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Termination Via Conditional Reductions

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Abstract

We generalize the notion of rewriting modulo an equational theory to include a special form of conditional reduction. We are able to show that this conditional rewriting relation restores the finite termination property which is often lost when rewriting in the presence of infinite congruence classes. In particular, we are able to handle the class of collapse equational theories which contain associative, commutative, and identity laws for one or more operators.

Introduction

In 1970 Knuth and Bendix established necessary and sufficient conditions for a set of reductions to be complete. These conditions are commonly referred to as the *finite termination property* and the *confluence property*. Based on these conditions, they were able to devise both an algorithm for testing the completeness of a set of reductions and a procedure which can take the equational axioms of an algebraic system and attempt to generate a complete set of reductions. The Knuth-Bendix procedure was able to generate complete sets of reductions for a limited number of algebraic systems, most notably free groups. Early completion procedures, however, were not able to handle any algebraic system whose definition included a commutativity axiom because inclusion of such axioms in the reduction set resulted in the loss of the finite termination property.

Peterson and Stickel [PS81](see also [LB77]) were able to overcome this limitation of completion procedures by splitting the equational axioms of an algebraic system into two sets: (1) equations which are incorporated into the pattern matching process used to apply reductions, and (2) equations which form the basis of a set of reductions to be completed. Their approach requires not only the finite termination and confluence properties, but also a linearity property for equations in the first set and a special compatibility property between the reductions and the first set of equations. Besides these properties, it is necessary to have a finite and complete unification algorithm for the equations which are incorporated into the pattern matching process. Peterson and Stickel were able to generate complete sets of reductions for algebraic systems which included both associativity and commutativity axioms, building these axioms into the pattern matching facility via associative-commutative unification. Such completion procedures are referred to as E-completion procedures, where E represents the set of equations incorporated into the pattern matching process. Using this E-completion procedure, Peterson and Stickel were able to generate complete sets of reductions for algebraic systems such as commutative groups, commutative rings, and distributive lattices.

Jouannaud and Kirchner [JK86] generalized the theory of E-completion sufficiently to account for all previous completion and E-completion theory. They were able to replace the compatibility requirement of Peterson and Stickel with a more general property which they call coherence and to remove the linearity requirement for E in favor of the more general requirement that the congruence classes generated by E must be finite. They regarded the problem of infinite congruence classes as a major open problem since many interesting cases such as equipotence and identity fall into this category. Bachmair and Dershowitz [BD87] generalized the theory of Jouannaud and Kirchner so as to remove the finite congruence class requirement, but still requiring the finite termination property. This generalization does not help if the termination property is lost for equational theories which generate infinite congruence classes.

Rather than attempting to solve this problem for all equational theories which generate infinite congruence classes, we consider only the class of equational theories which contain the associative, commutative, and identity laws for one or more operators (ACI). It is the presence of the identity law in these theories which causes them to generate infinite congruence classes and to lose the finite termination property which places them outside the scope of the previous E-completion theory. In [PB89] we describe the theory and implementation of a process which finds complete sets of reductions for ACI theories, omitting the details concerning the termination of the rewriting relation. In order to guarantee the finite termination property of the rewriting relation, we found it necessary to restrict the applicability of the reductions through the use of a special form of conditional reduction.

Preliminaries

We begin by stating some of the basic definitions and notation which will be used in our discussion. The notation is similar to that used in [JK86] or [PB89] and the interested reader should refer to those papers for more details.

Let S be a set of terms, E a set of equations, and R a set of reductions. The relation $=^1_E$ is a single application of an equality in E and the relation $\rightarrow_R$ is a single application of one of the rewrite rules in R. Let R/E(or $\rightarrow_{R/E}$) be the relation $=_E \circ \rightarrow_R \circ =_E$.

For any set of substitutions Θ, let $\iota_E\Theta$ be the set of all E instances of substitutions in Θ, where an E instance of σ is a substitution which is E equal to an instance of σ. Then $\sigma \notin \iota_E\Theta$ means that σ is not an E instance of a substitution in Θ. For a term t, $dom(t)$ is the set of all positions of subterms of t, ϵ is the element of $dom(t)$ at which t occurs, $sdom(t)$ is the subset of $dom(t)$ at which non variables occur, if $m \in dom(t)$, then t/m is the subterm occurring at m, and $t\,[m \leftarrow s]$ is the term obtained from t if the subterm at m is replaced with s.

Let the relation $t \rightarrow_{R,E} s$ mean there exist $\lambda \rightarrow \rho \in R$, a position $m \in sdom(t)$, and a substitution σ such that $t/m =_E \lambda\sigma$ and $s = t\,[m \leftarrow \rho\sigma]$.

Note that $\rightarrow_{R,E} \subseteq \rightarrow_{R/E}$, thus if $\rightarrow_{R,E}$ contains infinite chains $\rightarrow_{R/E}$ will contain them also. The following example demonstrates that $\rightarrow_{R,E}$ and $\rightarrow_{R/E}$ termination can, in fact, both be lost when rewriting relative to an ACI equational theory.

Let R contain the reduction $-(x+y) \rightarrow (-x)+(-y)$ and let E be the ACI equational theory for $+$. Then the term $(-a)$ can be rewritten as

$$\begin{aligned} -a \; &=_E \; -(a+0) \rightarrow_R (-a)+(-0) \\ &=_E \; -(a+0)+(-0) \rightarrow_R ((-a)+(-0))+(-0) \\ &=_E \; \cdots \end{aligned}$$

When rewritten as an $\rightarrow_{R,E}$ chain we have

$$\begin{aligned} -a \; &\rightarrow_{R,E} \; (-a)+(-0) \\ &\rightarrow_{R,E} \; ((-a)+(-0))+(-0) \\ &\rightarrow_{R,E} \; \cdots \end{aligned}$$

Clearly both $\rightarrow_{R,E}$ and $\rightarrow_{R/E}$ contain infinite chains in this example. It is very easy to find many other similar examples where termination is lost for ACI equational theories and for other equational theories which generate infinite congruence classes.

In order to develop any theory for rewriting relative to an ACI equational theory, we must first develop a rewriting relation which is provably terminating. In the following we develop a generalization of the $\rightarrow_{R/E}$ rewriting relation for rewriting relative to ACI equational theories and establish the criteria under which its termination is guaranteed.

Central to the development of the termination criteria for $\rightarrow_{R/E}$ is the notion of a core element of a congruence class generated by an ACI equational theory. We will say that a term t is a *core element* of $[t]_{ACI}$ if t is in normal form with respect to the rewriting relation $\rightarrow_{I,AC}$, where I is the set of reductions of the form $x+0 \rightarrow x$ and AC is the set of associative and commutative laws for each ACI operator, $+$, in the equational theory. The rewriting relation used here is precisely the same as $\rightarrow_{R,E}$ with I playing the role of R and AC playing the role of E. We will write $t \downarrow^I$ to mean the normal form of t with respect to $\rightarrow_{I,AC}$. Note that I is by itself a complete set of reductions with respect to AC, thus all core elements of $[t]_{ACI}$ are AC-equal to each other. Clearly this means that there are a finite number of terms in the core for any congruence class generated by an ACI theory. Furthermore, given any term of finite size, we can easily find the associated core element.

For example, consider the ACI congruence class which contains the terms $a + b, (a + b) + 0, (a + 0) + b, b+(0+a), (a+0)+(0+b), (0+0)+(b+a), (((a+b)+0)+0), \ldots$ The core for this congruence class contains only the two terms $a + b$ and $b + a$.

As is usually the case, our proof of termination for the $\rightarrow_{R/E}$ rewriting relation will be based on the use of a weighting function, W, such that $W(t)$ gives the weight of any term t. We will depend on the following six properties for W:

W1: $\forall t\ W(t) > 0$
W2: i is an identity for an ACI operator in $E \Rightarrow \forall t\ W(i) \leq W(t)$
W3: $s =_{AC} t \Rightarrow W(s) = W(t)$
W4: $W(s) > W(t) \Rightarrow W(T[m \leftarrow s]) > W(T[m \leftarrow t])$
W5: $W(s) > W(t)$ and θ is any substitution $\Rightarrow W(s\theta) > W(t\theta)$
W6: $t/m = s$, for some $m \in dom(t) \Rightarrow W(s) \leq W(t)$

These properties have been shown for a number of weighting functions. Since weighting functions are usually dependent on the actual operators allowable in s and t, we will assume that such a W exists. The required properties can then be demonstrated when the sets R and E have been given, making known the allowable operators for s and t. For problems which involve the ACI operators $+$ and $*$ and the unary operator $-$, the complexity measures of Lankford [La79] have been shown to meet the required properties.

Another possible approach to this problem would have been to develop a new weighting function which handles some of the problems which we encounter when dealing with infinite congruence classes. For instance, we could have attempted to develop a weighting function which assigns the same weight to all members of an ACI congruence class. In doing this, however, we would lose property W5, which seems to be more useful than the suggested property. Our present approach, therefore, is to work with weighting functions similar to those which have already been developed by those working with finite congruence classes under AC theories.

R/E Termination

In this section we establish sufficient conditions for the termination of $\rightarrow_{R/E}$. The basic approach is to demonstrate criteria under which the weight of a term strictly decreases on every $\rightarrow_{R/E}$ step. We first present and prove a theorem which indicates these requirements. This result is then used to redefine the notion of $\rightarrow_{R/E}$ rewriting. In order to accomplish our goal in this section we begin by proving a group of lemmas which allow us to reduce the problem to that of classifying the substitution involved in the rewriting.

The following property of ACI congruence classes was mentioned informally in our previous discussion of core elements. We state it more formally here for reference in a later proof.

Lemma 1 (L1) *If $t =_{ACI} s$ then $t\downarrow^I =_{AC} s\downarrow^I$.*

Proof: This is a direct consequence of the definition of $\downarrow^I$ and the fact that I is by itself a complete set of reductions with respect to AC.

We now show that coring a term can never increase its weight.

Lemma 2 (L2) *For every term, t, $W(t\downarrow^I) \leq W(t)$.*

Proof: It will suffice to show that if s is obtained from t by one application of an identity law, then $W(s) \leq W(t)$. We assume without loss of generality that the identity law is $x + 0 \rightarrow x$, for some ACI operator $+$, that there exist $m \in dom(t)$ such that $t/m = u + 0$ for some term u, and $s = t[m \leftarrow u]$. Since u is a term, it follows from W6 that $W(u) \leq W(u + 0)$, and hence we have using W4 that $W(s) = W(t[m \leftarrow u]) \leq W(t[m \leftarrow (u + 0)]) = W(t)$.

The next lemma makes it clear that we can preserve ACI-equality when we substitute equals for equals on both sides of the equality, provided that the subterm being replaced is in the same context in each term, relative to the ACI theory. This contextual requirement is assured by the added condition that the subterm occurs exactly once in each side. It is easy to see that the lemma is not true without this contextual requirement.

Lemma 3 (L3) *Given terms t and t', a constant c, and positions $x \in dom(t)$ and $x' \in dom(t')$ such that $t[x \leftarrow c] =_{ACI} t'[x' \leftarrow c]$, $c \neq_{ACI} Ident(\alpha)$ for any ACI operator α in E , and c occurs in neither t nor t', then for any term s, $t[x \leftarrow s] =_{ACI} t'[x' \leftarrow s]$.*

Proof: Since we are given that $t[x \leftarrow c] =_{ACI} t'[x' \leftarrow c]$, this means that there exists a sequence of terms $t[x \leftarrow c] \equiv t_1 =^1_{ACI} t_2 =^1_{ACI} \cdots =^1_{ACI} t_n \equiv$

$t'[x' \leftarrow c]$, where $=^1_{ACI}$ is used to mean a single application of one of the ACI equations. Since none of these equations can eliminate or duplicate c it follows that there is exactly one occurrence of c in each t_i. This means that a corresponding sequence of $=^1_{ACI}$ steps with each c replaced by s can be used to demonstrate that $t[x \leftarrow s] =_{ACI} t'[x' \leftarrow s]$.

We now establish the existence of a core term which is similar enough in structure to a given term that we can replace a subterm in each with ACI-equal terms and preserve ACI-equality. This lemma will provide the backbone for the proof of our main theorem in this section.

Lemma 4 (L4) *Given a term, t, and a position, $x \in dom(t)$, then there exists a core term, t', and a position, $x' \in dom(t')$, such that for any term $s, t[x \leftarrow s] =_{ACI} t'[x' \leftarrow s\downarrow^I]$.*

Proof: Let $t' = (t[x \leftarrow c])\downarrow^I$ where c is a special constant not previously appearing in t and $c \neq_{ACI} Ident(\alpha)$ for any ACI operator α in E. The special constant c will serve as a marker to mark position x in t and allow the determination of the corresponding position in t' after the coring process has taken place. Clearly the rewriting relation $\xrightarrow{*}_{I,AC}$ can move the position of c during the coring process, however, it can neither eliminate nor duplicate c since the AC equations can only serve to permute terms and the I reduction can only eliminate identities, which c is not. Thus there must be one and only one position $x' \in dom(t')$ such that $t'/x' = c$. We know that $t[x \leftarrow c] =_{ACI} (t[x \leftarrow c])\downarrow^I$ since $\downarrow^I$ preserves ACI-equality. But $(t[x \leftarrow c])\downarrow^I = t'$ by definition, and since $t'/x' = c$, we now have $t[x \leftarrow c] =_{ACI} t'[x' \leftarrow c]$. Since c occurs exactly once on each side of this equation, we can apply Lemma 3 and substitute any term s for the marker giving $t[x \leftarrow s] =_{ACI} t'[x' \leftarrow s]$. Finally, we can core s on one side since coring preserves ACI-equality and we have $t[x \leftarrow s] =_{ACI} t'[x' \leftarrow s\downarrow^I]$.

It is important to note that the term s can be changed arbitrarily after t' and x' have been found. This will allow us to find t' for a given t and then change the substituted subterm without having to find another $t - t'$ pair.

Our next lemma will be used later to establish that, under the conditions which we will assume, $\rightarrow_{R/E}$ cannot replace a subterm which is E-equal to an identity.

Lemma 5 (L5) *If $W(t\downarrow^I) > W(s\downarrow^I)$ then $t\downarrow^I \neq_{ACI} Ident(\alpha)$ for any ACI operator α in E.*

Proof: Assume $t\downarrow^I =_{ACI} Ident(\alpha)$. This implies that $t\downarrow^I =_{AC} Ident(\alpha)$ by Lemma 1 since both are core terms. Then by W3 we have $W(Ident(\alpha)) = W(t\downarrow^I)$. But this together with the given hypothesis allows us to conclude that $W(Ident(\alpha)) > W(s\downarrow^I)$, which contradicts W2. Thus the assumption that $t\downarrow^I =_{ACI} Ident(\alpha)$ must be false.

The following lemma shows that coring the subterm inserted into a cored term is equivalent to coring the resulting term, provided that the inserted subterm does not collapse down to an identity.

Lemma 6 (L6) *If $y \in dom(t\downarrow^I)$ and $s\downarrow^I \neq_{ACI} Ident(\alpha)$ for any ACI operator α in E, then $(t\downarrow^I [y \leftarrow s])\downarrow^I = t\downarrow^I [y \leftarrow s\downarrow^I]$.*

Proof: Clearly $t\downarrow^I [y \leftarrow s\downarrow^I]$ is in normal form with respect to $\rightarrow_{I,AC}$ unless $s\downarrow^I = Ident(\alpha)$ for some ACI operator α in $t\downarrow^I$ which has $s\downarrow^I$ in its scope. Since we are given that $s\downarrow^I \neq Ident(\alpha)$ for any ACI operator α in E, this cannot be the case. Thus $t\downarrow^I [y \leftarrow s\downarrow^I] = (t\downarrow^I [y \leftarrow s\downarrow^I])\downarrow^I$, which is equal to $(t\downarrow^I [y \leftarrow s])\downarrow^I$ by the definition of $\downarrow^I$.

Given a substitution $\sigma = \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$ and a term t, we can split σ into two disjoint portions by defining functions Σ_1 and Σ_2 as follows:

$$\begin{aligned}\Sigma_1(\sigma,t) &= \{(x_i \leftarrow t_i) \mid (x_i \leftarrow t_i) \in \sigma \text{ and } x_i \text{ is in} \\ &\quad\ \text{the scope of an ACI operator } \alpha \text{ in } t \\ &\quad\ \text{and } t_i =_E Ident(\alpha), \text{ the identity for } \alpha\} \\ \Sigma_2(\sigma,t) &= \sigma - \Sigma_1(\sigma,t)\end{aligned}$$

Clearly, if $\sigma_1 = \Sigma_1(\sigma,t)$ and $\sigma_2 = \Sigma_2(\sigma,t)$ then $\sigma = \sigma_1 \bigcup \sigma_2 = \sigma_1\sigma_2$. Our main theorem will show that the termination of $\rightarrow_{R/E}$ is dependent only on the Σ_1 portion of the term matching substitution which is used to apply each reduction. In order to show that the Σ_1 portion of a substitution plays the vital role in this process, we will first consider the role of the Σ_2 portion. For a substitution σ and a term t define a *cored substitution* $\sigma\downarrow^I$, by $\sigma\downarrow^I = \{(x_i \leftarrow t_i\downarrow^I) \mid (x_i \leftarrow t_i) \in \sigma\}$.

Lemma 7 (L7) *Given a substitution σ and a term t then $(t\sigma_2)\downarrow^I = (t\downarrow^I)\sigma_2\downarrow^I$, where $\sigma_2 = \Sigma_2(\sigma,t)$.*

Proof: The only way these terms can differ is if $\sigma_2 \downarrow^I$ can introduce a context for the application of $\downarrow^I$ into $t \downarrow^I$ causing $(t \downarrow^I)\sigma_2 \downarrow^I$ not to be a core term while $(t\sigma_2) \downarrow^I$ is clearly a core term. This cannot happen, however, because if the context for $\downarrow^I$ had been in t_i it has already been eliminated and if $t_i =_{ACI} Ident(\alpha)$ where x_i is in the scope of an ACI operator, α, in t, then $(x_i \leftarrow t_i) \notin \sigma_2 \downarrow^I$ by definition of Σ_2.

As the final piece which we will need in order to prove our main theorem for this section, we now define the *restricted substitution set*, $\Theta(\lambda \rightarrow \rho)$, as follows:

$$\Theta(\lambda \rightarrow \rho) = \{\sigma \mid \sigma = \{x_1 \leftarrow Ident(\alpha_1), \ldots x_n \leftarrow Ident(\alpha_n)\},$$

where
- $n \geq 0$,
- each α_i is an ACI operator,
- each x_i is a variable in λ which is in the scope of α_i, and
- $W((\lambda\sigma) \downarrow^I) \leq W((\rho\sigma) \downarrow^I)\}$.

We now state and prove our main result: that $\rightarrow_{R/E}$ must terminate when the conditions for each rule, represented by $\Theta(\lambda \rightarrow \rho)$, are enforced.

Theorem 1 *If the reduction $t \rightarrow_{R/E} s$ is allowed to take place only when $\sigma \notin \iota_E\Theta(\lambda \rightarrow \rho)$, then the rewriting relation $\rightarrow_{R/E}$ must terminate.*

Proof: It will be sufficient to show that, under the given conditions, $W(t \downarrow^I) > W(s \downarrow^I)$. If $t_1 \rightarrow_{R/E} t_2 \rightarrow_{R/E} \ldots$ is an infinite sequence of $\rightarrow_{R/E}$ reductions, then $t_1 \downarrow^I, t_2 \downarrow^I, \ldots$ is an infinite sequence of terms whose weights get strictly smaller, but this is impossible by W1. We proceed as follows: By the definition of $t \rightarrow_{R/E} s$, there exist terms t' and s' such that

$$t =_E t' \rightarrow_R s' =_E s.$$

Since $t =_E t \downarrow^I$, it follows that

$$t \downarrow^I =_E t' \rightarrow_R s' =_E s \downarrow^I .$$

By the definition of $\rightarrow_{R/E}$ there exists $m \in dom(t')$ such that $t'/m = \lambda\sigma$ and $\sigma' = t'[m \leftarrow \rho\sigma]$. By Lemma 4 there exists a core term t'' and a position $m' \in dom(t'')$ such that for every term u,

$$t'[m \leftarrow u] =_E t''[m' \leftarrow u \downarrow^I]. \qquad (1)$$

Let $\sigma_1 = \Sigma_1(\sigma, \lambda)$ and $\sigma_2 = \Sigma_2(\sigma, \lambda)$. From the definitions of Σ_1 and Σ_2 it is clear that $\sigma = \sigma_1\sigma_2$. From the definition of E-equality for substitutions, it follows that $\sigma =_E \sigma_1 \downarrow^I \sigma_2$. Since for each $(x_i \leftarrow t_i) \in \sigma_1$ the definition of Σ_1 gives that $t_i =_E Ident(\alpha_i)$, it follows that $t_i \downarrow^I = Ident(\alpha_i)$. From the given condition, $\sigma \notin \iota_E\Theta(\lambda \rightarrow \rho)$, we see that $\sigma_1 \downarrow^I \notin \Theta(\lambda \rightarrow \rho)$, giving

$W((\lambda\sigma_1) \downarrow^I)$	
$= W((\lambda\sigma_1 \downarrow^I) \downarrow^I)$	by L1 and W3
$> W((\rho\sigma_1 \downarrow^I) \downarrow^I)$	by def. of $\Theta(\lambda \rightarrow \rho)$
$= W((\rho\sigma_1) \downarrow^I)$	by L1 and W3,

and

$W((\lambda\sigma) \downarrow^I)$	
$= W((\lambda\sigma_1) \downarrow^I \sigma_2 \downarrow^I)$	by L7
$> W((\rho\sigma_1) \downarrow^I \sigma_2 \downarrow^I)$	by W5
$\geq W(((\rho\sigma_1) \downarrow^I \sigma_2 \downarrow^I) \downarrow^I)$	by L2
$= W((\rho\sigma_1\sigma_2) \downarrow^I)$	by L1 and W3
$= W((\rho\sigma) \downarrow^I)$	by defs. of Σ_1 and Σ_2.

Now Lemma 5 assures us that $(\lambda\sigma) \downarrow^I$ cannot be an identity. We conclude that

$W(t \downarrow^I)$	
$= W(t' \downarrow^I)$	by L1 and W3
$= W((t'[m \leftarrow \lambda\sigma]) \downarrow^I)$	since $t'/m = \lambda\sigma$
$= W((t''[m' \leftarrow (\lambda\sigma) \downarrow^I]) \downarrow^I)$	by (1) above
$= W(t''[m' \leftarrow (\lambda\sigma) \downarrow^I])$	by L6
$> W(t''[m' \leftarrow (\rho\sigma) \downarrow^I])$	by W4
$\geq W((t''[m' \leftarrow (\rho\sigma) \downarrow^I]) \downarrow^I)$	by L2
$= W((t'[m \leftarrow \rho\sigma]) \downarrow^I)$	by (1), L1 and W3
$= W(s' \downarrow^I)$	since $s' = t'[m \leftarrow \rho\sigma]$
$= W(s \downarrow^I)$	by L1 and W3.

We now propose to redefine the notion of $\rightarrow_{R/E}$ rewriting as follows:

$$t_1 \rightarrow_{R/E} t_2 \iff$$

$$t_1 =_E t_1' \rightarrow_R t_2' =_E t_2 \text{ and } \sigma \notin \iota_E\Theta(\lambda \rightarrow \rho)$$

Note that the conditional version of $\rightarrow_{R/E}$ can be thought of as a generalization of the normal definition of $\rightarrow_{R/E}$. All that is required is to have "empty" conditions on reductions of the normal $\rightarrow_{R/E}$ variety. When viewed as such, any theory developed around conditional reductions subsumes a similar theory developed around the usual unconditional reductions. Hereafter we will use $\rightarrow_{R/E}$ to refer to this generalization. Peterson et al. [PB89] present a procedure for testing the completeness of a set of reductions relative to an ACI equational theory, based on the conditional version of the $\rightarrow_{R/E}$ rewriting relation presented above. It was assumed in that study that $\rightarrow_{R/E}$ did terminate, subject to the conditions, and the main proofs were based on that assumption. Our termination result thus collaborates that assumption.

Applying the Termination Theorem

We now describe a simple procedure for calculating the conditions which are needed for each reduction in order to satisfy the termination property. Recall from the previous section that the conditions for each reduction are represented by the restricted substitution set $\Theta(\lambda \rightarrow \rho)$. We begin by finding the set $I(\lambda)$, where $I(t)$ is given by

$$I(t) = \{(x \leftarrow Ident(\alpha)) \mid x \text{ is a variable in } t \text{ in the scope of an ACI operator}, \alpha\}.$$

The set $I(\lambda)$ then forms the basis of identity substitution pairs from which all possible members of $\Theta(\lambda \rightarrow \rho)$ will be generated. We then generate potential substitutions, $P(\lambda)$, where $P(\lambda)$ is given by

$$P(\lambda) = \left\{ y \mid y \in 2^{I(\lambda)} \text{ and } y \text{ is a valid substitution} \right\}.$$

Clearly, the powerset, $2^{I(\lambda)}$, generates all possible combinations of identity substitution pairs. We must discard any substitution which assigns more than one identity to the same variable because these are not valid substitutions. Finally, we test each member, σ, of $P(\lambda)$ to see whether or not $W((\lambda\sigma)\downarrow^I) \leq W((\rho\sigma)\downarrow^I)$. If the test succeeds we place σ in $\Theta(\lambda \rightarrow \rho)$, otherwise we do not.

Example 1: The following example illustrates how the preceding procedure is applied to a set of reductions to ensure $\rightarrow_{R/E}$ termination when E is an ACI equational theory. Consider the following set of reductions where $+$ is an ACI operator and $-$ has none of the ACI properties:

R1: $x + (-x) \rightarrow 0$
R2: $-(-x) \rightarrow x$
R3: $-(x + y) \rightarrow (-x) + (-y)$

For each of the examples which we present in this section we will use the weighting function $W(t)$ which is defined as follows:

$$W(constant) = 2$$
$$W(variable) = 2$$
$$W(x * y) = W(x) * W(y)$$
$$W(x + y) = W(x) + W(y) + 5$$
$$W((-x)) = 2 + 2 * W(x)$$

For R1 the only variable in the scope of an ACI operator is x. The corresponding ACI operator is $+$ and the corresponding identity is 0. This gives $I(\lambda) = \{x \leftarrow 0\}$ and $P(\lambda) = \{\phi, \{x \leftarrow 0\}\}$. Using these substitutions for σ, we find that $\forall\sigma\ W(\lambda\sigma\downarrow^I) > W(\rho\sigma\downarrow^I)$, thus no restrictions are needed for R1. R2 has no variables in the scope of ACI operators, giving $I(\lambda) = P(\lambda) = \Theta(\text{R2}) = \phi$. Thus R2 must only satisfy the property $W(\lambda) > W(\rho)$, which it does. R3 has variables x and y in the scope of the ACI operator $+$ with the corresponding identity 0. This gives $I(\lambda) = \{x \leftarrow 0, y \leftarrow 0\}$, and $P(\lambda) = \{\phi, \{x \leftarrow 0\}, \{y \leftarrow 0\}, \{x \leftarrow 0, y \leftarrow 0\}\}$. Calling these substitutions σ_1, σ_2, σ_3, and σ_4, respectively, we find that $W(\lambda\sigma\downarrow^I) \leq W(\rho\sigma\downarrow^I)$ for all substitutions $\sigma = \sigma_i$ except $\sigma = \sigma_1$. Thus $\Theta(\text{R3})$ becomes $\{\sigma_2, \sigma_3, \sigma_4\}$. Since σ_4 is an instance of σ_2 and σ_3, any substitution which is an E-instance of σ_4 will also be an E-instance of σ_2 and σ_3. Because of this we will get the same result with $\Theta(\text{R3}) = \{\sigma_2, \sigma_3\}$ as with $\Theta(\text{R3}) = \{\sigma_2, \sigma_3, \sigma_4\}$. For the sake of simplicity we will use the more concise form. We now have the restrictions $\Theta(\text{R1}) = \phi$, $\Theta(\text{R2}) = \phi$, and $\Theta(\text{R3}) = \{\{x \leftarrow 0\}, \{y \leftarrow 0\}\}$. Equivalently, the set of reductions which guarantees $\rightarrow_{R/E}$ termination can be represented as the set of conditional reductions given below:

R1: $x + (-x) \rightarrow 0$
R2: $-(-x) \rightarrow x$
R3: If $x \neq 0$ and $y \neq 0$ then
$-(x + y) \rightarrow (-x) + (-y)$

This set has been shown to be a complete set of reductions for abelian groups relative to the ACI equational theory for $+$.

The preceding example suggests a better procedure for computing $\Theta(\lambda \rightarrow \rho)$. When λ contains at least one variable there will always be substitutions in $P(\lambda)$ which are instances of other substitutions in $P(\lambda)$ because the powerset of $I(\lambda)$ will contain members which are supersets of other members. For instance, as shown above, $P(x + y) = \{\phi, \{x \leftarrow 0\}, \{y \leftarrow 0\}, \{x \leftarrow 0, y \leftarrow 0\}\}$. Calling these substitutions σ_1, σ_2, σ_3, and σ_4, respectively, it is clear that σ_2, σ_3, and σ_4 are supersets of σ_1 making them instances of σ_1, and σ_4 is likewise an instance of both σ_2 and σ_3. This suggests that we generate and test the elements of the powerset from the smallest to the largest. If an element p of $P(\lambda)$ is placed in $\Theta(\lambda \rightarrow \rho)$ then no larger element q of $P(\lambda)$ which is a superset of p need even be tested as to whether or not $W((\lambda q) \downarrow^I) \leq W((\rho q) \downarrow^I)$. The test would indicate that q should be added to $\Theta(\lambda \rightarrow \rho)$, but we know that we can leave it out. Because of the manner in which the substitutions are used, this clearly will not change the effect of $\Theta(\lambda \rightarrow \rho)$ but will speed up its calculation while automatically providing the restrictions in the most concise form. An interesting result of the process is that $\Theta(\lambda \rightarrow \rho) = \phi$ represents a reduction with no restrictions while $\Theta(\lambda \rightarrow \rho) = \{\phi\}$ represents a reduction which is always restricted, since every substitution is an instance of the empty substitution.

Example 2: In this example we will calculate restrictions using the procedure just described, so as to obtain minimal restrictions. Consider the following set of reductions:

$$\begin{array}{ll} \text{R4:} & x * (y + z) \rightarrow (x * y) + (x * z) \\ \text{R5:} & x * 0 \rightarrow 0 \\ \text{R6:} & x * (-y) \rightarrow -(x * y) \end{array}$$

For R4, $I(\lambda) = \{x \leftarrow 1, y \leftarrow 0, z \leftarrow 0\}$ and when we generate the powerset elements from the smallest to the largest we find that the singleton sets $\{x \leftarrow 1\}$, $\{y \leftarrow 0\}$, and $\{z \leftarrow 0\}$ are all added to $\Theta(\lambda \rightarrow \rho)$. No larger members need be tested as all larger members are supersets of at least one of these sets. For both R5 and R6 we find $I(\lambda) = \{x \leftarrow 1\}$, $P(\lambda) = \{\phi, \{x \leftarrow 1\}\}$, and $\Theta(\lambda \rightarrow \rho) = \{x \leftarrow 1\}$. Viewing these restrictions as conditional reductions we now have:

$$\begin{array}{ll} \text{R4:} & \text{If } x \neq 1 \text{ and } y \neq 0 \text{ and } z \neq 0 \text{ then} \\ & \qquad x * (y + z) \rightarrow (x * y) + (x * z) \\ \text{R5:} & \text{If } x \neq 1 \text{ then } x * 0 \rightarrow 0 \\ \text{R6:} & \text{If } x \neq 1 \text{ then } x * (-y) \rightarrow -(x * y) \end{array}$$

The set { R1, R2, R3, R4, R5, R6 } has been shown to be a complete set of reductions for commutative rings with unit elements relative to the ACI equational theory for $+$ and $*$.

Example 3: As a final example let us examine a reduction which leads to a more complicated set of restrictions. Consider the following reduction which is an absorption law from the definition of a distributive lattice:

$$\text{R7:} \quad x + (x * y) \rightarrow x$$

$I(\lambda) = \{x \leftarrow 0, x \leftarrow 1, y \leftarrow 0, y \leftarrow 1\}$. Note that $y \leftarrow 0$ must be included because, under identity substitution and coring, it is possible for y to appear in the scope of the $+$ operator. $P(\lambda)$ with elements listed from smallest to largest is $\{\phi, \{x \leftarrow 0\}, \{x \leftarrow 1\}, \{y \leftarrow 0\}, \{y \leftarrow 1\}, \{x \leftarrow 0, y \leftarrow 0\}, \{x \leftarrow 0, y \leftarrow 1\}, \{x \leftarrow 1, y \leftarrow 0\}, \{x \leftarrow 1, y \leftarrow 1\}\}$. Note that several members of $2^{I(\lambda)}$ were discarded because they were not valid substitutions. Of the remaining substitutions, only $\{x \leftarrow 0, y \leftarrow 1\}$ and $\{x \leftarrow 1, y \leftarrow 0\}$ are placed in $\Theta(\lambda \rightarrow \rho)$. This restriction differs from the previous examples in that it allows for either x or y to take on an identity, but prevents both x and y from taking on identities at the same time. Represented as a conditional reduction, R7 now becomes:

$$\begin{array}{ll} \text{R7:} & \text{If } \neg((x = 0 \text{ and } y = 1) \text{ or } (x = 1 \text{ and } y = 0)) \\ & \qquad \text{then } x + (x * y) \rightarrow x, \end{array}$$

or, equivalently,

$$\begin{array}{ll} \text{R7:} & \text{If } (x \neq 0 \text{ or } y \neq 1) \text{ and } (x \neq 1 \text{ or } y \neq 0) \\ & \qquad \text{then } x + (x * y) \rightarrow x. \end{array}$$

Summary

Have we weakened the original rewriting relations by adding the conditions in the above examples? No, we have not. In Example 1 the most general form of a critical pair which could have been conflated by R3 before the conditions but cannot be conflated by R3 after the conditions must be $\langle -(t + 0), (-t) + (-0)\rangle$ or $\langle -(0 + t), (-0) + (-t)\rangle$. It is easy to see that R1 can be used to conflate all such pairs since $(-t) + (-0) =_E (-t) + ((-0) + 0) \rightarrow_{R1} (-t) + 0 =_E -(t + 0)$. Thus, taken together, the rewriting power of R1, R2, R3 has not been weakened by the introduction of the conditions needed for termination.

Likewise, in Example 2 we see that the most general form of pairs which could have been conflated by R4 were it not for the conditions must be either $\langle 1 * (y + z), 1 * y + 1 * z\rangle$, $\langle x * (0 + z), x * 0 + x * z\rangle$, or $\langle x * (y + 0), x * y + x * 0\rangle$. The pair $\langle 1 * (y + z), 1 * y + 1 * z\rangle$ conflates trivially since $1 * (y + z) =_E 1 * y + 1 * z$. The other two pairs are easily conflated via R5 since $x * 0 + x * z \rightarrow_{R5} 0 + x * z =_E x * (0 + z)$. As before we see that, taken together, the rewriting power of the entire reduction set has not been weakened by the conditions.

Finally, we see that in Example 3 the restriction on

R7 only prevents its application to a pair of the general form $\langle 0+(0*1),0\rangle$ or $\langle 1+(1*0),1\rangle$, which conflate trivially since $0+(0*1) =_E 0$ and $1+(1*0) =_E 1$. Thus the restriction only prevents its application when its application was not needed in the first place. These examples indicate that the conditions needed for termination may not weaken the rewriting strength of a reduction at all, and that when they do another reduction in the same set may still provide the same functionality as that which was removed. In such cases termination is achieved while the set of reductions as a whole loses no rewriting strength.

Since $\rightarrow_{R/E}$ is a very general form of a rewriting relation between congruence classes generated by an equational theory, it is clear that the conditions which give $\rightarrow_{R/E}$ termination also give the termination of many less general rewriting relations. For instance, any rewriting relation $\rightarrow_{R^E}$ such that $\rightarrow_R \subseteq \rightarrow_{R^E} \subseteq \rightarrow_{R/E}$ must terminate under these same conditions. This is important because the $\rightarrow_{R/E}$ rewriting relation is not conveniently implemented in a computer program, especially when E generates infinite congruence classes. We have found it useful to implement $\rightarrow_{R^E}$ for an ACI equational theory, E , in the following manner. Let $t \rightarrow_{R^E} s$ mean that there exist $\lambda \rightarrow \rho \in R$, $m \in dom(t)$, and σ such that

$$\begin{aligned}&\sigma \notin \iota_E \Theta(\lambda \rightarrow \rho),\\ &t/m =_E \lambda\sigma, \text{ and}\\ &s = t[m \leftarrow \rho\sigma].\end{aligned}$$

This rewriting relation is in the range between $\rightarrow_R$ and $\rightarrow_{R/E}$ and is very easy to implement. The conditions which give termination are enforced as a simple modification to the ACI term matching routine. The term matching routine receives a term, a pattern, and the conditions. Whereas the normal ACI term matching would return the first substitution which matches the pattern to the term, the modified routine returns the first such match which does not violate the conditions. If no such match can be found, the term and pattern are considered not to match.

The rewriting relation we have described is actually a rewriting relation from a core element of one congruence class to a core element of another congruence class. When rewriting in this manner, we begin with a core element, but we are allowed to leave the core during the ACI-matching step before we apply the reduction. We then push the result back down to the core. As a principle application of this idea, we are able to restore the finite termination property that is often lost when rewriting in the presence of infinite congruence classes, which includes the ACI equational theory.

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