig. 1 7 DOF spring-mass system

Table 1 Results of frequencies of a 7-DOF spring-mass system rad/s

Mode No.	1	2	3	4	5	6	7
Actual	26.175	52.157	77.199	100.00	103.56	116.1	163.09
Analytical	25.378	48.757	78.699	97.388	103.27	111.53	162.2

methods, the sparsity of the mass and stiffness matrices is used.

Two cases are considered in terms of the completeness of measured modes. In case 1, all seven modes are used in updating the model and in case 2, only the first three modes are used.

For Case 1, the frequency errors of the updated model when compared to the "test" model are given in Table 2, and the matching modal shapes are plotted in Fig. 2. (Note that "IM(GD)," which appears in the legends of Fig. 2 should read "IM(GT).")

The results show that the BB and IM (GT) methods produce the exact updated model as the actual model, while the IM (GA) provides almost exact results except for a very small discrepancy in the first frequency.

The relative errors of the diagonal elements in the updated mass and stiffness matrices are also listed in Tables 3 and 4, respectively, for Case 1. The BB method again reproduces the diagonals in mass and stiffness matrices as expected, and IM(GT) and IM(GA) produce the results with errors of less than 0.3%.

The results for Case 2, where only the first three modes are considered, are shown in Fig. 3 and Tables 5-7 (Note that "IM(GD)," which appears in the legends of Fig. 3 should read "IM(GT)."). The IM(GA) method provides the most accurate modal frequencies, and the modal shapes match very well with the actual ones. It is also superior to the BB and IM(GT) methods in updating the diagonals in the mass and stiffness matrices.

In practical applications, only truncated modes are

Table 2 Relative error of frequencies for the updated models (Case 1) %

Mode No.	1	2	3	4	5	6	7
BB	0	0	0	0	0	0	0
IM(GT)	0	0	0	0	0	0	0
IM(GA)	-0.02	-0.00	0.00	0.00	0.00	-0.01	-0.00

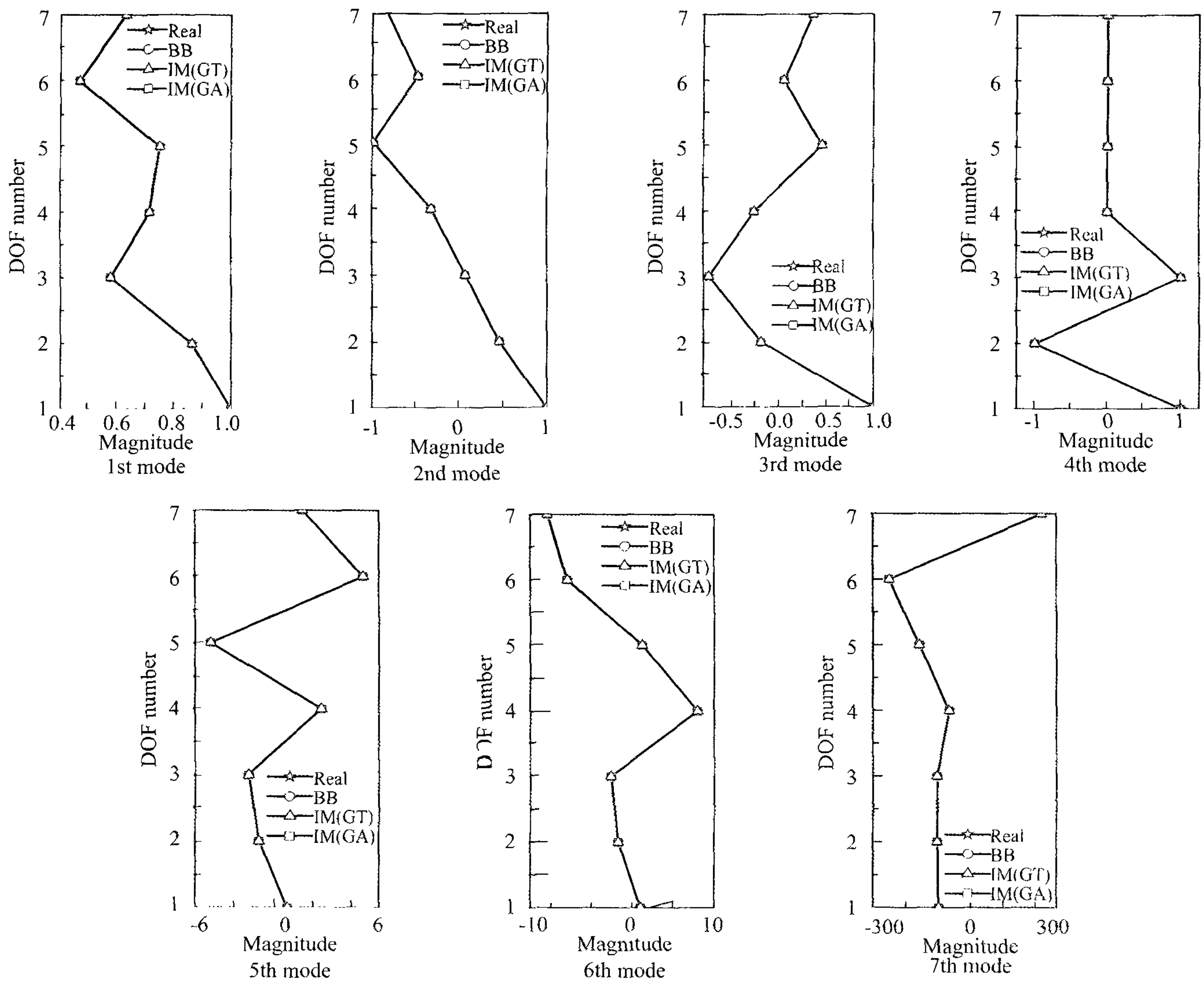

Fig. 2 Actual vs. simulated modal shapes (Case 1)

Table 3 Relative error of diagonals for the updated mass matrix (Case 1) %

DOF No.	1	2	3	4	5	6	7
BB	0	0	0	0	0	0	0
IM(GT)	-0.03	-0.03	-0.03	-0.03	-0.03	-0.03	-0.03
IM(GA)	0.23	0.22	0.22	0.16	0.16	0.09	-0.05

Table 4 Relative error of diagonals for the updated stiffness matrix (Case 1) %

DOF No.	1	2	3	4	5	6	7
BB	0	0	0	0	0	0	0
IM(GT)	-0.03	-0.03	-0.03	-0.03	-0.03	-0.03	-0.03
IM(GA)	0.20	0.22	0.20	0.11	0.13	0.06	0.02

Table 5 Relative error of frequencies for the updated models (Case 2) %

Mode No.	1	2	3	4	5	6
BB	0.00	0.00	-0.00	-12.50	-3.64	-6.45
IM(GT)	-6.77	-1.21	-3.06	-6.45	-2.80	-0.56
IM(GA)	0.03	-0.47	0.01	-0.11	0.28	-1.45

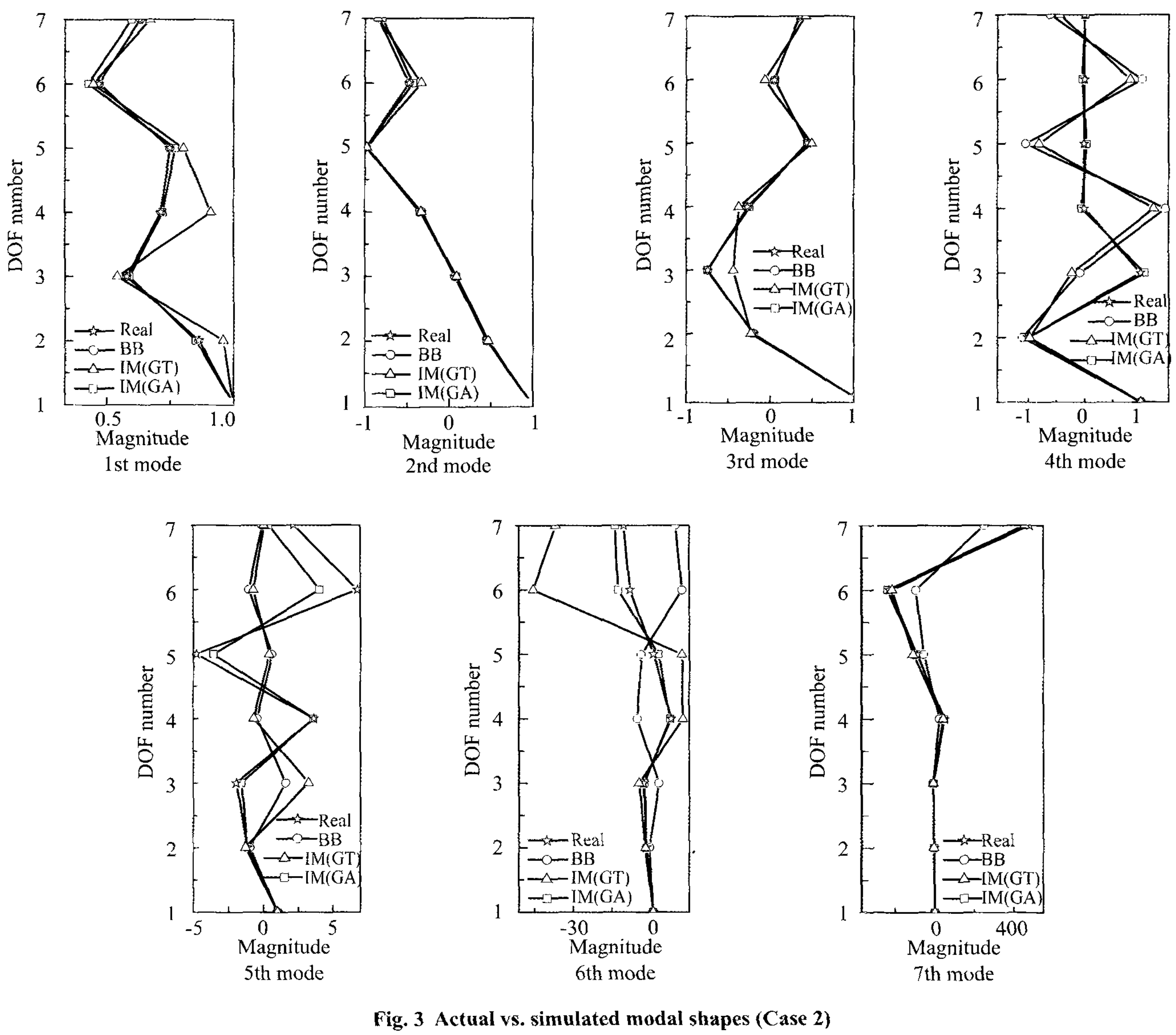

Fig. 3 Actual vs. simulated modal shapes (Case 2)

Table 6 Relative error of diagonals for the updated mass matrix (Case 2) %

DOF No.	1	2	3	4	5	6	7
BB	0.58	0.99	-10.38	12.52	-0.36	-0.19	-0.10
IM(GT)	-5.88	-1.15	-11.06	2.44	-4.12	-11.99	-6.58
IM(GA)	-0.35	3.38	-4.06	4.89	-1.75	0.43	8.30

Table 7 Relative error of diagonals for the updated stiffness matrix (Case 2) %

DOF No.	1	2	3	4	5	6	7
BB	-3.39	-12.55	-9.60	-18.55	-5.65	-19.62	-17.58
IM(GT)	-12.42	-15.72	2.84	-30.68	-6.39	-14.36	-20.26
IM(GA)	-0.69	2.309	-5.289	-3.399	-0.91	3.94	3.69

available, and the practical meaning of the proposed improved method with the genetic algorithm (IM(GA)) is demonstrated by case 2.

5.2 Example 2

To further validate the effectiveness of the improved method, consider a 14-bay simply supported planar steel truss as shown in Fig. 4. It is 5.6m long and 0.4m tall, and consists of 53 members (hollow steel tubes, 17.1mm outer diameter, 3.1mm wall thickness). The material properties are as follows: Young's Modulus is 1.999×10^{11}Pa, the Possion ratio is 0.3 and the mass density is 7827kg/m^3. The damping ratio is 1% for each mode. The finite element model for the structure is denoted as actual or "test" model. The analytical model is produced by reducing the sections of diagonal bars 2-5 and 4-7, and the vertical bars 4-5and 6-7 of the actual model by 90%, 70%, 90% and 50%, respectively.

The vertical acceleration responses of the actual model and analytical model under random white noise are simulated, and the modal parameters of the two models are identified by the eigen-system realization algorithm (ERA). The first six natural frequencies of the test model and the analytical model, as well the discrepancies between the two models before updating are shown in Table 8.

Then, the mass and stiffness matrices of the analytical model with the identified modal parameters were updated using IM(GA). The frequency errors after updating are also shown in Table 8. The corresponding modal shapes are shown in Fig.5, where, for simplicity, only the lower chord nodes are plotted. The model updated with IM(GA) matches the actual structure very well.

6 Conclusions

The objective function of the optimal matrix method and optimal elemental method used to update finite

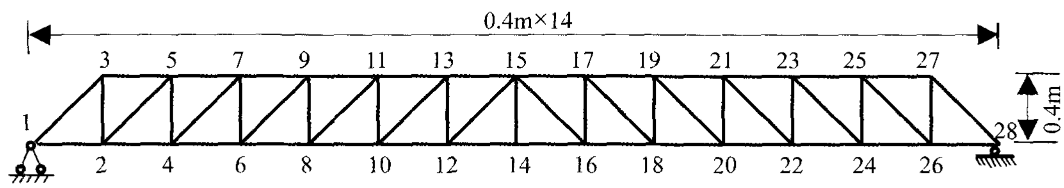

Fig. 4 A fourteen bays planar truss

Table 8 First six frequencies of the structure before and after updating

Mode No.	Actual (Hz)	Analytical (Hz)	Error of frequency before updating (%)	Error of frequency after updating (%)
1	8.79	8.34	5.12	-0.55
2	29.60	28.35	4.22	-1.48
3	43.39	37.70	13.11	2.23
4	59.10	56.06	5.14	-0.24
5	90.62	84.33	6.94	-0.40
6	119.81	92.08	23.14	-0.11

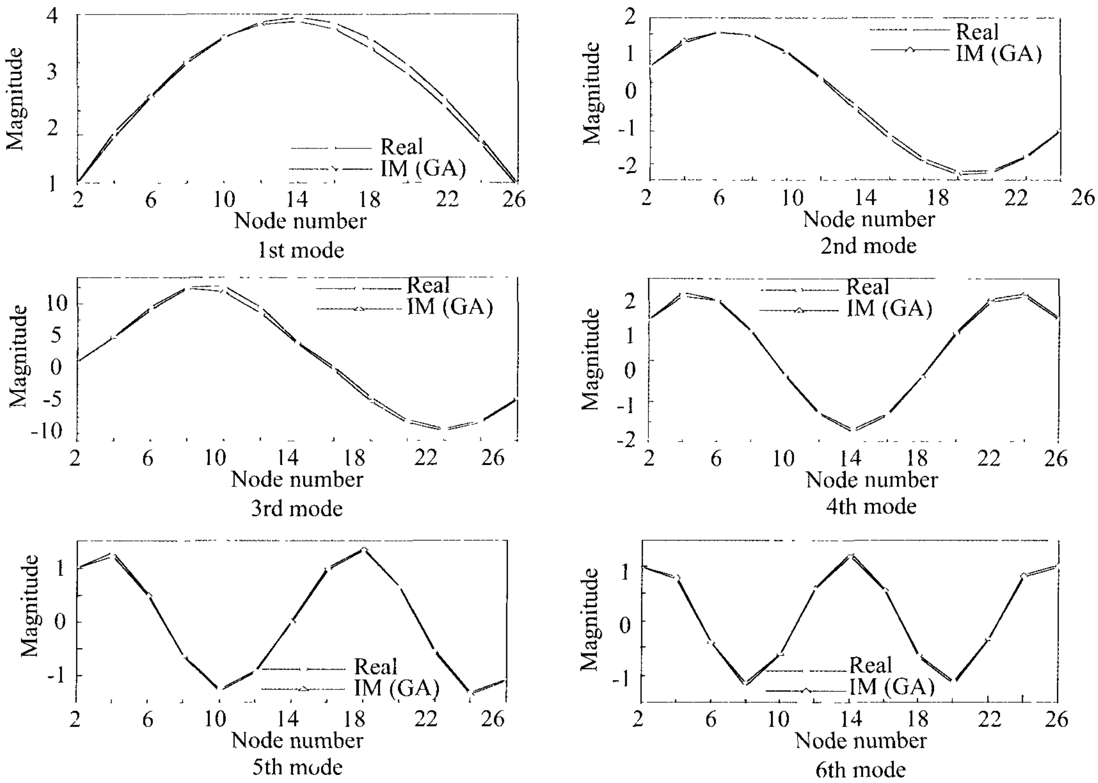

Fig. 5 Actual vs. updated modal shapes for the 14-bay structure

element models may misrepresent incomplete modes and differ significantly from those presented in the actual structure. The updated model that is closest to the analytical model under the constraints of orthogonality and eigenequation is not necessarily the actual model. An improved optimal elemental method is presented in this paper by defining a new objective function, which consistently represents the actual condition of the model. Both traditional optimization techniques based on gradients and the genetic algorithm are employed to solve the nonlinear optimization problem found in updating these models. Two examples are given that demonstrate the advantages of the improved model updating method, and the potential of this method to use the solving strategy found in the generic algorithm to update actual complex structures.

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