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Discounted Stochastic Games, the *3M* Property and Stationary Markov Perfect Equilibria

Jing Fu
Frank Page

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Abstract
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Keywords: *m*-tuples of Caratheodory functions, upper Caratheodory correspondences, the *3M* property, continuum valued upper Caratheodory sub-correspondences, weak star upper semicontinuous measurable selection valued correspondences, approximate Caratheodory selections, discounted stochastic games, stationary Markov perfect equilibria.

JEL Classification: C7

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Jing Fu, Fukuoka Institute of Technology and Systemic Risk Centre, London School of Economics

Frank Page, Systemic Risk Centre, London School of Economics

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Discounted Stochastic Games, the $3M$ Property, and Stationary Markov Perfect Equilibria

In memoriam

Nancy C. Page

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Jing Fu[1]
Department of System Management
Fukuoka Institute of Technology
3-30-1 Wajiro-higashi, Higashi-ku, Fukuoka, 811-0295
JAPAN
j.fu@fit.ac.jp

Frank Page[2]
Systemic Risk Centre
London School of Economics and Political Science
London WC2A 2AE
UK
fpage.supernetworks@gmail.com

March 25, 2022[3]

[1]Research Associate, Systemic Risk Centre, London School of Economics and Political Science, London WC2A 2AE, UK.

[2]Visiting Professor and Co-Investigator, Systemic Risk Centre, London School of Economics and Political Science, London WC2A 2AE, UK. Permanent address: Indiana University, Bloomington, IN 47405, USA

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Abstract

We show that all discounted stochastic games $DSGs$ satisfying the usual assumptions have Nash payoff selection correspondences having fixed points. Our fixed point result is surprising because it is well known that Nash payoff selection correspondences are badly behaved, being in general neither convex valued nor closed valued in the appropriate topologies (in this case the weak star topologies). Here we show that because all $DSGs$ satisfying the usual assumptions have upper Caratheodory (uC) Nash (equilibrium) correspondences *containing* uC *Nash sub-correspondences having the* $3M$ *property* (defined here), these uC Nash sub-correspondences are continuum valued and therefore induce interval-valued uC player payoff sub-correspondences - and therefore, Caratheodory approximable uC player payoff sub-correspondences. Finally, because these uC player payoff sub-correspondences are Caratheodory approximable, their induced Nash payoff selection sub-correspondences have fixed points - implying that the $DSGs$ to which they belong have stationary Markov perfect equilibria.

Key Words: m-tuples of Caratheodory functions, upper Caratheodory correspondences, the $3M$ property, continuum valued upper Caratheodory sub-correspondences, weak star upper semicontinuous measurable selection valued correspondences, approximate Caratheodory selections, discounted stochastic games, stationary Markov perfect equilibria.

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1 Introduction

One-shot games are the key to determining whether or not a discounted stochastic game ($\mathcal{DSG}$) has stationary Markov perfect equilibria ($SMPE$). Under the usual assumptions, we know that a $\mathcal{DSG}$ has $SMPE$ if and only if the collection of one-shot games ($\mathcal{OSG}s$) belonging to the $\mathcal{DSG}$ has a Nash payoff selection correspondence having fixed points.[1] But to date, no such fixed point result has been established. This is not surprising because Nash payoff selection correspondences are, in general, neither convex valued nor closed valued in the appropriate topology - nor are they upper semicontinuous. To make matters worse, Levy and McLennan (2015) have constructed examples of $\mathcal{DSG}s$ having no $SMPE$, and therefore, examples of $\mathcal{DSG}$ having Nash payoff selection correspondences having no fixed points. *Our contribution is to establish a general $SMPE$ existence result for uncountable-compact $\mathcal{DSG}s$ satisfying the usual assumptions, with players having convex, compact action sets, by proving that all such $\mathcal{DSG}'s$ have Nash payoff selection correspondences having fixed points.*[2]

Our approach has two parts: First, we show that the measurable selection valued Nash payoff selection correspondence induced by the composition of players' m-tuple of real-valued Caratheodory payoff functions with the upper Caratheodory (uC) Nash correspondence has fixed points *if the underlying uC Nash correspondence in the composition contains a* ***continuum valued*** *uC Nash sub-correspondence.* Second, we show that under the usual assumptions specifying a discounted stochastic game, this is always the case. In particular, we show that *a $\mathcal{DSG}$ always has a uC Nash correspondence containing a continuum valued uC Nash sub-correspondence* - implying that all such $\mathcal{DSG}s$ have Nash payoff selection correspondences with fixed points - and therefore implying that all $\mathcal{DSG}s$ satisfying the usual assumptions have $SMPE$.

There are two reasons why a $\mathcal{DSG}$ has a uC Nash correspondence containing a continuum valued uC Nash sub-correspondence. First, under the usual assumptions, all $\mathcal{DSG}s$, *with players having convex, compact action sets,* have uC Nash correspondences given by the composition of a *Ky Fan correspondence (the KFC)* and the *$\mathcal{OSG}'s$ collective security mapping (the CSM).*[3] The KFC is an upper semicontinuous correspondence defined on the compact metric hyperspace of Ky Fan sets taking nonempty compact values in the $\mathcal{OSG}'s$ set of Nash equilibria. The CSM is a uC correspondence, jointly measurable in states and value functions and upper semicontinuous in value functions, taking set values given by Ky Fan sets - with the Ky Fan values taken by the CSM being determined by the $\mathcal{OSG}'s$ Nikaido-Isoda function. Second, as we show here, under the usual assumptions, the $\mathcal{OSG}'s$ Ky Fan correspondence, as well as all of its sub-correspondences have the $3M$ property (the 3 misses property, defined below). As a consequence, the KFC has minimal USCOs (i.e., minimal $KFCs$) taking *connected*, minimally essential Nash equilibrium values (essential in the sense of Fort, 1950). Thus, each of these $3M$ minimal $KFCs$ when composed with the $\mathcal{OSG}'s$ collective security mapping delivers a continuum valued uC Nash sub-correspondence. Composing any one of these continuum-valued minimal $KFCs$ with the players' m-tuple of Caratheodory payoff functions, gives us an m-tuple of *interval-valued* players' uC Nash payoff sub-correspondences, and therefore, an m-tuple of *Caratheodory approximable* players' uC Nash payoff sub-correspondences

[1] We will refer to the parameterized *collection* of m-player, state-contingent, strategic form games as an $\mathcal{OSG}$. Thus, the collection of one-shot games underlying a $\mathcal{DSG}$ is an example of an $\mathcal{OSG}$.

[2] We will often times refer to discounted stochastic games ($\mathcal{DSG}s$) with uncountable state space and compact metrizable and convex player action sets, satisfying the usual assumptions as an uncountable-compact $\mathcal{DSG}$ (and as an uncountable-finite $\mathcal{DSG}$ if players' action sets are finite - and thus a sub-class of uncountable-compact $\mathcal{DSG}$s).

[3] For example if players' action sets consist of all probability measures over a finite set of pure actions, then players' action sets are convex and compact.

(see Kucia and Nowak, 2000). As a consequence of Caratheodory approximability, we are able to give a very direct proof that the induced Nash payoff selection sub-correspondence has fixed points - and therefore, that the $\mathcal{DSG}$ to which the Nash payoff selection sub-correspondence belongs has $SMPE$.

Some observations before we present our results. First, regarding Levy and McLennan (2015). To understand how our results here are related to those of Levy and McLennan, it is useful to divide all $\mathcal{DSG}s$ into two disjoint classes: those that are approximable and those that are not. We say that a $\mathcal{DSG}$ is approximable if it has a Nash payoff selection sub-correspondence that is approximable.[4] If a $\mathcal{DSG}$ is approximable, then because it's Nash payoff selection correspondence has fixed points, it follows that the $\mathcal{DSG}$ has stationary Markov perfect equilibria (SMPE) - *and conversely*, if the $\mathcal{DSG}$ has no SMPE, then because it's Nash payoff selection correspondence has no fixed points, the $\mathcal{DSG}$'s Nash payoff selection correspondence is not approximable (i.e., the $\mathcal{DSG}$ is not approximable). Page (2015, 2016) has shown that all approximable $\mathcal{DSG}s$ have stationary Markov perfect equilibria. Here we show that all $\mathcal{DSG}s$ satisfying the usual assumptions are approximable. In the Levy-McLennan counterexamples (2015) the cause of the nonexistence problem is Nash equilibria homeomorphic to the unit circle (i.e., circular Nash equilibria) causing the $\mathcal{DSG}$s in the Levy-McLennan examples to have Nash payoff selection correspondences without fixed points. In fact, Levy-McLennan (2015) start with a static strategic form base game with circular Nash equilibria and via a sequence of delicate modifications and additions to the base game construct a $\mathcal{DSG}$ that is not approximable and is without $SMPE$ - thus showing that in the class of non-approximable $\mathcal{DSG}s$, there exists $\mathcal{DSG}$s having no stationary Markov perfect equilibria. Thus, while *not all* non-approximable $\mathcal{DSG}s$ have $SMPE$, as shown by Levy and McLennan (2015), all approximable $\mathcal{DSG}s$ do, as shown by Page (2015, 2016). Moreover, because all $\mathcal{DSG}s$ satisfying the usual assumptions have $3M$ Nash sub-correspondences, and therefore, have Caratheodory approximable Nash payoff correspondences, all such $\mathcal{DSG}s$ escape the Levy-McLennan counterexamples, and as we show here, possess stationary Markov perfect equilibria.

Thinking about the SMPE existence problem for discounted stochastic games from the perspective of approximability provides a useful way of viewing some of the recent literature on existence. We can think of this literature as a record of the search for conditions on the primitives of a $\mathcal{DSG}$, over and above the usual assumptions which guarantee that the $\mathcal{DSG}$'s underlying one-shot game has as an approximable Nash payoff selection sub-correspondence. Most significant in this regard is the work of He and Sun (2017). Using results on the conditional expectations of correspondences due to Dynkin and Evstigneev (1976, Section 4, 4.1-4.4, pp.334-336), He and Sun (2017) show that if the $\mathcal{DSG}$ has a (decomposable) coarser transition kernel, then the game's Nash payoff selection correspondence contains a convex valued sub-correspondence - implying that this sub-correspondence is approximable, and therefore, has fixed points.[5] He and Sun (2017) also show that if a $\mathcal{DSG}$ is noisy in the sense of Duggan (2012), then it has a coarser transition kernel - again implying that it has a Nash payoff selection sub-correspondence

[4] A correspondence is approximable if it has a graph about which, for any $\varepsilon > 0$, an open ε-ball can be placed containing the graph of a continuous function. If the $\mathcal{DSG}$'s has a Nash payoff sub-correspondence that is Caratheodory approximable, then the induced Nash payoff selection sub-correspondence will be approximable (Kucia and Nowak, 2000).

[5] This sub-correspondence consists of payoff selections that are conditional expectations functions. Let $\mathcal{G}$ be a sub-σ-field of B_Ω and denote by $\mu^{\mathcal{G}}(\cdot)$ a regular $\mathcal{G}$-conditional probability given sub-σ-field $\mathcal{G}$. Following Dynkin and Evstigneev, $A \in B_\Omega$ is $\mathcal{G}$-atom if $\mu(A) > 0$ and for any $B \in B_\Omega$ such that $B \subset A$

$$\mu\left\{\omega \in \Omega : 0 < \mu^{\mathcal{G}}(B)(\omega) < \mu^{\mathcal{G}}(A)(\omega)\right\} = 0.$$

If a $\mathcal{DSG}$ is without $\mathcal{G}$-atoms (if it is $\mathcal{G}$-nonatomic) then it is a $\mathcal{DSG}$ with a courser transition kernel (see Page, 2016, and He and Sun, 2017).

that is approximable.[6] Moreover, He and Sun (2017) show that the nonatomic part of the Levy-McLennan (2015) $\mathcal{DSG}$ does not have a decomposable coarser transition kernel. Our analysis here supports the slightly stronger conclusion that the Levy-McLennan $\mathcal{DSG}$ is not approximable - and this in turn implies that the Levy-McLennan $\mathcal{DSG}$ does not have a decomposable coarser transition kernel. Nowak (2003, 2007) identifies smoothness, concavity, and transition kernel decomposability conditions which together with the usual assumptions imply that the Nash payoff selection correspondence is single-valued. Variations on Nowak's (2003, 2007) work enabled Jaskiewicz and Nowak (2016, 2018) to show, under some transition kernel decomposability conditions (see assumption (v) in Jaskiewicz and Nowak, 2016), that if the $\mathcal{DSG}$ satisfies the usual assumptions and each player's strategy set consists of behavioral action-valued measurable functions of the current *and last period's* state, then the $\mathcal{DSG}$ possesses stationary almost Markov perfect equilibria (SAMPE) - i.e., a stationary Nash equilibrium in behavioral action-valued measurable strategies (functions) of the current and last period's state. Under stronger assumptions about players' strategy sets, Barelli and Duggan (2014) show that if each player's strategy set consists of behavioral action-valued measurable functions of the current state, last period's state, *and* players' last period's action profile, then any $\mathcal{DSG}$ satisfying the usual assumptions has stationary semi-Markov perfect equilibria - i.e., a stationary Nash equilibrium in behavioral action-valued measurable functions of the current state, last period's state, and players' last period action profile. Here, *we show that all $\mathcal{DSG}$s satisfying the usual assumptions - with no additional assumptions - automatically have $3M$ minimal uC Nash sub-correspondences which induce Caratheodory approximable player uC Nash payoff sub-correspondences - which in turn induce Nash payoff selection sub-correspondences having fixed points.*

Now to the details.

Part I

One-Shot Games and Their Nash Correspondences

Our focus here will be the one-shot games,

$$\left\{(\Phi_d(\omega), u_d(\omega, v_d, \cdot))_{d\in D}\right\}_{(\omega,v)}$$

($\mathcal{OSG}s$) underlying discounted stochastic games.

[6] A $\mathcal{DSG}$ is noisy in the sense of Duggan if the state space is given by $\Omega := T \times S$ with typical element $t := (t, s)$, where both T and S are complete separable metric spaces with metrics ρ_T and ρ_S, equipped with the Borel σ-fields B_T and B_S. The law of motion is given by

$$(\underbrace{(t,s)}_{\omega}, a) \longrightarrow q(\cdot|\underbrace{(t,s)}_{\omega}, a)$$

where

$$q(d(t',s')|(t,s),a) := \varepsilon(ds'|t')\delta(dt'|(t,s),a),$$

or

$$q(d(t',s')|\omega,a) := \varepsilon(ds'|t')\delta(dt'|\omega,a),$$

where $\omega = (t, s)$ denotes the current state and $\omega' = (t', s')$ denotes the coming state - and depending on the regular state t' chosen by the probability measure, $\delta(dt'|\omega, a)$, in current state $\omega = (t, s)$ given action profile $a \in \Phi(\omega)$, the noisy state s' will be chosen according to the probability measure, $\varepsilon(ds'|t')$.

2 Primitives and Assumptions

To begin, an m-player, non-zero sum, discounted stochastic game, $\mathcal{DSG}$, is given by the following primitives:

$$\mathcal{DSG} := \left\{ \underbrace{(\Omega, B_\Omega, \mu)}_{\text{probability space}}, \underbrace{\left\{ (X_d, \Phi_d(\omega), \beta_d, u_d(\omega, v_d, \cdot))_{d \in D} \right\}_{(\omega, v)}}_{\text{the one-shot game}}, \underbrace{q(\cdot|\omega, \cdot)}_{\text{law of motion}} \right\}, \tag{1}$$

where Ω is the state space, B_Ω is the σ-field of events, and μ is a probability measure. For each player d, X_d is the set of all possible actions available to player d, while $\Phi_d(\omega)$ is the feasible set of actions available to player d in state ω. Finally, $\beta_d \in (0,1)$ is player $d's$ discount rate and $u_d(\omega, v_d, \cdot)$ is player $d's$ payoff function in state ω given valuations (or prices) v_d, and $q(\cdot|\omega, \cdot)$ is the law of motion in state ω. If players holding value function profile $v = (v_1 \ldots, v_m)$ choose feasible action profile,

$$x = (x_1, \ldots, x_m) \in \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) = \Phi(\omega),$$

in state ω then the next state ω' is chosen in accordance with probability measure $q(\cdot|\omega, x) \in \Delta(\Omega)$ and player $d's$ expected payoff is given by

$$u_d(\omega, v_d, x) := (1 - \beta_d) r_d(\omega, x) + \beta_d \int_\Omega v_d(\omega') q(\omega'|\omega, x). \tag{2}$$

We will denote by, $\mathcal{G}_{(\omega,v)}$, the m-player one-shot game in state ω underlying the $\mathcal{DSG}$ when players hold valuations $v := (v_1, \ldots, v_m)$.

Formally, the $\mathcal{DSG}$s we will consider here satisfy the following list of assumptions (a list we think of as the usual assumptions), labeled $[\mathcal{OSG}]$(1)-(11):

(1) $D =$ *the set of players, consisting of* m *players indexed by* $d = 1, 2, \ldots, m$ *and each having discount rate given by* $\beta_d \in (0,1)$.
(2) (Ω, B_Ω, μ), *the state space where* Ω *is a complete separable metric spaces with metric* ρ_Ω, *equipped with the Borel* σ*-field,* B_Ω, *upon which is defined a probability measure,* μ.
(3) $Y := Y_1 \times \cdots \times Y_m$, *the space of players' payoff profiles,* $U := (U_1, \ldots, U_m)$, *such that for each player* d, $Y_d := [-M, M]$, $M > 0$, *and is equipped with the absolute value metric,* $\rho_{Y_d}(U_d, U'_d) := |U_d - U'_d|$ *and* Y *is equipped with the sum metric,* $\rho_Y := \sum_d \rho_{Y_d}$.
(4) $X := X_1 \times \cdots \times X_m := \prod_d X_d \subset E := \prod_d E_d$, *the space of player action profiles,* $x := (x_1, \ldots, x_m)$, *such that for each player* d, X_d *is a* ***convex****, compact metrizable subset of a locally convex Hausdorff topological vector space* E_d *and is equipped with a metric,* ρ_{X_d}, *compatible with the locally convex topology inherited from* E_d, *and* X *is equipped with the sum metric,* $\rho_X := \sum_d \rho_{X_d}$.
(5) $\omega \longrightarrow \Phi_d(\omega)$, *is player* $d's$ *measurable action constraint correspondence, defined on* Ω *taking nonempty,* ***convex****,* ρ_{X_d}*-closed (and hence compact) values in* X_d.
(6) $\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$, *players' measurable action profile constraint correspondence, defined on* Ω *taking nonempty, convex, and* ρ_X*-closed (hence compact) values in* X.
(7) $\mathcal{L}^\infty_{Y_d}$, *the Banach space of all* μ*-equivalence classes of measurable (value) functions,* $v_d(\cdot)$, *defined on* Ω *with values in* Y_d *a.e.* $[\mu]$, *equipped with metric* $\rho_{w^*_d}$ *compatible with the weak star topology inherited from* $\mathcal{L}^\infty_R$.
(8) $\mathcal{L}^\infty_Y := \mathcal{L}^\infty_{Y_1} \times \cdots \times \mathcal{L}^\infty_{Y_m} \subset \mathcal{L}^\infty_{R^m}$, *the Banach space of all* μ*-equivalence classes of measurable (value) function profiles,* $v(\cdot) := (v_1(\cdot), \ldots, v_m(\cdot))$, *defined on* Ω *with values*

in Y a.e. $[\mu]$*, equipped with the sum metric* $\rho_{w^*} := \sum_d \rho_{w_d^*}$ *compatible with the weak star product topology inherited from* $\mathcal{L}^\infty_{R^m}$.
(9) $\mathcal{S}^\infty(\Phi_d(\cdot))$*, the set of all* μ*-equivalence classes of measurable functions (selections),* $x_d(\cdot) \in \mathcal{L}^\infty_{X_d}$*, defined on* Ω *such that in* $x_d(\omega) \in \Phi_d(\omega)$ *a.e.* $[\mu]$*, and*

$$\mathcal{S}^\infty(\Phi_d(\cdot)) = \mathcal{S}^\infty(\Phi_1(\cdot)) \times \cdots \times \mathcal{S}^\infty(\Phi_m(\cdot)) \tag{3}$$

the set of all μ*-equivalence classes of measurable profiles (selection profiles),* $x(\cdot) = (x_1(\cdot), \ldots, x_m(\cdot)) \in \mathcal{L}^\infty_X$*, defined on* Ω *such that*

$$x(\omega) \in \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) \text{ a.e. } [\mu].$$

(10) $r_d(\cdot,\cdot) : \Omega \times X \longrightarrow Y_d$ *is player* d'*s affine, Caratheodory stage payoff function (i.e., for each* ω*,* $r_d(\omega, \cdot)$ *is* ρ_X*-continuous on* X*, for each* x*,* $r_d(\cdot, x)$ *is* (B_Ω, B_{Y_d})*-measurable on* Ω*, and for each* (ω, x_{-d}) *and each* x_d^0 *and* x_d^1 *in* X_d*,*

$$r_d(\omega, \gamma x_d^0 + (1-\gamma)x_d^1, x_{-d}) = \gamma r_d(\omega, x_d^0, x_{-d}) + (1-\gamma) r_d(\omega, x_d^1, x_{-d}), \tag{4}$$

for all $\gamma \in [0,1]$.
(11) $q(\cdot|\cdot,\cdot) : \Omega \times X \longrightarrow \Delta(\Omega)$ *is the law of motion defined on* $\Omega \times X$ *taking values in the space of probability measures on* Ω*, having the following properties: (i) each probability measure,* $q(\cdot|\omega, x)$*, in the collection*

$$Q(\Omega \times X) := \{q(\cdot|\omega, x) : (\omega, x) \in \Omega \times X\} \tag{5}$$

is absolutely continuous with respect to μ *(denoted* $Q(\Omega \times X) << \mu$*), (ii) for each* $E \in B_\Omega$*,* $q(E|\cdot,\cdot)$ *is measurable on* $\Omega \times X$*, and (iii) the collection of probability density functions,*

$$H_\mu := \{h(\cdot|\omega, x) : (\omega, x) \in \Omega \times X\}, \tag{6}$$

of $q(\cdot|\omega, x)$ *with respect to* μ *is such that for each state* ω*, the function*

$$(x_d, x_{-d}) \longrightarrow h(\omega'|\omega, x_d, x_{-d}) \tag{7}$$

is continuous in x *and affine in* x_d *a.e.* $[\mu]$ *in* ω'.

A one-shot game then is a collection of strategic form games,

$$\mathcal{G}(\Omega \times \mathcal{L}^\infty_Y) := \left\{\mathcal{G}_{(\omega,v)} : (\omega, v) \in \Omega \times \mathcal{L}^\infty_Y\right\}, \tag{8}$$

where each (ω, v)-game in the collection is given by

$$\mathcal{G}_{(\omega,v)} := \left\{ \underbrace{\Phi_d(\omega)}_{\text{feasible actions}}, \underbrace{u_d(\omega, v_d, (\cdot,\cdot))}_{\text{payoff function}} \right\}_{d \in D}, \tag{9}$$

Under assumptions $[\mathcal{OSG}]$, in a (ω, v)-game player $d's$ payoff function, given by

$$x \longrightarrow u_d(\omega, v_d, x) := (1-\beta_d) r_d(\omega, x) + \beta_d \int_\Omega v_d(\omega') q(\omega'|\omega, x), \tag{10}$$

is jointly continuous in action profiles, $x = (x_1, \ldots, x_m)$, and for any sequence of value function-action profiles pairs, $\{(v^n, x^n)\}_n$, if $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $x^n \underset{\rho_X}{\longrightarrow} x^*$ then for each ω,

$u(\omega, v^n, x^n) \underset{\rho_Y}{\longrightarrow} u(\omega, v^*, x^*)$ (i.e., $u(\omega, \cdot, \cdot)$ is jointly continuous in (v, x)). Thus, the Y-valued players' payoff function, $u(\cdot, \cdot, \cdot)$, is a Caratheodory function: $\rho_{w^* \times X}$-continuous in (v, x) for each ω, and (B_Ω, B_Y)-measurable in ω on Ω for each (v, x). Moreover, for each ω, $u(\omega, \cdot, \cdot)$ is uniformly continuous on the compact product space, $\mathcal{L}_Y^\infty \times X$. As a consequence, for each ω, the v-parameterized collection of m-tuples of integrands given by $\{u(\omega, v, \cdot) : v \in \mathcal{L}_Y^\infty\}$ is uniformly equicontinuous.[7] Likewise, for the x-parameterized collection of m-tuples of integrands given by $\{u(\omega, \cdot, x) : x \in X\}$.

3 One-Shot Nash Correspondences

Any $\mathcal{OSG}$ satisfying assumptions $[\mathcal{OSG}]$ above, has a Nash correspondence given by an *upper Caratheodory* (uC) correspondence,

$$\mathcal{N}(\cdot, \cdot) : \Omega \times \mathcal{L}_Y^\infty \longrightarrow P_f(X), \tag{11}$$

jointly measurable in (ω, v) and upper semicontinuous in v for each ω (see 7.3.1 on uC correspondences in the Appendices). We call the collection of upper semicontinuous Nash correspondences, $\{\mathcal{N}(\omega, \cdot) : \omega \in \Omega\}$, the USCO part (Hola and Holy, 2015), and $\{\mathcal{N}(\cdot, v) : v \in \mathcal{L}_Y^\infty\}$ the measurable part of the uC Nash correspondence $\mathcal{N}(\cdot, \cdot)$. Denote by $\mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}P_f(X)}$ the collection of all such uC correspondences.

Next consider the Y-valued Caratheodory players' payoff function,

$$(\omega, v, x) \longrightarrow u(\omega, v, x) := (u_1(\omega, v, x), \dots, u_m(\omega, v, x)) \in Y, \tag{12}$$

measurable in ω and jointly continuous in (v, x), and let

$$\mathcal{P}(\cdot, \cdot) : \Omega \times \mathcal{L}_Y^\infty \longrightarrow P_f(Y), \tag{13}$$

denote the composition of uC Nash correspondence $\mathcal{N}(\cdot, \cdot)$ with the m-tuple of Caratheodory players' payoff functions, $(u_1(\cdot, \cdot, \cdot), \dots, u_m(\cdot, \cdot, \cdot))$. For each $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$ let

$$\mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v)). \tag{14}$$

The correspondence, $\mathcal{P}(\cdot, \cdot)$, is the $\mathcal{OSG}'s$ uC Nash payoff correspondence - a uC composition correspondence.

The uC Nash payoff correspondence, $\mathcal{P}(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}P_f(Y)}$, induces a measurable selection valued Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)) := \mathcal{S}^\infty(\mathcal{P}_v), \tag{15}$$

where for each $v \in \mathcal{L}_Y^\infty$, $\mathcal{S}^\infty(\mathcal{P}_v)$ is the collection of μ-equivalence classes of functions u in $\mathcal{L}_Y^\infty$ such that $u(\omega) \in \mathcal{P}(\omega, v)$ a.e. $[\mu]$. We will show that for all such Nash payoff selection

[7] The collection,

$$\{u(\omega, v, \cdot) : v \in \mathcal{L}_Y^\infty\},$$

is uniformly equicontinuous if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any a and a' in $\Phi(\omega)$ with $\rho_A(a, a') < \delta$,

$$\rho_Y(u(\omega, v, a), u(\omega, v, a')) < \varepsilon,$$

for all $v \in \mathcal{L}_Y^\infty$. By the uniform continuity of $u(\omega, \cdot, \cdot)$ on the compact metric space, $\mathcal{L}_Y^\infty \times A$, for each ω, the collection of functions $\{u(\omega, \cdot, a) : a \in A\}$ is also uniformly equicontinuous - so that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for $\rho_{w^*}(v, v') < \delta$,

$$\rho_Y(u(\omega, v, a), u(\omega, v', a) < \varepsilon$$

for all $a \in A$.

correspondences,

$$\left.\begin{array}{r} v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}(\cdot,v)) = \mathcal{S}^{\infty}(u(\cdot,v,\mathcal{N}(\cdot,v))) \\ = (\mathcal{S}^{\infty}(u_1(\cdot,v,\mathcal{N}(\cdot,v)),\ldots,\mathcal{S}^{\infty}(u_m(\cdot,v,\mathcal{N}(\cdot,v))), \end{array}\right\} \tag{16}$$

if the underlying uC Nash correspondence, $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^{\infty}_{Y}\text{-}P_f(X)}$, contains a *continuum valued Nash sub-correspondence*, $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^{\infty}_{Y}\text{-}P_f(X)}$ (i.e., a uC Nash correspondence $\eta(\cdot,\cdot)$ taking continuum values such that $Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot)$ for all ω) then its induced uC Nash payoff sub-correspondence, $(\omega,v) \longrightarrow u(\omega,v,\eta(\omega,v))$, induces a selection sub-correspondence,

$$v \longrightarrow \mathcal{S}^{\infty}(p(\cdot,v)) := \mathcal{S}^{\infty}(u(\cdot,v,\eta(\cdot,v))), \tag{17}$$

that is weak star upper semicontinuous and has fixed points. Thus while the original Nash selection correspondence, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v)$, may fail to be weak star upper semicontinuous, the selection sub-correspondence induced by a continuum valued uC Nash sub-correspondence will be weak star upper semicontinuous, and more importantly, will have fixed points.

Part II

All One-Shot Games Have Continuum Valued Nash Sub-Correspondences

Our objective is to show that all $\mathcal{OSG}s$ satisfying assumptions [$\mathcal{OSG}$] have Nash correspondences containing continuum valued Nash sub-correspondences. A key ingredient in our approach is a novel decomposition of the Nash correspondence. In particular, we show that all $\mathcal{OSG}s$ have Nash correspondences that can be written as the composition of two correspondences,

$$(\omega,v) \longrightarrow \mathcal{N}(\omega,v) = N \circ K(\omega,v),$$

a Ky Fan correspondence (a KFC),

$$N(\cdot): Ky\ Fan\ Sets \longrightarrow Sets\ of\ Nash\ Equilibria,$$

and the game's *collective security mapping (the CSM)*,

$$K(\cdot,\cdot):\ \textit{State-Parameter Pairs} \longrightarrow Ky\ Fan\ Sets.$$

In Part II, our objective is to show that the Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, belonging to any $\mathcal{OSG}$ satisfying assumptions [$\mathcal{OSG}$] contains a uC Nash sub-correspondence, $\eta(\cdot,\cdot)$, meaning that $Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot)$ for all ω. such that $\eta(\omega,v)$ is a continuum for each $(\omega,v) \in \Omega \times \mathcal{L}^{\infty}_{Y}$. We will show that just such a uC Nash sub-correspondence is given by, $(\omega,v) \longrightarrow n(K(\omega,v))$, where $(\omega,v) \longrightarrow K(\omega,v)$ is the $\mathcal{OSG}'s$ collective security mapping and $n(\cdot)$ is any minimal KFC (defined on the compact metric hyperspace of Ky Fan sets and taking Nash equilibria values) belonging to the KFC, $N(\cdot)$. Then in Part III, we will show that, as a consequence of the fact that the $\mathcal{OSG}'s$ Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, has a continuum valued sub-correspondence, $n(K(\cdot,\cdot))$, the $\mathcal{OSG}'s$ induced uC Nash payoff sub-correspondence, $p(\cdot,\cdot)$, given by

$$(\omega,v) \longrightarrow p(\omega,v) := (u_1(\omega,v_1,n(K(\omega,v))),\ldots,u_m(\omega,v_m,n(K(\omega,v)))),$$

is such that there exists $v^* \in \mathcal{L}_Y^\infty$, such that

$$v^*(\omega) \in p(\omega, v^*) \subset \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu],$$

implying that

$$v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*}).$$

Thus, the $\mathcal{OSG}'s$ Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$, has fixed points.

4 Ky Fan Sets

Throughout our discussion of Ky Fan sets, in order to simplify the notation, we will assume that the state, $\omega \in \Omega$, is fixed and we will let $\Phi(\omega) = X$. Later, when we discuss the collective security mapping, we release these assumptions.

Definition 1 *(Ky Fan Sets)*
Let E be a subset of $X \times X$ satisfying the following three properties:

(a) For each $x \in X$, $(x, x) \in E$.
(b) For each $y \in X$, $\{x \in X : (y, x) \in E\}$ is closed.
(c) For each $x \in X$, $\{y \in X : (y, x) \notin E\}$ is convex or empty.

Any such subset E satisfying properties (a), (b), and (c) is called a Ky Fan set.
The collection of all Ky Fan sets in $X \times X$ is denoted by

$$\mathbb{S} := \{E \subset X \times X : E \text{ has properties (a), (b), and (c)}\}.$$

Given $E \in \mathbb{S}$, let

$$\left.\begin{array}{c} E(y) := \{x \in X : (y, x) \in E\} \\ \text{and} \\ E(x) := \{y \in X : (y, x) \in E\}. \end{array}\right\} \tag{18}$$

The section of E at y, $E(y)$, is the set of action profiles, x, that deter noncooperative defection y, while the section of E at x, $E(x)$, is the set of noncooperative defections, y, deterred by actions x. Note that the deterrence mapping, $y \longrightarrow E(y)$, is an USCO defined on X taking values in the hyperspace, $P_f(X)$, of nonempty, closed subsets of X.

We will equip the set of Ky Fan sets with the Hausdorff metric, $h_{X \times X}$ ($:= h_{\rho_{X \times X}}$ where $\rho_{X \times X} := \rho_X + \rho_X$).

Theorem 1 *(The hyperspace of Ky Fan sets is a compact metric space with the Hausdorff metric)*
Let $\mathcal{G}(\Omega \times \mathcal{L}_Y^\infty)$ be a one-shot game satisfying assumptions [$\mathcal{OSG}$] with Ky Fan sets $\mathbb{S}$. Then $\mathbb{S}$ is a $h_{X \times X}$-closed subset of $P_f(X \times X)$.

Proof: Let $\{E^n\}_n \subset \mathbb{S}$ be a sequence of Ky Fan sets such that $h_{X \times X}(E^n, E^0) \longrightarrow 0$. We must show that $E^0 \in \mathbb{S}$. We have for each $x \in X$ that $(x, x) \in E^n$ for all n. Thus, $h_{X \times X}(E^n, E^0) \longrightarrow 0$ implies that $(x, x) \in E^0$ (i.e., (a) holds). Because $h_{X \times X}(E^n, E^0) \longrightarrow 0$, implies that all converging sequences $\{(y, x^n)\}_n$ with $(y, x^n) \in E^n$ for all n must have limit (y, x^0) in E^0, we must conclude that (b) holds. To see that (c) holds - i.e., that for all $x \in X$, $\{y \in X : (y, x) \notin E^0\}$ is convex and possibly empty - suppose not. Then for some x^0 in X, there exists y^1 and y^2 in X, such that for some $y^0 = \lambda^0 y^1 + (1 - \lambda^0) y^2 \in X$, $\lambda^0 \in (0, 1)$, x^0 *deters* y^0 *but does not deter* y^1 or y^2. Therefore, we have $x^0 \notin E^0(y^1) \Longrightarrow (y^1, x^0) \notin E^0$, $x^0 \notin E^0(y^2) \Longrightarrow (y^2, x^0) \notin E^0$, but

$x^0 \in E^0(y^0) \Longrightarrow (y^0, x^0) \in E^0$. But now because $h_{X\times X}(E^n, E^0) \longrightarrow 0$, and because $(y^i, x^0) \notin E^0$, for $\delta > 0$ sufficiently small, we have for some N_δ sufficiently large that for any y' on the line segment between y^1 and y^2 contained in each of the convex sets, $\{y \in X : (y, x^0) \notin E^n\}$,

$$[B_{\rho_X}(\delta, y') \times \{x^0\}] \cap E^n = \varnothing \text{ for all } n \geq N_\delta,$$

contradicting the assumption that for $y' = y^0$, $(y^0, x^0) \in E^0$. **Q.E.D.**

5 Ky Fan Correspondences

For Ky Fan sets, $E \in \mathbb{S}$, the Ky Fan correspondence is given by,

$$E \longrightarrow N(E) := \cap_{y \in X} \{x \in X : (y, x) \in E\}, \tag{19}$$

It follows from Lemma 4 in Ky Fan (1961) that for any $E \in \mathbb{S}$, $N(E)$ is nonempty.

Theorem 2 *(The KFC is an USCO)*
Let $\mathcal{G}(\Omega \times \mathcal{L}^\infty_Y)$ be a one-shot game satisfying assumptions [$\mathcal{OSG}$] with KFC, $N(\cdot)$. Then $N(\cdot)$ is an USCO, that is,

$$N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}P_f(X)} := \mathcal{U}(\mathbb{S}, P_f(X)). \tag{20}$$

Proof: By Ky Fan (1961) $N(E)$ is nonempty for all $E \in \mathbb{S}$ and it is easy to see that $N(E)$ is compact for all $E \in \mathbb{S}$. To see that $N(\cdot)$ is upper semicontinuous consider a sequence $\{(E^n, x^n)\}_n \subset GrN(\cdot)$ where $\{E^n\}_n \subset \mathbb{S}$ and WLOG assume that $E^n \underset{h_{X\times X}}{\longrightarrow} E^0$, and $x^n \underset{\rho_X}{\longrightarrow} x^0$. By (19) we have for each n, $(y, x^n) \in E^n$ *for all* $y \in X$. If we can show that $(y, x^0) \in E^0$ for all $y \in X$, the proof will be complete. Suppose there is some $y^0 \in X$ such that $(y^0, x^0) \notin E^0$. Let $\{y^n\}_n$ be any sequence in X converging to y^0. We have $(y^n, x^n) \in E^n$ for all n with $(y^n, x^n) \longrightarrow (y^0, x^0)$. Because $E^n \underset{h_{X\times X}}{\longrightarrow} E^0$, we must conclude that $(y^0, x^0) \in E^0$ - a contradiction. Thus, $(E^0, x^0) \in GrN(\cdot)$. By compactness, the fact that $GrN(\cdot)$ is closed implies that $N(\cdot)$ is upper semicontinuous - with nonempty, compact values. **Q.E.D.**

5.1 Essential Sets and the 3M Property in the Hyperspace of Ky Fan Sets

We begin by considering the notions of essential and minimally essential sets for KFCs.

Definition 2 *(Essential Sets and Minimal Essential Sets)*
Let $N(\cdot)$ be a KFC and let $E^0 \in \mathbb{S}$ be a given Ky Fan set.
(1) A nonempty, closed subset $e(E^0)$ of $N(E^0)$ is said to be essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$ if for any $\varepsilon > 0$ there exists $\delta^\varepsilon > 0$ such that for all $E \in B_{h_{X\times X}}(\delta^\varepsilon, E^0) \subset \mathbb{S}$,

$$N(E) \cap B_{\rho_X}(\varepsilon, e(E^0)) \neq \emptyset. \tag{21}$$

We will denote by $\mathcal{E}[N(E^0)]$ the collection of all nonempty, closed subsets of $N(E^0)$ essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$.
(2) A nonempty closed subset $m(E^0)$ of $N(E^0)$ is said to be minimally essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$ if (i) $m(E^0) \in \mathcal{E}[N(E^0)]$ and if (ii) $m(E^0)$ is a minimal element of

$\mathcal{E}[N(E^0)]$ ordered by set inclusion (i.e., if $e(E^0) \in \mathcal{E}[N(E^0)]$ and $e(E^0) \subseteq m(E^0)$ then $e(E^0) = m(E^0)$). We will denote by $\mathcal{E}^*[N(E^0)]$ the collection of all nonempty, closed subsets of $N(E^0)$ minimally essential for $N(\cdot)$ at $E^0 \in \mathbb{S}$.

Note that for any $E \in \mathbb{S}$, if B is a proper subset of $m(E)$, then $B \notin \mathcal{E}[N(E)]$. The $3M$ property for KFCs is defined as follows:

Definition 3 *(The 3M Property)*
Let $\mathcal{G}(\Omega \times \mathcal{L}^\infty_Y)$ be a one-shot game satisfying assumptions [$\mathcal{OSG}$], and let, $N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}P_f(X)}$, be the KFC.
We say that $N(\cdot)$ is $3M$ at $E^0 \in \mathbb{S}$ if, given any $\delta > 0$ and given any pair of nonempty, disjoint, closed sets, F^1 and F^2 in X, there exists Ky Fan sets E^1 and E^2 in $B_{h_{X\times X}}(\delta, E^0)$ such that

$$N(E^1) \cap F^1 = \emptyset \text{ and } N(E^2) \cap F^2 = \emptyset,$$

then there exists a third Ky Fan set, E^3, in the larger open ball,$B_{h_{X\times X}}(3\delta, E^0)$, such that

$$N(E^3) \cap [F^1 \cup F^2] = \emptyset,$$

We say that the KFC, $N(\cdot)$, is $3M$ if $N(\cdot)$ is $3M$ at E for all $E \in \mathbb{S}$. We will denote by $\mathcal{U}^{3M}_{\mathbb{S}\text{-}P_f(X)}$ the collection of all $3M$ KFCs.

5.2 All $KFCs$ are $3M$

Our next Theorem, the $3M$ Theorem, establishes a surprising fact: under assumptions [$\mathcal{OSG}$], *all* USCOs defined on the hyperspace of Ky Fan sets (i.e., all KFCs) are $3M$ - i.e., $\mathcal{U}_{\mathbb{S}\text{-}P_f(X)} = \mathcal{U}^{3M}_{\mathbb{S}\text{-}P_f(X)}$.

Theorem 3 *(The $3M$ Theorem)*
Let $\mathcal{G}(\Omega \times \mathcal{L}^\infty_Y)$ be a one-shot game satisfying assumptions [$\mathcal{OSG}$]. Then $\mathcal{U}_{\mathbb{S}\text{-}P_f(X)} = \mathcal{U}^{3M}_{\mathbb{S}\text{-}P_f(X)}$.

Proof: We have $\mathcal{U}^{3M}_{\mathbb{S}\text{-}P_f(X)} \subset \mathcal{U}_{\mathbb{S}\text{-}P_f(X)}$. Suppose $N(\cdot) \in \mathcal{U}_{\mathbb{S}\text{-}P_f(X)}$ does not have the $3M$ property at $E^0 \in \mathbb{S}$. Then for some $\delta^0 > 0$ and some pair of closed disjoint sets F^1 and F^2 in X, the open ball, $B_{h_{X\times X}}(\delta^0, E^0) \subset \mathbb{S}$ contains two Ky Fan sets, E^1 and E^2, such that

$$N(E^1) \cap F^1 = \emptyset \text{ and } N(E^2) \cap F^2 = \emptyset, \tag{22}$$

but such that for all $E \in B_{h_{X\times X}}(3\delta^0, E^0)$, $N(E) \cap [F^1 \cup F^2] \neq \emptyset$. We will show that this leads to a contradiction by exhibiting a Ky Fan set, $E^* \in B_{h_{X\times X}}(3\delta^0, E^0)$ such that if $N(E^*) \cap [F^1 \cup F^2] \neq \emptyset$, then $N(E^i) \cap F^i \neq \emptyset$ for $i = 1$ or 2 violating (22).

First, given that the KFC $N(\cdot)$ is an USCO, under [$\mathcal{OSG}$] there are disjoint open sets U^i such that $F^i \subset U^i$ and $N(E^i) \cap U^i = \emptyset$, $i = 1, 2$, and moreover, such that

$$\left.\begin{array}{c} N(E^*) \cap [F^1 \cup F^2] \neq \emptyset \text{ for all } E^* \in B_{h_{X\times X}}(3\delta^0, E^0) \cap \mathbb{S}, \\ \text{implies that} \\ N(E^*) \cap [U^1 \cup U^2] \neq \emptyset \text{ for all } E^* \in B_{h_{X\times X}}(3\delta^0, E^0) \cap \mathbb{S}. \end{array}\right\} \tag{23}$$

We will show that (23) leads to a contradiction by constructing a Ky Fan set, $E^* \in \mathbb{S}$ with $E^* \in B_{h_{X\times X}}(3\delta^0, E^0)$ such that

$$N(E^*) \cap [U^1 \cup U^2] \neq \emptyset \ (*),$$

implying that $N(E^i) \cap U^i \neq \emptyset$ for some $i = 1$ and/or 2. Our candidate for such a set is given by

$$E^* := [E^1 \cap (X \times U^2)^c] \cup [E^2 \cap (X \times U^1)^c] \tag{24}$$

where

$$(X \times U^i)^c := \left\{(y, x) \in X \times X : x \notin U^i\right\}.$$

We must show that,

$$\begin{gathered}\text{(a) } E^* \in \mathbb{S},\\ \text{(b) } E^* \in B_{h_{X\times X}}(3\delta^0, E^0),\\ \text{and}\\ \text{(c) } N(E^*) \cap \left[U^1 \cup U^2\right] \neq \emptyset \Rightarrow N(E^i) \cap U^i \neq \emptyset \text{ for some } i = 1, 2.\end{gathered}$$

(a) $E^* \in \mathbb{S}$: It is easy to see that $E^* \in P_f(X \times X)$. Moreover, because $E^i \in \mathbb{S}$ $i = 1, 2$, it is easy to see that Definition 1(a) holds for E^*. Also, it is easy to see that Definition 1(b) holds for E^*. Thus, $(y, y) \in E^*$ for all $y \in X$, and $E^*(y)$ is closed for all $y \in X$.

It remains to show that for all $x \in X$, $\{y \in X : (y, x) \notin E^*\}$ is convex or empty. Let $x \in U^1$, then because U^1 and U^2 are disjoint,

$$\{y \in X : (y, x) \notin E^*\} = \left\{y \in X : (y, x) \notin E^1\right\},$$

a convex or empty set because $E^1 \in \mathbb{S}$.

Let $x \in U^2$, then because U^1 and U^2 are disjoint,

$$\{y \in X : (y, x) \notin E^*\} = \left\{y \in X : (y, x) \notin E^2\right\},$$

a convex or empty set because $E^2 \in \mathbb{S}$.

Let $x \in X \backslash U^1 \cup U^2$. Then

$$\{y \in X : (y, x) \notin E^*\}$$

$$= \left\{y \in X : (y, x) \notin E^1\right\} \cap \left\{y \in X : (y, x) \notin E^2\right\},$$

the latter being the intersection of convex or empty sets. Thus, $\{y \in X : (y, x) \notin E^*\}$ is convex or empty.

(b) $E^* \in B_{h_{X\times X}}(3\delta^0, E^0)$: We have

$$E^* = [E^1 \cap (X \times U^2)^c] \cup [E^2 \cap (X \times U^1)^c] \tag{25}$$

and by the triangle inequality,

$$\begin{gathered}h_{X\times X}(E^1, E^2) \leq h_{X\times X}(E^1, E^0) + h_{X\times X}(E^0, E^2) < 2\delta^0,\\ \text{and}\\ h_{X\times X}(E^*, E^0) \leq h_{X\times X}(E^*, E^1) + h_{X\times X}(E^1, E^0).\end{gathered} \tag{26}$$

We know already that $h_{X\times X}(E^1, E^0) < \delta^0$. Consider $h_{X\times X}(E^*, E^1)$. We have

$$h_{X\times X}(E^*, E^1) := \max\left\{e_{X\times X}(E^*, E^1), e_{X\times X}(E^1, E^*)\right\}.$$

It is easy to check that,

$$e_{X\times X}(E^*, E^1) = \sup_{(y,x)\in E^*} dist_{X\times X}((y, x), E^1)$$

$$= \sup_{(y,x)\in[E^2\cap(X\times U^1)^c]} dist_{X\times X}((y, x), E^1)$$

$$\leq \sup_{(y,x)\in E^2} dist_{X\times X}((y, x), E^1) = e_{X\times X}(E^2, E^1).$$

To show that $e_{X\times X}(E^1,E^*)\leq e_{X\times X}(E^1,E^2)$ observe that

$$e_{X\times X}(E^1,E^*)=\sup_{(y,x)\in E^1} dist_{X\times X}((y,x),E^*)$$

$$=\sup_{(y,x)\in E^1} dist_{X\times X}((y,x),[E^1\backslash(X\times U^2)]\cup[E^2\backslash(X\times U^1)]).$$

Letting $E^1=[E^1\backslash(X\times U^2)]\cup\left[E^1\cap(X\times U^2)\right]$, we have for all $(y,x)\in E^1\backslash(X\times U^2)$,

$$dist_{X\times X}((y,x),E^*)$$

$$=dist_{X\times X}((y,x),[E^1\backslash(X\times U^2)]\cup[E^2\backslash X\times U^1)])$$

$$\leq dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)]\cup[E^2\cap(X\times U^1)])$$

$$=dist_{X\times X}((y,x),E^2).$$

Moreover, we have for all $(y,x)\in E^1\cap(X\times U^2)$,

$$dist_{X\times X}((y,x),E^*)$$

$$=dist_{X\times X}((y,x),[E^1\backslash(X\times U^2)]\cup[E^2\backslash(X\times U^1)])$$

$$=dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)]),$$

and

$$dist_{X\times X}((y,x),E^2)$$

$$=dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)]\cup[E^2\cap(X\times U^1)])$$

$$=dist_{X\times X}((y,x),[E^2\backslash(X\times U^1)]).$$

Thus, for all $(y,x)\in E^1$,

$$\rho_{X\times X}((y,x),E^*)\leq\rho_{X\times X}((y,x),E^2),$$

implying that $e_{X\times X}(E^1,E^*)\leq e_{X\times X}(E^1,E^2)$. Together,

$$e_{X\times X}(E^1,E^*)\leq e_{X\times X}(E^1,E^2)$$
and
$$e_{X\times X}(E^*,E^1)\leq e_{X\times X}(E^2,E^1)$$

imply that $h_{X\times X}(E^*,E^1)\leq h_{X\times X}(E^2,E^1)<2\delta^0$. Thus, we have

$$h_{X\times X}(E^*,E^0)\leq h_{X\times X}(E^*,E^1)+h_{X\times X}(E^1,E^0)$$

$$\leq h_{X\times X}(E^2,E^1)+h_{X\times X}(E^1,E^0)$$

$$<2\delta^0+\delta^0=3\delta^0.$$

(c) $N(E^*)\cap\left[U^1\cup U^2\right]\neq\emptyset\Rightarrow N(E^i)\cap U^i\neq\emptyset$ for some $i=1$ and/or 2:

Suppose that $x\in N(E^*)\cap U^1$. Given the definition of the KFC, $N(\cdot)$, We have for each $x\in N(E^*)$ and $y\in X$,

$$(y,x)\in\left(E^1\cap\left(X\times U^2\right)^c\right)\cup\left(E^2\cap\left(X\times U^1\right)^c\right),$$

and because $x \in U^1$, this implies that for each $y \in X$, $(y,x) \in E^1 \cap \left(X \times U^2\right)^c$, and specifically, that for each $y \in X$,

$$(y,x) \in E^1 \cap \left(X \times U^1\right). \ (*)$$

Thus, given that $x \in N(E^*)$ and $y \in X$, (*) implies that $x \in N(E^1) \cap U^1$, contradicting the fact that $N(E^1) \cap U^1 = \emptyset$. Thus we must conclude that $N(\cdot)$ has the 3M property. **Q.E.D.**

5.3 All Minimal $KFCs$ Take Minimally Essential and Connected Values

Our next result establishes a second surprising fact about $KFCs$: all minimal USCOs defined on the hyperspace of Ky Fan sets (i.e., all minimal $KFCs$), as a consequence of being $3M$, take minimally essential values.

Theorem 4 *(Minimal KFCs Belonging to Quasi-Minimal KFCs Are Minimally Essential Valued)8*
Let $\mathcal{G}(\Omega \times \mathcal{L}^{\infty}_{Y})$ be a one-shot game satisfying assumptions [$\mathcal{OSG}$] with KFC, $N(\cdot)$. The following statements are true:
(1) *If $N(\cdot)$ is quasi-minimal with $[N(\cdot)] = \{n(\cdot)\}$ for some minimal USCO, $n(\cdot)$, then $n(E) \in \mathcal{E}^*[N(E)]$ for all $E \in \mathbb{S}$.*
(2) If $n(\cdot)$ is a minimal USCO, then $\{n(E)\} = \mathcal{E}^*[n(E)]$ for all $E \in \mathbb{S}$.

Proof: Suppose that for some E^0 there is some nonempty, closed and proper subset $e(E^0)$ of $n(E^0)$ with $e(E^0) \in \mathcal{E}[N(E^0)]$. Fix some $x^0 \in n(E^0)\diagdown\overline{e(E^0)}$ and let $B_{\rho_X}(\varepsilon^0, e(E^0)) \subset X$ be an open enlargement of $e(E^0)$ such that $x^0 \notin \overline{B_{\rho_X}(\varepsilon^0, e(E^0))}$. Since $e(E^0) \in \mathcal{E}[N(E^0)]$ there is a $\delta^0 > 0$ such that for all $E \in B_{h_{X\times X}}(\delta^0, E^0)$, $N(E) \cap B_{\rho_X}(\varepsilon^0, e(E^0)) \neq \emptyset$. Define the mapping $\varphi(\cdot)$ as follows:

$$\varphi(E) := \begin{cases} N(E) \cap \overline{B_{\rho_X}(\varepsilon^0, e(E^0))} & E \in B_{h_{X\times X}}(\delta^0, E^0) \\ N(E) & z \in \mathbb{S}\diagdown B_{h_{X\times X}}(\delta^0, E^0). \end{cases}$$

By Lemma 2(ii) in Anguelov and Kalenda (2009), $\varphi(\cdot)$ is an USCO with $Gr\varphi \subset GrN$ and hence $Grn \subset Gr\varphi$. In particular, $x^0 \in \varphi(E^0)$, a contradiction. **Q.E.D.**

As the following example makes clear, the quasi-minimality of the USCO is critical to the above result.

Example 1 *(Quasi-Minimality is Critical)*
Let $Z = X = [-1,1]$ and define $N \in \mathcal{U}_{\rho_Z\text{-}\rho_X}$ as follows:

$$N(z) := \begin{cases} \{-1\} & z \in [-1, -\frac{1}{2}) \\ \{-1, 1\} & z \in [-\frac{1}{2}, \frac{1}{2}] \\ \{1\} & z \in (\frac{1}{2}, 1]. \end{cases}$$

While the mapping N is an USCO is not quasi-minimal. Next consider the following USCO:

$$n(z) := \begin{cases} \{-1\} & z \in [-1, 0) \\ \{-1, 1\} & z = 0 \\ \{1\} & z \in (0, 1]. \end{cases}$$

We have $n \in [N]$ but $n(0)$ is not minimally essential for N at $z = 0$ because $n(0) = \{-1, 1\}$ but the smaller sets $\{-1\}$ and $\{1\}$ are each minimally essential for N at $z = 0$.

The following Theorem establishes a third fundamental fact about minimal USCOs defined on Ky Fan sets: all minimal $KFCs$ take minimally essentially, and *connected* values.

Theorem 5 *(The Connection Between a KFC's Minimal USCOs and Connected Minimally Essential Sets)*
Let $\mathcal{G}(\Omega \times \mathcal{L}_Y^\infty)$ be a one-shot game satisfying assumptions [$\mathcal{OSG}$] with KFC, $N(\cdot)$. If $n(\cdot)$ is a minimal USCO belonging to $N(\cdot)$, then for each $E \in \mathbb{S}$, $\{n(E)\} = \mathcal{E}^*[n(E)]$ and $n(E)$ is connected.

Proof: Because $n(\cdot)$ is a minimal USCO belonging to $N(\cdot)$, $n(\cdot)$ is quasi-minimal. Thus by Theorem 4 and the minimality if $n(\cdot)$, $\{n(E)\} = \mathcal{E}^*[n(E)]$ for each $E \in \mathbb{S}$.

Next, suppose that for some $\widetilde{E} \in \mathbb{S}$, $n(\widetilde{E})$ is not connected. Then there are two nonempty, compact sets, $n^1(\widetilde{E})$ and $n^2(\widetilde{E})$, and two nonempty, *disjoint* open subsets, W^1 and W^2, in X such that (i) $n^1(\widetilde{E}) \subset W^1$ and $n^2(\widetilde{E}) \subset W^2$, and (ii) $n(\widetilde{E}) = n^1(\widetilde{E}) \cup n^2(\widetilde{E})$.

Therefore, neither $n^1(\widetilde{E})$ nor $n^2(\widetilde{E})$ are essential implying that there are two nonempty, open sets G^1 and G^2 with

$$n^1(\widetilde{E}) \subset G^1 \text{and } n^2(\widetilde{E}) \subset G^2$$

such that *for all* $\delta > 0$, *there exists Ky Fan* $E^{\delta 1}$ and $E^{\delta 2}$ in $B_{h_{X \times X}}(\delta, \widetilde{E})$ such that

$$n(E^{\delta 1}) \cap G^1 = \emptyset \text{ and } n(E^{\delta 2}) \cap G^2 = \emptyset.$$

Let $U^1 = W^1 \cap G^1$ and $U^2 = W^2 \cap G^2$. We have $U^1$and U^2 disjoint open sets such that $n^1(\widetilde{E}) \subset U^1$ and $n^2(\widetilde{E}) \subset U^2$ and for all $\delta > 0$, there exist

$$E^{\delta 1} \in B_{h_{X \times X}}(\delta, \widetilde{E}) \cap \mathbb{S} \text{ and } E^{\delta 2} \in B_{h_{X \times X}}(\delta, \widetilde{E}) \cap \mathbb{S}$$

such that

$$n(E^{\delta 1}) \cap U^1 = \emptyset \text{ and } n(E^{\delta 2}) \cap U^2 = \emptyset. \tag{27}$$

Given that the sets $n(E^{\delta i})$ are compact, under [$\mathcal{OSG}$], there exist open sets V^1 and V^2 such that for $i = 1, 2$,

$$n^i(\widetilde{E}) \subset V^i \subset \overline{V}^i \subset U^i.$$

Thus, we have for all $\delta > 0$, $E^{\delta i} \in B_{h_{X \times X}}(\delta, \widetilde{E}) \cap \mathbb{S}$ such that

$$n(E^{\delta 1}) \cap \overline{V}^1 = \emptyset \text{ and } n(E^{\delta 2}) \cap \overline{V}^2 = \emptyset. \tag{28}$$

Now we have a contradiction: First, because $n(\widetilde{E})$ is a minimal essential set of $n(\widetilde{E})$ and because $n(\widetilde{E}) \subset \left[V^1 \cup V^2\right]$, there exists a positive number $\delta^* > 0$ such that *for all Ky Fan sets* $E \in B_{h_{X \times X}}(\delta^*, \widetilde{E}) \cap \mathbb{S}$,

$$n(E) \cap \left[V^1 \cup V^2\right] \neq \emptyset. \tag{29}$$

But because $\delta > 0$ can be chosen arbitrarily, choosing $\delta = \frac{\delta^*}{3}$, we have by (28) and the 3M property, the existence of a

$$\overline{E} \in B_{h_{X \times X}}(3\tfrac{\delta^*}{3}, \widetilde{E}) \cap \mathbb{S} = B_{h_{X \times X}}(\delta^*, \widetilde{E}) \cap \mathbb{S},$$

such that $n(\overline{E}) \cap \left[\overline{V}^1 \cup \overline{V}^2 \right] = \emptyset$. **Q.E.D.**

6 The Collective Security Mapping - the CSM

In this section we will no longer assume that ω is fixed or that $\Phi(\omega) = X$.

With each (ω, v)-game,

$$\mathcal{G}_{(\omega,v)} := \{\Phi_d(\omega), u_d(\omega, v_d, (\cdot, \cdot))\}_{d \in D}\,, \tag{30}$$

we can associate a *Nikaido-Isoda function* (Nikaido and Isoda, 1955) given by

$$\left.\begin{aligned} &\varphi(\omega, v, (y, x)) := u(\omega, v, (y, x)) - u(\omega, v, (x, x)) \\ &:= \textstyle\sum_{d \in D} u_d(\omega, v_d, (y_d, x_{-d})) - \sum_{d \in D} u_d(\omega, v_d, (x_d, x_{-d})), \end{aligned}\right\} \tag{31}$$

for each $(y, x) \in \Phi(\omega) \times \Phi(\omega)$. We say that $x' \in \Phi(\omega)$ *is collectively secure against a feasible defection profile, $y' \in \Phi(\omega)$,* with player specific noncooperative player defections given by, (y'_d, x'_{-d}), for players $d = 1, 2, \ldots, m$, if and only if

$$(y', x') \in K(\omega, v) := \{(y, x) \in \Phi(\omega) \times \Phi(\omega) : \varphi(\omega, v, (y, x)) \leq 0\}.$$

Thus, for each one-shot game, $\mathcal{G}(\Omega \times \mathcal{L}^{\infty}_{Y})$, satisfying assumptions $[\mathcal{OSG}]$, there is collective security mapping - a CSM - given by

$$(\omega, v) \longrightarrow K(\omega, v) := \{(y, x) \in \Phi(\omega) \times \Phi(\omega) : \varphi(\omega, v, (y, x)) \leq 0\}. \tag{32}$$

The *collectively secure action mapping* (i.e., the CS action mapping) is given by,

$$y \longrightarrow K(\omega, v)(y) := \{x \in \Phi(\omega) : \varphi(\omega, v, (y, x)) \leq 0\}\,. \tag{33}$$

For each defection profile $y \in \Phi(\omega)$ with player specific defections of the form $y = (y_d, x_{-d})$, $K(\omega, v)(y)$ is the (closed) set of action profiles, $x = (x_d, x_{-d})$, in $\Phi(\omega)$ that are *collectively secure* against potential *noncooperative* defections represented by profile, y. Note that if, given state-value function profile pair (ω, v), x is contained in $K(\omega, v)(y)$ for *all* possible defection profiles $y \in \Phi(\omega)$, that is, if

$$x \in \cap_{y \in \Phi(\omega)} K(\omega, v)(y) \tag{34}$$

then for each player d, $x = (x_d, x_{-d})$ is secure against any defection of the form $y = (y_d, x_{-d})$. Thus, $x \in \cap_{y \in \Phi(\omega)} K(\omega, v)(y)$ implies that

$$u_d(\omega, v_d, (y_d, x_{-d})) \leq u_d(\omega, v_d, (x_d, x_{-d})),$$

for all players, d, and all pairs $y = (y_d, x_{-d})$ and $x = (x_d, x_{-d})$ - and conversely. Thus, the set of Nash equilibria given state-value function profile pair (ω, v) can be fully characterized as follows:

$$x \in \mathcal{N}(\omega, v) \text{ if and only if } x \in \cap_{y \in \Phi(\omega)} K(\omega, v)(y), \tag{35}$$

and therefore, the Nash correspondence is given by,

$$(\omega, v) \longrightarrow \mathcal{N}(\omega, v) = \cap_{y \in \Phi(\omega)} K(\omega, v)(y). \tag{36}$$

Under assumptions $[\mathcal{OSG}]$, the function, $\varphi(\cdot, \cdot, (\cdot, \cdot))$ which specifies for each $(\omega, v) \in \Omega \times \mathcal{L}^{\infty}_{Y}$ a particular Nikaido-Isoda function has the following properties:

(F1) for each ω, $\varphi(\omega, \cdot, (\cdot, \cdot))$ is continuous on the compact metric space, $\mathcal{L}_Y^\infty \times (X \times X)$;

(F2) for each $(v, (y, x))$, $\varphi(\cdot, v, (y, x))$ is (B_Ω, B_Y)-measurable; and

(F3) $y \longrightarrow \varphi(\omega, v, (\cdot, x))$ is affine in y on X.

For each state-value function profile pair (ω, v), the graph of the CS action mapping, $K(\omega, v)(\cdot)$, is given by

$$GrK(\omega, v)(\cdot) := \{(y, x) \in \Phi(\omega) \times \Phi(\omega) : \varphi(\omega, v, (y, x)) \leq 0\}. \tag{37}$$

Thus, for any $(y', x') \in GrK(\omega, v)(\cdot)$, strategy profile $x' \in \Phi(\omega)$ is secure against defection profile $y' \in \Phi(\omega)$ and we have $\varphi(\omega, v, (y', x')) \leq 0$. *Thus, for each* $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$*, the value of the* CSM*,* $K(\omega, v)$*, is given by the graph of the* CS *action mapping* $K(\omega, v)(\cdot)$ *- and we will show that* $K(\omega, v) = GrK(\omega, v)(\cdot)$ *is a Ky Fan set. Thus, the* CSM *is given by the Ky Fan set-valued mapping,*

$$(\omega, v) \longrightarrow K(\omega, v) := GrK(\omega, v)(\cdot) \in \mathbb{S} \text{ for all } (\omega, v) \in \Omega \times \mathcal{L}_Y^\infty. \tag{38}$$

Moreover, we will show that for each minimal KFC, $n(\cdot)$, the composition correspondence, $(\omega, v) \longrightarrow n(K(\omega, v))$, is upper Caratheodory and takes connected values.

Our main results regarding the collective security mapping are the following:

Theorem 6 *(The collective security function,* $K(\cdot, \cdot)$*, is Ky Fan valued and Caratheodory)*
Let $\mathcal{G}(\Omega \times \mathcal{L}_Y^\infty)$ *be a one-shot game satisfying assumptions [*$\mathcal{OSG}$*] with* KFC, $N(\cdot)$, and CSF, $K(\cdot, \cdot)$. *Then the following statements about the* CSM*,* $K(\cdot, \cdot)$*, and the collection of minimal* $KFCs$*,* $[N]$ *are true:*
(1) For each $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$*,* $K(\omega, v)$ *is a Ky Fan set.*
(2) For each minimal KFC*,* $n \in [N]$*,* $n(K(\cdot, \cdot))$*, is upper Caratheodory (i.e.,* $n(K(\cdot, \cdot))$ *is* $\left(B_\Omega \times B_{\mathcal{L}_Y^\infty}\text{-}B_X\right)$*-measurable and for each* ω*,* $n(K(\omega, \cdot))$ *is* ρ_{w^*}*-*ρ_X*-upper semicontinuous (i.e., for each* ω*,* $v^n \underset{\rho_{w^*}}{\longrightarrow} v$ and $x^n \underset{\rho_X}{\longrightarrow} x$ with $x^n \in n(K(\omega, v^n))$ *for all* n *implies that* $x \in n(K(\omega, v))$*).*

Proof of (1): It is easy to see that for each $y \in \Phi(\omega)$ and $v \in \mathcal{L}_Y^\infty$, $(y, y) \in K(\omega, v)$. Thus Definition 1(a) holds. It is also easy to see that for all $v \in \mathcal{L}_Y^\infty$ and $y \in \Phi(\omega)$, $K(\omega, v)(y)$ is closed, so that Definition 1(b) holds. To see that Definition 1(c) holds observe that because $\varphi(\omega, v, (\cdot, x))$ is affine in y, $y \in \Phi(\omega)$ such that $(y, x) \notin K(\omega, v)$ is given by the set, $\{y \in \Phi(\omega) : \varphi(\omega, v, (y, x)) > 0\}$, and this set is convex (or empty).

Proof of (2): Let ω be fixed and suppose that the sequence, $\{(v^n, x^n)\}_n$, is such that, $v^n \underset{\rho_{w^*}}{\longrightarrow} v$ and $x^n \underset{\rho_X}{\longrightarrow} x$ with $x^n \in n(K(\omega, v^n))$ for all n. First, we have for each n and any $y \in \Phi(\omega)$ that $\varphi(\omega, v^n, (y, x^n)) \leq 0$. Thus, in the limit, we have for any $y \in \Phi(\omega)$ that $\varphi(\omega, v^0, (y, x^0)) \leq 0$, implying that $x^0 \in n(K(\omega, v^0))$.

That $n(K(\cdot, \cdot))$ is $\left(B_\Omega \times B_{\mathcal{L}_Y^\infty}\text{-}B_X\right)$-measurable follows from Lemma 3.1 in Kucia and Nowak (2000). **Q.E.D.**

Because all minimal USCOs, $n(\cdot)$, belonging to a KFC, $N(\cdot)$, are such that for each Ky Fan set E, $n(E)$ is a continuum of minimally essential Nash equilibria and because the CSM, $K(\cdot, \cdot)$, is such that for each (ω, v), $K(\omega, v)$ is a Ky Fan set, the upper Caratheodory Nash sub-correspondence

$$(\omega, v) \longrightarrow n(K(\omega, v))$$

is a continuum-valued. Thus, we have for any one-shot game, $\mathcal{G}(\Omega \times \mathcal{L}_Y^\infty)$, satisfying $[\mathcal{OSG}]$ with Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, CSM, $K(\cdot,\cdot)$, and KFC, $N(\cdot)$, that

$$n(K(\cdot,\cdot)) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}C_f(X)},$$

with $n(K(\omega,v)) \subseteq \mathcal{N}(\omega,v)$ for all $(\omega,v) \in \Omega \times \mathcal{L}_Y^\infty$. It only remains to show that because the uC Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, contains a continuum-valued uC Nash sub-correspondence, $n(K(\cdot,\cdot))$, the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot,v)) := \mathcal{S}^\infty(u(\cdot,v,\mathcal{N}(\cdot,v)))$, has fixed points.

Part III

A Fixed Point Theorem for Nash Payoff Selection Correspondences

We are now ready to prove that all $\mathcal{OSG}s$ satisfying the usual assumptions (i.e., assumptions $[\mathcal{OSG}]$) - and therefore all $\mathcal{DSG}s$ satisfying the usual assumptions - have Nash payoff selection correspondences having fixed points - implying that all such $\mathcal{DSG}s$ have stationary Markov perfect equilibria. We have shown that while all Nash payoff selection correspondences belonging to $\mathcal{DSG}s$ satisfying the usual assumptions fail to be weak star upper semicontinuous, nonetheless, all such correspondences contain selection sub-correspondences which are approximable and have fixed points. The reason for this, as we have shown here, is that all Nash payoff correspondences belonging to $\mathcal{DSG}s$ satisfying the usual assumptions have upper Caratheodory (uC) players' Nash payoff correspondences that are made up of interval-valued uC players' Nash payoff sub-correspondences. And this in turn is a consequence of the $3M$ property of minimal $KFCs$ - implying that all $\mathcal{DSG}s$ satisfying the usual assumptions have upper Caratheodory (uC) Nash correspondences that are made up of continuum-valued uC Nash sub-correspondences. In particular, for all such $\mathcal{DSG}s$ the Nash payoff correspondence,

$$(\omega,v) \longrightarrow \mathcal{P}(\omega,v) := (u_1(\omega,v_1,\mathcal{N}(\omega,v)),\ldots,u_m(\omega,v_m,\mathcal{N}(\omega,v))),$$

is made up of Nash payoff sub-correspondences,

$$(\omega,v) \longrightarrow p(\omega,v) := (u_1(\omega,v_1,n(K(\omega,v))),\ldots,u_m(\omega,v_m,n(K(\omega,v)))),$$

where $n(\cdot)$ is a continuum-valued minimal KFC and $K(\cdot,\cdot)$ is the game's Ky Fan valued CSM. Because $n(K(\cdot,\cdot))$ is continuum-valued and upper Caratheodory each player's induced uC Nash payoff sub-correspondence, $(\omega,v) \longrightarrow u_d(\omega,v_d,n(K(\omega,v)))$, is closed interval-valued - and therefore, is Caratheodory approximable.

In our last result, we prove that as a consequence of Caratheodory approximability there exists $v^* \in \mathcal{L}_Y^\infty$ such that $v^*(\omega) \in p(\omega,v^*)$ a.e. $[\mu]$, or equivalently, that there exists $v^* \in \mathcal{L}_Y^\infty$ such that $v^* \in \mathcal{S}^\infty(p(\cdot,v^*))$.

Theorem 7 *(Nash payoff selection correspondences have fixed points)*
Let $\mathcal{G}(\Omega \times \mathcal{L}_Y^\infty)$ be a one-shot game satisfying assumptions $[\mathcal{OSG}]$ with upper Caratheodory Nash payoff correspondence, $(\omega,v) \longrightarrow \mathcal{P}(\omega,v)$. Then there exists $\widehat{v} \in \mathcal{L}_Y^\infty$ such that

$$\widehat{v}(\omega) \in \mathcal{P}(\omega,\widehat{v}) \text{ a.e. } [\mu].$$

Proof: By Theorem 6 above, we know that there is a continuum-valued, minimal uC Nash sub-correspondence, $(\omega, v) \longrightarrow n(K(\omega, v))$. Because $n(K(\cdot, \cdot))$ takes closed and connected values, the induced uC composition sub-correspondence,

$$\left.\begin{array}{r} (\omega, v) \longrightarrow p(\omega, v) := (p_1(\omega, v), \ldots, p_m(\omega, v)) \\ = (u_1(\omega, v, n(K(\omega, v)), \ldots, u_m(\omega, v, n(K(\omega, v))) := u(\omega, v, n(K(\omega, v)), \end{array}\right\} \quad (39)$$

is such that for each $d = 1, 2, \ldots, m$, $(\omega, v) \longrightarrow p_d(\omega, v)$, takes closed interval values in Y_d, implying via Corollary 4.3 in Kucia and Nowak (2000) that $p_d(\cdot, \cdot)$ is Caratheodory approximable. Thus, there is a sequence of m-tuples of Caratheodory functions,

$$\{g^n(\cdot, \cdot)\}_n := \{(g_1^n(\cdot, \cdot), \ldots, g_m^n(\cdot, \cdot))\}_n, \quad (40)$$

such that for each n and for each $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$ there exists for each d, $(\overline{v}^{nd}, \overline{u}_d^n) \in Grp_d(\omega, \cdot)$ such that,

$$\rho_{w^*}(v, \overline{v}^{nd}) + \rho_{Y_d}(g_d^n(\omega, v), \overline{u}_d^n) < \tfrac{1}{m \cdot n}. \quad (41)$$

Next, consider the mapping from $\mathcal{L}_Y^\infty$ to $\mathcal{L}_Y^\infty$ given by

$$v \longrightarrow T^n(v) := g^n(\cdot, v) := (g_1^n(\cdot, v), \ldots, g_m^n(\cdot, v)) \in \mathcal{L}_Y^\infty. \quad (42)$$

Observe that for each n, $T^n(\cdot)$ is continuous (i.e., $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$ implies that $T^n(v^k) \underset{\rho_{w^*}}{\longrightarrow} T^n(v^*)$). This is true because for each n, $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$ implies that for each $\omega \in \Omega$, as $k \longrightarrow \infty$, $g^n(\omega, v^k) \underset{\rho_Y}{\longrightarrow} g^n(\omega, v^*) \in Y$. Therefore, for $l \in \mathcal{L}_{R^m}^1$ chosen arbitrarily, $\left\langle g^n(\omega, v^k), l(\omega) \right\rangle \underset{R}{\longrightarrow} \left\langle g^n(\omega, v^*), l(\omega) \right\rangle$ a.e. $[\mu]$, implying that as $k \longrightarrow \infty$,

$$\int_\Omega \left\langle g^n(\omega, v^k), l(\omega) \right\rangle d\mu(\omega) \longrightarrow \int_\Omega \left\langle g^n(\omega, v^*), l(\omega) \right\rangle d\mu(\omega).$$

Since the choice of $l \in \mathcal{L}_{R^m}^1$ was arbitrary, we can conclude that if $v^k \underset{\rho_{w^*}}{\longrightarrow} v^*$, then $g^n(\cdot, v^k) \underset{\rho_{w^*}}{\longrightarrow} g^n(\cdot, v^*) \in \mathcal{L}_Y^\infty$. By the Brouwer-Schauder-Tychonoff Fixed Point Theorem (e.g., see Aliprantis-Border, 17.56, 2006), for each n, there exists $v^n \in \mathcal{L}_Y^\infty$ such that

$$v^n = T^n(v^n) := g^n(\cdot, v^n). \quad (43)$$

Thus, we have for each n a set, N^n, of μ-measure zero such that

$$v^n(\omega) = g^n(\omega, v^n) \text{ for all } \omega \in \Omega \backslash N^n, \ \mu(N^n) = 0. \quad (44)$$

Letting $N^\infty := \cup_n N^n$ - so that, $\mu(N^\infty) = 0$ - we have for each $n = 1, 2, \ldots$and for each $d = 1, 2, \ldots, m$, that

$$v_d^n(\omega) = g_d^n(\omega, v^n) \text{ for all } \omega \in \Omega \backslash N^\infty, \ \mu(N^\infty) = 0. \quad (45)$$

Call the equation (45), one for each n, the Caratheodory equation and call the sequence, $\{v^n\}_n$, in $\mathcal{L}_Y^\infty$ the *Caratheodory fixed point sequence*.

For each pair of m-tuples of Caratheodory approximating functions and fixed points, $(g^n(\cdot, \cdot), v^n)$, consider the measurable function,

$$\omega \longrightarrow \min_{(v, u_d) \in Grp_d(\omega, \cdot)}[\rho_{w^*}(v^n, v) + \rho_{Y_d}(g_d^n(\omega, v^n), u_d)], \quad (46)$$

By Lemma 3.1 in Kucia and Nowak (2000) the graph correspondence, $\omega \longrightarrow Grp_d(\omega, \cdot)$, is measurable, and therefore, by the continuity of the function

$$(v, u_d) \longrightarrow [\rho_{w^*}(v^n, v) + \rho_{Y_d}(g_d^n(\omega, v^n), u_d)]$$

on $\mathcal{L}_Y^\infty \times Y_d$, there exists for each n, a measurable (everywhere) selection of $Grp_d(\omega, \cdot)$,

$$\omega \longrightarrow (\overline{v}_\omega^{nd}, \overline{u}_{\omega d}^n) \in \mathcal{L}_Y^\infty \times Y_d \tag{47}$$

solving the minimization problem (46) state-by-state (see Himmelberg, Parthasarathy, and VanVleck, 1976). Moreover, we have by the Caratheodory approximability of uC Nash payoff sub-correspondence,

$$p(\cdot, \cdot) := (p_1(\cdot, \cdot), \ldots, p_m(\cdot, \cdot)),$$

and (41) above that for the sequences of optimal selections, $\{(\overline{v}_{(\cdot)}^{nd}, \overline{u}_{(\cdot)d}^n)\}_n$, $d = 1, 2, \ldots, m$, where for each n and for each ω, $\overline{v}_\omega^{nd} \in \mathcal{L}_Y^\infty$ and $\overline{u}_{\omega d}^n \in Y_d$, we have for each n and for each ω,

$$\underbrace{\rho_{w^*}(v^n, \overline{v}_\omega^{nd})}_{A} + \underbrace{\rho_{Y_d}(g_d^n(\omega, v^n), \overline{u}_{\omega d}^n)}_{B} < \tfrac{1}{m \cdot n}. \tag{48}$$

Given (44) and (48), we have for the sequences,

$$\{g^n(\cdot, \cdot), v^n\}_n \text{ and } \{\overline{v}_{(\cdot)}^{nd}, \overline{u}_{(\cdot)d}^n\}_n, d = 1, 2, \ldots, m, \tag{49}$$

that for all $\omega \in \Omega \backslash N^\infty$, $\mu(N^\infty) = 0$, and for all n,

$$\rho_{w^*}(v^n, \overline{v}_\omega^{nd}) + \underbrace{\rho_{Y_d}(v_d^n(\omega), \overline{u}_{\omega d}^n)}_{C} < \tfrac{1}{m \cdot n}, \tag{50}$$

where for each d and for each n, $\omega \longrightarrow \overline{v}_\omega^{nd}$ is $\mathcal{L}_Y^\infty$-valued, while $\omega \longrightarrow \overline{u}_{\omega d}^n$ is Y_d-valued, and

$$\overline{u}_\omega^n := (\overline{u}_{\omega 1}^n, \ldots, \overline{u}_{\omega m}^n) \in (p_1(\omega, \overline{v}_\omega^{n1}), \ldots, p_m(\omega, \overline{v}_\omega^{nm})) \text{ for all } \omega \in \Omega. \tag{51}$$

Next, because $(\mathcal{L}_Y^\infty, \rho_{w^*})$ is a compact metric space we can assume without loss of generality that the sequence of fixed points in $\mathcal{L}_Y^\infty$, $\{v^n\}_n$, K-converges to some $\widehat{v} \in \mathcal{L}_Y^\infty$, implying that $v^n \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$ and therefore implying via (48)A that $\overline{v}_\omega^{nd} \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$ uniformly in d and ω (see 7.4 in the Appendices for Komlos or K-convergence and weak star convergence). Moreover, by (50)C, we have that

$$\widehat{\overline{u}}_{\omega d}^n = \tfrac{1}{n} \sum_{k=1}^{n} \overline{u}_{\omega d}^k \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega) \text{ a.e. } [\mu], \tag{52}$$

where for each n, $\overline{u}_{\omega d}^n \in p_d(\omega, \overline{v}_\omega^{nd})$ for all ω. By the properties of K-convergence, for each $n = 1, 2, 3, \ldots$, there is a set, $\widehat{N}^n$, of μ-measure zero such that for all d and for all $\omega \in \Omega \backslash \widehat{N}^n$ as $q \longrightarrow \infty$

$$\left.\begin{array}{c} \widehat{\overline{u}}_{\omega d}^{n+q} = \tfrac{1}{q} \sum_{r=1}^{q} \overline{u}_{\omega d}^{n+r} \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega), \\ \text{and} \\ \widehat{v}_d^{n+q}(\omega) = \tfrac{1}{q} \sum_{r=1}^{q} v_d^{n+r}(\omega) \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega). \end{array}\right\} \tag{53}$$

Letting $\widehat{N}^{\infty} := \cup_{n=1}^{\infty} \widehat{N}^n$ we have that for any $n = 1, 2, 3, \ldots$, that for each player the truncated sequences, $\{\overline{u}_{(\cdot)d}^{n+q}\}_{q=1}^{\infty}$ and $\{v_d^{n+q}(\cdot)\}_{q=1}^{\infty}$, have arithmetic mean sequences, $\{\widehat{\overline{u}}_{(\cdot)d}^{n+q}\}_{q=1}^{\infty}$ and $\{\widehat{v}_d^{n+q}(\cdot)\}_{q=1}^{\infty}$, converging pointwise to $\widehat{v}_d(\cdot)$ off the set $\widehat{N}^{\infty}$ of μ-measure zero where the exceptional set $\widehat{N}^{\infty}$ is independent of n.

Because $p_d(\omega, \cdot)$ is ρ_{w^*}-ρ_{Y_d}-upper semicontinuous and because for each d, $\overline{v}_{\omega}^{nd} \underset{\rho_{w^*}}{\longrightarrow} \widehat{v}$ uniformly in d and ω, we have for each d and ω and for any sequence of $k_{\omega} = 1, 2, \ldots$, increasing to ∞, that there is a sequence $\{n_{k_{\omega}}\}_{k_{\omega}}$ increasing to ∞, such that for all $n \geq n_{k_{\omega}}$ the ρ_{Y_d}-open ball, $B_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v}))$, about $p_d(\omega, \widehat{v})$ of radius $\frac{1}{k_{\omega}}$ with closure given by the closed, convex ball, $\overline{B}_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v}))$, about $p_d(\omega, \widehat{v})$ of radius $\frac{1}{k_{\omega}}$, is such that for all $n \geq n_{k_{\omega}}$ and $q = 1, 2, \ldots$.

$$p_d(\omega, \overline{v}_{\omega}^{(n+q)d}) \subset B_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v})) \subset \overline{B}_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v})). \tag{54}$$

Moreover, for all $\omega \in \Omega \backslash (N^{\infty} \cup \widehat{N}^{\infty})$, $n \geq n_{k_{\omega}}$, and $q = 1, 2, \ldots$, we have for each d

$$\overline{u}_{\omega d}^{n+q} \in p_d(\omega, \overline{v}_{\omega}^{(n+q)d}) \subset \overline{B}_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v})). \tag{55}$$

Because $\overline{B}_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v}))$ is closed and convex, and because

$$\widehat{\overline{u}}_{\omega d}^{n+q} \in \overline{B}_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v})) \text{ for all } \omega \in \Omega \backslash (N^{\infty} \cup \widehat{N}^{\infty}),\ n \geq n_{k_{\omega}}, \text{ and } q = 1, 2, \ldots, \tag{56}$$

the fact that for each d, $\widehat{\overline{u}}_{\omega d}^{n+q} \underset{\rho_{Y_d}}{\longrightarrow} \widehat{v}_d(\omega)$ for each $\omega \in \Omega \backslash (N^{\infty} \cup \widehat{N}^{\infty})$ and for each $n \geq n_{k_{\omega}}$ as $q = 1, 2, \ldots$, goes to ∞, implies that for each d and for all $\omega \in \Omega \backslash (N^{\infty} \cup \widehat{N}^{\infty})$,

$$\widehat{v}_d(\omega) \in \overline{B}_{\rho_{Y_d}}(\frac{1}{k_{\omega}}, p_d(\omega, \widehat{v})) \text{ for all } k_{\omega}. \tag{57}$$

Thus, as $k_{\omega} \longrightarrow \infty$ we have in the limit for each d and for each $\omega \in \Omega \backslash (N^{\infty} \cup \widehat{N}^{\infty})$

$$\widehat{v}_d(\omega) \in p_d(\omega, \widehat{v}). \tag{58}$$

Thus, we have $\widehat{v} = (\widehat{v}_1, \ldots, \widehat{v}_m)$ such that $\widehat{v}(\omega) \in p(\omega, \widehat{v}) \subset \mathcal{P}(\omega, \widehat{v})$ a.e. $[\mu]$. **Q.E.D.**

Part IV

Stationary Markov Perfect Equilibria for Discounted Stochastic Games

Having established that any Nash payoff selection correspondence belonging to a $\mathcal{DSG}$ satisfying the usual assumptions has fixed points, it only remains to show that any such fixed point, $\widehat{v} = (\widehat{v}_1, \ldots, \widehat{v}_m) \in \mathcal{L}_Y^{\infty}$, incentivizes the emergence and persistence of a Nash equilibrium in stationary Markov perfect strategies. But for this we need only note that by implicit measurable selection (e.g., see Theorem 7.1 in Himmelberg, 1975), there exists a profile, $\widehat{x}(\cdot) = (\widehat{x}_1(\cdot), \ldots, \widehat{x}_m(\cdot))$, of a.e. measurable selections of $\omega \longrightarrow n(K(\omega, \widehat{v}))$, such that for each player $d = 1, 2, \ldots, m$,

$$\widehat{v}_d(\omega) = u_d(\omega, \widehat{v}_d, \widehat{x}(\omega)) \in u_d(\omega, \widehat{v}_d, n(K(\omega, \widehat{v}))) := p_d(\omega, \widehat{v}) \text{ a.e. } [\mu], \tag{59}$$

Thus, for each player d, the state-contingent prices given by value function, $\widehat{v}_d(\cdot) \in \mathcal{L}^{\infty}_{Y_d}$, incentivizes the continued choice by each player d, of action strategy, $\widehat{x}_d(\cdot)$, and we have for the value function-strategy profile pair, $(\widehat{v}, \widehat{x}(\cdot)) \in \mathcal{S}^{\infty}(p_{\widehat{v}}) \times \mathcal{S}^{\infty}(\eta_{\widehat{v}})$, that

$$\widehat{v}(\omega) = u(\omega, \widehat{v}, \widehat{x}(\omega)) \in p(\omega, \widehat{v}) \text{ and } \widehat{x}(\omega) \in \eta(\omega, \widehat{v}) \text{ a.e. } [\mu], \tag{60}$$

implying that

$$\widehat{v}(\omega) \in \mathcal{P}(\omega, \widehat{v}) \text{ and } \widehat{x}(\omega) \in \mathcal{N}(\omega, \widehat{v}) \text{ a.e. } [\mu]. \tag{61}$$

Thus, for value function-strategy profile pair, $(\widehat{v}, \widehat{x}(\cdot))$, we have for each player $d = 1, 2, \ldots, m$ and for ω a.e. $[\mu]$, that $(\widehat{v}, \widehat{x}(\cdot))$ satisfies the Bellman equation (1 below) as well as satisfies the Nash condition (2 below),

$$\left.\begin{array}{c}
(1)\ \widehat{v}_d(\omega) = (1-\beta_d) r_d(\omega, \widehat{x}_d(\omega), \widehat{x}_{-d}(\omega)) + \beta_d \int_{\Omega} \widehat{v}_d(\omega') q(\omega' | \omega, \widehat{x}_d(\omega), \widehat{x}_{-d}(\omega)), \\
\text{and} \\
(2)\ (1-\beta_d) r_d(\omega, \widehat{x}_d(\omega), \widehat{x}_{-d}(\omega)) + \beta_d \int_{\Omega} \widehat{v}_d(\omega') q(\omega' | \omega, \widehat{x}_d(\omega), \widehat{x}_{-d}(\omega)) \\
= \max_{x_d \in \Phi_d(\omega)} (1-\beta_d) r_d(\omega, x_d, \widehat{x}_{-d}(\omega)) + \beta_d \int_{\Omega} \widehat{v}_d(\omega') q(\omega' | \omega, x_d, \widehat{x}_{-d}(\omega)).
\end{array}\right\} \tag{62}$$

Thus, $\widehat{x}(\cdot) \in \mathcal{S}^{\infty}(\mathcal{N}_{\widehat{v}})$ is a stationary Markov perfect equilibrium of a $\mathcal{DSG}$ satisfying assumptions $[\mathcal{OSG}]$, incentivized by state-contingent prices, $\widehat{v} \in \mathcal{S}^{\infty}(\mathcal{P}_{\widehat{v}})$.

7 Appendices

7.1 Hausdorff Metric

Suppose (Z, ρ_Z) is a compact metric space. Consider the hyperspace of nonempty, ρ_Z-closed subsets $P_f(Z)$. The distance from a point $z \in Z$ to a set $C \in P_f(Z)$ is given by

$$dist_{\rho_Z}(z, C) := \inf_{z' \in C} \rho_Z(z, z'). \tag{63}$$

Given two sets B and C in 2^Z, the excess of B over C is given by

$$e_{\rho_Z}(B, C) := \sup_{z \in B} dist_{\rho_Z}(z, C). \tag{64}$$

Given two sets B and C in $P_f(Z)$, the Hausdorff distance in $P_f(Z)$ between B and C is given by

$$h_{\rho_Z}(B, C) = \max\{e_{\rho_Z}(B, C), e_{\rho_Z}(C, B)\}. \tag{65}$$

Often we will write h rather than h_{ρ_Z} - when the underlying metric is clear.

7.2 Connectedness

A metric space X is said to be connected if and only if there does not exist disjoint open subsets, G^0 and G^1 of X such that $X = G^0 \cup G^1$. If X is a connected compact metric space, it is a *continuum*. A product space $X \times Y$ is connected if and only if X and Y are connected. The importance of connectedness derives from the fact that the continuous image of a connected space is connected - and the continuous image of a continuum is a continuum. All continua in R are closed bounded intervals. We will denote by $C_f(X)$ the hyperspace of nonempty ρ_X-closed and *connected* subsets of X.

7.3 Correspondences

7.3.1 Upper Caratheodory (uC) Correspondences

Consider an upper Caratheodory (uC) correspondence,

$$\mathcal{N}(\cdot,\cdot):\Omega\times\mathcal{L}_Y^\infty\longrightarrow P_f(X), \tag{66}$$

jointly measurable in (ω, v) and upper semicontinuous in v, taking nonempty ρ_X-closed (and hence, ρ_X-compact) values in X. The uC correspondence, $\mathcal{N}(\cdot,\cdot)$, has graph

$$Gr\mathcal{N}(\cdot,\cdot):=Gr\mathcal{N}:=\{(\omega,v,x)\in\Omega\times\mathcal{L}_Y^\infty\times X:x\in\mathcal{N}(\omega,v)\}\,. \tag{67}$$

Given ω or v, for $S\subset X$ define,

$$\left.\begin{array}{c}\mathcal{N}_\omega^-(S):=\{v\in\mathcal{L}_Y^\infty:\mathcal{N}_\omega(v)\cap S\neq\varnothing\},\\ \text{and}\\ \mathcal{N}_v^-(S):=\{\omega\in\Omega:\mathcal{N}_v(\omega)\cap S\neq\varnothing\},\end{array}\right\} \tag{68}$$

where for fixed ω, $\mathcal{N}_\omega(\cdot):=\mathcal{N}(\omega,\cdot)$, and for fixed v, $\mathcal{N}_v(\cdot):=\mathcal{N}(\cdot,v)$. Finally, let

$$\mathcal{N}^-(S):=\{(\omega,v)\in\Omega\times\mathcal{L}_Y^\infty:\mathcal{N}(\omega,v)\cap S\neq\varnothing\}. \tag{69}$$

We have the following definitions (see Wagner, 1977). Given correspondence, $\mathcal{N}(\cdot,\cdot)$, we say that,

(a) $\mathcal{N}_v(\cdot)$ is weakly measurable (or measurable) if for all S open in X, $\mathcal{N}_v^-(S)\in B_\Omega$;

(b) $\mathcal{N}_\omega(\cdot)$ is upper semicontinuous if for all S closed X, $\mathcal{N}_\omega^-(S)$ is closed in $\mathcal{L}_Y^\infty$;

(c) $\mathcal{N}(\cdot,\cdot)$ is product measurable (i.e., jointly measurable in ω and v) if for all S open in X, $\mathcal{N}^-(S)\in B_\Omega\times B_{\mathcal{L}_Y^\infty}$.

(d) $\mathcal{N}(\cdot,\cdot)$ is upper Caratheodory if $\mathcal{N}(\cdot,\cdot)$ is product measurable and for each ω, $\mathcal{N}_\omega(\cdot)$ is upper semicontinuous.

Because X is a compact metric space and $\mathcal{N}_v(\cdot)$ is closed valued, weak measurability of $\mathcal{N}_v(\cdot)$ implies that for each v $\mathcal{N}_v^-(S)\in B_\Omega$ for S closed in X.

We will denote by,

$$\mathcal{N}^{USCO}:=\{\mathcal{N}(\omega,\cdot):\omega\in\Omega\}\,, \tag{70}$$

the nonempty, ρ_X-compact valued, upper semicontinuous part of $\mathcal{N}(\cdot,\cdot)$. Following the terminology of Hola and Holy (2015), we will refer to $\mathcal{N}^{USCO}$ as the USCO part of $\mathcal{N}$. We will denote the measurable part of $\mathcal{N}(\cdot,\cdot)$, by

$$\mathcal{N}^{B_\Omega}:=\{\mathcal{N}(\cdot,v):v\in\mathcal{L}_Y^\infty\}\,. \tag{71}$$

For each state-parameter pair, (ω, v), the graphs of the USCO part and the measurable part are given by,

$$\left.\begin{array}{c}Gr\mathcal{N}(\omega,\cdot):=Gr\mathcal{N}_\omega:=\{(v,x)\in\mathcal{L}_Y^\infty\times X:x\in\mathcal{N}(\omega,v)\}\,,\\ \text{and}\\ Gr\mathcal{N}(\cdot,v):=Gr\mathcal{N}_v:=\{(\omega,x)\in\Omega\times X:x\in\mathcal{N}(\omega,v)\}\,.\end{array}\right\} \tag{72}$$

Moreover, by Lemma 3.1(ii) in Kucia and Nowak (2000), the correspondence, $\omega\longrightarrow Gr\mathcal{N}_\omega$, is measurable.

We will denote the collection of all upper Caratheodory correspondences defined on $\Omega\times\mathcal{L}_Y^\infty$ with nonempty, compact values in X by $\mathcal{UC}_{\Omega\times\mathcal{L}_Y^\infty\text{-}P_f(X)}$.

7.3.2 USCOs

For compact metric spaces $(\mathcal{L}_Y^\infty, \rho_{w^*})$ and (X, ρ_X), let $\mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)} := \mathcal{U}(\mathcal{L}_Y^\infty, P_f(X))$ denote the collection of all upper semicontinuous correspondences taking nonempty, ρ_X-closed (and hence ρ_X-compact) values in X. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). A correspondence $\Psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$ is an USCO if and only if $Gr\Psi$ is compact, where

$$Gr\Psi := \{(v, x) \in \mathcal{L}_Y^\infty \times X : x \in \Psi(v)\}.$$

Given any $\Psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$, denote by $\mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\Psi]$ the collection of all sub-USCOs belonging to Ψ, that is, all USCOs $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$ whose graph,

$$Gr\phi := \{(v, x) \in \mathcal{L}_Y^\infty \times X : x \in \phi(v)\},$$

is contained in the graph of Ψ. We will call any sub-USCO, $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\Psi]$ a minimal USCO belonging to Ψ, if for any other sub-USCO, $\psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\Psi]$, $Gr\psi \subseteq Gr\phi$ implies that $Gr\psi = Gr\phi$. We will use the special notation, $[\Psi]$, to denote the collection of all minimal USCOs belonging to Ψ. We know that for any USCO Ψ, $[\Psi] \neq \varnothing$ (see Drewnowski and Labuda, 1990). In general, we say that ψ is a minimal USCO, if for any other USCO $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$, $Gr\phi \subseteq Gr\psi$ implies that $Gr\phi = Gr\psi$. Let $\mathcal{M}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$ denote the collection of all minimal USCOs. The following characterizations of minimal USCOs (gathered from Anguelov and Kalenda, 2009, and Hola and Holy, 2009) will be useful later.

Characterizations of Minimal USCOs *(Anguelov and Kalenda, 2009, and Hola and Holy, 2009)*
Suppose assumptions [$\mathcal{OSG}$] hold. The following statements are equivalent:
(1) $\eta(\cdot) \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$ is a minimal USCO.
(2) If $U \subset \mathcal{L}_Y^\infty$ and $V \subset X$ are open sets such that $\eta(U) \cap V \neq \emptyset$, then there is a nonempty open subset W of U such that $\eta(W) \subset V$.
(3) If $U \subset \mathcal{L}_Y^\infty$ is an open set and $F \subset X$ is a closed set such that $\eta(v) \cap F \neq \emptyset$ for each $v \in U$, then $\eta(U) \subset F$.
(4) There exists a quasi-continuous selection f of $\eta(\cdot)$ such that $\overline{Grf} = Gr\eta$.[8]
(5) Every selection f of $\eta(\cdot)$ is quasi-continuous and $\overline{Grf} = Gr\eta$.[9]

Finally, we say that an USCO, $\Psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$, is quasi-minimal if for some $\psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$, $[\Psi] = \{\psi\}$ (i.e., Ψ has one and only one minimal USCO). Let $\mathcal{Q}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$ denote the collection of all quasi-minimal USCOs. We will denote by

$$S_\Psi := \{v \in \mathcal{L}_Y^\infty : \Psi(v) \text{ is a singleton}\}, \tag{73}$$

the subset where Ψ takes singleton values. Under our primitives and assumptions, if $\Psi \in \mathcal{Q}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$, then by Lemma 7 in Anguelov and Kalenda (2009), S_Ψ is a residual set - and in particular, a G_δ set ρ_{w^*}-dense in $\mathcal{L}_Y^\infty$.

[8] A function $f^* : Z \to X$ is quasicontinuous at z^0 if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that inside the open ball, $B_{\rho_Z}(\delta, z^0)$, there is contained an open set, U, such that for all $z \in U$,

$$f^*(z) \in B_{\rho_X}(\varepsilon, f^*(z^0)).$$

[9] Note that if a function is continuous, it is automatically quasi-continuous.

7.4 w^*-Convergence and K-Convergence in $\mathcal{L}_Y^\infty$

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^\infty$, converges weak star to $v^* = (v_1^*(\cdot), \ldots, v_m^*(\cdot)) \in \mathcal{L}_Y^\infty$, denoted by $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$, if and only if

$$\int_\Omega \langle v^n(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \longrightarrow \int_\Omega \langle v^*(\omega), l(\omega)\rangle_{R^m}\, d\mu(\omega) \tag{74}$$

for all $l(\cdot) \in \mathcal{L}_{R^m}^1$.

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^\infty$, K-convergences (i.e., Komlos convergence - Komlos, 1967) to $\widehat{v} \in \mathcal{L}_Y^\infty$, denoted by $v^n \underset{K}{\longrightarrow} \widehat{v}$, if and only if every subsequence, $\{v^{n_k}(\cdot)\}_k, of\ \{v^n(\cdot)\}_n$ has an arithmetic mean sequence, $\{\widehat{v}^{n_k}(\cdot)\}_k$, where

$$\widehat{v}^{n_k}(\cdot) := \frac{1}{k}\sum_{q=1}^{k} v^{n_q}(\cdot), \tag{75}$$

such that

$$\widehat{v}^{n_k}(\omega) \underset{R^m}{\longrightarrow} \widehat{v}(\omega)\ a.e.\ [\mu]. \tag{76}$$

The relationship between w^*-convergence and K-convergence is summarized via the following results which follow from Balder (2000): For every sequence of value functions, $\{v^n\}_n \subset \mathcal{L}_Y^\infty$, and $\widehat{v} \in \mathcal{L}_Y^\infty$ the following statements are true:

$$\left.\begin{array}{c} \text{(i) If the sequence } \{v^n\}_n\ K\text{-converges to } \widehat{v}, \text{ then } \{v^n\}_n\ w^*\text{-converges to } \widehat{v}. \\ \\ \text{(ii) The sequence } \{v^n\}_n\ w^*\text{-converges to } \widehat{v} \text{ if and only if} \\ \text{every subsequence } \{v^{n_k}\}_k \text{ of } \{v^n\}_n \text{ has a further subsequence, } \{v^{n_{k_r}}\}_r, \\ K\text{-converging to } \widehat{v}. \end{array}\right\} \tag{77}$$

For any sequence of value function profiles, $\{v^n\}_n$, in $\mathcal{L}_Y^\infty$ it is automatic that

$$sup_n \int_\Omega \|v^n(\omega)\|_{R^m}\, d\mu(\omega) < +\infty. \tag{78}$$

Thus, by the classical Komlos Theorem (1967), any such sequence, $\{v^n\}_n$, has a subsequence, $\{v^{n_k}\}_k$ that K-converges to some K-limit, $\widehat{v} \in \mathcal{L}_Y^\infty$.

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THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

Economic and Social Research Council

Systemic Risk Centre

The London School of Economics
and Political Science
Houghton Street
London WC2A 2AE
United Kingdom

tel: +44 (0)20 7405 7686
systemicrisk.ac.uk
src@lse.ac.uk