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An Important Historical Milestone: The Classification of the Cubic Equations

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Abstract

This article investigates the use of the history of mathematics as a pedagogical tool for the teaching and learning of mathematics, using the history of the cubic equation as a specific example.

Cubic equations arise intrinsically in many applications in natural sciences and mathematics. For example, in physics, the solutions of the equations of state in thermodynamics, or the computation of the speed of seismic Rayleigh waves require the solutions of cubic equations. In mathematics, they are instrumental in solving the quartic equations, for in the process, these are reduced to cubic equations. The impossibility of trisecting an angle or doubling a cube using only a straightedge and compass is equivalent to solving some cubic equations. As the name implies, the cubic spline approximation, an important tool in numerical analysis, also entails working with cubic functions.

Although cubic equations were explored by the ancient Babylonian, Greek, Chinese, Indian, and Egyptian scholars, it took the collective work of many well-known mathematicians such as Diophantus, Archimedes, Fibonacci, del Ferro, Khayyam, Tartaglia, Cardano, Viète, Descartes, and Lagrange to finally obtain a full solution.

Our goal in this paper is to investigate one of the most formidable steps in this extensive and prolific history, namely the complete classification of the cubic equations by Omar Khayyam in eleventh century, who was the first scholar to classify cubic equation and hence facilitate a methodical and logical approach to obtaining a general solution.

Keywords

Cubic equation, monic polynomial, classification of cubic equations

1. Introduction

History of mathematics is more than a few amusing and enjoyable anecdotes and/or some unverifiable apocrypha. Indeed, it is an account of the process of development of the discipline over long periods of time and in different places using a variety of valid ways of constructing concepts and proofs. As such, it fosters flexibility and creativity in the learners, and provides them with crucial sources of inspiration, insight, and motivation. It also presents mathematics within a context that corroborates the notion that this discipline is a dynamic, continuously transmuting field of study that is open to refutation and revision; in other words, it humanizes mathematics (Bidwell, 1993).

It is now well accepted that inclusion of the history of mathematics in the classroom not only refines problem-solving skills and helps students make useful mathematical connections (Jankvist, 2009), but it also emphasizes the interaction between mathematics and society (Wilson & Chauvot, 2000).

To sum up, history, introduced in a dynamic and vigorous manner, has an essentially fundamental role to play in today's mathematics classrooms. It demystifies and clarifies mathematics by showing that it is the creation of human beings and thus broadens the knowledge that students construct in a mathematics class (Marshall & Rich, 2000).

2. Methodology

To accentuate these points, in this paper, we will deal with a specific example: the historical development of the solution of the cubic equation.

There are different ways of using history in the classroom (Bidwell, 1993). For example, we can inject anecdotal material as the course is presented, that is, we can make historical references to coursework while it is being covered. Our methodological approach is somewhat different: we aim to make the accurate historical developments of a topic (namely the categorization of the cubic equation) a part of the course (algebra or pre-calculus). This is what the example given in this article is trying to achieve.

As is well known, today, using basic algebraic tools, the roots of the cubic equation

$Ax^3 + Bx^2 + Cx + D = 0$

can easily be computed. In fact, using Cardano's formula we have

$$x = \sqrt[3]{\left(-\frac{B^3}{27A^3}+\frac{BC}{6A^2}-\frac{D}{2A}\right)+\sqrt{\left(-\frac{B^3}{27A^3}+\frac{BC}{6A^2}-\frac{D}{2A}\right)^2+\left(\frac{C}{3A}-\frac{B^2}{9A^2}\right)^3}}$$

$$+\sqrt[3]{\left(-\frac{B^3}{27A^3}+\frac{BC}{6A^2}-\frac{D}{2A}\right)-\sqrt{\left(-\frac{B^3}{27A^3}+\frac{BC}{6A^2}-\frac{D}{2A}\right)^2+\left(\frac{C}{3A}-\frac{B^2}{9A^2}\right)^3}}-\frac{B}{3A}$$

In case an approximate solution is needed, we let

$f(x) = Ax^3 + Bx^2 + Cx + D$

and apply Newton-Raphson recursion formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

that is, we iterate

$$x_{n+1} = \frac{2Ax_n^3 + Bx_n^2 - D}{3Ax_n^2 + 2Bx_n + C}$$

until $|x_{n+1} - x_n| < \epsilon$, where ϵ is a given tolerance.

In this paper, we are interested in one particular stage in the long journey that started with some cuneiform tablets and ended up with not only the general solution techniques, but also the invention of complex numbers: specifically, the important step of classification of the cubic equations.

3. A Brief History

Cubic equations were known to the ancient Babylonian, Greek, Chinese, Indian, and Egyptian mathematicians (Van der Waerden, 1983). Indeed, Babylonians had extensive tables for calculating cubes and cube roots (Cooke, 2012).

The Greek involvement with the cubic equations grew out of their desire to solve the problems of doubling of the cube and trisection of an angle (Kline, 1990).

Archimedes (c. 287 BCE-c. 212 BCE) in his ***On the Sphere and Cylinder II***, dealt with the doubling of the cube problem and reduced the problem to the solution of some cubic equations of the type

$$x^3 = a^2b$$

and

$$x^2(a - x) = bc^2$$

(Katz, 2004).

It was also shown that the angle trisection problem could be written as

$$\cos(\theta) = 4\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)$$

or equivalently as

$$8x^3 - 6x - 1 = 0$$

another cubic equation.

In the 3rd century AD, Diophantus (200-284), in his famous book ***Arithmetica***, proposed the following interesting problem:

Find a right-angled triangle such that its area added to one of the perpendicular sides makes a square, while its perimeter is a cube.

It was shown by Diophantus that the solution of this problem required finding the solution of the equation

$$m^2 + 2m + 3 = m^3 - 3m^2 + 3m - 1$$

Diophantus found the solution to this equation to be $m = 4$ (Heath, 2009).

We also know that in the 7th century, the Chinese mathematician Xiaotong solved some cubic equations of the form

$$x^3 + px^2 + qx = N$$

(Van der Waerden, 1985).

However, it should be noted that all of the above solutions were based on *ad hoc* arguments and applied only to specific problems and specific equations. No attempts were made to obtain a general method for solving the cubic equation.

This all changed in the 11th century, when the Persian poet-mathematician-astronomer-philosopher, Omar Khayyam (1048-1131), made significant progress in the theory of cubic equations. First of all, he realized that a cubic equation could have more than one solution. Secondly, in his ***Fil Birâhini-al-el-mesâil-el-Cebr vel Mukabele***, he wrote a complete classification of cubic equations. Thirdly, in the same book, he proposed some general geometric solutions that involved intersecting certain conic sections (Katz, 2004).

Khayyam's general geometric solution of the cubic as well as his contributions to literature, philosophy, astronomy, geometry, and sciences will be analyzed in detail in another paper. Here, we are only interested in his classification of the cubic equation. For, it is this classification that provided the much needed background for the works of Fibonacci (1170-1250), [19], Scipione del Ferro (1465-1526), Niccolò Fontana Tartaglia (1500-1557), Gerolamo Cardano (1501-1576), François Viète (1540-1603) and René Descartes (1596-1650) (Note 1) *en route* to the complete solution of the cubic.

4. Omar Khayyam's Life and Work

Omar Khayyam (whose full name was Ebu Hafs Omar Ibn-i Ibrahim el Khayyami) was born in Nishapur in northeastern Iran in 1048.

Khayyam came from a middle-class family. The patronymic Khayyam (or Khayyami) meaning "tent maker" implies that at least one of his ancestors must have been a tent maker.

He was educated in Nishapur by some of the leading scholars and scientists of time such as Nasîr-ed-din Sheih Muhammad Mansur and Muvaffak-üd-din Abdullatif Ibn-ül Lübâd, and the renowned theologian Hâce Ali.

He died in the same city in 1131. For more information on his life and major works, please see Dilgan (1964).

Khayyam was introduced to Western world in the first half of nineteenth century by the translations of two of his major works. One was an 1851 translation of his work in algebra (***Fil Birâhini-al-el-mesâil-el-Cebr vel Mukabele***) by Franz Woepcke (1826-1864) as ***L'Algèbre d'Omar Al Khayyamî*** and the other was the 1859 translation of his poetry (Rubaiyat) by the well-known English poet and writer Edward FitzGerald (1809-1883) as ***The Rubaiyat of Omar Khayyam***. His philosophical thoughts have been compared to Epicure, Voltaire, and Schopenhauer (Dilgan, 1964).

Khayyam's ***Fil Birâhini-al-el-mesâil-el-Cebr vel Mukabele*** comprised five sections. The first section contained a preface and gave some basic definitions. In the second section, Khayyam discussed first and second degree equations and in the third, he talked about cubic equations. Fourth and fifth sections involved equations that contained powers of unknown in the denominator and some additional remarks on basic rules of algebra.

It must be mentioned that Khayyam, like his contemporaries, considered only equations whose coefficients and roots were positive numbers and if an equation did not have positive roots, he classified them as unsolvable.

It is interesting to note that for the first time, Khayyam established some relationships between roots and the coefficients of an equation. For example, in the equation

$$x^3 + r = sx^2$$

where $r, s > 0$, he showed that there would be two positive roots if $\sqrt[3]{r} \leq \frac{1}{2}s$ and no positive roots (i.e., in his terms, no solutions) if $\sqrt[3]{r} \geq s$.

In the case $\frac{1}{2}s < \sqrt[3]{r} < s$, he showed that the equation may or may not have positive roots. Indeed, some simple examples show this to be the case. For example, it can easily be verified that with $s = 2 \text{ and } r = 3$, the equation has no positive roots; with $s = 2 \text{ and } r = 1.2$, it has one positive root; and with $s = 2 \text{ and } r = 1.1$, it has two positive roots.

5. Khayyam's Classification of the Cubic

In section three of ***Fil Birâhini-al-el-mesâil-el-Cebr vel Mukabele***, Khayyam divided cubic equations into four major groups.

The first group consisted 21 equations. Only two of those could not be solved by the geometric method Khayyam proposed and required other techniques. Thus, 19 could be solved by Khayyam's method.

The second group contained trinomial equations that had three successive powers (x^2, x, and constant term or x^3, x^2, and x). There were 15 such equations and these were all solvable by Khayyam's method.

The third group contained 24 equations that contained x^3, x^2, and x terms, but not necessarily successively—the only stipulation was that a cubic term had to be present. All equations in this group were solvable by Khayyam's method.

The fourth and the last group contained 28 equations with four terms containing x^3, x^2, and x in a successive manner.

Thus, Khayyam's method applied to

$$19+15+24+28=86$$

different type of cubic equations.

Below we give the members of these groups. Here, a, b, c are positive constants.

Group I

$a=x \quad a=x^2 \quad a=x^3 \quad bx=x^2 \quad bx=x^3 \quad cx^2=x^3$

$\frac{1}{x^3}=\frac{a}{x^2} \quad \frac{1}{x^2}=\frac{a}{x} \quad \frac{1}{x}=a$

$\frac{1}{x^3}=\frac{a}{x} \quad \frac{1}{x^2}=a \quad \frac{1}{x}=ax$

$\frac{1}{x^3}=a \quad \frac{1}{x^2}=ax \quad \frac{1}{x}=ax^2$

$\frac{1}{x^3}=ax \quad \frac{1}{x^2}=ax^2 \quad \frac{1}{x}=ax^3$

$\frac{1}{x^3}=ax^2 \quad \frac{1}{x^2}=ax^3$ (not solvable by Khayyam's method)

$\frac{1}{x^3}=ax^3$

Group II

$x^2+bx=a \quad x^2+a=bx \quad bx+a= x^2$

$x^3+cx^2=bx \quad x^3+bx= cx^2 \quad cx^2+bx= x^3$

$\frac{1}{x}+a=bx \quad \frac{1}{x}+bx=a \quad \frac{1}{x}=a+bx$

$\frac{1}{x^2}+\frac{a}{x}=b \quad \frac{1}{x^2}+b=\frac{a}{x} \quad \frac{1}{x^2}=\frac{a}{x}+b$

$\frac{1}{x^3}+\frac{a}{x^2}=\frac{b}{x} \quad \frac{1}{x^3}+\frac{b}{x}=\frac{a}{x^2} \quad \frac{1}{x^3}=\frac{a}{x^2}+\frac{b}{x}$

Group III

$x^3 + bx = a \quad x^3 + x = bx \quad bx + a = x^3$

$x^3 + cx^2 = a \quad x^3 + a = cx^2 \quad cx^2 + a = x^3$

$\frac{1}{x^3} + \frac{a}{x} = b \quad \frac{1}{x^3} + b = \frac{a}{x} \quad \frac{1}{x^3} = \frac{a}{x} + b$

$\frac{1}{x^2} + a = bx \quad \frac{1}{x^2} + bx = a \quad \frac{1}{x^2} = a + bx$

$\frac{1}{x} + ax = bx^2 \quad \frac{1}{x} + bx^2 = ax \quad \frac{1}{x} = ax + bx^2$

$\frac{1}{x^3} + \frac{a}{x^2} = b \quad \frac{1}{x^3} + b = \frac{a}{x^2} \quad \frac{1}{x^3} = \frac{a}{x^2} + b$

$\frac{1}{x^2} + \frac{a}{x} = bx \quad \frac{1}{x^2} + bx = \frac{a}{x} \quad \frac{1}{x^2} = \frac{a}{x} + bx$

$\frac{1}{x} + a = bx^2 \quad \frac{1}{x} + bx^2 = a \quad \frac{1}{x} = a + bx^2$

Group IV

$x^3 + cx^2 + bx = a \quad x^3 + cx^2 + a = bx$

$x^3 + bx + a = cx^2 \quad cx^2 + bx + a = x^3$

$x^3 + cx^2 = bx + a \quad x^3 + bx = cx^2 + a \quad x^3 + a = cx^2 + bx$

$\frac{1}{x^3} + \frac{a}{x^2} + \frac{b}{x} = c \quad \frac{1}{x^3} + \frac{a}{x^2} + c = \frac{b}{x}$

$\frac{1}{x^3} + \frac{b}{x} + c = \frac{a}{x^2} \quad \frac{1}{x^3} = \frac{a}{x^2} + \frac{b}{x} + c$

$$\frac{1}{x^3}+\frac{a}{x^2}=\frac{b}{x}+c \quad \frac{1}{x^3}+\frac{b}{x}=\frac{a}{x^2}+c \quad \frac{1}{x^3}+c=\frac{a}{x^2}+\frac{b}{x}$$

$$\frac{1}{x^2}+\frac{a}{x}+b=cx \quad \frac{1}{x^2}+\frac{a}{x}+cx=b \quad \frac{1}{x^2}+b+cx=\frac{a}{x}$$

$$\frac{1}{x^2}=\frac{a}{x}+b+cx \quad \frac{1}{x^2}+\frac{a}{x}=b+cx \quad \frac{1}{x^2}++b=\frac{a}{x}+cx$$

$$\frac{1}{x^2}+cx=\frac{a}{x}+b$$

$$\frac{1}{x}+a+bx=cx^2 \quad \frac{1}{x}+a+cx^2=bx \quad \frac{1}{x}+bx+cx^2=a$$

$$\frac{1}{x}=a+bx+cx^2 \quad \frac{1}{x}+a=bx+cx^2 \quad \frac{1}{x}+bx=a+cx^2$$

$$\frac{1}{x}+cx^2=a+bx$$

Let us give a specific example of Khayyam's method. As mentioned earlier, the general method will be analyzed in a different paper.

Say, for example, we want to solve the first equation of Group III

$$x^3+bx=a$$

Of course, since $a,b>0$, x cannot be zero.

Khayyam's solution consisted of constructing the parabola

$$x^2=\sqrt{b}y$$

and the circle with center at $\left(\frac{a}{2b},0\right)$ and radius $\frac{a}{2b}$ and determine the x-coordinate of the intersction.

For, if we do that, we get

$$x^2=\sqrt{b}y$$

$$\left(x-\frac{a}{2b}\right)^2+y^2=\frac{a^2}{4b^2}$$

The second equation becomes

$$bx^2-ax+by^2=0$$

that is,

$bx^2 - ax + x^4 = 0$

Since $x \neq 0$, this gives us

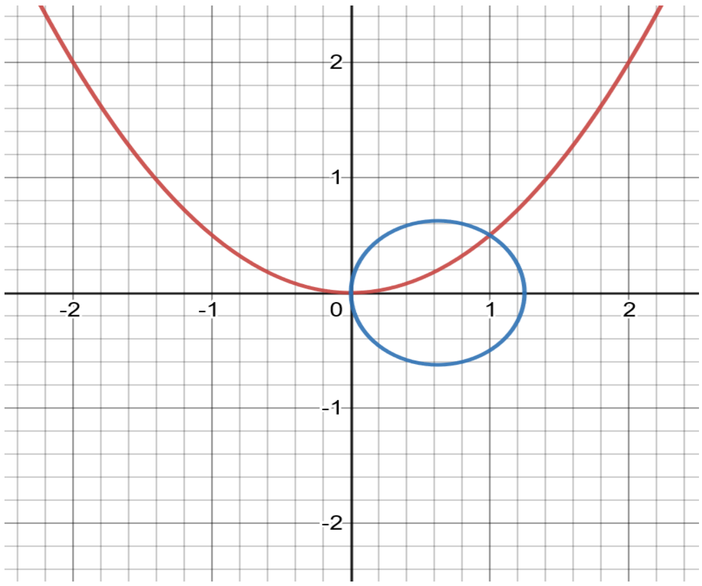
$x^3 + bx = a$

The root found by this method is the real and positive root since the length of a line segment cannot be negative or imaginary.

For example, to solve

$x^3 + 4x = 5$

we graph the parabola $x^2 = 2y$ and the circle $4x^2 - 5x + 4y^2 = 0$



and note that the x-coordinate of intersection of the two curves for $x > 0$ is at $x = 1$, which of course is the only real positive solution of the equation.

6. Conclusion

Were one to choose to adhere to the instrumentalist or mechanical view of mathematical knowledge and instruction, one would then equate computational proficiency with mathematical understanding. In other words, in this particular case, one would just give Cardano's formula (with or without proof) and use it to solve a few cubic equations. This, of course, would deny the students the true mathematical understanding of the problem: all of a sudden, somehow, a person discovered a formula, and all one has to do now is to substitute the right numbers in the right places, and then rely on the calculator. The instructor, with this approach, clearly would fail to provide the much needed insight to and motivation for the topic that is being covered.

In contrast, the historical approach, places problem solving at the heart of instruction. For instance, in our specific example, the students would realize that the solution of the cubic equation was developed over time and through the efforts of several scholars of different ethnic and cultural backgrounds. The reasoning behind the classification, the gist of our paper, would pave the path to students' ability to indulge in the study of a class of objects rather than a specific object. By taking a historical approach to the subject, students would learn that the classification of the cubic was indeed a mathematical necessity on the way to the discovery of the general solution.

As an added benefit, an analysis of the historical development of the solution of the cubic shows the internationalist character of mathematics. As teachers, if we are, as we should be, concerned about cultural chauvinism and parochial nationalism, and try to instill an unbiased perspective of mathematics, the historical approach is the needed panacea to convince learners that transnational collaboration, universal solidarity, and fellowship of scholars have always been the epitomes of mathematical development.

It also shows the interconnectedness of mathematical areas (geometry, algebra, etc.). Moreover, it proves that even masters were prone to erroneous thinking (denial of negative solutions) and hence shows the dangers of taking mathematics as a set of absolute truths as opposed to a set of fallible and transmutable ones.

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Note

Note 1. In his paper, *Réflexions sur la résolution algébrique des équations*, Joseph Louis Lagrange introduced a new method to solve equations of low degree in a uniform way, with the hope that he could generalize it for higher degrees. Of course, we now know of the non-existence of an algebraic formula for degrees 5 and higher (Abel's Theorem).