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The Double Eigenvalue Problem; Including Numerical Solutions

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I. INTRODUCTION

In this paper we will present a unified theory for the double eigenvalue problem whose differential equation is given as

$$L(x;\lambda,\mu)=(p(t)\,x'(t))'-l(t)\,x(t)+\lambda q(t)\,x(t)+\mu r(t)\,x(t)=0 \tag{1}$$

subject to $x(\alpha)=x(\beta)=x(\gamma)=0$, where $p(t)>0$ and $\alpha<\beta<\gamma$.

This theory will include qualitative and numerical results. The basic idea is to redefine the problem in (1) as a quadratic form problem

$$H(x;\lambda,\mu)=J(x)-\lambda K_1(x)-\mu K_2(x), \tag{2}$$

where

$$J(x)=\int_\alpha^\gamma [p(t)\,x'^2(t)+l(t)\,x^2(t)]\,dt, \tag{2a}$$

$$K_1(x)=\int_\alpha^\gamma q(t)\,x^2(t)\,dt, \tag{2b}$$

and

$$K_2(x)=\int_\alpha^\gamma r(t)\,x^2(t)\,dt, \tag{2c}$$

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and to apply the vast amount of mathematical ideas previously developed by the first author.

In a sense this problem is both an ending and a beginning of an approximation theory of quadratic forms. It is an ending since it includes the curent problems of interest in this setting, namely, the numerical focal, conjugate and oscillation point problems ($\lambda = \mu = 0$; $\beta$and γ ignored), the numerical eigenvalue problem and the one constraint problem ($\mu = 0$; γ ignored). It is a beginning since it includes a systematic theoretical and numerical development of the double eigenvalue problem occurring in physical problems (see [1] for references) and indicates how to handle multiconstraint problems and "mixed" eigenvalue-constraint problems.

In Section II we present the theoretical results and preliminaries necessary for the remainder of this paper. In particular, we give very general inequality results concerning the signature and nullity of quadratic forms in (2), relate this to differential equations in (1), and show how these results fit in a qualitative picture for our two parameter problem.

In Section III we show how to build finite dimensional Hilbert Spaces by use of splines and finite dimension quadratic forms which are approximations of the quadratic form in (2). We also give the "Euler–Lagrange solution" for this finite dimensional problem. A very strong approximation result relation to our finite dimension solution is then given. In Section IV we give a two dimension iteration scheme to find the proper values of λ and μ and in Section V test cases are given to show how efficient and numerically accurate our procedures are.

For convenience of the reader we present a diagram (Fig. 1) which indicates how (given fixed values of the parameters λ and μ) we numerically approximate this problem). Arrow (1) denotes ideas which were redeveloped to "fit" into the overall picture. Arrow (2) denotes approximation ideas given by the first author. Arrow (3) denotes the new ideas (algorithms) in this paper which include the iteration scheme, the oscillation vector (eigenvector)

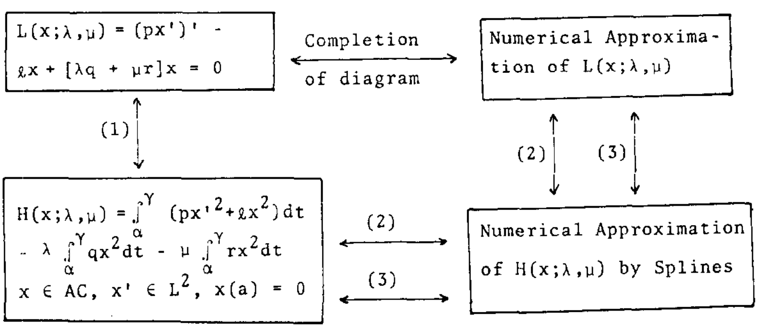

FIGURE 1

results, and the "Euler–Lagrange equations" for two parameter tridiagonal matrices.

In Section IV we will give a two dimensional iteration scheme which uses the idea of the diagram as a "subprogram." In Section V we give a numerical example which agrees with the example in [1].

II. Preliminary and Theoretical Results

In this section we present and derive basic results necessary for the remainder of this paper. We begin with a paraphrasing of the notation in [2] for the point $t=\beta$, relate solutions of (1) to the signature of a quadratic form, and finally give a parametric curve in the (λ, μ) axis which occurs in our numerical problems.

For each pair of real numbers (λ, μ) let the quadratic form $H(x;\lambda,\mu)=J(x)-\lambda K_1(x)-\mu K_2(x)$ given in (2) be defined on the interval $\alpha \leqslant t \leqslant \beta$. Let $\mathcal{A}(\beta)$ be the set of all arcs $x(t)$ defined on $\alpha \leqslant t \leqslant \beta$ such that $x(\alpha)=x(\beta)=0$ and such that $x(t)$ is absolutely continuous and $x'(t)$ is square integrable on $[\alpha, \beta]$. $\mathcal{A}(\beta)$ is a Hilbert Space with inner product

$$(x, y)=x(\alpha)\, y(\alpha)+\int_\alpha^\beta x'(t)\, y'(t)\, dt \tag{3}$$

and $\|x\|=\sqrt{(x,x)}$. Let $s_\beta(\lambda,\mu)$ denote the *signature* (*index*) of H on $\mathcal{A}(\beta)$, that is the dimension of a maximal subspace $B \subset \mathcal{A}(\beta)$ such that $x \neq 0$ in B implies $H(x;\lambda,\mu)<0$. Let $n_\beta(\lambda,\mu)$ denote the *nullity* of $H(x;\lambda,\mu)$ on $\mathcal{A}(\beta)$, that is, the dimension of the space of arcs in $\mathcal{A}(\beta)$ such that $H(x, y;\lambda,\mu)=0$ for all arcs $y(t)$ in $\mathcal{A}(\beta)$, where $H(x, y;\lambda,\mu)$ is the bilinear form associated with $H(x;\lambda,\mu)=H(x,x;\lambda,\mu)$ in (2).

We pause to review briefly some characteristics of these nonnegative integer valued functions which the reader should picture as the number of negative and zero eigenvalues of a real symmetric matrix. We assume that $s_\gamma(\lambda,\mu)$ and $n_\gamma(\lambda,\mu)$ is defined similarly to $s_\beta(\lambda,\mu)$ and $n_\gamma(\lambda,\mu)$ above on the interval $\alpha \leqslant t \leqslant \gamma$ and will "incorrectly" use the symbol H since there is no danger of confusion.

Theorem 1. *Assume $q(t) \geqslant 0$ and $r(t) \geqslant 0$; then $\lambda_1 \leqslant \lambda_2$ and $\mu_1 \leqslant \mu_2$ imply $s_\beta(\lambda_1,\mu_1) \leqslant s_\beta(\lambda_2,\mu_2)$. Similarly $s_\gamma(\lambda_1,\mu_1) \leqslant s_\gamma(\lambda_2,\mu_2)$. Finally $s_\beta(\lambda,\mu) \leqslant s_\gamma(\lambda,\mu)$.*

The first statement follows since $\lambda_1 \leqslant \lambda_2$ and $\mu_1 \leqslant \mu_2$ imply $H(x;\lambda_1,\mu_1)-H(x;\lambda_2,\mu_2)=J(x)-\lambda_1 K_1(x)-\mu_1 K_2(x)-J(x)+\lambda_2 K_1(x)+\mu_2 K_2(x) = (\lambda_2-\lambda_1)K_1(x) + (\mu_2-\mu_1)K_2(x) \geqslant 0$ or $H(x;\lambda_1,\mu_1) \geqslant H(x;\lambda_2,\mu_2)$ so that if $x_0(t)$ implies $H(x_0;\lambda_1,\mu_1)<0$ then $H(x_0;\lambda_2,\mu_2)<0$.

The final statement follows by defining $y_0(t)$ equal to $x_0(t)$ on $[\alpha, \beta]$ and $y_0(t) \equiv 0$ on $[\beta, \gamma]$. Then if $H(x_0; \lambda, \mu) < 0$ we have $H(y_0; \lambda, \mu) = H(x_0; \lambda, \mu) < 0$.

Of special importance is to note the connection between $s(\lambda, \mu)$ and the number of oscillation points of a nontrivial solution of (1) subject to $x(\alpha) = 0$. The next result is stated only for $t = \beta$ but holds equally well for β replaced by any t_0, $\alpha < t_0 \leqslant \gamma$. As above we would (incorrectly) use the same symbol H to denote a quadratic form with integration over the interval $[\alpha, t_0]$. Note that $s_{t_0}(\lambda, \mu)$ is a nondecreasing function of t_0 follows from Theorem 1.

THEOREM 2. *The value of $n_\beta(\lambda, \mu)$ is zero or one. It is one iff there exists a nontrivial solution $x_0(t)$ to (1) such that $x_0(\alpha) = x_0(\beta) = 0$. The value of $s_\beta(\lambda, \mu)$ equals m iff there exists a nontrivial solution $x_1(t)$ of (1) satisfying $x_1(\alpha) = 0$ and $x_1(t_j) = 0$ for $j = 1, 2, \ldots, m$, where $\alpha < t_1 < t_2 < \cdots < t_m < \beta$.*

These results have been given in more detail in Ref. [2]. Note that $s_\beta(\lambda, \mu)$ counts the number of points t on (α, β) for which $n_t(\lambda, \mu) = 1$.

Our final effort in this section is to note that for $t = \beta$, $q(t) \geqslant 0$, $r(t) \geqslant 0$ and q and r linearly independent functions we can separate the (λ, μ) plane into open sets

$$O_m = \{(\lambda, \mu) : s_\beta(\lambda, \mu) = m,\ n_\beta(\lambda, \mu) = 0\}$$

with boundary lines

$$\Gamma_m = \{(\lambda, \mu) : s_\beta(\lambda, \mu) = m, n_\beta(\lambda, \mu) = 1\}.$$

From Theorems 1 and 2 we note that Γ_m, for $m = 0, 1, 2, \ldots$, defines a function $\mu = g_m(\lambda)$ which is one to one and has negative slope. This follows immediately since for a fixed value of λ, we have shown in Ref. [4] that there exists an eigenvalue–eigenvector solution $(\mu, x_0(t))$ to Eq. (1), where $x_0(\alpha) = x_0(\beta) = 0$ and $x_0(t)$ at m points in the interval (α, β).
If we define $O_{j'}$ and $\Gamma_{j'}$ similar to above except that the "prime" denotes the γ situation, i.e.,

$$O_{j'} = \{(\lambda, \mu) : s_\gamma(\lambda, \mu) = j, n_\beta(\lambda, \mu) = 0\},$$

then our picture is Fig. 2, where (λ_0, μ_0), the solution of the double eigenvalue problem, lies at the intersection of the two lines Γ_m and $\Gamma_{j'}$ described above and the ordered pair, designated (m, m') at this intersection, denotes the fact that the corresponding eigensolution "crosses" the axis m times in (α, β) and $m' = j - m - 1$ times in (β, γ).

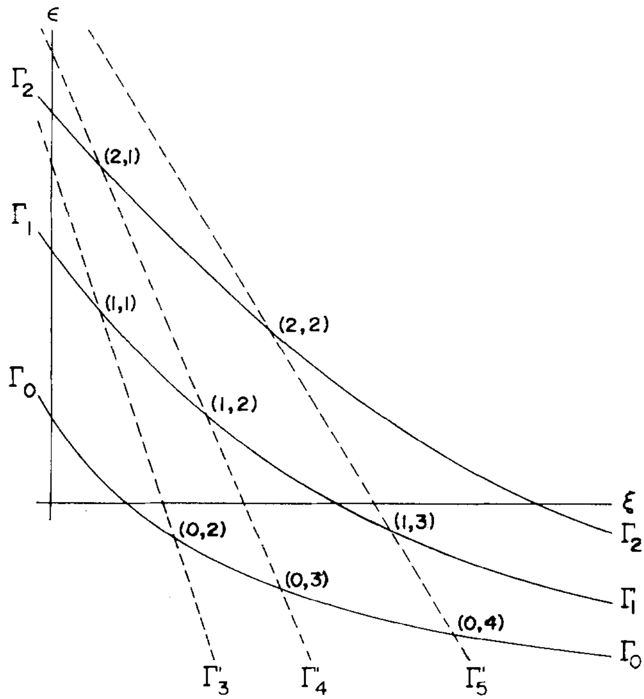

FIGURE 2

III. THE NUMERICAL APPROXIMATION PROBLEM

In this section we give the spline approximating setting associated with the differential equation–quadratic form problem by (1) and (2). The theoretical basis of these ideas is given in Ref. [2]. We ask the reader's indulgence while we introduce three more parameters and a "generalization" of the theory above with a "product" parameter $\eta = (\varepsilon, \sigma, \lambda, \mu)$.

Let $\mathcal{A}$ be the space of functions defined above, namely, the arcs $x(t)$ which are absolutely continuous with $x'(t)$ square integrable on $[\alpha, \gamma]$ such that $x(\alpha) = x(\gamma) = 0$ and norm given by

$$(x, y) = x(a)\, y(a) + \int_\alpha^\gamma x'(t)\, y'(t)\, dt.$$

Let Σ denote the set of real numbers of the form $\sigma = 1/n$ $(n = 1, 2, 3,...)$ and zero. The metric on Σ is the absolute value function. For $\sigma = 1/n$, define the partition

$$\pi(\sigma) = (a_0 = \alpha < a_1 < a_2 \cdots < a_n = \gamma),$$

where

$$a_k = k\frac{\gamma - a}{n} + a \qquad (k = 0, \ldots, n). \tag{4}$$

The space $\mathcal{O}(\sigma)$ is the set of continuous broken linear functions with vertices at $\pi(\sigma)$.

For each ε in $[\alpha, \gamma]$ let $\mathscr{H}(\varepsilon)$ denote the arcs $x(t)$ in $\mathcal{O}$ satisfying $x(a) = 0$ and $x(t) \equiv 0$ on $[\varepsilon, \gamma]$. Finally if $\eta = (\varepsilon, \sigma, \lambda, \mu)$ is an element in the metric space $\mathscr{M}(\eta) = [\alpha, \gamma] \times \Sigma \times \mathscr{R} \times \mathscr{R}$ with metric $d(\eta_1, \eta_2) = |\varepsilon_2 - \varepsilon_1| + |\sigma_2 - \sigma_1| + |\lambda_2 - \lambda_1| + |\mu_2 - \mu_1|$, where $\eta_i = (\varepsilon_i, \sigma_i, \lambda_i, \mu_i)$ for $i = 1, 2$, define $\mathscr{B}(\eta) = \mathcal{O}(\sigma) \cap \mathscr{H}(\varepsilon)$. Thus an arc $x(t)$ is in $\mathscr{B}(\eta)$ iff it is a spline function of degree 2 on $[\alpha, a_k]$, where $a_k \leqslant \varepsilon < a_{k+1}$, such that $x(a) = 0$ and $x(t) \equiv 0$ on $[a_k, \gamma]$.

Now that our Hilbert spaces are constructed we construct appropriate quadratic forms designated by $H(x; \eta) = H(x; \varepsilon, \sigma, \lambda, \mu)$ which are the approximating quadratic forms for (2). Thus define $p_\sigma(t) = p(a_k^*)$ if it is in $[a_k, a_{k+1})$, where $a_k^* = a_k + \sigma/2$ with similar definition of $l_\sigma(t)$, $q_\sigma(t)$, and $r_\sigma(t)$. For $\eta = (\varepsilon, \sigma, \lambda, \mu)$ let $H(x; \eta) = H(x, x; \eta)$, where

$$\begin{aligned} H(x, y; \eta) = &\int_\alpha^\gamma [p_\sigma(t)\, x'(t)\, y'(t) + l_\sigma(t)\, x(t)\, y(t)]\, dt \\ &- \lambda \int_\alpha^\gamma q_\sigma(t)\, x(t)\, t(t)\, dt - \mu \int_\alpha^\gamma r_\sigma(t)\, x(t)\, y(t)\, dt \end{aligned} \tag{5}$$

is defined for arcs $x(t)$, $y(t)$ in $\mathscr{M}(\eta)$. As above we define $s(\eta)$ and $n(\eta)$ to be the *signature* and *nullity* of the quadratic form $H(x; \eta)$ on the Hilbert space $\mathscr{M}(\eta)$.

The connection between $s(\eta)$, $n(\eta)$ and oscillation or conjugate points is now given. Let σ, λ and μ be given, a point ε at which $s(\varepsilon, \sigma, \lambda, \mu)$ is discontinuous is an *oscillation point* of $H(x; \varepsilon, \sigma, \lambda, \mu)$ *relative to* $\{\mathscr{H}(\varepsilon) : \varepsilon$ in $[a, b]\}$. In Ref. [2] we show that, as a consequence of a very general concept of approximation, the mth oscillation point $\varepsilon_m(\sigma, \lambda, \mu)$ is a continuous function for $m = 1, 2, 3, \ldots$ and $\varepsilon_m < \gamma$. Continuity is in the sense of the metric defined above. When $\sigma = 0$ we have the continuous problem given by (1) and (2) and our definition coincides with the usual definition of oscillation or conjugate points.

To numerically find $s(\varepsilon, \sigma, \lambda, \mu)$ for $\sigma \neq 0$ we follow the ideas in Ref. [4]. Choose $\sigma = 1/n$; $a_k = a + k/n$ for $k = 0, 1, 2, \ldots$;

$$z_k(t) = \begin{cases} 1 - n\,|t - a_k| & \text{if } t \text{ in } [a_{k-1}, a_{k+1}] \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

for $k = 1, 2, 3, \ldots$; and $x(t) = b_j z_j(t)$ (repeated indices are summed) in $\mathscr{B}(\eta)$ for $\eta = (\varepsilon, \sigma, \lambda, \mu)$. Note that $z_k(t)$ is the spline "hat" function and is the basis for our finite dimensional space. A straightforward calculation shows that $H(x; \eta) = b_i b_j e_{ij}(\eta) = B^T D(\eta) B$, where $B = (b_1, b_2, \ldots)^T$, $x = b_i z_i$ and $D(\eta)$ is a symmetric tridiagonal matrix "increasing" in ε so that the upper $k \times k$ submatrix of $D(a_{k+1}, \sigma, \lambda, \mu)$ is $D(a_k, \sigma, \lambda, \mu)$. The results of Theorem 3 are given in Theorems 2 and 3 of Ref. [4] with modification to this example (two additional parameters).

THEOREM 3. *The values $s(\varepsilon, \sigma, \lambda, \mu)$ and $n(\varepsilon, \sigma, \lambda, \mu)$ are respectively the number of negative and zero eigenvalues of the symmetric tridiagonal matrix $D(a_k, \sigma, \lambda, \mu)$, where $\sigma \neq 0$ and $a_k \leqslant \varepsilon < a_{k+1}$. The sum $s(a_{k+1}, \sigma, \lambda, \mu) + n(a_{k+1}, \sigma, \lambda, \mu)$ is the number of times the discrete solution $c(\sigma, \lambda, \mu)$, defined below, crosses the axis on the interval $(\alpha, a_{k+1}]$. There exists $\delta > 0$ such that if $|\varepsilon - a_{k+1}| + |\sigma - \sigma'| + |\lambda - \lambda'| + |\mu - \mu'| < \delta$ and a_{k+1} is not a conjugate point to $t = \alpha$, i.e., $x_0(t)$ a solution of* (1) *satisfying $x_0(\alpha) = x_0(a_{k+1}) = 0$ implies $x_0(t) \equiv 0$, the above sum is equal to* (i) *the number of oscillation points of* (1) *on (α, a_{k+1}),* (ii) *the sum $s(\varepsilon, \sigma', \lambda', \mu') + n(\varepsilon, \sigma', \lambda', \mu')$ and in particular* (iii) *the sum $s(\varepsilon, 0, \lambda, \mu) + n(\varepsilon, 0, \lambda, \mu)$.*

Our final effort will be to construct a finite dimensional approximation solution $c(\sigma, \lambda', \mu')$ to problem (1)–(2). That is if $x_0(t)$ is the solution to (1) and $c(\sigma, \lambda', \mu')$ is our approximate solution and if they are normalized to agree ar (say $t = a_1$) then

$$\int_\alpha^\gamma [c'(\sigma) - x_0'(t)]^2 \, dt \to 0 \qquad \text{as } \sigma \to 0, \lambda' \to \lambda, \mu' \to \mu. \tag{7}$$

This convergence is of course muc stronger than pointwise convergence.

Two steps are involved in this solution. The first is to construct the elements $e_{k,k}$ and $e_{k,k+1}$ of the symmetric tridiagonal matrix $D(\eta)$. The second is to give the Euler–Langrange equation of this matrix. This equation is the solution $c(\sigma, \lambda, \mu)$ referred to above.

A direct calculation in (5) leads to

$$\begin{aligned} e_{k,k} = H(z_k, z_k; \sigma, \lambda, \mu) &= \int_{a_{k-1}}^{a_{k+1}} [p_\sigma(t)\, z_k'^2(t) + l_\sigma(t)\, z_k^2(t)] \, dt \\ &\quad - \lambda \int_{a_{k-1}}^{a_{k+1}} q_\sigma(t)\, z_k^2(t) \, dt - \mu \int_{a_{k-1}}^{a_{k+1}} r_\sigma(t)\, z_k^2(t) \, dt \\ &= [p(a_{k-1}^*) + p(a_k^*)]/\sigma + \sigma/3[l(a_{k-1}^*) + l(a_k^*)] \\ &\quad - \lambda\sigma/3[q(a_{k-1}^*) + q(a_k^*)] - \mu\sigma/3[r(a_{k-1}^*) + r(a_k^*)], \end{aligned} \tag{8a}$$

$$\begin{aligned} e_{k,k+1} &= H(z_k, z_{k+1}; \sigma, \lambda, \mu) \\ &= \int_{a_{k=1}}^{a_{k+2}} [p_\sigma(t)\, z_k'(t)\, z_{k+1}'(t) + l_\sigma(t)\, z_k(t)\, z_{k+1}(t)]\, dt \\ &\quad - \lambda \int_{a_{k-1}}^{a_{k+2}} q_\sigma(t)\, z_k(t)\, z_{k+1}(t)\, dt \\ &\quad - \int_{a_{k-1}}^{a_{k+2}} r_\sigma(t)\, z_k(t)\, z_{k+1}(t)\, dt \\ &= -p(a_k^*)/\sigma + \sigma l(a_k^*)/6 - \lambda\sigma q(a_k^*)/6 - \mu\sigma r(a_k^*)/6. \end{aligned}$$

Finally, we show in [4] that for a given fixed value of λ and μ the finite dimensional approximation to the solution $x_0(t)$ in (1) is the vector $c(\sigma, \lambda, \mu) = c_i z_i(t)$ (repeated indices are summed) where the components c_i are defined recursively by

$$c_1 e_{11} + c_2 e_{12} = 0, \tag{9a}$$

$$c_1 e_{21} + c_2 e_{22} + c_3 e_{23} = 0, \tag{9b}$$

$$\vdots$$

$$c_{k-1} e_{k,k-1} + c_k e_{k,k} + c_{k+1} e_{k,k+1} = 0 \qquad kk = 3, 4, 5, \ldots). \tag{9c}$$

This vector $c(\sigma, \lambda, \mu)$ is the vector satisfying the limiting relationship in (7). In practice if the coefficient functions $p(t)$, $l(t)$, $q(t)$, $r(t)$ are at all "nice" our algorithm is easy to apply and converges quickly. This is due to the fact that we approximate an integration process using (2) and not a difference using (1). Furthermore for each choice of σ the values $p_\sigma(a_k^*)$, etc., need only be computed once in our two dimension iteration scheme unlike the case of differential equations. This results in relatively little computation time and allows us to compute all numerical eigensolutions $\lambda_n(\sigma)$, $\mu_n(\sigma)$ with one computation of $p_\sigma(a_k^*)$, etc.

IV. The Iteration Scheme

In this section we give a two dimensional iteration scheme which allows us to find $c(\sigma, \lambda, \mu)$ under the assumption that $p(t) > 0$, $q(t) \geqslant 0$, $r(t) \geqslant 0$. The condition $p(t) > 0$ is necessary to avoid singular theory; the nonnegative of $q(t)$ and $r(t)$ may be obtained by rewriting our equations slightly as in our example below and is not a requirement in our original problem. More precisely we find $c_{m,m'}(\sigma)$, where m and m' are (described above) respectively the number of crossings of $c(\sigma, \lambda, \mu)$ on the intervals (α, β) and (β, γ), respectively.

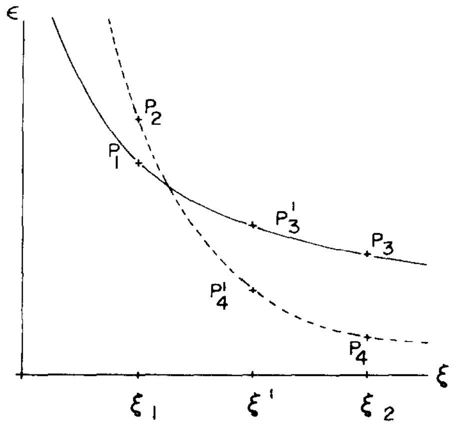

FIGURE 3

Our innermost subroutine computes a solution $c(\sigma, \lambda, \mu)$ from (9). If this solution "crosses" the axis "exactly" m times in the interval (α, β), $m+1$ times in the interval $(\alpha, \beta]$, $j = m + m' + 1$ times in (α, γ), and $j+1$ times in $(\alpha, \gamma]$ we are done. Call this solution $c_{m,m'}(\sigma)$. The word "exactly" means the crossing is within a predetermined ε-neighborhood of β and γ. If both λ and μ are too large the resulting solution $c(\sigma, \lambda, \mu)$ "crosses" too soon or is to the left of $c_{m,m'}(\sigma)$ and must be shifted to the right by decreasing λ and μ. Similarly if λ and μ are too small the curve $c(\sigma, \lambda, \mu)$ is to the right of $c_{m,m'}(\sigma)$ and must be shifted to the left by increasing λ and μ.

The second most inner loop is the single eigenvalue problem done twice, i.e., for given λ find $\mu_1 = \mu_1(\lambda)$ and $\mu_2 = \mu_2(\lambda)$, where μ_1 is the solution to the eigenvalue problem on (α, β) with m crossings and μ_2 is the solution of the eigenvalue problem on (α, γ) with $j = m + m' + 1$ crossings. This enables us to find the points P_1 and P_2 in Fig. 3. We assume without loss of generality that $\Delta\mu(\lambda_1) = \mu_2(\lambda_1) - \mu_1(\lambda_1) > 0$ and $\Delta\mu(\lambda_2) = \mu_2(\lambda_2) - \mu_1(\lambda_2) < 0$ have been found as in Fig. 3. Choosing $\lambda' = (\lambda_1 + \lambda_2)/2$ we compute $\Delta\mu(\lambda') = \mu_2(\lambda') - \mu_1(\lambda')$. If $|\Delta\mu(\lambda')| < \hat{\varepsilon}$ where $\hat{\varepsilon}$ is prescribed we are done. Otherwise if $\Delta\mu(\lambda')\,\Delta\mu(\lambda_1) < 0$ we set $\lambda_2 = \lambda'$ (if $\Delta\mu(\lambda')\,\Delta\mu(\lambda_1) > 0$ we set $\lambda = \lambda'$) and repeat this process. At each step the interval $[\lambda_1, \lambda_2]$ is halved. This process converges to the desired solution.

THEOREM 4. *The algorithm described above converges to a numerical solution $c_{m,m'}(\sigma)$ which is an eigenvector of* (1) *or* (2) *corresponding to the double eigenvalue $(\lambda, \mu(\lambda))$ found above. The numerical solution $c_{m,m'}(\sigma)$ is generated by* (9) *and satisfies the convergence criteria given by* (7).

V. A Numerical Example

In this section we consider the numerical example described in [1], namely,

$$x'' + (\lambda + \mu \cos t + e^t)x = 0, \qquad x(0) = x(2) = x(4) = 0 \tag{1'}$$

or

$$\begin{aligned} H(x;\lambda,\mu) = \int_0^4 (x'^2 - e^t x^2)\,dt - \lambda \int_0^4 x^2\,dt \\ - \mu \int_0^4 (\cos t)\,x^2\,dt. \end{aligned} \tag{2'}$$

That is, $\alpha = 0$, $\beta = 2$, $\gamma = 4$; $p(t) \equiv 1$, $l(t) = -e^t$, $q(t) \equiv 1$, $r(t) = \cos t$. Note that $r(t) \geqslant 0$ is not satisfied on $[0, 4]$. To correct this "deficiency" and to obtain (only) positive values of "λ and μ" for convenience we rewrite (1′) as

$$x'' + [(\lambda - \mu + 60) + \mu(1 + \cos t) + (e^t - 60)]x = 0 \tag{1''}$$

and

$$\begin{aligned} \bar{H}(x;\bar{\lambda},\bar{\mu}) = \int_0^4 [x'^2 + (60 - e^t)\,x^2]\,dt \\ - \bar{\lambda}\int_0^4 x^2\,dt - \bar{\mu}\int_0^4 (1 + \cos t)\,x^2\,dt, \end{aligned} \tag{2''}$$

where $\bar{\lambda} = \lambda - \mu + 60$, $\bar{\mu} = \mu$. Note that $\bar{H}(x;0,0)$ is positive definite as is $\bar{H}(x;\bar{\lambda},\bar{\mu})$ for $\bar{\lambda} \leqslant 0$ and $\bar{\mu} \leqslant 0$. The procedure described in Section IV yields $\bar{\lambda}$, $\bar{\mu}$ and hence $\lambda = \bar{\lambda} + \bar{\mu} - 60$, $\mu = \bar{\mu}$ with the same eigenvector solution.

The table below gives the values of λ and μ corresponding to m and m'.

m \ m'	0	1	2
0	−10.8034 17.1705	−4.6203 7.8785	−0.2913 −0.7143
1	−4.3164 24.3671	0.6342 13.6812	4.4521 4.9594
2	4.6384 34.3424	9.1117 23.0994	12.5428 14.6673

The values for $m = 0$, $m' = 1$ are given in [1] as $\lambda = -4.6204$, $\mu = 7.8787$; our values are $\lambda = -4.6203$, $\mu = 7.8785$. Additionally our eigenvector for these values crosses the axis (using linear interpolation) at $t_\beta = 1.99996$ and $t_\gamma = 3.999997$. For other values of m and m', Ref. [1] only gives answers to the nearest hundreds. We agree with their answers in all cases.

References

1. L. Fox, L. Hayes, and D. F. Mayers, The double eigenvalue problem, *in* "Topics in Numerical Analysis, Proceedings of the Royal Irish Academy Conference on Numerical Analysis, 1972 (John J. H. Miller, Ed.), pp. 93–112, Academic Press, New York, 1973.
2. J. Gregory, Generalized Fredholm quadratic forms and integral differential equations of the second kind, *J. Math. Anal. Appl.* **70** (1979), 120–130.
3. J. Gregory, Numerical algorithms for oscillation vectors of second order differential equations including the Euler–Lagrange equation for symmetric tridiagonal matrices, *Pacific J. Math.* **76**, No. 2 (1978), 397–406.
4. J. Gregory and R. Wilkerson, New numerical methods for eigenvalues and eigenvectors of second order differential equations, *in* "Applicable Analysis," in press.