

01 Jan 1994

## Order-Recursive Fir Smoothers

Jenq Tay Yuan

John A. Stuller

Missouri University of Science and Technology, [stuller@mst.edu](mailto:stuller@mst.edu)

Follow this and additional works at: [https://scholarsmine.mst.edu/ele\\_comeng\\_facwork](https://scholarsmine.mst.edu/ele_comeng_facwork)



Part of the [Electrical and Computer Engineering Commons](#)

---

### Recommended Citation

J. T. Yuan and J. A. Stuller, "Order-Recursive Fir Smoothers," *IEEE Transactions on Signal Processing*, vol. 42, no. 5, pp. 1242 - 1246, Institute of Electrical and Electronics Engineers, Jan 1994.

The definitive version is available at <https://doi.org/10.1109/78.295191>

This Article - Journal is brought to you for free and open access by Scholars' Mine. It has been accepted for inclusion in Electrical and Computer Engineering Faculty Research & Creative Works by an authorized administrator of Scholars' Mine. This work is protected by U. S. Copyright Law. Unauthorized use including reproduction for redistribution requires the permission of the copyright holder. For more information, please contact [scholarsmine@mst.edu](mailto:scholarsmine@mst.edu).

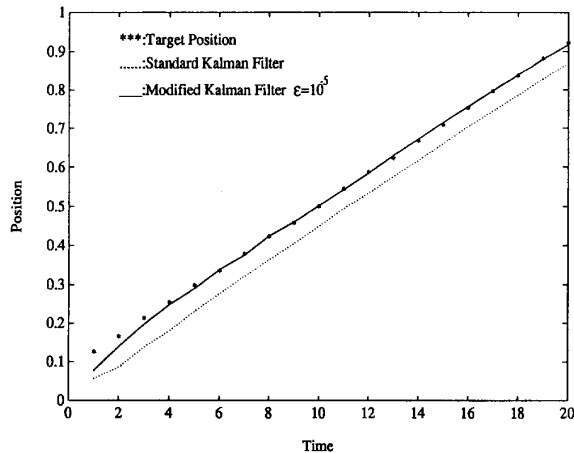


Fig. 2. Comparison of the standard and modified Kalman filter position trajectories for the conditions described in (29).

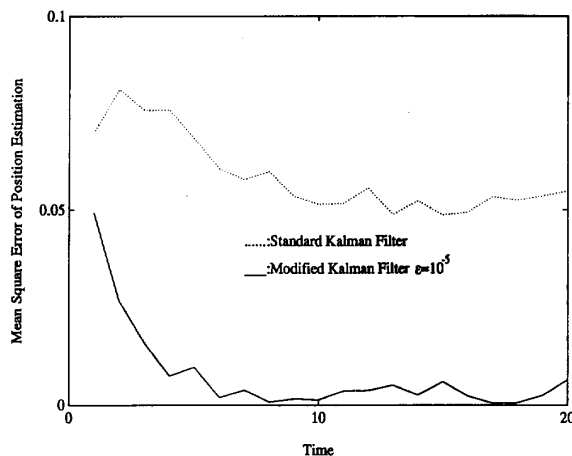


Fig. 3. Comparison of the standard and modified Kalman filter position estimation errors for the conditions described in (29).

not fully characterized. The algorithm includes a mismatch function that provides a modified value for the predicted state of the standard Kalman filter.

#### ACKNOWLEDGMENT

The authors would like to express their gratitude to the anonymous reviewers for many valuable comments.

#### REFERENCES

- [1] R. E. Kalman, "A new approach to linear filtering and prediction problems," *J. Basic Eng., Trans. ASME*, pp. 35–45, Mar. 1960.
- [2] C. K. Chui and G. Chen, *Kalman Filtering with Real-Time Applications*. New York: Springer-Verlag, 1987.
- [3] L. Ljung, "Asymptotic behavior of the extended Kalman filter as a parameter estimator for linear systems," *IEEE Trans. Automat. Contr.*, vol. 24, no. 1, pp. 36–50, Feb. 1979.

- [4] S. Haykin, *Adaptive Filter Theory*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1991.
- [5] Q. Wei, *Mathematics Programming and Optimization Design*. China Industry Press, 1984.
- [6] E. G. Philip and W. Murray, *Practical Optimization*. London: Academic, 1981.

## Order-Recursive FIR Smoothers

Jenq-Tay Yuan and John A. Stuller

**Abstract**—This paper introduces order-recursive FIR smoothers and shows that order-recursive FIR filters are special forms that occur when no future data values are used to estimate the signal. The formulation leads naturally to generalizations of the concepts of prediction-error basis and Cholesky factorization which are well known in FIR filter design.

### I. INTRODUCTION

Lattice realizations of FIR filters offer well-known advantages over tapped-delay-line realizations [1]–[5]. The fundamental property of the lattice realization of an FIR filter is that optimum higher order filters can be built from lower-order ones by simply adding on more lattice stages, leaving the original stages unchanged. This is called the *order-recursive property* [6]. An implication of the order-recursive property is that an  $N$ -stage lattice filter automatically generates all  $N$  of the outputs that would be provided from  $N$  separate tapped-delay-line filters of length  $1, 2, \dots, N$ .

We distinguish between the terms *smoother* and *filter*, ([7], pp. 156), ([8], pp. 273): An FIR filter estimates the present value of a signal sequence,  $x(n)$ , using a linear combination of only the present and previous values of a data sequence,  $y(n)$ . An FIR *smoother* noncausally estimates the present value of a signal sequence,  $x(n)$ , using a linear combination of the present, past and future values of a data sequence or observations,  $y(n)$ . Our interest in smoothers is motivated by the fact that their performance is potentially superior to filters ([8], pp. 279).

This paper introduces order-recursive FIR smoothers and shows that order-recursive FIR filters are special forms that occur when no future data values are used to estimate the signal. The formulation leads naturally to generalizations of the concepts of prediction-error basis and Cholesky factorization which are well known in FIR filter design.

### II. MMSE FIR SMOOTHERS

We consider an  $N$ th-order FIR smoother for which the signal is estimated from the current and  $p$ -past observations and  $f$ -future data

Manuscript received May 20, 1992; revised May 31, 1993. The associate editor coordinating the review of this paper and approving it for publication was D. Robert A. Gabel. This paper is based on work performed by J. T. Yuan in partial fulfillment of the Doctor of Philosophy degree at the University of Missouri-Rolla, Rolla, Missouri, USA.

J.-T. Yuan is with the Department of Electrical Engineering, Fu Jen Catholic University, Taipei, Taiwan, R.O.C.

J. A. Stuller is with the Department of Electrical Engineering, the University of Missouri-Rolla, Rolla, MO, USA 65401.

IEEE Log Number 9216666.

samples, where  $p > 0$ ,  $f > 0$  and

$$N = p + f. \quad (1)$$

We will occasionally refer to an  $N^{\text{th}}$  order smoother as a  $(p, f)$ th-order smoother where (1) is understood implicitly. The smoother output is the signal estimate

$$\hat{x}_{p,f}(n) = \sum_{i=-p}^f h_{(p,f),i} y(n+i) \quad (2)$$

where  $h_{(p,f),i}$  is the  $i$ th tap coefficient of order  $(p, f)$ . The coefficients  $h_{(p,f),i}$ ,  $-p \leq i \leq f$ , are chosen to minimize the mean squared value of the estimation error  $e_{p,f}(n) = x(n) - \hat{x}_{p,f}(n)$ . Insight to this minimization problem can be obtained by embedding it in a Hilbert space ([9], pp. 20). Here, the data or observations,  $y(n)$  and the signal,  $x(n)$ , will be assumed to be jointly stationary, zero mean random sequences. The Hilbert space,  $H$ , consists of all quadratic mean limits of sequences of finite linear combinations of the data and signal sequences. The inner product  $(u, v)$  of any two elements,  $u, v$ , in  $H$  is given by the expectation  $E\{uv\}$ . The squared norm of  $u$ ,  $\|u\|^2$ , is the mean square value of  $u$ ,  $E\{u^2\}$ . We define  $Y_{p,f}(n)$  as the Hilbert subspace of  $H$  generated by all linear combinations of the random variables  $y(n-p), y(n-p+1), \dots, y(n+f)$ , and use brackets  $[ \ ]$  to denote this subspace:  $Y_{p,f}(n) = [y(n-p), y(n-p+1), \dots, y(n+f)]$ .

To find the minimum mean square error (MMSE) estimate  $\hat{x}_{p,f}(n)$ , we apply the *Hilbert space projection theorem* ([9], pp. 26), which states that there is a unique element  $\hat{x}_{p,f}(n) \in Y_{p,f}(n)$  that minimizes  $\|x(n) - \hat{x}_{p,f}(n)\|$ . The element  $\hat{x}_{p,f}(n)$ , called the *orthogonal projection* of  $x(n)$  into  $Y_{p,f}(n)$ , is characterized by the requirement that  $e_{p,f}(n) = x(n) - \hat{x}_{p,f}(n)$  is orthogonal to  $Y_{p,f}(n)$ . An equivalent requirement is  $E\{(x(n) - \hat{x}_{p,f}(n))y(n+i)\} = 0$  for  $-p \leq i \leq f$ . The substitution of (2) into this equivalent requirement yields all but the last row of the following *augmented normal matrix equation*

$$\begin{bmatrix} r_{yy}(0) & r_{yy}(1) & & r_{yy}(N) & r_{yx}(f) \\ r_{yy}(1) & r_{yy}(0) & \dots & r_{yy}(N-1) & r_{yx}(f-1) \\ \vdots & \vdots & & \vdots & \vdots \\ r_{yy}(N) & r_{yy}(N-1) & \dots & r_{yy}(0) & r_{yx}(-p) \\ r_{yx}(f) & r_{yx}(f-1) & & r_{yx}(-p) & r_x(0) \end{bmatrix} \times \begin{bmatrix} -h_{(p,f),f} \\ -h_{(p,f),f-1} \\ \vdots \\ -h_{(p,f),2} \\ -h_{(p,f),1} \\ -h_{(p,f),0} \\ -h_{(p,f),-1} \\ -h_{(p,f),-2} \\ \vdots \\ -h_{(p,f),-p} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ E_{p,f,\min} \end{bmatrix} \quad (3a)$$

where  $r_{yy}(k) = E\{y(n+k)y(n)\}$  is the autocorrelation function of observations  $y(n)$  and  $r_{yx}(k) = E\{y(n+k)x(n)\}$  is the cross-correlation function of  $y(n)$  and  $x(n)$ . The tap coefficients satisfying the normal equations (given by the top  $N+1$  rows of (3a)) are the optimum tap coefficients sought. In the last row of (3a),  $E_{p,f,\min}$  is the minimum mean square (MMS) estimation (smoothing) error and  $r_x(0) = E\{x^2(n)\}$ . Using the obvious definitions, we can write (3a) compactly as

$$\mathbf{R}'_{p,f} \begin{bmatrix} -\mathbf{h}_{p,f} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N+1} \\ E_{p,f,\min} \end{bmatrix}. \quad (3b)$$

### III. ORDER-RECURSIVE FIR SMOOTHERS

The first step in the development of an order-recursive realization for (2) is to increase the order from  $N$  to  $N+1$ . Clearly, this may be done by increasing *either*  $f$  or  $p$  in (2) by one. We consider first the problem of increasing  $N$  by using one additional *future* data sample,  $y(n+f+1)$ . By the Hilbert space projection theorem, the MMSE estimate of  $\hat{x}_{p,f+1}(n)$  of  $x(n)$  is the orthogonal projection of  $x(n)$  onto  $Y_{p,f+1}(n) = [y(n-p), y(n-p+1), \dots, y(n+f), y(n+f+1)]$ . The augmented normal equations are

$$\mathbf{R}'_{p,f+1} \begin{bmatrix} -\mathbf{h}_{p,f+1} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N+2} \\ E_{p,f+1,\min} \end{bmatrix} \quad (4)$$

where  $E_{p,f+1,\min}$  is the MMS estimation error of the  $(p, f+1)$ st-order FIR smoother. It may be confirmed from (3) and (4) that the tap coefficients satisfy the recursion ([11], pp. 132)

$$\begin{bmatrix} -\mathbf{h}_{p,f+1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathbf{h}_{p,f} \\ 1 \end{bmatrix} - \frac{k_{p,f+1}^F}{P_{N+1}^F} \begin{bmatrix} \mathbf{a}_{N+1} \\ 0 \end{bmatrix}. \quad (5)$$

In (5)

$$k_{p,f+1}^F = r_{yx}(f+1) - \sum_{i=p}^f h_{(p,f),i} r_{yy}(f+1-i) \quad (6)$$

the vector  $\mathbf{a}_{N+1}^T = (1, a_{N+1,1}, a_{N+1,2}, \dots, a_{N+1,N+1})$  is the vector of tap coefficients in an  $(N+1)$ st order prediction error filter ([4], pp. 202) and  $P_{N+1}^F$  is the MMS prediction error. Superscript  $T$  denotes transpose of a vector or matrix. In general, for each  $m = 0, 1, 2, \dots$ , the  $m$ th-order prediction coefficient-vector  $\mathbf{a}_m$  minimizes the mean square value of the  $m$ th order *forward prediction error*

$$e_m^F(n) = y(n) - \hat{y}_m^F(n) = \sum_{i=0}^m a_{m,i} y(n-i) \quad (7)$$

where  $\hat{y}_m^F(n) \triangleq -\sum_{i=1}^m a_{m,i} y(n-i)$  and  $a_{m,0} = 1$ . The Levinson recursion ([9], pp. 49), ([4], pp. 202) provides a means to compute  $\mathbf{a}_{m+1}$  recursively from  $\mathbf{a}_m$ .

We can obtain an order recursion for the *delayed* signal estimate by premultiplying both sides of (5) by the row vector  $(y(n), y(n-1), \dots, y(n-N-1), 0)$ . The result is

$$\hat{x}_{p,f+1}(n-f-1) = \hat{x}_{p,f}(n-f-1) + g_{p,f+1}^F e_{N+1}^F(n) \quad (8)$$

where  $g_{p,f+1}^F = P_{N+1}^{-1} \rho_{p,f+1}^F$  with

$$\begin{aligned} \rho_{p,f+1}^F &= E\{x(n-f-1)e_{N+1}^F(n)\} \\ &= \sum_{i=0}^{N+1} a_{N+1,i} r_{yx}(f+1-i). \end{aligned} \quad (9)$$

We show in the appendix that  $k_{p,f+1}^F = \rho_{p,f+1}^F$ . The delayed  $(N+1)$ st-order estimate,  $\hat{x}_{p,f+1}(n-f-1)$ , in (8) is the projection of  $x(n-f-1)$  onto  $Y_{p,f+1}(n-f-1) = [y(n-N-1), y(n-N), \dots, y(n-1), y(n)]$  and the delayed  $N$ th-order estimate,  $\hat{x}_{p,f}(n-f-1)$ , is the projection of  $x(n-f-1)$  onto  $Y_{p,f}(n-f-1) = [y(n-N-1), y(n-N), \dots, y(n-1)]$ . Consider now the last term in (8). Since  $\hat{y}_{N+1}^F(n)$  is the MMSE estimate of  $y(n)$  from  $y(n-1), y(n-2), \dots, y(n-N-1)$ , then  $e_{N+1}^F(n)$  is orthogonal to  $Y_{p,f}(n-f-1)$ . It follows that  $E\{\hat{x}_{p,f}(n-f-1)e_{N+1}^F(n)\} = 0$ . Notice that the  $f+1$  units of delay in (8) make  $\hat{x}_{p,f+1}(n-f-1)$  a causal estimate of  $x(n-f-1)$ .

We can obtain an order recursion for the MMS smoothing error  $E_{p,f+1,\min}$ , by subtracting  $x(n-f-1)$  from both sides of (8) and

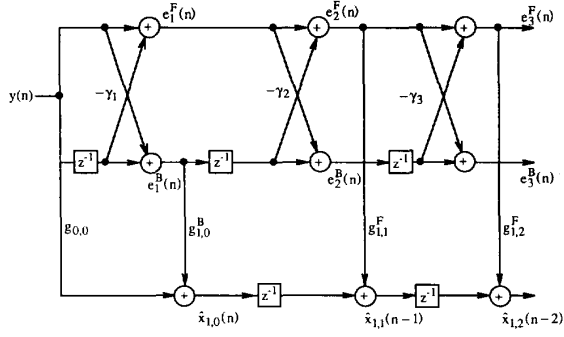


Fig. 1. BFF realization of an order (1, 2) smoother.

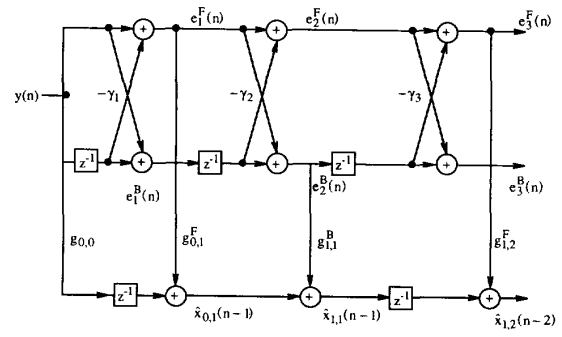


Fig. 2. FBF realization of an order (1, 2) smoother.

taking the mean square. This yields

$$E\{[x(n-f-1) - \hat{x}_{p,f+1}(n-f-1)]^2\} \\ = E\left\{\left[x(n-f-1) - \hat{x}_{p,f}(n-f-1) - g_{p,f+1}^F e_{N+1}^F(n)\right]^2\right\} \quad (10)$$

which reduces to

$$E_{p,f+1,\min} = E_{p,f,\min} - \left(\rho_{p,f+1}^F\right)^2 P_{N+1}^{-1}. \quad (11)$$

To conclude the development of the order-recursive lattice realization of a smoother, we consider the problem of increasing the order of (2) by using one additional *past* sample,  $y(n-p-1)$ . This yields, in a manner similar to the preceding development

$$\begin{bmatrix} 1 \\ -\mathbf{h}_{p+1,f} \end{bmatrix} = \begin{bmatrix} 1 \\ -\mathbf{h}_{p,f} \\ 0 \end{bmatrix} - \frac{\rho_{p+1,f}^B}{P_{N+1}} \begin{bmatrix} 0 \\ \mathbf{a}_{N+1}^R \end{bmatrix} \quad (12)$$

and

$$\hat{x}_{p+1,f}(n-f) = \hat{x}_{p,f}(n-f) + g_{p+1,f}^B e_{N+1}^B(n) \quad (13)$$

where  $g_{p+1,f}^B = P_{N+1}^{-1} \rho_{p+1,f}^B$  with

$$\rho_{p+1,f}^B = E\{x(n-f)e_{N+1}^B(n)\} \\ = \sum_{i=0}^{N+1} a_{N+1,N+1-i} r_{yx}(f-i) \quad (14)$$

and  $\mathbf{a}_{N+1}^{RT} = (a_{N+1,N+1}, a_{N+1,N}, \dots, a_{N+1,1}, 1)$ . The quantity  $e_{N+1}^B(n)$  appearing in (13) is the  $(N+1)$ st order *backward prediction error*. The  $m$ th-order backward prediction error is given by

$$e_m^B(n) = y(n-N-1) - \hat{y}_m^B(n-N-1) \\ = \sum_{i=0}^m a_{m,i} y(n-m-1+i) \quad (15)$$

for  $m = 0, 1, 2, \dots$ , where  $\hat{y}_m^B(n-m) = -\sum_{i=1}^m a_{m,i} y(n-m+i)$ . The terms in (13) satisfy  $E\{\hat{x}_{p,f}(n-f)e_{N+1}^B(n)\} = 0$ . The MMSE smoothing error is found by subtracting  $x(n-f)$  from both sides of (13) and taking the mean square. This yields

$$E_{p+1,f,\min} = E_{p,f,\min} - \left(\rho_{p+1,f}^B\right)^2 P_{N+1}^{-1}. \quad (16)$$

We are now prepared to construct order-recursive realizations for a MMSE FIR smoother. The forward and backward prediction errors  $e^F$  and  $e^B$  appearing in the recursions of (8) and (13) are all directly accessible from a prediction error lattice [12], [13]. To obtain a MMSE smoother of order  $(p, f)$ , (8) and (13) must be applied  $p$

and  $f$  times respectively, but any sequencing between these two equations is permissible. Consequently, the number of permissible lattice realizations of an order  $(p, f)$  smoother is equal to the number of length  $N = p + f$  sequences of  $p$  letters  $B$  and  $f$  letters  $F$ , namely;  $C_p^N = N!/p!f!$ .

We show two of the three possible lattice realizations of an order (1, 2) smoother in Figs. 1 and 2. The sequences of B's and F's in the figure titles denote the sequencing of (8) and (13) corresponding to each realization. Clearly, the smoother lattice structures have the order-recursive property: To increase the order of the smoother  $N \rightarrow N + 1$  by using one more *future* data value, one adds a new stage to the prediction error lattice and applies (8). To increase the order of the smoother using one more *past* data value, one adds a new stage to the prediction error lattice and applies (13). Either way, the original  $N$ th order smoother stages remain as is. Notice that each lattice realization automatically generates all the lower order smoothed estimates associated with its particular sequence of B's and F's.

In the preceding development, we assumed that  $p > 0$  and  $f > 0$  to conform to the definition of a smoother. The same development however, also applies to the *filtering* problem with  $f = 0$ . If we set  $f = 0$ , we obtain the well-known lattice realization of an FIR Wiener filter (see p. 251 of [4]). This realization is unique because there is only  $C_N^N = 1$  sequence of  $N = p$  B's and  $f = 0$  F's. We find, therefore, that the solution of any MMS smoothing or filtering problem can be embedded in the solution for the linear prediction error lattice and realized as an order-recursive lattice structure.

#### IV. ORTHOGONAL BASES FOR $N$ th-ORDER SMOOTHING AND FILTERING

In this section we describe the orthogonal basis sets that are directly accessible from an  $N$ -stage prediction-error lattice for use in order-recursive FIR smoothers and filters.

To develop insight, consider Table I which depicts the evolution of the Hilbert subspaces in the lattice for the (1, 2) smoother of Fig. 1. The top row in the table denotes the order of the estimate as the data progresses deeper into the lattice. The second row gives the sequence of B's and F's associated with the developing smoother. The third row gives the values  $(p, f)$  associated with the sequence of B's and F's. In the fourth row, the smoothed estimates having order 0, 1, 2, 3 are evaluated by means of (8) and (13). In the bottom row, the Hilbert subspaces are described both in terms of the data basis and in terms of a forward and backward prediction-error basis. It is easily verified that each prediction-error basis set shown is orthogonal. Notice that a total of  $C_1^3 = 3$  tables like that of Table I can be constructed for a (1, 2) order-recursive smoother because there are exactly three

TABLE I  
EVOLUTION OF HILBERT SUBSPACES

Order $N$	0	1	2	3
Sequence	B	F	F	F
$(p, f)$	(0, 0)	(1, 0)	(1, 1)	(1, 2)
$\hat{x}_{p,f}$	$\hat{x}_{0,0}(n)$ = $g_{0,0}y(n)$ = $g_{0,0}e_0(n)$	$\hat{x}_{1,0}(n)$ = $\hat{x}_{0,0}(n)$ + $g_{1,0}^B e_1^B(n)$ = $g_{0,0}e_0(n)$ + $g_{1,0}^B e_1^B(n)$	$\hat{x}_{1,1}(n-1)$ = $\hat{x}_{1,0}(n-1) + g_{1,1}^F e_2^F(n)$ = $g_{0,0}e_0(n-1)$ + $g_{1,0}^B e_1^B(n-1)$ + $g_{1,1}^F e_2^F(n)$	$\hat{x}_{1,2}(n-2)$ = $\hat{x}_{1,1}(n-2) + g_{1,2}^F e_3^F(n)$ = $g_{0,0}e_0(n-2)$ + $g_{1,0}^B e_1^B(n-2)$ + $g_{1,1}^F e_2^F(n-1)$ + $g_{1,2}^F e_3^F(n)$
$Y_{p,f}$	$Y_{0,0}(n)$ = $[y(n)]$ = $[e_0(n)]$	$Y_{1,0}(n)$ = $[y(n-1), y(n)]$ = $[e_0(n), e_1^B(n)]$	$Y_{1,1}(n-1)$ = $[y(n-2), y(n-1), y(n)]$ = $[e_0(n-1), e_1^B(n-1), e_2^F(n)]$	$Y_{1,2}(n-2)$ = $[y(n-3), y(n-2), y(n-1), y(n)]$ = $[e_0(n-2), e_1^B(n-2), e_2^F(n-1), e_3^F(n)]$

TABLE II  
BASIS SETS FOR  $Y_{2,2}(n-2)$

1	$\{e_0(n-2), e_1^B(n-2), e_2^B(n-2), e_3^F(n-1), e_4^F(n)\}$
2	$\{e_0(n-2), e_1^F(n-1), e_2^F(n), e_3^B(n), e_4^B(n)\}$
3	$\{e_0(n-2), e_1^F(n-1), e_2^B(n-1), e_3^F(n), e_4^B(n)\}$
4	$\{e_0(n-2), e_1^B(n-2), e_2^F(n-1), e_3^B(n-1), e_4^F(n)\}$
5	$\{e_0(n-2), e_1^F(n-1), e_2^B(n-1), e_3^B(n-1), e_4^F(n)\}$
6	$\{e_0(n-2), e_1^B(n-2), e_2^F(n-1), e_3^F(n), e_4^B(n)\}$

sequences of 1 B and 2 F's. Therefore, there are exactly three sets of orthogonal bases directly available from the lattice which can be used to construct a (1, 2) order-recursive smoother.

By generalizing the construction illustrated in Table I, one can show that there are  $C_p^N$  orthogonal basis sets for  $Y_{p,f}(n-f) = [y(n-N), y(n-N+1), \dots, y(n)]$  that are directly accessible from a prediction error lattice of length  $N$  for a  $(p, f)$  order-recursive smoother. Each set of bases errors has the form

$$\{e_0(n-i_0), e_1^{X_1}(n-i_1), \dots, e_{N-1}^{X_{N-1}}(n-i_{N-1}), e_N^{X_N}(n)\} \quad (17)$$

where  $X_1, X_2, \dots, X_N$  is one of the  $C_p^N$  possible sequences of  $p$  B's and  $f$  F's. In (17), the indices  $i_j$ , equal the number of F's in the sequence  $X_{j+1}, X_{j+2}, \dots, X_N$  for  $j = 0, 1, \dots, N-1$ . If we denote the number of F's and B's in a sequence  $v$  as  $N_F(v)$  or  $N_B(v)$ , respectively, we have  $i_j = N_F(X_{j+1}, X_{j+2}, \dots, X_N)$  for  $0 \leq j \leq N-1$ . We will refer to this result somewhat grandly as the *orthogonal basis theorem*.

To illustrate (17), we note that there are  $C_2^4 = 6$  orthogonal basis sets directly accessible from a prediction error lattice of length 4 for use in a (2, 2) order-recursive smoother. The basis sets are listed in Table II.

A second illustration of (17) is provided by the set of backward prediction errors  $\{e_0(n), e_1^B(n), \dots, e_N^B(n)\}$ . This orthogonal basis for  $Y_{N,0}(n)$  is unique for use in an order-recursive FIR filter ( $C_N^N = 1$ ).

#### V. GENERALIZED GRAM-SCHMIDT ORTHOGONALIZATION

Each orthogonal basis set of the form in (17) can be obtained from the data  $y(n), y(n-1), \dots, y(n-N)$  by a Gram-Schmidt procedure.

Consider the basis for  $Y_{1,1}(n-1)$  which appears in the  $N = 2$  column of Table I:  $Y_{1,1}(n-1) = [e_0(n-1), e_1^B(n-1), e_2^F(n)]$ . If we arrange the basis elements in a vector and apply (7) and (15), we obtain

$$\begin{bmatrix} e_0(n-1) \\ e_1^B(n-1) \\ e_2^F(n) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_{1,1} & 1 & 0 \\ a_{2,1} & a_{2,2} & 1 \end{bmatrix} \begin{bmatrix} y(n-1) \\ y(n-2) \\ y(n) \end{bmatrix} \quad (18)$$

which, with the obvious definitions, can be written as  $\mathbf{e}_{BF}(n-1) = \mathbf{L}_{BF} \mathbf{y}_{BF}(n)$ . Since  $\mathbf{L}_{BF}$  is lower triangular, the elements in  $\mathbf{e}_{BF}(n-1)$  are indeed obtainable using a Gram-Schmidt procedure provided that the data is selected in the order of the vector  $\mathbf{y}_{BF}(n)$  (from top to bottom).

In general, we can write

$$\mathbf{e}_{x_1, x_2, \dots, x_N}(n-f) = \mathbf{L}_{x_1, x_2, \dots, x_N} \mathbf{y}_{x_1, x_2, \dots, x_N}(n) \quad (19)$$

where the elements in  $\mathbf{e}_{x_1, x_2, \dots, x_N}(n-f)$  are obtained from the orthogonal basis theorem and  $\mathbf{L}_{x_1, x_2, \dots, x_N}$  is lower triangular. The data vector in (19) is given by

$$\mathbf{y}_{x_1, x_2, \dots, x_N}^T(n) = (y(n-f), y(n-f+i_1), y(n-f+i_2), \dots, y(n-f+i_N))$$

with  $i_j = N_F(X_1, X_2, \dots, X_j)$  if  $X_j = F$  and  $i_j = -N_B(X_1, X_2, \dots, X_j)$  if  $X_j = B$ , where  $1 \leq j \leq N$ .

#### VI. GENERALIZED CHOLESKY-FACTORIZATION

A generalization of Cholesky-factorization can also be accomplished using any basis set having the form in (17) if we rearrange the Gram-Schmidt equations (19) so the data vector appears in the conventional form  $\mathbf{y}_N^T(n) = (y(n), y(n-1), \dots, y(n-N))^T$ . For example, the Gram-Schmidt equations (18) rearrange to

$$\begin{bmatrix} e_2^F(n) \\ e_0(n-1) \\ e_1^B(n-1) \end{bmatrix} = \begin{bmatrix} 1 & a_{2,1} & a_{2,2} \\ 0 & 1 & 0 \\ 0 & a_{1,1} & 1 \end{bmatrix} \begin{bmatrix} y(n) \\ y(n-1) \\ y(n-2) \end{bmatrix} \quad (20)$$

or  $\mathbf{e}_{(2,0,1)}(n-1) = \mathbf{C}_{(2,0,1)} \mathbf{y}_3(n)$ . Notice that the matrix  $\mathbf{C}_{(2,0,1)}$  appearing in (20) is not triangular, and the elements in the vector  $\mathbf{e}_{(2,0,1)}(n-1)$  develop bidirectionally upwards and downwards as the dimensionality increases. This development can be understood by noting that  $n-1$  is the "time of the smoothed estimate,"  $\hat{x}_{1,1}(n-1)$

