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GROMOV-WITTEN INVARIANTS OF $\operatorname{Sym}^d \mathbb{P}^r$

ROB SILVERSMITH

Abstract. We give a graph-sum algorithm that expresses any genus-g Gromov-Witten invariant of the symmetric product orbifold $\operatorname{Sym}^d \mathbb{P}^r := [(\mathbb{P}^r)^d/S_d]$ in terms of "Hurwitz-Hodge integrals" – integrals over (compactified) Hurwitz spaces. We apply the algorithm to prove a mirror-type theorem for $\operatorname{Sym}^d \mathbb{P}^r$ in genus zero. The theorem states that a generating function of Gromov-Witten invariants of $\operatorname{Sym}^d \mathbb{P}^r$ is equal to an explicit power series $I_{\operatorname{Sym}^d \mathbb{P}^r}$, conditional upon a conjectural combinatorial identity. This is a first step in the direction of proving Ruan's Crepant Resolution Conjecture for the resolution $\operatorname{Hilb}^{(d)}(\mathbb{P}^2)$ of the coarse moduli space of $\operatorname{Sym}^d \mathbb{P}^2$.

1. Introduction

Over the last 20 years, following predictions from string theory [CdlOGP91], mathematicians have proven a series of results known as *mirror theorems*; an incomplete list is [Giv98b, LLY99, Giv98a, BCFKvS00, Zin09, Li11, JK02, CCIT15, CCFK15, FLZ20b, FLZ20a, CCIT14]. These theorems reveal elegant patterns and structures embedded in the collection of (usually genus-zero) Gromov-Witten invariants of a fixed target manifold or orbifold X. They also allow for easy computation of these invariants in certain cases where direct computation involves difficult combinatorial computations. However, the scope of these results, and much of Gromov-Witten theory in general, is limited to the world of toric geometry; in all cases above, X is a complete intersection in a toric variety or stack (or a deformation thereof). The essential reason for this is that computing a Gromov-Witten invariant of a toric variety can be reduced, via the Atiyah-Bott localization theorem, to evaluating a certain sum over labeled graphs.

In this paper, we study the Gromov-Witten invariants of $\operatorname{Sym}^d \mathbb{P}^r$, which has a torus action, but without a dense orbit. Some aspects of the theory remain similar to the toric case, many new obstacles must be dealt with, and some interesting new behaviors appear. In the first half of the paper we use localization to give an algorithm expressing any Gromov-Witten invariant of $\operatorname{Sym}^d \mathbb{P}^r$ explicitly in terms of *Hurwitz-Hodge integrals* (Theorem 4.5). Hurwitz-Hodge integrals are numerical invariants of a representation of a finite group G; they are defined as integrals over compactified Hurwitz spaces. Computing them in general is a main stumbling block in orbifold Gromov-Witten theory.

In order to apply localization to the case of $\operatorname{Sym}^d \mathbb{P}^r$, we must carefully describe the torus-invariant curves on $\operatorname{Sym}^d \mathbb{P}^r$ and their deformation theory. We do this in Sections 3 and 4. (These sections contain the main geometric content of the paper.)

In the second half of the paper, we apply the above algorithm in a recursive form (Theorem 5.5) to prove a genus-zero mirror-type theorem for $\operatorname{Sym}^d \mathbb{P}^r$ (Theorem 6.3), which was not possible using existing techniques. The theorem, which is conditional upon two explicit combinatorial identities we were unable to prove, gives a formula for a generating function of Gromov-Witten invariants of $\operatorname{Sym}^d \mathbb{P}^r$. The proof of Theorem 6.3 is notably combinatorial, and the specific combinatorics are of independent interest, see Remark 1.2. Theorem 6.3 is also the only known mirror theorem for a nonabelian orbifold, besides single points $[\bullet/G]$.

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Corollary 6.6 Assuming Identities 7.1 and 7.2, for any $d, r \geq 1$ there is an equality

$$I_{\mathrm{Sym}^d \mathbb{P}^r} = J_{\mathrm{Sym}^d \mathbb{P}^r} \mod (\mathbf{x})^2,$$

where $J_{\mathrm{Sym}^d \mathbb{P}^r}$ is a generating function of genus-zero Gromov-Witten invariants of $\mathrm{Sym}^d \mathbb{P}^r$ (see Section 2.4), and $I_{\mathrm{Sym}^d \mathbb{P}^r}$ is the explicit power series (29).

Remark 1.1. In Theorem 6.3 and Corollary 6.6, $I_{\mathrm{Sym}^d \mathbb{P}^r}$ is only defined up to first order in $\mathbf{x}$ — it would be very desirable to generalize this mirror theorem so that it involves a power series I' with arbitrary powers of $\mathbf{x}$. The primary obstacle is that one must first *produce* such a power series — and then check that it satisfies the conditions of Theorem 5.5. The power series (29) was produced after much computer experimentation, and we were unable to generalize it to arbitrary order in $\mathbf{x}$. Furthermore, the combinatorics required to prove that $I_{\mathrm{Sym}^d \mathbb{P}^r}$ satisfies the conditions of Theorem 5.5 are extremely complicated, and we were only able to establish them conditional upon the conjectural combinatorial identities in Section 7. While there are some systematic methods for producing such "I-functions" (e.g. [CFK16]), applying these methods to $\mathrm{Sym}^d \mathbb{P}^r$ (or any nonabelian orbifold) results in the *zeroth* order truncation of $I_{\mathrm{Sym}^d \mathbb{P}^r}$ in $\mathbf{x}$, losing *all* combinatorial structure.

We have three motivations for working with $\mathrm{Sym}^d \mathbb{P}^r$.

- $\mathrm{Sym}^d \mathbb{P}^r$ is very concrete, and is therefore a good starting point for studying both non-toric and non-abelian behavior. While the natural $(\mathbb{C}^*)^{r+1}$-action has infinitely many orbits, it also has finitely many fixed points; in this sense $\mathrm{Sym}^d \mathbb{P}^r$ is not too much more complicated than a toric variety. On the other hand, it is complicated enough that studying its Gromov-Witten invariants requires various new methods, which we expect to be useful for studying the Gromov-Witten theory of other non-toric and non-abelian
- *The crepant resolution conjecture.* Following physical predictions, Ruan [Rua06], Bryan-Graber [BG09], and Coates-Iritani-Tseng [CIT09] made a conjecture relating the Gromov-Witten invariants of an orbifold X to those of a crepant resolution of its coarse moduli space. This conjecture has been proven in the context of toric geometry [CIJ14]. However, the crepant resolution $\mathrm{Hilb}^{(d)}(\mathbb{P}^2)$ of the coarse moduli space of $\mathrm{Sym}^d \mathbb{P}^r$ was one of Ruan's motivating examples; this case has now been open for over a decade. Theorem 6.3 is a first step towards this case.
- *Higher genus invariants of projective space.* Costello's thesis expressed the genus g Gromov-Witten invariants of a smooth projective variety X in terms of the genus-zero Gromov-Witten invariants of $\mathrm{Sym}^{g+1} X$. Theorem 6.3 provides an efficient way of encoding the latter for $X = \mathbb{P}^r$. It may be possible to combine Costello's result with ours to find explicit formulas for genus-g Gromov-Witten invariants of $\mathbb{P}^r$.

We briefly describe the difficulties caused by the fact that $\mathrm{Sym}^d \mathbb{P}^r$ is not toric. To do so, we first broadly outline the proof of Coates-Corti-Iritani-Tseng of the mirror theorem for a toric stack X [CCIT15]. The two main ingredients are

(1) An algorithm for expressing Gromov-Witten invariants of X in terms of Hurwitz-Hodge integrals; this is supplied by localization calculations of Johnson [Joh14] and Liu [Liu13]. The localization technique roughly involves integrating over the moduli space of *torus-invariant* curves $C \subseteq X$, which is easy: this moduli space is a finite collection of points, in bijection with codimension-1 cones in the fan of X. The hardest part of the calculation is to find an explicit expression for the integrand, which is defined in terms of the deformation theory of the curves C.

(2) A technique of Brown [Bro14], which reinterprets the above algorithm as follows. To each torus-fixed point $\sigma \in X$ is associated a power series $\mathbf{f}_\sigma$ in a variable z; these power series together encode all genus-zero Gromov-Witten invariants of X. Each power series $\mathbf{f}_\sigma$ has a

collection of simple poles, and using the algorithm, one shows that the power series satisfy a recursion in the following sense: the residue of $\mathbf{f}_\sigma$ at a pole w is expressed as a linear combination of values $\mathbf{f}_{\sigma'}(w)$ for other fixed points $\sigma' \neq \sigma$ (such that $\mathbf{f}_{\sigma'}$ has no pole at w). The recursion uniquely defines $\mathbf{f}_\sigma$ for all σ, up to some change of variables.

The outline of the proof of Theorem 6.3 is similar, but with the following differences:

- (1′) As mentioned, Theorem 4.5 expresses any Gromov-Witten invariant of $\mathrm{Sym}^d \mathbb{P}^r$ in terms of Hurwitz-Hodge integrals. However, both parts of the calculation are substantially more difficult than in the toric case. The moduli space of torus-invariant curves is not finite — rather, it is positive dimensional, disconnected, and quite complicated. Luckily, we are able to give a complete characterization of the moduli space (Theorem 3.16). Our characterization is concrete enough to allow us to compute the requisite integrals. The deformation theory of torus-fixed curves is difficult for essentially the same reason, but again the computation can be carried out fully (Section 4).
- (2′) Theorem 5.5 is analogous to Brown's description above — we again have a power series $\mathbf{f}_\sigma$ attached to each torus-fixed point $\sigma \in \mathrm{Sym}^d \mathbb{P}^r$. However, these power series no longer have simple poles, but may have poles of arbitrarily high order. The algorithm again gives a recursion relation, this time expressing *any negative-power* Laurent coefficient of $\mathbf{f}_\sigma$ in terms of *nonnegative-power* Laurent coefficients of $\mathbf{f}_{\sigma'}$ for other fixed points σ'. We wish to highlight this feature, both because it is new, and because it is expected to appear in the Gromov-Witten theory of any nontoric variety with a nontrivial torus action. (The fact that there are only simple poles in the toric case should be viewed as exceptional.) We hope that this first example might provide clues for proving other nontoric mirror theorems.

Remark 1.2. We also wish to draw attention to the fact that the combinatorial structure encoded in Theorems 5.5 and (especially) 6.3 is much more intricate than in the toric case — so much so that we were not able to give an unconditional version of Theorem 6.3 despite the apparent fact that combinatorial complexity is the only hurdle — for example, the Chu-Vandermonde identity played a crucial role in the proof of Theorem 6.3. We hope that the combinatorics in this paper, though not quite complete, will be a useful case study in proving mirror theorems where high-order poles appear. The generating functions in this paper exhibit rich combinatorial structure, and are surely important for further understanding mirror symmetry for symmetric products, so we believe a more systematic study is worthwhile in the future. This is especially true of the generating functions appearing on pages 35–38, which are not specific to $\mathrm{Sym}^d \mathbb{P}^r$ but instead deal with twisted Gromov-Witten invariants of an orbifold *point.* (We note that some of the relevant framework may already exist, e.g. in the integrable systems literature — though we were unable to find anything that would imply Identities 7.1 and 7.2. The specific form of these identities, and the other combinatorial tools used in the proof of Theorem 6.3, are quite unlike anything appearing in the Gromov-Witten theory to our knowledge.)

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2. Notation, conventions, and background

This section sets up combinatorial conventions, and reviews Atiyah-Bott torus localization, orbifold Gromov-Witten theory, and moduli spaces of curves called *Losev-Manin spaces*, which are used in Section 3.5 to describe the torus invariant curves in $\mathrm{Sym}^d \mathbb{P}^r$.

We always work over $\mathbb{C}$. We write $H^*(X) := H^*(X, \mathbb{Q})$. For a point x of an orbifold X, we write G_x for the isotropy group of x.

2.1. Multipartitions and graphs.

It is convenient to use the language of multisets, denoted with parentheses, e.g. (a, a, b). We write $\mathrm{Mult}(\Pi, a)$ for the number of times that a appears in Π. We will refer to multiset unions and intersections, and sums indexed by multisets, without comment.

For an integer $d \geq 0$, $\mathrm{Part}(d)$ is the set of partitions of d, i.e. the set of multisets of positive integers that sum to d. A *weak composition of* d is an ordered tuple of nonnegative integers whose sum is d. The (finite) set of weak compositions of d of length r is denoted $\mathrm{ZPart}(d, r)$. If $D \in \mathrm{ZPart}(d, r)$, a *multipartition of* D is a multiset $(\Pi_d)_{d \in D}$, with Π_d a partition of d. The (finite) set of multipartitions of D is denoted $\mathrm{MultiPart}(D)$. For each partition $D \in \mathrm{ZPart}(d, r)$, there is a "trivial multipartition" of D, which we usually denote (abusing notation) by $(1, \ldots, 1)$, where every part of every Π_d is equal to 1. There is an "underlying partition" map $\mathrm{MultiPart}(D) \to \mathrm{Part}(\sum_{d \in D} d)$.

If Π is a partition, we write S_Π for the group of automorphisms of Π as a multiset (defined up to isomorphism); e.g. for $\Pi = (1, 1, 1, 2, 2)$ of 7, we have $S_\Pi = S_3 \times S_2$. For $\sigma = (\Pi_d)_{d \in D}$ a multipartition of $D \in \mathrm{ZPart}(d, r)$, we define $S_\sigma := \prod_{d \in D} S_{\Pi_d}$.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite graph. We denote by $E(\Gamma, v)$ the set of edges incident to v. The *valence* $\mathrm{val}(v)$ of $v \in V(\Gamma)$ is $|E(\Gamma, v)|$. (This is different from some Gromov-Witten theory literature, where $\mathrm{val}(v)$ includes contributions from certain decorations on Γ.) A *flag* of Γ is a pair $(v, e) \in V(\Gamma) \times E(\Gamma)$ with $e \in E(\Gamma, v)$. The set of flags of Γ is denoted $F(\Gamma)$.

2.2. Equivariant cohomology.

We will consider actions of the torus $T := (\mathbb{C}^*)^{r+1}$ on various spaces, e.g. $\mathbb{P}^r$, $\mathrm{Sym}^d \mathbb{P}^r$, and $\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. If T acts on a Deligne-Mumford stack X, the equivariant cohomology $H^*_T(X)$ is a module over $H^*_T(\mathrm{Spec}\,\mathbb{C}) \cong \mathbb{Q}[\alpha_0, \ldots, \alpha_r]$, where $-\alpha_i$ is the weight of the character $T \to \mathbb{C}^*$ defined by $(\lambda_0, \ldots, \lambda_r) \mapsto \lambda_i$. We write $H^*_{T,\mathrm{loc}}(\mathrm{Spec}\,\mathbb{C})$ for the localization $\mathbb{Q}(\alpha_0, \ldots, \alpha_r)$, and more generally $H^*_{T,\mathrm{loc}}(X) := H^*_T(X) \otimes_{H^*_T(\mathrm{Spec}\,\mathbb{C})} H^*_{T,\mathrm{loc}}(\mathrm{Spec}\,\mathbb{C})$. We will use the Atiyah-Bott *localization theorem*, as well as Graber-Pandharipande's generalization, the *virtual localization theorem.*

Theorem 2.1 ([AB84], see [EG98] for statement in the Chow ring)**.** *Let T be a torus acting on a smooth compact manifold X, with fixed point set F. Then the map $(\iota_F)_* : H^*_{T,\mathrm{loc}}(F) \to H^*_{T,\mathrm{loc}}(X)$ is an isomorphism, where $(\iota_F)_*$ is the Gysin map associated to the inclusion $F \hookrightarrow X$. The inverse map is $\iota_F^*/e_T(N_{F|X})$, where $e_T(N_F)$ is the equivariant Euler class of the normal bundle to F. In particular, for $\alpha \in H^*_{T,\mathrm{loc}}(X, \mathrm{Spec}\,\mathbb{C})$, we have*

$$\int_X \alpha = \int_X (\iota_F)_* \left(\frac{\iota_F^* \alpha}{e_T(N_F)} \right) = \int_F \frac{\iota_F^* \alpha}{e_T(N_F)}.$$

Theorem 2.2 ([GP99])**.** *Let X be a Deligne-Mumford stack with a T-action and a T-equivariant perfect obstruction theory $E^\bullet$. Again, let $\iota_F : F \hookrightarrow X$ denote the inclusion of the fixed locus. Let $[X]^{\mathrm{vir}}$ denote the virtual fundamental class associated to $E^\bullet$. The T-fixed part of $E^\bullet$ defines a perfect obstruction theory on F, with virtual fundamental class $[F]^{\mathrm{vir}}$. The* virtual normal bundle *N_F^{vir} to F is the T-moving part of $E^\bullet$. Then*

$$\int_{[X]^{\mathrm{vir}}} \alpha = \int_{[F]^{\mathrm{vir}}} \frac{\iota_F^* \alpha}{e_T(N_F^{\mathrm{vir}})}. \tag{1}$$

Remark 2.3. The proof in [GP99] requires that X have a global equivariant embedding into a smooth Deligne-Mumford stack, but this condition was removed in [CKL15].

2.3. **Symmetric product stacks.** Let X be a scheme over $\mathbb{C}$. There are two common (equivalent) definitions of $\operatorname{Sym}^d X$. The first is the stack quotient $[X^d/S_d]$, where S_d acts in the usual way on X^d. That is, objects and morphisms are described by

$$\text{Objects:}\quad \begin{tikzcd} \tilde{S} \arrow[r, "\tilde{f}"] \arrow[d, "\mathrm{pr}"] & X^d \\ S & \end{tikzcd} \qquad\qquad \text{Arrows:}\quad \begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] \arrow[rr, bend left, "\tilde{f}"] & \tilde{T} \arrow[r, "\tilde{g}"'] \arrow[d] & X^d \\ S \arrow[r] & T & \end{tikzcd}$$

where vertical maps are S_d-principal bundles, $\tilde{f}$ and $\tilde{g}$ are S_d-equivariant, and the square on the right is Cartesian. The second definition is given by

$$\text{Objects:}\quad \begin{tikzcd} S' \arrow[r, "f'"] \arrow[d, "\rho"] & X \\ S & \end{tikzcd} \qquad\qquad \text{Arrows:}\quad \begin{tikzcd} S' \arrow[r] \arrow[d] \arrow[rr, bend left, "f'"] & T' \arrow[r, "g'"'] \arrow[d] & X \\ S \arrow[r] & T & \end{tikzcd}$$

where vertical maps are degree d étale, and the square on the right is Cartesian. It is a straightforward exercise to show that the two stacks defined are naturally isomorphic. We will usually use the second, and we will consistently use the notations $S' \to S$ and $f' : S' \to X$ when referring to S-points of $\operatorname{Sym}^d X$. The two descriptions are related by the diagram:

$$\begin{tikzcd} \tilde{S} \times \{1,\ldots,d\} \arrow[rr] \arrow[dr] \arrow[dd] & & X^d \times \{1,\ldots,d\} \arrow[dr, "\tilde{\rho}"] \arrow[dd, "\mathrm{pr}'"] \arrow[ddrr, bend left] & & \\ & \tilde{S} \arrow[rr, "\tilde{f}"] \arrow[dd, "\mathrm{pr}"] & & X^d \arrow[dd, "\mathrm{pr}"] & \\ S' = \tilde{S} \times_{S_d} \{1,\ldots,d\} \arrow[rr] \arrow[dr, "\rho"] & & X^d \times_{S_d} \{1,\ldots,d\} \arrow[rr, "P"] \arrow[dr, "\rho"] & & X \\ & S \arrow[rr, "f"] & & \operatorname{Sym}^d X & \end{tikzcd} \tag{2}$$

Here the cube is Cartesian, and the left and right faces consist of étale maps. The composition $S' \to X^d \times_{S_d} \{1,\ldots,d\} \xrightarrow{P} X$ is f'.

Now assume X is smooth. We can understand the tangent bundle to $\operatorname{Sym}^d \mathbb{P}^r$ as follows:

Lemma 2.4. *There is a natural isomorphism* $T \operatorname{Sym}^d X \cong \rho_*(P^*TX)$, *where* ρ *and* P *are as in the diagram above.*

Proof. Since the square is cartesian and consists of étale maps, we have

$$\mathrm{pr}^*(\rho_*(P^*TX)) \cong \tilde{\rho}_*((\mathrm{pr}')^*(P^*TX)) = \tilde{\rho}_*((\mathrm{pr}' \circ P^*TX)).$$

Recall that $\mathrm{pr}' \circ P$ is simply the "universal coordinate map," so since $\tilde{\rho}$ is a trivial étale cover, there is a canonical isomorphism

$$\tilde{\rho}_*((\mathrm{pr}' \circ P^*TX)) \cong \bigoplus_{\ell=1}^{d} P_\ell^* TX \cong T(X^d).$$

Since $\tilde{\rho}$ is S_d-equivariant, there is an induced S_d-action on $T(X^d)$ which agrees with the usual one. Thus the isomorphism descends to give $\rho_*(P^*TX) \cong T\operatorname{Sym}^d X$. $\square$

Finally, we describe the cyclotomic inertia stack $I\operatorname{Sym}^d X \to \operatorname{Sym}^d X$, see Section 3 of [AGV08]. Assume X is connected. For each partition $\sigma \in \operatorname{Part}(d)$, there is a component $(\operatorname{Sym}^d X)_\sigma$ of $I\operatorname{Sym}^d X$, isomorphic to (a trivial gerbe over) $\prod_{\eta \geq 1} \operatorname{Sym}^{\operatorname{Mult}(\sigma,\eta)} X$, and the map $(\operatorname{Sym}^d X)_\sigma \to \operatorname{Sym}^d X$ is (a rigidification followed by) the obvious one. The generic point of $(\operatorname{Sym}^d X)_\sigma$ maps to a point in $\operatorname{Sym}^d X$ with isotropy group isomorphic to $\prod_{\eta\geq 1} S_\eta$.

Remark 2.5. The map $(\operatorname{Sym}^d X)_\sigma \to \operatorname{Sym}^d X$ (after rigidification) may not be an embedding. For example, consider $\operatorname{Sym}^4 X$, and let $\sigma = (2,1,1)$. By the above, $(\operatorname{Sym}^4 X)_\sigma$ is a trivial gerbe over $X \times \operatorname{Sym}^2 X$. The induced map $X \times \operatorname{Sym}^2 X \to \operatorname{Sym}^4 X$ sends points $(a,(b,c)) \mapsto (a,a,b,c)$, but this identifies the two distinct points $(a,(b,b))$ and $(b,(a,a))$ for all $a, b \in X$.

The (equivariant, nonorbifold) cohomology with rational coefficients may be computed explicitly by the Künneth decomposition, as the S_d-invariant part of $H^*_T(X^d,\mathbb{Q}) = \bigotimes_{j=1}^d H^*_T(X,\mathbb{Q})$. In particular, for $X = \mathbb{P}^r$, we will use the identification $H^2_T(\operatorname{Sym}^d \mathbb{P}^r,\mathbb{Q}) \cong H^2_T((\mathbb{P}^r)^d,\mathbb{Q})^{S_d} \cong H^2_T(\mathbb{P}^r,\mathbb{Q})$. We will abuse notation and write $[H_i] \in H^2_T(\operatorname{Sym}^d \mathbb{P}^r,\mathbb{Q})$ for the element that pulls back to $\sum_{j=1}^d \operatorname{pr}_j^*[H_i] \in H^2_T((\mathbb{P}^r)^d)$, where pr_j is the jth coordinate map and $[H_i]$ is the equivariant fundamental class of the ith coordinate hyperplane.

Fix a component $(I\operatorname{Sym}^d \mathbb{P}^r)_\sigma$ of $I\operatorname{Sym}^d \mathbb{P}^r$. For $\eta \in \sigma$, we denote by $[H_{\sigma,\eta,i}]$ the pullback of $[H_i]$ from the factor of $(I\operatorname{Sym}^d \mathbb{P}^r)_\sigma \cong \prod_{\eta\geq 1} \operatorname{Sym}^{\operatorname{Mult}(\sigma,\eta)} \mathbb{P}^r$ corresponding to η. We write $[H_{\sigma,i}]$ for $\sum_\eta [H_{\sigma,\eta,i}]$.

2.4. **(Orbifold) Gromov-Witten theory.** Our objects of study are the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of n-marked genus-g stable maps to a smooth proper Deligne-Mumford stack X of degree β, introduced in [CR02] and [AV02]. See [Liu13], Section 7 for an introduction to the subject (in all genera). Following [Liu13], we use the technical convention that all gerbes come with the data of a section.

In this paper we will have either $X = \operatorname{Sym}^d \mathbb{P}^r$ or $X = BG$ for some finite group G. We write $(f : C \to X)$ for a $\mathbb{C}$-point of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, and

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & X \\ \overline{\mathcal{M}}_{g,n}(X,\beta) & \end{tikzcd}$$

for the universal curve and universal map.

A *Gromov-Witten invariant* is an integral of the form

$$\langle \overline{\psi}_1^{a_1}\gamma_1, \ldots, \overline{\psi}_n^{a_n}\gamma_n \rangle^X_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{j=1}^n \overline{\psi}_j^{a_j} \operatorname{ev}_j^* \gamma_j \in \mathbb{Q}, \tag{3}$$

where

- $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ is the virtual fundamental class,
- $\overline{\psi}_j$ is the jth cotangent class on $\overline{\mathcal{M}}_{g,n}(X,\beta)$, coming from the cotangent space to the *coarse moduli space of* C,[1]
- the "insertions" γ_j are in the Chen-Ruan cohomology (see [CR04]) $H^*_{CR}(X)$, and

---

[1] Note that locally $\overline{\psi}_j = r_j\psi_j$, where r_j is the size of the isotropy group at the mark b_j, and ψ_j is the "stacky" cotangent class.

- $\mathrm{ev}_j : \overline{\mathcal{M}}_{g,n}(X, \beta) \to IX$ is the jth evaluation map.

If X has an action of a torus T, it induces a natural T-action on IX and $\overline{\mathcal{M}}_{g,n}(X, \beta)$, and $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}}$, $\overline{\psi}_j$, and $\mathrm{ev}_j^* \gamma_j$ are naturally equivariant classes (where $\gamma_j \in H^*_{CR,T}(X)$). In this case (3) defines an *equivariant Gromov-Witten invariant* (an element of $H^*_T(\operatorname{Spec} \mathbb{C})$, denoted by $\langle \cdots \rangle^{X,T}_{g,n,\beta}$) via T-equivariant integration.

We introduce some formalism for the case $g = 0$, which will be used to state and prove Theorems 5.5 and 6.3. Following [CCIT15], the *T-equivariant Novikov ring of* $\mathrm{Sym}^d \mathbb{P}^r$ is

$$\Lambda^{\mathrm{nov}}_T := H^*_{T,\mathrm{loc}}(\operatorname{Spec} \mathbb{C})[[Q]],$$

and *Givental's symplectic vector space* is

$$\mathcal{H} := H^*_{CR,T,\mathrm{loc}}(\mathrm{Sym}^d \mathbb{P}^r)[[Q]]((z^{-1})) = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

where $\mathcal{H}^+ = H^*_{CR,T,\mathrm{loc}}(\mathrm{Sym}^d \mathbb{P}^r)[[Q]][z]$ and $\mathcal{H}^- = z^{-1} H^*_{CR,T,\mathrm{loc}}(\mathrm{Sym}^d \mathbb{P}^r)[[Q]][[z^{-1}]]$. Inside $\mathcal{H}$, there is a special subscheme $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ — precisely, a formal germ of a subscheme over $\operatorname{Spec} \Lambda^T_{\mathrm{nov}}$, defined at $-1 \cdot z$, where $1 \in H^*_{CR,T,\mathrm{loc}}(\mathrm{Sym}^d \mathbb{P}^r)$ is the fundamental class of the untwisted sector — called the *Givental cone* of $\mathrm{Sym}^d \mathbb{P}^r$, which encodes the genus-zero Gromov-Witten invariants of $\mathrm{Sym}^d \mathbb{P}^r$.

Fix a basis γ_ϕ of $H^*_{CR,T,\mathrm{loc}}(\mathrm{Sym}^d \mathbb{P}^r)$, with Poincaré dual basis γ^ϕ. A $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point of $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ is defined to be a power series

$$-1z + \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \sum_{\phi} \frac{Q^\beta}{n!} \left\langle \mathbf{t}(\overline{\psi}), \ldots, \mathbf{t}(\overline{\psi}), \frac{\gamma_\phi}{-z - \overline{\psi}} \right\rangle^{\mathrm{Sym}^d \mathbb{P}^r, T}_{0,n+1,\beta} \gamma^\phi \in \mathcal{H}[[x]],$$

where $\mathbf{t}(z) \in \langle Q, x \rangle \subseteq \mathcal{H}^+[[x]]$.

Remark 2.6. This definition as stated is both confusing and slightly imprecise. The point is this: as a formal scheme over $\operatorname{Spec} \Lambda^T_{\mathrm{nov}}$, $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ is characterized (indeed, defined) not just by its $\mathbb{C}$-valued points or Λ^T_{nov}-valued points but by its points over arbitrary (topological) Λ^T_{nov}-algebras. The definition given is the most basic nontrivial example, and generalizes in an obvious way. See Appendix B of [CCIT09] for a complete discussion.

Remark 2.7. Another subtlety is that we may wish to take $\mathbf{t}(z)$ to be a power series in z, in which case it is not immediately obvious that the expression $\mathbf{t}(\overline{\psi})$ makes sense. In practice this is not a major concern; the key is that $\mathbf{t}(z)$ must be "topologically nilpotent," which will always be the case in practice. Again, see Appendix B of [CCIT09].

An important special case is

$$\mathbf{t}(z) = \theta = \sum_{\phi} x_\phi \gamma_\phi \in H^*_{CR,T,\mathrm{loc}}(\mathrm{Sym}^d \mathbb{P}^r)[[\{x\}_\phi]],$$

where $\{\gamma_\phi\}$ is the basis for $H^*_{CR,T,\mathrm{loc}}(\mathrm{Sym}^d \mathbb{P}^r)$ chosen above. The corresponding $\Lambda^T_{\mathrm{nov}}[[\{x\}_\phi]]$-valued point is called the J-function of $\mathrm{Sym}^d \mathbb{P}^r$ and is denoted $J_{\mathrm{Sym}^d \mathbb{P}^r}(Q, \theta, -z)$. Here $\mathbf{t}(z)$ has no nonzero powers of z, so the invariants appearing in $J_{\mathrm{Sym}^d \mathbb{P}^r}(Q, \theta, -z)$ have a single ψ-class.

$\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ has several important geometric properties that follow from relations between Gromov-Witten invariants: see Appendix B of [CCIT09], which also defines $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ rigorously as a non-Noetherian formal scheme. For example, it is a cone in a certain sense, hence the name (Proposition B.2 of [CCIT09]).

Given a vector bundle E on X, there is also a notion of an *E-twisted Gromov-Witten invariant* of X. We need this notion only when $X = BG$, with the trivial action of a torus T. Let E be a

$T \times G$ representation. Then $R\pi_* f^* E \in K^0_T(\overline{\mathcal{M}}_{g,n}(BG, 0))$. An E-twisted Gromov-Witten invariant of BG is known as a *Hurwitz-Hodge integral*, and is defined by

$$\langle \overline{\psi}_1^{a_1}\gamma_1, \dots, \overline{\psi}_n^{a_n}\gamma_n \rangle_{g,n,0}^{BG,T,E} := \int_{[\overline{\mathcal{M}}_{g,n}(BG,0)]^{\mathrm{vir}}} \prod_{j=1}^{n} \overline{\psi}_j^{a_j} \operatorname{ev}_j^* \gamma_j \cup e_T^{-1}(R\pi_* f^* E). \tag{4}$$

As above, in genus zero we can define the *twisted Lagrangian cone* $\mathcal{L}^E_{BG}$: a $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point of $\mathcal{L}^E_{BG}$ is defined to be

$$-1z + \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{\phi} \frac{1}{n!} \left\langle \mathbf{t}(\overline{\psi}), \dots, \mathbf{t}(\overline{\psi}), \frac{\gamma_\phi}{-z - \overline{\psi}} \right\rangle_{0,n+1,0}^{BG,T,E} \gamma^\phi, \tag{5}$$

for some $\mathbf{t}(z) \in \langle Q, x\rangle \subseteq \mathcal{H}^+[[x]]$. Here γ_ϕ and γ^ϕ are dual bases of $H^*_T(X)$ under the *twisted* Poincaré pairing, see [CCIT15].

Notation 2.8. In the important case where $\mu \cong BG$ is a T-fixed point of an ambient orbifold Y, and $E = T_\mu Y$, we write $\mathcal{L}^{\mathrm{tw}}_\mu := \mathcal{L}^{T_\mu Y}_\mu$.

2.5. **Losev-Manin spaces.** We recall certain moduli spaces of marked curves, studied originally by Losev and Manin [LM00].

Definition 2.9. Let $k \geq 1$, and fix a 2-element set $\{0, \infty\}$. An $(0|k|\infty)$*-marked Losev-Manin curve* is a connected genus zero $(k+2)$-marked nodal curve $(C, b_0, b_1, \dots, b_k, b_\infty)$, satisfying:

- The irreducible components of C form a chain, with two leaves C_0 and C_∞,
- The points $b_0, b_1, \dots, b_k, b_\infty$ are smooth points of C, with $b_0 \in C_0$ and $b_\infty \in C_\infty$,
- $b_i \neq b_0$ and $b_i \neq b_\infty$ for $i = 1, \dots, k$ (though it is possible that $b_i = b_j$ for $i \neq j$), and
- Each irreducible component of C contains at least one point of $b_1, \dots, b_k$.

Theorem 2.10 ([LM00], Theorems 2.2 and 2.6.3)**.** *The moduli space of* $(0|k|\infty)$*-marked Losev-Manin curves* $\overline{\mathcal{M}}_{0|k|\infty}$ *is a smooth projective (toric) variety, and there is a natural birational morphism* $\varphi : \overline{\mathcal{M}}_{0,k+2} \to \overline{\mathcal{M}}_{0|k|\infty}$.

Remark 2.11. The spaces $\overline{\mathcal{M}}_{0|k|\infty}$ is an example of a moduli space $\overline{\mathcal{M}}_{0,\mathcal{A}}$ of weighted stable curves, developed later by Hassett [Has03], and Theorem 2.10 is a special case of Theorems 2.1 and 4.1 of [Has03]. Specifically, there is a natural isomorphism $\overline{\mathcal{M}}_{0|k|\infty} \to \overline{\mathcal{M}}_{0,\mathcal{A}}$, where $\mathcal{A}$ is the weight datum $(1, \epsilon, \epsilon, \dots, \epsilon, 1)$ of length $k+2$, for $\epsilon \leq 1/k$.

Definition 2.12. Let $s \geq 1$ be an integer. An *order-s orbifold* $(0|k|\infty)$*-marked Losev-Manin curve* is a $(k+2)$-marked twisted curve $(C, b_0, b_1, \dots, b_k, b_\infty)$ (in the sense of [Ols07]) whose coarse moduli space is a k-marked Losev-Manin curve, such that C has orbifold structure only at b_0, b_∞, and the nodes of C, all of which have order s.

The moduli space $\overline{\mathcal{M}}^s_{0|k|\infty}$ of order-s orbifold k-marked Losev-Manin curves has a natural map $\overline{\mathcal{M}}^s_{0|k|\infty} \to \overline{\mathcal{M}}_{0|k|\infty}$ that comes from taking coarse moduli spaces of curves. Our calculations in Section 5 will use the following fact, a special case from Lemma 2.3 of [Moo11].

Lemma 2.13. *Let* $\psi_{0,LM}$ *and* $\psi_{\infty,LM}$ *denote the tautological cotangent classes at* b_0 *and* b_∞ *on* $\overline{\mathcal{M}}_{0|k|\infty}$. *The pullbacks* $\varphi^*\psi_{0,LM}$ *and* $\varphi^*\psi_{\infty,LM}$ *along the reduction morphism* $\overline{\mathcal{M}}_{0,k+2} \to \overline{\mathcal{M}}_{0|k|\infty}$ *are the cotangent classes* ψ_0 *and* ψ_∞*, respectively.*

Remark 2.14. Lemma 2.13 holds for order-s orbifold Losev-Manin spaces, either using the cotangent classes $\overline{\psi}$ (as we do in this paper), or replacing $\overline{\mathcal{M}}_{0,k+2}$ with a stacky replacement $\overline{\mathcal{M}}^s_{0,k+2}$. ($\overline{\mathcal{M}}^s_{0,k+2}$ parametrizes curves where b_0 and b_∞ have order-s orbifold structure, as do any nodes that separate b_0 from b_∞.)

3. THE ACTION OF $(\mathbb{C}^*)^{r+1}$ ON $\mathrm{Sym}^d\,\mathbb{P}^r$

There is a natural action of $T := (\mathbb{C}^*)^{r+1}$ on $\mathbb{P}^r$. This induces a diagonal action of $(\mathbb{C}^*)^{r+1}$ on $(\mathbb{P}^r)^d$, which commutes with the action of S_d, hence acts on $\mathrm{Sym}^d\,\mathbb{P}^r$. (The action on a diagram $S \xleftarrow{\rho} S' \xrightarrow{f'} \mathbb{P}^r$ as in Section 2.3 is by postcomposition of f'.) This T-action on $\mathrm{Sym}^d\,\mathbb{P}^r$ induces an action on $\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$ for all n and β.

The goal of this section is Theorem 3.16, which explicitly characterizes the T-fixed locus in $\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$. The building blocks of the construction are spaces $\overline{\mathcal{M}}_{g,n}(BG, 0)$ of admissible covers from [ACV03][2], the Losev-Manin spaces from Section 2.5, and combinatorial objects called *decorated graphs.*

3.1. T-fixed points and 1-dimensional orbits of $\mathrm{Sym}^d\,\mathbb{P}^r$.

We begin by fixing notation for points and lines in $\mathbb{P}^r$. We will denote the coordinate points of $\mathbb{P}^r$ by $P_0, P_1, \ldots, P_r$, where P_i is the point where the only nonzero coordinate is the ith one. We denote by $L_{(i_1,i_2)} = L_{(i_2,i_1)}$ the line through P_{i_1} and P_{i_2}. We write $P_{(i_1,i_2)}$ for the "midpoint" of this line, where the i_1-th and i_2-th coordinates are equal.

Recall from Section 2.3 that a map $f : S \to \mathrm{Sym}^d\,\mathbb{P}^r$ is the same as a degree-d étale cover $\rho : S' \to S$, and a map $f' : S' \to \mathbb{P}^r$. We use the notation $\bullet$ for $\operatorname{Spec}\mathbb{C}$, and $d(\bullet)$ for the union of d copies of $\operatorname{Spec}\mathbb{C}$. Note $d(\bullet)$ is the only degree-d étale cover of $\bullet$, so ($\mathbb{C}$-valued) points of $\mathrm{Sym}^d\,\mathbb{P}^r$ are in natural bijective correspondence with maps $f' : d(\bullet) \to \mathbb{P}^r$.

Proposition 3.1. *Points of* $\mathrm{Sym}^d\,\mathbb{P}^r$ *with 0- and 1-dimensional* T*-orbits are classified as follows:*

1. *A point* $(d(\bullet) \xrightarrow{f'} \mathbb{P}^r) \in \mathrm{Sym}^d\,\mathbb{P}^r$ *is* T*-fixed if and only if* $\mathrm{Im}(f') \subseteq \{P_0, \ldots, P_r\}$.
2. $(d(\bullet) \xrightarrow{f'} \mathbb{P}^r)$ *has a 1-dimensional* T*-orbit if and only if it is not* T*-fixed and* $\mathrm{Im}(f') \subseteq \{P_0, \ldots, P_r\} \cup L_{(i_1,i_2)}$ *for some* $0 \le i_1, i_2 \le r$.

Proof. (1) follows from the definition of the T-action by post-composition, and that fact that $\{P_0, \ldots, P_r\}$ is the T-fixed locus of $\mathbb{P}^r$.

The r-dimensional subtorus defined by $t_{i_1} = t_{i_2}$ acts trivially on $\{P_0, \ldots, P_r\} \cup L_{(i_1,i_2)}$, proving the backwards direction of (2). If $\mathrm{Im}(f') \not\subseteq \{P_0, \ldots, P_r\} \cup L_{(i_1,i_2)}$, then $\mathrm{Im}(f')$ contains either two points on different coordinate lines, or a point not on a coordinate line. In either case, is it is easy to check explicitly that the T-orbit is at least 2-dimensional. ☐

Remark 3.2. The T-fixed points of $\mathrm{Sym}^d\,\mathbb{P}^r$ are in natural bijection with the set $\mathrm{ZPart}(d, r+1)$ of length-$(r+1)$ weak compositions, where the ith part is the number of points of $d(\bullet)$ mapping to P_i. We will use this identification from now on.

3.2. T-fixed stable maps to $\mathrm{Sym}^d\,\mathbb{P}^r$ with irreducible source curve.

It is well-known (see [Liu13]) that if X is a Deligne-Mumford stack with an action of a torus T, then a stable map $f : C \to X$ is T-fixed if and only if each component C_ν of C maps into the fixed locus X^T, or maps to the closure $\overline{U}$ of a 1-dimensional T-orbit U, with special points (nodes and marks) and ramification points mapping to $\overline{U} \smallsetminus U$. (In the latter case it follows that C_ν is rational; we may regard $f|_{C_\nu}$ as a point of $\overline{\mathcal{M}}_{0,2}(X, \beta)$ for some β.) If T acts with isolated fixed points, we refer to the two types of components of C as *contracted* and *noncontracted*, since those of the first type map to a single point of X. On contracted components C_ν, f factors through BG for some G; thus $f|_{C_\nu}$ is an admissible G-cover in the sense of [ACV03]. The following lemma classifies noncontracted components of T-fixed stable maps to $\mathrm{Sym}^d\,\mathbb{P}^r$.

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[2] These stacks compactify Hurwitz spaces, and are now usually referred to as moduli spaces of admissible covers, though [ACV03] reserves that term for the related compactifications defined earlier by Harris-Mumford [HM82].

Lemma 3.3. *Let* $(f : C \to \mathrm{Sym}^d \mathbb{P}^r) \in \overline{\mathcal{M}}_{0,2}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ *be a stable map of degree* $\beta > 0$ *with irreducible source curve. Denote by* b_1 *and* b_2 *the two marked points of* C*. Denote by* $\rho : C' \to C$ *and* $f' : C' \to \mathbb{P}^r$ *the associated degree* d *étale cover and map to projective space, respectively. (See Section 2.3.) Then* $(f : C \to \mathrm{Sym}^d \mathbb{P}^r)$ *is* T*-fixed if and only if all of the following hold:*

- C' *is a disjoint union of rational connected components* C'_η*. (Since* C *has two orbifold points, this means that on coarse moduli spaces,* ρ *is a cover, fully ramified over* b_1 *and* b_2*.)*
- *There exist distinct indices* $0 \le i_1, i_2 \le r$ *such that* f' *maps each component* C'_η *either*
  - *(i) to the line* $L_{(i_1,i_2)}$*, or*
  - *(ii) to a* T*-fixed point of* $\mathbb{P}^r$*.*
- *On the level of coarse moduli spaces, the restriction* $f'|_{C'_\eta}$ *to any component of type (i) is a cover of* $L_{(i_1,i_2)}$*, fully ramified at the two points* $\rho^{-1}(b_1)$ *and* $\rho^{-1}(b_2)$*.*
- *For each component* C'_η*, write* c_η *for the degree of* $\rho|_{C'_\eta} : C'_\eta \to C$*. For components* C'_η *of type (i), write* β_η *for the degree of* $f'|_{C'_\eta} : C'_\eta \to L_{(i_1,i_2)}$*, and* $q_\eta := \beta_\eta / c_\eta$*. Then* $q := q_\eta$ *is independent of the type (i) component* C'_η*.*

Proof. The first three statements follow from the fact that C is genus zero with exactly two orbifold points, and from Proposition 3.1. It is a straightforward computation in coordinates to check that the last statement is equivalent to the fact that the T-action is compatible with the map ρ, i.e. that the action of $\lambda \in T$ is equivalent to a coordinate change on C. □

Remark 3.4. The same statement and proof apply to $\overline{\mathcal{M}}_{0,1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ and $\overline{\mathcal{M}}_{0,0}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ and in these cases we have a slightly stronger statement: since C has at most one orbifold point, it has no nontrivial étale cover. Thus $C' \cong C \times \{1, \ldots, d\}$ and $c_\eta = 1$ for all η.

From an irreducible T-fixed stable map as in Lemma 3.3, we may extract discrete data (see 2.1 for notation) as follows:

- The rational number q associated to type (i) components of C'.
- The two compositions $f(b_1), f(b_2) \in \mathrm{ZPart}(d, r+1)$. (See Remark 3.2.)
- A refinement of the above: for each $i \in \{0, \ldots, r\}$, the points of C' mapping to P_i are each counted with a multiplicity c_η. Whereas $f(b_1)$ remembers only the sum for each i, we could instead record the list of multiplicities c_η. The result is a multipartition $\mathrm{Mon}(b_1) \in \mathrm{MultiPart}(f(b_1))$. This multipartition describes the monodromy of f at b_1 as a conjugacy class in $G_{f(b_1)}$. Similarly $\mathrm{Mon}(b_2) \in \mathrm{MultiPart}(f(b_2))$.

3.3. **Decorated graphs.** Having classified irreducible components of T-fixed stable maps to $\mathrm{Sym}^d \mathbb{P}^r$, we will now describe how these components fit together. Following [Liu13], we introduce combinatorial objects called *decorated graphs*, which capture the combinatorial data of elements of $(\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T$.

Definition 3.5. An *n-marked genus-g $\mathrm{Sym}^d \mathbb{P}^r$-decorated graph* $(\Gamma, \mathrm{Mark}, \{g_v\}, \mathrm{VEval}, q, \overrightarrow{\mathrm{Mon}})$ is

- A graph Γ,
- A marking map $\mathrm{Mark} : \{1, \ldots, n\} \to V(\Gamma)$,
- A "vertex genus" map $V(\Gamma) \to \mathbb{Z}_{\ge 0}$ denoted $v \mapsto g_v$,
- A "vertex evaluation" map $\mathrm{VEval} = (\mathrm{VEval}_0, \ldots, \mathrm{VEval}_r) : V(\Gamma) \to \mathrm{ZPart}(d, r+1)$,
- An "edge degree ratio" map $q : E(\Gamma) \to \mathbb{Q}_{>0}$,
- A "monodromy map" $\mathrm{Mon} = (\mathrm{Mon}_0, \ldots, \mathrm{Mon}_r)$ that assigns to each $j \in \{1, \ldots, n\}$ an element of $\mathrm{MultiPart}(\mathrm{VEval}(\mathrm{Mark}(j)))$ (see Section 2.1), and assigns to each flag $(v, e) \in F(\Gamma)$ an element of $\mathrm{MultiPart}(\mathrm{VEval}(v))$,

subject to the conditions:

1. $h_1(\Gamma) + \sum_{v \in V(\Gamma)} g_v = g$.

(2) Let e be an edge of Γ connecting vertices v and v'. Then there exist two distinct indices $0 \leq i^{\mathrm{mov}}(v,e), i^{\mathrm{mov}}(v',e) \leq r$ such that:
  - $\mathrm{VEval}_{i^{\mathrm{mov}}(v,e)}(v) - \mathrm{VEval}_{i^{\mathrm{mov}}(v,e)}(v') > 0$.
  - If $i \notin \{i^{\mathrm{mov}}(v,e), i^{\mathrm{mov}}(v',e)\}$, then $\mathrm{VEval}_i(v) = \mathrm{VEval}_i(v')$ and $\mathrm{Mon}_i(v,e) = \mathrm{Mon}_i(v',e)$ (as partitions of $\mathrm{VEval}_i(v)$).
  - There are containments $\mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v',e) \subseteq \mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v,e)$ and $\mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v,e) \subseteq \mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v',e)$, and the relation between complements holds:

$$\mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v,e) \smallsetminus \mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v',e) = \mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v',e) \smallsetminus \mathrm{Mon}_{i^{\mathrm{mov}}(v',e)}(v,e).$$

  - For $\eta \in \mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v,e) \smallsetminus \mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v',e)$, we have $\eta \in \frac{1}{q(e)}\mathbb{Z}$.

(3) If $v \in V(\Gamma)$ with $g_v = 0$, $E(\Gamma, v) = \{e_v\}$, and $\mathrm{Mark}^{-1}(v) = \emptyset$, then $\mathrm{Mon}(v, e_v)$ is the "trivial" multipartition of $\mathrm{MultiPart}(\mathrm{VEval}(v))$ whose elements are all 1.

(4) If $v \in V(\Gamma)$ with $g_v = 0$, $E(\Gamma, v) = \{e_v\}$, and $\mathrm{Mark}^{-1}(v) = \{j\}$, then $\mathrm{Mon}(v, e_v) = \mathrm{Mon}(j)$.

(5) If $v \in V(\Gamma)$ with $g_v = 0$, $E(\Gamma, v) = \{e_v^1, e_v^2\}$, and $\mathrm{Mark}^{-1}(v) = \emptyset$, then $\mathrm{Mon}(v, e_v^1) = \mathrm{Mon}(v, e_v^2)$.

For brevity, we will write Γ instead of $(\Gamma, \mathrm{Mark}, \{g_v\}, \mathrm{VEval}, q, \overrightarrow{\mathrm{Mon}})$. For a fixed Γ, we introduce notation:

- Each part η of the multipartitions $\mathrm{Mon}(v,e)$ and $\mathrm{Mon}(j)$ is an element of one of the multisets $(\mathrm{Mon}_0, \dots, \mathrm{Mon}_r)$, and we write $i(\eta)$ for the element of $\{0, \dots, r\}$ such that $\eta \in \mathrm{Mon}_{i(\eta)}$.
- Let $\mathrm{Mov}(e)$ be the difference multiset $\mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v,e) \smallsetminus \mathrm{Mon}_{i^{\mathrm{mov}}(v,e)}(v',e)$, and let $\mathrm{Stat}(e) := \mathrm{Mon}(v,e) \smallsetminus \mathrm{Mov}(e)$ be its complement. By condition 2, $\mathrm{Mov}(e)$ and $\mathrm{Stat}(e)$ depend on e rather than (v,e). $\mathrm{Mov}(e)$ is the submultiset of "moving parts" of $\mathrm{Mon}(v,e)$ (or $\mathrm{Mon}(v',e)$), and $\mathrm{Stat}(e)$ is the submultiset of "stationary parts". Note that $\mathrm{Stat}(e)$ is a $\{0, \dots, r\}$-labeled multiset. We write $\mathrm{mov}(e) := |\mathrm{Mov}(e)|$.
- Let $\mathrm{Mon}(e)$ be the partition $\bigcup_k \mathrm{Mon}_k(v,e)$ of d, which again by condition 2 depends only on e. Note that unlike $\mathrm{Mon}(v,e)$ and $\mathrm{Mon}(j)$, $\mathrm{Mon}(e)$ is only a partition of d, rather than a multipartition.
- For v satisfying any one of conditions 3, 4, or 5, we write $\mathrm{Mon}(v)$ for $\mathrm{Mon}(v, e_v)$ or $\mathrm{Mon}(v, e_v^1) = \mathrm{Mon}(v, e_v^2)$.
- For an edge $e \in E(\Gamma)$, let $\beta(e) = \sum_{\eta \in \mathrm{Mov}(e)} \beta_\eta(e) := \sum_{\eta \in \mathrm{Mov}(e)} q(e)\eta$. Let $\beta(\Gamma) = \sum_{e \in E(\Gamma)} \beta(e)$.
- Denote by $\mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ the finite set of n-marked genus-g $\mathrm{Sym}^d \mathbb{P}^r$-decorated graphs Γ with $\beta(\Gamma) = \beta$. We refer to these as simply "decorated graphs" when no confusion is possible.

Lemma 3.6. *There is a natural map*

$$\Psi : (\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T \to \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta).$$

Proof. Let $(f : (C, b_1, \dots, b_n) \to \mathrm{Sym}^d \mathbb{P}^r) \in (\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T$. Define sets $V(\Gamma)$ equal to the set of connected components of $f^{-1}((\mathrm{Sym}^d \mathbb{P}^r)^T)$, and $E(\Gamma)$ the set of noncontracted irreducible components of C. By Lemma 3.3, associated to each noncontracted irreducible component of C are two T-fixed points P_{i_1} and P_{i_2}, so these define a graph Γ.

We now define the various decorations of Γ. Let $\mathrm{Mark}(j)$ be the connected component of $f^{-1}((\mathrm{Sym}^d \mathbb{P}^r)^T)$ containing b_j. Let $\mathrm{VEval}(v)$ be the $(r+1)$-tuple representing the T-fixed point $f(v)$, from Section 3.1. Let $q(e) = q$ be the rational number determined by Lemma 3.3. Let $\mathrm{Mon}(j)$ be the monodromy of f at b_j. This is a conjugacy class in the isotropy group $G_{f(b_j)}$, and these are in natural bijection with $\mathrm{MultiPart}(\mathrm{VEval}(\mathrm{Mark}(j)))$. Finally, let $\mathrm{Mon}(v,e)$ be the monodromy of

f at the point $\xi(v,e)$ where the connected component v meets the irreducible component e; this monodromy is naturally an element of $\mathrm{MultiPart}(\mathrm{VEval}(v))$.

Condition (2) for decorated graphs follows from the description in Lemma 3.3. Condition (3) follows from Remark 3.4. Condition (4) holds because for such v, $\xi(v,e_v)$ and b_j are the same point of C. Condition (5) is true for the same reason, together with the fact that the inverse of a conjugacy class in S_d is itself. $\square$

3.4. **Classifying the connected components of $(\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta))^T$.** The map in Lemma 3.6 gives a stratification of $(\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta))^T$ into (as we will see) locally closed substacks. In this section we describe how the strata fit together. To be precise, what we show does not quite classify connected components, but rather certain open and closed substacks — see Remark 3.18.

Notation 3.7. Let $(f : C \to \mathrm{Sym}^d\,\mathbb{P}^r) \in \Psi^{-1}(\Gamma)$. If $v \in V(\Gamma)$, then from Lemma 3.6, v corresponds to a subcurve of C. We denote this by C_v. Similarly, for $e \in E(\Gamma)$, we write C_e for the corresponding irreducible component of C. For $(v,e) \in F(\Gamma)$, we write $\xi(v,e)$ for the point $v \cap e \in C$, again using the notation of the proof of Lemma 3.6. We say (v,e) is a *special flag* if $\xi(v,e)$ is a special point, equivalently if $g_v > 0$ or $\mathrm{val}(v) > 1$ or $\mathrm{Mark}^{-1}(v) \neq \emptyset$. Note that the isotropy group at $\xi(v,e)$ (resp. b_j) has order $\mathrm{lcm}(\mathrm{Mon}(v,e))$ (resp. $\mathrm{lcm}(\mathrm{Mon}(j))$). For brevity we denote this by $r(v,e)$ (resp. r_j).

We adopt the following notation from [Liu13], corresponding to conditions 3, 4, and 5 in Definition 3.5:

$$\begin{aligned}
V^1(\Gamma) &= \{v \in V(\Gamma) | g_v = 0, \mathrm{val}(v) = 1, \left|\mathrm{Mark}^{-1}(v)\right| = 0\}\\
V^{1,1}(\Gamma) &= \{v \in V(\Gamma) | g_v = 0, \mathrm{val}(v) = 1, \left|\mathrm{Mark}^{-1}(v)\right| = 1\}\\
V^2(\Gamma) &= \{v \in V(\Gamma) | g_v = 0, \mathrm{val}(v) = 2, \left|\mathrm{Mark}^{-1}(v)\right| = 0\}\\
V^S(\Gamma) &= V(\Gamma) \smallsetminus (V^1(\Gamma) \cup V^{1,1}(\Gamma) \cup V^2(\Gamma)).
\end{aligned}$$

We call vertices in $V^S(\Gamma)$ *stable.* A vertex v is stable if and only if C_v is 1-dimensional (rather than a single point).

For $v \in V^1(\Gamma) \cup V^{1,1}(\Gamma)$, we always write $E(\Gamma, v) = \{e_v = (v, v')\}$. For $v \in V^2(\Gamma)$, we always write $E(\Gamma, v) = \{e_v^1 = (v, v_1), e_v^2 = (v, v_2)\}$.

Definition 3.8. Let $\Gamma \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$, and let $e_1, e_2 \in E(\Gamma)$. We say e_1 and e_2 are *combinable*, and write $e_1 \| e_2$, if there exists $v \in V^2(\Gamma)$ with $\{e_1, e_2\} = \{e_v^1, e_v^2\}$ and the following hold:

- $q(e_1) = q(e_2)$,
- $i^{\mathrm{mov}}(v_1, e_1) = i^{\mathrm{mov}}(v, e_2)$ and $i^{\mathrm{mov}}(v, e_1) = i^{\mathrm{mov}}(v_2, e_2)$.

Denote by $\mathcal{P} \subseteq \binom{E(\Gamma)}{2}$ the set of pairs $\{\{e_1, e_2\} : e_1 \| e_2\}$.

Definition 3.9. Let $(v,e) \in F(\Gamma)$. We say (v,e) is a *steady flag* if either of the following holds:

(1) $v \notin V^2(\Gamma)$, or
(2) $v \in V^2(\Gamma)$ and $\{e_v^1, e_v^2\} \notin \mathcal{P}$.

Definition 3.10. Let $\Gamma \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$ and let $e_1 \| e_2$ be a pair of combinable edges. We may define a new decorated graph $\mathrm{Comb}(\Gamma, e_1 \| e_2) \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d\,\mathbb{P}^r, \beta)$ by **combining e_1 and e_2**. In other words, we delete the vertex v and the edges e_1 and e_2, and add an edge $e_{12} = (v_1, v_2)$ with $q(e_{12}) = q(e_1) = q(e_2)$, $\mathrm{Mon}(v_1, e_{12}) = \mathrm{Mon}(v_1, e_1)$, and $\mathrm{Mon}(v_2, e_{12}) = \mathrm{Mon}(v_2, e_2)$. (See Figure 1.) It is easy to check that $\mathrm{Comb}(\Gamma, e_1 \| e_2)$ satisfies the two conditions of a decorated graph, and that $\mathrm{Mov}(e_{12}) = \mathrm{Mov}(e_1) \cup \mathrm{Mov}(e_2)$, and $\mathrm{Mon}(e_{12}) = \mathrm{Mon}(e_1) = \mathrm{Mon}(e_2)$. There is a natural map $\phi_{e_1,e_2} : E(\Gamma) \to E(\mathrm{Comb}(\Gamma, e_1 \| e_2))$ with $\phi_{e_1,e_2}(e_1) = \phi_{e_1,e_2}(e_2) = e_{12}$, and $\phi_{e_1,e_2}(e) = e$ for $e \in E(\Gamma) \smallsetminus \{e_1, e_2\}$.

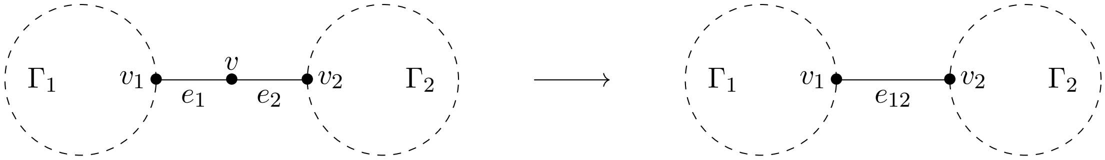

FIGURE 1. Combining edges

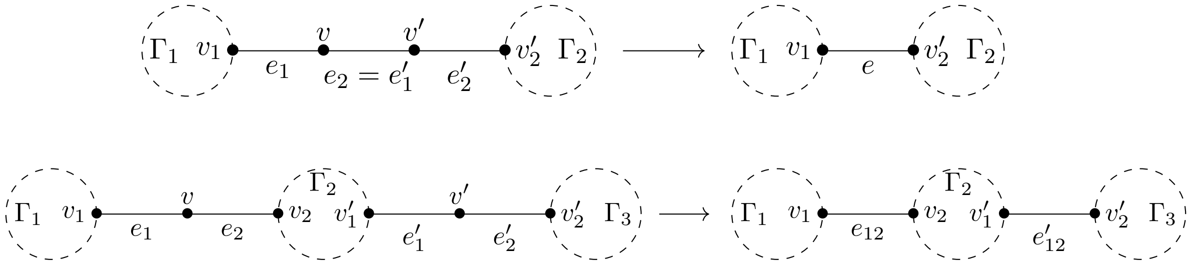

FIGURE 2. Combining two pairs of edges

Proposition 3.11. *Let* $\Gamma \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$*, and let* $e_1 \| e_2$ *and* $e_1' \| e_2'$ *be two distinct pairs of combinable edges of* Γ*. Then* $\phi_{e_1,e_2}(e_1') \| \phi_{e_1,e_2}(e_2')$ *as edges of* $\mathrm{Comb}(\Gamma, e_1 \| e_2)$ *and* $\phi_{e_1',e_2'}(e_1) \| \phi_{e_1',e_2'}(e_2)$ *as edges of* $\mathrm{Comb}(\Gamma, e_1' \| e_2')$*. Also, combining pairs commutes, i.e.*

$$\mathrm{Comb}(\mathrm{Comb}(\Gamma, e_1 \| e_2), e_1' \| e_2') \cong \mathrm{Comb}(\mathrm{Comb}(\Gamma, e_1' \| e_2'), e_1 \| e_2),$$

and this isomorphism identifies the maps $\phi_{e_1,e_2} \circ \phi_{e_1',e_2'}$ *and* $\phi_{e_1',e_2'} \circ \phi_{e_1,e_2}$*.*

Proof. There are two cases, pictured in the left side of Figure 2; either the pairs $e_1 \| e_2$ and $e_1' \| e_2'$ share an edge, or they do not. Suppose we are in the first case, i.e. the top line of Figure 2. By definition of ϕ_{e_1,e_2}, the edges $\phi_{e_1,e_2}(e_1')$ and $\phi_{e_1,e_2}(e_2')$ meet at v' (precisely, at the corresponding vertex in $\mathrm{Comb}(\Gamma, e_1 \| e_2)$), and satisfy the three conditions of Definition 3.8. Thus $\phi_{e_1,e_2}(e_1') \| \phi_{e_1,e_2}(e_2')$. Similarly $\phi_{e_1',e_2'}(e_1) \| \phi_{e_1',e_2'}(e_2)$. To see that $\mathrm{Comb}(\mathrm{Comb}(\Gamma, e_1 \| e_2), e_1' \| e_2') \cong \mathrm{Comb}(\mathrm{Comb}(\Gamma, e_1' \| e_2'), e_1 \| e_2)$, we note that both are obtained from the graph in Figure 2 by replacing the three edges shown with a single edge e connecting v_1 to v_2'. The decorations on this edge are:

- $q(e) := q(e_1) = q(e_2) = q(e_2')$,
- $\mathrm{Mon}(e) := \mathrm{Mon}(e_1) = \mathrm{Mon}(e_2) = \mathrm{Mon}(e_2')$,
- $i^{\mathrm{mov}}(v_1, e) := i^{\mathrm{mov}}(v_1, e_1) = i^{\mathrm{mov}}(v, e_2) = i^{\mathrm{mov}}(v', e_2')$, and
- $i^{\mathrm{mov}}(v_2', e) := i^{\mathrm{mov}}(v_2, e_2') = i^{\mathrm{mov}}(v', e_2) = i^{\mathrm{mov}}(v, e_1)$,

where the equalities follow from $e_1 \| e_2$ and $e_2 \| e_2'$. The maps $\phi_{e_1,e_2} \circ \phi_{e_1',e_2'}$ and $\phi_{e_1',e_2'} \circ \phi_{e_1,e_2}$ both send all of e_1, $e_2 = e_1'$, and e_2' to e.

The second case (the bottom line of 2) is a special case of this argument, so we omit it. ☐

Corollary 3.12. *Let* $\Gamma \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$*, and let* $\mathcal{E}$ *be any subset of the set* $\mathcal{P}(\Gamma)$ *of pairs of combinable edges in* Γ*. Then there is a well-defined graph* $\mathrm{Comb}(\Gamma, \mathcal{E}) \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ *obtained by combining all edge pairs in* $\mathcal{E}$*, in any order, and a well-defined associated map* $\phi_{\mathcal{E}} : E(\Gamma) \to E(\mathrm{Comb}(\Gamma, \mathcal{E}))$*. Furthermore,* $\mathcal{E}$ *is determined by the graphs* Γ *and* $\mathrm{Comb}(\Gamma, \mathcal{E})$*, and the map* $\phi_{\mathcal{E}}$*.*

Proof. The existence statement comes from repeatedly applying Proposition 3.11. The uniqueness statement amounts to the fact that if $e_1 \| e_2$ is a combinable pair of edges in Γ, then $\phi_{\mathcal{E}}(e_1) = \phi_{\mathcal{E}}(e_2)$

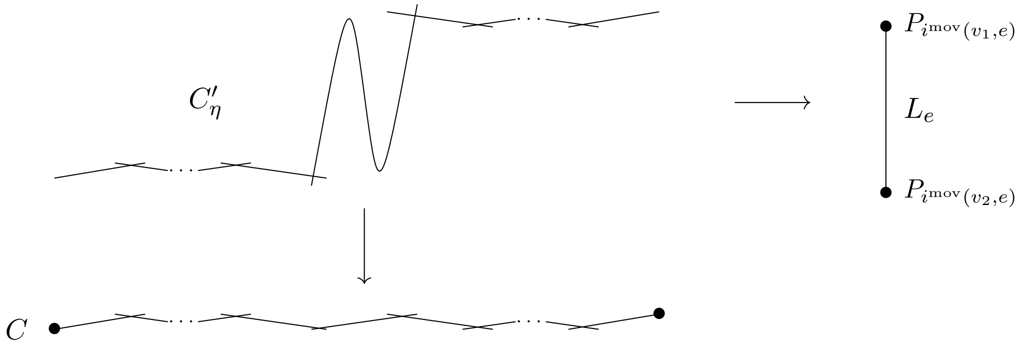

FIGURE 3. A portion of a map in $\overline{\Psi^{-1}(\Gamma_0)}$, with $\eta = 1$ and $q(e) = 3$

if and only if $(e_1, e_2) \in \mathcal{E}$. This follows from factoring $\phi_{\mathcal{E}}$ as a sequence of edge combination maps as in Definition 3.10. ☐

Corollary 3.12 may be restated as follows. Definition 3.10 determines a partial order $\leq$ on $\mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, where $\Gamma' \leq \Gamma$ if Γ' can be obtained from Γ by combining edges. Corollary 3.12 then states that for $\Gamma \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, there is a natural order-reversing bijection between $\{\Gamma' : \Gamma' \leq \Gamma\}$ and $\{\text{subsets of } \mathcal{P}(\Gamma)\}$, where the latter is partially ordered by inclusion. In particular, associated to Γ is a unique *minimal* decorated graph $\mathrm{Comb}(\Gamma, \mathcal{P}(\Gamma))$. Denote by $\mathrm{Graphs}^{\min}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ the set of $\leq$-minimal elements of $\mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$.

Theorem 3.13. *Let* $\Gamma_0 \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. *The closure of* $\Psi^{-1}(\Gamma_0)$ *is*

$$\bigcup_{\substack{\Gamma \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta) \\ \Gamma_0 \leq \Gamma}} \Psi^{-1}(\Gamma),$$

where Ψ *is the map from Lemma 3.6.*

Lemma 3.14. *Let* $\Gamma_0 = v_1 \overset{e}{\bullet\!\!-\!\!-\!\!-\!\!\bullet} v_2$, *where each of* v_1 *and* v_2 *contains a single marked point,* b_1 *and* b_2, *and* $g_{v_1} = g_{v_2} = 0$. *Let* $f : C \to \mathrm{Sym}^d \mathbb{P}^r$ *be in the closure of* $\Psi^{-1}(\Gamma_0)$, *and let* $\rho : C' \to C$ *and* $f' : C' \to \mathbb{P}^r$ *be the associated maps. Write* C'_η *for a noncontracted irreducible component of* C', *corresponding to* $\eta \in \mathrm{Mov}(e) \subseteq \mathrm{Mon}(e)$, *as described in Lemma 3.3. Denote by* $L_e := L_{(i^{\mathrm{mov}}(v_1,e), i^{\mathrm{mov}}(v_2,e))}$ *the line in* $\mathbb{P}^r$ *connecting* $P_{i^{\mathrm{mov}}(v_1,e)}$ *and* $P_{i^{\mathrm{mov}}(v_2,e)}$. *Then:*

(1) C *and* C'_η *are nodal chains of rational curves,*
(2) $f'|_{C'_\eta}$ *maps one irreducible component of* C'_η *to* L_e *with degree* $\beta_\eta(e) = q(e) \cdot \eta$ *(on coarse moduli spaces), and is fully ramified at the two special points of this component, and*
(3) $f'|_{C'_\eta}$ *contracts all other irreducible components of* C'_η *to one of the endpoints of* L_e.

That is, the restriction to C'_η *of a point in* $\overline{\Psi^{-1}(\Gamma_0)}$ *may be represented as in Figure 3 (where despite appearances we mean for the map to* L_e *to have a single preimage point over each of* $P_{i^{\mathrm{mov}}(v_1,e)}$ *and* $P_{i^{\mathrm{mov}}(v_2,e)}$*).*

Proof of Lemma. Let $f : \mathcal{C} \to \mathbb{P}^r$ be a family over S of stable maps whose generic fiber is in $\Psi^{-1}(\Gamma_0)$, and let $s \in S$ such that the fiber over s is the stable map $f : C \to \mathrm{Sym}^d \mathbb{P}^r$. After an étale base change $\tilde{S} \to S$, $\mathcal{C}'$ is a union of connected components $\mathcal{C}'_\eta$ indexed by $\mathrm{Mon}(e)$, and the maps $\mathcal{C}'_\eta \to \mathcal{C}$ have degrees determined by $\mathrm{Mon}(e)$. Fix $\eta \in \mathrm{Mov}(e)$.

Consider the Stein factorization of f' relative to S:

$$\begin{tikzcd}
\mathcal{C}'_\eta \arrow[r, "\mathrm{sf}"] \arrow[rrr, bend right, "f'"'] & \overline{\mathcal{C}'_\eta} \arrow[r] \arrow[rr, bend left, "\overline{f'}"] & \mathbb{P}^r \times S \arrow[r] & \mathbb{P}^r
\end{tikzcd}$$

(The map sf contracts connected components of fibers of $\mathcal{C}'_\eta$ over $\mathbb{P}^r \times S$.) On a generic fiber of $\overline{\mathcal{C}'_\eta}$ over S, the divisors $\overline{f'}^*(P_{i^{\mathrm{mov}}(v_1,e)}$ and $\overline{f'}^*(P_{i^{\mathrm{mov}}(v_2,e)}$ are each supported on a single point. By the definition of sf, on the special fiber $\overline{C'_\eta}$, these divisors are each supported on a connected locus, hence a single point — specifically, the points $\mathrm{sf}(\rho^{-1}(b_1))$ and $\mathrm{sf}(\rho^{-1}(b_2))$, respectively. As any component of $\overline{C'_\eta}$ maps surjectively to L_e, this implies that $\overline{C'_\eta}$ is irreducible. This proves claims (2) and (3).

Since f' is T-fixed, the above implies that a component of C'_η not contracted by f' has exactly two points that are nodes or are in $\rho^{-1}(b_1)$ or $\rho^{-1}(b_2)$.

If C is not a chain, then since it is genus zero, some component D has only one special point. By stability, there is a component of $\rho^{-1}(D)$ that is not contracted by f'. This contradicts the previous paragraph. Thus C is a chain, and it follows that each C'_η is a chain. This proves claim (1). ☐

Proof of Theorem 3.13. It is sufficient to consider the situation of Lemma 3.14. To see this, note that any $\Gamma_0 \in \mathrm{Graphs}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ may be decomposed into subgraphs of the form in the Lemma, together with single-vertex graphs, glued at marked points. There is a corresponding decomposition of $\Psi^{-1}(\Gamma_0)$ as a product (up to a finite morphism), and this decomposition extends to the closure (see [AGV08], Section 5.2, or [Liu13], Section 9.2). Thus we may treat each factor of the product separately.

First, we show

$$\overline{\Psi^{-1}(\Gamma_0)} \subseteq \bigcup_{\Gamma \geq \Gamma_0} \Psi^{-1}(\Gamma).$$

Let $(f : C \to \mathbb{P}^r) \in \overline{\Psi^{-1}(\Gamma_0)}$. By Lemma 3.14, we conclude:

- $\Psi(f : C \to \mathrm{Sym}^d \mathbb{P}^r)$ is a chain.
- The degree ratios $q(e)$ are equal for all edges e.
- The partitions $\mathrm{Mon}(e)$ are equal for all edges e.
- For any edge $e = (v, v')$, where v and Mark(1) are on the same connected component of $\Gamma \smallsetminus \{e\}$, we have $i^{\mathrm{mov}}(v, e) = i^{\mathrm{mov}}(v_1, e_{12})$ and $i^{\mathrm{mov}}(v', e) = i^{\mathrm{mov}}(v_2, e_{12})$. (This follows from the proof of Lemma 3.14.

Thus any pair of adjacent edges in $\Psi(f : C \to \mathrm{Sym}^d \mathbb{P}^r)$ is combinable. Combining them all yields Γ_0, i.e. $\Gamma_0 \leq \Psi(f : C \to \mathrm{Sym}^d \mathbb{P}^r)$.

For the reverse inclusion, first suppose $\Gamma \geq \Gamma_0$ has a single pair of combinable edges, i.e.

$$\Gamma = v_1 \bullet\!\overset{e_1}{\text{———}}\!\overset{v}{\bullet}\!\overset{e_2}{\text{———}}\!\bullet v_2.$$

Fix $(f : C \to \mathrm{Sym}^d \mathbb{P}^r) \in \Psi^{-1}(\Gamma)$. We will construct a family $f : \mathcal{C} \to \mathrm{Sym}^d \mathbb{P}^r$ over $\mathbb{C}$ whose restriction to $0 \in \mathbb{C}$ is the map $f : C \to \mathrm{Sym}^d \mathbb{P}^r$.

By Lemma 3.3 and by representability of $f : C \to \mathrm{Sym}^d \mathbb{P}^r$, the orbifold points and nodes of C have order $\mathrm{lcm}(\mathrm{Mon}(e_1)) = \mathrm{lcm}(\mathrm{Mon}(e_2))$. Thus C is isomorphic to $V(xy) \subseteq [\mathbb{P}^2/\mu_{\mathrm{lcm}(\mathrm{Mon}(e_1))}]$, where $\mathbb{P}^2$ has coordinates x, y, z, and $\mathrm{lcm}(\mathrm{Mon}(e_1))$ acts by multiplication by inverse roots of unity on the first two coordinates. Define $\mathcal{C}$ so that $\mathcal{C}_t = V(xy - tz^2)$ for $t \in \mathbb{C}$. Precisely, $\mathcal{C}$ is an open subset of $\left[\mathcal{B}\ell_{[1:0:0],[0:1:0]}\mathbb{P}^2/\mu_{\mathrm{lcm}(\mathrm{Mon}(e_1))}\right]$.

For $\eta \in \mathrm{Mon}(e_1)$ a part, there is an étale quotient map $\tilde{\rho} : [\mathbb{P}^2/\mu_\eta] \to [\mathbb{P}^2/\mu_{\mathrm{lcm}(\mathrm{Mon}(e_1))}]$. As above, define $(\mathcal{C}'_\eta)_t = V(xy - tz^2) \subseteq [\mathbb{P}^2/\mu_\eta]$.

We must now define a map $\tilde{f}' : \mathcal{C}'_\eta \to \mathbb{P}^r$ for each $\eta \in \mathrm{Mon}(e_1)$. As $\mathbb{P}^r$ is a variety, it is enough to define this on coarse moduli spaces. We choose isomorphisms of the fibers $(\mathcal{C}'_\eta)_0$ and $\mathcal{C}_0$ with C'_η and C respectively, such that the maps $\tilde{\rho}$ and ρ are identified. Then f' defines a map $\tilde{f}'_0 : (\mathcal{C}'_\eta)_0 \to L_{e_1} = L_{e_2}$. (The case where C'_η is contracted is trivial, so we assume it is not contracted.) By Lemma 3.14, after equivariantly identifying $L_{e_1} \cong \mathbb{P}^1$, $\tilde{f}'_0$ is given (without loss of generality, on coarse moduli spaces) by

$$\begin{aligned}
&[x : 0 : z] \mapsto [0 : 1] \\
&[0 : y : z] \mapsto [y^{\beta_\eta(e_1)} : z^{\beta_\eta(e_1)}].
\end{aligned}$$

It remains to extend this to a map $\tilde{f}' : \mathcal{C}'_\eta \to L_{e_1}$ that is fixed with respect to the T-action, i.e. fully ramified over the endpoints of L_{e_1}. We observe that the rational map

$$[x : y : z] \mapsto [y^{\beta_\eta(e_1)} : z^{\beta_\eta(e_1)}]$$

is regular after blowing up the point $[1 : 0 : 0]$. This defines a map $\tilde{f}'$ as desired. Doing this for all η shows that $f : C \to \mathrm{Sym}^d \mathbb{P}^r$ is in $\overline{\Psi^{-1}(\Gamma_0)}$.

If Γ has more than one pair of combinable edges, we apply this argument repeatedly. ☐

Corollary 3.15. *$(\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T$ is a disjoint union of open and closed substacks $\overline{\Psi^{-1}(\Gamma)}$, for $\Gamma \in \mathrm{Graphs}^{\min}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. We define $\overline{\mathcal{M}}_\Gamma := \overline{\Psi^{-1}(\Gamma)}$.*

3.5. Explicit description of $\overline{\mathcal{M}}_\Gamma$.

The rest of this section proves the following:

Theorem 3.16. *For a stable vertex v or edge $e = (v_1, v_2)$ of a minimal decorated graph $\Gamma = (\Gamma, \mathrm{Mark}, \{g_v\}, \mathrm{VEval}, q, \mathrm{Mon}) \in \mathrm{Graphs}^{\min}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$, we define*

$$\overline{\mathcal{M}}_v := \overline{\mathcal{M}}_{g_v, \overrightarrow{\mathrm{Mon}}(v)}(BS_{\mathrm{VEval}(v)}, 0)$$

$$\overline{\mathcal{M}}_e := \left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \Big/ \left(\prod_{\eta \in \mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \operatorname{wr} S_e \right) \right],$$

where:

- *$\overrightarrow{\mathrm{Mon}}(v)$ is the list of multipartitions $\{\mathrm{Mon}(i)\}_{i \in \mathrm{Mark}^{-1}(v)} \cup \{\mathrm{Mon}(v,e)\}_{e \in E(\Gamma, v)}$,*
- *$\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$ is the order $\mathrm{lcm}(\mathrm{Mon}(e))$ orbifold Losev-Manin space with $\mathrm{mov}(e)$ marked points $b_1, \ldots, b_{\mathrm{mov}(e)}$ and labeling set $\{v_1, v_2\}$, from Section 2.5,*
- *S_e is the group $C_{\mathrm{Stat}}(e) \times S_{\mathrm{Mov}(e)}$, where $C_{\mathrm{Stat}}(e)$ is the centralizer of any element of the conjugacy class $\mathrm{Stat}(e)$ in $\prod_{i=0}^r S_{|\mathrm{Stat}(e)_i|}$, and acts trivially on the Losev-Manin space,*
- *A generator of $\mu_{\beta_\eta(e)}$ acts by translating the marked point b_η by $e^{2\pi i/q(e)}$, and*
- *wr denotes the wreath product.*

Then the substack $\overline{\mathcal{M}}_\Gamma$ associated to Γ is isomorphic to a $\left(\prod_{(v,e)\ \mathit{steady}} \overline{C}_{\mathrm{VEval}(v)}(\mathrm{Mon}(v,e)) \right)$-gerbe over

$$\left[\left(\prod_{v \in V^S(\Gamma)} \overline{\mathcal{M}}_v \times \prod_{e \in E(\Gamma)} \overline{\mathcal{M}}_e \right) \Big/ \mathrm{Aut}(\Gamma) \right], \tag{6}$$

where $\overline{C}_{\mathrm{VEval}(v)}(\mathrm{Mon}(v,e))$ *is the centralizer in* $G_{\mathrm{VEval}(v)}$ *of any element of the conjugacy class* $\mathrm{Mon}(v,e)$*, modulo the subgroup generated by that element.*

Proof of 3.16. Using Theorem 3.13, Lemma 3.14, and the gluing morphisms for $\overline{\mathcal{M}}_{g,n}(X,\beta)$ (see [AGV08], Section 5.2), $\overline{\mathcal{M}}_\Gamma$ is a $\left(\prod_{(v,e)\text{ steady}} \overline{C}_{\mathrm{VEval}(v)}(\mathrm{Mon}(v,e))\right)$-gerbe over

$$\left[\left(\prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g_v,\overrightarrow{\mathrm{Mon}}(v)}(BS_{\mathrm{VEval}(v)},0) \times \prod_{e\in E(\Gamma)} \overline{\mathcal{M}}_{\Gamma_e}\right) \Big/ \mathrm{Aut}(\Gamma)\right],$$

where $\Gamma_e = v_1 \overset{e}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} v_2,$ and the decorations are inherited from Γ, with $g_{v_1} = g_{v_2} = 0$. (Note that the two vertices of Γ_e are labeled, i.e. $\mathrm{Aut}(\Gamma_e) = 1$.)

(The gerbe structure appears because gluing morphisms are fibered over the rigidified inertia stack $\overline{I}\,\mathrm{Sym}^d\,\mathbb{P}^r$, see [AGV08] or [Liu13]. The group $\overline{C}_{\mathrm{VEval}(v)}(\mathrm{Mon}(v,e))$ is the isotropy group of $\overline{I}\,\mathrm{Sym}^d\,\mathbb{P}^r$ at the point of $\overline{I}\,\mathrm{Sym}^d\,\mathbb{P}^r$ corresponding to $\mathrm{Mon}(v,e)$.)

We need to show that, for all $e = (v_1, v_2) \in E(\Gamma)$, we have

$$\overline{\mathcal{M}}_{\Gamma_e} \cong \left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \Big/ \left(\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \mathrm{wr}\, S_e\right)\right].$$

Write $P_e := P_{(i^{\mathrm{mov}}(v_1,e),\, i^{\mathrm{mov}}(v_2,e))}$ for the midpoint of L_e. For $(f : C \to \mathbb{P}^r) \in \overline{\mathcal{M}}_{\Gamma_e}$, consider the preimage of P_e under the associated map $f' : C' \to \mathbb{P}^r$. By Lemma 3.14, C' is a union of connected components C'_η for $\eta \in \mathrm{Mon}(e)$, and if $\eta \in \mathrm{Mov}(e)$ then the preimage of P_e on C'_η consists of $\beta_\eta(e)$ points on the single noncontracted component of C'_η. These points are $\mu_{\beta_\eta(e)}$-translates of each other, under the natural action that fixes the two special points.

After a principal $\left(\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \mathrm{wr}\, S_e\right)$-cover $\widetilde{\overline{\mathcal{M}}}_{\Gamma_e} \to \overline{\mathcal{M}}_{\Gamma_e}$, we may fix a labeling of the connected components C'_η, and label a distinguished preimage of P_e on C'_η for $\eta \in \mathrm{Mov}(e)$. (The S_e-cover removes all automorphisms of stable maps induced by automorphisms of the image curve that commute with the monodromy at b_{v_1} and b_{v_2}.) Remembering the images of these distinguished points under ρ yields a nodal chain of rational curves with $\mathrm{mov}(e)$ labeled marked points, none of which coincides with b_{v_1} or b_{v_2}. The stability condition for $\overline{\mathcal{M}}_{0,\{\mathrm{Mon}(e),\mathrm{Mon}(e)\}}(L_e, \beta(e))$ implies that this is a Losev-Manin curve, with orbifold points of order $\mathrm{lcm}(\mathrm{Mon}(e))$ at marked points and nodes. This construction works in families, so it defines a map $\widetilde{\overline{\mathcal{M}}}_{\Gamma_e} \to \overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$, which is equivariant by definition with respect to the action of $\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \mathrm{wr}\, S_e$. This gives a map

$$\Phi : \overline{\mathcal{M}}_{\Gamma_e} \to \left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \Big/ \left(\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \mathrm{wr}\, S_e\right)\right].$$

We now construct an inverse to this map. Let $(C, b_{v_1}, b_1, \ldots, b_{\mathrm{mov}(e)}, b_{v_2}) \in \overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$ be a Losev-Manin curve whose points are indexed by the multiset $\mathrm{Mov}(e)$. Fix a curve $C' = \bigsqcup_{\eta\in\mathrm{Mon}(e)} C'_\eta$ with étale maps $\rho_\eta : C'_\eta \to C$ of degree η. This may be done uniquely up to isomorphism. Also, uniquely up to isomorphism (of C' commuting with $\rho : C' \to C$), for each $\eta \in \mathrm{Mov}(e) \subseteq \mathrm{Mon}(e)$ we may choose a preimage point $b'_\eta \in C'_\eta$ of the corresponding marked point $b_\eta \in C$. Finally, there is a unique map $f' : C' \to \mathbb{P}^r$ that sends:

- C'_η to a T-fixed point, for $\eta \notin \mathrm{Mov}(e)$,

- C'_η to L_e with degree $\beta_\eta(e)$, with b'_η mapping to P_e, $\rho^{-1}(b_{v_1})$ mapping to $P_{i^{\mathrm{mov}}(v_1,e)}$ and $\rho^{-1}(b_{v_2})$ mapping to $P_{i^{\mathrm{mov}}(v_2,e)}$, for $\eta \in \mathrm{Mov}(e)$.

Again, this works in families, and defines a map $\tilde{\Theta} : \overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \to \overline{\mathcal{M}}_{\Gamma_e}$, which we claim is invariant under the action of $\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \operatorname{wr} S_e$. Indeed, acting by $e^{2\pi i/q(e)}$ on b_η translates the preimage b'_η by some power of $e^{2\pi i/\beta_\eta(e)}$, and commutes with f'. Thus $\tilde{\Theta}$ descends to a map

$$\Theta : \left[\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2} \Big/ \left(\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \operatorname{wr} S_e\right)\right] \to \overline{\mathcal{M}}_{\Gamma_e},$$

which is by construction an inverse to Φ. ☐

Corollary 3.17. *The* $\left(\prod_{\eta\in\mathrm{Mov}(e)} \mu_{\beta_\eta(e)} \operatorname{wr} S_e\right)$*-action on* $\overline{\mathcal{M}}^{\mathrm{lcm}(\mathrm{Mon}(e))}_{v_1|\mathrm{mov}(e)|v_2}$ *extends to the universal curve, so we have a universal curve on* $\overline{\mathcal{M}}_e$*, and by gluing, a universal curve on the left side of (6). The isomorphism of 3.16 naturally identifies this with the universal curve on* $\overline{\mathcal{M}}_\Gamma$.

Proof. The first statement is by definition of the action, and the second is immediate from the proof of Theorem 3.16. ☐

Remark 3.18. Theorem 3.16 shows in particular that $\overline{\mathcal{M}}_{\Gamma_e}$ is irreducible, so connected components of $(\overline{\mathcal{M}}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta))^T$ are indexed by minimal decorated graphs with the additional data of a connected component of $\overline{\mathcal{M}}_{g,\overrightarrow{\mathrm{Mon}(v)}}(BS_{\mathrm{VEval}(v)}, 0)$ for each v. (These connected components in turn can computed using elementary group theory.)

Notation 3.19. For a special flag $(v, e) \in F(\Gamma)$, we denote by $\psi_v^{\overline{\mathcal{M}}_e}$ the ψ-class on $\overline{\mathcal{M}}_e$ at the point labeled by v. If $v \in V^S(\Gamma)$, we denote by $\psi_e^{\overline{\mathcal{M}}_v}$ the ψ-class on $\overline{\mathcal{M}}_v$ at the marked point $\xi(v, e)$. We use the same notation for the $\overline{\psi}$-classes.

4. THE VIRTUAL NORMAL BUNDLE AND VIRTUAL FUNDAMENTAL CLASS OF $\overline{\mathcal{M}}_\Gamma$

In this section we compute the Euler class of the virtual normal bundle to $\overline{\mathcal{M}}_\Gamma$, and show that the virtual fundamental class of $\overline{\mathcal{M}}_\Gamma$ is equal to its fundamental class. Some of the arguments are "classical," and we will refer the reader to [Liu13] for these.

In this section we fix $\Gamma \in \mathrm{Graphs}^{\min}_{g,n}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_\Gamma$ and $\rho : \mathcal{C}' \to \mathcal{C}$ denote the universal curve and universal étale cover, respectively:

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "f'"] \arrow[d, "\rho"] & \mathbb{P}^r \\
\mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & \mathrm{Sym}^d \mathbb{P}^r \\
\overline{\mathcal{M}}_\Gamma &
\end{tikzcd}$$

By a standard argument (see [Liu13]), we have an exact sequence of T-equivariant sheaves on $\overline{\mathcal{M}}_{g,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ giving the perfect obstruction theory[3]

$$\begin{aligned} (7) \qquad & 0 \to \mathrm{Aut}(\mathcal{C}) \to R^0\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \mathrm{Def}(\mathcal{C}, f) \to \\ & \to \mathrm{Def}(\mathcal{C}) \to R^1\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \mathrm{Obs}(\mathcal{C}, f) \to 0, \end{aligned}$$

---

[3] We will always use the notation in (7) for higher direct image sheaves, writing e.g. $R^i\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r)$ instead of $R^i\pi_* f^*T\,\mathrm{Sym}^d \mathbb{P}^r$. This is because we will restrict π to various substacks of $\mathcal{C}$, and wish to avoid confusion.

where $\mathrm{Aut}(\mathcal{C})$ (resp. $\mathrm{Def}(\mathcal{C})$) is the sheaf on $\overline{\mathcal{M}}_{g,n+1}(\mathrm{Sym}^d \mathbb{P}^r)$ of infinitesimal automorphisms (resp. deformations) of the marked source curve $\mathcal{C}$. (See [Liu13] for rigorous definitions.) For $(f : C \to \mathrm{Sym}^d \mathbb{P}^r) \in \overline{\mathcal{M}}_\Gamma$, we also have a normalization exact sequence computing the fibers of the middle terms:

$$\begin{aligned} (8) \qquad & 0 \to H^0(C, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \bigoplus_\nu H^0(C_\nu, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \bigoplus_\xi H^0(\xi, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \\ & \to H^1(C, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to \bigoplus_\nu H^1(C_\nu, f^*T\,\mathrm{Sym}^d \mathbb{P}^r) \to 0, \end{aligned}$$

where ν runs over the set of irreducible components of C, and ξ runs over nodes of C. The sequences (7) and (8) each split as direct sums of two exact sequences: the T-fixed part and the T-moving part. We use the notations $\mathrm{Aut}(\mathcal{C})^{\mathrm{fix}}$ and $\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}$ (and similar) to denote the T-fixed subsheaf or subspace and its T-invariant complement. By definition (see [GP99]), the *Euler class of the virtual normal bundle* $e_T(N_\Gamma^{\mathrm{vir}})$ is

$$(9) \qquad \frac{e_T(\mathrm{Def}(\mathcal{C}, f)^{\mathrm{mov}})}{e_T(\mathrm{Obs}(\mathcal{C}, f)^{\mathrm{mov}})} = \frac{e_T(\mathrm{Def}(\mathcal{C})^{\mathrm{mov}}) e_T(R^0\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r)^{\mathrm{mov}})}{e_T(\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}) e_T(R^1\pi_*(\mathcal{C}, f^*T\,\mathrm{Sym}^d \mathbb{P}^r)^{\mathrm{mov}})} \in H_T^*(\overline{\mathcal{M}}_\Gamma),$$

and the *virtual fundamental class* $[\overline{\mathcal{M}}_\Gamma]^{\mathrm{vir}}$ of $\overline{\mathcal{M}}_\Gamma$ is $e_T(\mathrm{Obs}(\mathcal{C}, f)^{\mathrm{fix}})$. We compute the various terms of (7) and (8) one by one. It is convenient to compute by pulling back to the canonical $\mathrm{Aut}(\Gamma)$-cover $\overline{\mathcal{M}}_\Gamma^{\mathrm{rig}}$ of $\overline{\mathcal{M}}_\Gamma$, so that the correspondence between C and Γ is more concrete.

The sheaves $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Def}(\mathcal{C})$. In the toric case, from [Liu13] we have

$$(10) \qquad e_T(\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}) = \prod_{v \in V^1(\Gamma)} e_T(T_{\xi(v,e_v)} C) = \prod_{v \in V^1(\Gamma)} \psi_v^{\overline{\mathcal{M}}_{e_v}}.$$

The same argument and answer apply here, using (Theorem 3.13 and) the observation that combining edges gives a natural identification of $V^1(\Gamma)$. Briefly, moving automorphisms come from noncontracted components with only one special point, and correspond to vector fields on such a component that are nonvanishing at the nonspecial T-fixed point.

Similarly, in the toric case [Liu13] gives

$$(11) \qquad e_T(\mathrm{Def}(\mathcal{C})) = \left(\prod_{\substack{v \in V^2(\Gamma) \\ (v, e_v^1) \text{ steady}}} \left(-\psi_v^{\overline{\mathcal{M}}_{e_v^1}} - \psi_v^{\overline{\mathcal{M}}_{e_v^2}}\right) \right) \left(\prod_{\substack{(v,e) \in F(\Gamma) \\ v \in V^S(\Gamma)}} \left(-\psi_e^{\overline{\mathcal{M}}_v} - \psi_v^{\overline{\mathcal{M}}_e}\right) \right).$$

This is again correct in our case. The factors in (11) come from smoothing nodes. (Classically, the deformation space of a node is the tensor product of the tangent spaces to the two branches.) Therefore the observation we need is that the nodes that do not appear in (11) have T-fixed deformation space. We will use the following notation.

Definition 4.1. A node ξ is called *steady*[4] if $T_\xi C_1 \otimes T_\xi C_2$ has a nontrivial torus action, where C_1 and C_2 are the branches of ξ.

Remark 4.2. Steady nodes are exactly those of the form $\xi(v, e)$ for (v, e) a steady flag. By Theorem 3.13, if $\Psi(f : C \to \mathrm{Sym}^d \mathbb{P}^r) = \Gamma$ (i.e. it is minimal), then all nodes of C are steady nodes. Furthermore, the set of steady nodes is canonically identified for any two points of $\overline{\mathcal{M}}_\Gamma^{\mathrm{rig}}$.

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[4] This is similar, but not identical, to the definition of a *breaking node* from [OP10].

The factors in (11) are in correspondence with steady nodes.

The bundles $R^0\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$ and $R^1\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$. We use the sequence (8). The computation is similar to the original one by Kontsevich [Kon95] (and the orbifold computations of Johnson [Joh14] and Liu [Liu13]), but requires some care due to the edge moduli spaces.

Note that normalization does not commute with base change, so (8) cannot naively be applied to commute $R^i\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$. However, normalization of steady nodes does commute with base change on $\overline{\mathcal{M}}_\Gamma^{\mathrm{rig}}$, due to the canonical identification of nodes above. Thus we have the sequence

$$\begin{aligned}(12)\qquad 0 \to R^0\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \bigoplus_{\underline{\nu}} R^0\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \bigoplus_{\xi} R^0\pi_*(\xi, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \\ \to R^1\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to \bigoplus_{\underline{\nu}} R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \to 0,\end{aligned}$$

where $\underline{\nu}$ runs over closures of maximal subcurves of $\mathcal{C}$ containing only non-steady nodes, and ξ runs over steady nodes. Observe that either $\mathcal{C}_{\underline{\nu}}$ is contracted by f, or each fiber $C_{\underline{\nu}}$ of $\mathcal{C}_{\underline{\nu}}$ contains only noncontracted components.

By Section 2.3, we have

$$R^i\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r) = R^i\pi_*(\mathcal{C}_{\underline{\nu}}, \rho_*(f')^*T\mathbb{P}^r) = R^i(\pi\circ\rho)_*(\mathcal{C}_{\underline{\nu}}{}', (f')^*T\mathbb{P}^r).$$

(The second equality follows from the fact that ρ is étale, hence ρ_* is exact.) After an étale base change, we may distinguish the connected components of fibers of $\mathcal{C}'_{\underline{\nu}} \to \overline{\mathcal{M}}_\Gamma^{\mathrm{rig}}$. In other words, we may write

$$\mathcal{C}'_{\underline{\nu}} = \bigsqcup_{\eta} \mathcal{C}'_{\underline{\nu},\eta},$$

where $\mathcal{C}'_{\underline{\nu},\eta}$ has connected fibers. Then

$$R^i\pi_*(\mathcal{C}'_{\underline{\nu}}, (f')^*T\mathbb{P}^r) = \bigoplus_{\eta} R^i(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r). \tag{13}$$

If $\mathcal{C}_{\underline{\nu}} = \mathcal{C}_v$ is contracted, then $(f')^*T\mathbb{P}^r$ is trivial on $\mathcal{C}'_{\underline{\nu},\eta}$. Thus we have

$$R^i(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r) \cong R^i(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, \mathcal{O}_{\mathcal{C}'_{\underline{\nu},\eta}}) \otimes T_{P_{i(\eta)}}\mathbb{P}^r,$$

where as $i(\eta) \in \{0, \ldots, r\}$ is the label of η, i.e. $P_{i(\eta)} = f'(\mathcal{C}'_{\underline{\nu},\eta})$. In particular,

$$R^0\pi_*(\mathcal{C}_v, f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{fix}} = R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{fix}} = 0. \tag{14}$$

The bundle $R^1\pi_*(\mathcal{C}_v, f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{mov}}$ is nontrivial, and is isomorphic to a Hurwitz-Hodge bundle (see [Liu13], Section 7.5). However, note that $e_T(R\pi_*(\mathcal{C}_v, f^*T\operatorname{Sym}^d\mathbb{P}^r))$ is the inverse of the twisting class from (5). We will use this fact in Section 5 in our characterization of $\mathcal{L}_{\operatorname{Sym}^d\mathbb{P}^r}$, and in Section 6 to apply the orbifold quantum Riemann-Roch theorem.

Similarly for a steady node $\xi(v,e)$, we have

$$\begin{aligned}&R^0\pi_*(\xi(v,e), f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{fix}} = 0\\ (15)\qquad &R^0\pi_*(\xi(v,e), f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{mov}} = T_{(\mathrm{VEval}(v),\mathrm{Mon}(v,e))}I\operatorname{Sym}^d\mathbb{P}^r = \bigoplus_{\eta\in\mathrm{Mon}(v,e)} T_{P_{i(\eta)}}\mathbb{P}^r.\end{aligned}$$

Suppose $\mathcal{C}_{\underline{\nu}}$ is not contracted. The components $\mathcal{C}'_{\underline{\nu},\eta}$ are in bijection with $\mathrm{Mon}(e)$, where e is the edge of Γ corresponding to $\mathcal{C}_{\underline{\nu}}$.) First, we argue that $R^1(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta}, (f')^*T\mathbb{P}^r)$ vanishes for all η.

The normalization exact sequence for a fiber $C'_{\underline{\nu},\eta}$ reads:

$$\begin{aligned}0 \to H^0(C'_{\underline{\nu},\eta},(f')^*T\mathbb{P}^r) \to \bigoplus_{\nu\in\underline{\nu}} H^0(C'_{\nu,\eta},(f')^*T\mathbb{P}^r) \to \bigoplus_{\xi} H^0(\xi,(f')^*T\mathbb{P}^r) \to\\ \to H^1(C'_{\underline{\nu},\eta},(f')^*T\mathbb{P}^r) \to \bigoplus_{\nu\in\underline{\nu}} H^1(C'_{\nu,\eta},(f')^*T\mathbb{P}^r) \to 0,\end{aligned}$$

where we also denote by $\underline{\nu}$ the set indexing irreducible components C_ν of $C_{\underline{\nu}}$ (equivalently, irreducible components $C'_{\nu,\eta}$ of $C_{\underline{\nu},\eta}$). For each $\nu \in \underline{\nu}$, we have

$$H^1(C_\nu,(f')^*T\mathbb{P}^r) = 0 \tag{16}$$

by convexity of $\mathbb{P}^r$. We claim that the map

$$\bigoplus_{\nu\in\underline{\nu}} H^0(C'_{\nu,\eta},(f')^*T\mathbb{P}^r) \to \bigoplus_{\xi} H^0(\xi,(f')^*T\mathbb{P}^r)$$

is surjective, so that $H^1(C'_{\underline{\nu},\eta},(f')^*T\mathbb{P}^r) = 0$. (The map takes the difference of the sections on the two branches of a node.) If $C'_{\underline{\nu},\eta}$ has a component $C'_{\nu_0,\eta}$ not contracted by f', there is at most one, by Lemma 3.14. On any other component $C'_{\nu,\eta}$, we have $(f')^*T\mathbb{P}^r \cong \mathcal{O}_{C'_{\nu,\eta}} \otimes T\mathbb{P}^r$, i.e. $H^0(C'_{\nu,\eta},\mathcal{O}_{C'_{\nu,\eta}} \otimes T\mathbb{P}^r) \cong T\mathbb{P}^r$. Fix an arbitrary section $s \in H^0(C'_{\nu_0,\eta},(f')^*T\mathbb{P}^r)$. Then "working outward" from $C'_{\nu_0,\theta}$ shows that the map is surjective. The case where f' contracts $C'_{\underline{\nu},\eta}$ is similar and simpler.

Next, we compute $R^0(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta},(f')^*T\mathbb{P}^r)$. If $\mathcal{C}'_{\underline{\nu},\eta}$ is contracted, $(f')^*T\mathbb{P}^r$ is trivial and we have

$$R^0(\pi\circ\rho)_*(\mathcal{C}'_{\underline{\nu},\eta},(f')^*T\mathbb{P}^r) \cong T\mathbb{P}^r \otimes \mathcal{O}_{\overline{\mathcal{M}}_\Gamma^{\mathrm{rig}}}$$

by properness of $\pi\circ\rho$. Suppose $\mathcal{C}'_{\underline{\nu},\eta}$ is not contracted. Consider the Stein factorization of $f'|_{\mathcal{C}'_{\underline{\nu},\eta}}$ relative to $\pi\circ\rho$:

$$\begin{tikzcd}
\mathcal{C}'_{\underline{\nu},\eta} \arrow[r, "\mathrm{sf}"] \arrow[rr, bend left, "f'"] \arrow[d, "\pi\circ\rho"'] & \overline{\mathcal{C}'_{\underline{\nu},\eta}} \arrow[r, "f''"'] \arrow[dl, "\overline{\pi\circ\rho}"] & \mathbb{P}^r \\
\overline{\mathcal{M}}_\Gamma^{\mathrm{rig}} & &
\end{tikzcd}$$

If $(f : C \to \mathrm{Sym}^d\,\mathbb{P}^r)$ is in the dense open substack $\Psi^{-1}(\Gamma) \subseteq \overline{\mathcal{M}}_\Gamma^{\mathrm{rig}}$, then $C_{\underline{\nu}}$ is irreducible, hence so is $C'_{\underline{\nu},\eta}$. This, with the fact that $\mathcal{C}'_{\underline{\nu},\eta}$ is not contracted, implies that sf is birational. By the projection formula for coherent sheaves,

$$\begin{aligned}(\pi\circ\rho)_*(f')^*T\mathbb{P}^r &= (\pi\circ\rho)_*\mathrm{sf}^*(f'')^*T\mathbb{P}^r\\ &= (\overline{\pi\circ\rho})_*\mathrm{sf}_*\mathrm{sf}^*(f'')^*T\mathbb{P}^r\\ &= (\overline{\pi\circ\rho})_*((f'')^*T\mathbb{P}^r \otimes \mathrm{sf}_*\mathcal{O}_{C'_{\underline{\nu},\eta}})\\ &= (\overline{\pi\circ\rho})_*(f'')^*T\mathbb{P}^r.\end{aligned}$$

After an étale base change on $\overline{\mathcal{M}}_\Gamma^{\mathrm{rig}}$, the map f'' trivializes $\overline{\mathcal{C}'_{\underline{\nu},\eta}}$. Thus $R^0(\overline{\pi\circ\rho})_*(\overline{\mathcal{C}'_{\underline{\nu},\eta}},(f'')^*T\mathbb{P}^r)$ is a trivial vector bundle. Calculation of the T-weights of this vector bundle is identical to Kontsevich's calculation in Section 3.3.4 of [Kon95], which uses the Euler sequence on $\mathbb{P}^r$. The weights are

$$\frac{A}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_1,e)} + \frac{B}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_2,e)} - \alpha_i, \tag{17}$$

where $0 \leq A, B \leq \beta_\eta(e)$, $A + B = \beta_\eta(e)$, and $i \in \{0, \dots, r\}$. Note that this is zero exactly when $A = 0$ and $i = i^{\mathrm{mov}}(v_2, e)$, or $B = 0$ and $i = i^{\mathrm{mov}}(v_1, e)$. (These factors contribute to $e_T(R^0(\overline{\pi \circ \rho})_*(\overline{\mathcal{C}'_{\underline{\nu},\eta}}, (f'')^*T\mathbb{P}^r)^{\mathrm{fix}})$.) Putting together (15) and (17), for $\underline{\nu}$ noncontracted, the Euler class $e_T(R^0(\overline{\pi \circ \rho})_*(\overline{\mathcal{C}'_{\underline{\nu}}}, (f'')^*T\mathbb{P}^r)^{\mathrm{mov}})$ is equal to

(18)

$$\left(\prod_{\eta\in\mathrm{Stat}(e)} \prod_{i\neq i(\eta)} (\alpha_{i(\eta)} - \alpha_i)\right) \prod_{\eta\in\mathrm{Mov}(e)} \prod_{\substack{A+B=\beta_\eta(e)\\ 0\leq i\leq r\\ (A,i)\neq(0,i^{\mathrm{mov}}(v_2,e))\\ (B,i)\neq(0,i^{\mathrm{mov}}(v_1,e))}} \left(\frac{A}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_1,e)} + \frac{B}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_2,e)} - \alpha_i\right).$$

Summary. We collect the arguments of this section in the following two statements.

Proposition 4.3. *For any minimal decorated graph Γ, $\overline{\mathcal{M}}_\Gamma$ is smooth, and the virtual fundamental class is equal to the fundamental class.*

Proposition 4.4. *The equivariant Euler class $e_T(N^{\mathrm{vir}}_{\overline{\mathcal{M}}_\Gamma})$ of the virtual normal bundle to $\overline{\mathcal{M}}_\Gamma$ is*

$$\left(\frac{\prod_{v\in V^2(\Gamma)}(-\psi_v^{\overline{\mathcal{M}}_{e_v^1}} - \psi_v^{\overline{\mathcal{M}}_{e_v^2}})\prod_{\substack{(v,e)\in F(\Gamma)\\ v\in V^S(\Gamma)}}(-\psi_e^{\overline{\mathcal{M}}_v} - \psi_v^{\overline{\mathcal{M}}_e})}{\prod_{v\in V^1(\Gamma)}\psi_v^{\overline{\mathcal{M}}_{e_v}}}\right)$$

$$\cdot \prod_{e\in E(\Gamma)}\left(\left(\prod_{\substack{\eta\in\mathrm{Stat}(e)\\ i\neq i(\eta)}}(\alpha_{i(\eta)} - \alpha_i)\right) \prod_{\substack{\eta\in\mathrm{Mov}(e)\\ A+B=\beta_\eta(e)\\ 0\leq i\leq r\\ (A,i)\neq(0,i^{\mathrm{mov}}(v_2,e))\\ (B,i)\neq(0,i^{\mathrm{mov}}(v_1,e))}} \left(\frac{A}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_1,e)} + \frac{B}{\beta_\eta(e)}\alpha_{i^{\mathrm{mov}}(v_2,e)} - \alpha_i\right)\right)$$

$$\cdot\left(\frac{\prod_{v\in V^1(\Gamma)\cup V^{1,1}(\Gamma)\cup V^2(\Gamma)} e_T(T_{(\mathrm{VEval}(v),\mathrm{Mon}(v))}I\operatorname{Sym}^d\mathbb{P}^r)}{\prod_{(v,e)\in F(\Gamma)} e_T(T_{(\mathrm{VEval}(v),\mathrm{Mon}(v,e))}I\operatorname{Sym}^d\mathbb{P}^r)}\right)$$

$$\cdot\left(\prod_{v\in V^S(\Gamma)} e_T(R\pi_*(C_v, f^*T\operatorname{Sym}^d\mathbb{P}^r))^{\mathrm{mov}}\right).$$

Proof of Proposition 4.3. Recall from Theorem 2.2 that the virtual fundamental class of $\overline{\mathcal{M}}_\Gamma$ is obtained from the fixed part of the perfect obstruction theory on $\overline{\mathcal{M}}_{g,n}(\operatorname{Sym}^d\mathbb{P}^r, \beta)$. By (15), the fixed part of $\bigoplus_\xi R^0\pi_*(\xi, f^*T\operatorname{Sym}^d\mathbb{P}^r)$ is zero. Thus by (12),

$$R^1\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r) \cong \bigoplus_{\underline{\nu}} R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r).$$

But we showed, in (14) and (16), that $\bigoplus_{\underline{\nu}} R^1\pi_*(\mathcal{C}_{\underline{\nu}}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$ has no fixed part. Thus $R^1\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$ has no fixed part. By Proposition 5.5 of [BF97], the Proposition follows. (Smoothness already followed easily from Theorem 3.16.) □

Proof of Proposition 4.4. The first line is the contribution from $\mathrm{Def}(\mathcal{C})^{\mathrm{mov}}$ and $\mathrm{Aut}(\mathcal{C})^{\mathrm{mov}}$, from (10) and (11). The second line is the contribution of noncontracted components to $R\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$, from (18) and (16). The third line is the contribution of steady nodes to $R\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)$, from

(15). (The numerator corrects for the fact that $F(\Gamma)$ overcounts the steady nodes.) The last line is the contribution of contracted components to $R\pi_*(\mathcal{C}, f^*T\operatorname{Sym}^d\mathbb{P}^r)^{\mathrm{mov}}$, by definition. $\square$

Theorem 4.5. *The results of this section, together with Corollary 3.15 and Theorem 3.16, provide an algorithm to compute any Gromov-Witten invariant of* $\operatorname{Sym}^d\mathbb{P}^r$ *(for any d) in terms of Hurwitz-Hodge integrals, i.e. twisted Gromov-Witten invariants of BG for G a product of symmetric groups.*

Proof. Applying the virtual localization theorem 2.2, a genus-g Gromov-Witten invariant of $\operatorname{Sym}^d\mathbb{P}^r$ is expressed as a sum

$$\sum_{\Gamma\in G^{\min}_{g,n}(\operatorname{Sym}^d\mathbb{P}^r,\beta)}\int_{\overline{\mathcal{M}}_\Gamma}\frac{\iota^*\alpha}{e_T(N^{\mathrm{vir}}_{\overline{\mathcal{M}}_\Gamma})}.$$

By Theorem 3.16, $\overline{\mathcal{M}}_\Gamma$ is a finite cover of a product of Losev-Manin spaces $\overline{\mathcal{M}}_e$ (Section 2.5) and spaces $\overline{\mathcal{M}}_v = \overline{\mathcal{M}}_{g_v,\overrightarrow{\mathrm{Mon}(v)}}(BS_{\mathrm{VEval}(v)}, 0)$ of admissible covers. The factors $\overline{\mathcal{M}}_e$ can be integrated over using Lemma 2.13, since the only cohomology classes in the integrands are ψ classes at the two distinguished marked points (cf. (23) in the proof of Theorem 5.5). The remaining integrals are over the factors $\overline{\mathcal{M}}_v$. The integrand contains the factor

$$\prod_{v\in V^S(\Gamma)}\frac{1}{e_T(R\pi_*(C_v, f^*T\operatorname{Sym}^d\mathbb{P}^r))^{\mathrm{mov}}},$$

as well as ψ classes and classes pulled back along evaluation maps, and is thus a twisted Gromov-Witten invariant of $BS_{VEval(v)}$. $\square$

5. Characterization of the Givental cone $\mathcal{L}_{\operatorname{Sym}^d\mathbb{P}^r}$

In this section, we apply the results of Sections 3.2 and 4 to give a criterion (Theorem 5.5) that exactly determines whether a given power series lies on the Givental cone $\mathcal{L}_{\operatorname{Sym}^d\mathbb{P}^r}$. For the rest of the paper, we work only in genus zero, so we refer to "decorated trees" rather than "decorated graphs."

Definition 5.1. Fix $(\mu,\sigma)\in(I\operatorname{Sym}^d\mathbb{P}^r)^T$. Let $\Upsilon(\mu,\sigma)\subseteq\mathrm{Graphs}_{0,2}(\operatorname{Sym}^d\mathbb{P}^r,\beta)$ be the set of 1-edge decorated trees $\kappa = v_1\bullet\overset{e}{\rule{2em}{0.4pt}}\bullet v_2$, with $g_{v_1}=g_{v_2}=0$, marking set $\{b_{n+1}, b_\bullet\}$, with $\mathrm{Mark}(n+1)=v_1$ and $\mathrm{Mark}(\bullet)=v_2$, such that $\mu=\mathrm{VEval}(v_1)$ and $\sigma=\mathrm{Mon}(v_1,e)$.

Notation 5.2. For $\kappa\in\Upsilon(\mu,\sigma)$, we write (using the notation of Definition 3.5):

- $q(\kappa) := q(e)$,
- $\mathrm{Mov}(\kappa) := \mathrm{Mov}(e)$,
- $\mathrm{mov}(\kappa) := \mathrm{mov}(e)$,
- $\mathrm{Stat}(\kappa) := \mathrm{Stat}(e)$,
- for $\eta\in\mathrm{Mov}(\kappa)$, $\beta_\eta(\kappa) := \beta_\eta(e) = q(e)\cdot\eta$,
- $\beta(\kappa) = \sum_{\eta\in\mathrm{Mov}(\kappa)}\beta_\eta(\kappa)$
- $i_1^{\mathrm{mov}}(\kappa) := i^{\mathrm{mov}}(v_1,e)$,
- $i_2^{\mathrm{mov}}(\kappa) := i^{\mathrm{mov}}(v_2,e)$,
- $\mu'(\kappa) := \mathrm{VEval}(v_2)$,
- $\sigma'(\kappa) := \mathrm{Mon}(v_2,e)$, and
- $r(\kappa) := r(v_1,e) = r(v_2,e) = r_{n+1}$.

We also define:

$$w(\kappa) := \frac{\alpha_{i_1^{\mathrm{mov}}(\kappa)} - \alpha_{i_2^{\mathrm{mov}}(\kappa)}}{q(\kappa)} \in H^2_T(\operatorname{Spec}\mathbb{C}).$$

Remark 5.3. Note that $w(\kappa)$ is equal to the T-weight of the tangent space to the *coarse moduli space* of the source curve C at b_{n+1}; this is because $q(\kappa)$ is defined via coordinates on this coarse moduli space (see Lemma 3.3).

Definition 5.4. Let $\kappa \in \Upsilon(\mu, \sigma)$ and let $a \in \mathbb{Z}_{>0}$ We define the recursion coefficient

$$\begin{aligned}\mathbf{RC}(\kappa, a) = {} & \frac{(-1)^{\mathrm{mov}(\kappa)-a}}{q(\kappa)^{\mathrm{mov}(\kappa)}} \binom{\sigma_{i_1^{\mathrm{mov}}(\kappa)}}{\mathrm{Mov}(\kappa)} \binom{\mathrm{mov}(\kappa) - 1}{a-1} \\ & \cdot \frac{1}{\prod_{\eta \in \mathrm{Mov}(\kappa)} \prod_{\substack{1 \le B \le \beta_\eta(\kappa) \\ 0 \le i \le r \\ (B,i) \neq (\beta_\eta(\kappa), i_2^{\mathrm{mov}}(\kappa))}} \left(\frac{\beta_\eta(\kappa) - B}{\beta_\eta(\kappa)} \alpha_{i_1^{\mathrm{mov}}(\kappa)} + \frac{B}{\beta_\eta(\kappa)} \alpha_{i_2^{\mathrm{mov}}(\kappa)} - \alpha_i \right)},\end{aligned}$$

where $\binom{\sigma_{i_1^{\mathrm{mov}}(\kappa)}}{\mathrm{Mov}(\kappa)}$ is the number of ways of choosing $\mathrm{Mov}(\kappa)$ as a subpartition of $\sigma_{i_1^{\mathrm{mov}}(\kappa)}$ with specified parts.

The following theorem and its proof are in the same spirit as Theorem 41 of [CCIT15], which in turn is adapted from Theorem 2 of [Bro14].

Theorem 5.5. *Let* $\mathbf{f}$ *be an element of* $\mathcal{H}[[x]]$ *such that* $\mathbf{f}|_{Q=x=0} = -1z$*, where* 1 *denotes the fundamental class of* $\mathrm{Sym}^d \mathbb{P}^r \subseteq I \,\mathrm{Sym}^d \mathbb{P}^r$*. Then* $\mathbf{f}$ *is a* $\Lambda_{\mathrm{nov}}^T[[x]]$*-valued point of* $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$ *if and only if for each* T*-fixed point* $(\mu, \sigma) \in I \,\mathrm{Sym}^d \mathbb{P}^r$*, the following three conditions hold:*

(I) *The restriction* $\mathbf{f}_{(\mu,\sigma)}$ *along* $\iota_{(\mu,\sigma)} : (\mu, \sigma) \hookrightarrow I \,\mathrm{Sym}^d \mathbb{P}^r$ *is a power series in* Q *and* x*, such that each coefficient of this power series is an element of* $H^*_{T,\mathrm{loc}}(\bullet)(z)$*. Each coefficient is regular in* z *except for possible poles at* $z = 0$*,* $z = \infty$*, and*

$$z \in \{w(\kappa) : \kappa \in \Upsilon(\mu, \sigma)\}.$$

(II) *The Laurent coefficients of* $\mathbf{f}_{(\mu,\sigma)}$ *at the poles (other than* $z = 0$ *and* $z = \infty$*) satisfy the recursion relation:*

$$\mathrm{Coef}(\mathbf{f}_{\mu,\sigma}, (w-z)^{-a}) = \sum_{\substack{\kappa \in \Upsilon_{(\mu,\sigma)} \\ w(\kappa) = w \\ \mathrm{mov}(\kappa) \ge a}} Q^{\beta(\kappa)} \, \mathbf{RC}(\kappa, a) \, \mathrm{Coef}(\mathbf{f}_{(\mu'(\kappa), \sigma'(\kappa))}, (w-z)^{\mathrm{mov}(\kappa)-a}) \tag{19}$$

for $a > 0$*, and*

(III) *The restriction* $\mathbf{f}_\mu$ *along* $\iota_\mu : I\mu \hookrightarrow I \,\mathrm{Sym}^d \mathbb{P}^r$ *is a* $\Lambda_{\mathrm{nov}}^T[[x]]$*-valued point of* $\mathcal{L}_\mu^{\mathrm{tw}}$*.*

Remark 5.6. In (**III**), Λ_{nov}^T is the equivariant Novikov ring associated to $\mathrm{Sym}^d \mathbb{P}^r$, not μ. In other words, $\Lambda_{\mathrm{nov}}^T[[x]] = H^*_{CR,T,\mathrm{loc}}(\mu)[[Q, x]]$.

Remark 5.7. The major difference between Theorem 5.5 and the corresponding theorems in [CCIT15] and [Bro14] is that condition (**II**) gives a recursive relation for *all* negative-exponent Laurent coefficients at $z = w(\kappa)$, in terms of nonnegative-exponent ones. In [CCIT15] and [Bro14], only stacks with isolated 1-dimensional T-orbits are considered. Thus in that case, the poles at $z = w(\kappa)$ are simple, and a recursive relation is given for their residues.

Proof. Let $\mathbf{f}$ be a $\Lambda_{\mathrm{nov}}^T[[x]]$-valued point of $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$. By definition, we can write

$$\begin{aligned}\mathbf{f} &= -1z + \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \sum_{\phi} \frac{Q^\beta}{n!} \left\langle \mathbf{t}(\overline{\psi}), \ldots, \mathbf{t}(\overline{\psi}), \frac{\gamma_\phi}{-z - \overline{\psi}} \right\rangle_{0,n+1,\beta}^{\mathrm{Sym}^d \mathbb{P}^r, T} \gamma^\phi \\ &= -1z + \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{Q^\beta}{n!} (\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}) \cup \frac{1}{-z - \overline{\psi}} \cap [\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)]^{\mathrm{vir}} \right)\end{aligned}$$

for $\mathbf{t}(z) \in \mathcal{H}^+[[x]]$ with $\mathbf{t}|_{Q=x=0} = 0$. The restriction $\mathbf{f}_{(\mu,\sigma)}$ is then

$$-\delta_{\sigma=(1,\dots,1)} z + \iota^*_{(\mu,\sigma)} \mathbf{t}(z) + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{Q^\beta}{n!} \iota^*_{(\mu,\sigma)} \left((\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}) \cup \frac{1}{-z - \overline{\psi}} \cap [\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)]^{\mathrm{vir}} \right) \right).$$

Using the projection formula, we write

$$\begin{aligned}
&\iota^*_{(\mu,\sigma)} \left((\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}_j) \cup \frac{1}{-z - \overline{\psi}_{n+1}} \cap [\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)]^{\mathrm{vir}} \right) \right) \\
&= |C_\mu(\sigma)| \int_{\mathrm{Sym}^d \mathbb{P}^r} (\iota_{(\mu,\sigma)})_* \iota^*_{(\mu,\sigma)} \left((\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}_j) \cup \frac{1}{-z - \overline{\psi}_{n+1}} \cap [\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)]^{\mathrm{vir}} \right) \right) \\
&= |C_\mu(\sigma)| \int_{\mathrm{Sym}^d \mathbb{P}^r} [(\mu,\sigma)] \cup \left((\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}_j) \cup \frac{1}{-z - \overline{\psi}_{n+1}} \cap [\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)]^{\mathrm{vir}} \right) \right) \\
&= |C_\mu(\sigma)| \int_{\mathrm{Sym}^d \mathbb{P}^r} \left((\mathrm{ev}_{n+1})_* \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}_j) \cup \frac{\mathrm{ev}_{n+1}^*([(\mu,\sigma)])}{-z - \overline{\psi}_{n+1}} \cap [\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)]^{\mathrm{vir}} \right) \right) \\
&= |C_\mu(\sigma)| \int_{[\overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)]^{\mathrm{vir}}} \left(\prod_{j=1}^{n} \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}_j) \cup \frac{\mathrm{ev}_{n+1}^*([(\mu,\sigma)])}{-z - \overline{\psi}_{n+1}} \right) \\
&= |C_\mu(\sigma)| \left\langle \mathbf{t}(\overline{\psi}), \dots, \mathbf{t}(\overline{\psi}), \frac{[(\mu,\sigma)]}{-z - \overline{\psi}} \right\rangle_{0,n+1,\beta}^{\mathrm{Sym}^d \mathbb{P}^r, T}.
\end{aligned}$$

The first equality uses the identification of $\int_{\mathrm{Sym}^d \mathbb{P}^r} \circ \iota_{(\mu,\sigma)}$ with the identity map $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,\mathbb{C}$ on coarse moduli spaces, and the factor $|C_\mu(\sigma)|$ corrects for the isotropy at $(\mu,\sigma) \in I\,\mathrm{Sym}^d \mathbb{P}^r$. (Recall that $C_\mu(\sigma)$ denotes the centralizer of any element of σ in G_μ.) In summary,

$$\begin{aligned}
\mathbf{f}_{(\mu,\sigma)} &= -\delta_{\sigma=(1,\dots,1)} z + \mathbf{t}_{(\mu,\sigma)}(z) \\
&\quad + \sum_{n=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{|C_\mu(\sigma)|\, Q^\beta}{n!} \left\langle \mathbf{t}(\overline{\psi}), \dots, \mathbf{t}(\overline{\psi}), \frac{[(\mu,\sigma)]}{-z - \overline{\psi}} \right\rangle_{0,n+1,\beta}^{\mathrm{Sym}^d \mathbb{P}^r, T},
\end{aligned} \tag{20}$$

where $\mathbf{t}_{(\mu,\sigma)}(z) := \iota^*_{(\mu,\sigma)} \mathbf{t}(z)$. Now we calculate (20) by virtual torus localization (see Theorem 2.2). Namely, we may write

$$|C_\mu(\sigma)| \left\langle \mathbf{t}(\overline{\psi}), \dots, \mathbf{t}(\overline{\psi}), \frac{[(\mu,\sigma)]}{-z - \overline{\psi}} \right\rangle_{0,n+1,\beta}^{\mathrm{Sym}^d \mathbb{P}^r, T} = \sum_{\Gamma \in \mathrm{Graphs}^{\min}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)} \mathrm{Contr}_{(\mu,\sigma)}(\Gamma). \tag{21}$$

We can partition $\mathrm{Graphs}^{\min}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$ into three subsets:

(i) Γ such that $(\mathrm{VEval}(\mathrm{Mark}(n+1)), \mathrm{Mon}(n+1)) \neq (\mu,\sigma)$,
(ii) Γ such that $(\mathrm{VEval}(\mathrm{Mark}(n+1)), \mathrm{Mon}(n+1)) = (\mu,\sigma)$ and $\mathrm{Mark}(n+1) \in V^{1,1}(\Gamma)$, and
(iii) Γ such that $(\mathrm{VEval}(\mathrm{Mark}(n+1)), \mathrm{Mon}(n+1)) = (\mu,\sigma)$ and $\mathrm{Mark}(n+1) \in V^S(\Gamma)$.

In some literature, e.g. [CFK14], decorated trees of type (ii) are called *recursion type* and those of type (iii) are called *initial type*. (We will see below, however, that in our setup both types are used recursively.) Let $v_1 := \mathrm{Mark}(n+1)$.

For a tree Γ of type (i), the restriction $\mathrm{ev}_{n+1}^*([(\mu,\sigma)])$ vanishes, hence $\mathrm{Contr}_{(\mu,\sigma)}(\Gamma)=0$. For this reason, we may simplify our notation, and write $\mathrm{Contr}(\Gamma) := \mathrm{Contr}_{(\mu,\sigma)}(\Gamma)$, where $\mu = \mathrm{VEval}(\mathrm{Mark}(n+1))$ and $\sigma = \mathrm{Mon}(n+1)$.

If Γ is a tree of type (iii), then by Theorem 3.16 and Corollary 3.17, $\overline{\psi}_{n+1}$ is pulled back from $\overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}(v_1)}}(BG_\mu, 0)$, where G_μ is the isotropy group of μ. Since this stack parametrizes maps that factor through the fixed point μ, the action of T is trivial, hence

$$H^*_{T,\mathrm{loc}}(\overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}(v_1)}}(BG_\mu,0)) \cong H^*(\overline{\mathcal{M}}_{0,\overrightarrow{\mathrm{Mon}(v_1)}}(BG_\mu,0)) \otimes H^*_{T,\mathrm{loc}}(\bullet).$$

In particular, $\overline{\psi}_{n+1}$ is nilpotent. It follows that $\mathrm{Contr}(\Gamma)$ is a polynomial in z^{-1}, hence has a pole only at $z=0$.

Finally, let Γ be a tree of type (ii). By (1), we have

$$\mathrm{Contr}(\Gamma) = |C_\mu(\sigma)| \int_{[\overline{\mathcal{M}}_\Gamma]'} \frac{1}{e_T(N_\Gamma^{\mathrm{vir}})} \iota_\Gamma^* \left(\prod_{j=1}^n \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}) \cup \frac{\mathrm{ev}_{n+1}^*[(\mu,\sigma)]}{-z-\overline{\psi}_{n+1}} \right), \tag{22}$$

where ι_Γ is the inclusion $\overline{\mathcal{M}}_\Gamma \hookrightarrow \overline{\mathcal{M}}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta)$. Note that $\mathrm{ev}_{n+1} \circ \iota_\Gamma$ factors through (μ,σ), hence $\iota_\Gamma^* \, \mathrm{ev}_{n+1}^*[(\mu,\sigma)]$ is the weight $e_T(T_{(\mu,\sigma)} I \,\mathrm{Sym}^d \mathbb{P}^r)$.

Then Γ has a decorated subtree $\kappa \in \Upsilon(\mu,\sigma)$, obtained by removing all edges except for $e := e_{v_1}$ (and necessary vertices), and all marked points except b_{n+1}. Let $\Gamma \smallsetminus \kappa$ denote the tree obtained by *pruning* κ. That is, $\Gamma \smallsetminus \kappa \in \mathrm{Graphs}^{\min}_{0,n+1}(\mathrm{Sym}^d \mathbb{P}^r, \beta - \beta(\kappa))$ is defined by $V(\Gamma \smallsetminus \kappa) = V(\Gamma) \smallsetminus \{v_1\}$, $E(\Gamma \smallsetminus \kappa) = E(\Gamma) \smallsetminus e$, and decorations Mark, VEval, q, and Mon are unchanged, except $\mathrm{Mark}(n+1) := v_2$, where v_2 is the common vertex of κ and $\Gamma \smallsetminus \kappa$. Observe that an automorphism of Γ fixes b_{n+1}, and therefore fixes e, so we have $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma \smallsetminus \kappa)$. Thus by Theorem 3.16, up to a $\overline{C}_{\mathrm{VEval}(v_2)}(\mathrm{Mon}(v_2,e))$-gerbe, we may write

$$\overline{\mathcal{M}}_\Gamma \cong \overline{\mathcal{M}}_e \times \overline{\mathcal{M}}_{\Gamma \smallsetminus \kappa}.$$

We factor the T-equivariant map $\overline{\mathcal{M}}_\Gamma \to \mathrm{Spec}\,\mathbb{C}$ through the second projection, i.e. we integrate over $\overline{\mathcal{M}}_e$:

$$\mathrm{Contr}(\Gamma) = \frac{|C_\mu(\sigma)| \left|C_{\mu'(\kappa)}(\sigma'(\kappa))\right|}{r(\kappa)} \int_{[\overline{\mathcal{M}}_{\Gamma \smallsetminus \kappa}]'} \left(\int_{\overline{\mathcal{M}}_e} \frac{e_T(T_{(\mu,\sigma)} I \,\mathrm{Sym}^d \mathbb{P}^r)}{e_T(N_\Gamma^{\mathrm{vir}})} \iota_\Gamma^* \left(\prod_{j=1}^n \mathrm{ev}_j^* \, \mathbf{t}(\overline{\psi}) \cup \frac{1}{-z-\overline{\psi}_{n+1}} \right) \right).$$

The factor $\left|C_{\mu'(\kappa)}(\sigma'(\kappa))\right| / r(\kappa)$ is the order of $\overline{C}_{\mathrm{VEval}(v_2)}(\mathrm{Mon}(v_2,e))$. From Proposition 4.4, we may write

$$\frac{e_T(T_{(\mu,\sigma)} I \,\mathrm{Sym}^d \mathbb{P}^r)}{e_T(N_\Gamma^{\mathrm{vir}})} = \frac{1}{W} \cdot \frac{e_T(T_{(\mu'(\kappa),\sigma'(\kappa))} I \,\mathrm{Sym}^d \mathbb{P}^r)}{e(N^{\mathrm{vir}}_{\Gamma \smallsetminus \kappa})(-\psi_e^{\overline{\mathcal{M}}_{v_2}} - \psi_{v_2}^{\overline{\mathcal{M}}_e})},$$

where

$$\begin{aligned} W :&= \frac{\prod_{\eta \in \mathrm{Stat}(\kappa)} \prod_{i \neq i(\eta)} (\alpha_{i(\eta)} - \alpha_i)}{e_T(T_{(\mu,\sigma)} I \,\mathrm{Sym}^d \mathbb{P}^r)} \prod_{\eta \in \mathrm{Mov}(\kappa)} \prod_{\substack{A+B=\beta_\eta(\kappa) \\ 0 \le i \le r \\ (A,i) \neq (0, i^{\mathrm{mov}}(v_2,e)) \\ (B,i) \neq (0, i^{\mathrm{mov}}(v_1,e))}} \left(\frac{A}{\beta_\eta(\kappa)} \alpha_{i^{\mathrm{mov}}(v_1,e)} + \frac{B}{\beta_\eta(\kappa)} \alpha_{i^{\mathrm{mov}}(v_2,e)} - \alpha_i \right) \\ &= \prod_{\eta \in \mathrm{Mov}(\kappa)} \prod_{\substack{1 \le B \le \beta_\eta(\kappa) \\ 0 \le i \le r \\ (B,i) \neq (\beta_\eta(\kappa), i^{\mathrm{mov}}(v_2,e))}} \left(\frac{\beta_\eta(\kappa) - B}{\beta_\eta(\kappa)} \alpha_{i^{\mathrm{mov}}(v_1,e)} + \frac{B}{\beta_\eta(\kappa)} \alpha_{i^{\mathrm{mov}}(v_2,e)} - \alpha_i \right) \in H^*_{T,\mathrm{loc}}(\mathrm{Spec}\,\mathbb{C}) \end{aligned}$$

Note that the cancellation in the last step removes the factors where $B = 0$, and that $1/W$ is the product appearing in $\mathbf{RC}(\kappa, a)$.

To avoid confusion, we write $\overline{\psi}^{\Gamma}_{n+1}$ (resp. $\overline{\psi}^{\Gamma\setminus\kappa}_{n+1}$) for the $\overline{\psi}$-class at the $(n+1)$st marked point on $\overline{\mathcal{M}}_{\Gamma}$ (resp. $\overline{\mathcal{M}}_{\Gamma\setminus\kappa}$), recalling that on $\Gamma \setminus \kappa$ we defined $\operatorname{Mark}(n+1) = v_2$. We also have $\iota_{\Gamma}^{*}\overline{\psi}^{\Gamma}_{n+1} = \overline{\psi}^{\overline{\mathcal{M}}_e}_{v_1}$. The T-weight on $\overline{\psi}^{\overline{\mathcal{M}}_e}_{v_1}$ is $-w(\kappa)$ (see Notation 5.2), so we have

$$\overline{\psi}^{\overline{\mathcal{M}}_e}_{v_1} = \overline{\psi}^{\mathrm{ne}}_{v_1} - w(\kappa) \in H^*_T(\overline{\mathcal{M}}_{\Gamma}) \cong H^*(\overline{\mathcal{M}}_{\Gamma}) \otimes H^*_T(\operatorname{Spec}\mathbb{C}),$$

where $\overline{\psi}^{\mathrm{ne}}_{v_1}$ denotes the nonequivariant $\overline{\psi}$-class. Similarly $\overline{\psi}^{\overline{\mathcal{M}}_e}_{v_2} = \overline{\psi}^{\mathrm{ne}}_{v_2} + w(\kappa)$. Then since $\iota_{\Gamma}^{*}\operatorname{ev}_j^{*}\mathbf{t}(\overline{\psi})$ is pulled back from $\overline{\mathcal{M}}_{\Gamma\setminus\kappa}$,

$$\begin{aligned}\operatorname{Contr}(\Gamma) = {} & \frac{|C_{\mu}(\sigma)|\,\left|C_{\mu'(\kappa)}(\sigma'(\kappa))\right|}{r(\kappa)}\frac{e_T(T_{(\mu'(\kappa),\sigma'(\kappa))}I\operatorname{Sym}^d\mathbb{P}^r)}{W} \\ & \cdot \int_{[\overline{\mathcal{M}}_{\Gamma\setminus\kappa}]'}\left(\frac{\iota_{\Gamma}^{*}\left(\prod_{j=1}^{n}\operatorname{ev}_j^{*}\mathbf{t}(\overline{\psi})\right)}{e_T(N^{\mathrm{vir}}_{\Gamma\setminus\kappa})}\int_{\overline{\mathcal{M}}_e}\frac{1}{(-\psi^{\Gamma\setminus\kappa}_{n+1} - \psi^{\mathrm{ne}}_{v_2} - w(\kappa))}\frac{1}{(-z - \overline{\psi}^{\mathrm{ne}}_{v_1} + w(\kappa))}\right).\end{aligned}$$

We compute the last integral using the fact that $w(\kappa)$ is invertible, and Lemma 2.13, which says we may integrate on $\overline{\mathcal{M}}_{0,k+2}$ instead of $\overline{\mathcal{M}}_e$. We use

$$r(\kappa)(-\psi^{\Gamma\setminus\kappa}_{n+1} - \psi_{v_2}) = -\overline{\psi}^{\Gamma\setminus\kappa}_{n+1} - \overline{\psi}_{v_2} = \overline{\psi}^{\Gamma\setminus\kappa}_{n+1} - \overline{\psi}^{\mathrm{ne}}_{v_2} - w(\kappa).$$

It is well-known (see e.g. [Koc01], Lemma 1.5.1) that

$$\int_{\overline{\mathcal{M}}_{0,k}} \psi_1^{m}\psi_2^{k-3-m} = \binom{k-3}{m}. \tag{23}$$

By Lemma 2.13, this identity holds on $\overline{\mathcal{M}}_{0|k|\infty}$ also. Thus:

$$\begin{aligned}&\int_{\overline{\mathcal{M}}_e}\frac{1}{(-\overline{\psi}^{\Gamma\setminus\kappa}_{n+1} - \overline{\psi}^{\mathrm{ne}}_{v_2} - w(\kappa))}\frac{1}{(-z - \overline{\psi}^{\mathrm{ne}}_{v_1} + w(\kappa))} \\ &\quad = \frac{1}{|S_e|\prod_{\eta\in\operatorname{Mov}(\kappa)}\beta_\eta(\kappa)}\int_{\overline{\mathcal{M}}_{v_1|\operatorname{mov}(\kappa)|v_2}}\left(\sum_{m_1=0}^{\infty}\frac{(\overline{\psi}_{v_2})^{m_1}}{(-\overline{\psi}^{\Gamma\setminus\kappa}_{n+1} - w(\kappa))^{m_1+1}}\right)\left(\sum_{m_2=0}^{\infty}\frac{(\overline{\psi}_{v_1})^{m_2}}{(-z + w(\kappa))^{m_2+1}}\right) \\ &\quad = \frac{1}{|S_e|\prod_{\eta\in\operatorname{Mov}(\kappa)}\beta_\eta(\kappa)}\sum_{m_1+m_2=\operatorname{mov}(\kappa)-1}\frac{\binom{\operatorname{mov}(\kappa)-1}{m_1}}{(-\overline{\psi}^{\Gamma\setminus\kappa}_{n+1} - w(\kappa))^{m_1+1}(-z + w(\kappa))^{m_2+1}} \\ &\quad = \frac{1}{|S_e|\prod_{\eta\in\operatorname{Mov}(\kappa)}\beta_\eta(\kappa)}\frac{(-z - \overline{\psi}^{\Gamma\setminus\kappa}_{n+1})^{\operatorname{mov}(\kappa)-1}}{(-\overline{\psi}^{\Gamma\setminus\kappa}_{n+1} - w(\kappa))^{\operatorname{mov}(\kappa)}(-z + w(\kappa))^{\operatorname{mov}(\kappa)}}.\end{aligned} \tag{24}$$

The last equality in (24) comes from expanding

$$((-z + w(\kappa)) + (-\overline{\psi}^{\Gamma\setminus\kappa}_{n+1} - w(\kappa)))^{\operatorname{mov}(\kappa)-1}$$

via the binomial theorem. Altogether, we have

$$\mathrm{Contr}(\Gamma) = \frac{|C_\mu(\sigma)|\,\left|C_{\mu'(\kappa)}(\sigma'(\kappa))\right|}{|S_e|\prod_{\eta\in\mathrm{Mov}(\kappa)}\beta_\eta(\kappa)} \frac{e_T(T_{(\mu'(\kappa),\sigma'(\kappa))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{W\cdot(-z+w(\kappa))^{\mathrm{mov}(\kappa)}} \cdot \int_{[\overline{\mathcal{M}}_{\Gamma\smallsetminus\kappa}]'} \left(\frac{\iota_\Gamma^*\left(\prod_{j=1}^n \mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi})\right)}{e_T(N^{\mathrm{vir}}_{\Gamma\smallsetminus\kappa})} \frac{(-z-\overline{\psi}^{\Gamma\smallsetminus\kappa}_{n+1})^{\mathrm{mov}(\kappa)-1}}{(-\overline{\psi}^{\Gamma\smallsetminus\kappa}_{n+1}-w(\kappa))^{\mathrm{mov}(\kappa)}} \right). \tag{25}$$

For fixed β_0, and n_0, from (21), the coefficient of $Q^{\beta_0}x^{n_0}$ in $\mathbf{f}_{(\mu,\sigma)}$ only has contributions from $\Gamma\in\mathrm{Graphs}_{0,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta)$ for $\beta+n\le\beta_0+n_0$. This is because $\mathbf{t}(z)\in\langle Q,x\rangle$, so if $\mathcal{H}[[x]]$ is graded by giving Q and x degree 1, then the (n,β) term in (20) has degree at least $n+\beta$. In particular, $\bigcup_{\beta+n\le\beta_0+n_0}\mathrm{Graphs}_{0,n}(\mathrm{Sym}^d\,\mathbb{P}^r,\beta)$ is a finite set. Thus (21) and (25) realize the contribution to such a coefficient from trees of type (ii) as a finite sum of rational functions with poles at the weights κ. Together with the analysis above for types (i) and (iii), this proves that $\mathbf{f}_{(\mu,\sigma)}$ satisfies condition (**I**) of the Theorem.

We consider the Laurent coefficient $\mathrm{Coef}(\mathrm{Contr}(\Gamma),(w-z)^{-a})$. By (25), $\mathrm{Coef}(\mathrm{Contr}(\Gamma),(w-z)^{-a})$ is zero if $w\neq w(\kappa)$, or if $\mathrm{mov}(\kappa)<a$. Otherwise,

$$\begin{aligned}
&\mathrm{Coef}(\mathrm{Contr}(\Gamma),(w-z)^{-a})\\
&\quad= \frac{1}{(\mathrm{mov}(\kappa)-a)!}\left(\frac{d^{\mathrm{mov}(\kappa)-a}}{d(w(\kappa)-z)^{\mathrm{mov}(\kappa)-a}}(w(\kappa)-z)^{\mathrm{mov}(\kappa)}\,\mathrm{Contr}(\Gamma)\right)\Bigg|_{z\mapsto w(\kappa)}\\
&\quad= \frac{(-1)^{\mathrm{mov}(\kappa)-a}\,|C_\mu(\sigma)|\,\left|C_{\mu'(\kappa)}(\sigma'(\kappa))\right|\binom{\mathrm{mov}(\kappa)-1}{a-1}}{W\,|S_e|\prod_{\eta\in\mathrm{Mov}(\kappa)}\beta_\eta(\kappa)} \int_{[\overline{\mathcal{M}}_{\Gamma\smallsetminus\kappa}]'}\left(\frac{\iota_\Gamma^*\left(\prod_{j=1}^n\mathrm{ev}_j^*\,\mathbf{t}(\overline{\psi})\right)}{e_T(N^{\mathrm{vir}}_{\Gamma\smallsetminus\kappa})}\frac{e_T(T_{(\mu'(\kappa),\sigma'(\kappa))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{(-\overline{\psi}^{\Gamma\smallsetminus\kappa}_{n+1}-w(\kappa))^{\mathrm{mov}(\kappa)-a+1}}\right)
\end{aligned}$$

Now, summing over all Γ of type (ii) with associated subtree κ yields

$$\frac{(-1)^{\mathrm{mov}(\kappa)-a}\,|C_\mu(\sigma)|\,\left|C_{\mu'(\kappa)}(\sigma'(\kappa))\right|\binom{\mathrm{mov}(\kappa)-1}{a-1}}{W\,|S_e|\prod_{\eta\in\mathrm{Mov}(\kappa)}\beta_\eta(\kappa)}\left\langle \mathbf{t}(\overline{\psi}),\ldots,\mathbf{t}(\overline{\psi}),\frac{[(\mu'(\kappa),\sigma'(\kappa))]}{(-\overline{\psi}^{\Gamma\smallsetminus\kappa}_{n+1}-w(\kappa))^{\mathrm{mov}(\kappa)-a+1}}\right\rangle^{\mathrm{Sym}^d\,\mathbb{P}^r,T}_{0,n+1,\beta-\beta(\kappa)}. \tag{26}$$

On the other hand, the coefficient $\mathrm{Coef}(\mathbf{f}_{(\mu'(\kappa),\sigma'(\kappa))},(w(\kappa)-z)^{\mathrm{mov}(\kappa)-a})$ is

$$\sum_{\substack{\beta\ge0\\ n\ge0}}\frac{\left|C_{\mu'(\kappa)}(\sigma'(\kappa))\right|Q^\beta}{n!}\left\langle\mathbf{t}(\overline{\psi}),\ldots,\mathbf{t}(\overline{\psi}),\frac{[(\mu'(\kappa),\sigma'(\kappa))]}{(-\overline{\psi}^{\Gamma\smallsetminus\kappa}_{n+1}-w(\kappa))^{\mathrm{mov}(\kappa)-a+1}}\right\rangle^{\mathrm{Sym}^d\,\mathbb{P}^r,T}_{0,n+1,\beta} \tag{27}$$

We compute $\frac{|C_\mu(\sigma)|}{|S_e|\prod_{\eta\in\mathrm{Mov}(\kappa)}\beta_\eta(\kappa)}$ explicitly:

$$|C_\mu(\sigma)| = |S_\sigma|\prod_{\eta\in\sigma}\eta$$

$$|S_e| = |C_{\mathrm{Stat}}(\kappa)|\,\left|S_{\mathrm{Mov}(\kappa)}\right| = \left|S_{\mathrm{Stat}(\kappa)}\right|\,\left|S_{\mathrm{Mov}(\kappa)}\right|\prod_{\eta\in\mathrm{Stat}(\kappa)}\eta$$

$$\frac{|C_\mu(\sigma)|}{|S_e|\prod_{\eta\in\mathrm{Mov}(\kappa)}\beta_\eta(\kappa)} = \frac{|S_\sigma|\prod_{\eta\in\mathrm{Mov}(\kappa)}\eta}{\left|S_{\mathrm{Stat}(\kappa)}\right|\,\left|S_{\mathrm{Mov}(\kappa)}\right|\prod_{\eta\in\mathrm{Mov}(\kappa)}\beta_\eta(\kappa)} = \frac{1}{q(\kappa)^{\mathrm{mov}(\kappa)}}\binom{\sigma_{i_1^{\mathrm{mov}}}(\kappa)}{\mathrm{Mov}(\kappa)}$$

With (26) and (27), this proves (**II**). Note that the contribution from all graphs of type (ii) (and the term $\mathbf{t}_{(\mu,\sigma)}(z)$) is

$$\boldsymbol{\tau}_{(\mu,\sigma)}(z) := \mathbf{t}_{(\mu,\sigma)}(z) + \sum_{\substack{\kappa\in\Upsilon(\mu,\sigma)\\ a\leq\mathrm{mov}(\kappa)}} \frac{Q^{\beta(\kappa)}\,\mathbf{RC}(\kappa,a)}{(w(\kappa)-z)^a}\,\mathrm{Coef}(\mathbf{f}_{(\mu'(\kappa),\sigma'(\kappa))},(w(\kappa)-z)^{\mathrm{mov}(\kappa)-a}). \tag{28}$$

The proof of condition (**III**) is identical to that of condition (C3) in [CCIT15], and we reproduce the argument here for convenience.

Consider a decorated tree Γ of type (iii). We write $v := \mathrm{Mark}(n+1) \in V^S(\Gamma)$. The marked points of $\overline{\mathcal{M}}_v$ correspond to (1) elements of $\mathrm{Mark}^{-1}(v)$, and (2) edges $e \in E(\Gamma, v)$. To e is associated a maximal subtree Γ_e containing v, with $E(\Gamma_e, v) = e$. We decorate Γ_e so that $\mathrm{Mark}^{-1}(v) = b$, and the rest of the decorations inherited from Γ. We will then write $\mathrm{Contr}(\Gamma)$ in terms of $\mathrm{Contr}(\Gamma_e)$ for $e \in E(\Gamma, v)$, and integrals over the vertex moduli space $\overline{\mathcal{M}}_v$.

We apply (22) again. After an étale base change $\widetilde{\overline{\mathcal{M}}}_\Gamma \to \overline{\mathcal{M}}_\Gamma$, we may label the subtrees Γ_e. (Write M for the degree of this base change.) We then write $\widetilde{\overline{\mathcal{M}}}_\Gamma \cong \overline{\mathcal{M}}_v \times \prod_{e\in E(\Gamma,v)} \overline{\mathcal{M}}_{\Gamma_e}$. Now we again apply Proposition 4.4, to see that

$$\frac{1}{e_T(N_\Gamma^{\mathrm{vir}})} = e_T^{-1}(R\pi_*(C_v, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)) \prod_{e\in E(\Gamma,v)} \frac{r(v,e)e_T(T_{(\mu,\mathrm{Mon}(v,e))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{(-\overline{\psi}_e^{\overline{\mathcal{M}}_v} - \overline{\psi}_v^{\overline{\mathcal{M}}_e})e_T(N_{\Gamma_e}^{\mathrm{vir}})}$$

Observe that $\frac{e_T(T_{(\mu,\mathrm{Mon}(v,e))}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{(-\overline{\psi}_e^{\overline{\mathcal{M}}_v} - \overline{\psi}_v^{\overline{\mathcal{M}}_e})}$ is the insertion at b in $\mathrm{Contr}(\Gamma_e)|_{z\mapsto\overline{\psi}_e^{\overline{\mathcal{M}}_v}}$. Thus

$$\mathrm{Contr}(\Gamma) = \frac{1}{M}\int_{\overline{\mathcal{M}}_v}\left(\prod_{e\in E(\Gamma,v)} |C_\mu(\sigma)|\,Q^{\beta(\Gamma_e)}\,\mathrm{Contr}(\Gamma_e)|_{z\mapsto\overline{\psi}_e^{\overline{\mathcal{M}}_v}}\right) \cup \left(\prod_{i\in\mathrm{Mark}^{-1}(v)} \mathbf{t}(\overline{\psi}_i)\right)$$
$$\cup\, \frac{e_T(T_{(\mu,\sigma)}I\,\mathrm{Sym}^d\,\mathbb{P}^r)}{-z-\overline{\psi}_{n+1}} \cup e_T^{-1}(R\pi_*(C_v, f^*T\,\mathrm{Sym}^d\,\mathbb{P}^r)).$$

This is almost a twisted Gromov-Witten invariant of $\mathrm{VEval}(v)$, but not quite, since there are restrictions on the monodromies at the marked points. Summing over Γ_e for a single e, with everything else fixed, gives the insertion $\boldsymbol{\tau}_{(\mu,\mathrm{Mon}(v,e))}(\psi)$, where the initial term comes from replacing Γ_e with a marked point. Thus summing over all σ, and over all Γ of type (iii), gives

$$\sum_{m=2}^{\infty}\sum_{\sigma}\frac{1}{m!}\left\langle \boldsymbol{\tau}_\mu(\overline{\psi}),\ldots,\boldsymbol{\tau}_\mu(\overline{\psi}), \frac{[(\mu,\sigma)]}{-z-\overline{\psi}_{n+1}}\right\rangle_{0,m+1,0}^{\mathrm{VEval}(v),T,\mathrm{tw}} 1_{(\mu,\sigma)} \in H^*_{T,\mathrm{loc}}(I\mu),$$

where $1_{(\mu,\sigma)}$ is the fundamental class of $(\mu,\sigma) \in I\mu$, and $\boldsymbol{\tau}_\mu(z) = \sum_{\sigma'\in\mathrm{MultiPart}(\mu)} \boldsymbol{\tau}_{(\mu,\sigma')}(z)1_{(\mu,\sigma)}$. Adding in the contributions from type (ii) graphs, summing (20) over σ yields:

$$\mathbf{f}_\mu = \sum_\sigma \mathbf{f}_{(\mu,\sigma)}1_{\mu,\sigma} = -1_\mu z + \boldsymbol{\tau}_\mu(z) + \sum_{m=2}^{\infty}\sum_\sigma \frac{1}{m!}\left\langle \boldsymbol{\tau}_\mu(\overline{\psi}),\ldots,\boldsymbol{\tau}_\mu(\overline{\psi}), \frac{[(\mu,\sigma)]}{-z-\overline{\psi}_{n+1}}\right\rangle_{0,m+1,0}^{\mathrm{VEval}(v),T,\mathrm{tw}} 1_{(\mu,\sigma)},$$

where 1_μ is the untwisted fundamental class on $I\mu$. This shows that $\mathbf{f}_\mu$ is a $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point of $\mathcal{L}^{\mathrm{tw}}_\mu$.

The converse also requires no modification from [CCIT15]. Suppose $\mathbf{f}$ satisfies the conditions of the theorem. By conditions (**I**) and (**II**), we may uniquely write

$$\mathbf{f}_\mu = -1_\mu z + \sum_{\sigma \in \mathrm{MultiPart}(\mu)} \boldsymbol{\tau}_{(\mu,\sigma)} 1_{(\mu,\sigma)} + O(z^{-1}),$$

where $\boldsymbol{\tau}_{(\mu,\sigma)}(z)$ is the expression in (28), for some $\mathbf{t}_{(\mu,\sigma)}(z) \in \iota_\mu^*(\mathcal{H}^+)[[x]]$. We claim that the set $\{\mathbf{t}_{(\mu,\sigma)}(z)\}$ for all fixed points (μ,σ) determines $\mathbf{f}$. By the localization isomorphism, if suffices to show that it determines $\mathbf{f}_{(\mu,\sigma)}$ for all (μ,σ). We induct on the degree $\beta + k$, where k is the exponent of x. The base case $\beta = k = 0$ is taken care of by the assumption $\mathbf{f}|_{Q=x=0} = -1z$. Assume the coefficients of $\mathbf{f}_{(\mu,\sigma)}$ up to degree $\beta + k$ are determined by $\{\mathbf{t}_{(\mu,\sigma)}\}$. Consider the coefficients of degree $\beta + k + 1$. Some of these appear in $\mathbf{t}(z)$, but these are given. Some of them appear in $\boldsymbol{\tau}_{(\mu,\sigma)}(z)$, but these are determined since they are of the form: $Q^{\beta(\kappa)}$ multiplied by a factor determined by the inductive hypothesis. The sum of all of these terms is in $H^*_{CR,T,\mathrm{loc}}(\mu)[[Q,x]][[z]]$.

Finally, some of them appear in $O(z^{-1})$. However, condition (**III**) and (5) show that these are determined by terms of $-1z + \boldsymbol{\tau}_{(\mu,\sigma)}(z)$ of degree at most $\beta + k + 1$. Since all such terms are determined by $\mathbf{t}_{(\mu,\sigma)}$ and induction, the degree $\beta + k + 1$ coefficients of $\mathbf{f}_{(\mu,\sigma)}$ are determined. Thus in fact $\mathbf{f}$ is determined by $\{\mathbf{t}_{(\mu,\sigma)}(z)\}$.

Again by the localization isomorphism, the set $\{\mathbf{t}_{(\mu,\sigma)}(z)\}$ corresponds uniquely to an element $\mathbf{t}(z) \in \mathcal{H}^+[[x]]$ that restricts to each $\mathbf{t}_{(\mu,\sigma)}(z)$. This in turn corresponds uniquely to a $\Lambda^T_{\mathrm{nov}}[[x]]$-valued point $\mathbf{f}_{\mathrm{GW}}$ of $\mathcal{L}_x$. By the uniqueness argument above we have $\mathbf{f} = \mathbf{f}_{\mathrm{GW}}$. □

Remark 5.8. No modifications are required to replace $\Lambda^T_{\mathrm{nov}}[[x]]$ in the statement of Theorem 5.5 with a finitely generated graded Λ^T_{nov}-algebra.

6. The I-function and mirror theorem

In this section we introduce a function $I_{\mathrm{Sym}^d \mathbb{P}^r}(Q,t,\mathbf{x},-z)$, and show that it satisfies the conditions of Theorem 5.5, conditional upon two combinatorial identities that we checked extensively by computer, but were unable to prove. (See Section 7.) That is, we prove that these identities imply $I_{\mathrm{Sym}^d \mathbb{P}^r}(Q,t,\mathbf{x},-z)$ is a $\Lambda^T_{\mathrm{nov}}[[t,\mathbf{x}]]/(\mathbf{x})^2$-valued point of $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$, where $\mathbf{t} = \{t_0, \ldots, t_r\}$ and $\mathbf{x} = \{x_\pi\}_{\pi \in \mathrm{Part}(d)}$ are formal variables.

The (first order) I-function $I_{\mathrm{Sym}^d \mathbb{P}^r}(Q,\mathbf{t},\mathbf{x},z)$ is defined by its restrictions to the T-fixed points $\iota_{\mu,\sigma} : (\mu,\sigma) \hookrightarrow I\,\mathrm{Sym}^d \mathbb{P}^r$ as follows:

$$\begin{aligned}\iota^*_{(\mu,\sigma)} I_{\mathrm{Sym}^d \mathbb{P}^r}(Q,t,\mathbf{x},z) := \left(z\delta_{\sigma,(1,\ldots,1)} 1_\sigma + x_{\pi(\sigma)} 1_\sigma\right) \sum_{\beta \geq 0} \exp\left(\sum_{i=0}^r t_i \left(\beta + \sum_{j=0}^r \mu_j(\alpha_j - \alpha_i)/z\right)\right) Q^\beta \\ \cdot \sum_{\substack{(L_\eta)_{\eta\in\sigma} \\ L_\eta \geq 0 \\ \sum L_\eta = \beta}} \left(\prod_{j=0}^r \prod_{\eta \in \sigma_j} \frac{1}{\prod_{\gamma=1}^{L_\eta} \prod_{i=0}^r \left(\alpha_j - \alpha_i + \frac{\gamma}{\eta} z\right)}\right), \end{aligned} \tag{29}$$

where 1_σ is the fundamental class of $(\mu,\sigma) \in I\mu$. We will use the notations

$$I_{(\mu,\sigma)}(Q,\mathbf{t},\mathbf{x},z) := \iota^*_{(\mu,\sigma)} I_{\mathrm{Sym}^d \mathbb{P}^r}(Q,\mathbf{t},\mathbf{x},z)$$

and

$$I_\mu(Q,\mathbf{t},\mathbf{x},z) := \bigoplus_{\sigma \in \mathrm{MultiPart}(\mu)} I_{(\mu,\sigma)}(Q,\mathbf{t},\mathbf{x},z).$$

Remark 6.1. As in Remark 2.7, we have $I_{\mathrm{Sym}^d \mathbb{P}^r}(Q, \mathbf{t}, \mathbf{x}, z) \notin H^*(I \operatorname{Sym}^d \mathbb{P}^r, \mathbb{Q})[[Q, \mathbf{t}, \mathbf{x}]]((z^{-1}))$ due to the presence of arbitrarily high powers of z. The "topological nilpotence" condition we alluded to is simply to say that for a fixed monomial in the variables t_i and x_π, the powers of z in that monomial's coefficient are bounded above.

Remark 6.2. $I_{\mathrm{Sym}^d \mathbb{P}^r}$ may be decomposed into pieces that we might naturally regard as $I_{\mathrm{Sym}^d \mathbb{C}^r}$. If we introduce variables $x_{j,\pi}$, where $0 \le j \le r$ and π is a partition of μ_j, and set $\prod_{0 \le j \le r} x_{j,\pi_j} = x_{\cup \pi_j}$, then we may repeatedly switch orders of summation and products to write

$$I_\mu(Q, \mathbf{t}, \mathbf{x}, z) = I_\mu(Q, \mathbf{t}, 0, z) + \prod_{j=0}^{r} I_{j,\mu_j}(Q, \mathbf{t}, \mathbf{x}, z), \tag{30}$$

where

$$I_{j,d}(Q, \mathbf{t}, \mathbf{x}, z) = \sum_{\pi \in \mathrm{Part}(d)} x_{j,\pi} 1_{j,\pi} \prod_{\eta \in \pi} \sum_{\beta \ge 0} \frac{Q^\beta \exp\left(\sum_{i=0}^{r} t_i(\beta + \eta(\alpha_j - \alpha_i)/z)\right)}{\prod_{i=0}^{r} \prod_{\gamma=1}^{\beta} (\alpha_j - \alpha_i + \frac{\gamma}{\eta} z)}.$$

Here $1_{j,\pi}$ is the pullback of 1_π along the natural isomorphism $IBG_\mu \to \prod_{j=0}^{r} IBS_{\mu_j}$.

We now prove:

Theorem 6.3. *Assuming Identities 7.1 and 7.2 hold,* $I_{\mathrm{Sym}^d \mathbb{P}^r}(Q, \mathbf{t}, \mathbf{x}, -z)$ *is a* $\Lambda_{\mathrm{nov}}^T[[\mathbf{t}, \mathbf{x}]]/(\mathbf{x}^2)$*-valued point of* $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$.

Remark 6.4. This result is weaker than that in the original preprint, where $\Lambda_{\mathrm{nov}}^T[[\mathbf{t}, \mathbf{x}]]$ appeared without the quotient by the ideal $(\mathbf{x})^2$, and the dependence on Identities 7.1 and 7.2 was omitted. We do not know if it is possible to find an explicit formula for a (nontrivial) $\Lambda_{\mathrm{nov}}^T[[t, \mathbf{x}]]$-valued point of $\mathcal{L}_{\mathrm{Sym}^d \mathbb{P}^r}$.

Proof. We must prove that the criteria in Theorem 5.5 are satisfied. The form of (29) implies that the coefficient of $Q^\beta x_\pi \mathbf{t}^{\mathbf{a}}$ is a rational function in z with poles at $z = 0$, $z = \infty$, and $z = \frac{\alpha_{i_1} - \alpha_{i_2}}{q}$, where $i_1 = i(\eta)$ for some $\eta \in \sigma$, and $q \in \frac{1}{\eta}\mathbb{Z}$. This is exactly the set of values arising as $w(\kappa)$ for $\kappa \in \Upsilon(\mu, \sigma)$. This proves (**I**).

To prove (**II**), we fix $\mu \in \mathrm{ZPart}(d, r+1)$, $\sigma \in \mathrm{MultiPart}(\mu)$, $\beta \ge 0$, $L = (L_\eta)_{\eta \in \sigma}$ as in (29), $a \in \mathbb{Z}_{>0}$, distinct elements $i_1, i_2 \in \{0, \dots, r\}$ such that $\mu_{i_1} \ne 0$, and $q \in \mathbb{Q}$ such that $q \in \frac{1}{\eta}\mathbb{Z}$ for some $\eta \in \sigma_{i_1}$. Let $w = \frac{\alpha_{i_1} - \alpha_{i_2}}{q}$.

First, assume that σ is not the trivial multipartition of μ. The term of $I_{(\mu,\sigma)}(Q, \mathbf{t}, \mathbf{x}, -z)$ corresponding to L is $T_{\sigma,L}(z) x_{\pi(\sigma)} 1_\sigma Q^\beta$, where:

$$T_{\sigma,L}(z) = \prod_{j=0}^{r} \prod_{\eta \in \sigma_j} H_{L_\eta, j, \eta}(z) \tag{31}$$

and

$$H_{\beta,j,\eta}(z) = \frac{\exp\left(\sum_{i=0}^{r} t_i\left(\beta + \eta(\alpha_j - \alpha_i)/(-z)\right)\right)}{\prod_{\gamma=1}^{\beta} \prod_{i=0}^{r} \left(\alpha_j - \alpha_i - \frac{\gamma}{\eta} z\right)} \tag{32}$$

Let $\sigma_L = \{\eta \in \sigma_{i_1} : L_\eta \ge q\eta\}$, and recall Notation 5.2. Given a nonempty submultiset $M \subseteq \sigma_L$, there is a unique $\kappa_M \in \Upsilon(\mu, \sigma)$ such that $w(\kappa_M) = w$ and $\mathrm{Mov}(\kappa) = M$, and we may define $L'(\kappa_M) = (L'(\kappa_M)_\eta)_{\eta \in \sigma'(\kappa_M)}$ by letting $L'(\kappa_M)_\eta = L_\eta - q\eta$ for $\eta \in M$. Note that such η are parts of

$\sigma'_{i_2}(\kappa_M)$, and that we have $\sum_{\eta\in\sigma'(\kappa_M)} L'(\kappa_M)_\eta = \beta - \beta(\kappa_M)$. Therefore, to prove (**II**), it is sufficient to show that

$$\operatorname{Coef}(T_{\sigma,L}(z),(w-z)^{-a}) = \sum_{\substack{M\subseteq\sigma_L\\|M|\geq a}} \mathbf{RC}(\kappa_M,a)\operatorname{Coef}(T_{\sigma'(\kappa_M),L'(\kappa_M)}(z),(w-z)^{|M|-a}), \tag{33}$$

since adding up (33) over all L and β yields (19).

Note that $H_{L_\eta,j,\eta}(z)$ has a pole at w if and only if $j=i_1$ and $\eta\in\sigma_L$, and in this case $H_{L_\eta,j,\eta}(z)$ has a simple pole at w, coming from the factor $(\gamma,i)=(q\eta,i_2)$ in the denominator. Thus $T_{\sigma,L}(z)$ has a pole at w of order exactly $|\sigma_L|$. Define

$$\tilde{H}_{\mu,\sigma,L,j,\eta}(z) = \begin{cases}(w-z)H_{L_\eta,j,\eta}(z) & j=i_1,\eta\in\sigma_L\\ H_{L_\eta,j,\eta}(z) & \text{else.}\end{cases}$$

If $a>|\sigma_L|$, then both sides of (33) are zero, so assume $a\leq|\sigma_L|$. By the product rule, the left side of (33) is equal to

$$\frac{1}{(|\sigma_L|-a)!}\left(\frac{d^{|\sigma_L|-a}}{d(w-z)^{|\sigma_L|-a}}(w-z)^{|\sigma_L|}T_{\sigma,L}(z)\right)_{z\mapsto w} = \sum_{\substack{(k_{(j,\eta)})_{0\leq j\leq r,\eta\in\sigma_j}\\ \sum_{j,\eta}k_{(j,\eta)}=|\sigma_L|-a}} \prod_{j=0}^{r}\prod_{\eta\in\sigma_j}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta,j,\eta}(w)}{k_{(j,\eta)}!}. \tag{34}$$

Similarly, the right side of (33) is equal to

$$\sum_{\substack{M\subseteq\sigma_L\\|M|\geq a}}\mathbf{RC}(\kappa_M,a)\sum_{\substack{(k_{(j,\eta)})_{0\leq j\leq r,\eta\in\sigma'_j(\kappa_M)}\\ \sum_{j,\eta}k_{(j,\eta)}=|\sigma_L|-a}}\prod_{j=0}^{r}\prod_{\eta\in\sigma'_j(\kappa_M)}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L'(\kappa_M)_\eta,j,\eta}(w)}{k_{(j,\eta)}!}. \tag{35}$$

We may switch the order of summation in (35), using the natural bijection between the parts of σ and $\sigma'_{j_i}(\kappa_M)$. Note that this bijection identifies the parts of $M\subseteq\sigma_{i_1}$ with parts of $\sigma'_{i_2}(\kappa_M)$. For $\eta\notin M$ we have $L'(\kappa_M)_\eta = L_\eta$, so the result is:

$$\begin{aligned}
&\sum_{\substack{(k_{(j,\eta)})_{0\leq j\leq r,\eta\in\sigma_j}\\ \sum_{j,\eta}k_{(j,\eta)}=|\sigma_L|-a}}\sum_{\substack{M\subseteq\sigma_L\\|M|\geq a}}\mathbf{RC}(\kappa_M,a)\left(\prod_{\substack{0\leq j\leq r\\ \eta\in\sigma_j\\ \eta\notin M}}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta,j,\eta}(w)}{k_{(j,\eta)}!}\right)\prod_{\eta\in M}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta-q\eta,i_2,\eta}(w)}{k_{(i_2,\eta)}!} \qquad (36)\\
&= \sum_{\substack{(k_{(j,\eta)})_{0\leq j\leq r,\eta\in\sigma_j}\\ \sum_{j,\eta}k_{(j,\eta)}=|\sigma_L|-a}}\sum_{\substack{M\subseteq\sigma_L\\|M|\geq a}}(-1)^{|M|-a}\binom{\sigma_{i_1}}{M}\binom{|M|-1}{a-1}\left(\prod_{\substack{0\leq j\leq r\\ \eta\in\sigma_j\\ \eta\notin M}}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta,j,\eta}(w)}{k_{(j,\eta)}!}\right)\\
&\qquad\cdot\prod_{\eta\in M}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta-q\eta,i_2,\eta}(w)}{k_{(i_2,\eta)}!\cdot q\cdot\prod_{\substack{0\leq i\leq r\\ 1\leq\gamma\leq q\eta\\ (\gamma,i)\neq(q\eta,i_2)}}\left(\frac{q\eta-\gamma}{q\eta}\alpha_{i_1}+\frac{\gamma}{q\eta}\alpha_{i_2}-\alpha_i\right)}.
\end{aligned}$$

Consider a single summand $S(k_{(j,\eta)})$ of the leftmost sum in (36). Fix a subset

$$U\subseteq\sigma_L\cap\{(j,\eta):k_{(j,\eta)}>0\},$$

and consider the contribution $S((k_{(j,\eta)});U)$ to $S(k_{(j,\eta)})$ from all M such that

$$M\cap\{(j,\eta):k_{(j,\eta)}>0\}=U.$$

By definition, we have $S(k_{(j,\eta)}) = \sum_{U\subseteq \sigma_L \cap \{(j,\eta):k_{(j,\eta)}>0\}} S((k_{(j,\eta)});U)$. Explicitly,

$$
\begin{aligned}
S((k_{(j,\eta)});U) = \sum_{\substack{U\subseteq M\subseteq \sigma_L\\ |M|\geq a}} (-1)^{|M|-a}\binom{\sigma_{i_1}}{M}\binom{|M|-1}{a-1}\left(\prod_{\substack{0\leq j\leq r\\ \eta\in\sigma_j\\ \eta\notin M}} \frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta,j,\eta}(w)}{k_{(j,\eta)}!}\right)&\\
\cdot\left(\prod_{\eta\in U}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta-q\eta,i_2,\eta}(w)}{k_{(j,\eta)}!\cdot q\cdot \prod_{\substack{0\leq i\leq r\\ 1\leq\gamma\leq q\eta\\ (\gamma,i)\neq(q\eta,i_2)}}\left(\frac{q\eta-\gamma}{q\eta}\alpha_{i_1}+\frac{\gamma}{q\eta}\alpha_{i_2}-\alpha_i\right)}\right)&\\
\cdot\left(\prod_{\eta\in M\smallsetminus U}\frac{\tilde{H}_{L_\eta-q\eta,i_2,\eta}(w)}{k_{(j,\eta)}!\cdot q\cdot \prod_{\substack{0\leq i\leq r\\ 1\leq\gamma\leq q\eta\\ (\gamma,i)\neq(q\eta,i_2)}}\left(\frac{q\eta-\gamma}{q\eta}\alpha_{i_1}+\frac{\gamma}{q\eta}\alpha_{i_2}-\alpha_i\right)}\right)&.
\end{aligned}
\tag{37}
$$

From (32), a simple direct computation using the two identities

$$
L_\eta+\mu_{i_1}\frac{\alpha_{i_1}-\alpha_i}{-w}+\mu_{i_2}\frac{\alpha_{i_2}-\alpha_i}{-w} = (L_\eta-q\eta)+(\mu_{i_1}-q\eta)\frac{\alpha_{i_1}-\alpha_i}{-w}+(\mu_{i_2}+q\eta)\frac{\alpha_{i_2}-\alpha_i}{-w}. \tag{38}
$$

and

$$
\alpha_{i_2}-\alpha_i-\frac{\gamma}{\eta}w = \alpha_{i_1}-\alpha_i-\frac{\gamma+q\eta}{\eta}w. \tag{39}
$$

shows that

$$
\prod_{\eta\in M\smallsetminus U}\frac{\tilde{H}_{L_\eta-q\eta,i_2,\eta}(w)}{k_{(j,\eta)}!\cdot q\cdot \prod_{\substack{0\leq i\leq r\\ 1\leq\gamma\leq q\eta\\ (\gamma,i)\neq(q\eta,i_2)}}\left(\frac{q\eta-\gamma}{q\eta}\alpha_{i_1}+\frac{\gamma}{q\eta}\alpha_{i_2}-\alpha_i\right)} = \prod_{\eta\in M\smallsetminus U}\frac{\tilde{H}_{L_\eta,i_1,\eta}(w)}{k_{(i_1,\eta)}!}. \tag{40}
$$

(Specifically, (38) is used to show that the product of exponential factors appearing on both sides of (40) are identical, and (39) is used to show that for each η, the corresponding products of factors $(\alpha_j-\alpha_i-\frac{\gamma}{\eta}z)$ on each side of (40) are identical. The factor $\frac{1}{q}$ matches with $\frac{(w-z)}{\alpha_{i_1}-\alpha_i-\frac{\gamma}{\eta}z}$, where $(\gamma,i)=(q\eta,i_2)$, on the right side of (40).)

By (40), the product expressions in (37) are independent of M, i.e. we may rewrite (37):

$$
\begin{aligned}
S((k_{(j,\eta)});U) = \left(\prod_{\substack{0\leq j\leq r\\ \eta\in\sigma_j\\ \eta\notin U}} \frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta,j,\eta}(w)}{k_{(j,\eta)}!}\right)\left(\prod_{\eta\in U}\frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta-q\eta,i_2,\eta}(w)}{k_{(j,\eta)}!\cdot q\cdot \prod_{\substack{0\leq i\leq r\\ 1\leq\gamma\leq q\eta\\ (\gamma,i)\neq(q\eta,i_2)}}\left(\frac{q\eta-\gamma}{q\eta}\alpha_{i_1}+\frac{\gamma}{q\eta}\alpha_{i_2}-\alpha_i\right)}\right)&\\
\cdot\left(\sum_{\substack{U\subseteq M\subseteq \sigma_L\\ |M|\geq a}} (-1)^{|M|-a}\binom{\sigma_{i_1}}{M}\binom{|M|-1}{a-1}\right)&.
\end{aligned}
\tag{41}
$$

The last sum in (41) is equal to

$$
\sum_{m=0}^{|\sigma_L|-a}(-1)^m\binom{m+a-1}{a-1}\binom{|\sigma_L|-|U|}{m+a-|U|} = \sum_{m=0}^{|\sigma_L|-a}\binom{-a}{m}\binom{|\sigma_L|-|U|}{|\sigma_L|-a-m} = \binom{|\sigma_L|-a-|U|}{|\sigma_L|-a},
$$

where we have used the Chu-Vandermonde identity (as well as the usual conventions for binomial coefficients with negative first argument). Thus $S((k_{(j,\eta)});U)=0$ for $U\neq\emptyset$, i.e.

$$S(k_{(j,\eta)}) = S((k_{(j,\eta)});\emptyset) = \prod_{\substack{0\le j\le r\\ \eta\in\sigma_j}} \frac{\tilde{H}^{(k_{(j,\eta)})}_{L_\eta,j,\eta}(w)}{k_{(j,\eta)}!}.$$

This is precisely to say that (36) and (34) agree, proving (**II**) in the case when σ is nontrivial.

Lastly, we treat the case where σ is the trivial multipartition of μ, so $I_{(\mu,\sigma)}$ has a factor $z+x_\sigma$. We must therefore also prove the analog of (33) for $zT_{\sigma,L}(z)$. Very little modification is required. In fact, our argument never mentioned the exact form of $H_{\beta,j,\eta}(z)$ for $\eta\notin\sigma_L$ — using this, it is easy to see that *any* multiple $g(z)T_{\sigma,L}(z)$ satisfies (33). This completes the proof of (**II**).

Finally, we prove (**III**) using Tseng's orbifold quantum Riemann-Roch (OQRR) operator. It is sufficient to prove the statement for the specializations $I_\mu(Q,0,\mathbf{x},-z)$, since (1) Q may be rescaled to absorb $e^{t_i\beta}$, and (2) the string equation shows that $\mathcal{L}^{\mathrm{tw}}_\mu$ is invariant under multiplication by $e^{-\sum_{j=0}^r t_i\mu_j(\alpha_j-\alpha_i)/z}$.

The OQRR operator is expressed in terms of the tangent bundle $F=T_\mu\operatorname{Sym}^d\mathbb{P}^r$ over $\mu=(\mu_0,\dots,\mu_r)$. Note that F splits into subbundles $F_{j,i}$, where $0\le i,j\le r$ and $i\neq j$; here $F_{j,i}$ consists of tangent vectors along which the μ_j points at the coordinate point $P_j\in\mathbb{P}^r$ move along the coordinate line $L_{(j,i)}$. Note that $(t_0,\dots,t_r)\in(\mathbb{C}^*)^{r+1}$ acts on $F_{j,i}$ by multiplication by t_i/t_j. The isotropy group G_μ acts on $F_{j,i}$, and may prevent it from decomposing further.

For each multipartition $\sigma=(\sigma_0,\dots,\sigma_r)$ of μ, let $F_{j,i,\sigma}$ denote the pullback of $F_{j,i}$ along $\sigma\hookrightarrow I\mu\to\mu$.[5] In [Tse10], one must describe, for each $q\in\mathbb{Q}$, the eigenbundle $F^q_{j,i,\sigma}$ of $F_{j,i,\sigma}$ on which a representative $\alpha=(\alpha_0,\dots,\alpha_r)\in G_\mu\cong S_{\mu_0}\times\cdots\times S_{\mu_r}$ of σ acts with eigenvalue $q\,|G_\mu|$. By definition of $F_{j,i}$, α acts by a permutation matrix associated to the cycle type of α_j; the eigenvalues are therefore $\bigsqcup_{\eta\in\sigma_j}\{1,e^{2\pi i/\eta}\dots,e^{2\pi i(\eta-1)/\eta}\}$. Equivalently,

$$\mathrm{ch}_k(F^{(q\,\mathrm{lcm}(\sigma_j))}_{j,i,\sigma}) = \begin{cases}\#\{\eta\in\sigma_j : q\in\frac{1}{\eta}\mathbb{Z}\} & k=0\\ 0 & k>0,\end{cases}\tag{42}$$

Define a collection of formal variables $\mathbf{s}=(s^{(j,i)}_k)$ for $0\le i,j\le r$, $i\neq j$, and $k\ge 0$. These define a family of multiplicative characteristic classes

$$\mathbf{c}_{\mathbf{s}}(F)=\prod_{\substack{0\le i\le r\\ i\neq j}}\exp\left(\sum_{k\ge 0} s^{(j,i)}_k\,\mathrm{ch}_k(F_{j,i})\right),$$

with the specialization

$$s^{(j,i)}_k=\begin{cases}-\log(\alpha_j-\alpha_i) & k=0\\ (-1)^k(k-1)!(\alpha_j-\alpha_i)^{-k} & k\ge 1\end{cases}\tag{43}$$

giving the equivariant Euler class $\mathbf{c}_{\mathbf{s}}(F)=e_T(F)$ (see [CR10], Lemma 4.1.2). Under this specialization, $\mathbf{s}^{(j,i)}(x):=\sum_{k\ge 0}s^{(j,i)}_k\frac{x^k}{k!}$ satisfies $\exp(\mathbf{s}^{(j,i)}(x))=(\alpha_j-\alpha_i+x)^{-1}$.

---

[5] In fact $F_{j,i,\sigma}$ splits further, with a subbundle for each distinct integer appearing in σ_j. We will not need this splitting directly, but it is related to the choice of variables in the proof of Claim 6.5 below.

Let B_m denote the mth Bernoulli polynomial; recall that $B_m(0)$ is the m-th Bernoulli number. The OQRR operator for $F = \bigoplus_{j\geq 0}^{r} \bigoplus_{\substack{0\leq i\leq r\\ i\neq j}} F_{j,i}$ is

$$\begin{aligned}
\Delta &= \bigoplus_{\sigma\in \mathrm{MultiPart}(\mu)} \exp\left(\sum_{j=0}^{r} \sum_{\substack{0\leq i\leq r\\ i\neq j}} \sum_{q\in\mathbb{Q}\cap[0,1)} \sum_{k\geq 0} \sum_{m\geq 0} s_k^{(j,i)} \frac{B_m(q)}{m!} \mathrm{ch}_{k+1-m}(F_{j,i,\sigma}^{(q)}) z^{m-1} \right) \\
&= \bigoplus_{\sigma\in \mathrm{MultiPart}(\mu)} \exp\left(\sum_{\substack{0\leq i,j\leq r\\ i\neq j}} \sum_{\eta\in\sigma_j} \sum_{\ell=0}^{\eta-1} \sum_{m\geq 1} s_{m-1}^{(j,i)} \frac{B_m(\ell/\eta)}{m!} z^{m-1} \right) \\
&= \bigoplus_{\sigma\in \mathrm{MultiPart}(\mu)} \exp\left(\sum_{\substack{0\leq i,j\leq r\\ i\neq j}} \sum_{\eta\in\sigma_j} \sum_{m\geq 1} s_{m-1}^{(j,i)} \frac{B_m(0)}{m!} (z/\eta)^{m-1} \right),
\end{aligned}$$

where the second equality is by (42) and the third equality is from the following identity, easily proved via generating functions of Bernoulli polynomials:

$$\sum_{0\leq \ell\leq \eta-1} B_m(\ell/\eta) = \frac{B_m(0)}{\eta^{m-1}}.$$

Let

$$G^{(j,i)}(x,z) = \sum_{n,m\geq 0} s_{n+m-1}^{(j,i)} \frac{B_m(0)}{m!} \frac{x^n}{n!} z^{m-1}, \tag{44}$$

so that

$$\Delta = \bigoplus_{\sigma\in \mathrm{MultiPart}(\mu)} \exp\left(\sum_{\substack{0\leq i,j\leq r\\ i\neq j}} \sum_{\eta\in\sigma_j} G^{(j,i)}(0, z/\eta) \right).$$

By definition of the Bernoulli polynomials, the coefficient of $s_k^{(j,i)}$ in $G^{(j,i)}(x,z)$ is the degree k part of $\frac{e^x}{e^z-1}$. For $a\in\mathbb{C}$, the equation

$$\frac{e^{x+az}}{e^{az}-1} = \frac{e^x}{e^{az}-1} + e^x$$

implies the functional equation $G^{(j,i)}(x+az, az) - G^{(j,i)}(x, az) = \mathbf{s}^{(j,i)}(x)$. Applying this repeatedly shows that (after the specialization (43)) we have $I_\mu(Q,0,\mathbf{x},-z) = \Delta\left(I_\mu^{\mathrm{untw}}(Q,\mathbf{x},\mathbf{s},-z)\right)$, where

$$I_\mu^{\mathrm{untw}}(Q,\mathbf{x},\mathbf{s},-z) = \sum_{\sigma\in \mathrm{MultiPart}(\mu)} \left(z\delta_{\sigma,(1,\ldots,1)} 1_\sigma + x_{\pi(\sigma)} 1_\sigma\right) \prod_{j=0}^{r} \prod_{\eta\in\sigma_j} \sum_{\beta\geq 0} \frac{Q^\beta \eta^\beta}{\beta!(-z)^\beta} \exp\left(-\sum_{\substack{0\leq i\leq r\\ i\neq j}} G^{(j,i)}(-\beta z/\eta, z/\eta) \right).$$

Let $\nu(j) = (0,\ldots,0,1,0,\ldots,0) \in \mathrm{ZPart}(1,r+1)$, where the 1 is in the j-th position, and let $\rho(j)$ be the unique element of $\mathrm{MultiPart}(\nu(j))$. Note the relationship:

$$I_\mu^{\mathrm{untw}}(Q,\mathbf{x},\mathbf{s},-z) = \sum_{\sigma\in \mathrm{MultiPart}(\mu)} \left(z\delta_{\sigma,(1,\ldots,1)} 1_\sigma + x_{\pi(\sigma)} 1_\sigma\right) \prod_{j=0}^{r} \prod_{\eta\in\sigma_j} \frac{\eta}{z} I_{\nu(j)}^{\mathrm{untw}}(Q,\mathbf{s},-z/\eta). \tag{45}$$

It is now sufficient to prove:

Claim 6.5. Assuming Identities 7.1 and 7.2, $I_\mu^{\mathrm{untw}}(Q, \mathbf{x}, \mathbf{s}, -z)$ is a $\Lambda_{\mathrm{nov}}[[\mathbf{x}, \mathbf{s}]]/(\mathbf{x})^2$-valued point of the (untwisted) Lagrangian cone $\mathcal{L}_\mu$.

If we prove Claim 6.5, it will imply that $I_\mu(Q, 0, \mathbf{x}, -z)$ is a $\Lambda_{\mathrm{nov}}^T[[\mathbf{x}]]$-valued point of the $\mathbf{s}$-twisted Lagrangian cone of BG_μ for $\mathbf{s}$ as in (43) — which is precisely $\mathcal{L}_\mu^{\mathrm{tw}}$.

Proof of claim 6.5. We prove a slightly stronger statement, replacing the variables $x_{\pi(\sigma)}$ in I_μ^{untw} with new variables x_σ in the obvious way. Define $\deg(s_k^{(j,i)}) = k+1$. We will prove that I_μ^{untw} is on $\mathcal{L}_\mu$ by induction on the degree. For the base case, [JK02, Proposition 3.4] shows that the J-function $J(Q, \mathbf{x}, -z) \in \mathcal{L}_\mu$ is given by

$$J(Q, \mathbf{x}, -z) = -z \exp(Q/(-z)) \left(1 + \sum_{\sigma \in \mathrm{MultiPart}(\mu)} \frac{x_\sigma}{-z} 1_\sigma \right) \qquad \mathrm{mod}(\mathbf{x})^2. \tag{46}$$

That is, $I_\mu^{\mathrm{untw}}(Q, \mathbf{x}, 0, -z) = J(dQ, \mathbf{x}, -z) \in \mathcal{L}_\mu$.

For the inductive step, suppose that I_μ^{untw} lies on $\mathcal{L}_\mu$ up to degree M in the variables $s^{(j,i)}$. We will show that I_μ^{untw} lies on $\mathcal{L}_\mu$ up to degree $M+1$. It is sufficient to show that all derivatives $\frac{\partial I_\mu^{\mathrm{untw}}}{\partial s_k^{(j,i)}}$ lie in the tangent space $T_{I_\mu^{\mathrm{untw}}}\mathcal{L}_\mu$ up to degree M [CCIT09, p.393]. Let $-z + \mathbf{t}_\mu$ be the part of $I_\mu^{\mathrm{untw}}(Q, \mathbf{x}, \mathbf{s}, -z)$ with nonnegative z-exponents, and for $\sigma \in \mathrm{MultiPart}(\mu)$, define

$$\tau^\sigma(\mathbf{t}_\mu) = \sum_{n \geq 1} \frac{1}{n!} \langle 1, 1_\sigma, \mathbf{t}_\mu, \ldots, \mathbf{t}_\mu \rangle_{0,n+2}^\mu. \tag{47}$$

Then by [CCIT09, Prop. B.4], $T_{I_\mu^{\mathrm{untw}}}\mathcal{L}_\mu$ is freely generated as a $\Lambda_{\mathrm{nov}}[[\mathbf{x}, \mathbf{s}, z]]/(\mathbf{x})^2$-module by the derivatives $\partial_\sigma J(\tau, -z)|_{\tau = \tau(\mathbf{t}_\mu)}$ with respect to the variables Q and x_σ. From (46), we have:

$$\begin{aligned} \partial_{(1,\ldots,1)} J(\tau, -z)|_{\tau=\tau(\mathbf{t}_\mu)} &= \exp(\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)/(-z)) \left(1 + \sum_{\sigma \in \mathrm{MultiPart}(\mu)} \frac{\tau^\sigma(\mathbf{t}_\mu)}{-z} 1_\sigma \right) \\ \partial_\sigma J(\tau, -z)|_{\tau=\tau(\mathbf{t}_\mu)} &= \exp(\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)/(-z)) 1_\sigma \qquad \text{for } \sigma \neq (1, \ldots, 1). \end{aligned}$$

We must therefore show that $\frac{1}{\exp(\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)/(-z))} \frac{\partial I_\mu^{\mathrm{untw}}}{\partial s_{k_0}^{(i_0,j_0)}}$ is in the $\Lambda_{\mathrm{nov}}[[\mathbf{x}, \mathbf{s}, z]]/(\mathbf{x})^2$-module generated by:

$$1 + \sum_{\sigma \in \mathrm{MultiPart}(\mu)} \frac{\tau^\sigma(\mathbf{t}_\mu)}{-z} 1_\sigma \qquad \text{and} \qquad 1_\sigma \quad \text{for } \sigma \neq (1, \ldots, 1), \tag{48}$$

for any $0 \leq i_0, j_0 \leq r$ with $i_0 \neq j_0$ and any $k_0 \geq 0$. For convenience, define

$$f_{\mu,k_0}^{(i_0,j_0)}(-z) := \frac{1}{\exp(\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)/(-z))} \frac{\partial I_\mu^{\mathrm{untw}}}{\partial s_{k_0}^{(i_0,j_0)}} \quad \text{and} \quad g_\mu(-z) := \frac{I_\mu^{\mathrm{untw}}/z}{\exp(\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)/(-z))}. \tag{49}$$

By (45) and the product rule, we have

$$\frac{\partial I_\mu^{\mathrm{untw}}}{\partial s_{k_0}^{(i_0,j_0)}} = \sum_{\sigma \in \mathrm{MultiPart}(\mu)} \left(z \delta_{\sigma,(1,\ldots,1)} 1_\sigma + x_\sigma 1_\sigma \right) \sum_{\substack{0 \leq j_1 \leq r \\ \eta_1 \in \sigma_{j_1}}} \frac{\eta_1}{z} \frac{\partial I_{\nu(j_1)}^{\mathrm{untw}}(Q, \mathbf{s}, -z/\eta_1)}{\partial s_{k_0}^{(i_0,j_0)}} \prod_{\substack{0 \leq j \leq r \\ \eta \in \sigma_j \\ (j,\eta) \neq (j_1,\eta_1)}} \frac{\eta}{z} I_{\nu(j)}^{\mathrm{untw}}(Q, \mathbf{s}, -z/\eta). \tag{50}$$

Observe that Identity 7.1 in Section 7 implies

$$\tau^{(1,\dots,1)}(\mathbf{t}_\mu) = \sum_{0\le j\le r} \mu_j \tau^{\rho(j)}(\mathbf{t}_{\nu(j)}),$$

where $\nu(j)$ and $\rho(j)$ are as in (45). Thus (50) implies:

(51)

$$\begin{aligned} f^{(i_0,j_0)}_{\mu,k_0}(-z) &= \sum_{\sigma\in\mathrm{MultiPart}(\mu)} \left(z\delta_{\sigma,(1,\dots,1)}1_\sigma + x_\sigma 1_\sigma\right) \sum_{\substack{0\le j_1\le r\\ \eta_1\in\sigma_{j_1}}} \frac{\eta_1}{z} f^{(i_0,j_0)}_{\nu(j_1),k_0}(-z/\eta_1) \prod_{\substack{0\le j\le r\\ \eta\in\sigma_j\\ (j,\eta)\ne(j_1,\eta_1)}} g_{\nu(j)}(-z/\eta) \\ &= \sum_{\sigma\in\mathrm{MultiPart}(\mu)} \left(z\delta_{\sigma,(1,\dots,1)}1_\sigma + x_\sigma 1_\sigma\right) \left(\prod_{\substack{0\le j\le r\\ \eta\in\sigma_j}} g_{\nu(j)}(-z/\eta)\right) \sum_{\eta\in\sigma_{j_0}} \frac{\eta}{z} \frac{f^{(i_0,j_0)}_{\nu(j_0),k_0}(-z/\eta)}{g_{\nu(j_0)}(-z/\eta)}. \end{aligned}$$

(In the second equality, we have used the fact that $f^{(i_0,j_0)}_{\nu(j_1),k_0}(-z/\eta) = 0$ if $j_0 \ne j_1$, which is immediate from the definition of I^{untw}_μ.)

By the mirror theorem for $\mathrm{Sym}^1 \mathbb{P}^r = \mathbb{P}^r$ (specifically, the proof on [CCIT15, p.31]), we have $I^{\mathrm{untw}}_{\nu(j)} \in \mathcal{L}_{\nu(j)}$. Thus $\frac{1}{z} I^{\mathrm{untw}}_{\nu(j)} \in T_{I^{\mathrm{untw}}_{\nu(j)}}\mathcal{L}_{\mu'}$, by the tangent space property of $\mathcal{L}_{\nu(j)}$ [Giv04, Thm. 1]. In particular, $\frac{1}{z} I^{\mathrm{untw}}_{\nu(j)}(Q,\mathbf{s},-z/\eta)$ is divisible by $\exp(-\eta\tau^{\rho(j)}(\mathbf{t}_{\nu(j)})/z)$, where by "divisible", we mean that the ratio contains only nonnegative powers of z; in other words, $g_{\nu(j)}(-z/\eta)$ contains only nonnegative powers of z. Similarly, because $I^{\mathrm{untw}}_{\nu(j)} \in \mathcal{L}_{\nu(j)}$, we have that $f^{(i_0,j_0)}_{\nu(j_1),k_0}(-z/\eta_1)$ contains only nonnegative powers of z. That is, the first line of (51) implies $f^{(i_0,j_0)}_{\mu,k_0}(-z)$ is of the form:

$$1\cdot(\text{power series in } z) + \sum_{\sigma\in\mathrm{MultiPart}(\mu)} \frac{x_\sigma 1_\sigma}{-z}\cdot(\text{power series in } z) + O(\mathbf{x})^2.$$

In order to show $f^{(i_0,j_0)}_{\mu,k_0}(-z)$ is in the $\Lambda_{\mathrm{nov}}[[\mathbf{x},\mathbf{s},z]]/(\mathbf{x})^2$-module generated by (48), it remains to show that for all $\sigma\in\mathrm{MultiPart}(\mu)$ with $\sigma\ne(1,\dots,1)$, the coefficient of $z^{-1}\cdot 1_\sigma$ in (51) is equal to $-\tau^\sigma(\mathbf{t}_\mu)$ times the coefficient of $z^0\cdot 1$ in (51). In other words, we must show:

$$x_\sigma \left(\prod_{\substack{0\le j\le r\\ \eta\in\sigma_j}} g_{\nu(j)}(0)\right) \sum_{\eta\in\sigma_{j_0}} \eta \frac{f^{(i_0,j_0)}_{\nu(j_0),k_0}(0)}{g_{\nu(j_0)}(0)} = \tau^\sigma(\mathbf{t}_\mu) \left(\prod_{\substack{0\le j\le r\\ 1\le a\le\mu_j}} g_{\nu(j)}(0)\right) \sum_{1\le a_1\le\mu_{j_0}} \frac{f^{(i_0,j_0)}_{\nu(j_0),k_0}(0)}{g_{\nu(j_0)}(0)}.$$

After cancellation, this is precisely the statement of Identity 7.2. □

Claim 6.5 completes the proof of (**III**), and hence of Theorem 6.3. □

From (29), we compute that (assuming $r > 0$) we have

$$I_{\mathrm{Sym}^d\mathbb{P}^r}(Q,t,\mathbf{x},z) = 1\cdot z + \sum_{i=0}^r t_i H_i + \sum_{\sigma\in\mathrm{MultiPart}(\mu)} x_\sigma 1_\sigma + O(z^{-1}),$$

where $[H_i]$ is as in Section 2.3. By definition of $J_{\mathrm{Sym}^d\mathbb{P}^r}$ (from Section 2.4), Theorem 6.3 implies:

Corollary 6.6. *Assuming Identities 7.1 and 7.2, we have*

$$I_{\mathrm{Sym}^d\mathbb{P}^r}(Q,t,\mathbf{x},z) = J_{\mathrm{Sym}^d\mathbb{P}^r}(Q,\theta,z) \mod(\mathbf{x})^2,$$

where $\theta = \sum_{i=0}^{r} t_i H_i + \sum_{\sigma \in \mathrm{MultiPart}(\mu)} x_\sigma 1_\sigma$.

7. APPENDIX: CONJECTURAL COMBINATORIAL IDENTITIES

The Mirror Theorem 6.3 is conditional upon the following two conjectural combinatorial identities. Let $\mathbf{t}_\mu$ denote the part of I_μ^{untw} with nonnegative powers of z, and let $\tau^\sigma(\mathbf{t}_\mu)$ be as in (47). We conjecture the following:

Identity 7.1. *For all* $\mu \geq 1$, *we have*

$$\tau^{(1,\ldots,1)}(\mathbf{t}_\mu) = \sum_{0 \leq j \leq r} \mu_j \sum_{n_0, n_1, \ldots \geq 0} \frac{\left(1 + \sum_{k \geq 0} k n_k\right)^{-2 + \sum_{k \geq 0} n_k}}{\prod_{k \geq 0} n_k! (k!)^{n_k}} Q^{1 + \sum_{k \geq 0} k n_k} \prod_{k \geq 0} \left(\sum_{\substack{0 \leq i \leq r \\ i \neq j}} s_k^{(j,i)} \right)^{n_k} + O(\mathbf{x}^2).$$

Identity 7.2. *Let* $\mu \geq 1$, *and let* σ *be a partition of* μ *that is* not *equal to* $(1, \ldots, 1)$. *Then*

$$\tau^\sigma(\mathbf{t}_\mu) = x_\sigma \prod_{0 \leq j \leq r} g_{\nu(j)}(0)^{|\sigma_j| - \mu_j} + O(\mathbf{x})^2,$$

where $g_\mu(-z)$ *is as in* (49). *(Recall that* $\nu(j) \in \mathrm{ZPart}(1, r+1)$ *is the composition with 1 in the* j*th entry, and* $\rho(j)$ *is the unique multipartition of* $\nu(j)$*.)*

Both identities are entirely combinatorial in nature, as $\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)$, $\tau^\sigma(\mathbf{t}_\mu)$, and $g_{\nu(j)}(0)$ are entirely explicit. Specifically, one uses the formulas [Koc01, Lemma 1.5.1] and [JK02, Prop. 3.4], both of which we have already used extensively in this paper, to evaluate the integrals appearing in $\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)$ and $\tau^\sigma(\mathbf{t}_\mu)$ in terms of multinomial coefficients. The resulting expressions for $\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)$ and $\tau^\sigma(\mathbf{t}_\mu)$ are iterated sums over partitions.

We expect that both identities can be proved via cleverly switching the order of summation, and applying basic multinomial coefficient identities or generating function techniques. (In the introduction we speculated that tools from integrable systems might also yield a less-hands-on proof.) However, due to the complicatedness of the generating functions involved, we were not able to complete either proof. We instead conclude with some relevant observations and experimental verifications of both identities to small order.

Notes about Identity 7.1:

1. The variables x_σ are entirely absent from Identity 7.1, as follows. The invariants appearing in $\tau^{(1,\ldots,1)}(\mathbf{t}_\mu)$ are of the form

   $$\langle 1, 1, \mathbf{t}_\mu, \ldots, \mathbf{t}_\mu \rangle_{0,n+2}^{\mu}.$$

   For $\sigma \neq (1, \ldots, 1)$, the class 1_σ always appears with an x_σ factor in $\mathbf{t}_\mu$. As we are working modulo $(\mathbf{x})^2$, the only contributions are from invariants with at most one nontrivial class 1_σ. By [JK02, Prop. 3.4], invariants with exactly one nontrivial class 1_σ vanish. Thus we may replace $\mathbf{t}_\mu$ in Identity 7.1 with $\mathbf{t}_\mu^0$, where $\mathbf{t}_\mu^0$ denotes the coefficient of 1 in $\mathbf{t}_\mu$.
2. The difficulty in proving Identity 7.1 is not due to the presence of Bernoulli numbers in the definition of I_μ^{untw}; in fact the Bernoulli numbers appear to play no role whatsoever, as Identity 7.1 is a special case of the following more general formula. Let

$$\mathbf{f} = -z \prod_{0 \leq j \leq r} \left(\sum_{\beta \geq 0} \frac{Q^\beta}{\beta! (-z)^\beta} \exp\left(\sum_{\substack{k \geq 0 \\ 0 \leq \ell \leq k+1}} c_{k,\ell}^{(j)} z^k \beta^\ell \right) \right)^{\mu_j}. \tag{52}$$

Then experimentally we appear to have

(53)

$$\tau^{(1,\dots,1)}(\mathbf{t}_{\mathbf{f}}) = -\sum_{0\leq j\leq r} \mu_j \sum_{n_0,n_1,\dots\geq 0} \left(1+\sum_{k\geq 0} kn_k\right)^{-2+\sum_{k\geq 0} n_k} (-Q)^{1+\sum_{k\geq 0} kn_k} \prod_{k\geq 0} \frac{((k+1)c^{(j)}_{k,k+1})^{n_k}}{n_k!},$$

where $-z+\mathbf{t}_{\mathbf{f}}$ denotes the part of $\mathbf{f}$ with nonnegative powers of z. Using Note (1), Identity 7.1 is the special case where

$$c^{(j)}_{k,\ell} = \sum_{i\neq j} \frac{B_{k-\ell+1}(0)}{\ell!(k-\ell+1)!} s^{(j,i)}_k .$$

(Indeed, the absence of $c_{k,\ell}$ for $\ell \leq k$ in (53) shows that the Bernoulli numbers $B_m(0)$ for $m > 0$ are entirely irrelevant.) Note that we require $\ell \leq k+1$ in (52) because, in the expression $G^{(j,i)}(-\beta z/\eta, z/\eta)$ from (44), the power of β is always at most one more than the power of z.

(3) Writing

$$\mathbf{t}^0_\mu = 1\cdot(y_0+y_1 z+y_2 z^2+\cdots),$$

and evaluating the integrals

$$\left\langle 1,1,\mathbf{t}^0_\mu,\dots,\mathbf{t}^0_\mu\right\rangle^\mu_{0,n+2},$$

gives the expression

$$\tau^{(1,\dots,1)}(\mathbf{t}^0_\mu) = \sum_{n=1}^{\infty} \sum_{\zeta\in \mathrm{ZPart}(n-1,n)} \frac{1}{|S_\zeta|}\binom{n-1}{\zeta} \prod_{\eta\in\zeta} y_\eta .$$

We may compute each y_i explicitly. For example, if we set every $c_{k',\ell'}$ to zero *except for one*, say $c_{k,\ell}$, and we have $\mu_j = 0$ for all but one j, then we have:

$$y_\eta = (-1)^\eta \mu_j! \sum_{N\geq \eta/k} (c^{(j)}_{k,\ell})^N \frac{Q^{kN-\eta+1}}{N!(kN-\eta+1)!} \sum_{\pi\in \mathrm{ZPart}(kN-\eta+1,\mu_j)} \binom{kN-\eta+1}{\pi} \frac{\left(\sum_{\lambda\in\pi}\lambda^\ell\right)^N}{|S_\pi|}.$$

(4) Some straightforward combinatorial identities can be used to expand $\tau^{(1,\dots,1)}(\mathbf{t}_\mu)$ as a polynomial in μ, whose coefficients are sums over partitions. Experimentally, the coefficient of μ^m "miraculously" cancels to give zero whenever $m > 1$. (Indeed, proving this "linearity" would suffice for the purpose of Theorem 6.3; we do not need the explicit formula.)

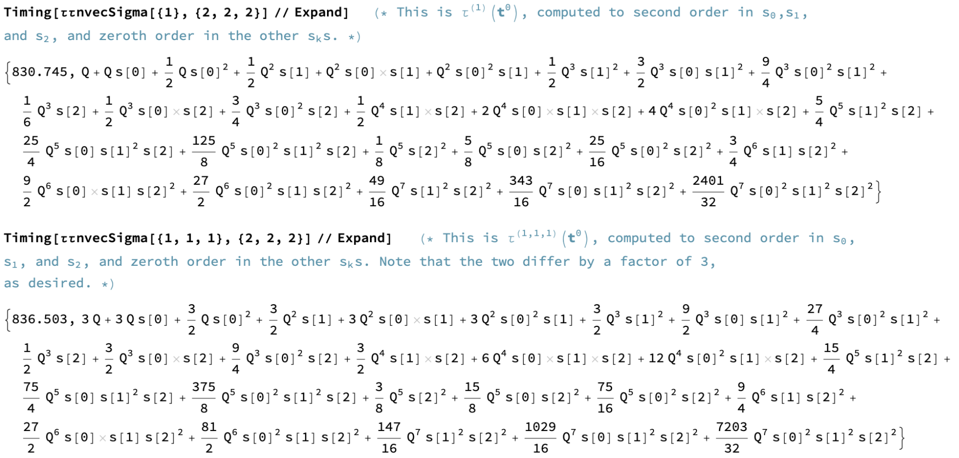
(5) Figure 4 verifies Identity 7.1 for $\mu = (1,0,\dots)$ and $\mu = (3,0,\dots)$, to second order in $s_k = \sum_{1\leq i\leq r} s^{(i,0)}_k$ for $k \leq 2$ (and to zeroth order in s_k for $k > 2$).

Notes about Identity 7.2:

(6) The same argument as in Note (1) shows that $\tau^\sigma(\mathbf{t}_\mu) \in x_\sigma \cdot \mathbb{C}[[Q,\{s^{(j,i)}_k\}]] + O(\mathbf{x}^2)$. As in Note (3), it is straightforward to expand $\tau^\sigma(\mathbf{t}_\mu)$ as an explicit sum over partitions.

(7) Unlike in Note (2), the Bernoulli numbers do appear to play a nontrivial role in Identity 7.2.

(8) Figure 5 verifies Identity 7.2 for $\sigma = \{4\}$, $\sigma = \{4,1\}$, and $\sigma = \{3,2\}$, to the same orders in s_k as above. Observe in particular that Identity 7.2 predicts that $\tau^\sigma(\mathbf{t}_\mu)$ is identical for these three choices of σ. (By $\sigma = \{4,1\}$, we really mean $\mu = (5,0\dots,)$ and $\sigma = (\{4,1\},\{\},\dots)$.)



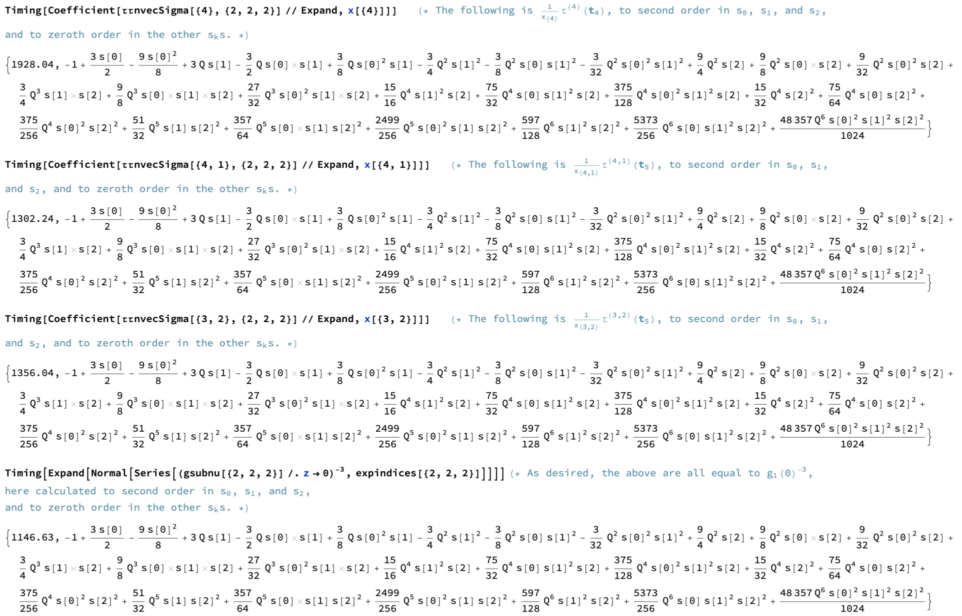
Timing[ττnvecSigma[{1}, {2, 2, 2}] // Expand] (* This is $\tau^{(1)}(\mathbf{t}^{0})$, computed to second order in s_0,s_1, and s_2, and zeroth order in the other s_ks. *)

$$\Big\{830.745,\ Q + Q s[0] + \frac{1}{2} Q s[0]^2 + \frac{1}{2} Q^2 s[1] + Q^2 s[0] \times s[1] + Q^2 s[0]^2 s[1] + \frac{1}{2} Q^3 s[1]^2 + \frac{3}{2} Q^3 s[0] s[1]^2 + \frac{9}{4} Q^3 s[0]^2 s[1]^2 + \frac{1}{6} Q^3 s[2] + \frac{1}{2} Q^3 s[0] \times s[2] + \frac{3}{4} Q^3 s[0]^2 s[2] + \frac{1}{2} Q^4 s[1] \times s[2] + 2 Q^4 s[0] \times s[1] \times s[2] + 4 Q^4 s[0]^2 s[1] \times s[2] + \frac{5}{4} Q^5 s[1]^2 s[2] + \frac{25}{4} Q^5 s[0] s[1]^2 s[2] + \frac{125}{8} Q^5 s[0]^2 s[1]^2 s[2] + \frac{1}{8} Q^5 s[2]^2 + \frac{5}{8} Q^5 s[0] s[2]^2 + \frac{25}{16} Q^5 s[0]^2 s[2]^2 + \frac{3}{4} Q^6 s[1] s[2]^2 + \frac{9}{2} Q^6 s[0] \times s[1] s[2]^2 + \frac{27}{2} Q^6 s[0]^2 s[1] s[2]^2 + \frac{49}{16} Q^7 s[1]^2 s[2]^2 + \frac{343}{16} Q^7 s[0] s[1]^2 s[2]^2 + \frac{2401}{32} Q^7 s[0]^2 s[1]^2 s[2]^2\Big\}$$

Timing[ττnvecSigma[{1, 1, 1}, {2, 2, 2}] // Expand] (* This is $\tau^{(1,1,1)}(\mathbf{t}^{0})$, computed to second order in s_0, s_1, and s_2, and zeroth order in the other s_ks. Note that the two differ by a factor of 3, as desired. *)

$$\Big\{836.503,\ 3 Q + 3 Q s[0] + \frac{3}{2} Q s[0]^2 + \frac{3}{2} Q^2 s[1] + 3 Q^2 s[0] \times s[1] + 3 Q^2 s[0]^2 s[1] + \frac{3}{2} Q^3 s[1]^2 + \frac{9}{2} Q^3 s[0] s[1]^2 + \frac{27}{4} Q^3 s[0]^2 s[1]^2 + \frac{1}{2} Q^3 s[2] + \frac{3}{2} Q^3 s[0] \times s[2] + \frac{9}{4} Q^3 s[0]^2 s[2] + \frac{3}{2} Q^4 s[1] \times s[2] + 6 Q^4 s[0] \times s[1] \times s[2] + 12 Q^4 s[0]^2 s[1] \times s[2] + \frac{15}{4} Q^5 s[1]^2 s[2] + \frac{75}{4} Q^5 s[0] s[1]^2 s[2] + \frac{375}{8} Q^5 s[0]^2 s[1]^2 s[2] + \frac{3}{8} Q^5 s[2]^2 + \frac{15}{8} Q^5 s[0] s[2]^2 + \frac{75}{16} Q^5 s[0]^2 s[2]^2 + \frac{9}{4} Q^6 s[1] s[2]^2 + \frac{27}{2} Q^6 s[0] \times s[1] s[2]^2 + \frac{81}{2} Q^6 s[0]^2 s[1] s[2]^2 + \frac{147}{16} Q^7 s[1]^2 s[2]^2 + \frac{1029}{16} Q^7 s[0] s[1]^2 s[2]^2 + \frac{7203}{32} Q^7 s[0]^2 s[1]^2 s[2]^2\Big\}$$

FIGURE 4. Experimental verification of Identity 7.1



Timing[Coefficient[ττnvecSigma[{4}, {2, 2, 2}] // Expand, x[{4}]]] (* The following is $\frac{1}{x_{\{4\}}}\tau^{\{4\}}(\mathbf{t}_4)$, to second order in s_0, s_1, and s_2, and to zeroth order in the other s_ks. *)

$$\Big\{1928.04,\ -1 + \frac{3 s[0]}{2} - \frac{9 s[0]^2}{8} + 3 Q s[1] - \frac{3}{2} Q s[0] \times s[1] + \frac{3}{8} Q s[0]^2 s[1] - \frac{3}{4} Q^2 s[1]^2 - \frac{3}{8} Q^2 s[0] s[1]^2 - \frac{3}{32} Q^2 s[0]^2 s[1]^2 + \frac{9}{4} Q^2 s[2] + \frac{9}{8} Q^2 s[0] \times s[2] + \frac{9}{32} Q^2 s[0]^2 s[2] + \frac{3}{4} Q^3 s[1] \times s[2] + \frac{9}{8} Q^3 s[0] \times s[1] \times s[2] + \frac{27}{32} Q^3 s[0]^2 s[1] \times s[2] + \frac{15}{16} Q^4 s[1]^2 s[2] + \frac{75}{32} Q^4 s[0] s[1]^2 s[2] + \frac{375}{128} Q^4 s[0]^2 s[1]^2 s[2] + \frac{15}{32} Q^4 s[2]^2 + \frac{75}{64} Q^4 s[0] s[2]^2 + \frac{375}{256} Q^4 s[0]^2 s[2]^2 + \frac{51}{32} Q^5 s[1] s[2]^2 + \frac{357}{64} Q^5 s[0] \times s[1] s[2]^2 + \frac{2499}{256} Q^5 s[0]^2 s[1] s[2]^2 + \frac{597}{128} Q^6 s[1]^2 s[2]^2 + \frac{5373}{256} Q^6 s[0] s[1]^2 s[2]^2 + \frac{48\,357 Q^6 s[0]^2 s[1]^2 s[2]^2}{1024}\Big\}$$

Timing[Coefficient[ττnvecSigma[{4, 1}, {2, 2, 2}] // Expand, x[{4, 1}]]] (* The following is $\frac{1}{x_{\{4,1\}}}\tau^{\{4,1\}}(\mathbf{t}_5)$, to second order in s_0, s_1, and s_2, and to zeroth order in the other s_ks. *)

$$\Big\{1302.24,\ -1 + \frac{3 s[0]}{2} - \frac{9 s[0]^2}{8} + 3 Q s[1] - \frac{3}{2} Q s[0] \times s[1] + \frac{3}{8} Q s[0]^2 s[1] - \frac{3}{4} Q^2 s[1]^2 - \frac{3}{8} Q^2 s[0] s[1]^2 - \frac{3}{32} Q^2 s[0]^2 s[1]^2 + \frac{9}{4} Q^2 s[2] + \frac{9}{8} Q^2 s[0] \times s[2] + \frac{9}{32} Q^2 s[0]^2 s[2] + \frac{3}{4} Q^3 s[1] \times s[2] + \frac{9}{8} Q^3 s[0] \times s[1] \times s[2] + \frac{27}{32} Q^3 s[0]^2 s[1] \times s[2] + \frac{15}{16} Q^4 s[1]^2 s[2] + \frac{75}{32} Q^4 s[0] s[1]^2 s[2] + \frac{375}{128} Q^4 s[0]^2 s[1]^2 s[2] + \frac{15}{32} Q^4 s[2]^2 + \frac{75}{64} Q^4 s[0] s[2]^2 + \frac{375}{256} Q^4 s[0]^2 s[2]^2 + \frac{51}{32} Q^5 s[1] s[2]^2 + \frac{357}{64} Q^5 s[0] \times s[1] s[2]^2 + \frac{2499}{256} Q^5 s[0]^2 s[1] s[2]^2 + \frac{597}{128} Q^6 s[1]^2 s[2]^2 + \frac{5373}{256} Q^6 s[0] s[1]^2 s[2]^2 + \frac{48\,357 Q^6 s[0]^2 s[1]^2 s[2]^2}{1024}\Big\}$$

Timing[Coefficient[ττnvecSigma[{3, 2}, {2, 2, 2}] // Expand, x[{3, 2}]]] (* The following is $\frac{1}{x_{\{3,2\}}}\tau^{\{3,2\}}(\mathbf{t}_5)$, to second order in s_0, s_1, and s_2, and to zeroth order in the other s_ks. *)

$$\Big\{1356.04,\ -1 + \frac{3 s[0]}{2} - \frac{9 s[0]^2}{8} + 3 Q s[1] - \frac{3}{2} Q s[0] \times s[1] + \frac{3}{8} Q s[0]^2 s[1] - \frac{3}{4} Q^2 s[1]^2 - \frac{3}{8} Q^2 s[0] s[1]^2 - \frac{3}{32} Q^2 s[0]^2 s[1]^2 + \frac{9}{4} Q^2 s[2] + \frac{9}{8} Q^2 s[0] \times s[2] + \frac{9}{32} Q^2 s[0]^2 s[2] + \frac{3}{4} Q^3 s[1] \times s[2] + \frac{9}{8} Q^3 s[0] \times s[1] \times s[2] + \frac{27}{32} Q^3 s[0]^2 s[1] \times s[2] + \frac{15}{16} Q^4 s[1]^2 s[2] + \frac{75}{32} Q^4 s[0] s[1]^2 s[2] + \frac{375}{128} Q^4 s[0]^2 s[1]^2 s[2] + \frac{15}{32} Q^4 s[2]^2 + \frac{75}{64} Q^4 s[0] s[2]^2 + \frac{375}{256} Q^4 s[0]^2 s[2]^2 + \frac{51}{32} Q^5 s[1] s[2]^2 + \frac{357}{64} Q^5 s[0] \times s[1] s[2]^2 + \frac{2499}{256} Q^5 s[0]^2 s[1] s[2]^2 + \frac{597}{128} Q^6 s[1]^2 s[2]^2 + \frac{5373}{256} Q^6 s[0] s[1]^2 s[2]^2 + \frac{48\,357 Q^6 s[0]^2 s[1]^2 s[2]^2}{1024}\Big\}$$

Timing[Expand[Normal[Series[(gsubnu[{2, 2, 2}] /. z → 0)$^{-3}$, expindices[{2, 2, 2}]]]]] (* As desired, the above are all equal to $g_1(0)^{-3}$, here calculated to second order in s_0, s_1, and s_2, and to zeroth order in the other s_ks. *)

$$\Big\{1146.63,\ -1 + \frac{3 s[0]}{2} - \frac{9 s[0]^2}{8} + 3 Q s[1] - \frac{3}{2} Q s[0] \times s[1] + \frac{3}{8} Q s[0]^2 s[1] - \frac{3}{4} Q^2 s[1]^2 - \frac{3}{8} Q^2 s[0] s[1]^2 - \frac{3}{32} Q^2 s[0]^2 s[1]^2 + \frac{9}{4} Q^2 s[2] + \frac{9}{8} Q^2 s[0] \times s[2] + \frac{9}{32} Q^2 s[0]^2 s[2] + \frac{3}{4} Q^3 s[1] \times s[2] + \frac{9}{8} Q^3 s[0] \times s[1] \times s[2] + \frac{27}{32} Q^3 s[0]^2 s[1] \times s[2] + \frac{15}{16} Q^4 s[1]^2 s[2] + \frac{75}{32} Q^4 s[0] s[1]^2 s[2] + \frac{375}{128} Q^4 s[0]^2 s[1]^2 s[2] + \frac{15}{32} Q^4 s[2]^2 + \frac{75}{64} Q^4 s[0] s[2]^2 + \frac{375}{256} Q^4 s[0]^2 s[2]^2 + \frac{51}{32} Q^5 s[1] s[2]^2 + \frac{357}{64} Q^5 s[0] \times s[1] s[2]^2 + \frac{2499}{256} Q^5 s[0]^2 s[1] s[2]^2 + \frac{597}{128} Q^6 s[1]^2 s[2]^2 + \frac{5373}{256} Q^6 s[0] s[1]^2 s[2]^2 + \frac{48\,357 Q^6 s[0]^2 s[1]^2 s[2]^2}{1024}\Big\}$$

FIGURE 5. Experimental verification of Identity 7.2

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