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SRC Discussion Paper No 111
June 2021

Systemic Risk Centre

Discussion Paper Series

ISSN 2054-538X

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JEL Codes: C63, C73, D40; D81; G11, G14

Keywords: Dark pool; limit order market; execution priority rules; liquidity; welfare

This paper is published as part of the Systemic Risk Centre's Discussion Paper Series. The support of the Economic and Social Research Council (ESRC) in funding the SRC is gratefully acknowledged [grant number ES/R009724/1].

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Published by
Systemic Risk Centre
The London School of Economics and Political Science
Houghton Street
London WC2A 2AE

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1 Introduction

In modern financial markets, traders can perform transactions not only in traditional exchanges (i.e., lit venues) that are in general organized as limit order markets, but also in alternative trading systems such as dark pools, where anonymous and undisplayed orders are executed. For instance, the average market share of dark pools in the United States accounted for approximately 44.9% of the trading volume in the stock market in June 2017 (as reported by TABB Group).[1] Moreover, Farley *et al*. (2018) show that, in the United States, there are stocks reaching a dark market share of above 70%. In Europe, as reported by the Best Execution Magazine, the market share of dark pools accounted for approximately 20% of the stock trading volume, while the market share of dark pools had increased to around 50% for large-in-scale orders by August 2018.[2] Nevertheless, despite the importance of dark pools in financial markets, we still lack clear understanding of the effect of this type of trading activity on market quality. In fact, different empirical studies have shown opposing results on the effect of dark pools on market performance.[3]

Is dark trading activity beneficial—or harmful—for the market quality of the whole system? How do traders decide whether to trade in traditional lit markets or dark pools? What is the impact on market quality of a migration of trading activity from lit markets to dark pools? In this paper, we answer these questions by using a dynamic equilibrium model that characterizes a multi-market environment, with a lit trading venue (organized as a limit order market) and a dark pool.[4] We show that the lit-dark trading activity—and its impact on market quality—depends on market conditions, which are affected by three intertwined elements: (i) execution priority rules in the dark market (e.g., size priority or time priority); (ii) adverse selection in the trading process; and

[1] See https://research.tabbgroup.com/report/v15-034-tabb-equity-digest-q2-2017
[2] An order is considered to be large-in-scale when its size is large compared with the market size threshold.
[3] See, e.g., Ray (2010), Ready (2014), Brugler (2015), Degryse *et al*. (2015), Kwan *et al*. (2015), Buti *et al*. (2016), Foley and Putniņš (2016), Gresse (2017), Hatheway *et al*. (2017), Menkveld *et al*. (2007), and Farley *et al*. (2018).
[4] In general, lit markets are organized as limit order markets. For instance, limit order markets in diverse countries represent 85% of the stock exchanges (Jain, 2005).

(iii) competition in order submission in each market when traders face execution costs (waiting and immediacy costs).

To understand how execution priority rules, adverse selection, and traders' competition affect lit-dark trading activity, it is important to identify the benefits and drawbacks of dark orders. Their advantages and disadvantages may be thought of as a mix between those of market orders and those of limit orders in lit markets. First, dark orders have implicit waiting costs that fall between the waiting costs of market and limit orders. For instance, if the undisplayed orders that are waiting in the dark pool are sell dark orders, a new buy dark order will be executed immediately (effectively as a market order). Conversely, if the undisplayed orders that are waiting in the dark pool are buy dark orders, a new buy dark order will have to wait in the execution queue as a limit order. Nevertheless, waiting costs can be lower for large-in-scale orders in the dark pool when there is a *size* priority rule, than when there is a *time* priority rule for the execution of orders in the dark market. It is important to consider the effect of execution priority rules on market quality, because several dark pools (e.g., Turquoise Plato, MS Pool, and Nordic@Mid, among other dark pools) use a *size* priority for the execution of orders, which has attracted a large volume of market trading activity. For instance, the trading activity of Turquoise Plato, which uses the size execution priority rule, increased its trading volume from £2,001M in July 2012 to £16,297M in July 2020 (i.e., an increase of 814.4%).

Second, dark orders are executed at better prices than market orders, but at worse prices than limit orders, as dark orders are in general executed at a price somewhere between the best quotes of the prevailing limit order market (e.g., at the midquote or the volume-weighted average price, VWAP). Third, unexecuted dark orders waiting in the dark pool also have the disadvantage that they can be ‘picked off’, in a similar way to unexecuted limit orders. This is because, due to cognitive limits, traders cannot instantaneously modify unexecuted dark orders when the asset value moves against them. Thus, other *competing* traders can profit by picking-off unexecuted dark orders in unfavorable positions (i.e., there is *adverse selection*).[5]

Consequently, the effects of execution priority rules, traders' competition, and adverse selection are all important aspects to consider when examining the impact of the introduction of a dark pool running in parallel to a limit order market. Moreover, it is important to note that these three elements endogenously interact with each other. For instance, a *size priority* rule may naturally induce a migration of traders with large orders to the dark pool, which may reduce *traders' competition* in order submission in the lit market. A reduction in traders' competition should increase the bid-ask spread in the lit venue. However, a growth in the bid-ask spread should diminish the chances of limit orders in the best quotes being mispriced after changes in the asset value; hence, there will be a reduction in the *picking-off risk* in the lit market. This reduction in the picking-off risk in the lit market may again attract trading activity back to the lit venue from the dark pool. Consequently, the effects of execution priority rules, adverse selection, and traders' competition should all be taken into account simultaneously in analyzing the impact of dark pools on market quality.

In this paper, we introduce a dynamic equilibrium model for a dark pool running in parallel to a limit order market, in which we simultaneously include different execution priority rules for dark orders, adverse selection, and traders' competition in order submission for all types of orders and all types of traders. This study is distinct from, and *complements*, the current theoretical literature on the interaction between a dark pool and a limit order market, since previous studies do not consider all three of these elements together.[6]

Our model describes a multi-market environment with an asset whose fundamental value evolves stochastically, and which can be traded in either a limit order market or a dark pool. Our

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[5] One may argue that cognitive limits do not exist for high-frequency trading (HFT) firms. However, HFT firms still suffer from cognitive limits since computers also require processing time to analyze new information, there are delays in order transmissions, and there are processing times in the exchanges.

[6] See, e.g., Buti *et al*. (2017), Beyona *et al*. (2017), and Brolley (2019).

framework is based on the model of a single limit order market of Goettler *et al*. (2005, 2009), which is extended to a multi-market setup that includes two market types: a dark pool and a limit order market. There are multiple traders who compete in order submission (i.e., there is *trader competition*). Traders can submit orders to the lit market (limit orders or market orders) or to the dark pool (undisplayed dark orders). In addition, traders face a cost of delaying that represents the cost of monitoring the market until an order is executed (i.e., there is a waiting cost), which also incentivizes competition for quick executions.

Traders can modify unexecuted limit orders and unexecuted dark orders. However, traders have cognitive limits, and therefore cannot instantaneously modify their unexecuted limit and dark orders when market conditions change. Cognitive limits are modeled by allowing traders to only re-enter at random times to modify their unexecuted orders. Thus, other traders in the market can pick-off unexecuted dark orders in unfavorable positions, because these cognitive limits induce short periods of asymmetric information among different participants (i.e. there is *adverse selection*). This exposure to picking-off risk is applicable even in today's automated markets because there is no way to ensure that a limit order trader will be able to cancel or modify her order before the fastest marketable order arrives.

Traders have different private reasons to trade, which define their liquidity needs in terms of the order direction (i.e., to buy or to sell the asset) and how important it is to them to trade quickly. Traders also have different order sizes, and therefore different potential liquidity impacts on the market. Moreover, the model allows us to specify the *execution priority rules* in the dark pool; thus, we implement model setups with either a time execution priority or a size execution priority for dark orders.

We find that asset volatility is the main determinant of the submission decision regarding limit, market, and dark orders. This is because asset volatility affects traders' competition, adverse selection, and execution speeds in the lit and dark markets. Primarily, we find that the migration of the trading activity from the lit market to the dark pool has a humped shape as a function of the asset volatility, with the maximum value of dark trading activity occurring at an intermediate level of volatility. This is generated by the strategic behavior of different trader types when the asset volatility increases.

We show that speculators (i.e., traders *without* large private reasons for trading quickly) strategically prefer limit orders when the asset volatility is low. However, in the case of a medium level of asset volatility, speculators prefer to use the dark pool to execute picking-off strategies through dark orders, as they provide higher payoffs than limit orders. Moreover, speculators are less likely to profit from a similar picking-off strategy carried out through market orders in the lit market under a medium level of volatility. As the asset volatility is not high enough, it is unlikely that speculators will find limit orders in unfavorable positions in relation to the asset value, since they have less competitive prices than waiting dark orders. When asset volatility is high, we show that speculators change their order submission strategies and prefer to submit more market orders. In this case, speculators make higher profits by searching for unexecuted limit orders in the lit market that are highly exposed to picking-off risk when the asset changes against them.

We also observe that the strategic behavior of other trader types also depends on volatility. Specifically, liquidity traders (i.e., traders *with* large private reasons for trading as soon as possible) have preferences for market orders when the asset volatility is low. In the case of a moderate level of volatility, liquidity traders change their trading behavior and begin to prefer the dark pool. This is because the asset can often take values that make market orders less attractive than dark orders, since dark orders provide better execution prices than market orders. Moreover, in this scenario, there is an increase in the execution probability of dark orders, which induces liquidity traders to migrate more to the dark pool. This increase in the execution probability of dark orders is due to the competition amongst the same liquidity traders in the dark pool, and also due to the additional competition amongst speculators generated by their migration to the dark market under a moderate asset volatility (i.e., liquidity attracts liquidity). When the asset volatility is high, there are high chances of the asset value moving against the prices of both dark orders and market orders, which can be compensated by the better prices offered by limit orders. Consequently, liquidity traders prefer to submit more limit orders under a high-volatility scenario. Moreover, the use of limit orders is encouraged by the trading behavior of speculators in the lit market when volatility is high, which increases trader competition and thus the execution probability of limit orders.[7,8]

These changes in order preferences made by different trader types under changes in asset volatility are consistent with several studies of limit order markets, which show that there is no single type of agent who provides or takes liquidity (e.g., Goettler *et al*., 2009; Li *et al*., 2019; Bernales, 2019; and Ladley, 2020). In contrast to these studies, we show that changes in order preferences are not only observed in a pure limit order market, but also in a lit-dark market environment.[9]

Furthermore, we show that the migration of trades to the dark pool also has a humped shape, as a function of both the bid-ask spread in the lit market and liquidity costs in the whole system.[10] This result is due to two main effects: Firstly, as expected, we find that the bid-ask spread in the lit market and liquidity costs in the whole system are positively related to the asset volatility. In both markets, the picking-off losses are larger when the asset volatility increases, since traders cannot react quickly to large asset movements. Secondly, as explained previously, there is a non-linear (hump-shaped) relationship between the migration of trades to the dark pool and asset volatility. Therefore, the interaction of these two effects results in the migration of trades to the

[7] This is interesting because liquidity traders are paying a waiting cost and a picking-off cost with dark orders (limit orders) so as to obtain liquidity when the asset volatility is moderate (high), instead of paying the bid-ask spread with market orders.

[8] This result is consistent with Zhu (2014), who also finds hump-shaped dark pool participation as a function of the asset volatility, but importantly, in his model, the dark pool interacts with a lit market organized as a dealer market, as opposed to the limit order book we use in our model.

[9] We also show that this behavior is observed in other types of traders who fall somewhere between speculators and liquidity traders in terms of their private values.

[10] Thus, empirical studies that use linear regressions may fail to capture the relationship between dark market activity, and asset volatility and market liquidity.

dark pool also being hump-shaped as a function of liquidity costs. This result is consistent with Ray (2010), who empirically shows that there is a hump-shaped association between dark trading activity and the bid-ask spread. Most importantly, this result may explain the mixed findings in the empirical literature regarding the relationship between the level of trading activity in the dark pool and market liquidity (see, e.g., Ray, 2010; Degryse *et al*., 2015; Kwan *et al*., 2015; Buti *et al*., 2016; Foley and Putniņš, 2016; Gresse, 2017; Hatheway *et al*., 2017; Menkveld *et al*., 2017; and Farley *et al*., 2018).

The bid-ask spread of the lit market is larger when the dark pool is introduced than when there is only the lit market. This is because the migration of trading activity from the lit market to the dark pool increases market fragmentation and negatively affects the liquidity of the lit market. Previous theoretical studies describe different effects (negative or positive) of dark pool activity on the lit market liquidity, depending on whether the migration is due to specific types of traders assumed to submit limit or respectively market orders in the lit market (e.g., Buti *et al*., 2017; Beyona *et al*., 2017; and Brolley, 2019). In contrast, in our model, any trader type (i.e., speculators or liquidity traders) can choose to submit limit or market orders in the lit market, where traders' submission preferences endogenously depend on the asset volatility. Therefore, any type of migration of trading activity to the dark pool induces damage to the liquidity of the lit market, since order migrations always involve liquidity providers and liquidity takers.

Welfare is reduced in a lit-dark market environment compared to when there is a lit market alone, independent of the asset volatility. This result is consistent with previous studies (e.g., Pagano, 1989), in which market fragmentation diminishes welfare in the presence of market frictions. A higher fragmentation of trading activity between lit and dark venues increases the liquidity costs of the whole system, since traders' competition is reduced in each individual market, in relation to the case of a single trading venue where all trading activity is concentrated.

Most importantly, in terms of policy implications, we show that a size execution priority rule in the dark pool has several benefits. The size execution priority rule for dark orders improves liquidity and welfare, and reduces the picking-off risk, of the whole lit-dark system relative to the case of a time execution priority rule. This is due to, as expected, the reduction in the waiting time (i.e., liquidity costs) of large traders who submit orders to the dark pool, under a size execution relative to a time execution priority rule for dark orders.

Speculators are better off when a dark pool is introduced than when there is just a lit market, whereas liquidity traders are worse off. On the one hand, the dark pool offers speculators an additional venue in which to operate a picking-off strategy against other traders. On the other hand, the market fragmentation observed when there is a dark pool and a lit market increases the liquidity costs of the whole system, which reduces the payoffs of liquidity traders. In addition, we find that the payoffs of speculators go down when there a size execution priority rule in relation to a time execution priority rule for dark orders, since a size execution priority reduces speculators' chances of picking off orders from large traders.

Interestingly, we show that large traders are worse off when the dark pool is introduced, relative to the case where there is only a lit market, even if there is a size execution priority rule in the dark pool. This is due to the higher liquidity costs that fragmentation induces in the whole market, which strongly affects large traders who want to trade large orders as soon as possible. This result is striking, since it goes against the traditional view of the benefits of dark pools for large traders, whereby the dark market should reduce the price impact of large orders, as dark orders are anonymous and not displayed in the dark pool.[11]

[11] For instance, the SEC, in the Regulation of NMS Stock Alternative Trading System (Release No. 34-76474; File No. S7-23-15]), states, "*Dark pools originally were designed to offer certain market participants, particularly institutional investors, the ability <u>to minimize transaction costs when executing trades in large size</u> by completing their trades without prematurely revealing the full extent of their trading interest to the broader market*".

Finally, despite the fact that a dark pool reduces the price impact of large orders as they are undisplayed and large orders may have higher execution priorities, we show that large traders do not necessarily always prefer to send orders to the dark pool over the lit market. For instance, large traders prefer the lit market to the dark pool when the level of trading activity in the dark pool is low (i.e., when the asset volatility is low or high), as in this case the waiting costs would be larger in the dark pool.

It is important to note that our focus is on providing a relatively simple but realistic model, in order to understand the behaviour of the dark-lit trading activity under different execution priority rules in the dark pool, adverse selection, and traders' competition. We recognize that our model is far from perfect (as any theoretical model), thus the dark-lit trading activity may be affected by other elements outside the scope of our modeling setup such as behavioral issues and market regulations, amongst other factors. Therefore, the objective of our study is not to propose a better model for the dark-lit trading activity, but to provide some light to understand the impact of dark pools under changes in market conditions.

Our paper is structured as follows. We conduct a literature review in Section 2. We outline our model in Section 3. In Section 4, we describe the traders' behavior in different scenarios. In Section 5, we analyze traders' payoffs and market welfare. In Section 6, we show empirical implications of dark pools in terms of market liquidity. Finally, Section 7 concludes.

2 Literature Review

This study is part of a growing literature on the effect of dark pools on financial markets. Empirical studies have shown that the trading activity in the dark pool is negatively related to market liquidity (see, e.g., Weaver, 2014; Ready, 2014; Degryse *et al*., 2015; Kwan *et al*., 2015; and Hatheway *et al*., 2017). At the same time, other studies report a positive relationship between the trading activity in the dark pool and market liquidity (Brugler, 2015; Buti *et al*., 2016; and Gresse, 2017). In addition, there are studies showing a mixed relationship between dark trading activity and liquidity (see, e.g., Ray, 2010; Foley and Putniņš, 2016; Menkveld *et al*., 2017; and Farley *et al*., 2018). Our paper provides a potential explanation for these conflicting empirical results, as we present evidence that the trading activity in the dark pool is hump-shaped as a function of market liquidity. This is because the migration from the lit market to the dark pool has a non-linear relationship with the main variables that describe the market conditions.

Our paper is particularly related to the theoretical literature that examines the interaction between the trading activity in a dark pool and that in a lit market organized as a limit order book. However, these studies do not simultaneously consider: (i) different execution priority rules; (ii) adverse selection; and (iii) traders' competition in order submission for all types of orders and all types of traders. For example, Buti *et al*. (2017) develop a theoretical model of a limit order market with a dark pool, in which there is competition among traders in order submission, with small and large traders who can split their orders. They find that liquidity and welfare decrease when a dark pool is added to a limit order market. However, they do not consider either the impact of asset volatility (which may generate a picking-off risk) or the effect of the size execution priority rules in the dark market that is particularly relevant for large traders.

Beyona *et al*. (2017) and Brolley (2019) present models in a dark-lit environment under adverse selection. These models, however, do not consider different execution priority rules. Moreover, in both models, some trader types are constrained in terms of the types of orders they can submit, which affects the traders' competition in the system. For instance, in Beyona *et al*. (2017), liquidity traders can only submit market orders, while other traders cannot provide liquidity in the dark pool. Thus, the dark pool has an exogenous probability of execution, which is not obtained in equilibrium. In Brolley (2019), 'professional' liquidity providers can only submit limit orders in the lit market and in the dark pool, while other investors cannot provide liquidity in the dark pool. Thus, the liquidity costs of each market do not completely capture modifications in the level of competition in order submission in each market, due to a potential migration of trading activity (of different trader types) between the dark pool and the limit order market. One interesting feature of Brolley (2019) is that he presents a model with non-midpoint dark pool trading in which the dark pool offers an improvement upon the prevailing displayed quote by a percentage of the bid-ask spread. Thus, our study *complements* the current theoretical literature on the interaction between dark pools and limit order markets. We contribute to this literature by introducing a theoretical model simultaneously including different execution priority rules, adverse selection, and trader competition. Including these elements is important, since they interact with each other, and hence the effect of dark pools on market quality is not straightforward to determine.

Our paper is also related to theoretical studies in which a dark pool interacts with a lit market organized as a dealer market (see, e.g., Hendershott and Mendelson, 2000; Degryse *et al.*, 2009; Ye, 2012; Zhu, 2014; Iyer *et al.*, 2018; and Ye, 2016). From this literature, our paper is particularly close to, and complements, the findings of Zhu (2014). Similarly to our study, he finds a hump-shaped dark pool participation as a function of asset volatility, but in his model the reason for this hump-shaped relationship is different. In his model, there is a dark pool together with a dealer market, in which there is a single risk-neutral liquidity provider who sets competitive bid and ask prices in a two-period setup. In this framework, he shows that asset volatility increases the participation of liquidity traders in the dark pool up to an intermediate level of volatility $\sigma^\diamond$. When the asset volatility is higher than $\sigma^\diamond$, informed traders also begin to trade in the dark pool, which results in many liquidity traders with low delay costs migrating out of the dark pool, thus reducing the dark pool participation. In contrast, in our model, the dark pool interacts with a lit venue that is characterized as limit order market, where any trader can be a liquidity provider or a liquidity taker in the lit market (by submitting limit or market orders, respectively) or a dark trader in the dark pool (by submitting a dark order). Thus, potential liquidity providers in the lit market also make optimal order submissions and modifications in a multi-period trading game. Most importantly, and complementary to Zhu (2014), the hump-shaped dark pool participation as a function of asset volatility is obtained in our model, even if there are no explicit informed investors in the system. This is because such a hump-shaped relationship is generated through non-linear strategic behaviors by different trader types (including speculators and liquidity traders) in terms of their order submissions, depending on levels of picking-off risk and traders' competition in each venue, which is affected by the asset volatility.

Our paper is also related to theoretical studies on the dark-lit trading activity, in which lit markets are organized either as a sequential-double-auction market (see, e.g., Antill and Duffie, 2017), or as an *infinite* trading crowd of competitive and infinitesimal liquidity providers (see, e.g., Menkveld *et al*., 2017). In this context, our study is particularly connected to Menkveld *et al*. (2007), who analyze how immediacy requirements of traders affect the dark-lit trading activity. Menkveld *et al*. (2007) argue that if traders have reduced immediacy requirements, traders would prefer dark pools since dark orders have the lowest execution costs although also lowest immediacy. Conversely, if traders have large immediacy requirements they would prefer lit markets because lit orders have the highest execution costs but the highest immediacy. Our work complements Menkveld *et al*. (2007) given that we also include different immediacy requirements for the agents in our model (which are modeled by the use of three elements: the cost of delaying; traders' private values; and trader order sizes), but we aim to answer different questions since our modeling setup includes additional features. Thus, differently but complementary to Menkveld *et al*. (2007), we examine in our study the dark-lit trading activity under different execution priority rules in the dark pool, where there is adverse selection and a *finite* number of liquidity providers that face traders' competition, where dark trading activity has a non-linear relationship with asset volatility and liquidity.[12]

Our paper is connected to theoretical studies that analyze the effect of fragmentation on lit markets (i.e., without dark pools) which are partially or completely organized as limit order markets. For instance, Parlour and Seppi (2003) analyze competition between a pure limit order book and a specialist market, while Foucault and Menkveld (2008) present a model to examine a fee-based competition between limit order markets. However, differently to our model, Parlour and Seppi (2003) and Foucault and Menkveld (2008) do not consider adverse selection in order submission when they analyze fragmentation. Baldauf and Mollner (2018) show that fragmentation intensifies adverse selection, despite the fact that exchange competition induces a downward pressure on trading fees. Nevertheless, Baldauf and Mollner's (2018) model imposes restrictions on who can submit limit orders, which affects traders' competition in liquidity provision. This is different to our study, where any trader type can submit limit and market orders (and even dark orders), thus any trader type can migrate between the lit market and the dark pool.

Our paper is also associated with empirical studies in which fragmentation is analyzed, but without considering markets organized as dark pools. These studies similarly report mixed findings on the effect of fragmentation on market quality. There are empirical studies that show that fragmentation increases liquidity (see, e.g., Branch and Freed, 1977; Hamilton, 1979; Battalio, 1997; Fink *et al*., 2006; Foucault and Menkveld, 2008; O'Hara and Ye, 2011; Menkveld, 2013; and He *et al*., 2015), whilst there are other empirical studies that report reductions in liquidity due to fragmentation (see, e.g., Bessembinder and Kaufman, 1997; Arnold *et al*., 1999; Amihud *et al*., 2003; Hendershott and Jones, 2005; Bennett and Wei, 2006; and Nielsson, 2009). In addition, there are

[12] In addition, our paper is associated with the literature on 'workups' trading mechanisms in Treasury markets, in which buyers and sellers successively increase, or 'work up,' the quantities of an asset that are exchanged at a fixed price (see, e.g., Duffie and Zhu, 2017; and Back and Barton, 2019).

empirical studies that show a mixed relationship between fragmentation and liquidity (see, e.g., Boneva *et al*., 2016; and Haslag and Ringgenberg, 2017).

Finally, our paper is also related to studies that examine different execution priority rules. For instance, Cohen *et al*. (1985), Angel and Weaver (1998), Panchapagesan (1998), and Cordella and Foucault (1999) show that a price/time priority rule induces price competition, which generates smaller spreads. However, to the best of our knowledge, there are no studies that explore different priority rules in dark pools.

3 The Model

3.1 Trading environment

We consider an economy containing an asset that can be traded in either a limit order market (also referred to as a lit market, henceforth, LM) or a dark pool (henceforth, DP).[13] The economy reflects a dynamic trading game in continuous time, with several traders who asynchronously arrive and compete in order submissions in each market. The arrival of traders follows a Poisson process with intensity λ. The fundamental value of the asset, v_t, is stochastic, with changes in value that are also described by a Poisson process, with intensity λ_v. The asset value can go up or down by one tick with equal probability, when a value change is observed in the Poisson process.

3.2 The limit order market (LM)

Traders can submit limit and market orders (in the LM), and dark orders (in the DP). In the case of the LM, limit orders wait to be executed in a limit order book. The limit order book associated with the LM, L_t, is characterized by a discrete set of prices $\{p^i\}_{i=-N}^{N}$, where $p^i < p^{i+1}$ and N is a finite number. As in real markets, $d = p^i - p^{i-1}$ is the tick size. In L_t, there is a potential

[13] All variables used in our study are described in Table A1 of Appendix A.

queue of shares from unexecuted limit orders, l_t^i, at time t for each price p^i. A positive number for l_t^i denotes buy limit orders, while a negative number denotes sell limit orders. The best bid price and the best ask price are then given by B_t= $\max\{p^i |\ l_t^i > 0\}$ and A_t= $\min\{p^i |\ l_t^i < 0\}$, respectively.[14]

3.3 The dark pool (DP)

In the case of the DP, traders can submit dark orders that are executed at the midquote of the lit market's best bid and ask prices, $M_t = 0.5(A_t + B_t)$. In the DP, there is a single queue of shares from unexecuted dark orders, k_t, at time t, for the single price M_t.[15] A positive value of k_t denotes buy dark orders, and a negative value of k_t indicates sell dark orders, which have not yet been executed. Thus, for instance, if a new trader at time t submits a sell dark order for one share and k_t is positive (i.e., there are unexecuted buy dark orders that are waiting in the DP), the new sell dark order is immediately executed at price M_t. Conversely, if k_t is negative, the new sell dark order for one share will enter the queue of unexecuted sell dark orders of the DP. If $k_t = 0$, the dark queue is empty and any order arriving will enter the queue. Moreover, as a robustness check, we also use the volume-weighted average price (VWAP) as the execution price in the dark pool in unreported results. The findings observed under the VWAP in the dark pool are qualitatively similar to the results reported here.

3.4 Opacity and execution priorities

Traders can observe the complete limit order book, L_t, and its changes. However, the DP is opaque in the sense that traders cannot observe the queue of shares from unexecuted dark orders, k_t. In the LM, limit orders are executed according to a price-time execution priority. This means

[14] In the market, $B_t = -\infty$ or $A_t = +\infty$ represents the case where the limit order book either has no unexecuted buy limit orders or no unexecuted sell limit orders, respectively.

[15] In the case that the limit order book is empty on either the buy side or the sell side, then $M_t = 0.5(A_{t-t^A} + B_{t-t^B})$, where $t - t^A$ and $t - t^B$ are the last times at which the buy and sell sides of the book respectively were not empty.

that buy limit orders at higher prices, and sell limit orders at lower prices, have priority for their execution, while unexecuted limit orders at the same price have a higher execution priority if they were submitted earlier. In contrast, in the DP, the waiting unexecuted dark orders are executed through either a size-time priority (which is referred to simply as 'size priority'), or a pure-time priority (which is simply termed 'time priority'). Under the size priority rule, dark orders of a larger size (i.e., dark orders that involve more shares) have higher execution priority, and between any two dark orders of the same size, the one submitted earlier has higher execution priority. Under the time priority rule, dark orders submitted earlier always have higher priority, regardless of their size.

3.5 Order modifications, picking-off risk and waiting costs

As in real markets, traders have cognitive limits, in the sense that they cannot observe the current asset value, and cannot continuously monitor market conditions. Thus, the asset value, v_t, is observed by all traders with a time lag Δ_t. In addition, since traders cannot continuously monitor market conditions, traders can only re-enter at random times to modify their unexecuted orders. Thus, the re-entry process, which traders follow to perform potential modifications of their unexecuted orders (unexecuted limit orders and unexecuted dark orders), is described by a Poisson process with intensity λ_r.

Due to cognitive limits, traders are exposed to picking-off risk regarding their unexecuted orders that are waiting in either the LM (i.e., unexecuted limit orders) or the DP (i.e., unexecuted dark orders). For instance, suppose trader i submits a sell limit order for one share to the LM at a new, more aggressive price A° (i.e., A° is the new ask price). Suppose that, immediately after the limit order submission from trader i, the fundamental asset value v° goes up to a level above A° (i.e., $A^\circ < v^\circ$). Then, the sell limit order is now in an incorrect position in relation to the asset value, as trader i is trying to sell the asset at a price lower than v°. However, given that trader i has cognitive limits, she does not instantaneously realize that her limit order is at an unfavorable price. Thus, a second trader j can submit a buy market order, with the objective of picking off the sell limit order from trader i. As a result, trader i has a loss equal to $(A^{\circ} - v^{\circ})$, and trader j receives a profit equal to $(v^{\circ} - A^{\circ})$, from the transaction.

In the case that a trader re-enters to modify an unexecuted order, potential order modifications imply benefits and drawbacks. In term of benefits, modifications of unexecuted orders provide traders with the opportunity of submitting at a better price, if they are aware of changes in market conditions. For instance, modifications reduce the picking-off risk, given that traders can modify their unexecuted mispriced orders (of course, only if the trader detects in time that the asset value has changed against their orders). Conversely, modifications to unexecuted orders have the drawback that traders may lose their priority in the queue for execution, which should increase their execution waiting time. This entails a cost for traders. Thus, to consider such waiting costs, traders' profits are subject to a discount rate ρ from the time of the trader's first entry to the point of order execution. This discount rate is not the money time-value but represents the costs of monitoring unexecuted orders.

3.6 Trader types

Traders are risk-neutral, but heterogeneous in relation to two features: the traders' exogenous reasons for trading, and their order size requirements. In relation to the heterogeneity in the traders' exogenous reasons for trading, each trader arrives with a private value, α, which will be obtained after they have traded each asset share. The private value α is known by each trader and is constant over time. The private value reflects their intrinsic reasons for trading (exogenous to the potential benefits of the transaction), such as liquidity needs, hedging requirements, and/or tax benefits. The private value of each trader is drawn from a discrete and finite vector $\{\alpha_1, \alpha_2, ..., \alpha_g\}$ with a probability distribution F_{α}. The private value of a trader influences both the direction and aggressiveness of her order submission. If α is positive (negative), a trader is willing to submit a buy (sell) order to obtain her private value after the transaction. Meanwhile, the absolute level of the private value (i.e., $|\alpha|$) also affects how quickly a trader wants to trade, and hence the aggressiveness of her order submission. If $|\alpha|$ is very large, a trader will want to trade the asset without delay, with the objective of reducing a potentially large waiting cost due to the discount rate ρ, which would be applied to her large private value.

In terms of the traders' order size requirements, when a trader with private value α arrives in the economy, there is a probability γ_α that she will be a small trader, and a probability $1 - \gamma_\alpha$ that she will be a large trader. A small trader submits an order for one share, while a large trader submits a bulk order for $Q > 1$ shares. Thus, large traders have larger orders (which are also referred to as 'large-in-scale' orders). Upon arrival, each small (large) trader stays in the economy until her share (shares) is (are) traded; after that, she leaves the economy permanently.

It is important to notice that, if a large trader submits an order for Q shares, this order is not necessarily executed as one whole transaction. For instance, a large trader with a sell limit order for Q shares at the ask price could see their order partially executed if a small trader were to submit a buy market order for one share. After that transaction, the sell limit order of the large trader would have $Q - 1$ shares available to trade, and the trader would stay in the market until she had traded the remaining shares in her order.

3.7 The trader's Bellman equation

A trader with given characteristics (i.e., her private value and whether she is a small or a large trader) makes a trading decision after observing market conditions, where the trader's characteristics and the market conditions are described by a set of states $s \in \{1,2,\dots,S\}$. Each state at time t is defined by (i) the agent's private value; (ii) whether the trader is small or large in terms of her order size requirements; (iii) the contemporaneous limit order book, L_t, in the LM (which also includes the price in the DP, since $M_t = 0.5(A_t + B_t)$); (iv) if the trader has shares that have not been traded from a previously submitted order (in the LM or the DP), the status of her current order (i.e., in which market the unexecuted order is waiting, whether it is a buy or sell order, the submission price, and the order priority of each remaining share in the order book in the case of unexecuted limit orders). The asset value observed by the trader, $v_{t-\Delta\mathrm{t}}$, is not an element that characterizes a given state because, similarly to Goettler *et al*. (2009), we center the limit order book of the LM on the asset value observed by traders (i.e., $p^0 = v_{t-\Delta\mathrm{t}}$ in L_t). Thus, the prices of all orders (including dark orders, as the transaction price in the DP is the midquote of the LM) are relative prices, with respect to the trader's beliefs about the asset value. Hence, the information about $v_{t-\Delta\mathrm{t}}$ is implicit in the information about the limit order book. Most importantly, by using relative prices, we greatly decrease the dimensionality of the state space, because (as explained above) traders think in relative values rather than in terms of absolute price levels.

Suppose that the optimal decision of a trader in state s is $a \in \mathcal{A}(s)$, where $\mathcal{A}(s)$ is the set of feasible trading decisions that the trader can take in state s. When a trader arrives at the market for the first time, that decision includes (i) the venue preferred for an order submission (LM or DP); (ii) the direction of the order (buy order or sell order); (iii) the type of order if the LM is selected (market order or limit order); and (iv) the submission price if the trader submits a limit order. In addition, when traders re-enter to potentially modify unexecuted orders, they then have to decide (i) to modify or keep unchanged the unexecuted order; (ii) if they decide to modify the unexecuted order, the venue in which to submit the new order; (iii) the direction of the new order (new buy order or new sell order); (iv) in the case of a new submission in the LM, the type of the new order (market order or limit order); and (v) the submission price in the case of a new limit order.

To simplify the notation, let the trader's time of arrival be zero. Additionally, let $\eta(h_q|a, s)$ be the probability density for share q of the order being traded at time h_q, where q is a share indicator, with $q = \{1,2, \dots, Q^*\}$ and Q^* is the number of remaining shares of the order to be traded, with $Q^* \leq Q$. The probability $\eta(\cdot)$ takes into account potential future states and optimal potential decisions performed by competing traders until the execution of the order. Thus, the expected payoff of decision a in state s, if the shares of the trader are traded before her re-entry at time h_r, is given by

$$\pi(h_r, a\ s) = \sum_{q=1}^{Q^*} \int_0^{h_r} \int_{-\infty}^{\infty} e^{-\rho h_q} \left[\left(\alpha + v_{h_q} - p_q \right) x \right] \gamma \left(v_{h_q} | h_q \right) \eta\left(h_q \middle| a, s\right) dv_{h_q} dh_q \tag{1}$$

where p_q is the order's price that is a component of the trading decision (which is equal to the midquote for dark orders), x is also an element of the trading decision and describes the order direction ($x = 1$ if the order is a buy order, and $x = -1$ if it is a sell order). In addition, $\gamma \left(v_{h_q} | h_q \right)$ is the density function of the asset value at time of execution $h_q \in [0, h_r]$ of share q of the trader's order.[16]

In equation (1), $\left(\alpha + v_{h_q} - p \right) x$ is the instantaneous payoff obtained when share q is traded.[17] Thus, using equation (1), we can construct the Bellman equation of the optimization problem faced by each trader:

$$V(s) = \max_{a \in \mathcal{A}(s)} \int_0^{\infty} \left[\pi(h_r, a\ s) + e^{-\rho_d h_r} \int_{s_{h_r} \in \mathcal{S}} V\left(s_{h_r}\right) \psi\left(s_{h_r} \middle| a, s, h_r\right) ds_{h_r} \right] dR(h_r). \tag{2}$$

Here, $\psi\left(s_{h_r} \middle| a, s, h_r\right)$ reflects the probability that state s_{h_r} is observed at h_r, where $\mathcal{S}$ is the set of potential states. Furthermore, $R(h_r)$ is the cumulative probability distribution of the time at which the trader re-enters the economy.

[16] It is important to note that the value of the submission price, p, may change in the case of the DP, between the submission time and time h_q, when share q of the order is traded, which is taken into account in our model. This is because, in the DP, p is the midquote of the LM's best bid and ask prices, $p = 0.5(A_t + B_t)$; thus, the values of A_t and B_t may change between the time of order submission and the time of the transaction involving share q of the order.

[17] The asset value used is the value at the transaction time, despite the fact that the trader observes the asset value with a lag (i.e., we use v_{h_q} rather than $v_{h_q - \Delta t}$). This is because the trader realizes what the *real* asset value in the transaction is after a lag Δt, and thus also realizes what the real payoff obtained from the transaction is, which allows the trader to understand the real impact of her trading decision.

3.8 Model equilibrium and solution

In equilibrium, traders behave optimally by taking decisions that maximize their expected discounted utility in each state of the economy (as in equation (2)). Thus, trading decisions are state-dependent. In addition, an optimal decision in a state s is a consequence of previous optimal decisions taken in previous states. Thus, the process of optimal decisions is Markovian. Moreover, the trading game reflected in the model is a Bayesian game, because traders observe their own private values and their own order size requirements, which are unknown to the rest of the traders participating in the economy. Therefore, we obtain a Markov-perfect Bayesian equilibrium (see, e.g., Maskin and Tirole, 2001).

As in Doraszelski and Pakes (2007), we consider an equilibrium that is stationary and symmetric, where optimal decisions are time-independent. Thus, two identical traders who observe the same state, one trader observing it in the present and the other observing it in the future, take the same decision. Furthermore, as mentioned before, the set $\mathcal{A}(s)$ of feasible trading alternatives is discrete and finite, and each state s is defined by variables that are discrete. Hence, the decision set is finite and the state space is countable; thus, there is a Markov-perfect Bayesian equilibrium (see Rieder, 1979).

In order to obtain the Markov-perfect Bayesian equilibrium, we use the Pakes and McGuire (2001) algorithm due to the extremely large state space. The intuition behind this algorithm is that traders learn to take optimal decisions under different market conditions by repeatedly playing the trading game. First, initial beliefs are set regarding the expected payoff of each trading decision. Traders then play the game and take their optimal decision (the one with the highest expected payoff) given the state observed. After each transaction, traders update their beliefs about the expected payoffs of their trading decision based on the observed realized payoffs that result from their actions.

The equilibrium is achieved as soon as there is no longer any learning occurring. This means that the equilibrium is reached when (i) the same type of trader takes the same optimal decision a^* in state s^* in the present or in the future and (ii) the expected payoff of optimal decision a^* in state s^* for the same type of trader does not change over time. We use the same procedure to determine whether the equilibrium has been reached as Goettler *et al.* (2009). Appendix D describes the implementation details of the Pakes and McGuire (2001) algorithm. We fix the beliefs of traders when the model equilibrium is obtained, and then simulate 100 million further events. All results presented in the following sections regarding the complete model are computed using these last 100 million events.

3.9 The model parameterization

The majority of the parameters used in this paper are the same as those of Goettler *et al*. (2009), who present a similar model for a single limit order book. Goettler *et al*. (2009) report evidence that such parameters are consistent with the behavior of the real market, and they are also in line with previous empirical findings (e.g., Bandi *et al*., 2006; Hansen and Lunde, 2006; Hollifield *et al*., 2006; and Aït-Sahalia *et al*., 2011).

One might still believe, however, that our findings were driven by the selected parameters, which is an important concern for theoretical studies. To deal with this concern, we perform robustness checks across parameter combinations (presented below). Additional results from the model using alternative sets of parameters are in line with the findings presented here. Moreover, we show in Section 4.1 that the intuitions behind the behavior of traders are independent of any parameter setup. Thus, we find that any potential new parameter setups only intensify (or weaken) the main behaviors of traders.

Furthermore, similar parameters have been used in other related theoretical studies, with dynamic models for a single limit order book (e.g., Chiarella and Ladley, 2016; and Bernales, 2019) and dynamic models with two limit order books (e.g., Bernales *et al.*, 2020). In particular, Bernales *et al.* (2020) analyze the suitability of the use of Goettler *et al.* (2009)'s parameters by computing the trader arrival rate, the fundamental value volatility, and the distribution of private values using message-level market data from 2015 for stocks on the London Stock Exchange. They show that these estimates are not substantively different from the model parameter choices in Goettler *et al.* (2009). This is because the parameters in Goettler *et al.* (2009) are presented in relative terms (e.g., the parameters concerning time periods are *relative* to the investors' arrival rate, while the parameters related to asset values are relative to the tick size of the market). Thus, even if financial markets have evolved over time, the relative relationships between trader arrival rates, fundamental value volatility, and the distribution of private values are stable.

Consequently, and following Goettler *et al.* (2009), the average time period between two consecutive events of the same type in our study (e.g., the average time period between two consecutive changes in the asset value) is defined relative to the average time periods observed for other types of events (e.g., the arrival of two consecutive traders). Thus, the Poisson process that describes the arrival of traders is parameterized such that new traders arrive, on average, every one unit of time (i.e., $\lambda = 1.0$), whilst each trader re-enters on average every four units of time (i.e., $\lambda_r = 0.25$).[18] Traders observe the asset value with a lag of 16 units of time (i.e., $\Delta = 16$). We use different values of the parameter that determines changes in the asset value in order to evaluate the impact of the asset volatility. Hence, the asset value changes from every 16 units of time to every single unit of time (i.e., λ_v takes values from 0.0625 to 1.0).

Similarly to in Goettler *et al.* (2009), profits and trading costs are measure in ticks. The value of one tick, d, is equal to one (i.e., $d = 1$). Private values of traders, α, are drawn from the discrete vector $\{-8, -4, 0, 4, 8\}$, with these values also measured in ticks. Like in Goettler *et al.* (2009), the probability that a traded share involves a trader with private value -8 or 8 is 15% in

[18] Thus, one unit of time can be one minute (which was appropriate 20 years ago in the real financial markets) or one microsecond (which is the speed of trading activity observed now).

each case. The probability that a traded share involves a trader with private value -4 or 4 is 20% in each case, and the probability that a traded share involves a trader with a private value of 0 is 30%.

We assume an asymmetric probability of being a large or a small trader across different private values, because large traders represent investors with large exogenous reasons to trade who want to execute quickly without a large price impact (which is the case for traders with high *absolute* private values). Thus, we assume that a third of the traded shares from traders with private values of -4 and 4 are submitted by large traders. We also assume that all traded shares from traders with private values of -8 and 8 are submitted by large traders, and that all traded shares from traders with a private value of 0 are submitted by small traders. We assume that large traders submit orders for $Q = 3$ shares.[19] Finally, in terms of the remaining model parameters, we assume that the number of discrete prices available on each side of the limit order book, N, is equal to 31, while the discount rate from delaying a trade, ρ, is equal to 0.05 per unit of time.

In unreported results, as mentioned above, we re-run the model with additional parameter setups as a robustness check. Specifically, we multiply the following parameters by 0.8 and 1.2 to generate new parameter combinations: the traders' arrival rate, λ; the traders' re-entry rate, λ_r; the time lag with which traders observe the asset value, Δ; and the delaying discount rate, ρ. We also re-run the model with a different distribution of agent types. Firstly, we assume a homogeneous probability of being a large trader across agents with different private values. Thus, we assume that 50% of the traded shares from all trader types in terms of their private values (i.e., all traders with private values −8, −4, 0, 4, and 8) are submitted by large traders. Secondly, we change the distribution of agents' private values. Thus, we assume that we only have traders with private values {-8, 0, 8}. In this scenario, 25% of the traded shares are from traders with a private value of -8, 25% from traders with a private value of 8, and 50% from traders with a private value of 0. Here we assume that half of the traded shares from traders with private values of -8 and 8 are submitted

[19] We could include additional shares in the orders of large traders. However, we assume $Q = 3$ to make the model computationally tractable.

by large traders, and all traded shares from traders with a private value of 0 are submitted by small traders. The findings observed under all of these additional parameter setups are qualitatively similar to the results reported here.

4 Traders' Behavior

The introduction of a dark pool (DP) offers additional trading alternatives to agents who traditionally trade in a lit market (LM) organized as a limit order book. Thus, traders can trade not only through limit and market orders in the lit venue, but also through undisplayed dark orders in the dark market. The question that we want to answer in this section is the following: How do traders trade optimally when they can submit orders to either the LM or the DP? The answer depends on the advantages and disadvantages of the trading alternatives under different market conditions. Thus, as a first step, we provide two simple examples to demonstrate the intuitions behind the trading decisions of agents, when the DP is running in parallel to the LM.

4.1 Two simple examples

The purpose of this section is to provide two simple examples, to help explain intuitively how agents make trading decisions when they have access to the LM and the DP. In the first example, suppose a trader (who is referred to as a speculator, or trader S), wants to buy one share of an asset. The speculator has a positive, but very low, private value of immediately trading the asset, α^S. In the second example, there is a different type of trader (called a liquidity trader, or trader L), who also wants to buy one share of the same asset. The liquidity trader has a positive private value of trading the asset without delay, α^L, which is larger than α^S (i.e., $0 \leq \alpha^S < \alpha^L$).

The only difference between the two examples is the trader type (i.e., there is either a speculator or a liquidity trader). In both examples, the asset value is stochastic and can be traded in either the LM or the DP. Suppose, also in both examples, that the asset value is v when each trader decides where to buy the asset share (i.e., in the LM or in the DP). The volatility of the asset can take three values: low (σ^*), medium (σ^{**}), or high (σ^{***}).

In both examples, the trader can buy the asset share by using either a buy market order (henceforth, MO) or a limit order (henceforth, LO) in the LM, or through a dark order (henceforth, DO) in the DP.[20] Each trader optimally chooses one of the three trading alternatives for buying the asset, depending on the market conditions.

Firstly, each trader has the option to submit a market order that is executed immediately at price A. The expected payoff of the market order will be $(v + \alpha^S - A)$ for the speculator and $(v + \alpha^L - A)$ for the liquidity trader.[21]

Secondly, each trader has the option to submit a buy limit order at price B (with $B < A$), which will wait in the limit order book until executed. In this case, the expected payoff is $e^{-\rho h_{LO}}(v_{LO} + \alpha^S - B)$ for the speculator and $e^{-\rho h_{LO}}(v_{LO} + \alpha^L - B)$ for the liquidity trader. Here, ρ is the discount rate that reflects the waiting cost, h_{LO} is the expected execution time of the buy limit order at price B, and v_{LO} is the expected value of the asset at time h_{LO}, where $B < v_{LO}$. In this example, for illustrative purposes, we assume that these values are fixed and the expected value of the asset at the time of execution of the limit order is independent of the current value. We will relax these assumptions below.

Thirdly, each trader has the option to submit a buy dark order at price M, where M is the midquote of the LM (i.e., $M = 0.5(A + B)$). There is a probability ϕ (with $0 \leq \phi \leq 1$) that the dark order is not immediately traded because undisplayed orders waiting to be executed in the DP may also be buy dark orders. Conversely, with probability $(1 - \phi)$, the dark order is traded immediately, since there is a chance that there are non-displayed sell dark orders waiting to be executed in the DP. Therefore, in the case of dark orders, the expected payoff for the speculator is

[20] The situation in which both traders want to sell one share of the asset is analogous.

[21] In these two examples, each agent can also be a large trader rather than a small trader, because the intuitions provided for the behaviors of traders do not change regardless of whether we consider small or large traders.

$(1-\phi)[v+\alpha^S-M]+\phi e^{-\rho\cdot h_{DO}}[v_{DO}+\alpha^S-M]$, while that for the liquidity provider is $(1-\phi)[v+\alpha^L-M]+\phi e^{-\rho\cdot h_{DO}}[v_{DO}+\alpha^L-M]$. Here, h_{DO} is the expected execution time of the buy dark order in the case where that order is not executed immediately, and v_{Do} is the expected value of the asset at time h_{DO}.

Figure 1 shows the expected payoffs of the traders as functions of the asset value, v, at the time of the order submission decision for the different order types. Panel A (Panel B) reflects the expected payoff of the speculator (liquidity trader). The range of potential values of the asset, which can be observed under different asset volatility levels, is also depicted in Figure 1. Suppose that the asset value was v_0 the instant before the traders' decision. The asset value, however, is stochastic and so the asset value v can change from v_0 in this instant. The potential range of values of v depends on the level of the asset volatility (i.e., σ^*, σ^{**}, or σ^{***}).

We can observe in Figure 1 that, for both trader types, the slope of the payoff line for a market order is one, while the slope of the payoff line for a dark order is $1-\phi$. The payoff line for a limit order is horizontal, since it does not depend on v.

[Insert Figure 1 here]

Figure 1 shows that the optimal decision depends on two main elements: (i) the private value of each trader (i.e., α^S and α^L, for speculators and liquidity traders, respectively) and (ii) the value v that traders observe when they make their trading decisions. The speculator (liquidity trader), who has a private value of α^S (α^L), strictly prefers the buy limit order when $v<v_{\alpha^S}^{LoDo}$ ($v<v_{\alpha^L}^{LoDo}$). The speculator (liquidity trader) strictly prefers the buy dark order when $v_{\alpha^S}^{LoDo}<v<v_{\alpha^S}^{DoMo}$ ($v_{\alpha^L}^{LoDo}<v<v_{\alpha^L}^{DoMo}$), while the speculator (liquidity trader) strictly prefers buy market orders when $v_{\alpha^S}^{DoMo}<v$ ($v_{\alpha^L}^{DoMo}<v$). In addition, the speculator (liquidity trader) is indifferent between the limit order and the dark order when $v=v_{\alpha^S}^{LoDo}$ ($v=v_{\alpha^L}^{LoDo}$), and the speculator

(liquidity trader) is indifferent between the dark order and the market order when $v = v_{\alpha^S}^{DoMo}$ ($v = v_{\alpha^L}^{DoMo}$).

It is interesting to see in Figure 1 that the chances of observing a given asset value, and therefore the chances of the various order decisions being made by each trader, depend on the asset volatility. In the scenario where the asset volatility is low, in Panel A, the speculator more often prefers a buy limit order (they behave as liquidity providers) as this means she receives the 'immediacy' cost paid by future market orders. Dark and market orders are not attractive for speculators since they provide lower profits. This is because dark and market orders have worse prices and speculators do not have large exogenous reasons (i.e., a large private value) for trading immediately. Conversely, Panel B shows that, when the asset volatility is low, the liquidity trader more often prefers buy market orders (they behave as liquidity takers) due to her private reasons for trading as soon as possible. The liquidity trader does not prefer limit or dark orders since they have associated waiting costs.

In the scenario with a medium level of asset volatility, depicted in Figure 1, both the speculator and the liquidity trader begin to prefer buy dark orders more often. In this scenario, the speculator starts to prefer dark orders regularly when v reaches values above $v_{\alpha^S}^{LoDo}$, due to the increase in the asset volatility (see Panel A). The speculator starts to use dark orders, because she can make a profit by picking off a potential sell dark order waiting in the DP if v is larger than $v_{\alpha^S}^{LoDo}$. However, there is a probability ϕ that the dark orders waiting in the DP are buy dark orders. Thus, with probability ϕ, the speculator may not be able to carry out her picking-off strategy, and will wait in the execution queue of the DP. In this scenario, with a medium asset volatility, the speculator cannot often perform a similar picking-off strategy through market orders, since the asset volatility is not high enough to place sell limit orders in the wrong position in relation to the asset value (i.e., it is rare to observe v larger than $v_{\alpha^S}^{DoMo}$).

Turning to the liquidity trader, as shown in Panel B of Figure 1, in the scenario with a medium level of asset volatility, she also starts to select buy dark orders regularly. This is because the asset value can decrease and take values below $v_{\alpha^L}^{DoMo}$, making her expected payoff from the market order lower than that from the dark order. This is because the large price A that would be paid for the market order (to buy the asset immediately) is not attractive enough to the liquidity trader when the asset value decreases below $v_{\alpha^L}^{DoMo}$. However, given that the buy dark order has a better price than the buy market order (i.e., M is lower than A), the dark order becomes more valuable than the market order if the dark order is executed immediately (which happens with probability $1-\phi$). There is, though, a chance that the dark order is not executed (with probability ϕ), but this is compensated for by the better price conditions of the dark order. Furthermore, in this scenario, the liquidity trader still does not prefer the buy limit order since the asset volatility is not high enough for very low levels of v, which would generate lower expected payoffs from both the dark and market orders than from the limit order, to be observed.

In the third scenario, with a high level of asset volatility, the speculator prefers market orders more often. This is because the speculator can obtain large payoffs by searching for unexecuted limit orders, which can be picked off when the asset value is larger than $v_{\alpha^S}^{DoMo}$ due to the high asset volatility (see Figure 1 Panel A). A picking-off strategy is executed with surety with a market order, which is different to the potential picking-off strategy in the DP, as explained above.

Liquidity traders, when the asset volatility is high, begin to employ more buy limit orders. This is because there is a chance that the asset value will decrease below $v_{\alpha^L}^{LoDo}$, which makes the expected payoff to the liquidity trader lower from both the buy dark order and the buy market order than from the buy limit order. Moreover, the use of the limit order is even more attractive due to the high likelihood of there being several speculators waiting in the LM to submit market orders, who will pick off limit orders in unfavorable positions (as previously described). Thus, any limit orders from liquidity traders should be traded relatively quickly because of their high exposure to the picking-off risk when the asset volatility is high.

Figure 1, therefore, shows that the trading activity in the DP should be higher when the asset volatility has an intermediate level. This is because, in this scenario, both speculators and liquidity traders will more often select dark orders. Of course, the two simple examples described in this section are special cases, with the purpose of explaining the intuitions behind the complete model described in Section 3. Thus, these two examples do not consider several endogenous interactions between the state variables, which will be affected by the level of the asset volatility. For instance, in this example, bid and ask prices are exogenous, expected execution times of limit and dark orders are constant, and the probability of the immediate execution of dark orders is static. However, the traders' behavior in the complete model (presented in Section 3) does consider such endogenous interactions.

4.2 Preferences of traders for limit, market, and dark orders

In this section, we present the traders' behavior under the full model, described in Section 3, for three market setups. Firstly, we report the results for the lit market (LM) without the dark pool (DP). This first market setup is called 'only LM' and represents the main benchmark model in our study. Secondly, we report the results for a model setup that considers both the LM and the DP, with a time execution priority for dark orders, which is called 'LM + DP(time priority)'. Thirdly, we show the results for a model setup in which there is again both the LM and the DP, but in this case dark orders follow a size execution priority. This third model setup is called 'LM + DP(size priority)'.[22]

[22] Table B1 in Appendix B presents a summary of the main results of our study, in which we show the impacts on trading activity, liquidity, and welfare, of changes in the execution priority rules in the DP, adverse selection, and traders' competition.

Figure 2 shows the percentage of executions of each order type under the three model setups described above, as a function of the asset volatility. We proxy the asset volatility by the rate of change of the asset value, λ_v, given that the asset value follows a Poisson process.[23] As a first step in this section, we present the order execution preferences for two types of traders, shown in Figure 2: speculators who are traders with zero private value from trading immediately (see Panels A-C); and liquidity traders with absolute private value equal to 8, who want to trade as soon as possible (see Panels D-F).

[Insert Figure 2 here]

Observation 1. (i) *Speculators and liquidity traders change their order preferences with the asset volatility under the three model setups.*

(ii) *Both speculators and liquidity traders use dark orders more often when there is a medium level of asset volatility.*

Figure 2 shows that the asset volatility affects the order preferences of traders, and thus their decisions regarding whether to use the LM or the DP, which is consistent with the intuitions about the traders' behavior that were explained in Section 4.1. Let us first analyze the order preferences of traders in a scenario where the asset volatility is low (i.e., when λ_v is small). Panels A-C show that speculators more often prefer to compete in the market by trading through limit orders (they behave as liquidity providers), whereas Panels D-F show that liquidity traders more often prefer market orders (they behave as liquidity takers) when the asset volatility is low. Liquidity traders want to trade quickly, and hence are willing to pay the immediacy cost through market orders to obtain their private value as soon as possible. Thus, when the asset volatility is low, speculators receive through limit orders the immediacy costs paid on the market orders submitted by liquidity traders.

[23] The variance of a Poisson distribution with parameter λ is equal to λ.

When the asset volatility increases, traders change their order preferences. In the case of the model setups involving both the lit and dark markets, we observe that speculators and liquidity traders begin to prefer dark orders when the asset volatility increases to a medium level (see Panels B-C and E-F in Figure 2). Consistent with our arguments presented in Section 4.1, speculators begin to make profits through a picking-off strategy in the DP when the asset volatility is at the medium level, as it increases the chances of the asset value moving against waiting dark orders submitted previously. A picking-off strategy is possible since traders have cognitive limits; hence, traders cannot modify unexecuted dark orders instantaneously when there are changes in the asset value. In fact, we show in Appendix C (see Figure C1) that speculators more often prefer to use the DP to execute their picking-off strategies through dark orders when there is a medium level of asset volatility.

Liquidity traders also prefer dark orders under a medium level of volatility because the asset can often take values that make market orders less attractive than dark orders, since the latter provide better conditions in terms of the execution price. Therefore, if the asset value falls (rises) far enough below (above) the current ask (bid) price of market orders, the liquidity trader might be willing to select the better price offered in the dark market (even if dark orders imply potential waiting costs if not immediately executed).

In the scenario with high asset volatility and both the LM and the DP, we observe that speculators begin increasingly to prefer market orders (see Panels B and C).[24] This is because speculators make higher profits by searching for unexecuted limit orders in the LM that are highly exposed to picking-off risk, when the asset changes against them due to the high levels of asset volatility. This is consistent with Figure C1 Panel C in Appendix C, which shows that speculators begin to more regularly prefer to use market orders in the LM to perform their picking-off strategies when the asset volatility is high.

[24] In the case of the benchmark model setup without the DP (only LM), speculators also begin to prefer market orders to limit orders when the asset volatility increases (see Figure 2 Panel A).

Liquidity traders, meanwhile, begin to use limit orders more often when the asset volatility is high (see Panels E and F).[25] This is because there is a chance of the asset value moving against the prices of dark orders and market orders, which can be compensated by the better prices offered by limit orders. Moreover, under this volatility scenario, the use of limit orders is helped by the fact that there will be several speculators waiting in the LM to submit market orders, which will promptly pick off any limit orders submitted by the liquidity traders (i.e., the liquidity traders can execute their limit orders quickly). This is an important point, with liquidity traders paying a waiting cost and a picking-off cost on limit orders so as to obtain liquidity when the asset volatility is high, instead of paying the bid-ask spread on market orders.[26]

Now understanding the trading behaviors of speculators and liquidity traders, as described in Figure 2, we present in Figure 3 the market shares of the LM and the DP as functions of the asset volatility. Panels A-C show the market shares for each trader type in terms of their private values (Panels A-C), for traders with small and large orders (Panels D-E), and for all traders (Panel F).

[Insert Figure 3 here]

Observation 2. (i) *The migration of the trading activity from the lit market to the dark pool has a humped shape as a function of the asset volatility, independent of the trader type and independent of the execution priority rule used in the dark pool.*

(ii) *The dark trading activity is higher when the dark pool has a size execution priority than when it has a time execution priority, especially in the case of moderate asset volatility.*

(iii) *Large traders prefer the lit market to the dark pool to a greater extent when the level of asset volatility is low or high, even if the dark pool reduces the waiting costs for large orders in the case that dark orders are executed according to a size execution priority, and even if the dark pool reduces the price impact of large orders.*

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[25] Liquidity traders also begin to prefer limit orders in the benchmark model setup without the DP (only LM) when the asset volatility goes up (see Figure 2 Panel D).

[26] Interestingly, we also show in Appendix C (Figure C1) that the size execution priority rule reduces the picking-off risk of the whole lit-dark system relative to the case of a time execution priority for dark orders.

Figure 3 shows that the trading activity in the DP has a humped shape as a function of the asset volatility, with a maximum value when the asset volatility has a medium level, for each agent type and for the whole market. Most importantly, the trading activity in the DP is higher when dark orders follow the size execution priority rule than when they follow the time execution priority rule, especially when volatility is at an intermediate level. This is because, as expected, the dark trading activity of traders with *large* orders is much higher when dark orders follow a size execution priority than when they follow a time execution priority, with a much more pronounced migration to the dark pool at a medium level of volatility. The higher trading activity of large traders in the dark market augments trader competition there and, as consequence, the execution probability of dark orders increases, which even attracts *other* trader types to the dark market.

It is interesting to observe in Panel E that, conditional on there being a DP running together with the LM, large traders prefer the LM to the DP to a greater extent when the level of asset volatility is low or high, independent of the execution priority rule for dark orders. This is because, when the level of asset volatility is low or high, the total trading activity is lower in the DP than in the LM, which increases the potential waiting costs for dark orders. Thus, despite the fact that the DP reduces the price impact of large orders, the waiting costs are high in the DP for large traders when the level of asset volatility is low or high. For instance, in the model setup 'LM + DP(size priority)', the waiting costs paid by traders in the DP are 38.3% and 13.4% *larger* than those in the LM when λ_v is 0.06 and 1.00, which represent low and high levels of volatility, respectively. At the same time, in the same model setup, the waiting costs paid by traders in the DP are 9.5% *lower* than those in the LM when λ_v is 0.25, which is an intermediate level of volatility (a similar result is obtained in the model setup 'LM + DP(time priority)').

5 Empirical Implications of Dark Pools, for Market Liquidity

In this section, we analyze the effect of the introduction of a dark pool (DP), which runs in parallel to the lit market (LM), on market liquidity. Our first measure of market liquidity in the DP and the LM is their relative levels of trading activity, which were reported earlier as the market share of each venue (see Figure 3 in Section 4.2). However, the relative trading activity in each market does not necessarily reflect potential liquidity costs (e.g., immediacy costs, waiting costs, and/or picking-off costs) that may affect the trading process. In this sense, the bid-ask spread is frequently used as a proxy for liquidity in lit markets, since it reflects such liquidity costs. Thus, our second liquidity measure is the bid-ask spread in the limit order market.

The bid-ask spread, however, does not exist in the DP, where transactions happen at the midquote of the LM. Thus, to analyze the market liquidity of the whole *lit-dark system*, we need a measure of liquidity (similar to the bid-ask spread) not only for the lit venue, but also for the DP and the complete system. Consequently, we create a third liquidity measure that can be used in all market combinations we consider.

This third liquidity measure should capture the liquidity cost paid by traders in their order submission. Thus, our liquidity measure simultaneously includes immediacy costs, waiting costs, and picking-off costs (captured by the bid-ask spread), since all affect the value obtained by traders in executing. Suppose that a trader arrives in the economy at time t_0, and has private value α. The trader submits an order for one share to either of the markets (i.e., the LM or the DP); thus, she can submit a limit order, a market order, or a dark order. The submission price of the order is p_0, and the order has direction x_0 ($x_0 = 1$ for a buy order and $x_0 = -1$ for a sell order). Suppose that the order is executed at time t_1. Then, the trader's *realized* payoff (i.e., not the expected payoff), π, is given by

$$\pi = x_0\big(\alpha + v_{t_1} - p_0\big)e^{-\rho(t_1 - t_0)}, \tag{3}$$

where v_{t_1} is the asset value at time t_1. It is important to notice that the realized payoff, π, in equation (3) includes potential waiting costs, since there is a chance that $t_1 \neq t_0$ in the case of limit orders and dark orders. In addition, π includes immediacy costs and picking-off costs, given that, in general, $p_0 \neq v_{t_1}$.[27]

Consider now a scenario of a lit-dark system with *infinite* liquidity, such that there are no liquidity costs (i.e., there are no immediacy costs, waiting costs, or picking-off costs). In this scenario with infinite liquidity, all orders are executed immediately since there are no waiting costs, so $t_1 = t_0$ in equation (3). In addition, under infinite liquidity, $p_0 = v_{t_1} = v_{t_0}$, given that there are no immediacy costs or picking-off costs. Thus, in this scenario without liquidity costs, the trader's realized payoff, $\pi^{NoLiqCosts}$, is

$$\pi^{NoLiqCosts} = x_0\alpha. \tag{4}$$

Then, the liquidity costs can easily be obtained by taking the realized payoff without liquidity costs, $\pi^{NoLiqCosts}$, minus the realized payoff with liquidity costs, π, which is equal to

$$Liquidity\ costs = x_0\alpha - \pi. \tag{5}$$

Figure 4 Panel A shows the bid-ask spread as a function of the asset volatility in the LM (Panel A). In addition, we present the liquidity costs as a function of the asset volatility in the LM (Panel B), in the DP (Panel C), and in the whole system (Panel D).

[Insert Figure 4 here]

Observation 3. (i) *The liquidity costs (in the lit market, the dark pool, and the whole system) are positively related to the asset volatility.*

(ii) *The liquidity costs of the whole system are larger when the dark pool is introduced than when there is only a lit market.*

[27] In the explanation of our liquidity measure for the DP and the complete system, we treat all traders as the same size. The agent, however, can be a trader with a large order; in this case, we consider the average payoff per share of her order.

(iii) *The size execution priority rule improves the liquidity of the whole lit-dark system relative to the case of a time execution priority for dark orders.*

In Figure 4, Panels A and B show that the bid-ask spread and the liquidity costs in the LM are larger when the DP is introduced. This is because there is a migration of trading activity from the LM to the DP, as described in Figure 3. This finding is important yet differs from previous theoretical studies, which have shown diverse effects (negative or positive) of dark trading activity on the LM liquidity, depending on whether the migration is due to specific types of traders assumed to submit limit orders or those assumed to submit market orders in the LM. For instance, Zhu (2014) and Buti *et al*. (2017) show that market liquidity is reduced in the LM after the introduction of a DP, while Brolley (2019) finds the opposite result.

As explained in Section 4, in our model, any trader type can decide to submit limit or market orders, with traders' decisions depending on the asset volatility. Therefore, any type of migration to the DP induces damage to the liquidity of the LM, since order migrations always involve liquidity providers and liquidity takers.

In addition, this figure shows that the bid-ask spread (Panel A) and liquidity costs in each market and in the whole system (Panels B-D) increase with the asset volatility. This is mainly due to the picking-off risk that increases when the asset volatility goes up.

Interestingly, Figure 4 Panel D shows that the liquidity costs of the *whole* system are larger when the DP is introduced than when there is just the LM. This result is consistent with Pagano (1989), who shows that there are negative externalities of market fragmentation in the presence of market frictions.

Figure 4 also reports that the bid-ask spread and the liquidity costs in the LM are larger (Panels A and B), and the liquidity costs in the DP are lower (Panel C), when the DP has a size execution priority, compared to when it has a time execution priority. This is consistent with Figure 3, showing that there is a larger migration of trading activity from the LM to the DP (mainly of traders with large orders, as expected) under a size execution priority than under a time execution priority for dark orders.

Most importantly, in terms of policy implications, Panel D shows that the liquidity costs of the *whole* system are lower when the DP has a size execution priority than when it has a time execution priority. This result mainly comes from the reduction in the waiting time for large traders who submit orders to the DP.

The findings regarding liquidity, from Figure 4, together with the market preferences of traders from Figure 3 (Panel F), provide the following implication: the trading activity of the DP is hump-shaped as a function of the bid-ask spread in the LM and as a function of liquidity costs in the whole system, which is reported in Figure 5. In this figure, Panel A (Panel B) shows the market share of the DP as a function of the bid-ask spread (liquidity costs in the whole system).

[Insert Figure 5 here]

Observation 4. (i) *The trading activity in the dark pool has a hump shape as a function of both the bid-ask spread in the lit market and as a function of the liquidity costs of the whole system.*

(ii) *Thus, the trading activity in the dark pool can be both positively and negativity related to the level of liquidity.*

The finding presented in Figure 5 is due to the fact that, on the one hand, there is a positive association between the bid-ask spread in the LM (see Figure 4 Panel A) – or the liquidity costs in the whole system (see Figure 4 Panel D) – and the asset volatility. On the other hand, there is a hump-shaped relationship between the DP activity and the asset volatility (see Figure 3 Panel F).[28] This result is consistent with Ray (2010), who shows that the dark trading activity has a hump-shaped relationship with the bid-ask spread, using data from three different DPs (POSIT, Liquidnet, and Pipeline Trading).

[28] In unreported results, we also find that the trading activity of the DP is hump-shaped as a function of the liquidity costs in the LM and as a function of the liquidity costs in the DP.

Furthermore, the results presented in Figure 5 suggest that empirical studies that use linear regressions may fail to capture the non-linear relationship between dark trading activity and liquidity measures. In addition, the results presented in Figure 5 may explain the mixed empirical results on the association between the level of trading activity in the DP and market liquidity (see, e.g., Ray, 2010; Degryse *et al.*, 2015; Kwan *et al.*, 2015; Buti *et al.*, 2016; Foley and Putniņš, 2016; Gresse, 2017; Hatheway *et al.*, 2017; and Farley *et al.*, 2018).

6 Traders' Payoffs and Market Welfare

In equilibrium, agents make decisions with the objective of maximizing their expected payoffs (i.e., by solving equation (2)). Let $G_z(\lambda_v)$ be the expected payoff *per traded share* of trader type z (where the trader type refers to whether she is a small or large trader, and her private value), as a function of the asset volatility, λ_v.[29,30] Similarly to Biais *et al.* (2015), we set utilitarian welfare equal to:

$$U(\lambda_v) = \sum_{z \in Z} \omega_z \, G_z(\lambda_v), \tag{6}$$

where ω_z is the proportion of trader type z in the market. Figure 6 Panel A presents the welfare of the complete system as a function of the asset volatility, under different model setups. Panel B reports the contributions that the LM and the DP make to welfare. For instance, for a given level of asset volatility, in the case of the model setup with the LM and the DP (with dark orders executed according to size priority), the sum of the black and grey dashed lines in Panel B is equal to the value of the dashed black line in Panel A.

[Insert Figure 6 here]

[29] The value of $G_z(\lambda_v)$ for a large trader is her total expected payoff divided by her number of shares to be traded, Q.

[30] It is important to highlight that the expected payoff per traded share of trader type k already includes the discounted private value of the trader, but also includes the benefits of the transaction *per se* (i.e., $e^{-\rho h_q}[(\alpha + v_{h_q} - p)x]$ in equation (1)).

Observation 5. (i) *There is a negative relationship between the welfare of the system and the asset volatility, independent of whether the dark pool is introduced.*

(ii) *The welfare of the system is lower when the dark pool is introduced than when there is only a lit market, independent of the level of asset volatility and the execution priority rule in the dark market.*

(iii) *The size execution priority rule improves the welfare of the whole lit-dark system relative to the case of time execution priority for dark orders.*

Figure 6 Panel A shows that the welfare of the system decreases as the asset volatility grows, for all model setups. In particular, welfare is reduced in a lit-dark market environment compared to when there is just the LM, consistent with market fragmentation diminishing welfare in the presence of market frictions.

Most importantly, Panel A also shows that the size execution priority in the DP improves the welfare in the system, because it naturally improves the execution process for traders with large orders in the DP. Furthermore, the contributions from the LM and the DP to the welfare in Panel B are in line with the market preferences of traders described in Figure 3 of Section 4.

We further examine welfare for different trader types. Figure 7 reports a similar analysis to the one presented in Figure 6, but conditional on the trader type in terms of their private values.

[Insert Figure 7 here]

Observation 6. (i) *Traders with* $|\alpha| = 0$ *(i.e., speculators) are better off in terms of their welfare when a dark pool is introduced. Moreover, there is a positive relationship between the speculators' welfare and the asset volatility.*

(ii) *Traders with* $|\alpha| \neq 0$ *(i.e., liquidity traders) are worse off in terms of their welfare when a dark pool is introduced. In addition, there is a negative relationship between the welfare of liquidity traders and the asset volatility.*

Figure 7 Panel A shows that agents with $|\alpha| = 0$ (i.e., speculators) are better off when the asset volatility increases. This is due to speculators being able to perform a picking-off strategy more frequently when the asset volatility is higher, which offers speculators higher profits than a pure liquidity provision strategy through limit orders (see the explanations provided in Figure 1).

Interestingly, Panel A shows that traders with $|\alpha| = 0$ have higher profits when the DP is introduced than when there is only the LM, regardless of the asset volatility and of the execution priority used in the DP. This is consistent with Figure C1 Panel B in Appendix C, which reports that traders with $|\alpha| = 0$ performed a picking-off strategy more often when the DP was introduced than when there was only the LM. As dark orders offer an additional alternative for carrying out a picking-off strategy, with prices that are more exposed to such risk, there is a higher picking-off risk when there is a DP in the system (see Figure C1 Panel A in Appendix C).

Conversely, Panels C and E show that agents with $|\alpha| \neq 0$ (i.e., with exogenous reasons to trade) receive lower profits when the asset volatility increases. Moreover, agents with $|\alpha| \neq 0$ have lower profits when the DP is introduced into the market. Such traders have to pay higher liquidity costs when the asset volatility increases (see Figure 4). In addition, the extra profits obtained by agents with $|\alpha| = 0$ when the DP is introduced are provided by agents with exogenous reasons to trade, with $|\alpha| \neq 0$.

Furthermore, Figure 7 shows that a size execution priority rule in the DP reduces (improves) the expected payoffs of traders with $|\alpha| = 0$ (traders with $|\alpha| \neq 0$) relative to their payoffs under a time execution priority rule for dark orders.

Finally, in Figure 8, we analyze the welfare of traders with different order sizes.

[Insert Figure 8 here]

Observation 7. (i) *The welfare of traders with small and large orders is negatively related to the asset volatility.*

(ii) *The welfare of traders with large orders is lower when a dark pool is introduced than when there is a lit market alone, despite the benefits that the dark pool provides to large traders in terms of both reductions in the waiting costs when dark orders have a size execution priority and a decrease in the price impact of large orders.*

(iii) *The contribution of the dark pool to the welfare of traders with large orders is lower than that of the lit market, in the case of either low or high asset volatility.*

Figure 8 shows that welfare decreases for traders with small and large orders when asset volatility goes up (Panels A and C, respectively). The figure also reveals two important findings. Firstly, Panel C shows that the welfare of traders with large orders is always lower when the DP is introduced than when there is the LM alone, regardless of both the asset volatility and the execution priority rule. The higher liquidity costs that fragmentation induces in the system lead to a detrimental effect on welfare for traders with large orders. This result goes in opposition to the traditional vision of the benefits of DPs for large traders, according to which it is thought that a dark market reduces the price impact of large orders as dark orders are anonymous and undisplayed.

Conditional on the DP being introduced (see Panel D), the contribution of the DP to the welfare of traders with large orders is larger than that of the LM when there is a medium level of asset volatility. However, the contribution of the DP to the welfare of large traders is lower than that of the LM when the asset volatility is low or high. This is consistent with the results regarding the preferences of large traders reported in Section 4 (see Figure 3 Panel E), where we explained that, when the asset volatility was low or high, the total trading activity was lower in the DP than in the LM, which increased the liquidity costs in the dark market.

7 Conclusions

In this paper, we analyze the impact of the introduction of a dark pool when there is a limit order market running in parallel. We present a model that describes a multi-market environment with an asset that can be traded in either a limit order market (through limit and market orders) or a dark pool (through dark orders). Differently to previous theoretical studies, we simultaneously include different execution priority rules for dark orders, adverse selection (i.e., picking-off risk), and traders' competition in order submission for all types of orders and all types of traders (where trader competition changes depending on the migration of trading activity between the lit market and the dark pool).

We show that the migration of trading activity from the lit market to the dark pool has a humped shape as a function of the asset volatility, with a maximum value of dark trading activity when the asset volatility has an intermediate level. As expected, the dark trading activity of traders with large orders is higher when the dark pool has a size execution priority than when it has a time execution priority. The liquidity costs of the whole system are larger when the dark pool is introduced than when there is a lit market alone. In addition, the liquidity costs (in the lit market, in the dark pool, and in the whole system) are positively related to the asset volatility. Moreover, the migration of trades to the dark pool also has a humped shape as a function of the liquidity costs in the whole system. This last result is particularly relevant because it may explain the mixed findings in the empirical literature about the relationship between the level of trading activity in the dark pool and market liquidity.

The welfare of the system is lower when the dark pool has been introduced than when there is a lit market alone, regardless of the level of asset volatility and the execution priority rule in the dark market. The introduction of a dark pool increases welfare only for speculators, while other traders are worse off. Interestingly, we show that large traders are worse off when there is a dark pool and a lit market, compared to when there is only a lit market, even when there is a size execution priority in the dark pool. This is due to the higher liquidity costs that fragmentation induces in the system, which strongly affect traders with large orders who want to trade those large orders as soon as possible. This result goes against the traditional view of the benefits of dark pools for large traders, in the sense that dark orders should reduce the price impact of large orders, since orders in the dark pool are anonymous and undisplayed.

Most importantly, in terms of policy implications, we show that the size execution priority rule improves liquidity and welfare, and reduces the picking-off risk, of the whole lit-dark system relative to the case of a time execution priority for dark orders. Moreover, the size execution priority generates a larger migration to the dark pool in the case of moderate asset volatility. Therefore, a size execution priority rule induces more benefits in the whole system when there is a larger migration to the dark pool with an intermediate level of volatility.

Whilst the intuitions behind the model are simple, they give rise to additional questions that we would like to answer in future research, for example, examining the impact of a dark pool under additional execution priority rules, analyzing a dark pool when there are high-frequency trading firms, and examining dark pools that do not function during the same working hours as lit markets. These questions have been left for future studies.

Appendix A: Variable definitions

In this appendix, Table A1 describes all the variables used in our study.

[Insert Table A1 here]

Appendix B: Summary of the main results

In this appendix, we present in Table B1 a summary of the main results of our study, comparing the impacts on trading activity, liquidity, and welfare of changes in the execution priority rules in the dark pool, adverse selection, and traders' competition.

[Insert Table B1 here]

Appendix C: Picking-off risk and percentage of orders from speculators that follow a picking-off strategy as a function of the asset volatility

In this appendix, we show the picking-off risk and percentage of orders from speculators that follow a picking-off strategy as a function of the asset volatility. Figure C1 Panel A reports the picking-off risk as a function of the asset volatility, where the picking-off risk reflects the ratio of the submitted waiting orders (i.e., waiting orders in the dark pool and the lit market) that are picked off, to the total number of submitted waiting orders. Panel B shows the proportion of executed orders in the whole system from speculators (i.e., traders with private value equal to zero) that were used to pick off waiting orders that had previously been submitted by other traders. Thus, Panel B reports the proportion of executed orders from speculators that are part of a picking-off strategy in the whole financial system (i.e., in the LM and DP). In addition, Panel C shows the proportion of executed orders from speculators in *each* market that were submitted as part of a picking-off strategy (see Panel C).

[Insert Figure C1 here]

Figure C1 supports the results of Section 4.2, by showing that speculators begin to prefer dark orders more often as the asset volatility increases to the medium level. As expected, Panel A shows that the picking-off risk increases as the asset volatility increases. In addition, this figure reports three important results. Firstly, Panel A shows that there is a higher picking-off risk when the dark pool is introduced than when there is only the lit market, which is in line with the intuitions explained in Figure 1. This is because dark orders offer better prices than market orders, which means that dark orders waiting in the dark pool are more exposed to being picked off than market orders when the asset value changes. Secondly, the size execution priority rule reduces the picking-off risk of the whole lit-dark system relative to the case of a time execution priority for dark orders. Thirdly, the preferences of speculators for picking-off strategies (see Figure C1 Panel B) are greater when the dark pool is introduced (i.e., the black dashed line and the black solid line are above the grey solid line).

Figure C1 Panel B also shows that the preference of speculators for a picking-off strategy increases with the asset volatility. Moreover, when there is a medium level of asset volatility, Panel C shows that speculators more often prefer to use the dark pool to perform their picking-off strategies through dark orders, than market orders in the lit market. Thus, the results of Figure C1 are consistent with the arguments of Section 4.1 and the results reported in Figure 2 of Section 4.2.[31]

Appendix D: Pakes and McGuire algorithm

As we explain in Section 3.8, the intuition behind the Pakes and McGuire (2001) algorithm in our model is that traders learn to take optimal decisions under different market conditions by repeatedly playing the trading game described in Section 3. At the beginning of the algorithm, initial beliefs are set regarding the expected payoff of all trading decisions in each state. Traders then play the game and take their optimal decision (the one with the highest expected payoff) given the state observed. After each transaction, traders update their beliefs about the expected payoffs of their trading decision based on the observed realized payoffs that result from their actions.

Initial beliefs: Suppose that $W(a|s)$ is the expected payoff at time t that is associated with action $a \in \mathcal{A}(s)$ that a trader can take when she faces state s. We set the initial belief about the expected payoff, $W(a|s)$, as follows. Suppose one of the possible actions for a trader with private value α in state s is to submit a limit order at price $\hat{p}$ for her remaining Q^* shares, when the fundamental value is $\hat{v}$. We assume that the initial expected payoff of this action is $(\alpha + \hat{v} - \hat{p})xQ^*$

[31] It is interesting to observe in Figure C1 Panel B that, when the asset volatility is small, speculators still perform picking-off strategies (i.e., around 30% on the left-hand side of the plot in Panel B). This is because, despite the low level of asset volatility, some liquidity traders will still submit very aggressive limit orders which may even be in the 'wrong position' in relation to the asset value. These limit orders are like market orders, since they are executed very quickly due to being in the wrong position (i.e., they are exposed to being picked off), and they still have better prices than market orders.

discounted by ρ until the expected time t_N that a new trader arrives in the market, and where $x = 1$ if the order is a buy order and $x = -1$ if it is a sell order; thus, $W(a|s) = e^{-\rho t_N}(\alpha + \hat{v} - \hat{p})xQ^*$. This value is only a first approximation since the next trader may not trade with the first agent. In the case of a market order, the expected payoff for a trader with Q^* shares remaining is simply $W(a|s) = (\alpha + \hat{v} - \hat{p})xQ^*$, without any discount. In the case of a dark order with Q^* shares, the initial expected payoff is the average of the expected payoffs of the limit and market orders. Thus, the initial expected payoff for a dark order is $W(a|s) = 0.5e^{-\rho t_N}\big(\alpha + \hat{v} - \widehat{M}\big)xQ^* + 0.5(\alpha + \hat{v} - \widehat{M})xQ^*$, where $\widehat{M}$ is the midquote of the lit market.

The updating process used to reach equilibrium: Suppose that the trader decides at time t to take the optimal action a° that provides the maximum expected payoff out of all possible actions. As a first case, suppose that the optimal decision a° is a market order in which all remaining shares, Q^*, are immediately executed. Then, the updating process of the expected payoff of the optimal action a° in this case can be expressed as:

$$W(a^\circ|s) = \frac{n_{a^\circ,s}}{n_{a^\circ,s}+1} W(a^\circ|s) + \frac{1}{n_{a^\circ,s}+1} \sum_{q=1}^{Q^*} (\alpha + v_t - p_q)x, \tag{D1}$$

where $n_{a^\circ,s}$ is a counter that increases by one when action a° is taken in state s.[32]

As a second case, suppose that the optimal decision a° at time t is a limit order in the lit market, for Q^* shares. Later, at time t_r the same trader re-enters the market and observes that her order has not been executed, but the market conditions have changed. The trader observes a new state s_{t_r} in which she follows the optimal strategy $a^{\circ\circ}$ that gives a maximum payoff under the new market conditions. Consequently, the original decision a° induces a realized continuation of optimal actions and expected payoffs, and thus the updating process of beliefs can be written as:

[32] The value of $n_{a^\circ,s}$ affects how quickly we reach the model equilibrium (a large value of $n_{a^\circ,s}$ is associated with a slow convergence). Therefore, we reset $n_{a^\circ,s}$ intermittently to improve the convergence speed.

$$W(a^{\circ}|s) = \frac{n_{a^{\circ},s}}{n_{a^{\circ},s}+1} W(a^{\circ}|s) + \frac{1}{n_{a^{\circ},s}+1} e^{-\rho(t_r - t)} W\left(a^{\circ\circ}\middle|s_{t_r}\right). \tag{D2}$$

As a third case, suppose that the optimal decision a° at time t is a limit order in the lit market for Q^* shares. Later on, at time t_r, a trade of one share is executed. Thus, albeit the trader has not taken a new decision (i.e., the original decision a° still holds), the trader faces a new state $s_{t_r}^{Q^*-1}$, in which she has trades of $Q^* - 1$ shares waiting to be executed. The updating process for the trader with the optimal action a° can be reflected in the following equation:

$$W(a^{\circ}|s) = \frac{n_{a^{\circ},s}}{n_{a^{\circ},s}+1} W(a^{\circ}|s) + \frac{1}{n_{\tilde{a}^*,s}+1} e^{-\rho_d(t_r - t)} \left[\left(\alpha + v_{t_r} - \tilde{p}\right)\tilde{x} + W\left(a^{\circ}\middle|s_{t_r}^{Q^*-1}\right)\right]. \tag{D3}$$

As a fourth case, suppose that the optimal decision a° is a dark order in the dark pool for Q^* shares at time t. Immediately after submission, the dark order can be (i) executed as a whole (i.e., the Q^* shares are traded); (ii) partially executed (i.e., not all Q^* shares are traded); or (iii) not executed at all (i.e., none of the Q^* shares are traded). Thus, despite the trader not taking a new decision (i.e., the original decision a° still holds), the trader may face a new state s^q, in which she has q shares waiting to be executed. The updating process for the trader with the optimal action a° is reflected in the following equation:

$$W(a^{\circ}|s) = \frac{n_{a^{\circ},s}}{n_{a^{\circ},s}+1} W(a^{\circ}|s)$$

$$+\begin{cases} \dfrac{1}{n_{a^{\circ},s}+1} \displaystyle\sum_{q=1}^{Q^*} (\alpha + v_t - M_t)x & if\ Q^*\ shares\ are\ trad.at\ t \\ \dfrac{1}{n_{a^{\circ},s}+1} \left[\displaystyle\sum_{q=1}^{Q^*-1} (\alpha + v_t - M_t)x - W(a^{\circ}|s^1) \right] & if\ Q^* - 1\ shares\ are\ trad.at\ t \\ \vdots & \vdots \\ \vdots & \vdots \\ \dfrac{1}{n_{a^{\circ},s}+1} \left[(\alpha + v_t - M_t)x - W\left(a^{\circ}\middle|s^{Q^*-1}\right) \right] & if\ one\ share\ is\ trad.at\ t \\ \dfrac{1}{n_{a^{\circ},s}+1} [W(a^{\circ}|s)] & if\ zero\ shares\ are\ trad.at\ t \end{cases}, \tag{D4}$$

where M_t is the midquote of the lit market's best bid and ask prices.[33]

Convergence criteria: We check for convergence after running the trading game for a couple of billion trading events. Afterwards, we check the evolution of agents' beliefs every 500 million simulations for convergence. Suppose that the first group of 500 million simulations after we start checking for convergence finishes at time t_1. Suppose that the subsequent group of 500 million simulations finishes at time t_2. Let $W_{t_1}(a|s)$ and $W_{t_2}(a|s)$ be the *expected* payoffs that are associated with action a when state s is present at times t_1 and t_2, respectively. In addition, suppose that $m_{a,s}^{t_1,t_1}$is the number of times that action a was taken between t_1 and t_2 when traders faced state s. We evaluate the change in the expected value of the expression $|W_{t_1}(a|s) - W_{t_2}(a|s)|$ for all pairs (a, s) weighted by $m_{a\,s}^{t_1,t_1}$ every 500 million simulations. Once this weighted absolute difference is smaller than 0.01 (which suggests that the model has converged), we apply two further convergence criteria in line with Goettler *et al*. (2009).

As in Goettler *et al*. (2009), after reaching a small weighted absolute difference in the change in the expected values, we fix the agents' beliefs concerning the *expected* payoffs, $W^*(\cdot)$, and simulate the trading game for another 500 million events. Then, we calculate the *realized* payoffs of all order submissions after they have been executed. Let $\tilde{J}(\cdot)$ be the *realized* payoffs of the 500 million events. It is important to observe that $\tilde{J}(\cdot)$ is a direct measure of the benefits of trading, which is not 'averaged' as in equations (D1) to (D4). First, we require that the correlation between $W^*(\cdot)$ and $\tilde{J}(\cdot)$ is higher than 0.99. Second, we calculate the mean absolute difference between $W^*(\cdot)$ and $\tilde{J}(\cdot)$, weighted by the number of times that a specific action is selected in a given state in the last 500 million simulated events, which for convergence we also require to be below 0.01 (i.e., like in the previous paragraph when we evaluated the change in the expected value between $W(\tilde{a}|s)$ and

[33] We induce trembles in the traders' decisions to ensure that the updating process considers all possible actions in each state when we run the trading game to solve for the equilibrium. Specifically, we disturb the traders' decisions with a small probability ξ that they select actions that are suboptimal while the algorithm converges. We set ξ equal to 0.5%. In the case of a tremble, the trader selects among all suboptimal actions with equal probability. Once we reach the equilibrium of the model, we make ξ equal to zero.

$W_{t_2}(\tilde{a}|s)$ weighted by $m_{\tilde{a},s}^{t_1,t_1}$). If any convergence criterion is not reached, we continue simulating the trading game and updating the beliefs until all convergence criteria are satisfied.

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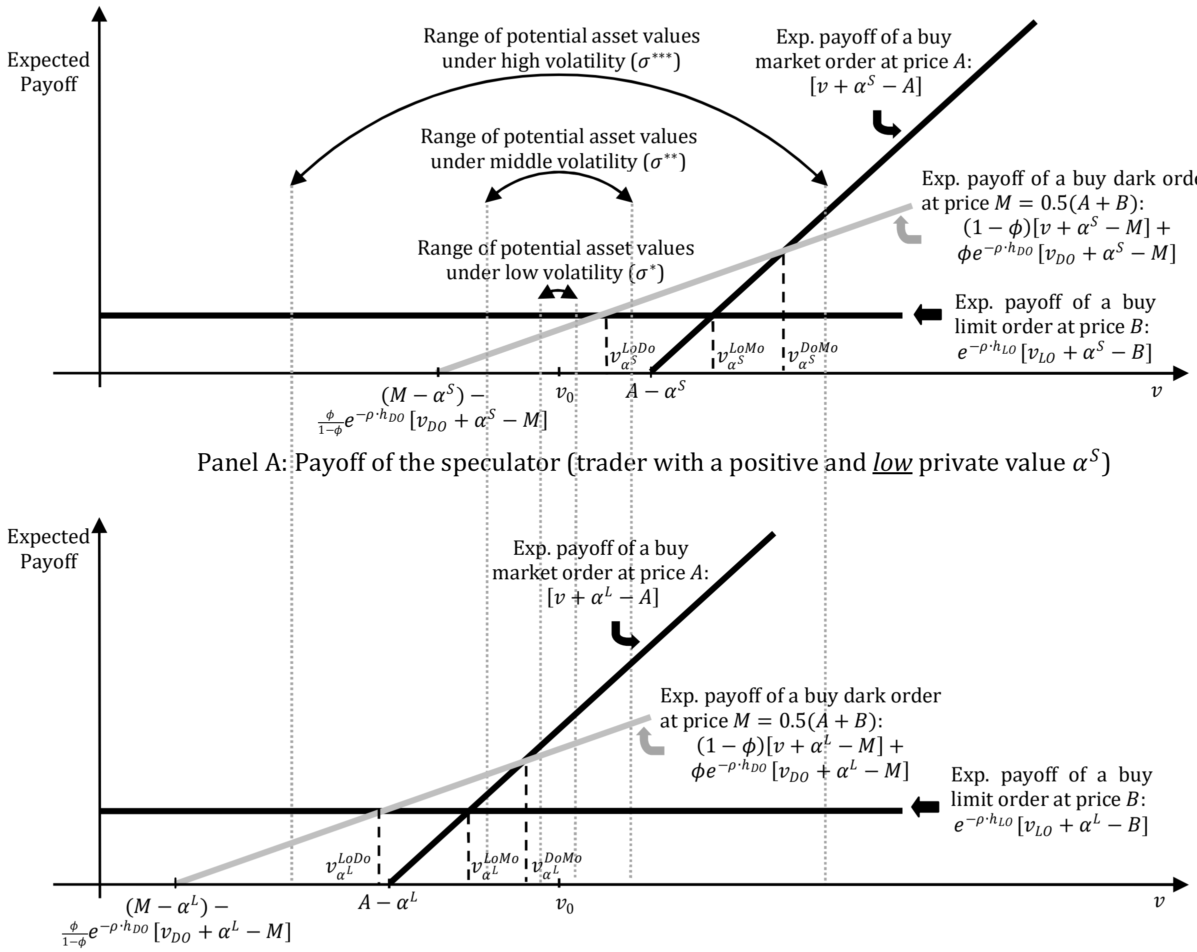

Panel A: Payoff of the speculator (trader with a positive and <u>low</u> private value α^S)

Panel B: Payoff of the liquidity trader (trader with a positive and <u>big</u> private value α^L)

Figure 1. Expected payoffs of limit, market, and dark orders in two simple examples (one example with a speculator and the other with a liquidity trader). This figure shows the expected payoffs of two traders as a function of the asset value, v, for the different order types. Panel A reflects the first simple example, where there is a speculator (i.e., a trader with a positive and small private value α^S). Panel B shows the second simple example, in which there is a liquidity trader (i.e., a trader with a positive and large private value α^L). The range of potential values of the asset depends on the level of the asset volatility (which is low (σ^*), medium (σ^{**}), or high (σ^{***})). In both examples, the trader can buy the asset share by using either a market order, MO, or a limit, LO, in the limit order market, or through a dark order, DO, in the dark pool.

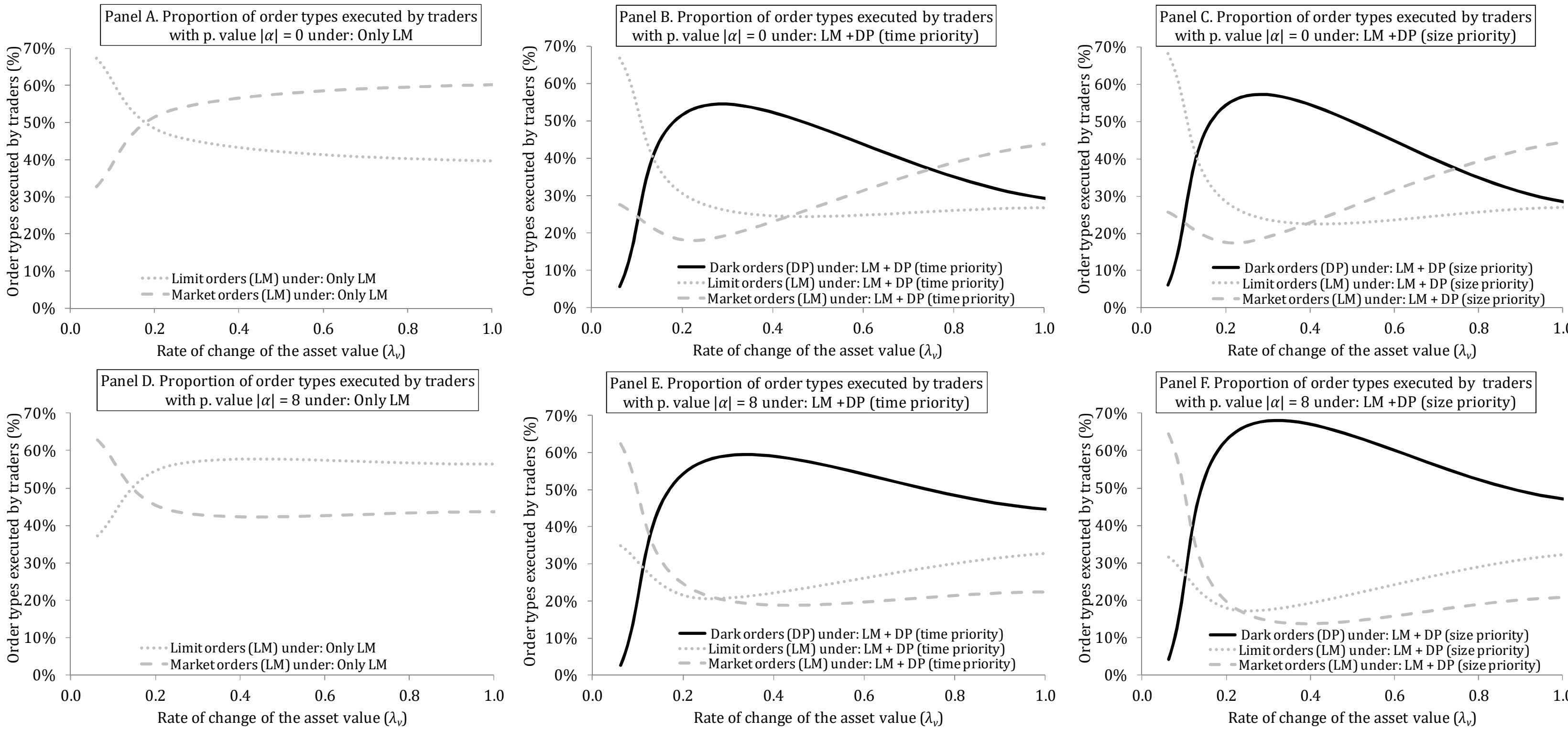

Figure 2. Percentages of order type executions as a function of the asset volatility (for two groups of traders under different market setups). This figure shows the percentages of executions of limit, market, and dark orders, as a function of the asset volatility. Thus, traders can submit orders in either the limit order market, LM, or the dark pool, DP. Panels A-C show the order preferences for speculators (i.e., agents with zero private value gained from trading immediately). Panels D-F present the order preferences for liquidity traders (i.e., agents with absolute private value equal to eight, who want to trade as soon as possible to obtain such private value). Order preferences are presented under three market setups: only a limit order market (Only LM), a limit order market and a dark pool under a time execution priority for dark orders (LM + DP(time priority)), and a limit order market and a dark pool under a size execution priority for dark orders (LM + DP(size priority)). The volatility of the asset is proxied by the rate of change of the asset value, λ_V.

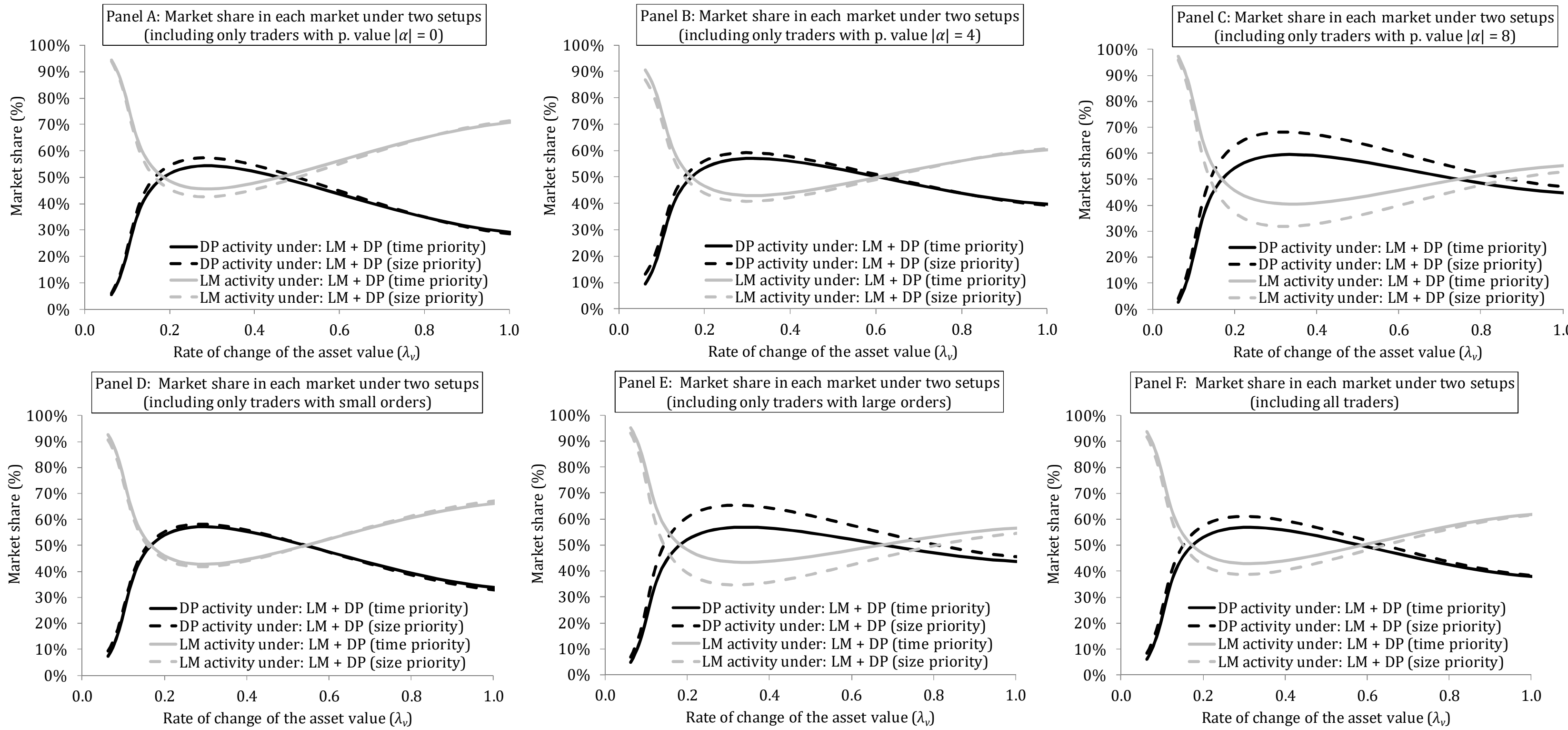

Figure 3. Market shares of the lit market and the dark pool as functions of the asset volatility. This figure reports the market shares of the limit order market, LM, and the dark pool, DP, as functions of the asset volatility, for each trader type in relation to their private values (Panels A-C), for traders with small and large orders (Panels D-E), and for all traders (Panel F). The results are presented under different market setups, which were described in Figure 2. The volatility of the asset is proxied by the rate of change of the asset value, λ_V.

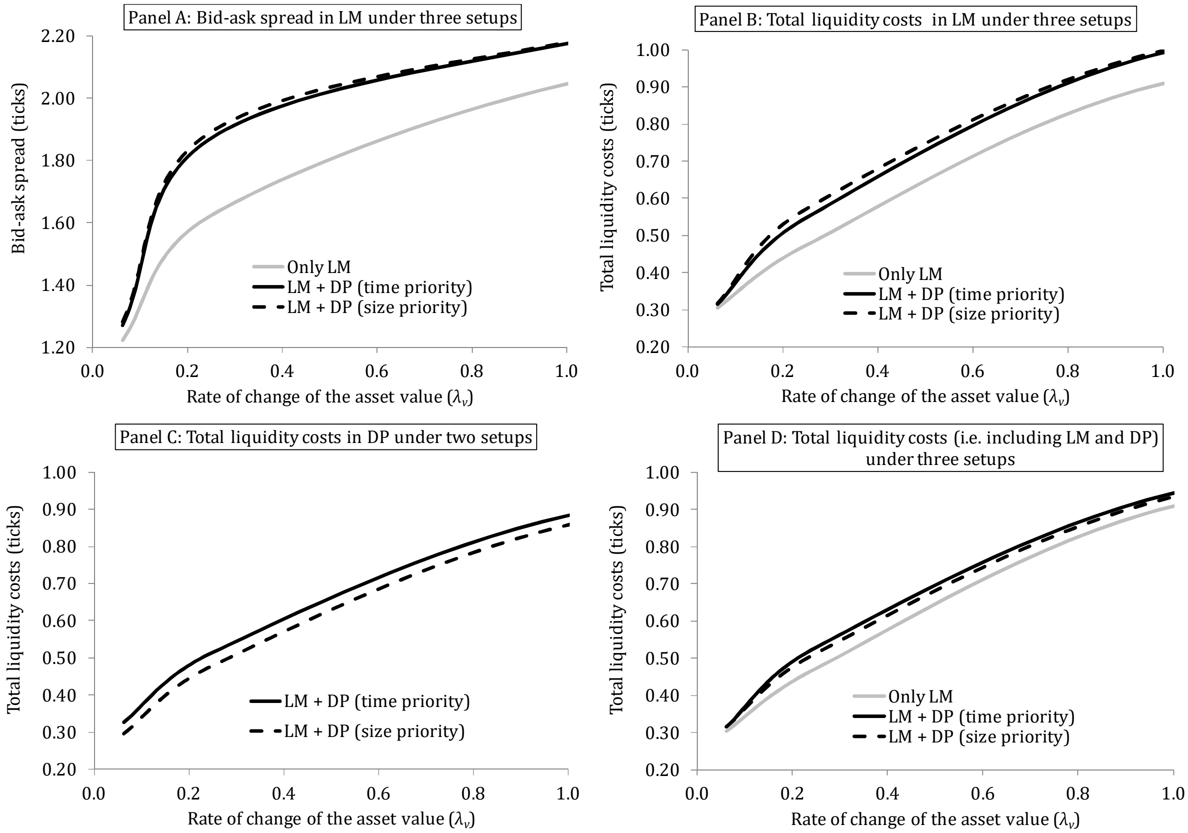

Figure 4. The effect of asset volatility on market liquidity. Panel A shows the bid-ask spread in the lit market, LM, as a function of the asset volatility. This table also presents the total liquidity costs in the lit market (Panel B), in the dark pool (Panel C), and in the whole system (Panel D). The liquidity costs are obtained by taking the difference between the realized payoff without liquidity costs and the realized payoff with liquidity costs, as in equation (5). The bid-ask spread and the liquidity costs are expressed in ticks. The results are presented under different market setups, which were described in Figure 2. The volatility of the asset is proxied by the rate of change of the asset value, λ_V.

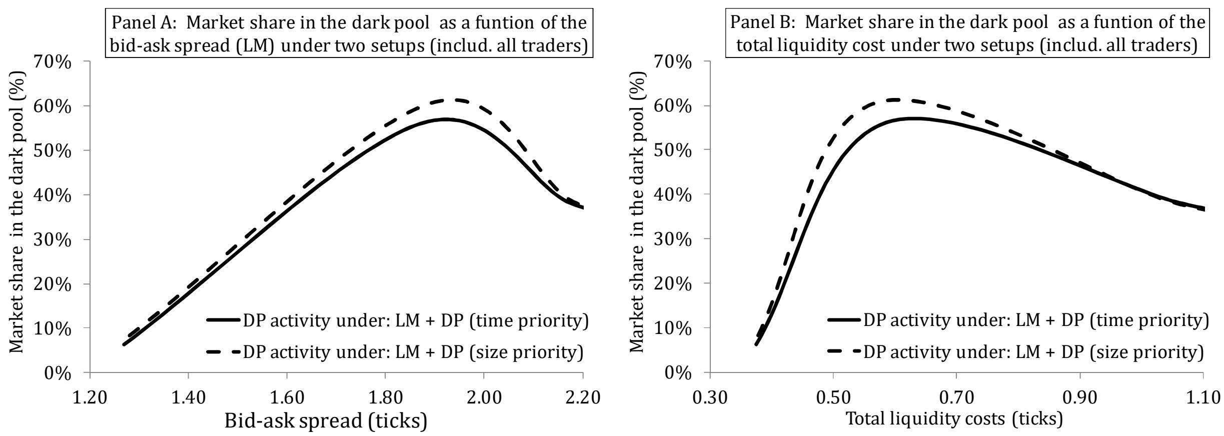

Figure 5. The effect of market liquidity on dark trading activity. This figure shows the market share of the dark pool as a function of the bid-ask spread in the lit market (Panel A), and as a function of the liquidity costs of the whole system (Panel B). Traders can submit orders in either the limit order market, LM, or the dark pool, DP. The liquidity costs are obtained by taking the difference between the realized payoff without liquidity costs and the realized payoff with liquidity costs, as in equation (5). The bid-ask spread and the liquidity costs are expressed in ticks. The results are presented under different market setups, which were described in Figure 2.

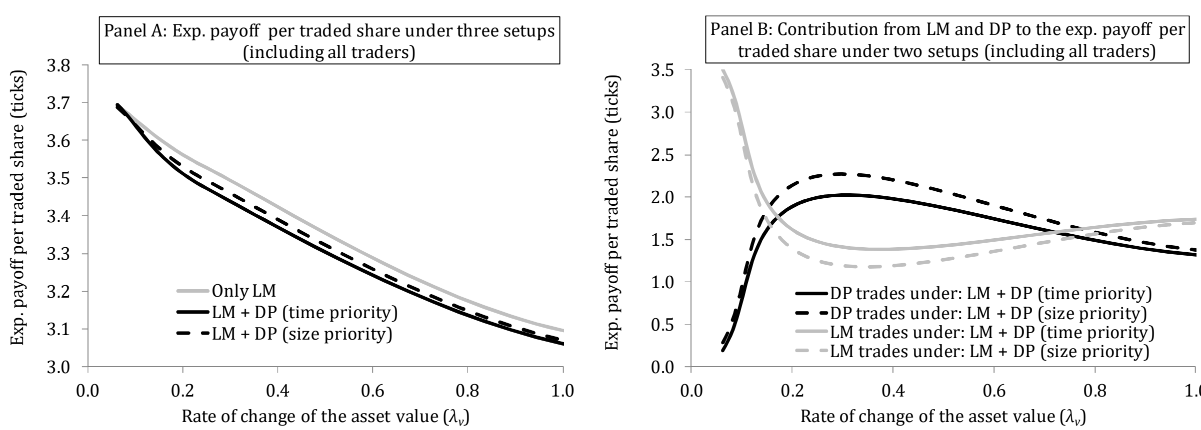

Figure 6. The effect of asset volatility on the system's welfare. In this figure, Panel A presents the welfare of the complete system as a function of the asset volatility. Panel B shows the contributions of the lit market, LM, and the dark pool, DP, to the welfare. We measure welfare as the expected payoff per traded share as in equation (6), which implicitly includes a selection of different trader types and order sizes. The welfare of the system is expressed in ticks. The results are presented under different market setups, which were described in Figure 2. The volatility of the asset is proxied by the rate of change of the asset value, λ_V.

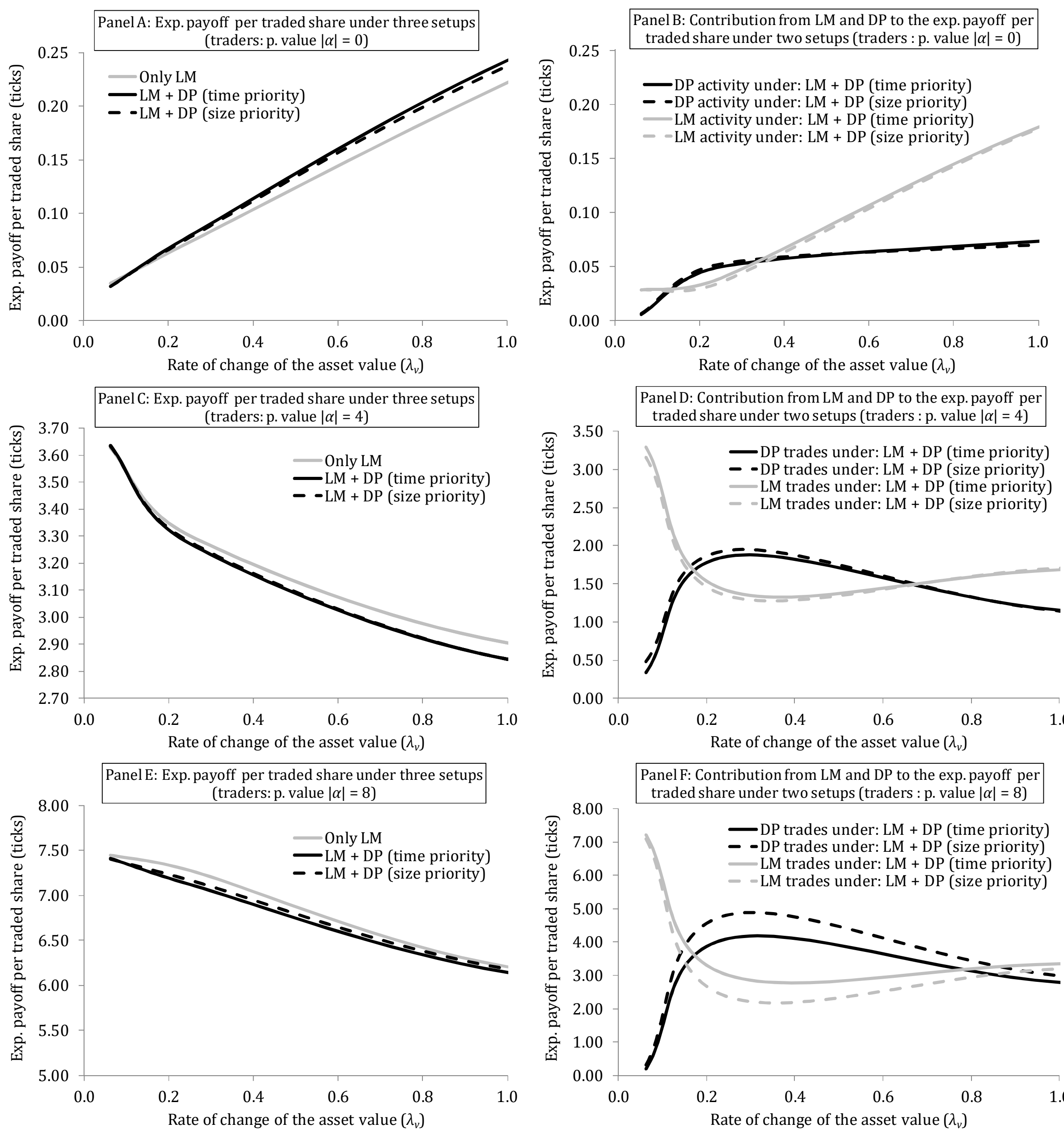

Figure 7. The effect of asset volatility on the welfare of agents with different private values. In this figure, Panels A, C, and D show the welfare of agents with private value equal to zero (Panel A), with private values -4 and 4 (Panel C), and with private values -8 and 8 (Panel D), as functions of the asset volatility. Panels B, D, and F show the contributions of the lit market, LM, and the dark pool, DP, to the welfare as functions of the asset volatility (conditional on the trader type in terms of their private values). We measure welfare as the expected payoff per traded share. The welfare is expressed in ticks. The results are presented under different market setups, which were described in Figure 2. The volatility of the asset is proxied by the rate of change of the asset value, λ_V.

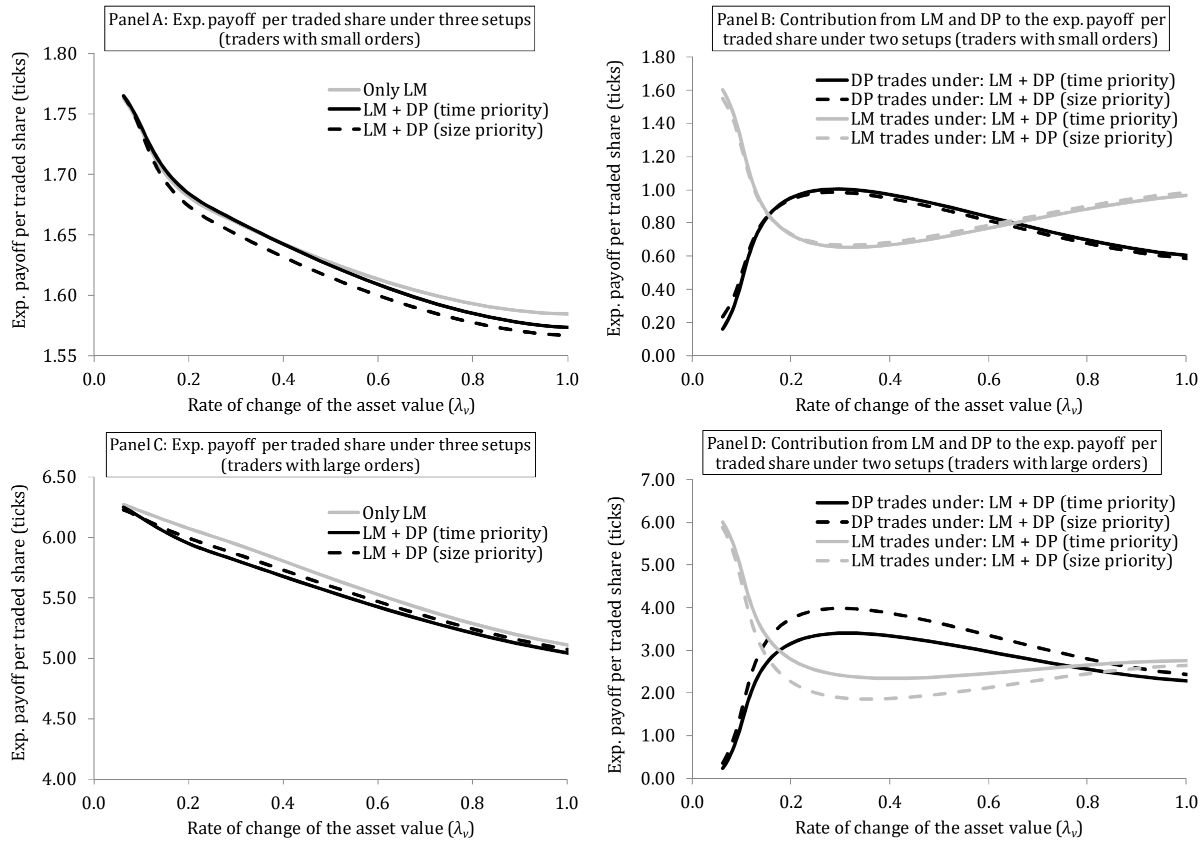

Figure 8. The effect of asset volatility on the welfare of traders with different order sizes. This figure shows the welfare of agents with small orders (Panel A) and with large orders (Panel C), as functions of the asset volatility. Panels B and D present the contributions of the lit market, LM, and the dark pool, DP, to the welfare as functions of the asset volatility (conditional on the trader type in terms of different order sizes). We measure welfare as the expected payoff per traded share. The welfare is expressed in ticks. The results are presented under different market setups, which were described in Figure 2. The volatility of the asset is proxied by the rate of change of the asset value, λ_V.

Table A1. Variable definitions. This table provides a glossary of all the variables that are used in our study.

Variable	Description
	Variables in Section 3
LM	Trading venue: lit market
DP	Trading venue: dark pool
λ	The arrival rate (Poisson process) of new traders
v_t	The fundamental value of the asset at any instant t
λ_v	Innovations (Poisson process) in the fundamental value
L_t	The limit order book associated with the lit market which is characterized by a finite set of N discrete prices $\{p^i\}_{i=-N}^{N}$, where $p^i < p^{i+1}$
d	The tick size defined as the distance between two consecutive prices
l_t^i	The queue of shares from unexecuted limit orders at price $p^i$and time t. A positive (negative) number for l_t^i denotes buy (sell) limit orders
B_t, A_t	The best bid and the best ask price at any instant t
M_t	The midquote of the lit market's best bid and best ask prices; the dark pool's price setting rule
k_t	The single queue of shares from unexecuted dark orders, k_t, at time t, for the single price M_t
Δ_t	The time lag at which all the traders observe the fundamental value
λ_r	The re-entry rate (Poisson process) of traders with more shares to trade
ρ	The rate at which payoffs from trades are discounted back to the time of entry (discount rate)
α	A trader's private value that represents reasons for trading
F_α	Private values' discrete distribution
γ_α $(1-\gamma_\alpha)$	The probability that the market participant is a small (large) trader
Q	The maximum number of shares a new large trader can trade
s	The state observed on a particular entry to the market
a	A trader's optimal action in state s
$\mathcal{A}(s)$	The set of feasible trading decisions a trader can take in state s
q	The share indicator with $q = \{1,2, \dots, Q^* \leq Q\}$, where $Q^*$is the number of remaining shares of the order
$\eta(\cdot)$	The probability density that a share is traded at some future time
$\pi(\cdot)$	A trader's expected payoff of taking an optimal decision in a given state
p_q	The order's price
x	The order's direction
$\gamma(\cdot)$	The density function of the asset value at time of execution
$\psi(\cdot)$	The probability that a state is observed at some future time after a trader's re-entry
$\mathcal{S}$	The set of potential states at re-entry
$R(\cdot)$	The cumulative probability distribution of the time at which the trader re-enters the economy
$V(\cdot)$	The value to an agent of being in any given state
	Variables in Section 4
S, α^S	A speculator with private value α^S
L, α^L	A liquidity trader with private value α^L
$\sigma^*, \sigma^{**}, \sigma^{***}$	The asset's volatility, which can take three values: low (σ^*), medium (σ^{**}), or high (σ^{***})
MO, A	Buy market order at price A
LO, B	Buy limit order at price B
DO, M	Dark order at price M
ϕ	The probability that the DO is not immediately traded
	Variables in Section 6
$G_z(\lambda_v)$	The expected payoff per traded share for trader type z as a function of the asset's volatility λ_v
$U(\cdot)$	Utilitarian welfare
ω_z	Proportion of trader type z in the market

	Variables in Appendix D
$W(a\|s)$	The agents' belief concerning the expected payoff that is associated with action $\tilde{a}$ when state s is observed at time t_1
$n_{a,s}$	A counter that increases by one when (optimal) action a - that offers the maximum expected payoff out of all possible actions - is taken in state s
$\tilde{J}(\cdot)$	An order's realized payoff

Table B1. Summary of the main results of our study. This figure shows a summary of the main results of our study comparing the impacts on trading activity, liquidity, and welfare of changes in the execution priority rules in the dark pool, adverse selection, and traders' competition.

Summary table of results				
		Lit market	Dark pool	Lit-dark system (whole market)
		Size execution priority rule for dark orders		
Execution priority rules	Trading activity	↓	↑	-
	Liquidity	↓	↑	↑
	Welfare	↓	↑	↑
		Time execution priority rule for dark orders		
	Trading activity	↑	↓	-
	Liquidity	↑	↓	↓
	Welfare	↑	↓	↓
		Asset volatility ↑ (adverse selection ↑)		
Adverse selection	Trading activity	↑↓	↑↓	-
	Liquidity	↓	↓	↓
	Welfare	↑↓	↑↓	↓
		One market: the lit market (traders' competition ↑)		
Traders' competition	Trading activity	↑	-	-
	Liquidity	↑	-	-
	Welfare	↑	-	-
		Two markets: the lit and dark markets (traders' competition ↓)		
	Trading activity	↓	↑	-
	Liquidity	↓	↑	↓
	Welfare	↓	↑	↓

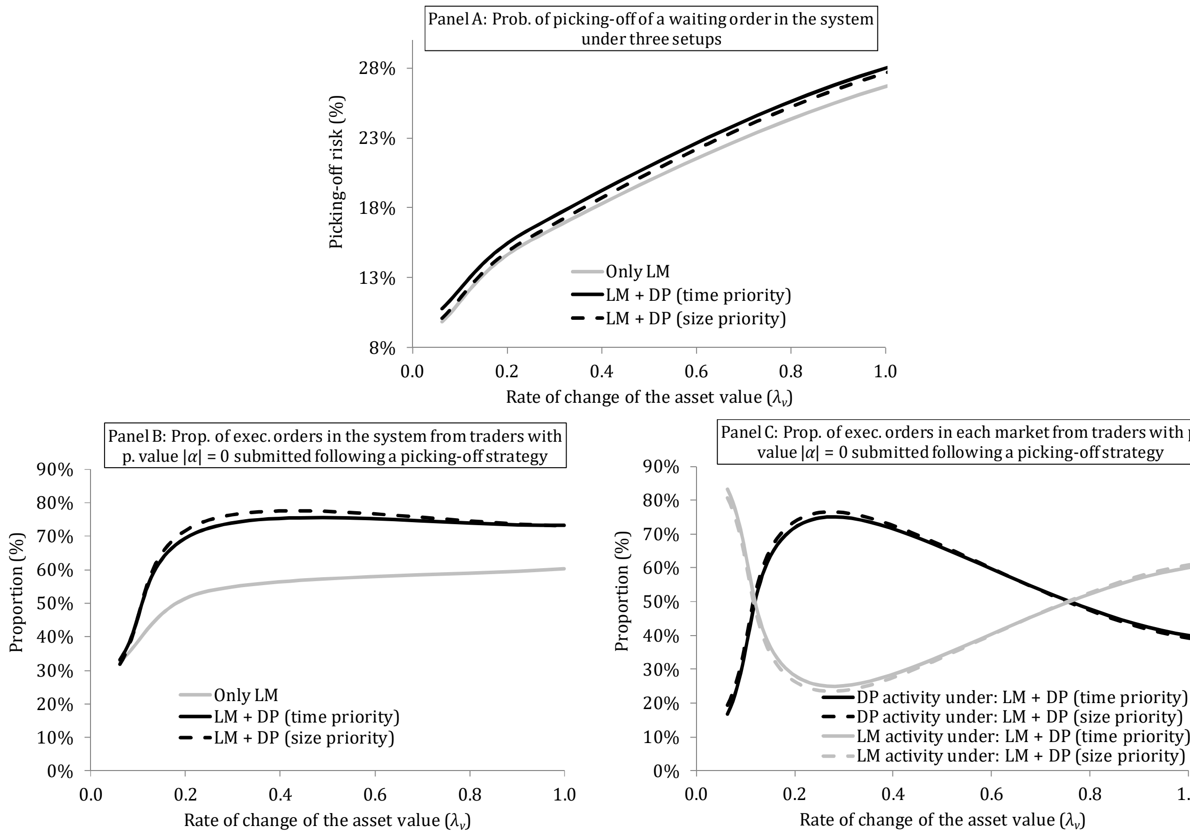

Figure C1. Picking-off risk and percentage of orders from speculators that follow a picking-off strategy as a function of the asset volatility. In this figure, Panel A reports the picking-off risk of all waiting orders (i.e., unexecuted dark orders and unexecuted limit orders) as a function of the asset volatility. Panel B presents the proportion of executed orders in the whole system (i.e., in the lit market, LM, and the dark pool, DP) from speculators (i.e., traders with private value equal to zero) that were submitted by following a picking-off strategy. Panel C shows the proportion of executed orders from speculators in *each* market that were submitted by following a picking-off strategy. The results are presented under different market setups, which were described in Figure 2. The volatility of the asset is proxied by the rate of change of the asset value, λ_V.

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