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From CFTs to theories with Bondi-Metzner-Sachs symmetries: Complexity and out-of-time-ordered correlators

Aritra Banerjee,[1,*] Arpan Bhattacharyya,[2,†] Priya Drashni,[2,‡] and Srinidhi Pawar[2,§]

[1]*Okinawa Institute of Science and Technology, 1919-1 Tancha, Onna-son, Okinawa 904-0495, Japan*
[2]*Indian Institute of Technology, Gandhinagar, Gujarat-382355, India*

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We probe the contraction from $2d$ relativistic CFTs to theories with Bondi-Metzner-Sachs (BMS) symmetries, or equivalently conformal Carroll symmetries, using diagnostics of quantum chaos. Starting from an ultrarelativistic limit on a relativistic scalar field theory and following through at the quantum level using an oscillator representation of states, one can show the CFT_2 vacuum evolves smoothly into a BMS_3 vacuum in the form of a squeezed state. Computing circuit complexity of this transmutation using the covariance matrix approach shows clear divergences when the BMS point is hit or equivalently when the target state becomes a boundary state. We also find similar behavior of the circuit complexity calculated from methods of information geometry. Furthermore, we discuss the Hamiltonian evolution of the system and investigate out-of-time-ordered correlators and operator growth complexity, both of which turn out to scale polynomially with time at the BMS point.

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I. INTRODUCTION

The holy grail of quantum gravity is one of the most sought after treasures in modern theoretical physics, and many avenues to achieve that exist, with string theory being arguably the most successful one. A very fruitful effort towards the same has taken shape over the past two decades, famously known as the holographic duality, whose most well-known avatar is the AdS/CFT correspondence [1]. The AdS/CFT correspondence states that gravity in asymptotically anti–de Sitter (AdS) spacetime is equivalent to a quantum field theory with conformal invariance living on the boundary of the AdS space and observables calculated from either theory can be matched up using this equivalence. The advent of AdS/CFT has given rise to various multidisciplinary research fields involving many seemingly disconnected branches of physics, including string theory, black hole physics, condensed matter physics, and, more recently, quantum information theory.

However, our visible Universe seems to have nothing to do with AdS, as experimental signatures argue against a negative value of the cosmological constant. An extension of this duality to de Sitter space (dS) has not been satisfactorily formulated yet. Hence, the prospect of extending the duality to asymptotically flat spacetime seems exciting enough to pursue. We recall that a central idea in the so-called holographic dictionary is the asymptotic symmetry of the bulk $(d+1)$ dimensional gravity theory agreeing precisely with the global symmetry of the dual field theory living in one lower dimension, and both are the conformal symmetry in d dimensions. The asymptotic symmetry for four-dimensional flat Minkowski spacetime containing Einstein gravity, first studied by Bondi-van der Burg-Metzner-Sachs (BMS) [2,3], is the so-called BMS symmetry. Hence the putative dual theory living on the boundary of flat spacetime, via this "flat holography," should also be invariant under the same BMS symmetries, which is the main message of the whole program. Consequently, many investigations on BMS invariant field theories (BMSFTs) in various dimensions have appeared in the literature over the last few years [4–20].

In another development related to flat holography, known as "celestial holography," the idea of holography for asymptotically flat spacetimes has been formulated as a correspondence between gravity on $4d$ flat space and a $2d$ CFT living on the celestial sphere. This way of thinking has attracted a lot of attention in the last few years in terms of linking asymptotic symmetries and scattering amplitudes and one can have a look at the excellent reviews [21–23] for a detailed understanding of these exciting structures.

*aritra.banerjee@oist.jp
†abhattacharyya@iitgn.ac.in
‡priyad@iitgn.ac.in
§srinidhip@iitgn.ac.in

What is more intriguing, there seems to be a very recently discovered reconciliation between these two seemingly different avenues put forward in [24]. This nicely packages ideas from celestial holography in the language of correlation functions of Carrollian CFT (equivalently, BMSFTs) and links them naturally to $4d$ scattering amplitudes. For further intriguing ideas linking the two approaches, see also [25].

In lower dimensions, especially in the case of three, the BMS group has a particularly simple structure and is isomorphic to the Galilean conformal algebra in two dimensions (GCA_2) [26]. While the GCA_2 can be obtained from the nonrelativistic (NR) contraction of the two-dimensional conformal algebra, which is two copies of the Virasoro algebra, the BMS_3 algebra is the ultrarelativistic (UR) contraction of the same, and can be shown to be isomorphic to a conformal Carrollian algebra in two dimensions (CCA_2) [27–29]. All of this turns out to be a part of the general equivalence between BMS algebra in $(d+1)$ dimensions and conformal Carrollian algebra in d dimensions. Carrollian symmetries occur whenever we encounter a null surface and Riemannian structures degenerate [30–32] due to the closure of light cones. Two-dimensional field theories with these symmetries are a very active research area, and various results have been obtained [33–35]. Despite the thorough investigations into entanglement entropy of such theories [9–11,36], many other information-theoretic structures of this theory remain in the shadows. However, recently some more investigations into the quantum chaotic structure in BMSFTs have been detailed in [37]. In this work, we will be trying to shed light on some unexplored issues, especially how certain information-theoretic markers change as a physical system goes through the contraction of conformal symmetries into BMS symmetries. Our focus will be on a free scalar field theory, which has appeared in the literature in various guises, including in the study of null string theories [38–41], as a BMSFT action [42], and as deformations of $2d$ CFT actions [43,44].

In quantum information theory, quantum circuit complexity is a very useful tool to probe into the structure of an inherently quantum theory. The idea of complexity in quantum information theory is simple. Given a suitable basis, it is a quantity that determines the minimum number of operations needed to perform the desired task. Specifically for a quantum system, the notion of complexity is associated with an efficient quantum circuit that takes a reference state (usually a state that can be prepared relatively easily in the "lab") into the desired target state given a set of quantum gates. In recent times, the notion of complexity has appeared extensively in the context of holography. In the context of AdS/CFT, certain geometrical objects have been interpreted as gravity dual of the circuit complexity of the dual field theory state. These proposals go by the names of $complexity = volume$ [45] (maximal volume of codimension one bulk slice) and $complexity = action$ [46] (gravitational action defined on a certain Wheeler-De Witt patch inside the bulk spacetime). This has spurred lots of studies of circuit complexity in the context of quantum field theory [47–77].[1] However, the ramifications of such constructions are far from well explored. Several methods of quantifying complexity in a QFT exist, and they all have their own advantages, see [50] for a detailed discussion.

It has recently been proposed [51,59], that the circuit complexity can be used as a useful probe of flows between different quantum field theories (more specifically, as a probe of renormalization group flow) and quantum phase transitions. Motivated by this, in this paper, we will use circuit complexity to probe the purported "flow" from CFT to BMS invariant theories [44]. Besides exploring circuit complexity, we will also discuss the Hamiltonian dynamics of the system. Intriguing new research has unearthed that quantum chaos in quantum many-body systems plays an important role in understanding some of the important open questions, e.g., thermalization, transport in quantum many-body systems, black hole information loss etc. [85,86]. In this paper, we will also compute out-of-time-ordered correlators (OTOCs) for our system. It has been shown [87–89][2] that OTOC gives pertinent information about the Lyapunov exponent and the scrambling time.[3] We will also study the nature of operator evolution in the Heisenberg picture for such a flow from CFT to BMS. The complexity associated to this process has been termed Krylov complexity in the literature, and has been examined thoroughly [91–99][4] in recent times.

The organization of the paper is as follows. In Sec. II, we discuss the underlying model based on the massless scalar field and the limiting procedure to obtain the BMS vacuum from the CFT vacuum. In Sec. III, we explore circuit complexity as a function of the contraction parameter from CFT to BMS. We observe that the complexity becomes divergent when the system hits the BMS point. To get further intuition about this diverging complexity, we study the associated information geometry in Sec. IV. Specifically, we study the Fubini-Study metric and the geodesic that connects the CFT and BMS vacuum on the state manifold. We also comment on the Berry curvature for this process. Lastly, in Sec. V, we study the Hamiltonian of

[1]This list is by no means exhaustive. Readers are referred to the reviews [78–80], and references therein for more details. Also, there are several other proposals for holographic complexity e.g., the ones discussed in [81–84]. Again this list is also not exhaustive at all.

[2]For computation of OTOC in quantum mechanical systems interested readers are referred to [90].

[3]For a detailed review one can look at [86] and the references therein.

[4]This list again does not do justice to the literature that has discussed related topics in recent times. Interested readers are referred to the references and citations of these papers.

the system and the behavior of the OTOCs. We find that the OTOC is a polynomial function of time in the BMS limit. A close study of the Krylov complexity finds a similar polynomial scaling associated with operator evolution in this limit. We conclude in Sec. VI by summarizing our results and proposing future directions.

II. REVISITING BMS_3 INVARIANT SCALAR FIELD

A. The intrinsic model

As discussed in the Introduction, our core model concerns an Inn-Wigner contraction from $2d$ relativistic conformal field theories to theories with BMS_3 as their symmetry algebra. A very well-studied example of this appears in the study of null or tensionless string theories [38–41]. In this limit, the worldsheet of the string becomes null, endowed with a degenerate metric and acquires a Carrollian structure, where the residual symmetry algebra coincides with that of BMS_3. From a CFT point of view, BMSFTs generically occur as a limit of the $2d$ conformal algebra, which is isomorphic to two copies of the Virasoro algebra. For completeness, these Virasoro generators on a cylinder parametrized by $(\sigma \sim \sigma + 2\pi, \tau)$ are given by the following vector fields:

$$\mathcal{L}_k = \frac{i}{2} e^{ik(\tau+\sigma)}(\partial_\tau + \partial_\sigma), \qquad \bar{\mathcal{L}}_k = \frac{i}{2} e^{ik(\tau-\sigma)}(\partial_\tau - \partial_\sigma). \quad (1)$$

At the level of mode expansions, these correspond to two independent sets of oscillators corresponding to holomorphic and antiholomorphic sectors in the CFT. These generators satisfy the classical part of the Virasoro algebra

$$[\mathcal{L}_k, \mathcal{L}_{k'}] = (k-k')\mathcal{L}_{k+k'}, \quad [\bar{\mathcal{L}}_k, \bar{\mathcal{L}}_{k'}] = (k-k')\bar{\mathcal{L}}_{k+k'}, \quad (2)$$

where one can add Virasoro central charges $c, \bar{c}$ to the algebra when quantized. Given the two Virasoro generators $\mathcal{L}_k$ and $\bar{\mathcal{L}}_k$, the contraction of the algebra is given by

$$L_k = \bar{\mathcal{L}}_k - \bar{\mathcal{L}}_{-k}, \qquad M_k = \epsilon(\bar{\mathcal{L}}_k + \bar{\mathcal{L}}_{-k}), \qquad \epsilon \to 0. \quad (3)$$

This is often known as an ultrarelativistic contraction since the effective speed of light goes to zero in this construction. The resulting algebra is that of BMS_3, which is isomorphic to the Galilean conformal algebra (GCA) in two dimensions,

$$\begin{aligned}
[L_k, L_{k'}] &= (k-k')L_{k+k'} + c_L \delta_{k+k',0}(k^3 - k), \\
[L_k, M_{k'}] &= (k-k')M_{k+k'} + c_M \delta_{k+k',0}(k^3 - k), \\
[M_k, M_{k'}] &= 0,
\end{aligned} \quad (4)$$

where $c_{L,M}$ are central charges to be determined. At the level of coordinates and coupling constants, these correspond to singular scalings, viz.

$$\sigma \to \sigma, \qquad \tau \to \epsilon\tau, \qquad \epsilon \to 0. \quad (5)$$

If one wants to relate the central charges, they also scale accordingly:

$$c_L = c - \bar{c}, \qquad c_M = \epsilon(c + \bar{c}). \quad (6)$$

Starting from a CFT_2 action on flat spacetime and performing the above contraction leads one to the action

$$S = \frac{1}{4\pi} \int d\tau d\sigma (\partial_\tau \Phi)^2. \quad (7)$$

We note that only the temporal derivative of the field $\Phi(\sigma, \tau)$ survives under the contraction procedure. But there still survives the notion of space and time in the (σ, τ) coordinates; however, they are not on equal footing as is the case of relativistic theories. One could explicitly check that this action is invariant under the BMS transformations,

$$\sigma \to f(\sigma), \qquad \tau \to f'(\sigma)\tau + g(\sigma). \quad (8)$$

Here f, g are arbitrary functions and prime denotes a derivative with respect to σ. It is easy to see that these transformations are generated by the symmetry generators,

$$\begin{aligned}
L(f) &= f'(\sigma)\tau\partial_\tau + f(\sigma)\partial_\sigma \\
&= \sum_k c_k e^{ik\sigma}(\partial_\sigma + ik\tau\partial_\tau) = -i\sum_k c_k L_k,
\end{aligned} \quad (9)$$

$$M(g) = g(\sigma)\partial_\tau = \sum_k d_k e^{ik\sigma}\partial_\tau = -i\sum_k d_k M_k, \quad (10)$$

where $f = \sum c_k e^{ik\sigma}$, $g = \sum d_k e^{ik\sigma}$ have been expanded in Fourier modes. The modes L_n and M_n generate the classical part of the BMS_3 algebra. The equations of motion for the scalar takes the form

$$\ddot{\Phi} = 0. \quad (11)$$

Subject to periodic boundary conditions on a cylinder $\Phi(\tau, \sigma) = \Phi(\tau, \sigma + 2\pi)$, the above equation of motion is solved by the following mode expansion:

$$\Phi(\sigma, \tau) = A_0 + B_0\tau + \sum_k \frac{i}{k}(A_k - ik\tau B_k)e^{-ik\sigma}. \quad (12)$$

Here A, B are purely Hermitian operators, and the conjugate momentum is given by

$$\Pi = \frac{\partial S}{\partial \dot{\Phi}} = \frac{1}{2\pi}\dot{\Phi}, \quad (13)$$

and the canonical Poisson bracket which reads

$$\{\Pi(\sigma',\tau),\Phi(\sigma,\tau)\} = \delta(\sigma-\sigma') \tag{14}$$

implies the following algebra for the oscillators:

$$\begin{aligned}&\{A_k, A_{k'}\}_{P.B.} = \{B_k, B_{k'}\}_{P.B.} = 0,\\ &\{A_k, B_{k'}\}_{P.B.} = -2ik\delta_{k+k',0}.\end{aligned} \tag{15}$$

These are clearly not usual CFT oscillators as is evident from the brackets, and they more look like quantum mechanical oscillators $\{X,P\}$. However we can always go to a basis where these oscillators act as decoupled set of (anti)holomorphic oscillators [41],

$$C_k = \frac{1}{2}(A_k + B_k), \qquad \tilde{C}_k = \frac{1}{2}(-A_{-k} + B_{-k}). \tag{16}$$

Now, the Poisson brackets take the canonical form

$$\begin{aligned}&\{C_k, C_{k'}\} = -ik\delta_{k+k',0}, \qquad \{\tilde{C}_k, \tilde{C}_{k'}\} = -ik\delta_{k+k',0},\\ &\{C_k, \tilde{C}_{k'}\} = 0.\end{aligned} \tag{17}$$

Starting from the generators

$$L_k = \frac{1}{2}\sum_{k'} A_{-k'}B_{k'+k} \quad \text{and} \quad M_k = \frac{1}{2}\sum_{k'} B_{-k'}B_{k'+k}, \tag{18}$$

we can now write them in terms of the C oscillators,

$$L_k = \frac{1}{2}\sum_{k'}[C_{-k'}C_{k'+k} - \tilde{C}_{-k'}\tilde{C}_{k'-k}], \tag{19}$$

$$M_k = \frac{1}{2}\sum_{k'}[C_{-k'}C_{k'+k} + \tilde{C}_{-k'}\tilde{C}_{k'-k} + 2C_{-k'}\tilde{C}_{-k'-k}]. \tag{20}$$

These generators again span the BMS_3 algebra. However, one can spot that these generators are the same as the null string ones mentioned in [41] but with the spacetime indices stripped off. Many of our physical intuitions in subsequent sections will be borrowed from that of null strings, and we will mention that in particular places.

B. Canonical quantization

Let us try to understand the Hilbert space of the BMS invariant scalar theory. As usual in quantized theory, all Poisson brackets go to Dirac brackets, and we can have the canonical commutation relations,

$$[C_k, C_{k'}] = [\tilde{C}_k, \tilde{C}_{k'}] = k\delta_{k+k',0}. \tag{21}$$

And a CFT-like *oscillator vacuum* $|0\rangle_c$ can be defined by the following:

$$C_k|0\rangle_c = \tilde{C}_k|0\rangle_c = 0 \quad \forall\ k>0. \tag{22}$$

Here, one can clearly notice that $|0\rangle_c$ is not a pure state anymore but an entangled state of these new left and right oscillator sectors:

$$|0\rangle_c = |0\rangle_R \otimes |0\rangle_L. \tag{23}$$

In this case, in terms of the C oscillators, we can write down the relevant zero modes of the BMS generators:

$$L_0 = \frac{1}{2}\sum_k [C_{-k}C_k - \tilde{C}_{-k}\tilde{C}_k], \tag{24}$$

$$M_0 = \frac{1}{2}\sum_k [C_{-k}C_k + \tilde{C}_{-k}\tilde{C}_k + 2C_{-k}\tilde{C}_{-k}]. \tag{25}$$

These can be thought of as analogues of angular momentum operator and Hamiltonian for usual relativistic CFT. However, note that M_0 here is seemingly not diagonalizable. This structure is central to defining the quantum nature of a BMS invariant theory. For details related to quantum structures and vacuum classifications of this theory, the reader is directed to [42,100].

C. Limiting perspective

As we have emphasized earlier, BMS invariant theories can be discussed either from an intrinsic point of view, or equivalently by taking limits on their relativistic counterparts. To start along the second avenue, consider the relativistic free conformal scalar model on the cylinder,

$$S = \frac{1}{4\pi}\int d\sigma dt((\partial_t\Phi)^2 - (\partial_\sigma\Phi)^2), \quad (\sigma,t)\sim(\sigma+2\pi,t). \tag{26}$$

Under the UR limit (5) together with the corresponding rescaling of the field,

$$t = \epsilon\tau, \qquad \Phi = \sqrt{\epsilon}\phi, \qquad \epsilon\to 0, \tag{27}$$

the action (26) becomes the BMS scalar action (7) on the cylinder $(\sigma,\tau)\sim(\sigma+2\pi,\tau)$, which we reproduce here,

$$S = \frac{1}{4\pi}\int d\sigma d\tau (\partial_\tau\phi)^2. \tag{28}$$

The equation of motion of the relativistic scalar field [coming from (26)] can be solved in terms of the mode expansion

$$\begin{aligned}\Phi(\sigma,t) = \phi_0 + \pi_0 t\\ + \frac{i}{\sqrt{2}}\sum_{k\neq 0}\frac{1}{k}(a_k e^{-ik(\sigma+t)} - \bar{a}_{-k}e^{-ik(\sigma-t)}),\end{aligned} \tag{29}$$

where $a_k^\dagger = a_{-k}$ etc., with the canonical commutation relations

$$[a_k, a_{k'}] = [\bar{a}_k, \bar{a}_{k'}] = k\delta_{k+k',0},$$
$$[a_k, \bar{a}_{k'}] = 0, [\phi_0, \pi_0] = i. \tag{30}$$

The CFT vacuum is defined by these oscillators

$$a_k|0\rangle_a = \bar{a}_k|0\rangle_a = 0 \quad \forall \ k > 0. \tag{31}$$

Comparing with the mode expansion of the BMS free scalar on the cylinder (12) we obtain the relation between modes before and after the UR limit

$$A_k = \lim_{\epsilon\to 0} \frac{1}{\sqrt{\epsilon}}(a_k - \bar{a}_{-k}),$$
$$B_k = \lim_{\epsilon\to 0} \sqrt{\epsilon}\,(a_k + \bar{a}_{-k}), \quad k \neq 0, \tag{32}$$

$$A_0 = \frac{\phi_0}{\sqrt{\epsilon}}, \qquad B_0 = -i\sqrt{\epsilon}\pi_0. \tag{33}$$

Following these limits and from (16), we can now see the relation between C oscillators and the CFT oscillators in the limit read:

$$C_k = \frac{1}{2}\left(\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}}\right)a_k + \frac{1}{2}\left(\sqrt{\epsilon} - \frac{1}{\sqrt{\epsilon}}\right)\bar{a}_{-k},$$
$$\tilde{C}_k = \frac{1}{2}\left(\sqrt{\epsilon} - \frac{1}{\sqrt{\epsilon}}\right)a_{-k} + \frac{1}{2}\left(\sqrt{\epsilon} + \frac{1}{\sqrt{\epsilon}}\right)\bar{a}_k. \tag{34}$$

The general transformation between C and a oscillators turns out to be a Bogoliubov transformation since the canonical structure remains intact under the generic transformation[5]:

$$C_k(\epsilon) = \cosh\theta a_k - \sinh\theta \bar{a}_{-k},$$
$$\tilde{C}_k(\epsilon) = -\sinh\theta a_{-k} + \cosh\theta \bar{a}_k. \tag{36}$$

And quantum mechanically, the parameter changing from $\epsilon = 1$ to $\epsilon = 0$ describes the contraction from a scalar CFT to a BMS scalar theory. At $\epsilon = 0$ the oscillators explicitly belong to that of the BMS algebra; however, the above relations hold even for $\epsilon = 1$ where it goes back to the CFT oscillators [41]. Hence we can extrapolate these definitions for the whole range of validity for the parameter ϵ. It goes without saying, this is an approximation, but this helps us to understand the underlying structures better. The associated transformation can be generated using

$$C_k = e^{-iG} a_k e^{iG}, \qquad \tilde{C}_k = e^{-iG} \bar{a}_k e^{iG}, \tag{37}$$

where the unitary transformation operator is a two-mode squeezing operator that can be written as

$$G(\theta(\epsilon)) = i\sum_{k=1}^{\infty} \frac{\theta}{k}[a_k^\dagger \bar{a}_k^\dagger - a_k \bar{a}_k], \tag{38}$$

Remember that the new vacuum is defined by $C_k|0\rangle_c = \tilde{C}_k|0\rangle_c = 0 \ \forall \ k > 0$, and this condition, using (35) translates into

$$(a_k - \tanh\theta \bar{a}_{-k})|0\rangle_c = 0, \qquad k > 0;$$
$$(\bar{a}_k - \tanh\theta a_{-k})|0\rangle_c = 0. \tag{39}$$

Now we are in a position to write down the mapping between the two vacua $|0\rangle_a$ and $|0\rangle_c$. This is given by the following two mode squeezed state:

$$|0\rangle_c = e^{-iG(\theta(\epsilon))}|0\rangle_a$$
$$= \sqrt{\cosh\theta} \prod_{k=1}^{\infty} \exp\left[\frac{\tanh\theta}{k} a_k^\dagger \bar{a}_k^\dagger\right]|0\rangle_a. \tag{40}$$

Similarly, the inverse transformation to relate the two vacua reads

$$a_k|0\rangle_a = (C_k - \tanh\theta\, \tilde{C}_{-k})|0\rangle_a = 0, \qquad k > 0;$$
$$\bar{a}_k|0\rangle_a = (\tilde{C}_k - \tanh\theta\, C_{-k})|0\rangle_a = 0, \tag{41}$$

which can be thought to be generated by the inverse displacement operator:

$$\bar{G}(\theta(\epsilon)) = -i\sum_{k=1}^{\infty} \frac{\theta}{k}[C_k^\dagger \tilde{C}_k^\dagger - C_k \tilde{C}_k] \tag{42}$$

with $C_k^\dagger = C_{-k}$ etc. Here we have $\tanh\theta = \frac{\epsilon-1}{\epsilon+1}$, which makes sure that (40) is valid at $\epsilon = 1$. The solution in this case is given as

$$|0\rangle_a = \frac{1}{\mathcal{N}} \prod_{k=1}^{\infty} \exp\left[-\frac{\tanh\theta}{k} C_k^\dagger \cdot \tilde{C}_k^\dagger\right]|0\rangle_c. \tag{43}$$

Note that at the BMS (or the tensionless) point $\epsilon = 0$, the CFT vacuum turns out to be a special state with respect to the BMS oscillators:

[5]Note that we could also have had

$$C_k(\epsilon) = \cosh\theta e^{-i\chi} a_k + \sinh\theta e^{i\chi} \bar{a}_{-k},$$
$$\tilde{C}_k(\epsilon) = \sinh\theta e^{-i\chi} a_{-k} + \cosh\theta e^{i\chi} \bar{a}_k \tag{35}$$

which still would respect the canonical commutations as χ is just a pure phase. The squeezing operator in this case has to be changed accordingly. Since our ϵ is considered purely real, we omit this extra phase factor.

$$|0\rangle_a = \frac{1}{\mathcal{N}'}\prod_{k=1}^{\infty}\exp\left[-\frac{1}{k}C_k^\dagger\tilde{C}_k^\dagger\right]|0\rangle_c. \tag{44}$$

At the level of wave functions, the question is which way we want to evolve in ϵ. In a sense, this is a "thermal" evolution and may be thought of as a Euclidean time evolution. More details on this can be found in [101].

D. "Position space" representation of the vacuum

In the present section, we will be computing circuit complexity for the state (40). We start by solving for the "position-space" wave function. To do that, first let us define the following "position" and "momentum" operators out of the C oscillators,

$$q_k = \frac{1}{\sqrt{2k}}(C_k^\dagger + C_k), \qquad p_k = \frac{i}{\sqrt{2k}}(C_k^\dagger - C_k),$$
$$\tilde{q}_k = \frac{i}{\sqrt{2k}}(\tilde{C}_k^\dagger - \tilde{C}_k), \qquad \tilde{p}_k = -\frac{1}{\sqrt{2k}}(\tilde{C}_k^\dagger + \tilde{C}_k), \qquad k > 0. \tag{45}$$

It is easy to check that they satisfy the canonical commutation relations i.e.,

$$[q_k, p_{k'}] = i\delta_{k,k'} = [\tilde{q}_k, \tilde{p}_{k'}]. \tag{46}$$

To do that, we first write (41) in terms of these position and momentum operators and then they give us the following first-order differential equations in position-space, which we can easily solve[6]

$$\begin{aligned}&(q_k + i\tanh\theta_k\tilde{q}_k)\psi_c(q_k,\tilde{q}_k)\\&\quad + (\partial_{q_k} - i\tanh\theta_k\partial_{\tilde{q}_k})\psi_c(q_k,\tilde{q}_k) = 0,\\&(i\tilde{q}_k - \tanh\theta_k q_k)\psi_c(q_k,\tilde{q}_k)\\&\quad + (i\partial_{\tilde{q}_k} + \tanh\theta_k\partial_{q_k})\psi_c(q_k,\tilde{q}_k) = 0.\end{aligned} \tag{47}$$

Note that, in the position space representation,

$$p_k = -i\frac{\partial}{\partial q_k}, \qquad \tilde{p}_k = -i\frac{\partial}{\partial\tilde{q}_k}.$$

Solving (47) we get the wave function,

$$\Psi_c(q_k,\tilde{q}_k) = \langle q_k;\tilde{q}_k|0_c\rangle = \prod_{k=1}^{\infty}\frac{e^{-A(q_k^2+\tilde{q}_k^2)-iBq_k\tilde{q}_k}}{\sqrt{\pi\cosh 2\theta_k}}, \tag{48}$$

where the constants are

$$A = \frac{1}{2\cosh 2\theta_k}, \qquad B = \tanh 2\theta_k. \tag{49}$$

[6] We add a generic k subscript to θ. However, our Bogoliubov coefficients are not directly mode dependent, so all θ_k's are the same.

Using the definition of θ_k i.e., $\tanh\theta_k = \frac{\epsilon-1}{\epsilon+1}$ we can rewrite these in the following manner,

$$A = \frac{\epsilon}{1+\epsilon^2}, \qquad B = \frac{\epsilon^2-1}{\epsilon^2+1}. \tag{50}$$

Further, we can introduce a new set of canonical variables to decouple the system into two sectors,

$$q_k^+ = \frac{q_k + \tilde{q}_k}{\sqrt{2}}, \qquad q_k^- = \frac{q_k - \tilde{q}_k}{\sqrt{2}},$$
$$p_k^+ = \frac{p_k + \tilde{p}_k}{\sqrt{2}}, \qquad p_k^- = \frac{p_k - \tilde{p}_k}{\sqrt{2}}. \tag{51}$$

Then the wave function mentioned in (48) becomes

$$\begin{aligned}\psi_c(q_k^+,q_k^-) &= \prod_{k=1}^{\infty}\frac{e^{-\frac{1}{2}(2A+iB)(q_k^+)^2-\frac{1}{2}(2A-iB)(q_k^-)^2}}{\sqrt{\pi\cosh 2\theta_k}}\\&= \prod_{k=1}^{\infty}\psi_{k,c}^+(q_k^+)\psi_{k,c}^-(q_k^-).\end{aligned} \tag{52}$$

In the next section, we will be using (52) for the computation of circuit complexity.

III. CIRCUIT COMPLEXITY: FROM CFT TO BMS

A. Circuit complexity: A brief introduction

As mentioned before, our goal is to probe the transition from CFT to BMS using tools of quantum information. Particularly, we will focus on *circuit complexity*. We have already introduced this quantity in the Introduction, but here let us give a brief technical review. We will mainly follow the approach pioneered by Nielsen, and his collaborators [102–104]. For more details, interested readers are referred to [47]. Operationally, given a set of elementary gates, it quantifies the minimal number of operations needed to build a circuit which will take a suitable reference state $|\psi_R\rangle$ as input and generate the desired target state $|\psi_T\rangle$ as an output. Formally, given a reference state and set of gates, a quantum circuit starts at the reference state (at $s = 0$) and terminates at a target state ($s = 1$)

$$|\psi_T(s=1)\rangle = U(s=1)|\psi_T(s=0)\rangle. \tag{53}$$

Here U is the unitary operator that takes the reference state to the target state. It takes the following form:

$$U(s) = \overleftarrow{\mathcal{P}}\exp\left[-i\int_0^s ds' H(s')\right]. \tag{54}$$

The s parametrizes a path in the space of the unitaries and given a set of elementary gates M_I, the *control* Hamiltonian $[H(s)]$ can be written as

$$H(s) = Y^I(s) M_I. \quad (55)$$

The coefficients $Y^I(s)$ counts the number of times that a particular gate acts at a given value of s. It can be easily shown that [47]

$$\frac{dU(s)}{ds} = -iY^I(s) M_I U(s). \quad (56)$$

Then we define a cost functional $\mathcal{F}(U, \dot{U})$ as follows:

$$\mathcal{C}(U) = \int_0^1 \mathcal{F}(U, \dot{U}) ds. \quad (57)$$

The dot defines the derivative with respect to s. Minimizing this cost functional gives the optimal Y^I's and hence it gives us the optimal circuit. There are different choices for the cost functional [47,104,105]. In this paper we will consider the following:

$$\mathcal{F}_2(U, Y) = \sqrt{\sum_I (Y^I)^2}. \quad (58)$$

Here $\mathcal{F}_2(U, Y)$ corresponds to standard distance measure over a Riemannian geometry, here the one associated to the state space, on which (57) defines the length functional.[7]

For our case, a natural choice of the reference state $|\Psi_R\rangle$ is the CFT ground state which is a Gaussian state. So the reference wave function in $(q_k, \tilde{q}_k)$ basis takes the following form:

$$\langle q_k; \tilde{q}_k | \psi \rangle_R = \psi_c(q_k, \tilde{q}_k)|_{\epsilon=1} = \prod_{k=1}^{\infty} \frac{e^{-\frac{1}{2}(q_k^2 + \tilde{q}_k^2)}}{\sqrt{\pi}}. \quad (59)$$

$\psi_c(q_k, \tilde{q}_k)$ is defined in (52) and $\epsilon = 1$ corresponds to the CFT ground state. Then for the target state we choose the state mentioned in (52) but for $\epsilon \neq 1$. In this way the circuit complexity will be the function of the flow parameter ϵ and thereby will help us to probe the CFT to BMS flow.

B. Behavior of circuit complexity as a function of ϵ

Given the target and reference state we follow [50,69,106] to compute circuit complexity. Note that, both the target and reference state in our cases are Gaussian states. The Gaussian states are equivalently described by their corresponding *covariance matrix*. The covariance matrix for each mode k is defined in the following way:

$$G_k(\epsilon) = \langle \psi_c(q_k^+, q_k^-) | \Psi_k \Psi_k^\dagger | \psi_c(q_k^+, q_k^-) \rangle, \quad (60)$$

where

$$\Psi^T = \{q_k^+, p_k^+, q_k^-, p_k^-\}.$$

For our case we will have the following two covariance matrices for reference ($\epsilon = 1$) and target state ($\epsilon \neq 1$) wave functions:

$$G_k^{s=0}(\epsilon = 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G_k^{s=1}(\epsilon) = \begin{pmatrix} \frac{1}{2A} & -\frac{B}{2A} & 0 & 0 \\ -\frac{B}{2A} & \frac{4A^2+B^2}{2A} & 0 & 0 \\ 0 & 0 & \frac{1}{2A} & \frac{B}{2A} \\ 0 & 0 & \frac{B}{2A} & \frac{4A^2+B^2}{2A} \end{pmatrix}. \quad (61)$$

We note that, from (49) $4A^2 + B^2 = 1$. We can compute the circuit complexity in terms of these covariance matrices [50,106]. We want to construct the optimal circuit such that

$$G_k^{s=1} = U(s=1) \cdot G_k^{s=0} \cdot U(s=1)^T. \quad (62)$$

Note that, these covariance matrices are of block diagonal form. Each of the blocks are an element of the $SU(1,1)$ group. So we can take the generators M_I's as generators $SU(1,1) \times SU(1,1)$. For details interested readers are referred to [50]. Finally the complexity per mode k takes the following form due to the structure of the covariance matrix [50][8]:

$$\mathcal{C}_k = \frac{1}{\sqrt{2}} \left| \operatorname{arccosh}\left(\frac{1 + 4A^2 + B^2}{4A}\right) \right| = \frac{1}{\sqrt{2}} \left| \operatorname{arccosh} \frac{1}{2A} \right| = \frac{1}{\sqrt{2}} \left| \operatorname{arccosh} \frac{1+\epsilon^2}{2\epsilon} \right|. \quad (63)$$

It is evident that $\mathcal{C}_k$ is a monotonically *increasing* function of the parameter ϵ. It starts from zero at $\epsilon = 1$, i.e., at the CFT ground state and diverges at $\epsilon = 0$ i.e., at the BMS vacuum. This is illustrated in the Fig. 1.

Some comments are in order after this result. The divergence in circuit complexity indicates that the target state may not be reachable from the reference state via a combination of unitary operations. But this can also be interpreted as nonanalyticity corresponding to some critical

[7]This is a natural choice for our study as we will compare it with the Fubini-Study distance in a subsequent section.

[8]Note that, the total complexity will be $\mathcal{C} = \sqrt{\sum_{k=1}^N (\operatorname{arccosh}(\frac{1}{2A}))^2}$. As the argument of arccosh is independent of k, we will get $\mathcal{C} = \frac{1}{\sqrt{2}} |\operatorname{arccosh} \frac{1}{2A}| \sqrt{V}$, where V is the momentum space volume i.e., $V = \sum_{k=1}^N$. This overall factor of V does not affect our conclusions, hence we focus on the complexity per volume to avoid unnecessary clutter.

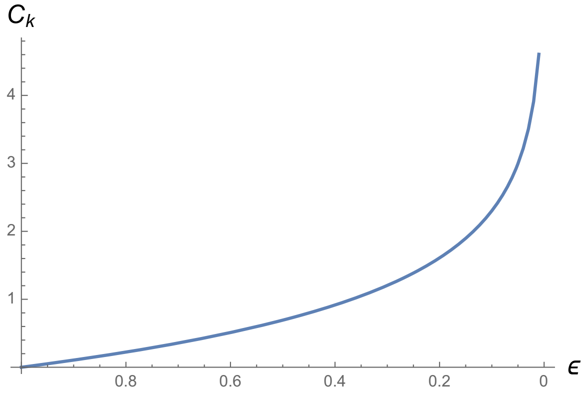

FIG. 1. Complexity as function of flow parameter $\epsilon.\epsilon = 1$ and $\epsilon = 0$ corresponds to the CFT and BMS point respectively. It clearly diverges at the point $\epsilon = 0$. We have rescaled the complexity by a factor of $\sqrt{2}$.

points [107] and signals the presence of a quantum phase transition. Looking at the system at hand, it makes perfect sense to assume there is a phase transition in going from CFT to BMS at the very extreme point, where a notion of ultralocality sets in. For the tensionless string case, this phase transition was interpreted as a Bose-Einstein like condensation that gives rise to open strings degrees of freedom from closed strings [108], as the target state is essentially a boundary state along with all spacetime directions. We can safely assume a related interpretation for our case as well. However, the actual physical perspective may be different here.

IV. INFORMATION GEOMETRY

A. Fubini-Study metric

To get further insight into the diverging complexity at $\epsilon = 0$, as we have uncovered in the last section, we will first try to associate a Riemannian structure to the space of wave functions (40). We identify the coherent state we have been working with a point on a group manifold, and the complexity for the target state is defined as the geodesic distance between the state and the reference one on the group manifold. As noted before the state mentioned in (40) is a $SU(1,1)$ coherent state. So we start with a generic state of the form

$$|\psi\rangle = \mathcal{N}\prod_{k=1}^{N} e^{z_k K_+}|0,0\rangle. \tag{64}$$

Here K_+ is a $SU(1,1)$ generator, with the set of generators given by combination of oscillators (without the k subscripts):

$$K_+ = \beta^\dagger\tilde{\beta}^\dagger, \qquad K_- = \beta\tilde{\beta}, \qquad K_z = \frac{1}{2}(\beta^\dagger\beta + \tilde{\beta}\tilde{\beta}^\dagger), \tag{65}$$

which generates the familiar algebra:

$$[K_+, K_-] = -2K_z, \qquad [K_z, K_\pm] = \pm K_\pm. \tag{66}$$

Then the state associated to (64) can be given a Riemannian structure [109]. The infinitesimal distance in this state space, also know as "Fubini-Study" metric, can be written as

$$ds^2 = g_{ij}dx^i dx^j, \tag{67}$$

where the metric tensor is given by

$$g_{ij} = \langle\partial_i\psi|\partial_j\psi\rangle - \langle\partial_i\psi|\psi\rangle\langle\psi|\partial_j\psi\rangle. \tag{68}$$

Finally we get [48,56]

$$ds^2 = \sum_k \frac{|dz_k|^2}{(1-|z_k|^2)^2}. \tag{69}$$

Further we can parametrize the complex function z_k in (64) as

$$z = |z_k|e^{i\phi_k},$$

where we take $|z_k| = \tanh(\bar{\theta}_k/2)$. Then we get

$$ds^2 = \sum_{k=1}^{N}\frac{1}{4}(d\bar{\theta}_k^2 + \sinh(\bar{\theta}_k)^2 d\phi_k^2). \tag{70}$$

Then the geodesic distance between two point $(\bar{\theta}_{1,k}, \phi_{1,k})$ and $(\bar{\theta}_{2,k}, \phi_{2,k})$ is given by

$$d_{FS} = \frac{1}{2}\sqrt{\sum_{k=1}^{N}(\text{arccosh}[\cosh(\bar{\theta}_{1,k})\cosh(\bar{\theta}_{2,k}) - \sinh(\bar{\theta}_{1,k})\sinh(\bar{\theta}_{2,k})\cos(\phi_{1,k}-\phi_{2,k})])^2}. \tag{71}$$

From (40) it is evident that $z_k = \frac{(\epsilon-1)}{(\epsilon+1)}$. Also, we can clearly see that for CFT ($\epsilon = 1$) $z_k = 0$ as $\bar{\theta}_k = 0$ and for BMS ($\epsilon = 0$) $z_k = -1$. Hence the length of the geodesic connecting the following two points[9]

[9]Note that, although the phases ϕ_k are zero for the in initial and final state, the shortest geodesic connecting them could pass through states with nonvanishing phase. So for computations pertaining to an intermediate state, we should keep track of the phase factor.

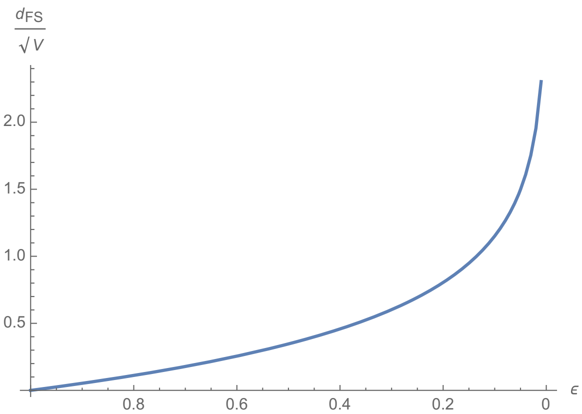

FIG. 2. Fubini-Study distance as function of flow parameter ϵ.$\epsilon = 1$ and $\epsilon = 0$ correspond to the CFT and BMS points respectively.

$$(\bar{\theta}_{1,k} = 0, \phi_{1,k} = 0),$$
$$\left(\bar{\theta}_{2,k} = 2 \text{ arc tanh}\left(\frac{(\epsilon - 1)}{(\epsilon + 1)}\right), \phi_{2,k} = 0\right) \quad [\epsilon < 1],$$

turns out to be

$$d_{FS} = \frac{1}{2}\sqrt{\sum_{k=1}^{N} \bar{\theta}_{2,k}^2} = \sqrt{\sum_{k=1}^{N} \text{arctanh}\left(\frac{(\epsilon-1)}{(\epsilon+1)}\right)^2} = \text{arctanh}\left(\frac{(\epsilon-1)}{(\epsilon+1)}\right)\sqrt{V}. \tag{72}$$

Here $V = \sqrt{\sum_{k=1}^{N}}$ denotes the phase-space volume. It is easy to check that it is a monotonically increasing quantity and diverges at $\epsilon = 0$. This is shown in Fig. 2. Also, note that, for $\epsilon = 0$, $z_k = 1$ and from (70) it is evident that the information metric becomes degenerate as we approach the BMS point, i.e., the geodesic never reaches the BMS point staying on the same coordinate chart.

So we could again see that the complexity diverges at the special point of $\epsilon = 0$ as before, showing similar qualitative behavior. Although the geometric notion associated with this intriguing observation is still unclear, one could recall that $|z_k| = 1$ corresponds to a degenerate point on the projective space hyperbola on the Fubini-Study metric. This also corresponds to the phase transition point in the physical space, especially with the ones associated with ground state degeneracies. As seen in the literature [108] this particular point with $\epsilon = 0$ has been interpreted as an infinitely degenerate vacuum with all excitations in the tensile string condensing into just one state. This could be the real reason behind this diverging geodesic distance. However, a true CFT notion is beyond the scope of this work.

B. Berry curvature

Furthermore, we can compute the Berry curvature [110] associated with these kinds of $SU(1,1)$ coherent states (64), transforming finally into a boundary state in this process. The Berry curvature is a measure to quantify a path in a group representation that connects our initial and final points. The components of Berry connections are defined by

$$A_i = \langle\psi|\partial_i|\psi\rangle. \tag{73}$$

For our case, we get the following components for each k mode [111]:

$$A_{z_k} = \frac{\bar{z}_k}{2(1-|z_k|^2)}, \qquad A_{\bar{z}_k} = -\frac{z_k}{2(1-|z_k|^2)}. \tag{74}$$

The bar on z_k denotes the complex conjugate for generic parametrizations. Then we can define a two-form, the Berry curvature, for each mode k as follows:

$$F = dA \tag{75}$$

where d is the exterior derivative and the one-form A is Berry connection components which are defined in (73). For our case, we will have

$$F = \frac{i}{2}\sinh(\bar{\theta}_k) d\theta_k \wedge d\phi_k. \tag{76}$$

Here we have used the fact that $z = \tanh(\bar{\theta}_k/2)e^{i\phi_k}$. For the state (40), using $\bar{\theta}_k = 2\text{arctanh}(\frac{(\epsilon-1)}{(\epsilon+1)})$ as before we get

$$F_{\theta\phi} = \frac{i}{2}\sinh\left[2 \text{ arc tanh}\left(\frac{(\epsilon-1)}{(\epsilon+1)}\right)\right] = \frac{i}{4}\left(\frac{\epsilon^2 - 1}{\epsilon}\right). \tag{77}$$

It is easy to see that the Berry curvature diverges at $\epsilon = 0$, i.e., at the BMS point. So the behavior of both the complexity and the Berry curvature is the same at the critical point $\epsilon = 0$.

V. HAMILTONIAN EVOLUTION

For the last couple of sections, we have been focusing on the evolution of our system only in ϵ, presumably at an initial time slice. Our discussion clearly shows that the quantities we are interested in could be ill defined at the pure BMS point $\epsilon = 0$. In this sense, ϵ acts as a cutoff in the system. However, the story could be different at a finite time slice, which we will deliberate on in this section.

A. Diagonalization

Let us now consider the temporal dynamics associated with our system's Hamiltonian that also changes with the parameter ϵ. One can remember that for relativistic $2d$ CFT, the Hamiltonian and angular momentum operators were tentatively given by combinations of Virasoro zero modes,

$$H_M = \mathcal{L}_0 + \bar{\mathcal{L}}_0, \qquad J_M = \mathcal{L}_0 - \bar{\mathcal{L}}_0. \tag{78}$$

Upon quantization, the operators $\mathcal{L}_0$, $\bar{\mathcal{L}}_0$ are given in terms of CFT oscillators (31) by

$$\mathcal{L}_0 = \frac{1}{2}\sum_k a_{-k}a_k, \qquad \bar{\mathcal{L}}_0 = \frac{1}{2}\sum_k \bar{a}_{-k}\bar{a}_k. \tag{79}$$

Now remember that under contraction, the oscillators change with a Bogoliubov transformation, which are the inverse transforms of (34),

$$\begin{aligned} a_k &= \Omega_+ C_k - \Omega_- \tilde{C}_{-k}, \\ \bar{a}_k &= \Omega_+ \tilde{C}_k - \Omega_- C_{-k}. \end{aligned} \tag{80}$$

Using the above, we can see that action of the operator $(\mathcal{L}_0^{(\epsilon)} - \bar{\mathcal{L}}_0^{(\epsilon)})$ on a state remains invariant throughout the ϵ evolution since $\Omega_+^2 - \Omega_-^2 = 1$ by definition of Bogoliubov transformations. But the other combination reads

$$\begin{aligned} \mathcal{L}_0^{(\epsilon)} + \bar{\mathcal{L}}_0^{(\epsilon)} = \sum_k [&(\Omega_+^2 + \Omega_-^2)(C_k^\dagger C_k + \tilde{C}_k^\dagger \tilde{C}_k) \\ &- 4\Omega_+\Omega_- C_k \tilde{C}_k]. \end{aligned} \tag{81}$$

Note that this is the Hamiltonian that appeared in [101] in the null string theory context. So the action of $\bar{\mathcal{L}}_0^{(e)} + \bar{\mathcal{L}}_0^{(e)}$ combination does not remain invariant as we move to more and more in ϵ, and an extra "perturbation" term generates a deformation.[10] This is the extra seemingly nondiagonal term in the above equation.

A possible way out of the problem is to consider the Hamiltonian arbitrary near the null surface, where the BMS symmetry arises. Here, we can write an appropriately scaled and finite perturbative normal ordered Hamiltonian near $\epsilon \to 0$ (but not exactly) with next to leading correction in ϵ,[11]

$$\begin{aligned} H_\epsilon = \sum_{k=0}^{\infty} [&(1+\epsilon^2)(C_{-k}C_k + \tilde{C}_{-k}\tilde{C}_k) \\ &+ (1-\epsilon^2)(C_{-k}\tilde{C}_{-k} + C_k\tilde{C}_k)]. \end{aligned} \tag{82}$$

We can see that this Hamiltonian consists of two normal ordered number operators and two nondiagonal parts. As we go to $\epsilon = 0$, we get back the exact BMS answer for M_0 in (24). However, as we saw for the bogoliubov transformations, this definition can also be extrapolated to the CFT point $\epsilon = 1$, where only the two number operators remain (with the identification $a = C$ etc.). Note that, with the definitions of the basis we use in (45), the perturbation term in the Hamiltonian can be written as

$$C_k^\dagger \tilde{C}_k^\dagger + C_k \tilde{C}_k = -k(q_k \tilde{p}_k + p_k \tilde{q}_k). \tag{83}$$

We can further notice that these commutation relations (46) are invariant up to a discrete transformation,

$$\tilde{p}_k \to -\tilde{q}_k, \qquad \tilde{q}_k \to \tilde{p}_k, \tag{84}$$

or similarly for nontilde variables, which basically gives another set of basis oscillators for our wave function, where the tilde and nontilde set of (q, p)s in (45) are treated on the same footing. These transformations can also be achieved by a "flipping" map of $\tilde{C}$ oscillators $\tilde{C}_k \to \tilde{C}'_k = i\tilde{C}_{-k}$. Evidently, the perturbation term in the Hamiltonian also changes under this map:

$$k(q_k \tilde{q}_k - p_k \tilde{p}_k) = i(C_k^\dagger \tilde{C}_k^\dagger - C_k \tilde{C}_k), \tag{85}$$

which is just the displacement operator, and usually appears in the Hamiltonian for a thermofield double (TFD).

Next we time-evolve (48) with the Hamiltonian (82) written in the position-momentum basis. After neglecting a constant additive term (82) becomes

$$\begin{aligned} H_\epsilon = \sum_{k=0}^{\infty} \bigg[&\frac{k}{2}(1+\epsilon^2)(q_k^2 + p_k^2 + \tilde{q}_k^2 + \tilde{p}_k^2) \\ &- k(1-\epsilon^2)(q_k \tilde{p}_k + p_k \tilde{q}_k)\bigg]. \end{aligned} \tag{86}$$

Furthermore we can diagonalize (86) by using the transformations as below:

$$\begin{aligned} q_k^+ &= \frac{\tilde{q}_k - p_k}{\sqrt{2}}, \qquad q_k^- = \frac{\tilde{q}_k + p_k}{\sqrt{2}}, \\ p_k^+ &= \frac{q_k + \tilde{p}_k}{\sqrt{2}}, \qquad p_k^- = -\frac{q_k - \tilde{p}_k}{\sqrt{2}}, \end{aligned} \tag{87}$$

which are related to our earlier expression in (51) via the identifications in (84). Then in terms of these new ($\pm$) variables the Hamiltonian becomes

$$H_\epsilon = \sum_{k=0}^{\infty} k[\epsilon^2(p_k^{+^2} + q_k^{-^2}) + p_k^{-^2} + q_k^{+^2}]. \tag{88}$$

One can also see here that imposing (84) into the Hamiltonian and diagonalizing it gives rise to the same structure as above. As one can see from (88), in the strict $\epsilon = 0$ limit, both oscillators "freeze out," i.e., there are no

[10] See [44] for some physical insight into the nature of this term as a current-current deformation to the CFT.

[11] There is an implicit ϵ multiplying the whole Hamiltonian to make it finite, i.e., $H \to \epsilon H$.

dynamics at all. One may be tempted to call this a true Carrollian situation, where light cones close down, and there is no movement in space at all. We should moreover note from (88) that there are two different sets of eigenvalues of this Hamiltonian. One set scales as ϵ^2 and vanishes in the limit $\epsilon = 0$, i.e., at the BMS point, and the other set scales as $\frac{1}{\epsilon^2}$ which survives at the BMS point, effectively leading to one remaining set of oscillators.

B. Out-of-time-ordered correlators

Let us now actually focus on a particular observable diagnostic of quantum chaos, namely out-of-time-ordered correlators for this system, armed with our diagonal Hamiltonian. This will also require us to talk about how operators time evolve in this system as ϵ changes. In general OTOCs in a quantum system are defined as $C_T(t) = -\langle[W(t), V(0)]^2\rangle$, where $W(t)$ and $V(0)$ are some generic operators in Heisenberg representation at time "t" and some initial time. Let us then start with the diagonlized Hamiltonian mentioned in (88). Time evolution with this at $\epsilon = 0$ is tricky as there is no apparent dynamics, hence we need to calculate our OTOCs at finite (but small) ϵ, at finite time, and take a suitable limit.

We choose the position and momentum operators in the $\pm$ basis (51) as of interest. Under time evolution, the operators change as follows [90]:

$$q_k^{\pm}(t) = \cos(2k\epsilon t) q_k^{\pm}(0) + \epsilon^{\pm 1} \quad \sin(2k\epsilon t) p_k^{\pm}(0), \tag{89}$$

$$p_k^{\pm}(t) = \cos(2k\epsilon t) p_k^{\pm}(0) - \epsilon^{\mp 1} \quad \sin(2k\epsilon t) q_k^{\pm}(0). \tag{90}$$

The OTOCs in this case is then given by

$$\begin{aligned}
[q_k^{\pm}(t), q_k^{\pm}(0)] &= i\epsilon^{\pm 1}\sin(2k\epsilon t),\\
[p_k^{\pm}(t), p_k^{\pm}(0)] &= i\epsilon^{\mp 1}\sin(2k\epsilon t),\\
[q_k^{\pm}(t), p_k^{\pm}(0)] &= i\cos(2k\epsilon t).
\end{aligned} \tag{91}$$

One can observe that while at finite values of ϵ the OTOCs scale sinusoidally, in the strict limit of $\epsilon \to 0$ they either go to zero or scale *polynomially* with time ($k^2 t^2$ to be exact), signaling the freeze-out we just discussed.[12] Note also that while the bracket $[q(t), p(0)]$ gives the canonical commutation relation at $t = 0$, the same behavior comes back at $\epsilon = 0$ too. This is an intriguing dynamical behavior, as the Lyapunovian exponential behavior gives way to this polynomial growth. However, this phenomenon and its consequences need to be understood in a better physical way which we leave for future work.

C. Krylov complexity

In this section, to get further insight into the dynamics of the system, we sketch the idea of the complexity of Hamiltonian evolution. In this context, a natural notion of complexity which has been investigated in recent times in various contexts is *Krylov complexity* [91]. In recent times, *operator growth* has played an important role in the context many-body system [114–117]. An operator grows under the Liouvillian superoperator, and Krylov complexity captures the notion of the spread of the operators.

To proceed, let us think of our coherent state in the context of operator evolution under $SU(1,1) \approx SL(2,R)$, where the states are written as

$$|\psi\rangle = D(\xi)|0\rangle, \qquad D(\xi) = \prod_k e^{\xi_k L_{-1} - \bar{\xi}_k L_1}. \tag{92}$$

For our case, $\xi = \bar{\xi}$ is a constant real parameter, and the two-mode squeezed state representation of L operators are analogous to (65) i.e., we can identify $L_{\mp 1} = K_{\pm}, L_0 = K_z$. In the case where $\xi = \bar{\xi}$ is complex and proportional to time, this signifies unitary evolution with the Hamiltonian. However, in this case, our total Hamiltonian (82) is given by

$$H = \gamma_1(L_1 + L_{-1}) + \gamma_2 L_0, \tag{93}$$

where the coefficients are real, $\gamma_2 = 2(1 + \epsilon^2)$ and $\gamma_1 = (1 - \epsilon^2)$. This is a generic $SL(2,R)$ Hamiltonian, albeit we do not have a unity component as it would just contribute an overall phase. It can always be restored by suitably normal ordering the Hamiltonian. Time evolution under this can be thought of as producing generalized time-dependent coherent states. Notice again that at $\epsilon = 1$, i.e., at the CFT point, $\gamma_1 = 0$, and there is no generic displacement operator at work.

This being said, we can consider this evolution as the time-dependent evolution of the thermofield double state. Here two copies of the Virasoro CFT were disjoint at first, but they start talking to each other once $\epsilon < 1$ and produce a maximally entangled (boundary) state at $\epsilon = 0$. It has been argued [41] that the interpolating vacuum during ϵ evolution, i.e., $|0\rangle_c$ (40) signifies a thermal phase of the CFT. This was further corroborated in recent works [101,118] concerning null strings where this vacuum was interpreted as the vacuum for an analog of the worldsheet Unruh effect, driven by the Bogoliubov transformations (34) near the extreme. Here the parameter ϵ sets the scale for inverse acceleration and hence the same for inverse temperature.

[12] Following [112,113], one can also calculate the entire Lyapunov spectrum. One first constructs the matrix $L = M^{\dagger}M$, where $M_{ij} = i[z_i(t), z_j(0)]$ with i, $j = 1, 2, 3, 4$ and $z = \{q^+, q^-, p^+, p^-\}$ for each value of the mode k. Then the eigenvalues of L give the information about the entire Lyapunov spectrum. For a chaotic system, these eigenvalues typically behave as the exponential of t, and the exponents give the quantum Lyapunov spectrum. For our case, we can easily check that for $\epsilon = 0$, the eigenvalues of L are polynomials of t, indicating the absence of chaotic behavior.

If this interpretation withstands, the generic thermal evolving state at an initial time can be written as

$$|0\rangle_c = \mathcal{N}\sum_k e^{-\beta\omega_k/2}|k\rangle \otimes |\tilde{k}\rangle, \qquad \omega_k = k + \frac{1}{2}. \tag{94}$$

Here β is as usual the inverse temperature. It can then be shown using the Lanczos algorithm [96] by assuming a particular representation of the state space that unitary time evolution of the above state under (93) requires the strength parameters to have a form

$$\gamma_1 = \frac{\omega}{2\sinh\frac{\beta\omega}{2}}, \qquad \gamma_2 = \frac{\omega}{\tanh\frac{\beta\omega}{2}}. \tag{95}$$

Note here, since the frequencies are generally k dependent, the γs should also be k dependent. However, the explicit one-parameter form of the coefficients prohibits that, and we take $\omega_k = \omega$, i.e., we concentrate on a single mode, without any k dependence. Now the time evolved TFD state is analogous to $e^{iHt}|0\rangle_c$, which we can compute using the Baker-Campbell-Hausdorff formula, and that is the target state we are looking for. The characteristic oscillation frequency for our case is then set as

$$\frac{\omega^2}{4} = -\gamma_1^2 + \frac{\gamma_2^2}{4} = 4\epsilon^2. \tag{96}$$

Now one can see that there are clearly three dynamical regimes for our system. For $\gamma_2 > 2\gamma_1$, the frequency is real, while for $\gamma_2 < 2\gamma_1$, the frequency is imaginary. One can think of these two regimes respectively corresponding to the standard and inverted harmonic oscillators. The transition point between these two regimes $\gamma_2 = 2\gamma_1$ is intriguing for us as $\epsilon = 0$ at this point. This makes sense as the frequency becomes zero at this point, reinforcing our comments on the freeze-out of dynamics at the onset of Carrollian physics.

Following the discussion in [96], we can find that the Krylov basis is the standard two-oscillator Fock space and Krylov complexity is proportional to the average particle number in the time evolved state

$$C(t) \propto \frac{1}{(1-\frac{\gamma_2^2}{4\gamma_1^2})}\sinh^2\left(\sqrt{\gamma_1^2 - \frac{\gamma_2^2}{4}}t\right) = \gamma_1^2\left(\frac{\sin(2\epsilon t)}{2\epsilon}\right)^2 \tag{97}$$

which grows exponentially when $\gamma_2 < 2\gamma_1$ and has the usual sinusoidal behavior when $\gamma_2 > 2\gamma_1$. For our system, notice that when $0 < \epsilon \leq 1$ this quantity will always vary as a sinusoid. Explicitly at the BMS point $\epsilon = 0$, we have a vanishing frequency and hence the complexity varies quadratically with time, i.e.,

$$\lim_{\gamma_1 \to 2\gamma_2} C(t) \sim \gamma_1^2 t^2, \tag{98}$$

which has a similar scaling as our OTOCs in (91) for the system at the BMS point. This result remains finite even at $\epsilon = 0$ when we take the limit carefully.

Before ending this discussion, let us also notice an intriguing fact about the physical significance of the parameter space region $\gamma_2 < 2\gamma_1$ for our system. This explicitly points to the situation where $\epsilon \to i\epsilon$, i.e., a complex contraction of the conformal algebra.[13] In this regime, one could have an exponentially growing behavior of the complexity, commensurate with an unstable phase of the oscillator, and consequently, a Lyapunov exponent can be read off [79,97,113,119].[14]

VI. DISCUSSIONS AND CONCLUSIONS

In this paper, we discussed information-theoretic probes for the transition from a CFT_2 to a BMS_3 invariant scalar field theory. It is well known that these two theories are related via an Innu-Wigner contraction, which we explicitly used to construct quantum states that flow from one theory to another. It is also well documented that the endpoint in this path, the BMS invariant theory, presents singularities and degeneracies associated with the Carrollian manifold it inherently lives on. We took our reference state as the Gaussian ground state of the CFT and the target state as the entangled ground state of BMS, which turned out to be related to each other via a squeezing operation. Since at the exact BMS point, the state evolves into a boundary state, the underlying physics is expected to change drastically, and our computations bear witness to this fact. We explicitly showed that the complexity diverges at this critical point, signaling a quantum phase transition into a unitarily inequivalent theory. We proceeded to show that these two states are connected via an infinite length geodesic on the state manifold, which proves what was said before. The same behavior was reproduced when we extracted the Berry curvature associated with this process, effectively indicating the cutoff nature of the contraction parameter.

To understand this transition better, we then quantified the time evolution under the total Hamiltonian of the system, which continuously varies with the contraction parameter. Time-dependent markers of quantum chaos turn out to be much better controlled when a careful $\epsilon \to 0$ limit is taken on them. It turns out that dialing the contraction parameter from CFT to BMS changes the OTOCs of the system from oscillatory to polynomial behaviors. We also looked at the operator complexity associated with this

[13]One may also be tempted to interpret the scaling $t \to i\epsilon\tau$ as equivalent to contracting a Euclidean theory.

[14]Several other works have investigated whether complexity can detect the scrambling time and Lyapunov exponent, e.g., [120–125]. This list is by no means exhaustive, and a thorough look at the reference and citations of these papers is recommended.

transition and found it to scale polynomially with time as well. It was very important to note that when $0 < \epsilon \leq 1$ the complexity varies sinusoidally, while at the transition point $\epsilon = 0$ it reduces to scale with t^2, signifying two completely different phase structures associated with these realms. All of these results point to the apparent absence of chaotic behavior in this transition.

It would be nice to understand the origin of the polynomial behavior for the time-dependent quantities we talked about in this work. It is intriguing to see that such systems have been discussed in the literature (see, e.g., [126]), and point out a regime where fast scrambling may not be present for the system. However, a connection is merely speculation at this point, and a concrete mathematical link has to be established rigorously.

One could also go ahead and ask whether similar physics appears in the study of higher dimensional BMS invariant field theories. The study of BMSFTs is a very nascent activity as of now, and a lot of corners has not really been explored yet. Although works have appeared studying classical symmetries of higher dimensional BMS scalar fields [24,34], the systematic quantization and related vacuum structure of such theories are still mysterious. Since the transformation between ground states of $2d$ free scalar CFT and a BMS_3 invariant free scalar field offers such a unique connection, one could hope that similar structures also work out in higher dimensions, but concrete proposals are yet to materialize. One related state independent approach in this regard would be to use field theoretic techniques centered around symmetry algebras for BMS invariant theories, after modifying the approach for CFTs widely discussed in recent times [58,75].

Another interesting thing to note is the authors of [37] found clear Lyapunovian behavior in studying chaos for Carrollian conformal field theories in two dimensions. The situation there does not pertain to a transition from a relativistic CFT, but, intriguingly, actual intrinsic Carrollian dynamics does produce a chaotic spectrum. One may want to investigate the contradiction between these two approaches and learn more about such theories. We can also conjecture that something exciting is happening if one can make the parameter ϵ purely imaginary in a particular setting and perhaps compute the Lyapunov index, but we would come back to these questions in a separate work.

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