# Areas of areas generate the shuffle algebra 

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#### Abstract

We consider the anti-symmetrization of the half-shuffle on words, which we call the 'area' operator, since it corresponds to taking the signed area of elements of the iterated-integral signature. The tensor algebra is a so-called Tortkara algebra under this operator. We show that the iterated application of the area operator is sufficient to recover the iterated-integral signature of a path. Just as the "information" that the second level adds to the first one is known to be equivalent to the area between components of the path, this means that all the information added by subsequent levels is equivalent to iterated areas. On the way to this main result, we characterize (homogeneous) generating sets of the shuffle algebra. We finally discuss compatibility between the area operator and discrete integration and stochastic integration, and conclude with some results on the linear span of the areas of areas.


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## 1. Introduction

We give a concise introduction here and spell out the notation more fully in the next section. The shuffle algebra $T\left(\mathbb{R}^{d}\right)$ over $d$ letters is the vector space spanned by words in the letters $1, \ldots, \mathrm{~d}$ with the commutative shuffle product. This is a free commutative algebra over the Lyndon words. Put differently, it can be viewed a polynomial algebra in new commuting variables $x_{w}$, where $w$ ranges over all Lyndon words. That is, as commutative algebras,

$$
\mathbb{R}\left[x_{w}: w \text { is Lyndon }\right] \cong T\left(\mathbb{R}^{d}\right)
$$

The isomorphism is given by $x_{w} \mapsto w \in T\left(\mathbb{R}^{d}\right)$. There are many more (free) generators known: any basis for the Lie algebra, coordinates of the first kind, ... (compare Corollary 4.5).
The relevance for iterated integrals is as follows. Let $X:[0, T] \rightarrow \mathbb{R}^{d}$ be a (piecewise smooth) curve and let $f_{i} \in T\left(\mathbb{R}^{d}\right), i \in I$, be a generating set of the shuffle algebra. Then: any term in the iterated-integrals signature $S(X)_{0, T}$ ([Che1954], [Lyo2007, Chapter 2]) is a polynomial in the real numbers

$$
\left\langle f_{i}, S(X)_{0, T}\right\rangle, i \in I
$$

Indeed, by assumption, any word $w$ can be written as

$$
w=P_{\mathrm{w}}\left(f_{i}: i \in I\right),
$$

where $P_{\mathrm{\amalg}}$ is some shuffle polynomial in finitely many of the $f_{i}$. By the shuffle identity we then get

$$
\left\langle w, S(X)_{0, T}\right\rangle=\left\langle P_{\amalg}\left(f_{i}: i \in I\right), S(X)_{0, T}\right\rangle=P\left(\left\langle f_{i}, S(X)_{0, T}\right\rangle: i \in I\right),
$$

where $P$ is the corresponding polynomial expression in the real numbers $\left\langle f_{i}, S(X)_{0, T}\right\rangle, i \in I$. The latter numbers then contain all the information of the iterated-integrals signature, since every iterated integral is a polynomial expression in them.
We are interested in whether there is a shuffle generating set in terms of "areas of areas". Define the following bilinear operation on $T\left(\mathbb{R}^{d}\right)$

$$
\operatorname{area}(x, y):=x \succ y-y \succ x,
$$

where $\succ$ denotes the half-shuffle. For $v, w \in T\left(\mathbb{R}^{d}\right)$, let

$$
V_{t}=\left\langle v, S(X)_{0, t}\right\rangle \quad \text { and } \quad W_{t}=\left\langle w, S(X)_{0, t}\right\rangle
$$

and define

$$
\operatorname{Area}(V, W)_{t}:=\int_{0}^{t} \int_{0}^{s} d V_{r} d W_{s}-\int_{0}^{t} \int_{0}^{s} d W_{r} d V_{s}
$$

We then have

$$
\operatorname{Area}(V, W)_{t}=\left\langle\operatorname{area}(v, w), S(X)_{0, t}\right\rangle
$$

Our naming of area and Area stems from the fact that $\operatorname{Area}(V, W)$ is (two times) the signed area (see Figure 1) enclosed by the two-dimensional curve ( $V, W$ ), [LP2006]. Note that the antisymmetrization $\operatorname{Area}(V, W)_{t}$ of the Riemann-Stieltjes integral (where the Riemann-Stieltjes integral forms a Zinbiel algebra on a suitable space of functions $V$ with $V_{0}=0$ ) has already been looked at as an algebraic operation by Rocha in 2003 in [Roc2003, Equation (7)], in [Roc2003b, Equation (6.11)] and in 2005 in [Roc2005, Equation (2.4)], it was even already noted by Rocha [Roc2003, page 321], [Roc2005, page 3] that the operation Area except being antisymmetric does not satisfy any additional identity of order three.
The following question is inspired by a remark made by T.L. during a talk in 2011:
Is repeated application of the Area operator enough to get the whole signature of a path $X$ ?
For $d=2$ and the first two levels, this is quickly verified. We start with the increments themselves, which we assume to be given (we think of them as " 0 -th order" areas), which are $\int d X^{1}$ and $\int d X^{2}$. Then we can write, using integration-by-parts,

$$
\begin{aligned}
& \iint d X^{1} d X^{1}=\frac{1}{2} \int d X^{1} \cdot \int d X^{1} \\
& \iint d X^{2} d X^{2}=\frac{1}{2} \int d X^{2} \cdot \int d X^{2} \\
& \iint d X^{1} d X^{2}=\frac{1}{2}\left(\iint d X^{1} d X^{2}-\iint d X^{2} d X^{1}+\int d X^{1} \cdot \int d X^{2}\right) \\
& \iint d X^{2} d X^{1}=\frac{1}{2}\left(-\left(\iint d X^{1} d X^{2}-\iint d X^{2} d X^{1}\right)+\int d X^{1} \cdot \int d X^{2}\right)
\end{aligned}
$$

and hence get all iterated integrals up to order 2 .
Products of integrals become, on the algebra side, $\boldsymbol{\text { -products. This reads as }}$

$$
\begin{array}{rlrl}
11 & =\frac{1}{2} 1 \text { ш } 1 & 22 & =\frac{1}{2} 2 \text { Ш } 2 \\
12 & =\frac{1}{2}(\operatorname{area}(1,2)+1 \text { ш } 2) & 21 & =\frac{1}{2}(-\operatorname{area}(1,2)+1 \text { ш } 2)
\end{array}
$$

In general, however, the expansion is non-unique, as the following example illustrates:

$$
\begin{aligned}
123= & \frac{1}{3} \operatorname{area}(1, \operatorname{area}(2,3))+\frac{1}{6} \operatorname{area}(\operatorname{area}(1,3), 2)+\frac{1}{3} 1 \amalg \operatorname{area}(2,3) \\
& -\frac{1}{6} 2 \amalg \operatorname{area}(1,3)+\frac{1}{2} 3 \amalg \operatorname{area}(1,2)+\frac{1}{6} 1 \amalg 2 \amalg 3 \\
= & \frac{1}{12} \operatorname{area}(1, \operatorname{area}(2,3))-\frac{1}{12} \operatorname{area}(\operatorname{area}(1,3), 2)+\frac{1}{4} \operatorname{area}(\operatorname{area}(1,2), 3) \\
& +\frac{1}{12} 1 \amalg \operatorname{area}(2,3)+\frac{1}{12} 2 \amalg \operatorname{area}(1,3)+\frac{1}{4} 3 \amalg \operatorname{area}(1,2)+\frac{1}{6} 1 \amalg 2 \amalg 3
\end{aligned}
$$

To formulate the problem algebraically, let $\mathscr{A} \subset T\left(\mathbb{R}^{d}\right)$ be the smallest linear space containing the letters $1, \ldots, \mathrm{~d}$ that is closed under the (bilinear, non-associative) operation area. The question then becomes:

$$
\text { Is } \mathscr{A} \text { a generating set for the shuffle algebra } T\left(\mathbb{R}^{d}\right) \text { ? }
$$

The affirmative answer to this question is given in this paper.
What we really have in mind here is a two-stage numerically-stable procedure for calculating the signature of a physical path. In the first stage one calculates areas, areas of areas and so forth, possibly using an analog physical apparatus. ${ }^{1}$ The second stage uses these measurements, say on a digital computer, and computes polynomial expressions in these.
The rest of the paper is structured as follows. In the next subsection we fix notation. In Section 2 we revisit results by Rocha [Roc2003] in purely algebraic terms. The outcome of this is a formula for the Dynkin operator applied to the signature. This make the areas operator appear naturally. Together with Section 4 this will prove the generating property of areas-of-areas.

For completeness, we show in Section 3 how to express coordinates of the first kind using only areas-of-areas. Again, this is basically a purely algebraic revisiting of results by Rocha, in which we also correct some of the expressions he gives.
In Section 4 we state a general condition for a set of polynomials to be (free) generators of the shuffle algebra $T\left(\mathbb{R}^{d}\right)$. We then show how a couple of well-known generators fall into this formulation and, how using Section 2 (or 3), the generating property of areas-of-areas is established in Corollary 4.8. The proof of that Corollary can also serve as a good roadmap for exploring the entire paper.

Apart from its geometric interpretation, the area operation possesses some interesting properties. Some of them we present in Section 5, where it is shown that it is nicely compatible with discrete integration as well as stochastic integration. In Section 6 we collect some results on the linear span generated by the area operator, as it is of interest in its own right.

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We would like to thank Cristopher Salvi for extended discussions, valuable presentations and new insights on the area operator, in particular its Jacobian bracketing (which helped shape the interpretation as signed volume) and the interplay with the shuffle product, and on coordinates of the second kind.

[^1]Tensor algebra and tree computations for this project have been done in python and sage, where besides standard packages and further custom code by the authors the python package free_lie_algebra.py [Rei2021], which implements a lot of the definitions in [Reu1993], has been of central use.


Figure 1.: The signed area of a curve $X$, shown at points $t=t_{1}$ (shaded blue) and at $t=T$ (shaded red).

### 1.1. Notation

Denote by $T\left(\left(\mathbb{R}^{d}\right)\right)$ the space of formal infinite linear combinations of words in the letters $1, . ., \mathrm{d}$. Equip it with the concatenation product • (often we write $b \cdot b^{\prime}=b b^{\prime}$ ).
Denote by $T\left(\mathbb{R}^{d}\right)$ its dual, the space of finite linear combinations of words. Equip it with the shuffle product $\amalg$. It decomposes as

$$
a \amalg a^{\prime}=a \succ a^{\prime}+a^{\prime} \succ a,
$$

where $\succ$ is the half-shuffle. The half-shuffle is defined on words $a=a_{1} \ldots a_{m}, b=b_{1} \ldots b_{n}$, where $b$ is not the empty word, as

$$
a \succ b=\left(a \amalg b_{1} \ldots b_{n-1}\right) \cdot b_{n} .
$$

The dual pairing is written for $a \in T\left(\mathbb{R}^{d}\right), b \in T\left(\left(\mathbb{R}^{d}\right)\right)$ as

$$
\langle a, b\rangle .
$$

Denote the grouplike elements of $T\left(\left(\mathbb{R}^{d}\right)\right)$ by $G$. Denote the primitive elements, or Lie elements, of $T\left(\left(\mathbb{R}^{d}\right)\right)$, i.e. the free Lie algebra, by $\mathfrak{g}$.
Denote by $\operatorname{proj}_{n}, \operatorname{proj}_{\geq n}$, etc, the projection on $T\left(\left(\mathbb{R}^{d}\right)\right)$ to level $n$, to levels larger equal to $n$, $\ldots$ We write $T_{n}\left(\left(\mathbb{R}^{d}\right)\right)=\operatorname{proj}_{n} T\left(\left(\mathbb{R}^{d}\right)\right), T_{\geq n}\left(\left(\mathbb{R}^{d}\right)\right)=\operatorname{proj}_{\geq n} T\left(\left(\mathbb{R}^{d}\right)\right)$, etc. Denote the empty word by $e$.
Denote by $\mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle$ the free tensor algebra over $d$ generators with coefficients in the ring $\mathcal{R}$, where $\mathcal{R}:=\left(T\left(\mathbb{R}^{d}\right), ш\right)$, and by $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ the corresponding space of tensor series. We then
canonically have $\mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle \subsetneq \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$, and identify the $\mathcal{R}$-algebra $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ with

$$
\begin{equation*}
\mathcal{W}:=\prod_{n=1}^{\infty} T\left(\mathbb{R}^{d}\right) \otimes T_{n}\left(\mathbb{R}^{d}\right) \tag{1.1}
\end{equation*}
$$

where we use the shuffle product on the left and the concatenation product on the right. We denote the product on both $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ and $\mathcal{W}$, which are isomorphic as $\mathbb{R}$-algebras, by $\mathbf{\square}$. The $\mathcal{R}$-subalgebra $(\mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle, \mathbf{\square})$ is then $\mathbb{R}$-algebra-isomorphic to $\left(T\left(\mathbb{R}^{d}\right) \otimes T\left(\mathbb{R}^{d}\right), \mathbf{■}\right)$.
We use the usual grading on $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$, that is in the representation (1.1), for $a, b$ words, $|a \otimes b|:=|b|$. Then, the projection $\operatorname{proj}_{n}$ makes also sense on $\mathcal{W}$.
We furthermore introduce an $\mathcal{R}$-linear coproduct on $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$, which maps to the graded completion of the $\mathcal{R}$-module tensor product $\boxtimes$ :

$$
\Delta_{\underline{\Perp}}: \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle \rightarrow \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle \hat{\otimes} \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle:=\prod_{m, n=1}^{\infty} \operatorname{proj}_{m} \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle \boxtimes \operatorname{proj}_{n} \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle,
$$

where the unshuffle coproduct on $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ is defined via the usual unshuffle coproduct as

$$
\Delta_{\underline{\Perp}}\left(\sum_{w} a_{w} \underline{w}\right):=\sum_{w} a_{w} \Delta_{\underline{\Perp}} \underline{w}:=\sum_{w} a_{w} \sum_{(w)}^{\amalg} \underline{w_{1}} \boxtimes \underline{w_{2}},
$$

where the last Sweedler summation is well defined by the unshuffle coproduct on $T\left(\mathbb{R}^{d}\right)$ because there is a unique $\mathbb{R}$-linear map $T\left(\mathbb{R}^{d}\right) \otimes T\left(\mathbb{R}^{d}\right) \rightarrow \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle \hat{\otimes} \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ characterized by sending each tensor pair of words $w \otimes v$ to $\underline{w} \boxtimes \underline{v}$ (which is however non-surjective). We have the isomorphism

$$
\prod_{m, n=1}^{\infty} T\left(\mathbb{R}^{d}\right) \otimes T_{m}\left(\mathbb{R}^{d}\right) \otimes T_{n}\left(\mathbb{R}^{d}\right) \cong \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle \hat{\otimes} \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle
$$

as $\mathbb{R}$ vector spaces given by the map

$$
\sum_{w, v} a_{w, v} \otimes w \otimes v \mapsto \sum_{w, v} a_{w, v}(\underline{w} \boxtimes \underline{v})=\sum_{w, v}\left(a_{w, v} \underline{w}\right) \boxtimes \underline{v}=\sum_{w, v} \underline{w} \boxtimes\left(a_{w, v} \underline{v}\right) .
$$

The unshuffle coproduct on $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ is an $\mathcal{R}$-algebra homomorphism as a consequence of the homomorphism property of the usual unshuffle coproduct, as for words $w, v$ we have

$$
\begin{aligned}
& \Delta_{\underline{\Perp}}(p \underline{w} ■ q \underline{v})=(p \amalg q) \Delta_{\underline{\amalg}} \underline{w \cdot v}=(p \amalg q) \sum_{(w),(v)}^{\amalg} \underline{w_{1} \cdot v_{1}} \boxtimes \underline{w_{2} \cdot v_{2}}
\end{aligned}
$$

$$
\begin{align*}
& =\left(\Delta_{\underline{\Perp}} p \underline{w}\right) \tilde{\mathbf{m}^{2}}\left(\Delta_{\underline{\underline{m}} q \underline{v}}\right), \tag{1.2}
\end{align*}
$$

where

$$
\left(\sum_{w_{1}, v_{1}} a_{w_{1}, v_{1}} \underline{w_{1}} \boxtimes \underline{v_{1}}\right) \tilde{■}\left(\sum_{w_{2}, v_{2}} b_{w_{2}, v_{2}} \underline{w_{2}} \boxtimes \underline{v_{2}}\right):=\sum_{w_{1}, v_{1}, w_{2}, v_{2}}\left(a_{w_{1}, v_{1}} \amalg b_{w_{2}, v_{2}}\right)\left(\underline{w_{1}} \llbracket \underline{w_{2}}\right) \boxtimes\left(\underline{v_{1}} \underline{\boxminus} \underline{v_{2}}\right)
$$

is the usual induced product on the tensor product. When restricting to $\mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle$, we have $\Delta_{\underline{\underline{m}}}$ : $\mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle \rightarrow \mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle \boxtimes \mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle$ and the other compatibility relations of a Hopf algebra are checked along the same lines, so we indeed get an $\mathcal{R}$-Hopf algebra ( $\mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle, \boldsymbol{\Perp}, \Delta_{\underline{\underline{w}}}, \underline{\alpha}$ ), a Hopf algebra in the category of $\mathcal{R}$-modules, with antipode

$$
\underline{\alpha}\left(\sum_{w} a_{w} \underline{w}\right)=\sum_{w}(-1)^{|w|} a_{w} \underline{\overleftarrow{w}},
$$

where $\overleftarrow{w}$ is $w$ written backwards.
Now, since we have the homomorphism property of the unshuffle $\Delta_{\underline{\Perp}}$ on $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ according to Equation (1.2), and furthermore

$$
\Delta_{\underline{\underline{m}}}(\underline{\mathrm{i}})=\underline{e} \boxtimes \underline{\mathrm{i}}+\underline{\mathrm{i}} \boxtimes \underline{e}
$$

for any letter $\underline{\underline{i}}$ in $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$, our $\Delta_{\underline{\underline{~}}}$ is exactly the coproduct from [Reu1993] for the choice of $K$ as the unital commutative ring $\mathcal{R}$ with characteristic zero. Thus, we may apply all the theory in Reutenauer's book valid for the general setting of a unital commutative ring of characteristic zero to $\mathcal{R}\langle\langle 1, \ldots, d\rangle\rangle$. In particular, we get that the group

$$
\underline{G}=\left\{g \in \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle \mid \Delta_{\underline{\Perp}} g=g \boxtimes g, g \neq 0\right\}
$$

with product $\quad$ [Reu1993, Corollary 3.3] and the $\mathcal{R}$-Lie-algebra

$$
\underline{\mathfrak{g}}=\left\{x \in \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle \mid \Delta_{\underline{\underline{\bullet}}} x=\underline{e} \boxtimes x+x \boxtimes \underline{e}\right\}
$$

with Lie bracket $[x, y]:=x \llbracket y-y \llbracket x$ are in a one-to-one correspondence [Reu1993, Theorems 3.1 and 3.2 ] via the exponential map [Reu1993, Equation (3.1.2)]

$$
\exp _{\mathbf{\bullet}}: \underline{\mathfrak{g}} \rightarrow \underline{G}, \quad \exp _{\mathbf{a}}(x)=\underline{e}+\sum_{n=1}^{\infty} \frac{x^{\mathbf{n}}}{n!}
$$

with inverse the logarithm [Reu1993, Equation (3.1.1)]

$$
\log _{\mathbf{\bullet}}: \underline{G} \rightarrow \underline{\mathfrak{g}}, \quad \log _{\mathbf{\bullet}}(g)=\sum_{n=1}^{\infty}(-1)^{n+1} \frac{(g-\underline{e})^{\boldsymbol{n}}}{n}
$$

Analogous to $G$ and $\mathfrak{g}$, we call the elements of $\underline{G}$ grouplike and the elements of $\underline{\mathfrak{g}}$ primitive.
Note however that $\left(\mathcal{R}\langle 1, \ldots, \mathrm{~d}\rangle, \mathbf{■}, \Delta_{\underline{\Perp}}\right)$ does not form a $\mathbb{R}$ Hopf algebra.
Fixing $x \in T\left(\left(\mathbb{R}^{d}\right)\right)$, define for any $F=\sum_{w} a_{w} \otimes w \in \mathcal{W}$, where $a_{w} \in T\left(\mathbb{R}^{d}\right)$ for all words $w$,

$$
\operatorname{eval}_{x}(F):=\sum_{w}\left\langle x, a_{w}\right\rangle w \in T\left(\left(\mathbb{R}^{d}\right)\right)
$$

This operation now forms an associative algebra isomorphism from $(\mathcal{W}, \mathbf{\square})$ to $\left(\mathcal{L}\left(T\left(\left(\mathbb{R}^{d}\right)\right), T\left(\left(\mathbb{R}^{d}\right)\right)\right), *\right)$, where $\mathcal{L}\left(T\left(\left(\mathbb{R}^{d}\right)\right), T\left(\left(\mathbb{R}^{d}\right)\right)\right)$ denotes the linear maps from $T\left(\left(\mathbb{R}^{d}\right)\right)$ to $T\left(\left(\mathbb{R}^{d}\right)\right)$ which are continuous in the product topology and $*$ denotes the convolution product of the Hopf algebra $\left(T\left(\mathbb{R}^{d}\right), \cdot, \Delta_{\mathrm{w}}\right)$
extended to $T\left(\left(\mathbb{R}^{d}\right)\right)$. Indeed, for any $F, G \in \mathcal{W}$, we have eval $(F)$, eval $(G) \in \mathcal{L}\left(T\left(\left(\mathbb{R}^{d}\right)\right), T\left(\left(\mathbb{R}^{d}\right)\right)\right)$ by definition with

$$
\operatorname{eval}(F ■ G)=\operatorname{eval}(F) * \operatorname{eval}(G),
$$

since for $F=\sum_{w} a_{w} \otimes w, G=\sum_{w^{\prime}} b_{w^{\prime}} \otimes w^{\prime}, a_{w}, b_{w} \in T\left(\mathbb{R}^{d}\right)$ for all words $w$, and $x \in T\left(\left(\mathbb{R}^{d}\right)\right)$,

$$
\begin{aligned}
\operatorname{eval}_{x}(F ■ G) & =\sum_{w, w^{\prime}}\left\langle x, a_{w} \amalg b_{w^{\prime}}\right\rangle w \cdot w^{\prime}=\sum_{w, w^{\prime}}\left\langle\Delta_{\amalg} x, a_{w} \otimes b_{w^{\prime}}\right\rangle w \cdot w^{\prime} \\
& =\sum_{(x)}^{\amalg} \sum_{w}\left\langle x_{1}, a_{w}\right\rangle w \cdot \sum_{w^{\prime}}\left\langle x_{2}, b_{w^{\prime}}\right\rangle w^{\prime}=\sum_{(x)}^{\amalg} \operatorname{eval}_{x_{1}}(F) \cdot \operatorname{eval}_{x_{2}}(G) \\
& =\operatorname{conc}(\operatorname{eval}(F) \otimes \operatorname{eval}(G)) \Delta_{\uplus} x=(\operatorname{eval}(F) * \operatorname{eval}(G))(x),
\end{aligned}
$$

where $\sum_{(x)}^{\amalg} x_{1} \otimes x_{2}:=\Delta_{\amalg} x$ is Sweedler's notation and conc : $T\left(\left(\mathbb{R}^{d}\right)\right) \hat{\otimes} T\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow T\left(\left(\mathbb{R}^{d}\right)\right)$ is the continuous linear map corresponding to the bilinear map $\cdot$
Likewise, for arbitrary $y \in T\left(\mathbb{R}^{d}\right)$ and $F=\sum_{w} a_{w} \otimes w \in \mathcal{W}, a_{w} \in T\left(\mathbb{R}^{d}\right)$, we define

$$
\operatorname{coeval}^{y}(F):=\sum_{w}\langle w, y\rangle a_{w} \in T\left(\mathbb{R}^{d}\right) .
$$

Then, coeval forms an isomorphism from $(\mathcal{W}, \mathbf{\square})$ to $\left(\mathrm{L}\left(T\left(\mathbb{R}^{d}\right), T\left(\mathbb{R}^{d}\right)\right), \star\right)$, where $\mathrm{L}\left(T\left(\mathbb{R}^{d}\right), T\left(\mathbb{R}^{d}\right)\right)$ denotes all linear maps from $T\left(\mathbb{R}^{d}\right)$ to itself and $\star$ is the convolution product of the Hopf algebra $\left(T\left(\mathbb{R}^{d}\right), \amalg, \Delta.\right)$.
We refer to [Reu1993] for more details on all of this, except for the half-shuffle, for which a nice entry point to the literature is for example [FP2013].

### 1.2. Objectives

### 1.2.1. Revisiting the work of Rocha on coordinates of the first kind

In Sections 2 and 3, we show how the area operation appears naturally in a purely algebraic formulation of the work of Rocha on coordinates of the first kind. What we may take over from Rocha here is a very interesting network of bilinear operations on $\mathcal{W}$ refining the basic product. It is based on a dendriform structure, as the following diagram (cf. [Roc2003, diagram page 320], [Roc2003b, Diagram 5]) and definition show:


For $A=p \otimes q, B=p^{\prime} \otimes q^{\prime}$,

$$
\begin{aligned}
& A \succeq B:=\left(p \succ p^{\prime}\right) \otimes\left(q \cdot q^{\prime}\right), \\
& A \preceq B:=\left(p^{\prime} \succ p\right) \otimes\left(q \cdot q^{\prime}\right) \text {, } \\
& A \_B:=A \succeq B+A \preceq B=\left(p \amalg p^{\prime}\right) \otimes\left(q \cdot q^{\prime}\right), \\
& A \triangleright B:=A \succeq B-B \preceq A=\left(p \succ p^{\prime}\right) \otimes\left[q, q^{\prime}\right], \\
& A \triangleright_{\mathrm{Sym}} B:=A \triangleright B+B \triangleright A=A \succeq B+B \succeq A-A \preceq B-B \preceq A=\operatorname{area}\left(p, p^{\prime}\right) \otimes\left[q, q^{\prime}\right] \text {, } \\
& {[A, B]:=A \backsim B-B ■ A=A \triangleright B-B \triangleright A=A \succeq B+A \preceq B-B \succeq A-B \preceq A} \\
& =\left(p ш p^{\prime}\right) \otimes\left[q, q^{\prime}\right] .
\end{aligned}
$$

In fact, one could describe this network for any dendriform structure, but Rocha's and our work offer a promising first usecase for talking about all of these operations together, while this system of operations without $\triangleright_{\text {Sym }}$ has been explored before e.g. in [KM2009]. The symmetrized preLie operation $\triangleright_{\text {Sym }}$ stays the most mysterious also to us, we may only point to the discovery of Bergeron and Loday in [BL2011] that the symmetrization of pre-Lie does not in general satisfy any further identities except non-associative commutativity, though since the pre-Lie product $\triangleright$ certainly isn't free we except some kind of relations for $\triangleright_{\text {Sym }}$ also, but this is still a question of future work.

With the area operation forming the left part of the symmetrized pre-Lie operation $\triangleright_{\text {Sym }}$, we obtain our main argument to show that the set of all areas of areas forms a shuffle generating set, albeit not a minimal one.

### 1.2.2. Areas of areas and further shuffle generating sets

With our main focus being shuffle-generating sets in terms of areas of areas, in Section 4 we first give a general criterion Lemma 4.2 for (homogeneous) subsets to form a shuffle generating set (resp. a free shuffle generating set). The condition being that the set contains (resp. forms) a dual basis to some basis of the free Lie algebra $\mathfrak{g} \subsetneq T\left(\left(\mathbb{R}^{d}\right)\right)$. While our actual hands-on proof is based on the characterization of the annihilator of the free Lie algebra which we cite from [Reu1993], we sketch a more abstract argument in Remark 4.3 related to the Milnor-Moore theorem.

We continue by illustrating how our statement can be applied to some known shuffle generating sets, as well as to the image of $\rho$ (the dual of the Dynkin map, Section 2), which concludes one of our proofs that areas-of-areas generate the shuffle algebra.

### 1.2.3. The area Tortkara algebra

We study the smallest linear subspace $\mathscr{A} \subsetneq T\left(\mathbb{R}^{d}\right)$ closed under the area operation and containing the letters in Section 6. Thanks to the work by Dzhumadil'daev, Ismailov and Mashurov, we can use the categorial framework of Tortkara algebras, where the objects are characterized as vector spaces with a bilinear antisymmetric operation which furthermore satisfy the Tortkara identity, and the morphisms are homomorphisms of the bilinear operations as usual. Also thanks to [DIM2018], we have a simple linear basis of $\mathscr{A}$ in terms of linear combinations of words, see

Lemma 6.1. We continue by a very important conjecture that the left bracketings of the area operation yield another basis of $\mathscr{A}$, which was shown for dimension two in both [DIM2018] and [Rei2018]. The rest of the section is dedicated to some interesting observations we made while, so far unsuccessfully, trying to prove that conjecture for any dimension.

### 1.2.4. Applications and characterizations

In Section 5 we are connecting the purely algebraic considerations of this paper with the world of (deterministic and probabilistic) path spaces and iterated-integrals signatures on these path spaces, as they have been the motivation for this work to begin with. What we are generally looking at are characterizations of the area Tortkara algebra $\mathscr{A}$ in terms of special properties for given path spaces, like the space of piecewise linear paths (Subsection 5.1). The case of piecewise linear paths promises in fact to develop into the main application of the study of areas of areas. They form the most common discretization of general continuous paths that one works with when actually computing iterated-integrals signature numerically, in machine learning for example. It turns out that for piecewise linear paths, the computation of discrete areas is much simpler and better behaved than the computation of discrete integrals.
However, besides the discrete deterministic setting, the study of signatures has, since Lyons' theory of rough paths, been intimately related with stochastic analysis, and we observe how areas of areas preserve the martingale property central in stochastic analysis, while general iterated Stratonovich integrals fail to do so.

## 2. The Dynkin operator

We recall the linear maps $r, D: T\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow T\left(\left(\mathbb{R}^{d}\right)\right)$ from [Reu1993, Section 1, p.20]. The linear right-bracketing map or Dynkin operator $r$ is given on a word $w=l_{1} \ldots l_{n}$ as

$$
\begin{equation*}
r\left(1_{1} \ldots 1_{n}\right):=\left[1_{1},\left[1_{2}, \ldots\left[1_{n-1}, 1_{n}\right]\right]\right], \tag{2.1}
\end{equation*}
$$

with $r(e)=0$ and $r(\mathrm{i})=\mathrm{i}$ for any letter i . The map $D$ (for derivation) is given on a word $w$ as

$$
D(w):=|w| w,
$$

where $|w|$ is the length of the word. On $T_{\geq 1}\left(\left(\mathbb{R}^{d}\right)\right), D$ is invertible with inverse $D^{-1}(w)=\frac{1}{|w|} w$.
Remark 2.1. 1. The seemingly simple Dynkin operator $r$ has found several applications. It for example characterizes Lie elements of $T\left(\left(\mathbb{R}^{d}\right)\right.$ ) [Reu1993, Theorem 3.1 (vi)]: $x \in T\left(\left(\mathbb{R}^{d}\right)\right)$ is a Lie series if and only if $\langle e, x\rangle=0$ and $r(x)=D(x)$. See also [PR2002], [Gar1990] and references therein. In the backward error analysis of numerical schemes it is used for example in [LMK2013].
2. Truncated at a fixed level, the grouplike elements / signatures of tree-reduced paths, form a Lie group. The Dynkin operator $r$ is a logarithmic derivative, i.e. the derivative pulled-back to the tangent space at the identity, of an endomorphism of this group in the following sense (see [MP2013] for more on this). Let $\delta_{\epsilon}$ be the dilation operator, i.e. the operation on tensors which
multiplies each level $m$ by $\epsilon^{m}$, which corresponds to dilating or scaling a path by the factor $\epsilon$. For $g \in G$, let $g^{\epsilon}:=\delta_{\epsilon} g$. Then

$$
\begin{aligned}
\left(\frac{d}{d \epsilon} g^{\epsilon}\right) \cdot\left(g^{\epsilon}\right)^{-1} & =\left(\frac{d}{d \epsilon} g^{\epsilon}\right) \cdot \alpha\left[g^{\epsilon}\right]=\left(\frac{1}{\epsilon} D\left[g^{\epsilon}\right]\right) \cdot \alpha\left[g^{\epsilon}\right]=\frac{1}{\epsilon}(\operatorname{conc} \circ(D \otimes \alpha))\left[g^{\epsilon} \otimes g^{\epsilon}\right] \\
& =\frac{1}{\epsilon}\left(\operatorname{conc} \circ(D \otimes \alpha) \circ \Delta_{ш}\right)\left[g^{\epsilon}\right] \quad \text { as } g^{\epsilon} \in G,[\text { Reu1993, Theorem 3.2] } \\
& =\frac{1}{\epsilon} r\left[g^{\epsilon}\right], \quad \text { see [Reu1993, p32 or Lemma 1.5] }
\end{aligned}
$$

where $\alpha$ is the antipode on $T\left(\left(\mathbb{R}^{d}\right)\right.$ ) (which is the inverse in the Lie group, and corresponds to reversing a path), $\otimes$ is the external tensor product, conc is the linear map taking $a \otimes b$ to $a \cdot b$, and $\Delta_{\mathrm{w}}$ is the unshuffle coproduct, which [Reu1993] denotes with $\delta$.

Let $\underline{r}, \underline{D}, \underline{D}^{-1}$ act on $\mathcal{W}$ by letting $r, D, D^{-1}$ act on the right side of the tensor, i.e.

$$
\begin{aligned}
\underline{r}(a \otimes b) & :=a \otimes r(b) \\
\underline{D}(a \otimes b) & :=a \otimes D(b) \\
\underline{D}^{-1}(a \otimes b) & :=a \otimes D^{-1}(b) .
\end{aligned}
$$

Define ${ }^{2}$

$$
\begin{align*}
& S:=\sum_{w} w \otimes w \\
& R:=\underline{r}(S)=\sum_{w} w \otimes r(w)=\sum_{v} \rho(v) \otimes v . \tag{2.2}
\end{align*}
$$

Both are elements of $\mathcal{W}$. The last equality implicitly defines $\rho$. There also exists a recursive definition given by $\rho(e)=0, \rho(i)=i$ for any letter $i$ and

$$
\begin{equation*}
\rho(\mathrm{i} w \mathrm{j})=\mathrm{i} \rho(w \mathrm{j})-\mathrm{j} \rho(\mathrm{i} w) \tag{2.3}
\end{equation*}
$$

for any (empty or non-empty) word $w$ and letters $\mathbf{i}, \mathbf{j}$, see [Reu1993, p.32]. Based on this recursion, we derive an expansion of $\rho$ via an action of elements of the symmetric group algebra in Proposition 6.11. We repeat that $r(e)=\rho(e)=0$, so the sum in (2.2) is actually only taken over words of strictly positive length.
We record the following for future use ([Reu1993, Theorem 1.12]). For any word $w^{3}$

$$
\begin{equation*}
D w=\sum_{u v=w} \rho(u) \amalg v=\sum_{u v=w,|u| \geq 1} \rho(u) ш v . \tag{2.4}
\end{equation*}
$$

Note that this yields yet another recursive definition of $\rho$ :

$$
\rho(e)=0, \quad \rho(w)=|w| w-\sum_{\substack{u v=w \\|u|,|v| \geq 1}} \rho(u) ш v,
$$

where $w$ is an arbitrary non-empty word.

[^2]Proposition 2.2. The map $r: G \rightarrow \mathfrak{g}$ is invertible. To be specific, define for $x \in T_{\geq 1}\left(\left(\mathbb{R}^{d}\right)\right)$ the linear map

$$
\begin{aligned}
A_{x}: T\left(\left(\mathbb{R}^{d}\right)\right) & \rightarrow T\left(\left(\mathbb{R}^{d}\right)\right) \\
z & \mapsto D^{-1}(x z)
\end{aligned}
$$

Then for $x \in \mathfrak{g}$

$$
\begin{align*}
r^{-1}[x] & =\sum_{\ell \geq 0} A_{x}^{\ell} e  \tag{2.5}\\
& =e+D^{-1}(x)+D^{-1}\left(x D^{-1}(x)\right)+D^{-1}\left(x D^{-1}\left(x D^{-1}(x)\right)\right)+. .
\end{align*}
$$

Equivalently, with $\underline{A}_{R^{z}}:=\underline{D}^{-1}[R ■ z]$,

$$
\begin{align*}
S & =\sum_{\ell \geq 0}\left(\underline{A}_{R}\right)^{\ell}(e \otimes e)  \tag{2.6}\\
& =e \otimes e+\underline{D}^{-1}[R]+\underline{D}^{-1}\left[R ■ \underline{D}^{-1}[R]\right]+\underline{D}^{-1}\left[R ■ \underline{D}^{-1}\left[R ■ \underline{D}^{-1}[R]\right]\right]+\ldots
\end{align*}
$$

Remark 2.3. Compare [EGP200', Theorem 4.1] for a statement in a more general setting.

Proof of Proposition 2.2. The claimed equivalence is shown as follows. For $t \in \mathcal{W}$, with zero coefficient for $e \otimes e$,

$$
\operatorname{eval}_{g}\left(\underline{D}^{-1}(t)\right)=D^{-1}\left(\operatorname{eval}_{g}(t)\right)
$$

Hence

$$
\begin{aligned}
g & =e+D^{-1}(r(g))+D^{-1}\left(r(g) D^{-1}(r(g))\right)+. . \quad \forall g \in G \\
& \Leftrightarrow \\
\operatorname{eval}_{g}(S) & =\operatorname{eval}_{g}(e)+D^{-1}\left(\operatorname{eval}_{g}(R)\right)+D^{-1}\left(\operatorname{eval}_{g}(R) D^{-1}\left(\operatorname{eval}_{g}(R)\right)\right)+. . \quad \forall g \in G \\
& =\operatorname{eval}_{g}(e)+\operatorname{eval}_{g}\left(\underline{D}^{-1}(R)\right)+\operatorname{eval}_{g}\left(\underline{D}^{-1}\left(R ■ \underline{D}^{-1}(R)\right)\right)+. . \\
& \Leftrightarrow \\
S & =e+\underline{D}^{-1}(R)+\underline{D}^{-1}\left(R ■ \underline{D}^{-1}(R)\right)+. .
\end{aligned}
$$

where we used the homomorphism property of $\mathrm{eval}_{g}$ and the fact that grouplike elements linearly span $T\left(\left(\mathbb{R}^{d}\right)\right)$ projectively (i.e. truncated, at level $n$, grouplike elements linearly span $\left.T_{\leq n}\left(\left(\mathbb{R}^{d}\right)\right)\right)$. We now show (2.5). Write $x:=r[g]=D[g] \cdot g^{-1}$ (compare Remark 2.1.2). Then

$$
g=e+D^{-1}[x \cdot g]
$$

i.e.

$$
\begin{equation*}
g=e+A_{x} g \tag{2.7}
\end{equation*}
$$

Now since $x$ does not contain a component in the empty word, this actually amounts to a recursive formula,

$$
\operatorname{proj}_{0} g=e, \quad \operatorname{proj}_{n} g=\sum_{m=1}^{n} A_{\operatorname{proj}_{m} x}\left(\operatorname{proj}_{n-m} g\right), \quad n \geq 1
$$

and thus Equation (2.7) has a unique solution. Hence

$$
g=\sum_{\ell \geq 0} A_{x}^{\ell} e
$$

since the series converges due to being a finite sum for each homogeneous component and obviously provides a solution for Equation (2.7).

This shows that (2.5) gives a left-inverse.
It is also a right inverse. Indeed, first note that for $x \in \mathfrak{g}$ and $n \geq 2$ we have $r\left[A_{x}^{n} e\right]=0$. For $n=2$, using Lemma 3.3, this follows from

$$
\begin{aligned}
r\left[A_{x}^{2} e\right] & =r\left[D^{-1}\left(x D^{-1}(x)\right)\right]=D^{-1}\left(r\left[x D^{-1}(x)\right]\right)=D^{-1}\left(r\left[r\left[D^{-1} x\right] D^{-1}(x)\right]\right) \\
& =D^{-1}\left(\left[r\left[D^{-1} x\right], r\left[D^{-1}(x)\right]\right]\right)=0
\end{aligned}
$$

Assume it is true for $A_{x}^{n-1}$, then

$$
\begin{aligned}
r\left[A_{x}^{n} e\right] & =r\left[D^{-1}\left(x A_{x}^{n-1} e\right)\right]=D^{-1}\left(r\left[x A_{x}^{n-1} e\right]\right)=D^{-1}\left(r\left[r\left[D^{-1} x\right] A_{x}^{n-1} e\right]\right) \\
& =D^{-1}\left(\left[r\left[D^{-1} x\right], r\left[A_{x}^{n-1} e\right]\right]\right)=0
\end{aligned}
$$

Hence

$$
r\left[e+D^{-1}(x)+D^{-1}\left(x D^{-1}(x)\right)+. .\right]=x
$$

so that the Lemma indeed provides a right inverse.

Definition 2.4. Define the following product on $\mathcal{W}$,

$$
(p \otimes q) \triangleright\left(p^{\prime} \otimes q^{\prime}\right):=\left(p \succ p^{\prime}\right) \otimes\left[q, q^{\prime}\right]
$$

where [.,.] is the Lie bracket in $T\left(\left(\mathbb{R}^{d}\right)\right)$ and $\succ$ is the half-shuffle in $T\left(\mathbb{R}^{d}\right)$.
Remark 2.5. This product is pre-Lie, as the tensor product of a Zinbiel algebra and a Lie algebra is always a pre-Lie algebra (this is shown in Rocha's thesis as [Roc2003b, Proposition 4.13 and Corollary 4.14], though there the terminology 'chronological algebra' is used to mean what we call pre-Lie algebra), although we will not use this fact. It comes from the dendriform structure

$$
\begin{aligned}
& (p \otimes q) \succeq\left(p^{\prime} \otimes q^{\prime}\right):=\left(p \succ p^{\prime}\right) \otimes q q^{\prime} \\
& (p \otimes q) \preceq\left(p^{\prime} \otimes q^{\prime}\right):=\left(p^{\prime} \succ p\right) \otimes q q^{\prime}
\end{aligned}
$$

i.e. $x \triangleright y=x \succeq y-y \preceq x$. Indeed, the operations $\succeq$ and $\preceq$ together satisfy the three dendriform identities (e.g. [KM2009, Equations (8)-(10)]), which is a straightforward consequence of the Zinbiel identity of the halfshuffle and the associativity of the concatenation,

$$
\begin{aligned}
(A \preceq B) \preceq C & =\left(p_{3} \succ\left(p_{2} \succ p_{1}\right)\right) \otimes q_{1} q_{2} q_{3}=\left(\left(p_{3} \succ p_{2}\right) \succ p_{1}\right) \otimes q_{1} q_{2} q_{3}+\left(\left(p_{2} \succ p_{3}\right) \succ p_{1}\right) \otimes q_{1} q_{2} q_{3} \\
& =A \preceq(B \preceq C)+A \preceq(C \preceq B), \\
A \succeq(B \succeq C) & =\left(p_{1} \succ\left(p_{2} \succ p_{3}\right)\right) \otimes q_{1} q_{2} q_{3}=\left(\left(p_{1} \succ p_{2}\right) \succ p_{3}\right) \otimes q_{1} q_{2} q_{3}+\left(\left(p_{2} \succ p_{1}\right) \succ p_{3}\right) \otimes q_{1} q_{2} q_{3} \\
& =(A \succeq B) \succeq C+(B \succeq A) \succeq C, \\
(A \succeq B) \preceq C & =\left(p_{3} \succ\left(p_{1} \succ p_{2}\right)\right) \otimes q_{1} q_{2} q_{3}=\left(\left(p_{3} \succ p_{1}+p_{1} \succ p_{3}\right) \succ p_{2}\right) \otimes q_{1} q_{2} q_{3} \\
& =\left(p_{1} \succ\left(p_{3} \succ p_{2}\right)\right) \otimes q_{1} q_{2} q_{3}=A \succeq(B \preceq C),
\end{aligned}
$$

for $A=p_{1} \otimes q_{1}, B=p_{2} \otimes q_{2}, C=p_{2} \otimes q_{3}$.
For more background on pre-Lie products and this relation to dendriform algebras see for example [KM2009] and references therein.

The object $R$ satisfies a quadratic fixed-point equation.
Lemma 2.6.

$$
\begin{equation*}
(\underline{D}-\mathrm{id}) R=R \triangleright R . \tag{2.8}
\end{equation*}
$$

Proof. Let $|w| \geq 1$. Starting from (2.4) and concatenating a letter a from the right on both sides, we get

$$
\sum_{u v=w,|u| \geq 1}(\rho(u) ш v) \mathrm{a}=(D w) \mathrm{a}=(D-\mathrm{id})(w \mathrm{a}) .
$$

Hence

$$
\sum_{u v=w,|u| \geq 1} \rho(u) \succ v \mathrm{a}=(D-\mathrm{id})(w \mathrm{a}),
$$

which means, for $|\bar{w}| \geq 2$,

$$
\begin{equation*}
\sum_{u v=\bar{w},|u| \geq 1,|v| \geq 1} \rho(u) \succ v=(D-\mathrm{id}) \bar{w} . \tag{2.9}
\end{equation*}
$$

Recall

$$
\begin{aligned}
\operatorname{ad}_{v} w & =[v, w] \\
\operatorname{Ad}_{v} w & =\left[\mathrm{k}_{1},\left[\mathrm{k}_{2}, . .,\left[\mathrm{k}_{n}, w\right] . .\right]\right],
\end{aligned}
$$

where $v=\mathrm{k}_{1} \cdots \mathrm{k}_{n}$. By [Reu1993, Theorem 1.4], for a Lie polynomial $P$ one has

$$
\begin{equation*}
\operatorname{ad}_{P}=\operatorname{Ad}_{P} . \tag{2.10}
\end{equation*}
$$

For a word $w$ define the linear map $I_{w}$ as

$$
I_{w} x:=w \succ x,
$$

and extend linearly to the whole tensor algebra. The map

$$
I_{\bullet} \otimes \operatorname{ad}: \mathcal{W} \rightarrow \operatorname{Hom}_{\mathbb{R}}(\mathcal{W}, \mathcal{W}),
$$

is defined as

$$
\left(I_{x} \otimes \operatorname{ad}_{y}\right) a \otimes b=\left(I_{x} a\right) \otimes\left(\operatorname{ad}_{y} b\right) .
$$

Now

$$
\begin{aligned}
\left(I_{\bullet} \otimes \mathrm{ad}_{\bullet}\right) R & =\left(I_{\bullet} \otimes \operatorname{ad}_{\bullet}\right) \sum w \otimes r(w)=\left(I_{\bullet} \otimes \operatorname{Ad}_{\bullet}\right) \sum w \otimes r(w)=\left(I_{\bullet} \otimes \operatorname{Ad}_{\bullet}\right) \sum \rho(v) \otimes v \\
& =\sum_{|v| \geq 1} I_{\rho(v)} \otimes \operatorname{Ad}_{v}
\end{aligned}
$$

where we used (2.10) and then (2.2). Then

$$
\begin{aligned}
R \triangleright R & =\left(\left(I_{\bullet} \otimes \mathrm{ad}_{\bullet}\right) R\right) R=\left(\sum_{|v| \geq 1} I_{\rho(v)} \otimes \operatorname{Ad}_{v}\right) \sum_{|w| \geq 1} w \otimes r(w)=\sum_{|v|,|w| \geq 1}(\rho(v) \succ w) \otimes r(v w) \\
& =\sum_{|x| \geq 2} \sum_{v w=x,|v|,|w| \geq 1}(\rho(v) \succ w) \otimes r(v w)=\sum_{|x| \geq 2}(|x|-1) x \otimes r(x) \\
& =\sum_{|x| \geq 2} x \otimes((|x|-1) r(x))=\sum_{|x| \geq 2} x \otimes r[(D-\mathrm{id}) x]=(\underline{D}-\mathrm{id}) R .
\end{aligned}
$$

Remark 2.7. We sketch the connection to the ODE approach of [Roc2003]. Let $S_{t}^{\epsilon}:=\delta_{\epsilon} S(X)_{t}$ be the signature at time $t$, dilated by a factor $\epsilon>0$. Define

$$
Z_{t}^{\epsilon}:=\frac{d}{d \epsilon} S_{t}^{\epsilon} \cdot\left(S_{t}^{\epsilon}\right)^{-1}
$$

which, as we have seen in Remark 2.1.2, is equal to $\epsilon^{-1} r\left[S_{t}^{\epsilon}\right]$. One can show (see [AGS1989, (1.8)], in the language of 'chronological algebras'), that $Z_{t}^{\epsilon}$ satisfies

$$
\begin{equation*}
\partial_{\epsilon} Z_{t}^{\epsilon}=\int_{0}^{t}\left[Z_{r}^{\epsilon}, \dot{Z}_{r}^{\epsilon}\right] d r \tag{2.11}
\end{equation*}
$$

where [.,.] is the Lie bracket in $\mathfrak{g}$. We may give an alternative proof of (2.11) based on the quadratic fixed-point equation (2.8). For the left hand side,

$$
\begin{aligned}
\partial_{\epsilon} Z_{t}^{\epsilon} & =\partial_{\epsilon}\left(\epsilon^{-1} r\left[S_{t}^{\epsilon}\right]\right)=-\epsilon^{-2} r\left[S_{t}^{\epsilon}\right]+\epsilon^{-1} r\left[\partial_{\epsilon} S_{t}^{\epsilon}\right]=-\epsilon^{-2} r\left[S_{t}^{\epsilon}\right]+\epsilon^{-2} r\left[D S_{t}^{\epsilon}\right]=\epsilon^{-2} r\left[(D-\mathrm{id}) S_{t}^{\epsilon}\right] \\
& =\epsilon^{-2} \sum_{w}\left\langle S_{t}^{\epsilon}, w\right\rangle r[(D-\mathrm{id}) w]=\epsilon^{-2} \operatorname{eval}_{S_{t}^{\epsilon}}[(\underline{D}-\mathrm{id}) R],
\end{aligned}
$$

Aiming at the right hand side, we first note that in general for $p \otimes q, p^{\prime} \otimes q^{\prime} \in \mathcal{W}$, we have

$$
\begin{aligned}
\int_{0}^{t}\left[\left\langle S_{s}, p\right\rangle q,\left\langle\dot{S}_{s}, p^{\prime}\right\rangle q^{\prime}\right] d s & =\int_{0}^{t}\left\langle S_{s}, p\right\rangle d\left\langle S_{s}, p^{\prime}\right\rangle\left[q, q^{\prime}\right]=\left\langle S_{t}, p \succ p^{\prime}\right\rangle\left[q, q^{\prime}\right] \\
& =\operatorname{eval}_{S_{t}}\left[(p \otimes q) \triangleright\left(p^{\prime} \otimes q^{\prime}\right)\right]
\end{aligned}
$$

This implies

$$
\begin{aligned}
\int_{0}^{t}\left[Z_{r}^{\epsilon}, \dot{Z}_{r}^{\epsilon}\right] d r & =\epsilon^{-2}\left[r\left[S_{t}^{\epsilon}\right], r\left[\dot{S}_{t}^{\epsilon}\right]\right]=\epsilon^{-2} \sum_{w} \sum_{w^{\prime}} \int_{0}^{t}\left[\left\langle S_{s}^{\epsilon}, w\right\rangle r[w],\left\langle\dot{S}_{s}^{\epsilon}, w^{\prime}\right\rangle r\left[w^{\prime}\right]\right] d s \\
& =\epsilon^{-2} \sum_{w} \sum_{w^{\prime}} \operatorname{eval}_{S_{t}^{\epsilon}}\left[(w \otimes r[w]) \triangleright\left(w^{\prime} \otimes r\left[w^{\prime}\right]\right)\right]=\epsilon^{-2} \operatorname{eval}_{S_{t}^{\epsilon}}[R \triangleright R] .
\end{aligned}
$$

Putting things together, we thus have that (2.11) is equivalent to

$$
\operatorname{eval}_{S_{t}}[(\underline{D}-\mathrm{id}) R]=\operatorname{eval}_{S_{t}^{\epsilon}}[R \triangleright R] .
$$

which is of course an immediate consequence of (2.8).
By symmetrizing the pre-Lie product in the quadratic fixed point equation (2.8), we make the area-operator appear. Define

$$
\begin{aligned}
(p \otimes q) \triangleright \operatorname{Sym}\left(p^{\prime} \otimes q^{\prime}\right) & :=(p \otimes q) \triangleright\left(p^{\prime} \otimes q^{\prime}\right)+\left(p^{\prime} \otimes q^{\prime}\right) \triangleright(p \otimes q) \\
& =\operatorname{area}\left(p, p^{\prime}\right) \otimes\left[q, q^{\prime}\right] .
\end{aligned}
$$

This product was introduced exactly like this already by Rocha in [Roc2003b, Lemma 6.5] and is the tensor algebra analogue of the vector field product also introduced by Rocha in [Roc2003b, Equation (6.13) and Proposition 6.3].

Corollary 2.8. We have

$$
(\underline{D}-\mathrm{id}) R=\frac{1}{2} R \triangleright_{\mathrm{Sym}} R .
$$

Let $R_{n}:=\operatorname{proj}_{n} R=\sum_{|w|=n} w \otimes r[w]$ be the $n$-th level of $R$. Then for $n \geq 2$ this spells out as

$$
(n-1) R_{n}=\frac{1}{2} \sum_{\ell=1}^{n} R_{\ell} \triangleright_{\mathrm{Sym}} R_{n-\ell}= \begin{cases}\sum_{\ell=1}^{(n-1) / 2} R_{\ell} \triangleright_{\mathrm{Sym}} R_{n-\ell} & n \text { odd } \\ \sum_{\ell=1}^{n / 2} R_{\ell} \triangleright_{\mathrm{Sym}} R_{n-\ell}+\frac{1}{2} R_{n / 2} \triangleright_{\mathrm{Sym}} R_{n / 2} & n \text { even }\end{cases}
$$

with $R_{n} \in \mathfrak{R}:=\left\langle\mathrm{i} \otimes \mathrm{i}, \mathrm{i}=1 \ldots \mathrm{~d} ; \triangleright_{\mathrm{Sym}}\right\rangle$.
Proof. This follows immediately from Lemma 2.6.
Remark 2.9. Note that [Roc2003b, Proposition 6.8] has a slightly more complicated recursion. This stems from the facts that the $Z^{n}$ there relates to our $\frac{R_{n}}{n!}$ here.

Example 2.10. Let $R_{n}:=\operatorname{proj}_{n} R$ be the $n$-th level of $R$. Then Lemma 2.6 and Corollary 2.8 give

$$
\begin{aligned}
R_{2} & =R_{1} \triangleright R_{1}=\frac{1}{2} R_{1} \triangleright_{\mathrm{Sym}} R_{1} \\
2 R_{3} & =R_{1} \triangleright R_{2}+R_{2} \triangleright R_{1}=\frac{1}{2} R_{1} \triangleright_{\mathrm{Sym}} R_{2}+\frac{1}{2} R_{2} \triangleright R_{1}=R_{1} \triangleright_{\mathrm{Sym}} R_{2} \\
3 R_{4} & =R_{1} \triangleright R_{3}+R_{2} \triangleright R_{2}+R_{3} \triangleright R_{1}=\frac{1}{2} R_{1} \triangleright_{\mathrm{Sym}} R_{3}+\frac{1}{2} R_{2} \triangleright_{\mathrm{Sym}} R_{2}+\frac{1}{2} R_{3} \triangleright_{\mathrm{Sym}} R_{1} \\
& =R_{1} \triangleright_{\mathrm{Sym}} R_{3}+\frac{1}{2} R_{2} \triangleright_{\mathrm{Sym}} R_{2}
\end{aligned}
$$

For $d=2$ this becomes

$$
\begin{aligned}
R_{1}= & 1 \otimes 1+2 \otimes 2 \\
R_{2}= & \frac{1}{2}(\operatorname{area}(1,2) \otimes[1,2]+\operatorname{area}(2,1) \otimes[2,1]) \\
R_{3}= & \frac{1}{4}(\operatorname{area}(1, \operatorname{area}(1,2)) \otimes[1,[1,2]]+\operatorname{area}(1, \operatorname{area}(2,1)) \otimes[1,[2,1]] \\
& +\operatorname{area}(2, \operatorname{area}(1,2)) \otimes[2,[1,2]]+\operatorname{area}(2, \operatorname{area}(2,1)) \otimes[2,[2,1]]) \\
R_{4}= & \frac{1}{12}(\operatorname{area}(1, \operatorname{area}(1, \operatorname{area}(1,2))) \otimes[1,[1,[1,2]]]+\operatorname{area}(1, \operatorname{area}(1, \operatorname{area}(2,1))) \otimes[1,[1,[2,1]]] \\
& +\operatorname{area}(1, \operatorname{area}(2, \operatorname{area}(1,2))) \otimes[1,[2,[1,2]]]+\operatorname{area}(1, \operatorname{area}(2, \operatorname{area}(2,1))) \otimes[1,[2,[2,1]]] \\
& +\operatorname{area}(2, \operatorname{area}(1, \operatorname{area}(1,2))) \otimes[2,[1,[1,2]]]+\operatorname{area}(2, \operatorname{area}(1, \operatorname{area}(2,1))) \otimes[2,[1,[2,1]]] \\
& +\operatorname{area}(2, \operatorname{area}(2, \operatorname{area}(1,2))) \otimes[2,[2,[1,2]]]+\operatorname{area}(2, \operatorname{area}(2, \operatorname{area}(2,1))) \otimes[2,[2,[2,1]]])
\end{aligned}
$$

In general this looks as follows.
Definition 2.11. Denote by $\mathrm{BPT}_{n}$ the set of (complete, rooted) binary planar trees with $n$ leaves labelled with the letters $1, \ldots$, d. Given $\tau \in \mathrm{BPT}_{n}$ we define area. $(\tau)$ (resp. lie. $(\tau)$ ) as the bracketing-out using area (resp. [., .]). For example

$$
\begin{aligned}
& \text { area. }\left({\underset{0}{1}}_{\bigvee_{0}}^{3}\right)=\operatorname{area}(1, \operatorname{area}(2,3)) \\
& \operatorname{lie}_{\cdot}\left({ }^{1} \vee^{2}{ }^{3} \zeta^{4}\right)=[[1,2],[3,4]] \text {. }
\end{aligned}
$$

Define a function $c: \mathrm{BPT}_{n} \rightarrow \mathbb{R}$, which does not depend on the specific letter labels, recursively as follows

$$
\begin{aligned}
c(\mathrm{i}) & =1 \quad \text { for any i in } 1, \ldots, \mathrm{~d} \\
c\left(\tau_{1} \tau^{2}\right) & =2 c\left(\tau_{1}\right) c\left(\tau_{2}\right)\left(\left|\tau_{1}\right|_{\text {leaves }}+\left|\tau_{2}\right|_{\text {leaves }}-1\right)
\end{aligned}
$$

where $|\tau|_{\text {leaves }}$ denotes the number of leaves of the tree $\tau$. For example

$$
\begin{aligned}
c(2) & =1 \\
c\left(\mathbf{1}^{2}\right) & =2 \cdot 1 \cdot 1 \cdot(2-1)=2 \\
c\left(3 \bigvee^{2}\right) & =2 \cdot 1 \cdot 2 \cdot(3-1)=8
\end{aligned}
$$

## Lemma 2.12.

$$
R_{n}=\sum_{\tau \in \mathrm{BPT}_{n}} \frac{1}{c(\tau)} \text { area. }(\tau) \otimes \operatorname{lie}_{\bullet}(\tau)
$$

Remark 2.13. We note that [Roc2003, Lemma 1] has a slightly more complicated expression for $R_{n}$, since in that work some of terms are factored out, owing to antisymmetry. We do not
pursue this here since the end result, also in [Roc2003], still contains redundant terms, which we do not know how to explicitly get rid of. In fact, due to antisymmetry alone, we already know that $c(\tau)$ is not a unique choice for this equation to hold, however it remains an interesting and more involved question if it is the only choice which is symmetric, i.e. well-defined on non-planar trees, and invariant under change of the leaf labels.
A further very interesting question is to find a modified $c^{\prime}$ which may not be symmetric and may depend on the leaf labels, such that the equations still holds, but such that the number of non-zero summands in the equation is minimized for each $n$.

Proof of 2.12. For the purpose of this proof, let $R_{n}$ be defined as in Corollary 2.8 and

$$
R_{n}^{\prime}:=\sum_{\tau \in \mathrm{BPT}_{n}} \frac{1}{c(\tau)} \text { area. }(\tau) \otimes \operatorname{lie}_{\mathbf{0}}(\tau) .
$$

We proceed by induction over $n$. We have

$$
R_{1}=\sum_{|w|=1} w \otimes r(w)=\sum_{|w|=1} w \otimes w=\sum_{|w|=1} \frac{1}{c(w)} \text { area. }(w) \otimes \text { lie. }(w)=R_{1}^{\prime} .
$$

Assuming $R_{n}=R_{n}^{\prime}$ holds for some $n \in \mathbb{N}$, we get

$$
\begin{aligned}
& R_{n+1}=\frac{1}{2 n} \sum_{l=1}^{n+1} R_{l} \triangleright_{\text {Sym }} R_{n-l} \\
& =\frac{1}{2 n} \sum_{l=1}^{n+1} \sum_{\substack{\tau_{1} \in \mathrm{BPT}_{l}, \tau_{2} \in \mathrm{BP}_{n+1-l}}} \frac{1}{c\left(\tau_{1}\right) c\left(\tau_{2}\right)} \operatorname{area}\left(\text { area. }\left(\tau_{1}\right), \text { area. }\left(\tau_{2}\right)\right) \otimes\left[\text { lie. }\left(\tau_{1}\right), \text { lie. }\left(\tau_{2}\right)\right] \\
& =\sum_{l=1}^{n+1} \sum_{\substack{\tau_{1} \in \mathrm{BP}_{l}, \tau_{2} \in \mathrm{BPT}_{n+1},}} \frac{1}{c\left(\tau_{1} \tau_{2} \tau_{2}\right)} \text { area. }\left(\tau_{1} \curlyvee^{\tau_{2}}\right) \otimes \text { lie. }_{\mathbf{\bullet}}\left(\tau_{1} \mho^{\tau_{2}}\right) \\
& =\sum_{\tau \in \mathrm{BPT}_{n+1}} \frac{1}{c(\tau)} \text { area. }(\tau) \otimes \text { lie. }_{\mathbf{\bullet}}(\tau)=R_{n+1}^{\prime} .
\end{aligned}
$$

Rather than working with the recursion for $R_{n}$ from Corollary 2.8 directly on $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$, in the following theorem we will pursue the alternative approach of first applying the coeval ${ }^{S_{h}}$ to work on $T\left(\mathbb{R}^{d}\right)$, or, to be more specific, on $\mathscr{A}$ as we will see.

Theorem 2.14. We have $R=\sum_{h} \mathfrak{r}_{h} \otimes P_{h}$ where $\mathfrak{r}_{h}:=\rho\left(S_{h}\right)$ satisfies the recursion

$$
\begin{equation*}
\mathfrak{r}_{h}=\frac{1}{|h|-1} \sum_{h_{1}<h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \operatorname{area}\left(\mathfrak{r}_{h_{1}}, \mathfrak{r}_{h_{2}}\right) . \tag{2.12}
\end{equation*}
$$

More explicitly, we have

$$
\mathfrak{r}_{h}=\sum_{\tau} \frac{1}{b(\tau)} q_{\tau}^{h} \text { area. }(\tau)=\sum_{\tau} \frac{1}{c(\tau)} p_{\tau}^{h} \text { area. }(\tau)
$$

with

$$
\begin{aligned}
& q_{\tau}^{h}=\sum_{h_{1}<h_{2}} q_{\tau^{\prime}}^{h_{1}} q_{\tau^{\prime \prime}}^{h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle, \\
& p_{\tau}^{h}=\left\langle S_{h}, \text { lie. }(\tau)\right\rangle=\sum_{h_{1}, h_{2}} q_{\tau^{\prime}}^{h_{1}} q_{\tau^{\prime \prime}}^{h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle
\end{aligned}
$$

for $|\tau|_{\text {leaves }},|h| \geq 2$ and

$$
\begin{aligned}
q_{\mathrm{i}}^{h} & =p_{\mathrm{i}}^{h}=\delta_{h, \mathrm{i}}, \\
q_{\tau}^{\mathrm{i}} & =p_{\tau}^{\mathrm{i}}
\end{aligned}=\delta_{\tau, \mathrm{i}},
$$

with $b(\tau)=b\left(\tau^{\prime}\right) b\left(\tau^{\prime \prime}\right)\left(\left|\tau^{\prime}\right|_{\text {leves }}+\left|\tau^{\prime \prime}\right|_{\text {leaves }}-1\right), b(i)=1$.
Proof. We have $r=\operatorname{eval}(R)$ with $\operatorname{Im} r=\mathfrak{g}$ and thus

$$
r=r \circ \operatorname{eval}\left(\sum_{h} S_{h} \otimes P_{h}\right)=\operatorname{eval}\left(\sum_{w, h}\left\langle w, S_{h}\right\rangle \rho(w) \otimes P_{h}\right)=\operatorname{eval}\left(\sum_{h} \rho\left(S_{h}\right) \otimes P_{h}\right),
$$

which means $R=\sum_{h} \rho\left(S_{h}\right) \otimes P_{h}$ since eval is bijective. Putting $\mathfrak{r}_{h}:=\rho\left(S_{h}\right)$, we get

$$
\sum_{h}(|h|-1) \mathfrak{r}_{h} \otimes P_{h}=(\underline{D}-\mathrm{id}) R=\frac{1}{2} R \triangleright_{\mathrm{Sym}} R=\frac{1}{2} \sum_{h_{1}, h_{2}} \operatorname{area}\left(\mathfrak{r}_{h_{1}}, \mathfrak{r}_{h_{2}}\right) \otimes\left[P_{h_{1}}, P_{h_{2}}\right],
$$

which by applying coeval ${ }^{S_{h}}$ on both sides yields

$$
(|h|-1) \mathfrak{r}_{h}=\frac{1}{2} \sum_{h_{1}, h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \operatorname{area}\left(\mathfrak{r}_{h_{1}}, \mathfrak{r}_{h_{2}}\right)=\sum_{h_{1}<h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \operatorname{area}\left(\mathfrak{r}_{h_{1}}, \mathfrak{r}_{h_{2}}\right) .
$$

Since due to $q_{\tau}^{i}=\delta_{\tau, \mathrm{i}}$ and $b(\mathrm{i})=1$ we have

$$
\mathfrak{r}_{\mathrm{i}}=\rho(\mathrm{i})=\mathrm{i}=\operatorname{area}_{\mathbf{\bullet}}(\mathrm{i})=\sum_{\tau} \frac{1}{b(\tau)} q_{\tau}^{\mathrm{i}} \mathrm{area}_{\mathbf{0}}(\tau),
$$

we obtain by induction

$$
\begin{aligned}
\mathfrak{r}_{h} & =\frac{1}{|h|-1} \sum_{h_{1}<h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \operatorname{area}\left(\mathfrak{r}_{h_{1}}, \mathfrak{r}_{h_{2}}\right) \\
& =\frac{1}{|h|-1} \sum_{h_{1}<h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \sum_{\tau_{1}, \tau_{2}} \frac{1}{b\left(\tau_{1}\right) b\left(\tau_{2}\right)} q_{\tau_{1}}^{h_{1}} q_{\tau_{2}}^{h_{2}} \operatorname{area}\left(\text { area. }\left(\tau_{1}\right), \text { area. }\left(\tau_{2}\right)\right) \\
& =\sum_{\tau_{1}, \tau_{2}} \frac{1}{(|h|-1) b\left(\tau_{1}\right) b\left(\tau_{2}\right)} \sum_{h_{1}<h_{2}} q_{\tau_{1}}^{h_{1}} q_{\tau_{2}}^{h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \text { area. } \\
& =\sum_{\tau} \frac{\tau_{1} \tau^{\prime} \tau_{2}}{\left(\left|\tau^{\prime}\right|_{\text {leaves }}+\left|\tau^{\prime \prime}\right|_{\text {leaves }}-1\right) b\left(\tau^{\prime}\right) b\left(\tau^{\prime \prime}\right)} \sum_{h_{1}<h_{2}} q_{\tau^{\prime}}^{h_{1}} q_{\tau^{\prime \prime}}^{h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \text { area. }_{\bullet}(\tau) \\
& =\sum_{\tau} \frac{1}{b(\tau)} q_{\tau}^{h} \operatorname{area}_{\bullet}(\tau)
\end{aligned}
$$

Furthermore, due to Lemma 2.12, we have

$$
\mathfrak{r}_{h}=\operatorname{coeval}^{S_{h}}\left(\mathfrak{r}_{|h|}\right)=\sum_{\tau \in \mathrm{BPT}_{|h|}} \frac{1}{c(\tau)}\left\langle S_{h}, \text { lie. }(\tau)\right\rangle \text { area. }(\tau)=\sum_{\tau} \frac{1}{c(\tau)} p_{\tau}^{h} \operatorname{area}_{\mathbf{\bullet}} .(\tau),
$$

where we have $p_{\mathrm{i}}^{h}=\left\langle S_{h}, \operatorname{lie} .(\mathrm{i})\right\rangle=\left\langle S_{h}, \mathrm{i}\right\rangle=\delta_{h, \mathrm{i}}$ as well as $p_{\tau}^{\mathrm{i}}=\left\langle S_{\mathrm{i}}, \operatorname{lie} .(\tau)\right\rangle=\langle\mathrm{i}, \operatorname{lie} .(\tau)\rangle=\delta_{\tau, \mathrm{i}}$ and by induction over $|\tau|_{\text {leaves }} \geq 2$

$$
\begin{aligned}
p_{\tau}^{h} & =\left\langle S_{h}, \text { lie. }(\tau)\right\rangle=\left\langle S_{h},\left[\operatorname{lie} .\left(\tau^{\prime}\right), \text { lie. }\left(\tau^{\prime \prime}\right)\right]\right\rangle=\sum_{h_{1}, h_{2}}\left\langle S_{h},\left[\left\langle S_{h_{1}}, \text { lie. }\left(\tau^{\prime}\right)\right\rangle P_{h_{1}},\left\langle S_{h_{2}}, \text { lie. }\left(\tau^{\prime}\right)\right\rangle P_{h_{2}}\right]\right\rangle \\
& =\sum_{h_{1}, h_{2}} p_{\tau^{\prime}}^{h_{1}} p_{\tau^{\prime \prime}}^{h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle .
\end{aligned}
$$

Remark 2.15. Let $h(\tau)$ be the Hall word corresponding to the Hall tree $\tau$. Then,

$$
q_{\tau}^{h_{0}}=p_{\tau}^{h_{0}}=\delta_{h(\tau), h_{0}} .
$$

This is immediate by definition of $q$ for $|h(\tau)|=|\tau|_{\text {leaves }}=1$, and then by induction over $|\tau|_{\text {levees }}$ if $\tau$ is a Hall tree, then $\tau^{\prime}, \tau^{\prime \prime}$ are Hall trees and thus

$$
\begin{aligned}
q_{\tau}^{h} & =\sum_{h_{1}<h_{2}} q_{\tau^{\prime}}^{h_{1}} q_{\tau^{\prime \prime}}^{h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle=\sum_{h_{1}<h_{2}} \delta_{h\left(\tau^{\prime}\right), h_{1}} \delta_{h\left(\tau^{\prime \prime}\right), h_{2}}\left\langle S_{h},\left[P_{h_{1}}, P_{h_{2}}\right]\right\rangle \\
& =\left\langle S_{h},\left[P_{h\left(\tau^{\prime}\right)}, P_{h\left(\tau^{\prime \prime}\right)}\right]\right\rangle=\delta_{h(\tau), h}
\end{aligned}
$$

due to $\left(P_{h}\right)_{h}$ and $\left(S_{h}\right)_{h}$ being dual bases. For $p_{\tau}^{h}=\left\langle S_{h}\right.$, lie. $\left.(\tau)\right\rangle$ the claim is immediate. Note furthermore that we could also have derived

$$
\mathfrak{r}_{h}=\sum_{\tau} \frac{1}{b(\tau)} q_{\tau}^{h} \text { area. }(\tau)
$$

from

$$
\mathfrak{r}_{h}=\sum_{\tau} \frac{1}{c(\tau)} p_{\tau}^{h} \text { area. }(\tau)
$$

by looking at how $q_{\tau}^{h}$ and $p_{\tau}^{h}$ relate to each other.
Example 2.16. In the case of $T\left(\mathbb{R}^{2}\right)$ and the Lyndon words $H$, the values of $\mathfrak{r}_{h}$ up to level five
are

$$
\begin{aligned}
\mathfrak{r}_{1} & =1, \quad \mathfrak{r}_{2}=2, \\
\mathfrak{r}_{12} & =12-21=\operatorname{area}(1,2), \\
\mathfrak{r}_{112} & =112-121=\frac{1}{2} \text { area }(1, \text { area }(1,2)), \\
\mathfrak{r}_{122} & =-212+221=\frac{1}{2} \operatorname{area}(\operatorname{area}(1,2), 2), \\
\mathfrak{r}_{1112} & =1112-1121=\frac{1}{6} \text { area }(1, \operatorname{area}(1, \text { area }(1,2))) \\
\mathfrak{r}_{1122} & =-1212+1221-2112+2121=\frac{1}{6} \text { area }(1, \text { area }(\operatorname{area}(1,2), 2))+\frac{1}{6} \text { area }(\operatorname{area}(1, \text { area }(1,2)), 2), \\
\mathfrak{r}_{1222} & =2212-2221=\frac{1}{6} \text { area }(\operatorname{area}(\operatorname{area}(1,2), 2), 2), \\
\mathfrak{r}_{11112} & =11112-11121, \\
\mathfrak{r}_{11122} & =-11212+11221-12112+12121-21112+21121, \\
\mathfrak{r}_{11222} & =12212-12221+21212-21221+22112-22121, \\
\mathfrak{r}_{12122} & =21212-21221+22112-22121, \\
\mathfrak{r}_{11212} & =21112-21121, \\
\mathfrak{r}_{12222} & =-22212+22221,
\end{aligned}
$$

where we gave the area bracketings according to the recursion Equation (2.12) up to level four. The trend of the values being just $-1,0,1$ combinations of words does not continue to higher levels, e.g.

$$
\mathfrak{r}_{112212}=-211212+211221-212112+212121-3221112+3221121 .
$$

Example 2.17. In the case of $T\left(\mathbb{R}^{3}\right)$ and the Lyndon words $H$, the values of $\mathfrak{r}_{h}$ up to level four which are not immediate from the previous example are

$$
\begin{aligned}
\mathfrak{r}_{123} & =123-132-312+321, \\
\mathfrak{r}_{132} & =-213+231-312+321, \\
\mathfrak{r}_{1123} & =1123-1132-1312+1321-3112+3121, \\
\mathfrak{r}_{1132} & =-1213+1231-1312+1321-2113+2131-3112+3121, \\
\mathfrak{r}_{1213} & =-2113+2131+3112-3121, \\
\mathfrak{r}_{1223} & =1223-1232+3212-3221, \\
\mathfrak{r}_{1232} & =-2123+2132+2312-2321+23212-23221, \\
\mathfrak{r}_{1233} & =-1323+1332-3123+3132+3312-3321, \\
\mathfrak{r}_{1322} & =2213-2231+2312-2321+3212-3221, \\
\mathfrak{r}_{1323} & =-3123+3132+3213-3231+23312-23321, \\
\mathfrak{r}_{1332} & =2313-2331+3213-3231+3312-3321 .
\end{aligned}
$$

Example 2.18. In the case of $T\left(\mathbb{R}^{2}\right)$ and the standard Hall words $H$, the values of $\mathfrak{r}_{h}$ up to level
five are

$$
\begin{aligned}
\mathfrak{r}_{1} & =1, \quad \mathfrak{r}_{2}=2, \\
\mathfrak{r}_{12} & =12-21, \\
\mathfrak{r}_{121} & =-112+121, \\
\mathfrak{r}_{122} & =-212+221, \\
\mathfrak{r}_{1211} & =1112-1121, \\
\mathfrak{r}_{1221} & =1212-1221+2112-2121, \\
\mathfrak{r}_{1222} & =2212-2221, \\
\mathfrak{r}_{12111} & =-11112+11121, \\
\mathfrak{r}_{12211} & =-11212+11221-12112+12121-21112+21121, \\
\mathfrak{r}_{12221} & =-12212+12221-21212+21221-22112+22121, \\
\mathfrak{r}_{12222} & =-22212+22221, \\
\mathfrak{r}_{12112} & =-21112+21121, \\
\mathfrak{r}_{12212} & =-21212+21221-22112+22121,
\end{aligned}
$$

where once again the trend of the values being just $-1,0,1$ combinations of words does not continue to higher levels, e.g.

$$
\begin{aligned}
\mathfrak{r}_{122112}= & 121212-121221+122112-122121+2211212-2211221 \\
& +2212112-2212121+3221112-3221121 .
\end{aligned}
$$

## 3. Coordinates of the first kind

Let $\left(P_{h}\right)_{h \in H}$ be a basis for the free Lie algebra $\mathfrak{g}$. For the index set $H$ we have a Hall set in mind ([Reu1993, Section 4]), but this is not necessary at this stage. Any grouplike element $g \in G$ can be written as the exponential of a Lie series,

$$
\begin{equation*}
g=\exp \left(\sum_{h \in H} c_{h}(g) P_{h}\right) \tag{3.1}
\end{equation*}
$$

for some uniquely determined $c_{h}(g) \in \mathbb{R}$. In fact, there exist unique $\zeta_{h} \in T\left(\mathbb{R}^{d}\right), h \in H$, such that $c_{h}(g)=\left\langle\zeta_{h}, g\right\rangle$. The $\zeta_{h}$ are called the coordinates of the first kind (corresponding to $\left.\left(P_{h}\right)_{h \in H}\right)$, see for example [Kaw2009].
We now formulate this in a way, where we do not have to test against $g \in G$. Recall the product - on $\mathcal{W}$ : shuffle product on the left and concatenation product on the right.

For words $a, b$

$$
\begin{aligned}
\operatorname{eval}_{g}(a \otimes b) \cdot \operatorname{eval}_{g}\left(a^{\prime} \otimes b^{\prime}\right) & =\langle a, g\rangle\left\langle a^{\prime}, g\right\rangle b b^{\prime}=\left\langle a ш a^{\prime}, g\right\rangle b b^{\prime} \\
& =\operatorname{eval}_{g}\left(\left(a \amalg a^{\prime}\right) \otimes\left(b \cdot b^{\prime}\right)\right)=\operatorname{eval}_{g}\left((a \otimes b) \bullet\left(a^{\prime} \otimes b^{\prime}\right)\right)
\end{aligned}
$$

Both expressions are bilinear, so this is true for general elements in $\mathcal{W}$. Hence eval ${ }_{g}$ is an algebra homomorphism from $(\mathcal{W}, \mathbf{\square})$ to $\left(T\left(\left(\mathbb{R}^{d}\right)\right), \cdot\right)$. Then on one hand, using first (3.1) and then the homomorphism property

$$
\begin{aligned}
g & =\exp \left(\sum_{h \in H} c_{h} P_{h}\right)=\exp \left(\sum_{h \in H}\left\langle\zeta_{h}, g\right\rangle P_{h}\right)=\exp \left(\operatorname{eval}_{g}\left(\sum_{h \in H} \zeta_{h} \otimes P_{h}\right)\right) \\
& =\operatorname{eval}_{g}\left(\exp _{\mathbf{\bullet}}\left(\sum_{h \in H} \zeta_{h} \otimes P_{h}\right)\right) .
\end{aligned}
$$

Here, of course, for $x \in \mathcal{W}$,

$$
\exp _{\mathbf{n}}(x):=\sum_{n \geq 0} \frac{x^{\mathbf{n}}}{n!}:=\sum_{n \geq 0} \frac{\overbrace{x \text { п. }}^{n \text { times }} x}{n!} .
$$

On the other hand, trivially

$$
g=\sum_{w}\langle w, g\rangle w=\operatorname{eval}_{g}\left(\sum_{w} w \otimes w\right) .
$$

Since grouplike elements linearly span ${ }^{4}$ all of $T\left(\left(\mathbb{R}^{d}\right)\right)$ we get that for all $x \in T\left(\left(\mathbb{R}^{d}\right)\right)$

$$
\operatorname{eval}_{x}\left(\exp _{\mathbf{\bullet}}\left(\sum_{h \in H} \zeta_{h} \otimes P_{h}\right)\right)=\operatorname{eval}_{x}\left(\sum_{w} w \otimes w\right)
$$

which is equivalent to

$$
\begin{equation*}
\sum_{w} w \otimes w=\exp _{\mathbf{\bullet}}\left(\sum_{h \in H} \zeta_{h} \otimes P_{h}\right) \tag{3.2}
\end{equation*}
$$

respectively

$$
\log _{\mathbf{\bullet}} \sum_{w} w \otimes w=\sum_{h \in H} \zeta_{h} \otimes P_{h} .
$$

We have arrived at a definition of coordinates of the first kind which does not rely on testing against grouplike elements.

Remark 3.1. Considering $S$ as an element of $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$, it is grouplike. Indeed, for $a, b \in$ $T\left(\mathbb{R}^{d}\right)$,

$$
\begin{aligned}
\left\langle\underline{a} \boxtimes \underline{b}, \Delta_{\underline{\underline{W}}} S\right\rangle & =\left\langle\underline{a} \boxtimes \underline{b}, \Delta_{\underline{\Perp}} \sum_{w} w \underline{w}\right\rangle=\sum_{w} w\left\langle\underline{a} \boxtimes \underline{b}, \Delta_{\underline{\uplus}} \underline{w}\right\rangle=\sum_{w} w\langle\underline{a} \underline{\varpi} \underline{b}, \underline{w}\rangle \\
& =a \varpi b=\left\langle\underline{a} \boxtimes \underline{b}, \sum_{w, v} w ш v \underline{w} \boxtimes \underline{v}\right\rangle=\left\langle\underline{a} \boxtimes \underline{b}, \sum_{w} w \underline{w} \boxtimes \sum_{v} v \underline{v}\right\rangle \\
& =\langle\underline{a} \boxtimes \underline{b}, S \boxtimes S\rangle .
\end{aligned}
$$

[^3]Here for a word $w \in T\left(\mathbb{R}^{d}\right)$ we write $\underline{w}$ as its realization in $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$. Then

$$
\Lambda:=\log _{\mathbf{n}} S
$$

is primitive. The search for coordinates of the first kind then amounts to finding a "simple" expression for this primitive element.

One can construct the coordinates $\zeta_{h}$ as follows. Pick any $S_{h} \in T\left(\mathbb{R}^{d}\right), h \in H$, such that

$$
\left\langle S_{h}, P_{h^{\prime}}\right\rangle=\delta_{h, h^{\prime}} .
$$

Using [Reu1993, Theorem 5.3], if $P_{h}$ is a Hall basis, one can actually pick the $S_{h}$ in such a way that they extend to the dual of the corresponding PBW basis of $T\left(\left(\mathbb{R}^{d}\right)\right)$, and while this is not necessary here, in the rest of the paper we really mean that the $S_{h}$ are chosen in this specific way. Then

$$
\left\langle S_{h}, \log \left(\exp \left(\sum_{h^{\prime} \in H} c_{h^{\prime}} P_{h^{\prime}}\right)\right)\right\rangle=\left\langle S_{h}, \sum_{h^{\prime} \in H} c_{h^{\prime}} P_{h^{\prime}}\right\rangle=c_{h}
$$

We want "to put the logarithm on the other side". This is indeed possible, since the logarithm on grouplike elements extends to a linear map $\pi_{1}$ on all of $T\left(\left(\mathbb{R}^{d}\right)\right.$ ) (see [Reu1993, Section 3.2] and also [MNT2013] for a general overview on idempotents), given as

$$
\begin{equation*}
\pi_{1}(u):=\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sum_{v_{1}, ., v_{n}}\left\langle v_{1} ш . . ш v_{n}, u\right\rangle v_{1} \cdot . . \cdot v_{n} . \tag{3.3}
\end{equation*}
$$

Denote its dual map by $\pi_{1}^{\top} .{ }^{5}$ It is given as

$$
\pi_{1}^{\top}(v)=\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} \sum_{u_{1}, .,, u_{n} \text { non-empty }}\left\langle v, u_{1} \cdot . . \cdot u_{n}\right\rangle u_{1} \amalg . . Ш u_{n} .
$$

Then for all $h \in H$

$$
\left\langle\pi_{1}^{\top} S_{h}, \exp \left(\sum_{h^{\prime} \in H} c_{h^{\prime}} P_{h^{\prime}}\right)\right\rangle=c_{h},
$$

that is, the coordinate of first kind are given by (compare [GK2008, Theorem 1])

$$
\begin{equation*}
\zeta_{h}=\pi_{1}^{\top} S_{h} \quad h \in H \tag{3.4}
\end{equation*}
$$

We note that $\zeta_{h}$ must of course be independent of the choice of the $S_{h}$ and this is indeed the case, since ker $\pi_{1}^{\top}=\left(\operatorname{im} \pi_{1}\right)^{\perp}=\mathfrak{g}_{n}^{\perp}$.

Example 3.2. Let $\left(P_{h}\right)_{h \in H}$ be the Lyndon basis (which is a Hall basis, [Reu1993, Section 5]). In the case $d=2$, we give in Table 3.1 the first few elements for $P_{h}, S_{h}$ and $\pi_{1}^{\top} S_{h}$, where we take $S_{h}$ as in [Reu1993, Theorem 5.3].

[^4]| Lyndon word $h$ | $P_{h}$ | $S_{h}$ | $\zeta_{h}=\pi_{1}^{\top} S_{h}$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 |
| 12 | [1, 2] | 12 | $+\frac{1}{2} 12-\frac{1}{2} 21$ |
| 112 | $[1,[1,2]]$ | 112 | $+\frac{1}{6} 112-\frac{1}{3} 121+\frac{1}{6} 211$ |
| 122 | [[1, 2], 2] | 122 | $+\frac{1}{6} 122-\frac{1}{3} 212+\frac{1}{6} 221$ |
| 1112 | $[1,[1,[1,2]]]$ | 1112 | $-\frac{1}{6} 1121+\frac{1}{6} 1211$ |
| 1122 | [1, [[1, 2], 2]] | 1122 | $+\frac{1}{6} 1122-\frac{1}{6} 1212+\frac{1}{6} 2121-\frac{1}{6} 2211$ |
| 1222 | [[[1, 2], 2], 2] | 1222 | $-\frac{1}{6} 2122+\frac{1}{6} 2212$ |
| 11112 | $[1,[1,[1,[1,2]]]]$ | 11112 | $\frac{1}{30}[-11112-11121+411211-12111-21111]$ |
| 11122 | $[1,[1,[[1, ~ 2], 2]]]$ | 11122 | $\frac{1}{30}\left[\begin{array}{c} 211122-311212-311221+212112+212121 \\ -312211+221112+221121-321211+22111 \end{array}\right]$ |
| 11222 | [1, [[[1, 2], 2], 2]] | 11222 | $\frac{1}{30}\left[\begin{array}{c}211222-312122+212212+212221-321122 \\ +221212+221221-322112-322121+22211 ~\end{array}\right]$ |
| 12122 | [[1, 2], [[1, 2], 2]] | $12122+311222$ | $\frac{1}{30}\left[\begin{array}{c} 311222-212122-212212+3122212-221122 \\ +321212 \end{array}-22121-22112-22121+32211\right]$ |
| 11212 | $[[1, ~[1, ~ 2]],[1,2]]$ | $11212+211122$ | $\frac{1}{30}\left[\begin{array}{c}11122+11212+11221-412112+12121 \\ +12211+21112-421121+21211+22111\end{array}\right]$ |
| 12222 | [[[[1, 2], 2], 2], 2] | 12222 | $\frac{1}{30}[-12222-21222+422122-22212-22221]$ |

Table 3.1.: Example values for the Lyndon basis on two elements. The first column shows the Lyndon words, which are the Hall words for this basis. For each Lyndon word h, we show element $P_{h}$ of the Hall basis which is also the PBW basis element labelled by $h$. Next we show the corresponding element $S_{h}$ of the dual PBW basis, which also serves as $S_{h}$ described above. Finally we show the corresponding coordinate of the second kind.

| Lyndon <br> word $h$ | $P_{h}$ | $S_{h}$ | $\zeta_{h}=\pi_{1}^{\top} S_{h}$ |
| :--- | :--- | :--- | :--- |
| 1 | 1 | 1 | 1 |
| 12 | $[1,2]$ | 12 | $\frac{1}{2} 12-\frac{1}{2} 21$ |
| 112 | $[1,[1,2]]$ | 112 | $\frac{1}{6}[112-2121+211]$ |
| 122 | $[[1,2], 2]$ | 122 | $\frac{1}{6}[122-2212+221]$ |
| 123 | $[1,[2,3]]$ | 123 | $\frac{1}{6}[2123-132-213-231-312+2321]$ |
| 132 | $[[1,3], 2]$ | $123+132$ | $\frac{1}{6}[123+132-2213+231-2312+321]$ |
| 1123 | $[1,[1,[2,3]]]$ | 1123 | $\frac{1}{6}\left[\begin{array}{l}1123-1213-1231 \\ 1122+321-321]\end{array}\right]$ |
| 1132 | $[1,[[1,3], 2]]$ | $1123+1132$ | $\frac{1}{6}[123+1132-21213-1312$ |
| 1213 | $[[1,2],[1,3]]$ | $1123+1132+1213$ | $\left.\frac{1}{6}[1213-1312-211+3121-321]+2131+3112-3121\right]$ |

Table 3.2.: Example values for the Lyndon basis on three elements. The first column shows the Lyndon words, which are the Hall words for this basis. For each Lyndon word h, we show element $P_{h}$ of the Hall basis which is also the PBW basis element labelled by $h$. Next we show the corresponding element $S_{h}$ of the dual PBW basis, which also serves as $S_{h}$ described above. Finally we show the corresponding coordinate of the second kind.

| $\begin{gathered} \text { Hall } \\ \text { word } h \end{gathered}$ | $P_{h}$ | $S_{h}$ | $\zeta_{h}=\pi_{1}^{\top} S_{h}$ |
| :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 |
| 12 | [1,2] | 12 | $\frac{1}{2} 12-\frac{1}{2} 21$ |
| 121 | [[1, 2], 1] | $112+121$ | $\frac{1}{6}[2121-112-211]$ |
| 122 | [ [1, 2], 2] | 122 | $\frac{1}{6}[122+221-2212]$ |
| 1211 | [[[1, 2], 1], 1] | $1112+1121+1211$ | $\frac{1}{6}[1211-1121]$ |
| 1221 | [[[1, 2], 2], 1] | $1122+1212+1221$ | $\frac{1}{6}[1212-1122-2121+2211]$ |
| 1222 | [[[1, 2], 2], 2] | 1222 | $\frac{1}{6}[2212-2122]$ |
| 12111 | $[[[[1,2], 1], 1], 1]$ | $\begin{aligned} & 11112+11121 \\ & +11211+12111 \end{aligned}$ | $\frac{1}{30}{ }^{[11112+11221-411211+12111+21111]}$ |
| 12211 | $[[[[1,2], 2], 1], 1]$ | $\begin{aligned} & 11122+11212+11221 \\ & \quad+12112+12121+12211 \end{aligned}$ | $\frac{1}{30}\left[\begin{array}{c}211122-311212-311221+2121112 \\ -312211\end{array}+221112+221121-321211+222111\right]$ |
| 12221 | [[[[1, 2], 2], 2], 1] | $\begin{aligned} & 11222+12122 \\ & +12212+12221 \end{aligned}$ | $\left.\frac{1}{30}\left[\begin{array}{c} -211222+311112-212212-212221+321122 \\ -221212 \\ -221221+32112+32121 \end{array}\right]-222211\right]$ |
| 12222 | [[[[1, 2], 2], 2], 2] | 12222 | $\frac{1}{30}[-12222-21222+422122-22212-2221]$ |
| 12112 | [[[1, 2], 1], [1, 2]] | $\begin{aligned} & 411122+311212+211221 \\ & \quad+212112+12121 \end{aligned}$ | $\frac{1}{30}\left[\begin{array}{c} -11122-11212-11221+412122-12121 \\ -12211 \\ -21112 \end{array}+421121-21211-22111\right]$ |
| 12112 | [[[1, 2], 2], [1, 2]] | $311222+212122+12212$ |  |

Table 3.3.: Example values for the standard Hall basis on two elements. The first column shows the Hall words. For each Hall word h, we show element $P_{h}$ of the Hall basis which is also the PBW basis element labelled by $h$. Next we show the corresponding element $S_{h}$ of the dual PBW basis, which also serves as $S_{h}$ described above. Finally we show the corresponding coordinate of the second kind.

The expressions given by (3.4) can become quite unwieldy. This motivated Rocha to look for more tractable expressions in [Roc2003]. We will now reproduce his results using purely algebraic arguments.

### 3.1. Coordinates of first kind in terms of areas-of-areas

As in Remark 3.1 we consider the grouplike element $S \in \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$. The goal is to find a "simple expression" for

$$
\Lambda:=\log _{\bullet} S
$$

Following Rocha, we obtain

$$
R=\underline{r}(S)=\left(\underline{r} \circ \exp _{\mathbf{\bullet}}\right)[\Lambda]
$$

The last step consists now in inverting $\underline{r} \circ \exp$. here. We shall need the following version of Baker's identity [Reu1993, (1.6.5)].

Lemma 3.3. Let $x, q \in T\left(\left(\mathbb{R}^{d}\right)\right.$ ) (resp. $L, Q \in \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ ) with $q$ (resp. Q) having no coefficient in the empty word $e(r e s p . \underline{e})$ and $x$ (resp. L) primitive. Then

$$
r(x \cdot q)=[x, r(q)], \quad \underline{r}(L ■ Q)=[L, \underline{r}(Q)]_{■} .
$$

Proof. For $x, L$ Lie, by [Reu1993, Theorem 1.4], $\operatorname{ad}_{x}=\operatorname{Ad}_{x}$ on $T\left(\left(\mathbb{R}^{d}\right)\right)$ and $\operatorname{ad}_{L}=\operatorname{Ad}_{L}$ on $\mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$. Hence for $q \in T\left(\left(\mathbb{R}^{d}\right)\right)$ and $Q \in \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ any polynomial having no coefficient in the empty word,

$$
\begin{aligned}
r[x \cdot q] & =\operatorname{ad}_{x} r[q]=\operatorname{Ad}_{x} r[q]=[x, r[q]], \\
\underline{r}[L ■ Q] & =\operatorname{ad}_{L} \underline{r}[Q]=\operatorname{Ad}_{L} \underline{r}[Q]=[L, \underline{r}[Q]]_{\mathbf{■}}
\end{aligned}
$$

We denote by $[., .]_{■}$ the Lie bracket on $\mathcal{W}$ coming from the product $■$. Note that

$$
\left[p \otimes p^{\prime}, q \otimes q^{\prime}\right]_{\bullet}=(p \amalg q) \otimes\left[p^{\prime}, q^{\prime}\right]
$$

Remark 3.4. This is the Lie structure for the pre-Lie structure $\triangleright$ (Remark 2.5), i.e.

$$
[x, y]_{■}=x ■ y-y ■ x=x \triangleright y-y \triangleright x=x \succeq y-y \succeq x-y \preceq x+x \preceq y
$$

For $x \in \mathcal{W}$ denote by ad $_{\mathbf{\bullet} ; x}$ the corresponding adjunction operator, i.e. $\operatorname{ad}_{\mathbf{\bullet} ; x} y:=[x, y]_{\mathbf{\bullet}}$.
Let $\Lambda \in \mathcal{R}\langle\langle 1, \ldots, \mathrm{~d}\rangle\rangle$ be primitive. Then, using Lemma 3.3,

$$
\underline{r}\left(\Lambda^{\boldsymbol{n}}\right)=\left[\Lambda, \underline{r}\left(\Lambda^{n-1}\right)\right]_{\mathbf{\bullet}}
$$

Iterating this, we get

$$
\underline{r}\left(\Lambda^{n}\right)=\left(\operatorname{ad}_{\mathbf{\bullet}} ; \Lambda\right)^{n-1} \underline{D} \Lambda .
$$

Hence

$$
\begin{equation*}
R=\underline{r}\left[\exp _{\mathbf{\bullet}}(\Lambda)\right]=\underline{r}\left[\sum_{n \geq 0} \frac{\Lambda^{\bullet n}}{n!}\right]=\sum_{n \geq 1} \frac{\left(\operatorname{ad}_{\mathbf{\bullet}} ; \Lambda\right)^{n-1}}{n!} \underline{D} \Lambda . \tag{3.5}
\end{equation*}
$$

This can now be used to recursively construct $\Lambda$ from $R$. Put

$$
\left[x_{1}, \ldots, x_{n}\right]_{\mathbf{\bullet}}:=\left[x_{1},\left[\ldots,\left[x_{n-1}, x_{n}\right] \ldots\right]_{\mathbf{\bullet}}, \quad\left[x_{1}, x_{2}\right]_{\boldsymbol{\bullet}}:=\left[x_{1}, x_{2}\right]_{\bullet}, \quad[x]_{\mathbf{\bullet}}:=x\right.
$$

Proposition 3.5. We have $\Lambda_{1}=R_{1}$ and

$$
\Lambda_{n}=\frac{1}{n} R_{n}-\frac{1}{n} \sum_{i=2}^{n} \frac{1}{i!} \sum_{\substack{n_{1}, \ldots, n_{i}=n \\ n_{1}+\cdots+n_{i}=n}} n_{i}\left[\Lambda_{n_{1}}, \ldots, \Lambda_{n_{i}}\right] .
$$

Proof. Rewriting Equation (3.5) for the homogeneous part $R_{n}$ yields

$$
\begin{aligned}
R_{n} & =\underline{D} \Lambda_{n}+\frac{1}{2} \sum_{m=1}^{n-1}\left[\Lambda_{m}, \underline{D} \Lambda_{n-m}\right]+\sum_{i=3}^{n} \frac{1}{i!} \sum_{\substack{n_{1}, \ldots, n_{1} \\
n_{1}+\ldots+n_{1}=n}}\left[\Lambda_{n_{1}}, \ldots, \Lambda_{n_{i-1}}, \underline{D} \Lambda_{n_{i}}\right]_{\mathbf{\bullet}} \\
& =n \Lambda_{n}+\sum_{m=1}^{n-1}(n-m)\left[\Lambda_{m}, \Lambda_{n-m}\right]_{\mathbf{\bullet}}+\sum_{i=3}^{n} \frac{1}{i!} \sum_{\substack{n_{1}, \ldots, n_{1} \\
n_{1}+\ldots+n_{1}=n}} n_{i}\left[\Lambda_{n_{1}}, \ldots, \Lambda_{n_{i-1}}, \Lambda_{n_{i}}\right]_{\mathbf{\bullet}} \\
& =n \Lambda_{n}+\sum_{i=2}^{n} \frac{1}{i!} \sum_{\substack{n_{1}, \ldots, n_{1} \\
n_{1}+\ldots+n_{1}=n}} n_{i}\left[\Lambda_{n_{1}}, \ldots, \Lambda_{n_{i}}\right]_{\mathbf{\bullet}},
\end{aligned}
$$

which shows the claim.
Example 3.6. Let us spell out the first few summands of (3.5),

$$
R=\underline{D} \Lambda+\frac{1}{2!}[\Lambda, \underline{D} \Lambda]_{\mathbf{\bullet}}+\frac{1}{3!}\left[\Lambda,[\Lambda, \underline{D} \Lambda]_{\mathbf{\bullet}}\right]_{\mathbf{■}}+\frac{1}{4!}\left[\Lambda,\left[\Lambda,[\Lambda, \underline{D} \Lambda]_{\mathbf{\bullet}}\right]_{\mathbf{\bullet}}\right]_{\mathbf{\bullet}}+. .
$$

Level by level (remember that $\Lambda_{0}=0$ ), we see

$$
\begin{aligned}
& R_{1}=\Lambda_{1} \\
& R_{2}=2 \Lambda_{2}+\frac{1}{2!}\left[\Lambda_{1}, \Lambda_{1}\right]_{\bullet} \\
& R_{3}=3 \Lambda_{3}+\frac{1}{2!}\left(\left[\Lambda_{1}, 2 \Lambda_{2}\right]_{\bullet}+\left[\Lambda_{2}, \Lambda_{1}\right]_{\bullet}\right)+\frac{1}{3!}\left[\Lambda_{1},\left[\Lambda_{1}, \Lambda_{1}\right]_{\bullet}\right]_{\bullet} \\
& R_{4}=4 \Lambda_{4}+\frac{1}{2!}\left(\left[\Lambda_{1}, 3 \Lambda_{3}\right]_{\bullet}+\left[\Lambda_{2}, 2 \Lambda_{2}\right]_{\bullet}+\left[\Lambda_{3}, \Lambda_{1}\right]_{\bullet}\right) \\
& +\frac{1}{3!}\left(\left[\Lambda_{1},\left[\Lambda_{1}, 2 \Lambda_{2}\right]_{\bullet}\right]{ }_{\bullet}+\left[\Lambda_{1},\left[\Lambda_{2}, \Lambda_{1}\right]_{\bullet}\right]_{\bullet}+\left[\Lambda_{2},\left[\Lambda_{1}, \Lambda_{1}\right]_{\bullet}\right]_{\bullet}\right) \\
& +\frac{1}{4!}\left[\Lambda_{1},\left[\Lambda_{1},\left[\Lambda_{1}, \Lambda_{1}\right]_{\bullet}\right]_{\boldsymbol{n}}\right]_{\Perp}
\end{aligned}
$$

Plugging in the expressions from Example 2.10 in for $R$, we get for $d=2$

$$
\begin{aligned}
& \Lambda_{1}=R_{1}=1 \otimes 1+2 \otimes 2 \\
& \Lambda_{2}=\frac{1}{2} R_{2}=\frac{1}{4}(\operatorname{area}(1,2) \otimes[1,2]+\operatorname{area}(2,1) \otimes[2,1])=\frac{1}{2} \operatorname{area}(1,2) \otimes[1,2] \\
& \Lambda_{3}=\frac{1}{3}\left(R_{3}-\frac{1}{2}\left[\Lambda_{1}, 2 \Lambda_{2}\right]_{\bullet}-\frac{1}{2}\left[\Lambda_{2}, \Lambda_{1}\right]_{\bullet}-\frac{1}{3}\left[\Lambda_{1},\left[\Lambda_{1}, \Lambda_{1}\right]_{\bullet}\right]_{\bullet}\right)=\frac{1}{3}\left(R_{3}-\frac{1}{2}\left[\Lambda_{1}, \Lambda_{2}\right]_{\bullet}\right) \\
& =\frac{1}{6} \operatorname{area}(1, \operatorname{area}(1,2)) \otimes[1,[1,2]]+\frac{1}{6} \operatorname{area}(2, \operatorname{area}(1,2)) \otimes[2,[1,2]] \\
& -\frac{1}{12}(1 \text { ш area }(1,2)) \otimes[1,[1,2]]-\frac{1}{12}(2 \text { ш area }(1,2)) \otimes[2,[1,2]] \\
& \Lambda_{4}=\frac{1}{4}\left\{R_{4}-\frac{1}{2!}\left(\left[\Lambda_{1}, 3 \Lambda_{3}\right]_{\bullet}+\left[\Lambda_{2}, 2 \Lambda_{2}\right]_{\bullet}+\left[\Lambda_{3}, \Lambda_{1}\right]_{\bullet}\right)\right. \\
& -\frac{1}{3!}\left(\left[\Lambda_{1},\left[\Lambda_{1}, 2 \Lambda_{2}\right]_{\Perp}\right] \_+\left[\Lambda_{1},\left[\Lambda_{2}, \Lambda_{1}\right]_{\bullet}\right] \_+\left[\Lambda_{2},\left[\Lambda_{1}, \Lambda_{1}\right]_{\bullet}\right]\right) \\
& \left.-\frac{1}{4!}\left[\Lambda_{1},\left[\Lambda_{1},\left[\Lambda_{1}, \Lambda_{1}\right]_{\bullet}\right]_{\Perp}\right]_{\bullet}\right\} \\
& =\frac{1}{4}\left(R_{4}-\left[\Lambda_{1}, \Lambda_{3}\right]_{\bullet}-\frac{1}{3!}\left[\Lambda_{1},\left[\Lambda_{1}, \Lambda_{2}\right]_{\bullet}\right]_{\bullet}\right) \\
& =\frac{1}{24}(\operatorname{area}(1, \operatorname{area}(1, \operatorname{area}(1,2))) \otimes[1,[1,[1,2]]]+\operatorname{area}(1, \operatorname{area}(2, \operatorname{area}(1,2))) \otimes[1,[2,[1,2]]] \\
& +\operatorname{area}(2, \operatorname{area}(1, \operatorname{area}(1,2))) \otimes[2,[1,[1,2]]]+\operatorname{area}(2, \operatorname{area}(2, \operatorname{area}(1,2))) \otimes[2,[2,[1,2]]] \\
& -(1 \text { ш area }(1, \text { area }(1,2))) \otimes[1,[1,[1,2]]]-(2 \text { ш area }(1, \text { area }(1,2))) \otimes[2,[1,[1,2]]] \\
& -(1 \text { Ш } \operatorname{area}(2, \operatorname{area}(1,2))) \otimes[1,[2,[1,2]]]-(2 \amalg \operatorname{area}(2, \text { area }(1,2))) \otimes[2,[2,[1,2]]])
\end{aligned}
$$

Remark 3.7. Comparing with [Roc2003, p.322] we note that we correct some of the coefficients appearing in $\Lambda_{3}$ and $\Lambda_{4}$ there.

Definition 3.8. Let $\widetilde{\mathrm{BPT}}_{n}$ be binary planar trees, with two types of inner nodes, $\cdot$ and $\boldsymbol{\bullet}$, and such that the subset of all $\cdot$ nodes is either empty or forms a subtree with the same root as the tree itself. In other words the square nodes are all connected to the root.
Define $e:{\widetilde{\mathrm{BPT}_{n}}}_{n} \rightarrow \mathbb{R}$ as follows. If the root of $\tau$ is $\boldsymbol{\bullet}$, then

$$
e(\tau):=\frac{1}{n c(\tau)}
$$

where $c$ was defined in Lemma 2.12. Otherwise, we can write $\tau$ uniquely as
for some $\ell=\ell(\tau) \geq 2, \tau^{(1)}, \ldots, \tau^{(\ell-1)} \in \widetilde{\mathrm{BPT}}$ and $\tau^{(\ell)} \in \mathrm{BPT}$. Here $\sigma \rightarrow_{\mathbf{\imath}} \rho$ is the grafting, to a new root of type $\llbracket$, with $\sigma$ on the left and $\rho$ on the right. Then

$$
e(\tau):=-\sum_{j=2}^{\ell(\tau)} \frac{\left|\tau^{(\geq j)}\right|_{l_{\text {eaves }}} e\left(\tau^{(\geq j)}\right)}{j!|\tau|_{\text {leves }}}\left(\prod_{i=1}^{j-1} e\left(\tau^{(i)}\right)\right)
$$

where

$$
\begin{aligned}
& \tau^{(\geq j)}:={\underset{\substack{\left.(j+1) \\
\tau^{(j)}\right)}}{\tau^{(\ell-1) \tau^{(\ell)}}}=\left(\tau^{(j)} \rightarrow_{\mathbf{\bullet}}\left(\tau^{(j+1)} \rightarrow_{\mathbf{\bullet}}\left(\cdots \rightarrow_{\mathbf{\bullet}}\left(\tau^{(\ell-1)} \rightarrow_{\mathbf{!}} \tau^{(\ell)}\right)\right)\right)\right), \quad j=1, \ldots, \ell-1}_{\tau^{(\geq \ell)}:=\tau^{(\ell)} .} .
\end{aligned}
$$

Finally, for a tree $\tau \in \widetilde{\mathrm{BPT}}_{n}$ and a word $w$ of length $n$, define $\widetilde{\operatorname{area}}(\tau)$ as bracketing out using area if a node of type $\bullet$ is encountered and multiplying using $\amalg$ when a node $■$ is encountered.

Example 3.9. The trees in $\widetilde{\mathrm{BPT}}_{2}$ are

$$
i_{\bigvee}^{j}, i_{\bigvee}^{j}
$$

for letters i and j, and the trees in $\widetilde{\mathrm{BPT}}_{3}$ are
for letters i, j and k.
We have

$$
\begin{aligned}
e(2) & =c(2)=1 \\
e\left({ }^{2} \bigvee^{3}\right) & =-\frac{1}{2 \cdot 2} e(2)|3|_{\text {leaves }} e(3)=-\frac{1}{4} \\
e\left(1^{2} \bigvee^{3}\right) & =-\frac{1}{3}\left(\left.\left.\frac{1}{2!} e(1)\right|^{2} \bigvee^{3}\right|_{\text {leaves }} e\left({ }^{2} \bigvee^{3}\right)+\frac{1}{3!} e(2) e(2)|3|_{\text {leaves }} e(3)\right)=-\frac{1}{3}\left(-\frac{1}{4}+\frac{1}{6}\right)=\frac{1}{36}
\end{aligned}
$$

And

$$
\text { area. }\left(1^{2} \bigvee^{3}\right)=1 \text { ш area }(2,3)
$$

Theorem 3.10. Then

$$
\Lambda_{n}=\sum_{\tau \in \widetilde{\mathrm{BPT}_{n}}} e(\tau) \widetilde{\operatorname{area}_{\bullet}}(\tau) \otimes \operatorname{lie}_{\bullet}(\tau) \in \mathfrak{L}:=\langle\mathfrak{R} ;[\cdot, \cdot] \cdot \mathbf{\bullet}
$$

Proof. Define

$$
\begin{aligned}
\tau \in \widetilde{\mathrm{BPT}}_{n ; \geq i} & :=\left\{\tau \in{\left.\widetilde{\mathrm{BPT}_{n}} \mid \ell(\tau) \geq i\right\}}^{\left[x_{1}, \ldots, x_{n}\right]}:=\left[x_{1},\left[\ldots,\left[x_{n-1}, x_{n}\right] \ldots\right], \quad\left[x_{1}, x_{2}\right]:=\left[x_{1}, x_{2}\right], \quad[x]:=x\right.\right.
\end{aligned}
$$

Due to $e(i)=c(i)=1$ for all letters $i$, we have

$$
\Lambda_{1}=R_{1}=\sum_{\mathrm{i}=1}^{\mathrm{d}} \mathrm{i} \otimes \mathrm{i}=\sum_{\tau \in \mathrm{BPT}_{1}} e(\tau) \widetilde{\text { area }_{\bullet}}(\mathrm{i}) \otimes \mathrm{lie}_{\bullet}(\mathrm{i})
$$

and then via induction over $n$

$$
\begin{aligned}
& \Lambda_{n}=\frac{1}{n} R_{n}-\frac{1}{n} \sum_{i=2}^{n} \frac{1}{i!} \sum_{\substack{n_{1}, \ldots, n_{i} \\
n_{1}+\cdots+n_{i}=n}} n_{i}\left[\Lambda_{n_{1}}, \ldots, \Lambda_{n_{i}}\right]_{\boldsymbol{\bullet}} \\
& =\sum_{\tau \in \mathrm{BPT}_{n}} \frac{1}{n c(\tau)} \text { area. }_{\bullet}(\tau) \otimes \operatorname{lie}_{.}(\tau) \\
& -\frac{1}{n} \sum_{i=2}^{n} \frac{1}{i!} \sum_{\substack{n_{1}, \ldots, n_{i} \\
n_{1}+\cdots+n_{i}=n}} \sum_{\substack{\tau_{1}, \ldots \tau_{i} \\
\tau_{j} \in \operatorname{BPT}_{n_{j}}}}\left|\tau_{i}\right|_{\text {leaves }} \prod_{j=1}^{i} e\left(\tau_{j}\right) \widetilde{\operatorname{area}} .\left(\tau_{1}\right) ш \cdots ш \widetilde{\operatorname{area}_{\bullet}}\left(\tau_{i}\right) \otimes\left[\text { lie. }\left(\tau_{1}\right), \ldots, \text { lie. }\left(\tau_{i}\right)\right] \\
& =\sum_{\tau \in \mathrm{BPT}_{n}} e(\tau) \text { area. }(\tau) \otimes \operatorname{lie} .(\tau) \\
& -\frac{1}{n} \sum_{i=2}^{n} \frac{1}{i!} \sum_{\tau \in \mathrm{BPT}_{n ; i}}\left|\tau^{(\geq i)}\right|_{\left.\right|_{\text {eaves }}} e\left(\tau^{(\geq i)}\right) \prod_{j=1}^{i-1} e\left(\tau^{(j)}\right) \widetilde{\operatorname{area}_{.}}(\tau) \otimes \operatorname{lie} .(\tau) \\
& =\sum_{\tau \in \mathrm{BPT}_{n}} e(\tau) \text { area. }(\tau) \otimes \operatorname{lie} .(\tau)-\sum_{\tau \in \widetilde{\mathrm{BPT}_{n}}{ }_{n}^{\geq 2}} \sum_{i=2}^{\ell(\tau)} \frac{\left|\tau^{(\geq i)}\right|_{\text {eaves }} e\left(\tau^{(\geq i)}\right)}{i!|\tau|_{\text {leves }}} \prod_{j=1}^{i-1} e\left(\tau^{(k)}\right) \widetilde{\operatorname{area}_{\mathbf{0}}}(\tau) \otimes \text { lie. }_{\mathbf{e}}(\tau) \\
& =\sum_{\tau \in \widetilde{\mathrm{BPT}_{n}}} e(\tau) \widetilde{\operatorname{area}_{\mathbf{a}}}(\tau) \otimes \operatorname{lie} .(\tau) .
\end{aligned}
$$

Remark 3.11. Recall, from Corollary 2.8,

$$
\mathfrak{R}=\left\langle i \otimes i, i=1 \ldots d ; \triangleright_{S y m}\right\rangle
$$

Define

$$
\begin{aligned}
& \mathfrak{P}:=\langle i \otimes i, i=1 \ldots d ; \triangleright\rangle \\
& \mathfrak{D}:=\langle i \otimes i, i=1 \ldots d ; \succeq, \preceq\rangle .
\end{aligned}
$$

Then, we have $S_{n}, R_{n}, \Lambda_{n} \in \mathfrak{D}$, and the chain of inclusions

$$
\mathfrak{R} \subsetneq \mathfrak{L} \subseteq \mathfrak{P} \subsetneq \mathfrak{D}
$$

Indeed, the mere inclusions are clear since $\triangleright_{\text {Sym }}$ and $[\cdot, \cdot]$, are symmetrization and antisymmetrization of $\triangleright$, and $\triangleright$ itself is defined as a combination of $\succeq$ and $\preceq$. Regarding the strictness of two of the inclusions, on the one hand for any $d \geq 2$, the only anagram axis of $12 \otimes 12$ contained in $\mathfrak{R}$ is spanned by

$$
(1 \otimes 1) \triangleright_{\text {Sym }}(2 \otimes 2)=(2 \otimes 2) \triangleright_{\text {Sym }}(1 \otimes 1)=\operatorname{area}(1,2) \otimes[1,2]=(12-21) \otimes(12-21),
$$

and thus the $\mathfrak{L}$ element

$$
[1 \otimes 1,2 \otimes 2]_{\bullet}=(1 \amalg 2) \otimes[1,2]=(12+21) \otimes(12-21)
$$

is not contained in $\mathfrak{R}$. On the other hand, the anagram space of $12 \otimes 12$ in $\mathfrak{P}$ is spanned by the two vectors

$$
\begin{aligned}
& (1 \otimes 1) \triangleright(2 \otimes 2)=(1 \succ 2) \otimes[1,2]=12 \otimes(12-21) \\
& (2 \otimes 2) \triangleright(1 \otimes 1)=(2 \succ 1) \otimes[2,1]=-21 \otimes(12-21)
\end{aligned}
$$

and is thus easily seen to not contain the $\mathfrak{D}$ element

$$
(1 \otimes 1) \succeq(2 \otimes 2)=(1 \succ 2) \otimes(1 \cdot 2)=12 \otimes 12
$$

However, it remains an open problem whether $\mathfrak{L}$ and $\mathfrak{P}$ conincide.
Finally, we note the inclusion $\mathfrak{D} \subseteq \mathfrak{A}$, where $(\mathfrak{A}, \succeq, \preceq)$ is the dendriform algebra with linear basis given by all $w \otimes v$ such that $w$ is a word and $v$ is an anagram of $w$, and leave as a further question for future work whether $\mathfrak{D}$ and $\mathfrak{A}$ actually conincide.

Since the expansion in this theorem is not in terms of a basis of the Lie algebra, these are not yet coordinates of the first kind. But, by a straightforward projection procedure we get

## Corollary 3.12.

$$
S=\exp _{\mathbf{n}}(\Lambda)
$$

with

$$
\Lambda=\sum_{h} \zeta_{h} \otimes P_{h}
$$

where $h$ runs over Hall words, $P_{h}$ are the corresponding Lie Hall basis elements, and the $\zeta_{h}$ are expressed as linear combinations of shuffles of areas-of-areas,

$$
\zeta_{h}=\sum_{\substack{\tau \in \widehat{\mathrm{BPT}}_{|h|}, \\ \text { foliage of } \tau \in \operatorname{Anagrams}(h)}} e(\tau)\left\langle S_{h}, \text { lie. }(\tau)\right\rangle \widetilde{\operatorname{area}}_{\bullet}(\tau) .
$$

Remark 3.13. Again, this result is not satisfying because the $\zeta_{h}$ are expensive to calculate due to the large number of summands, which are not even linearly independent. We mention it only for completeness.

Proof of Corollary 3.12. Let $P_{h}$ be Lie basis and $S_{h}$ its dual basis. Then

$$
\begin{aligned}
\Lambda & =\sum_{h} \sum_{\tau \in \widetilde{\mathrm{BPT}}_{n}} e(\tau) \widetilde{\operatorname{area}_{\bullet}}(\tau)\left\langle S_{h}, \text { lie. }_{\bullet}(\tau)\right\rangle \otimes P_{h} \\
& =: \sum_{h} \zeta_{h} \otimes P_{h}
\end{aligned}
$$

where $S_{h}$ can be expressed as an element of $T\left(\mathbb{R}^{d}\right)$ which is a linear combination of anagrams of $h$, thus $\left\langle S_{h}\right.$, lie $\left._{\bullet}(\tau)\right\rangle=0$ if the foliage of $\tau$ is not an anagram of $h$.

## 4. Shuffle generators

For a countable index set $I$ consider the free commutative algebra $\mathbb{R}\left[x_{i}: i \in I\right]$ over the indeterminates $x_{i}, i \in I$ ([Row1988, Definition 1.2.12]). If $V$ is a vector space with a countable basis, we also write $\mathbb{R}[V]$ for $\mathbb{R}\left[x_{i}: i \in I\right]$ where $I$ is some basis of $V$. A commutative algebra $\mathcal{A}$ is generated by some elements $z_{i} \in \mathcal{A}, i \in I$, if the commutative algebra morphism

$$
\mathbb{R}\left[x_{i}: i \in I\right] \rightarrow \mathcal{A},
$$

extended from $x_{i} \mapsto z_{i}$, is surjective. If it is also injective, the algebra is freely generated by the elements $z_{i}$. The goal of this section is to find a simple condition on a countable family $z_{i} \in T\left(\mathbb{R}^{d}\right), i \in I$, to be (freely) generating.
Before stating the general results, let us begin with the example of the image of $\rho$.
Proposition 4.1. Any basis for the image of $\operatorname{Im} \rho$ is generating. More explicitly, for any nonempty word $w$, we have

$$
\begin{equation*}
w=\sum_{\substack{w_{1}, \ldots, w_{n} \\ w_{1} \cdots w_{n}=w}} \frac{1}{k_{\left|w_{1}\right|, \ldots,\left|w_{n}\right|}} \rho\left(w_{1}\right) \text { Ш } \cdots \omega \rho\left(w_{n}\right), \tag{4.1}
\end{equation*}
$$

where $k_{m_{1}, \ldots, m_{n}}=\left(m_{1}+\cdots+m_{n}\right) k_{m_{2}, \ldots, m_{n}}$, with $k_{m}=m$.
Proof. For any letter i, we have $i=\rho(i)$ in accordance with Equation (4.1). Assume the equation holds for all non-empty words $v$ with $|v| \leq \ell$ for some $\ell \geq 1$, and let $w$ be a word with $|w|=\ell+1$. Then, by Equation (2.4) we have

$$
\begin{aligned}
|w| w & =D w=\sum_{u v=w} \rho(u) ш v=\sum_{u v=w} \rho(u) ш \sum_{\substack{v_{1}, \ldots, v_{n} \\
v_{1}, v_{n}=v}} \frac{1}{k_{\left|v_{1}\right|, \ldots,\left|v_{n}\right|}} \rho\left(v_{1}\right) ш \ldots ш \rho\left(v_{n}\right) \\
& =\sum_{\substack{w_{1}, \ldots, w_{n} \\
w_{1}, \ldots w_{n}=w}} \frac{1}{k_{\left|w_{2}\right|, \ldots,\left|w_{n}\right|}} \rho\left(w_{1}\right) ш \ldots \amalg \rho\left(w_{n}\right),
\end{aligned}
$$

again in accordance with Equation (4.1).
Note that in order for the induction to work, we made use again of the fact that $\rho(e)=0$, so we only sum over non-empty words.

Lemma 4.2. For each $n \geq 1$, let $X_{n} \subset T_{n}\left(\mathbb{R}^{d}\right)$ be a subset of the shuffle algebra at level $n$. Let $X:=\bigcup_{n \geq 1} X_{n}$. Then:

For all $n \geq 1$, for all nonzero $L \in \mathfrak{g}_{n}$ there is an $x \in X_{n}$ such that $\langle x, L\rangle \neq 0$
if and only if $X$ generates the shuffle algebra $T\left(\mathbb{R}^{d}\right)$.

If moreover $\left|X_{n}\right|=\operatorname{dim} \mathfrak{g}_{n}, n \geq 1$, then $X$ is freely generating.

Remark 4.3. This lemma can also be seen as a consequence of (the proof of) the Milnor-Moore theorem, see for example [Car2007, p.48]. Let us sketch this. Let $T\left(\mathbb{R}^{d}\right)^{* g r}$ be the graded dual of $T\left(\mathbb{R}^{d}\right)$, the subspace of $T\left(\left(\mathbb{R}^{d}\right)\right.$ ) consisting of only finite linear combinations of words. Endowed with the unshuffle coproduct, the dual of the shuffle product, this is a cocommutative, conilpotent coalgebra. Then, by (the proof of) [Car2007, Theorem 3.8.1], there exists an isomorphism of cocommutative coalgebras

$$
e_{T\left(\mathbb{R}^{d}\right)^{* g r}}: \Gamma[\mathfrak{g}] \rightarrow T\left(\mathbb{R}^{d}\right)^{* \mathrm{gr}}
$$

Here $\Gamma[\mathfrak{g}] \subset T(\mathfrak{g})$ are the symmetric tensors over $\mathfrak{g}$, generated, as a vector space, by the elements $\underbrace{v \otimes \cdots \otimes v}_{n \text { times }}, v \in \mathfrak{g}, n \geq 0$, and endowed with the deconcatenation coproduct. The map $e_{T\left(\mathbb{R}^{d}\right)^{* \mathrm{gr}}}$ acts on these elements as

$$
e_{T\left(\mathbb{R}^{d}\right)^{* \mathrm{gr}}}(\underbrace{v \otimes \cdots \otimes v}_{n \text { times }})=\frac{v^{n}}{n!}
$$

where the $n$-th power on the right hand side is taken with respect to the concatenation product (under which $T\left(\mathbb{R}^{d}\right)^{* g r}$ is closed). The grading on $T\left(\mathbb{R}^{d}\right)^{* g r}$ induces a grading on $\Gamma[\mathfrak{g}]$ via the isomorphism $e_{T\left(\mathbb{R}^{d}\right)^{* g r}}$. The graded dual (with respect to this induced grading) of $\Gamma[\mathfrak{g}]$ is then given by $\mathbb{R}\left[\mathfrak{g}^{* g r}\right]$, i.e. the symmetric algebra over $\mathfrak{g}^{* g r}$, where $\mathfrak{g}^{* g r}$ is the graded dual of $\mathfrak{g}$. Since $e_{T\left(\mathbb{R}^{d}\right)^{*} \mathrm{gr}}$ is an isomorphism of cocommutative coalgebras, the dual map

$$
e_{T\left(\mathbb{R}^{d}\right)^{* \mathrm{gr}}}^{* \mathrm{gr}}: T\left(\mathbb{R}^{d}\right) \rightarrow \mathbb{R}\left[\mathfrak{g}^{* \mathrm{gr}}\right]
$$

is an isomorphism of commutative algebras. $X$ (freely) generating $T\left(\mathbb{R}^{d}\right)$ is then equivalent to $e_{T\left(\mathbb{R}^{d}\right)^{* \mathrm{gr}}}^{* \mathrm{gr}}(X)$ (freely) generating $\mathbb{R}\left[\mathfrak{g}^{* \mathrm{gr}}\right]$, which is equivalent to our condition, using Lemma A.1.

Proof. We show for every level $N$ :

$$
\begin{gathered}
\forall n \leq N \forall 0 \neq L \in \mathfrak{g}_{n} \text { there is } x \in X_{n} \text { with }\langle x, L\rangle \neq 0 \\
\text { if and only if } \\
\cup_{1 \leq n \leq N} X_{n} \text { shuffle generates } T_{\leq N}\left(\mathbb{R}^{d}\right)
\end{gathered}
$$

It is clearly true for $N=1$. Let it be true for some $N$. We show it for $N+1$.
Let shuff ${ }_{N+1} \subset T_{N+1}\left(\mathbb{R}^{d}\right)$ denote the linear space of shuffles of everything "from below", i.e.

$$
\operatorname{shuff}_{N+1}:=\bigcup_{n=1}^{N}\left\{T_{n}\left(\mathbb{R}^{d}\right) \amalg T_{N-n}\left(\mathbb{R}^{d}\right)\right\}
$$

By [Reu1993, Theorem 3.1 (iv)]

$$
\left\langle\operatorname{shuff}_{N+1}, L\right\rangle=0
$$

for all $L \in \mathfrak{g}_{N+1}$. In other words, shuff ${ }_{N+1}$ is contained in the annihilator of $\mathfrak{g}_{N+1}$. By [Reu1993, Theorem 6.1], the shuffle algebra is freely generated by the Lyndon words in $1, \ldots, \mathrm{~d}$, which have dimension $\operatorname{dim} \mathfrak{g}_{n}$ on level $n$. Hence

$$
\operatorname{dim} \operatorname{shuff}_{N+1}=\operatorname{dim} T_{N+1}\left(\mathbb{R}^{d}\right)-\operatorname{dim} \mathfrak{g}_{N+1} .
$$

By dimension counting we hence have that shuff ${ }_{N+1}$ must actually be equal to the annihilator of $\mathfrak{g}_{N+1}$. Then, a fortiori, $\mathfrak{g}_{N+1}$ is the annihilator of shuff ${ }_{N+1}$.
By Lemma 4.4,

$$
\begin{gathered}
T_{N+1}\left(\mathbb{R}^{d}\right)=\operatorname{shuff}_{N+1}+\operatorname{span}_{\mathbb{R}} X_{N+1} \\
\text { if and only if } \\
\forall 0 \neq L \in \mathfrak{g}_{N+1} \text { there is } x \in \operatorname{span}_{\mathbb{R}} X_{N+1} \text { with }\langle x, L\rangle \neq 0 .
\end{gathered}
$$

But this is the case if and only if $\forall 0 \neq L \in \mathfrak{g}_{N+1}$ there is $x \in X_{N+1}$ with $\langle x, L\rangle \neq 0$. This finishes the proof regarding the generating property.

Regarding freeness: denote $\iota: \mathbb{R}\left[x_{v}: v \in X\right] \rightarrow T\left(\mathbb{R}^{d}\right)$ the extension, as a commutative algebra morphism, of the map $x_{v} \mapsto v$. Denote $\iota_{\text {Lyndon }}: \mathbb{R}\left[y_{w}: w \in L\right] \rightarrow T\left(\mathbb{R}^{d}\right)$ the extension, as a commutative algebra morphism, of the map $y_{w} \mapsto w$, where $L$ are the Lyndon words. By [Reu1993, Theorem 6.1], $\iota_{\text {Lyndon }}$ is an isomorphism. By what we have shown so far, $\iota$ is surjective. Since $X$ consists of homogeneous elements, we can grade $\mathbb{R}\left[x_{v}: v \in X\right]$ induced from the grading of $T\left(\mathbb{R}^{d}\right)$ and analogously for $\mathbb{R}\left[y_{w}: w \in L\right]$. By assumption, the graded dimensions match. Hence, there is an isomorphism of graded, commutative algebras

$$
\Phi: \mathbb{R}\left[x_{v}: v \in X\right] \rightarrow \mathbb{R}\left[y_{w}: w \in L\right]
$$

Since $\iota_{\text {Lyndon }}$ is an isomorphism of graded, commutative algebras and $\iota$ is epimorphism of graded, commutative algebras (where each homogeneous subspace is finite dimensional!) we must have that $\iota$ is in fact an isomorphism.

We used the following simple lemma.
Lemma 4.4. Let $V$ be a finite dimensional vector space with dual $W:=V^{*}$. We denote the pairing by $\langle w, v\rangle$, for $w \in W, v \in V$. Let $W_{1}, W_{2}$ be subspaces of $W$ and let

$$
W_{1}^{\perp}:=\left\{v \in V:\left\langle w_{1}, v\right\rangle=0 \forall w_{1} \in W_{1}\right\},
$$

be the annihilator of $W_{1}$. Then:

$$
\begin{gathered}
\forall 0 \neq v_{1} \in W_{1}^{\perp} \text { there is } w_{2} \in W_{2} \text { with }\left\langle w_{2}, v_{1}\right\rangle \neq 0 \\
\text { if and only if } \\
W_{1}+W_{2}=W
\end{gathered}
$$

Proof. Recall the well-known identity ([Hal2017, Exercise 17.8.c)])

$$
\left(W_{1}+W_{2}\right)^{\perp}=W_{1}^{\perp} \cap W_{2}^{\perp} .
$$

Then

$$
W_{1}+W_{2}=W \Leftrightarrow W_{1}^{\perp} \cap W_{2}^{\perp}=\{0\},
$$

which is the claim.
Corollary 4.5. Let $X$ be a set of homogeneous elements of $T\left(\mathbb{R}^{d}\right)$. Then, the following are equivalent:
(i) $X$ freely shuffle generates $T\left(\mathbb{R}^{d}\right)$,
(ii) $X$ is a homogeneous realization of a dual basis to a homogeneus basis of $\mathfrak{g}$,
(iii) $X$ is a homogeneous basis for the image of a projection $\pi^{\top}$, where $\pi$ is a graded projection $\pi: T\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow \mathfrak{g} \subset T\left(\left(\mathbb{R}^{d}\right)\right)^{6}$.
Examples include:

1. $\pi:=\pi_{1}$, the Eulerian idempotent (3.3)
$\rightsquigarrow$ Coordinates of the first kind.
2. A rescaling of the Dynkin map $r$ (2.1) (to make it a projection)
$\rightsquigarrow A$ basis for the image of $\rho$, for example $\mathfrak{r}_{h}$ from Theorem 2.14, which by Corollary 2.8 can be expressed as areas-of-areas.
3. $\pi$ the orthogonal projection (with respect to the inner product in the ambient space $T\left(\left(\mathbb{R}^{d}\right)\right)$ ) onto $\mathfrak{g}$ (the Garsia idempotent, [Duc1991])
$\rightsquigarrow A n y$ (homogeneous) basis for the Lie algebra $\mathfrak{g} \subset T\left(\left(\mathbb{R}^{d}\right)\right)$, identified as elements of $T\left(\mathbb{R}^{d}\right)$.

Remark 4.6. 1. Point 3. is shown in [Reu1993, Section 6.5.1]. We include it here, as it falls nicely into the setting of Lemma 4.2.
2. Coordinates of the first kind must - by definition - contain all the information of the signature, so it is reasonable that they shuffle generate $T\left(\mathbb{R}^{d}\right)$. For the other sets this is not immediately evident. The basis for the Lie algebra is one such example and it does not even live in the correct space (formally, it is an element of the concatenation algebra $T\left(\left(\mathbb{R}^{d}\right)\right)$ not of the shuffle algebra $T\left(\mathbb{R}^{d}\right)$ ).

Proof. (iii) $\Rightarrow$ (i): Assume first that we have given a graded projection $\pi$ with image $\mathfrak{g}$, and a homogeneous basis $\left(x_{i}\right)_{i}$ of $\operatorname{Im} \pi^{\top}$. Because of the grading it makes sense to speak of the component $\pi_{n}: T_{n}\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow T_{n}\left(\left(\mathbb{R}^{d}\right)\right)$. Then $\pi_{n}^{\top}: T_{n}\left(\mathbb{R}^{d}\right) \rightarrow T_{n}\left(\mathbb{R}^{d}\right)$ and

$$
\operatorname{im}\left(\pi_{n}^{\top}\right)^{\perp}=\operatorname{ker}\left(\pi_{n}\right)
$$

Since $\pi_{n}$ itself is also a projection, we have that

$$
T_{n}\left(\mathbb{R}^{d}\right)=\operatorname{ker}\left(\pi_{n}\right) \oplus \operatorname{im}\left(\pi_{n}\right) .
$$

Hence, for every $L \in \operatorname{im}(\pi)$ there is $x \in \operatorname{im}\left(\pi^{\top}\right)$ with $\langle x, L\rangle \neq 0$. Then Lemma 4.2 applies.

[^5](i) $\Rightarrow\left(\right.$ ii): Let now $X$ be a homogeneous free shuffle generating set. Then $\left\{\langle x, \cdot\rangle \mid x \in X_{n}\right\}$ spans the whole dual space of $\mathfrak{g}_{n}$. Indeed, assume this is not the case, then a comparison with some $\mathbb{R}^{n}$ shows that there is a nonempty annihilator of $X_{n}$ inside $\mathfrak{g}_{n}$, but this contradicts the criterion from Lemma 4.2. Hence, $X$ does span the dual space of $\mathfrak{g}$, thus contains a dual basis to some basis of $\mathfrak{g}$, and since $X$ is freely generating, $X$ is actually that dual basis (otherwise, a subset of $X$ would already generate, which contradicts the assumption that $X$ freely generates).
(ii) $\Rightarrow$ (iii): Let $P_{h} \in T\left(\left(\mathbb{R}^{d}\right)\right), h \in H$, be some homogeneous basis for the Lie algebra. Let $D_{h} \in T\left(\mathbb{R}^{d}\right), h \in H$ be a realization of a dual basis. That is
$$
\left\langle D_{h}, P_{h^{\prime}}\right\rangle=\delta_{h, h^{\prime}} .
$$

Then choose $\pi$ such that ker $\pi=\left(\operatorname{span}_{\mathbb{R}}\left\{D_{h}: h \in H\right\}\right)^{\top}$.
Proposition 4.7. If $X$ is a homogeneous set and $\pi: T\left(\left(\mathbb{R}^{d}\right)\right) \rightarrow \mathfrak{g} \subset T\left(\left(\mathbb{R}^{d}\right)\right)$ is a graded projection, then $\pi^{\top} X$ is a shuffle generating set if and only if $\pi^{\top} X$ spans $\operatorname{Im} \pi^{\top}$ if and only if $X$ is a shuffle generating set. If $X$ is freely generating, then so is $\pi^{\top} X$.

Proof. Since for any $x \in X$ and $p \in \mathfrak{g}$ we have

$$
\left\langle\pi^{\top} x, p\right\rangle=\langle x, \pi p\rangle=\langle x, p\rangle,
$$

the condition for being a shuffle generating set in Lemma 4.2 is fullfilled for $X$ if and only if it is fullfilled for $\pi^{\top} X$. Since any basis of $\operatorname{Im} \pi^{\top}$ is a free and thus also a minimal shuffle generating set by Corollary $4.5, \pi^{\top} X \subseteq \operatorname{Im} \pi^{\top}$ shuffle generates if and only if it linearly spans $\operatorname{Im} \pi^{\top}$. If $X$ freely shuffle generates, than the shuffle generating set $\pi^{\top} X$ must also have minimal dimension for each homogeneity, and thus freely generate due to the freeness of the shuffle algebra.

Point 3.2 in Corollary 4.5 proves, using Corollary 2.8, what we set out to prove: areas-of-areas do shuffle generate $T\left(\mathbb{R}^{d}\right)$.

Corollary 4.8. The set $\mathscr{A}$ of the Introduction is a generating set for $T\left(\mathbb{R}^{d}\right)$. A free generating set is given e.g. by any basis for the image of $\rho$.

Proof. We give three proofs.
Via $\Lambda$ and coordinates of the first kind
Using $D w=\sum_{u v=w,|u| \geq 1} \rho(u) ш v, \operatorname{ad}_{P}=\operatorname{Ad}_{P}$ for any Lie polynomial $P$ and $\sum_{w} w \otimes r(w)=$ $\sum_{w} \rho(w) \otimes w$, we showed $(\underline{D}-\mathrm{id}) R=R \triangleright R$ (Lemma 2.6).
We then have $R \in \mathfrak{R}$ due to $R_{1}=\sum_{\mathrm{i}=1}^{\mathrm{d}} \mathrm{i} \otimes \mathrm{i}$ and $R_{n}=\frac{1}{2(n-1)} \sum_{l=1}^{n-1} R_{l} \triangleright_{\mathrm{Sym}} R_{n-l}$ (Corollary 2.8).

Using $R=\underline{r} \exp _{\mathbf{a}} \Lambda$ and Baker's identity for $\underline{r}$ (Lemma 3.3), we show that $\Lambda_{n}$ is a linear combination of $R_{n}$ and $\llbracket$-Lie-bracketings of lower order $\Lambda_{i}$. Thus $\Lambda \in \mathfrak{L}$, and together with the fact $\zeta_{h}=\operatorname{coeval}^{S_{h}}(\Lambda)$ following from the definition of $\Lambda$ we conclude that each $\zeta_{h}$ is a linear combination of shuffles of areas of areas.
Via $R$ and $\rho$

From $R \in \mathfrak{R}$ we conclude that the image of $\rho$ lies in $\mathscr{A}$, since $\rho(v)=\operatorname{coeval}^{v}(R)$ for any word $v$. Since the image of $\rho$ shuffle generates the shuffle algebra, Point 3.2 in Corollary 4.5, so does $\mathscr{A}$.
Via [DIM2018] and $\rho$
Via a combinatorial expression for $\rho(12 \ldots \mathrm{~d})$ for any $d$ (Proposition 6.11 ) we conclude (Corollary 6.12) that the image of $\rho$ is a subspace of

$$
\operatorname{span}_{\mathbb{R}}\{i: i \text { a letter }\} \oplus \operatorname{span}_{\mathbb{R}}\{w(\mathrm{ij}-\mathrm{ji}): w \text { a word, } \mathrm{i}, \mathrm{j} \text { letters }\},
$$

which is nothing but $\mathscr{A}$, according to [DIM2018]. Again, since $\rho$ shuffle generates the shuffle algebra, so do areas of areas.

Remark 4.9. Corollary 4.8 is an a priori stronger statement than the following easy-to-prove statement, with which it is occasionally confused.
(A) Any word is a linear combination of shuffles of letters and areas of arbitrary words.

An illustration of $(A)$ is as follows.

$$
\begin{aligned}
123 & =(1 \succ 2) \succ 3=\frac{1}{2}\{1 \text { Ш } 2+\operatorname{area}(1,2)\} \succ 3 \\
& =\frac{1}{4}[\{1 \text { Ш } 2+\operatorname{area}(1,2)\} \text { Ш } 3+\operatorname{area}(\{1 \text { Ш } 2+\operatorname{area}(1,2)\}, 3)] \\
& =\frac{1}{4}[1 \text { Ш } 2 \text { Ш } 3+\operatorname{area}(1,2) \text { Ш } 3+\operatorname{area}(1 \text { Ш } 2,3)+\operatorname{area}(\operatorname{area}(1,2), 3)]
\end{aligned}
$$

Corollary 4.8 implies that this can be done with all the shuffles outside all the areas, namely
(B) Any word is a linear combination of shuffles of letters and iterated areas of letters.

For example

$$
\begin{aligned}
123= & \frac{1}{3} \operatorname{area}(1, \operatorname{area}(2,3))+\frac{1}{6} \operatorname{area}(\operatorname{area}(1,3), 2)+\frac{1}{3} 1 \text { Ш area }(2,3) \\
& -\frac{1}{6} 2 \text { Ш area }(1,3)+\frac{1}{2} 3 \text { ш area }(1,2)+\frac{1}{6} 1 \text { ш } 2 \text { ш } 3 .
\end{aligned}
$$

## 5. Applications

The antisymmetrizing feature of the area operation leads to pleasant properties for piecewise linear paths and semimartingales.

### 5.1. Piecewise linear paths: computational aspects

For two time series $a_{0}, \ldots, a_{n}, b_{0}, \ldots, b_{n} \in \mathbb{R}$ define the new time series

$$
\begin{aligned}
\operatorname{DiscreteArea}(a, b)_{\ell} & :=\operatorname{Corr}_{1}(a, b)_{\ell}-\operatorname{Corr}_{1}(b, a)_{\ell} \\
& :=\sum_{i=0}^{\ell-1} a_{i+1} b_{i}-\sum_{i=0}^{\ell-1} b_{i+1} a_{i}, \quad \ell=0, \ldots, n,
\end{aligned}
$$

set to be 0 for $\ell=0$. It is known ([DR2018, Section 3.2]), that for a piecewise linear curve $X$ through the points $0, x_{1}, \ldots, x_{n} \in \mathbb{R}^{2}$, one has

$$
\begin{equation*}
\left\langle\operatorname{area}(1,2), S(X)_{0, n}\right\rangle=\text { DiscreteArea }\left(x^{1}, x^{2}\right)_{n} . \tag{5.1}
\end{equation*}
$$

We will show that this iterates nicely.
Lemma 5.1. If $X, Y$ are piecewise linear then $\operatorname{Area}(X, Y)$ is piecewise linear.
Proof.

$$
\begin{aligned}
\frac{d}{d t} \operatorname{Area}(X, Y)_{t} & =\int_{0}^{t} d X_{r} \dot{Y}_{t}-\int_{0}^{t} d Y_{r} \dot{X}_{t} \\
\frac{d^{2}}{d t^{2}} \operatorname{Area}(X, Y)_{t} & =\int_{0}^{t} d X_{r} \ddot{Y}_{t}+\dot{X}_{t} \dot{Y}_{t}-\int_{0}^{t} d Y_{r} \ddot{X}_{t}-\dot{Y}_{t} \dot{X}_{t} \\
& =\int_{0}^{t} d X_{r} \ddot{Y}_{t}-\int_{0}^{t} d Y_{r} \ddot{X}_{t}=0
\end{aligned}
$$

since $X, Y$ are piecewise linear. Hence, $\operatorname{Area}(X, Y)$ is indeed piecewise linear.
In particular, for $\phi \in \mathscr{A}$ (defined in the Introduction) and $X$ piecewise linear

$$
t \mapsto\left\langle\phi, S(X)_{0, t}\right\rangle
$$

is piecewise linear. Note that by Lemma 6.1, $\phi$ can be written as linear combination of elements of the form $w(i j-j i)$. One can also see directly that such elements yield something piecewise linear:

$$
\begin{aligned}
& \frac{d^{2}}{d t^{2}}\left\langle w(i j-j i), S(X)_{0, t}\right\rangle \\
&= \frac{d^{2}}{d t^{2}}\left\{\int_{0}^{t} \int_{0}^{s}\left\langle w, S(X)_{0, r}\right\rangle d X_{r}^{(i)} d X_{s}^{(j)}-\int_{0}^{t} \int_{0}^{s}\left\langle w, S(X)_{0, r}\right\rangle d X_{r}^{(j)} d X_{s}^{(i)}\right\} \\
&= \int_{0}^{t}\left\langle w, S(X)_{0, r}\right\rangle d X_{r}^{(i)} \ddot{X}_{t}^{(j)}+\left\langle w, S(X)_{0, t}\right\rangle \dot{X}_{t}^{(i)} \dot{X}_{t}^{(j)} \\
&-\int_{0}^{t}\left\langle w, S(X)_{0, r}\right\rangle d X_{r}^{(j)} \ddot{X}_{t}^{(i)}-\left\langle w, S(X)_{0, t}\right\rangle \dot{X}_{t}^{(i)} \dot{X}_{t}^{(j)} \\
&= 0
\end{aligned}
$$

Lemma 5.2. For all nonzero $z \in T\left(\mathbb{R}^{d}\right)$, there is a piecewise linear path $X$ such that $\langle z, S(X)\rangle \neq$ 0 .

Proof. Let $z \in T\left(\mathbb{R}^{d}\right) \backslash\{0\}$ be arbitrary and let $n$ be its degree (the length of the longest word in the word-expansion of $z$ ). Since $G_{\leq n}:=\operatorname{proj}_{\leq n} G$ spans $T_{\leq n}\left(\mathbb{R}^{d}\right)$ (see e.g. [DR2018, Lemma 8]), there are $g_{1}, \ldots, g_{k} \in G$ and $r_{1}, \ldots, r_{k} \in \mathbb{R}$ such that

$$
\left\langle z, r_{1} g_{1}+\cdots+r_{k} g_{k}\right\rangle=\langle z, z\rangle \neq 0
$$

and hence there is $g_{i} \in G$ such that $\left\langle z, g_{i}\right\rangle \neq 0$. Now, due to Chow's theorem according to [FV2010, Theorem 7.28], there is a piecewise linear $X$ such that $\operatorname{proj}_{\leq n} g_{i}=\operatorname{proj}_{\leq n} S(X)$, which implies $\langle z, S(X)\rangle=\left\langle z, g_{i}\right\rangle \neq 0$.

Theorem 5.3. $\left\langle S(X)_{0, t}, \phi\right\rangle$ is piecewise linear for all piecewise linear paths $X$ if and only if $\phi \in \mathbb{R} \oplus \mathscr{A}$.

Proof. We already showed in Lemma 5.1 that for piecewise linear $X,\left\langle S(X)_{0, t}, \phi\right\rangle$ is again piecewise linear for all $\phi \in \mathscr{A}$. Since the whole tensor space $T\left(\mathbb{R}^{d}\right)=\mathbb{R} \oplus \mathscr{A} \oplus B$, where $B$ is $\operatorname{span}_{\mathbb{R}}\{w i j, w$ a word, $\mathrm{i} \leq \mathrm{j}$ letters $\}$, and since the sum of a function which is not piecewise linear with a piecewise linear function is again not piecewise linear, it only remains to show that for any $b \in B \backslash\{0\}$, there is a piecewise linear $X$ such that $t \mapsto\left\langle b, S(X)_{0, t}\right\rangle$ is not piecewise linear.
To this end, let $b=\sum_{i \leq j} d_{i j} \mathrm{ij} \in B \backslash\{0\}$ be arbitrary. If there is a letter 1 such that $d_{l l} \neq 0$ (case 1), choose a piecewise linear path $X:[0,2] \rightarrow \mathbb{R}^{d}$ such that $\left\langle d_{l l}, S(X)_{0,1}\right\rangle \neq 0$ and such that $X{ }_{[1,2]}$ is linear with $x_{l}=1$ and $x_{i}=0$ for $i \neq l$, where $x_{i}:=\dot{X}_{3 / 2}^{i}$. Otherwise, since $b$ is nonzero, there are letters $\mathrm{k}<1$ such that $d_{k l} \neq 0$ (case 2 ), and in this case, choose a piecewise linear path $X:[0,2] \rightarrow \mathbb{R}^{d}$ such that $\left\langle d_{k l}, S(X)_{0,1}\right\rangle \neq 0$ and such that $X \upharpoonright_{[1,2]}$ is linear with $x_{k}=1, x_{l}=1$ and $x_{i}=0$ for $i \notin\{k, l\}$, where $x_{i}:=\dot{X}_{3 / 2}^{i}$. In both cases, such a piecewise linear $X$ exists due to Lemma 5.2.
Since $X_{[1,2]}$ is linear, we have for arbitrary $z \in T\left(\mathbb{R}^{d}\right.$ that $t \mapsto\left\langle z, S(X)_{0, t}\right\rangle$ is polynomial on $[1,2]$, and thus arbitraryly often continuously differentiable on $(1,2)$. Thus, since $\ddot{X}=0$ and $\dot{X}$ constant on $(1,2)$, we have

$$
\lim _{t \searrow 1} \frac{d^{2}}{d t^{2}}\left\langle\sum_{i \leq j} d_{i j} \mathrm{ij}, S(X)_{0, t}\right\rangle=\sum_{i \leq j}\left\langle d_{i j}, S(X)_{0,1}\right\rangle x_{i} x_{j}= \begin{cases}\left\langle d_{l l}, S(X)_{0,1}\right\rangle \neq 0, & \text { case 1 } \\ \left\langle d_{k l}, S(X)_{0,1}\right\rangle \neq 0, & \text { case 2 }\end{cases}
$$

In both cases, we conclude that $t \mapsto\left\langle b, S(X)_{0, t}\right\rangle$ is not piecewise linear on any interval $[1, s]$, $1<s \leq 2$, which finishes the proof.

The fact that "being linear" is preserved under the Area-operation immediately leads to the following theorem.

Theorem 5.4. Let $X$ in $\mathbb{R}^{d}$ be a piecewise linear curve through the points $0, x_{1}, \ldots, x_{n} \in \mathbb{R}^{d}$. Then: for every tree $\tau$,

$$
\left\langle\operatorname{area}_{\bullet}(\tau), S(X)_{0, n}\right\rangle=\text { DiscreteArea. }(\tau, x)_{n} .
$$

Here, area. is defined in Lemma 2.12 and DiscreteArea. is defined similarly, as iterated bracketing using the DiscreteArea-operator.

Example 5.5. For $\tau={ }_{1}^{1}{ }^{\gamma^{2}}{ }^{3}$ the statement reads as

$$
\begin{aligned}
\left\langle\operatorname{area} \cdot\left(\begin{array}{l}
1 \\
\gamma^{2} \\
\gamma^{3}
\end{array}\right), S(X)_{0, n}\right\rangle & :=\left\langle\operatorname{area}(\operatorname{area}(1,2), 3), S(X)_{0, n}\right\rangle \\
& = \\
\text { DiscreteArea. }\left({ }^{1} \vartheta^{2}, x, x\right)_{n} & :=\operatorname{DiscreteArea}\left(\operatorname{DiscreteArea}\left(x^{(1)}, x^{(2)}\right), x^{(3)}\right)_{n},
\end{aligned}
$$

which one can verify by a direct, but tedious, calculation.
Remark 5.6. This is not obvious at all. Indeed, if we just look at the discrete integration operator (still assuming $x_{0}=0$ )

$$
\left\langle 12, S(X)_{0, n}\right\rangle=\sum_{i=0}^{n-1} \frac{1}{2}\left(x_{i}^{1}+x_{i+1}^{1}\right)\left(x_{i+1}^{2}-x_{i}^{2}\right)=: \text { Discretelntegral }\left(x^{1}, x^{2}\right)_{n},
$$

this does not iterate. Indeed,

$$
\begin{aligned}
\left\langle 123, S(X)_{0, n}\right\rangle & =\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i-1} \frac{1}{2}\left(x_{j}^{1}+x_{j+1}^{1}\right) x_{j, j+1}^{2}+\left(\frac{1}{2} x_{i}^{1}+\frac{1}{3!} x_{i, i+1}^{1}\right) x_{i, i+1}^{2}\right) x_{i, i+1}^{3} \\
& \left.\neq \text { DiscreteIntegral (Discretelntegral }\left(x^{1}, x^{2}\right), x^{3}\right)_{n} .
\end{aligned}
$$

Proof. If $Y, Z$ are piecewise linear between the points $0, y_{1}, .$. and $0, z_{1}, \ldots$, then $\operatorname{Area}(Y, Z)$ is piecewise linear between the points

$$
0, \operatorname{DiscreteArea}(y, z)_{1}, . ., \text { DiscreteArea }(y, z)_{n} .
$$

We can hence iterate (5.1).

### 5.2. Martingales

Another pleasant property of the area-operation presents itself when working with a continuous semimartingale M. One has (see [IW1988, Chapter III])

$$
\begin{equation*}
\int_{0}^{T} M_{0, r}^{i} d_{\mathrm{Strat}} M_{r}^{j}=\int_{0}^{T} M_{0, r}^{i} d_{\mathrm{It}} M_{r}^{j}+\frac{1}{2}\left[M^{i}, M^{j}\right]_{T} \tag{5.2}
\end{equation*}
$$

where $d_{\text {Strat }}$ denotes Stratonovich integration, $d_{\text {Itō }}$ denotes Itō integration and [., .] denotes the quadratic covariation.

Proposition 5.7. Let $\left(\Omega, \mathcal{F}, \mathcal{F}_{t}, \mathbb{P}\right)$ be a filtered probability space. Let $M$ be a continuous, $\mathcal{F}_{t^{-}}$ martingale such that all iterated Itō integrals are martingales. Then $t \mapsto\left\langle\phi, S_{\text {Strat }}(M)_{0, t}\right\rangle$ is an $\mathcal{F}_{t}$-martingale for all $\phi \in \mathscr{A}$.

Proof. Let $X, Y$ be as in the statement. Assume for simplicity $X_{0}=Y_{0}=0$ almost surely. Then, using Equation (5.2),

$$
\text { Area }_{\operatorname{Strat}}(X, Y)_{t}:=\int_{0}^{t} X_{r} d_{\mathrm{Strat}} Y_{r}-\int_{0}^{T} Y_{r} d_{\mathrm{Strat}} X_{r}=\int_{0}^{t} X_{r} d_{\mathrm{It} \bar{o}} Y_{r}-\int_{0}^{t} Y_{r} d_{\mathrm{It} \bar{o}} X_{r}
$$

is again an $\left(\mathcal{F}_{t}\right)_{t}$-martingale. Hence for every $\phi \in \mathscr{A},\left\langle\phi, S_{\text {Strat }}(M)_{0, t}\right\rangle$ is a $\mathcal{F}_{t}$-martingale.

Proposition 5.8. Let $M$ be the piecewise linear interpolation of a time discrete martingale whose moments all exist. Then $t \mapsto\left\langle\phi, S(M)_{0, t}\right\rangle$ is again the piecewise linear interpolation of a time discrete martingale.

Proof. Let $\left(a_{n}\right)_{n},\left(b_{n}\right)_{n}$ be two time discrete $L^{2}$ martingales for the filtration $\mathcal{F}_{k}$ whose moments all exist. Then,

$$
\begin{aligned}
& \mathbb{E}\left[\text { DiscreteArea }(a, b)_{k+1} \mid \mathcal{F}_{k}\right]-\operatorname{DiscreteArea}(a, b)_{k} \\
& \quad=\mathbb{E}\left[\operatorname{DiscreteArea}(a, b)_{k+1}-\operatorname{DiscreteArea}(a, b)_{k} \mid \mathcal{F}_{k}\right]=\mathbb{E}\left[a_{k+1} b_{k}-b_{k+1} a_{k} \mid \mathcal{F}_{k}\right] \\
& \quad=b_{k} \mathbb{E}\left[a_{k+1} \mid \mathcal{F}_{k}\right]-a_{k} \mathbb{E}\left[b_{k+1} \mid \mathcal{F}_{k}\right]=b_{k} a_{k}-a_{k} b_{k}=0
\end{aligned}
$$

thus (DiscreteArea $\left.(a, b)_{n}\right)_{n}$ is again an $\left(\mathcal{F}_{n}\right)_{n}$ martingale. If $\left(c_{n}\right)_{n},\left(d_{n}\right)_{n}$ are $\tau_{n}$ local $L^{2}$ martingales, then

$$
\text { DiscreteArea }(a, b)^{\tau_{k}}=\operatorname{DiscreteArea}\left(a^{\tau_{k}}, b^{\tau_{k}}\right)^{\tau_{k}}=\operatorname{DiscreteArea}\left(a^{\tau_{k}}, b^{\tau_{k}}\right)
$$

is a martingale for any $k$ and thus $\operatorname{DiscreteArea~}(a, b)$ is a $\tau_{k}$ local martingale.
The previous results imply that for $M$ a martingale with all iterated integrals being in $L^{1}(\Omega)$, or for $M$ a linear interpolation of a time discrete martingale, all expectancies of areas of areas vanish for $M$. This naturally leads to the very interesting question of what is the class of all semimartingale paths such that all area expectancies vanish?

In particular, for all these paths, we have, by Corollary 6.12, that the expected Stratonovich signature lies in the kernel of $r$,

$$
\begin{aligned}
r\left(\mathbb{E}\left[S_{\text {Strat }}(M)_{0, T}\right]\right) & =\mathbb{E}\left[r\left(S_{\text {Strat }}(M)_{0, T}\right)\right]=\sum_{w} \mathbb{E}\left[\left\langle w, S_{\text {Strat }}(M)_{0, T}\right\rangle\right] r(w) \\
& =\sum_{w} \mathbb{E}\left[\left\langle\rho(w), S_{\text {Strat }}(M)_{0, T}\right\rangle\right] w=0
\end{aligned}
$$

In fact this last property obviously holds for any Semimartingale with $\left\langle\rho(w), S_{\text {Strat }}\right\rangle=0$ for any $w$, which is a priori a larger class than just those paths where the expectancy of all areas of areas vanishes since $\operatorname{Im} \rho$ does not linearly $\operatorname{span} \mathscr{A}$.

## 6. Linear span of area expressions

In [Dzh2007] the antisymmetric, non-associative operation area was studied in detail. It was shown that

- area does not satisfy any new identity of degree 3 ; in particular it does not satisfy the Jacobi identity.
- On degree 4 there is exactly one new identity, the Tortkara identity. Over a field of characteristic zero, we have the following equivalent formulations:

$$
\begin{aligned}
& \operatorname{area}(\operatorname{area}(a, b), \operatorname{area}(c, b))=\operatorname{area}(\operatorname{vol}(a, b, c), b), \\
& \operatorname{area}(\operatorname{area}(a, b), \operatorname{area}(c, d))+\operatorname{area}(\operatorname{area}(a, d), \operatorname{area}(c, b))=\operatorname{area}(\operatorname{vol}(a, b, c), d)+\operatorname{area}(\operatorname{vol}(a, d, c), b), \\
& 2 \cdot \operatorname{area}(\operatorname{area}(a, b), \operatorname{area}(c, d)) \\
& =\operatorname{area}(\operatorname{vol}(a, b, c), d)+\operatorname{area}(\operatorname{vol}(a, d, c), b)+\operatorname{area}(\operatorname{vol}(b, a, d), c)+\operatorname{area}(\operatorname{vol}(b, c, d), a)
\end{aligned}
$$

where $\operatorname{vol}(x, y, z):=\operatorname{area}(\operatorname{area}(x, y), z)+\operatorname{area}(\operatorname{area}(y, z), x)+\operatorname{area}(\operatorname{area}(z, x), y)$.
We chose the notation vol because $\left\langle\operatorname{vol}(u, v, w), S(X)_{0, T}\right\rangle$ is six times the signed volume ([DR2018]) of the curve $(U, V, W)$, where

$$
U_{t}=\left\langle u, S(X)_{0, t}\right\rangle, \quad V_{t}=\left\langle v, S(X)_{0, t}\right\rangle, \quad W_{t}=\left\langle w, S(X)_{0, t}\right\rangle .
$$

The Tortkara identity is readily verified on all of $T^{\geq 1}\left(\mathbb{R}^{d}\right)$ by computing

$$
\begin{aligned}
\operatorname{area}(\operatorname{area}(1,2), \operatorname{area}(3,2)) & =-21223+21232+22213-22231-23212+23221 \\
& =\operatorname{area}(\operatorname{vol}(1,2,3), 2)
\end{aligned}
$$

where

$$
\operatorname{vol}(1,2,3)=123-132-213+231+312-321
$$

Indeed, this computation suffices to show the Tortkara identity on $T^{\geq 1}\left(\mathbb{R}^{d}\right)$ due to the universal property of the free Zinbiel algebra $\left(T^{\geq 1}\left(\mathbb{R}^{3}\right), \succ\right)$, i.e. for any $a, b, c \in T^{\geq 1}\left(\mathbb{R}^{d}\right)$, there is a unique Zinbiel homomorphism $\left(T^{\geq 1}\left(\mathbb{R}^{3}\right), \succ\right) \rightarrow\left(T^{\geq 1}\left(\mathbb{R}^{d}\right), \succ\right)$ with $1 \mapsto a, 2 \mapsto b, 3 \mapsto c$, and then the Tortkara identity follows for $a, b, c$ from the above computation by the homomorphism property and the fact that area is nothing but the antisymmetrization of $\succ$.

Proof of equivalence of the tortkara identities. Let $\mathscr{T}$ be a vector space over an arbitrary field with a bilinear antisymmetric operation $t$ and

$$
\mathrm{J}(x, y, z):=\mathrm{t}(\mathrm{t}(x, y), z)+\mathrm{t}(\mathrm{t}(y, z), x)+\mathrm{t}(\mathrm{t}(z, x), y) .
$$

1. Assume first that for all $x, y, z \in \mathscr{T}$ we have

$$
\mathrm{t}(\mathrm{t}(x, y), \mathrm{t}(z, y))=\mathrm{t}(\mathrm{~J}(x, y, z), y) .
$$

Then, for all $a, b, c, d \in \mathscr{T}$, due to bilinearity, we have

$$
\begin{aligned}
& \mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, b))+\mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d))+\mathrm{t}(\mathrm{t}(a, d), \mathrm{t}(c, b))+\mathrm{t}(\mathrm{t}(a, d), \mathrm{t}(c, d)) \\
& =\mathrm{t}(\mathrm{t}(a, b+d), \mathrm{t}(c, b+d))=\mathrm{t}(\mathrm{~J}(a, b+d, c), b+d) \\
& =\mathrm{t}(\mathrm{~J}(a, b, c), b)+\mathrm{t}(\mathrm{~J}(a, b, c), d)+\mathrm{t}(\mathrm{~J}(a, d, c), b)+\mathrm{t}(\mathrm{~J}(a, d, c), d) .
\end{aligned}
$$

Since $\mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, b))=\mathrm{t}(\mathrm{J}(a, b, c), b)$ and $\mathrm{t}(\mathrm{t}(a, d), \mathrm{t}(c, d))=\mathrm{t}(\mathrm{J}(a, d, c), d)$, we obtain the identity

$$
\mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d))+\mathrm{t}(\mathrm{t}(a, d), \mathrm{t}(c, b))=\mathrm{t}(\mathrm{~J}(a, b, c), d)+\mathrm{t}(\mathrm{~J}(a, d, c), b)
$$

for all $a, b, c, d \in \mathscr{T}$. Using antisymmetry, we furthermore get

$$
\begin{aligned}
& \mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d))+\mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d)) \\
& =\mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d))+\mathrm{t}(\mathrm{t}(a, d), \mathrm{t}(c, b))+\mathrm{t}(\mathrm{t}(b, a), \mathrm{t}(d, c))+\mathrm{t}(\mathrm{t}(b, c), \mathrm{t}(d, a)) \\
& =\mathrm{t}(\mathrm{~J}(a, b, c), d)+\mathrm{t}(\mathrm{~J}(a, d, c), b)+\mathrm{t}(\mathrm{~J}(b, a, d), c)+\mathrm{t}(\mathrm{~J}(b, c, d), a)
\end{aligned}
$$

If the field is of characteristic different from two, this reads as

$$
2 \mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d))=\mathrm{t}(\mathrm{~J}(a, b, c), d)+\mathrm{t}(\mathrm{~J}(a, d, c), b)+\mathrm{t}(\mathrm{~J}(b, a, d), c)+\mathrm{t}(\mathrm{~J}(b, c, d), a)
$$

for all $a, b, c, d \in \mathscr{T}$.
2. Assume now that for all $a, b, c, d \in \mathscr{T}$ we have

$$
\mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d))+\mathrm{t}(\mathrm{t}(a, d), \mathrm{t}(c, b))=\mathrm{t}(\mathrm{~J}(a, b, c), d)+\mathrm{t}(\mathrm{~J}(a, d, c), b) .
$$

This immediately implies

$$
\mathrm{t}(\mathrm{t}(x, y), \mathrm{t}(z, y))+\mathrm{t}(\mathrm{t}(x, y), \mathrm{t}(z, y))=\mathrm{t}(\mathrm{~J}(x, y, z), y)+\mathrm{t}(\mathrm{~J}(x, y, z), y)
$$

for all $x, y, z \in \mathscr{T}$, which is an empty statement in characteristic two, but in characteristic different from two reduces to

$$
\begin{equation*}
\mathrm{t}(\mathrm{t}(x, y), \mathrm{t}(z, y))=\mathrm{t}(\mathrm{~J}(x, y, z), y) \tag{6.1}
\end{equation*}
$$

3. For the last implication we want to show, assume that the characteristic of the underlying field is different from two and for all $a, b, c, d \in \mathscr{T}$ we have

$$
2 \mathrm{t}(\mathrm{t}(a, b), \mathrm{t}(c, d))=\mathrm{t}(\mathrm{~J}(a, b, c), d)+\mathrm{t}(\mathrm{~J}(a, d, c), b)+\mathrm{t}(\mathrm{~J}(b, a, d), c)+\mathrm{t}(\mathrm{~J}(b, c, d), a) .
$$

This implies

$$
2 \mathrm{t}(\mathrm{t}(x, y), \mathrm{t}(y, z))=2 \mathrm{t}(\mathrm{~J}(x, y, z), y)+\mathrm{t}(\mathrm{~J}(y, x, y), z)+\mathrm{t}(\mathrm{~J}(y, z, y), x)=2 \mathrm{t}(\mathrm{~J}(x, y, z), y),
$$

since due to antisymmetry

$$
\mathrm{J}(y, x, y)=\mathrm{J}(y, z, y)=0
$$

Since the characteristic is different from two, we can divide by two, and thus again arrive at (6.1).

In [DIM2018, Section 6] it is shown that in $d=2,\left(\mathscr{A}\right.$, area) is the free tortkara algebra. ${ }^{7}$
This linear space has a surprisingly simple description. The following is [DIM2018, Theorem 2.1] (see also [Rei2018, Section 3.2, Theorem 31] for another proof in $d=2$ ).

## Lemma 6.1.

$$
\begin{equation*}
\mathscr{A}=\operatorname{span}_{\mathbb{R}}\{\mathrm{i}: \mathrm{i} \text { a letter }\} \oplus \operatorname{span}_{\mathbb{R}}\{w(\mathrm{ij}-\mathrm{ji}): w \text { a word, } \mathrm{i}, \mathrm{j} \text { letters }\} . \tag{6.2}
\end{equation*}
$$

[^6]Example 6.2. We have that [DR2018, Equation (4)]

$$
\operatorname{Inv}_{n}:=\sum_{\sigma \in S_{n}} \operatorname{sign}(\sigma) \sigma(1) \cdots \sigma(\mathrm{i})
$$

where we interpret $\sigma$ as a permutation of the letters, is in $\mathscr{A}$ for $d \geq n \geq 2$ by Lemma 6.1, an element which plays an important role as the lowest order SL invariant component of the signature in dimension $d=n$, see [DR2018, Section 3.3], and can be interpreted as the $d=n$ dimensional signed volume of the path underlying the signature. In particular, we recover

$$
\begin{aligned}
& \operatorname{Inv}_{2}=\operatorname{area}(1,2), \\
& \operatorname{Inv}_{3}=\operatorname{vol}(1,2,3),
\end{aligned}
$$

and in fact this can be generalized by defining the multilinear map

$$
\operatorname{vol}^{n}: T^{\geq 1}\left(\mathbb{R}^{d}\right)^{n} \rightarrow T^{\geq 1}\left(\mathbb{R}^{d}\right)
$$

such that $\operatorname{vol}^{n}\left(a_{1}, \ldots, a_{n}\right)$ is the image of $\operatorname{Inv}_{n}$ under the unique Zinbiel homomorphism (unique due to freeness of the halfshuffle algebra as a Zinbiel algebra) that maps $\mathrm{i} \mapsto a_{i}$ for $\mathrm{i}=1, \ldots, \mathrm{n}$. Written out, this means

$$
\operatorname{vol}^{n}\left(a_{1}, \ldots, a_{n}\right)=\sum_{\sigma \in S_{n}} \operatorname{sign}(\sigma)\left(\left(\left(a_{\sigma(1)} \succ a_{\sigma(2)}\right) \succ a_{\sigma(3)}\right) \succ \cdots\right) \succ a_{\sigma(n)} .
$$

Through the fact that $\operatorname{Inv}_{n} \in \mathscr{A}$ for $d=n$, it is immediate that we obtain the restriction

$$
\mathrm{vol}^{n}: \mathscr{A}^{n} \rightarrow \mathscr{A}
$$

for any $d \geq 2$.
We note the following conjecture, which was shown to hold true in the case $d=2$ in [DIM2018, Section 6] as well as in [Rei2018, Section 3.2, Theorem 31]. The case $d \geq 3$ is still open.

Conjecture 6.3. $\mathscr{A}$ is linearly generated by strict left-bracketings of the area operation. In particular, a linear basis for $\mathscr{A}$ (without the single letters) is given by

$$
\operatorname{area}\left(\operatorname{area}\left(\operatorname{area}\left(\mathbf{i}_{1}, \dot{i}_{2}\right), \dot{i}_{3}\right), . ., \dot{i}_{n}\right), \quad n \geq 2, \dot{i}_{1}, \ldots, \dot{i}_{n} \in\{1, \ldots, \mathrm{~d}\}, \dot{i}_{1}<\dot{i}_{2} .
$$

Example 6.4. For example, with $d=2$, the tensor $12(12-21)$, which is in $\mathscr{A}$, can be written as

$$
12(12-21)=\frac{1}{6}[2 \operatorname{area}(\operatorname{area}(\operatorname{area}(1,2), 1), 2)-\operatorname{area}(\operatorname{area}(\operatorname{area}(1,2), 2), 1)] .
$$

It turns out that for a bilinear, antisymmetric operation, showing that all bracketings can be rewritten as linear combination of left-bracketings reduces to showing that this is possible for a small subset of bracketings. We have not been able to show that this subset of bracketings can be rewritten, but want to record this general fact nonetheless. We formulate the statement imprecisely here, and leave the exact statement Proposition A. 2 and its proof to the appendix.

Proposition 6.5. To be able to rewrite any bracketing to a linear combination of left-brackets, it is enough to verify this for bracketings of the form

$$
\operatorname{area}\left(\operatorname{area}\left(\ldots, \operatorname{area}\left(\operatorname{area}\left(a_{1}, a_{2}\right), a_{3}\right), \ldots, a_{n-2}\right), \operatorname{area}\left(a_{n-1}, a_{n}\right)\right), \quad a_{i} \in \mathscr{A} .
$$

While trying to find a proof for Conjecture 6.3 for $d \geq 3$, we investigated in detail the operator area given by the following definition.

Definition 6.6. If $w=l_{1} \ldots l_{n}$ is a word, we define $\overleftarrow{\text { area }}(w)$ to be the left-bracketing expression

$$
\operatorname{area}\left(\ldots \operatorname{area}\left(\operatorname{area}\left(\operatorname{area}\left(l_{1}, 1_{2}\right), 1_{3}\right), 1_{4}\right), \ldots, 1_{n}\right) .
$$

This is expanded linearly to an operation on the tensor algebra with ârea $(e)=0$ and ârea $(1)=1$ for any letter 1 .

We came across some interesting properties.
First, we show that there is an expansion formula for $\overleftarrow{\text { area }(w) \text { in terms of permutations of the }}$ letters in the word $w$. To this end, define the right action of a permutation $\sigma \in S_{n}$ on words of length $n$ as

$$
\sigma:=l_{\sigma(1)} . .1_{\sigma(n)},
$$

where $w=l_{1} \cdots l_{n}$.
Proposition 6.7. We have

$$
\overleftarrow{\text { area }}\left(l_{1} \cdots l_{n}\right)=l_{1} \cdots l_{n} \theta_{n},
$$

where

$$
\theta_{n}:=\sum_{\sigma \in S_{n}} f_{n}(\sigma) \sigma
$$

and $f_{n}: S_{n} \rightarrow\{-1,1\}$ is given as

$$
f_{n}(\sigma)=\prod_{i=1}^{n} g_{i}(\sigma)
$$

with

$$
g_{i}(\sigma)= \begin{cases}+1, & \text { if } \sigma^{-1}(j)<\sigma^{-1}(i) \text { for all } j \in \mathbb{N} \text { with } j<i, \\ -1, & \text { else } .\end{cases}
$$

Proof. For $n=1$, there is only the identity permutation and $f_{1}(\mathrm{id})=g_{1}(\mathrm{id})=1$, thus the statement is obviously true. For $n=2$, we have $S_{2}=\{\mathrm{id},(12)\}, f_{2}(\mathrm{id})=-f_{2}((12))=1$ and

$$
\operatorname{area}\left(l_{1} l_{2}\right)=l_{1} l_{2}-l_{2} l_{1}=f_{2}(\mathrm{id}) 1_{1} l_{2}+f_{2}((12)) 1_{2} l_{1} .
$$

Assume the statement holds for some $n \in \mathbb{N} \backslash\{1\}$. Then,

$$
\begin{aligned}
& \overleftarrow{\operatorname{area}}\left(\mathrm{l}_{1} \cdots 1_{n+1}\right)=\operatorname{area}\left(\overleftarrow{\operatorname{area}}\left(\mathrm{l}_{1} \cdots 1_{n}\right), 1_{n+1}\right) \\
& =\sum_{\sigma \in S_{n}} f_{n}(\sigma) 1_{\sigma(1)} \cdots 1_{\sigma(n)} 1_{n+1}-\sum_{\sigma \in S_{n}} f_{n}(\sigma) 1_{n+1} \succ\left(1_{\sigma(1)} \cdots 1_{\sigma(n)}\right) \\
& =\sum_{\substack{\tilde{\sigma} \in S_{n+1}: \\
g_{n+1}(\tilde{\sigma})=1}} f_{n+1}(\tilde{\sigma}) l_{\tilde{\sigma}(1)} \cdots l_{\tilde{\sigma}(n+1)}-\sum_{\sigma \in S_{n}} f_{n}(\sigma)\left(1_{n+1} Ш\left(1_{\sigma(1)} \cdots 1_{\sigma(n-1)}\right)\right) 1_{\sigma(n)} \\
& =\sum_{\substack{\tilde{\sigma} \in S_{n+1}: \\
g_{n+1}(\tilde{\sigma})=1}} f_{n+1}(\tilde{\sigma}) l_{\tilde{\sigma}(1)} \cdots l_{\tilde{\sigma}(n+1)}+\sum_{\substack{\tilde{\sigma} \in S_{n+1}: \\
g_{n+1}(\tilde{\sigma})=-1}} f_{n+1}(\tilde{\sigma}) l_{\tilde{\sigma}(1)} \cdots l_{\tilde{\sigma}(n+1)} \\
& =\sum_{\tilde{\sigma} \in S_{n+1}} f_{n+1}(\tilde{\sigma}) l_{\tilde{\sigma}(1)} \cdots l_{\tilde{\sigma}(n)} .
\end{aligned}
$$

We furthermore have the following surprising identity.
Proposition 6.8. For all integers $n>2$, we have

$$
\begin{equation*}
\overleftarrow{\text { area }}(1 l(2 \cdots \mathrm{n}))=1 \overleftarrow{\text { area }}(l(2 \cdots \mathrm{n})), \tag{6.3}
\end{equation*}
$$

where $l$ is the left Lie bracketing. This implies (by freeness of the half-shuffle algebra)

$$
\overleftarrow{\operatorname{area}}(v l(w))=\overleftarrow{\operatorname{area}}(v) \overleftarrow{\operatorname{area}}(l(w))
$$

for words $v, w$ such that $|w| \geq 2$.
Remark 6.9. Note that due to the well known fact that strict left Lie bracketings linearly generate the free Lie algebra, more generally formulated, it holds that

$$
\overleftarrow{\text { area }}(v x)=\overleftarrow{\text { area }}(v) \overleftarrow{\text { area }}(x)
$$

for any $v \in T\left(\mathbb{R}^{d}\right)$ and any Lie polynomial $x$ with $\langle x, \mathrm{i}\rangle=0$ for all letters i .
Remark 6.10. In particular, we have

$$
\overleftarrow{\text { area }}\left(l_{1} \cdots l_{n} l_{n+1}\right)-\overleftarrow{\text { area }}\left(l_{1} \cdots l_{n+1} 1_{n}\right)=2 \overleftarrow{\text { area }}\left(l_{1} \cdots l_{n-1}\right)\left(l_{n} 1_{n+1}-l_{n+1} l_{n}\right)
$$

for any letters $1_{1}, \ldots, l_{n+1}$.
Proof. For the base case, we compute

$$
\overleftarrow{\text { area }}(123-132)=2(123-132)=1 \overleftarrow{\text { area }}(23-32) .
$$

Assume (6.3) holds for some integer $n>2$ and let $w$ be a word of length $n-1$. Then, for any
letter i,

$$
\begin{aligned}
\overleftarrow{\operatorname{area}}(1 l(w \mathrm{i})) & =\overleftarrow{\mathrm{area}}(1 l(w) \mathrm{i}-1 \mathrm{i} l(w))=\overleftarrow{\operatorname{area}}(1 l(w) \mathrm{i})-\overleftarrow{\operatorname{area}}(1 \mathrm{i}) \overleftarrow{\text { area }}(l(w)) \\
& =\overleftarrow{\operatorname{area}}(1 l(w)) \mathrm{i}-\mathrm{i} \succ \overleftarrow{\operatorname{area}}(1 l(w))-\overleftarrow{\text { area }}(1 \mathrm{i}) \overleftarrow{\text { area }}(l(w)) \\
& =1 \overleftarrow{\text { area }}(l(w)) \mathrm{i}-\mathrm{i} \succ(1 \overleftarrow{\text { area }}(l(w)))-1 \mathrm{i} \overleftarrow{\text { area }}(l(w))+\mathrm{i} 1 \overleftarrow{\text { area }}(l(w)) \\
& =1 \overleftarrow{\text { area }}(l(w)) \mathrm{i}-1(\mathrm{i} \succ \overleftarrow{\mathrm{area}}(l(w)))-1 \mathrm{i} \overleftarrow{\text { area }(l(w))} \\
& =1 \overleftarrow{\text { area }}(l(w) \mathrm{i})-1 \mathrm{i} \overleftarrow{\text { area }}(l(w)) \\
& =1 \overleftarrow{\text { area }}(l(w) \mathrm{i})-1 \overleftarrow{\text { area }}(\mathrm{i} l(w)) \\
& =1 \overleftarrow{\text { area }}(l(w \mathrm{i}))
\end{aligned}
$$

where we used that $i \succ(1 \overleftarrow{\operatorname{arrea}}(l(w)))=$ i $1 \overleftarrow{\mathrm{area}}(l(w))+1(i \succ \overleftarrow{\mathrm{area}}(l(w)))$ due to the combinatorial expansion formula for the half-shuffle.

Interestingly, $\rho$ admits a permutation expansion which is quite similar to that of area (), in fact again via $f_{n}$, just that now only a subset $T_{n}$ of all permutations $S_{n}$ is involved.

Proposition 6.11. We have

$$
\rho\left(l_{1} \cdots 1_{n}\right)=1_{1} \cdots l_{n} \vartheta_{n}
$$

where

$$
\vartheta_{n}:=\sum_{\sigma \in T_{n}} f_{n}(\sigma) \sigma
$$

$f_{n}: S_{n} \rightarrow\{-1,1\}$ is as in Lemma 6.7 and $T_{n}$ is the set of all $\sigma \in S_{n}$ such that $\{\sigma(i), \ldots, \sigma(n)\}$ is an interval of integers ( $a$ set of the form $[a, b] \cap \mathbb{N}$ ) for all $i \in\{1, \ldots, n-1\}$.

Proof. For $n=1$, there is only the identity permutation and $f_{1}(\mathrm{id})=g_{1}(\mathrm{id})=1$, thus the statement is obviously true. For $n=2$, we have $T_{2}=\{\mathrm{id},(12)\}, f_{2}(\mathrm{id})=-f_{2}((12))=1$ and

$$
\rho\left(l_{1} l_{2}\right)=l_{1} l_{2}-l_{2} l_{1}=f_{2}(\mathrm{id}) 1_{1} 1_{2}+f_{2}((12)) 1_{2} l_{1}
$$

Assume the statement holds for some $n \in \mathbb{N} \backslash\{1\}$. Then, using the recursive definition of $\rho$ from Equation (2.3),

$$
\begin{aligned}
& \rho\left(l_{1} \cdots l_{n+1}\right)=l_{1} \rho\left(1_{2} \cdots l_{n+1}\right)-l_{n+1} \rho\left(l_{1} \cdots l_{n}\right) \\
& =\sum_{\sigma \in T_{n}} f_{n}(\sigma) l_{1} l_{\sigma(1)}^{\prime} \cdots l_{\sigma(n)}^{\prime}-\sum_{\sigma \in T_{n}} f_{n}(\sigma) l_{n+1} l_{\sigma(1)} \cdots l_{\sigma(n)} \\
& =\sum_{\substack{\tilde{\sigma} \in T_{n+1}: \\
\tilde{\sigma}(1)=1}} f_{n+1}(\tilde{\sigma}) l_{\tilde{\sigma}(1)} \cdots l_{\tilde{\sigma}(n+1)}+\sum_{\substack{\tilde{\sigma} \in T_{n}: \\
\tilde{\sigma}(1)=n+1}} f_{n+1}(\tilde{\sigma}) l_{\tilde{\sigma}(1)} \cdots l_{\tilde{\sigma}(n+1)} \\
& =\sum_{\tilde{\sigma} \in T_{n+1}} f_{n+1}(\tilde{\sigma}) l_{\tilde{\sigma}(1)} \cdots l_{\tilde{\sigma}(n)},
\end{aligned}
$$

where $l_{i}^{\prime}=l_{i+1}$.
Via the recursive formula for $\rho$, we also get an alternative proof of the following.

Corollary 6.12. $\operatorname{Im} \rho \subset \mathscr{A}$
Proof. It suffices to show $\rho(w) \in \mathscr{A}$ for any word $w$. We have $\rho(e)=0, \rho(\mathrm{i})=\mathrm{i} \in \mathscr{A}$ and $\rho(\mathrm{ij})=\mathrm{ij}-\mathrm{ji} \in \mathscr{A}$ for any letters $\mathrm{i}, \mathrm{j}$. Let $n \geq 2$ and assume that $\rho(w) \in \mathscr{A}$ for any word $w$ with $|w|=n$. Then, for any word $v$ with $|v|=n-1$ and any letters $\mathrm{i}, \mathrm{j}$, we have

$$
\rho(\mathrm{i} v \mathrm{j})=\mathrm{i} \rho(v \mathrm{j})-\mathrm{j} \rho(\mathrm{i} v) \in \mathscr{A}
$$

since $\rho(v \mathrm{j}), \rho(\mathrm{i} v) \in \mathscr{A}$ with $|\rho(v \mathrm{j})|=|\rho(\mathrm{i} v)|=n \geq 2$ due to the induction hypothesis, and by Lemma 6.1, the non-letter part of $\mathscr{A}$ is stable under concatenation of any element of the tensor algebra from the left. Thus, the induction hypothesis also holds for all words of length $n+1$.

## 7. Conclusion

We have linked the area operation in the tensor algebra to work in control theory and more abstract work on Tortkara algebras. We have shown that starting from letters and applying the area operation, one obtains enough elements to shuffle-generate the tensor algebra.
There are many open directions for research. We have not identified a minimal set of areas-of-areas which is just enough to shuffle-generate the tensor algebra - i.e. to shuffle-generate it exactly. The linear span of the areas-of-areas has been identified, but a basis for it in terms of areas-of-areas has not.

### 7.1. Open combinatorial problems

1. What is span $\left\{\operatorname{area}\left(\mathrm{i}_{1} \amalg \cdots \mathrm{i}_{n}, \mathrm{j}_{1} \amalg \cdots \amalg \mathrm{j}_{m}\right), n, m \in \mathbb{N}, \mathrm{i}_{1}, \ldots \mathrm{i}_{n}, \mathrm{j}_{1}, \ldots \mathrm{j}_{m}\right.$ letters $\}$ ? Does it shuffle generate together with the letters, and if not what is the smallest subalgebra of the associative shuffle algebra containing it and the letters? This is the algebraic formulation of the question "what do we know about a path if we are only allowed to collect its increment and the values of the first area of any two dimensional polynomial image of the path"?
2. Give linear bases for $\mathscr{A} \amalg \mathscr{A}, \mathscr{A} \amalg \mathscr{A} ш \mathscr{A}, \ldots$ Does $\sum_{m=1}^{n} \mathscr{A}^{Ш m}$ already arrive at $T\left(\mathbb{R}^{d}\right)$ for a finite $n$ ?
3. Give a minimal generating set for $T\left(\mathbb{R}^{d}\right)$ as a Tortkara algebra. Is it free?
4. In light of Proposition 6.8, look at $\langle x \in \mathfrak{g}:\langle x, \mathrm{i}\rangle=0 \forall \mathrm{i} ; \cdot\rangle$ and its image under ărea ()
5. What are the eigenspaces of $\overleftarrow{\text { area }()}$ ?

## A. Appendix

Lemma A.1. Let $V=\bigoplus_{n \geq 1} V_{n}$ be a graded vector space, each $V_{n}$ finite dimensional, and denote the grading $|\cdot| V$.
Consider $\mathbb{R}[V]$, the symmetric algebra over $V$ (see Section 4), with two different gradings, defined on monomials as follows

- $\left|x^{m}\right|_{\mathrm{deg}}:=m$ (denote the corresponding projection onto degree 1 by $\mathrm{proj}_{1}^{\mathrm{deg}}$ ).
- $\left|x^{m}\right|_{\text {weight }}:=m \cdot|x|_{V}$.

Let $Y \subset \mathbb{R}[V]$, countable, be such that every $y \in Y$ is homogeneous with respect to $|\cdot|_{\text {weight }}$. Then:

$$
Y \text { generates } \mathbb{R}[V] \text { (as commutative algebra) }
$$

if and only if
$\operatorname{span}_{\mathbb{R}} \operatorname{proj}_{1}{ }^{\operatorname{deg}} Y=V$
If moreover proj $_{1}^{\text {deg }} y, y \in Y$, are linearly independent, then $Y$ freely generates $\mathbb{R}[V]$ (as commutative algebra).

Proof. We show the first statement.
$\Rightarrow:$ Assume $v \in V \subset \mathbb{R}[V]$ is not in the span of $\operatorname{proj}_{1}^{\mathrm{deg}} Y$. Then it is clearly not in the algebra generated by $Y$. Hence $Y$ does not generate $\mathbb{R}[V]$. This proves the contrapositive.
$\Leftarrow$ : Denote, local to this proof, by $\langle M\rangle$ the subalgebra generated by $M \subset \mathbb{R}[V]$.
Claim: $\left\langle V_{1}\right\rangle \subset\langle Y\rangle$. Indeed, $v \in V_{1}$ can, by assumption be written as linear combination of some

$$
\operatorname{proj}_{1}^{\mathrm{deg}} y_{i},
$$

where $y_{i} \in Y$. Since the $y_{i}$ are homogeneous they must be of weight 1 . Hence $\operatorname{proj}_{1}^{\operatorname{deg}} y_{i}=y_{i}$, hence $V_{1} \subset\langle Y\rangle$, hence $\left\langle V_{1}\right\rangle \subset\langle Y\rangle$, which proves the claim.
Now let $\left\langle V_{1} \oplus \cdots \oplus V_{n}\right\rangle \subset\langle Y\rangle$. Claim: $V_{n+1} \subset\langle Y\rangle$. Indeed, $v \in V_{n+1}$ can be written as linear combination of some

$$
\operatorname{proj}_{1}^{\mathrm{deg}} y_{i}
$$

where $y_{i} \in Y$, of weight $n+1$. Then

$$
y_{i}=\operatorname{proj}_{1}^{\operatorname{deg}} y_{i}+r_{i},
$$

with $r_{i}$ monomials (of order 2 an higher) in terms from $V_{1} \oplus \cdots \oplus V_{n}$, i.e. $r_{i} \in\left\langle V_{1} \oplus \cdots \oplus V_{n}\right\rangle \subset\langle Y\rangle$. Hence $v \in\langle Y\rangle$.
Hence $\left\langle V_{1} \oplus \cdots \oplus V_{n} \oplus V_{n+1}\right\rangle \subset\langle Y\rangle$. Iterating, we see that $\mathbb{R}[V]=\langle V\rangle \subset\langle Y\rangle$, which proves the first claim.

We finally give a precise statement and proof of Proposition 6.5
Let $V$ be an $\mathbb{R}$-vector space and let

$$
\mathfrak{B}: V \times V \rightarrow V,
$$

be a bilinear map. ${ }^{8}$ We encode bracketings as planar trees. Define the complete left-bracketed tree with $n$ leaves as

```
LeftBracketTree \(_{1}:=\) •
LeftBracketTree \({ }_{n}:=\) LeftBracketTree \(_{n-1} \rightarrow \bullet \bullet, \quad n \geq 2\),
```

[^7]her $\rightarrow_{\text {• }}$ denotes grafting to a new root.
Define
$$
\text { SpecialTree }_{n}:=\text { LeftBracketTree }_{n-2} \rightarrow_{\bullet} \text { LeftBracketTree }{ }_{2} .
$$

For example


For any tree $\tau$ with $n$ leaves, and $a_{1}, . ., a_{n} \in V$ write

$$
\tau\left(a_{1}, . ., a_{n}\right)
$$

as the corresponding bracketing. We extend this definition to the case where (some of) the $a_{i}$ are planar trees (with labeled leaves) themselves, by just replacing the respective leaf of $\tau$ with $a_{i}$. (This is consistent, when considering $a \in V$ as the tree with exactly on vertex, labeled $a$.)
On every new level $n+1$, it is enough to check that Special $_{\text {Tree }}^{n+1}$ can be expressed in terms of left brackets:

Proposition A.2. Assume that $\mathfrak{B}$ is symmetric or anti-symmetric.
Assume, for some $n$, that all trees $\tau$ with $|\tau|_{\text {leaves }} \leq n$ can be expressed in terms of left brackets, i.e. for some $c(\tau, \sigma) \in \mathbb{R}$,

$$
\tau\left(a_{1}, . ., a_{n}\right)=\sum_{\sigma \in S_{n}} c(\tau, \sigma) \text { LeftBracketTree }_{n}\left(a_{\sigma(1)}, \ldots, a_{\sigma(n)}\right) . \quad \forall a_{1}, . ., a_{n} \in V
$$

Assume that (every labeling of) SpecialTree ${ }_{n+1}$ can be expressed in terms of left brackets. Then:
(every labeling of) every tree $\sigma$ with $|\sigma|_{\text {leves }}=n+1$ can be expressed in terms of left brackets.

Proof. Consider

$$
\tau=\underbrace{}_{T_{1} \quad T_{2}},
$$

with

$$
\begin{aligned}
& T_{1}=\tau_{1}\left(a_{1}, . ., a_{m}\right) \\
& T_{2}=\tau_{2}\left(a_{m+1}, . ., a_{m+\ell}\right)
\end{aligned}
$$

with $|\tau|_{\text {leaves }}=n+1=m+\ell$. By using symmetry/antisymmetry, we can assume $\left|\tau_{1}\right|_{\text {leaves }}=m \geq$ $\left|\tau_{2}\right|_{\text {eaves }}=\ell$.

By assusmption, we can write both $\tau_{1}$ and $\tau_{2}$ in terms of left-bracketings. It is hence enough to consider

$$
\begin{aligned}
& \tau_{1}=\text { LeftBracketTree }_{m} \\
& \tau_{2}=\text { LeftBracketTree }_{\ell},
\end{aligned}
$$

with $m+\ell=n+1$ and $m \geq \ell$.
Claim: we can reduce to $\ell=1$ and $\ell=2$.
Indeed, write

$$
\begin{aligned}
\tau_{1} & =\left(\left(t_{1}, t_{2}\right), . ., t_{m-1}\right) \\
\tau_{2} & =\left(t_{m}, t_{m+1}\right)
\end{aligned}
$$

with

$$
\begin{aligned}
t_{1} & =\left(a_{1}, a_{2}\right) \\
t_{2} & =a_{3} \\
\quad . & \\
t_{m-1} & =a_{m} \\
t_{m} & =\left(\left(a_{m+1}, a_{m+2}\right), . ., a_{m+\ell-1}\right) \\
t_{m+1} & =a_{m+\ell}
\end{aligned}
$$

If $\ell \geq 2$ we have $m+1 \leq n$. Hence by assumption

$$
\tau=\sum \text { left bracketings }\left(t_{1}, . ., t_{m+1}\right)
$$

Now, consider the rightmost spot in each leftbracketing.

- If it is taken by a letter: $\rightsquigarrow \ell=1$.
- If it is taken by $t_{1}: \rightsquigarrow \ell=2$.
- If it is taken by $t_{m}:\left|t_{m}\right|_{\text {eaves }}=\left|\tau_{2}\right|_{\text {eaves }}-1=\ell-1$. So we go from $\ell$ to $\ell-1$.

We can finish by induction.

## Symbol index

■, 6
$\langle\cdot\rangle,$,
$\triangleright_{\text {Sym }}, 16$
$\succ, 5$
ャ, 13
$[.,$.$] , 27$
$\mathscr{A}, 4$
$\operatorname{ad}_{v}$ and $\mathrm{Ad}_{v}, 14$
$\operatorname{ad}_{\mathbf{■} ; x}, 27$
Area, 3
area, 2
àrea(), 46
area., 17
area., 30
$\mathrm{BPT}_{n}, 17$
${\widetilde{\mathrm{BPT}_{n}}}_{n}, 29$
lie., 17
$c, 17$
D, 10
D, 11
$D^{-1}, 10$
$\underline{D}^{-1}, 11$
$e, 5$
$G, 5$
$\mathfrak{g}, 5$
$I_{w}, 14$
$P_{h}, 22$
$\pi_{1}$ and $\pi_{1}^{\top}, 24$
$\operatorname{proj}_{n}$ and $\operatorname{proj}_{\geq n}, 5$
$R, 11$
$r, 10$
$\rho, 11$
$\underline{r}, 11$
$R_{n}, 16$
S, 11
$S_{h}, 24$
ш, 5
shuff, 34
$T_{n}\left(\left(\mathbb{R}^{d}\right)\right)$ and $T_{\geq n}\left(\left(\mathbb{R}^{d}\right)\right), 5$
$T\left(\left(\mathbb{R}^{d}\right)\right), 5$
$T\left(\mathbb{R}^{d}\right), 5$
$\mathcal{W}, 5$
$\zeta_{h}, 22$

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[^1]:    ${ }^{1}$ One physical device that has historically been used to measure area is a planimeter [FS2007]. In general, this is related to nonholonomic control, see for example [BD2015].

[^2]:    ${ }^{2}$ From now on, if we sum over a variable with no given index set, we sum over all words in the alphabet $1, .$. , d, including the empty word $e$.
    ${ }^{3}$ If $|w|=0$ then both sides are equal to zero.

[^3]:    ${ }^{4}$ This is true level by level, i.e. projectively. See for example [DR2018, Lemma 8].

[^4]:    ${ }^{5}$ [Reu1993, Section 6.2] uses the notation $\pi_{1}^{*}$.

[^5]:    ${ }^{6}$ Identifying $\mathfrak{g}$ as a subset of $T\left(\left(\mathbb{R}^{d}\right)\right)$.

[^6]:    ${ }^{7}$ Recall the definition of $\mathscr{A}$ from the introduction: the smallest linear space containing the letters $1, .$. d and being closed under the area operation.

[^7]:    ${ }^{8}$ This section would be most comfortably be formulated in the language of operads. But this would require more mathematical setup, which we want to avoid.

