3

Expanding student perception of linear algebra

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Introduction

This report concerns itself with a project related to the course *Linear algebra and classical mechanics* ("Linear algebra and classical mechanics - Course website", 2020) (referred to as *MatN* from this point). The course is a mandatory first year course for students in the nanoscience study programme at the University of Copenhagen located in the block four before the summer break; as shown in Figure 3.1. The main purpose of the course is to supply the students with the tools and mathematical understanding needed for the subsequent courses in quantum mechanics and statistical physics in the second year of the study programme. The course consists of two separate parts: linear algebra and classical mechanics. This project concerns itself with the former.

A recurring problem in the course - or perhaps rather in the following courses relying on the skills and materials taught in this course - is the students' lack of ability to deploy the concepts taught in this course in the context of other courses and contexts. To use the SOLO taxonomy (J. B. Biggs & Collis, 1982): most students are not elevated from a unistructural understanding of the material to a multistructural understanding. This is problematic, as an abstract, multistructural understanding of certain central topics in linear algebra is a prerequisite for the subsequent courses in quantum mechanics.

	Blok 1	Blok 2	Blok 3	Blok 4
1. år	Nano 1 - introduktion til nanovidenskab	Introduktion til matematik for de kemiske fag	Sandsynligheds-regning, dataanalyse og indledende ellære	Elektromagnetisme og elektronik
	Organisk kemi i naturvidenskab		Almen og uorganisk kemi	Lineær algebra og klassisk mekanik
2. år	Nanotermodynamik	Nanokvant	Kvantefænomener i nanosystemer	Molekylær statistik
	Nanobio 1	Nanobio 2		
3. år			Nanobio 3	
			Bachelorprojekt	

Fig. 3.1. The 2020 course plan for the B.Sc. programme in nanoscience at University of Copenhagen (“The nanoscience study programme at University of Copenhagen”, 2020). MatN is highlighted in dark grey, whereas white and grey spaces are elective and restricted, elective courses, respectively.

Among other initiatives taken this year, the 2020 edition of the course featured a new teaching method: so-called reflection exercises. Simply put, the idea was to expose the students to problems of a more theoretical and abstract nature rather than the more calculation-intense and algorithmic problems usually found in first-year mathematics courses; in particular courses in linear algebra. Or perhaps rather problems demanding a more lateral and intuitive understanding and approach to the topics in linear algebra.

As a second motivation, the students generally struggle appreciating the objective relevance of the material, as it is presented to them. On several occasions, I have had students asking me: *“Why do we have to learn [matrix inversion/eigensystems/change-of-basis]?*” The relevance of the various mathematical topics in the course will definitely become evident for them during later courses, and I do attempt to expose them to illustrative examples of scientific applications of linear algebra. By stressing the importance of intuition in linear algebra, my hope was that the general applicability of the material, algorithms, and methodology covered in the course might be more appreciated by the students.

The 2020 edition of the course was taught entirely on-line due to the Covid-19 outbreak; which greatly influenced the course planning, teaching methods, and overall student experience; as well as the execution and expected results of the planned changes and their impact. For this project, it is particularly important to note that all classroom sessions were conducted using the on-line meeting tool Zoom (Zoom Video Communications, 2020).

Format and Intentions

The exercises were designed and structured around the TDS model (Brousseau, 1996) as described below:

1. Devolution: The exercises were introduced by a slide shown in class. An example of these slides is shown in Figure 3.2; the full set of exercise problems can be found in the Appendix.

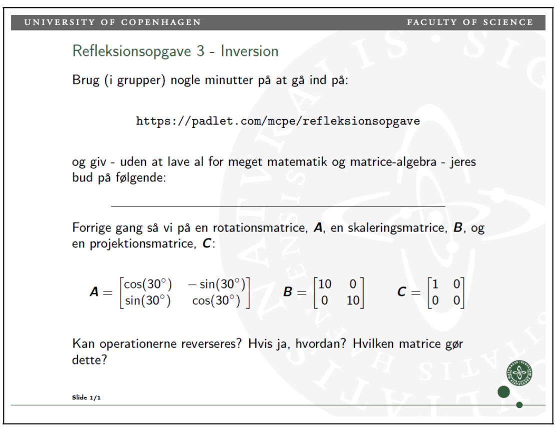
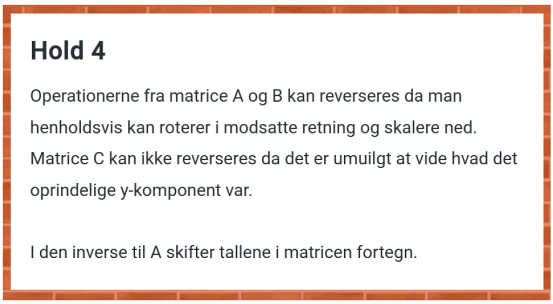


Fig. 3.2. The third reflection exercise which focuses on the intuition behind matrix inversion. The students are asked to assess, whether or not the presented matrix operations, A, B, and C can be inverted. Preferably by using intuition rather than relying on calculations.

2. Activation: The students were separated into groups of 3 to 5 using the break-out room function in Zoom.
3. Formulation: Each group prepared their arguments and formulated solutions to the presented problem in their respective break-out room.

4. Validation: These arguments and solutions were on a shared, public Padlet website (Wallwisher Inc., 2020). Students were encouraged to read the other replies and consider their soundness and supporting argumentation. An example of a reply is shown in Figure 3.3.



Hold 4

Operationerne fra matrice A og B kan reverseres da man henholdsvis kan roterer i modsatte retning og skalere ned. Matrice C kan ikke reverseres da det er umuilgt at vide hvad det oprindelige y-komponent var.

I den inverse til A skifter tallene i matricen fortegn.

Fig. 3.3. Screenshot of a student group reply from Padlet.

5. Institutionalisation: After the exercise, the students were called back to the main Zoom classroom, and the various arguments and solutions were outlined, discussed, and assessed. As shown in Figure 3.4, this was done in an attempted mimicking of traditional "blackboard" teaching.

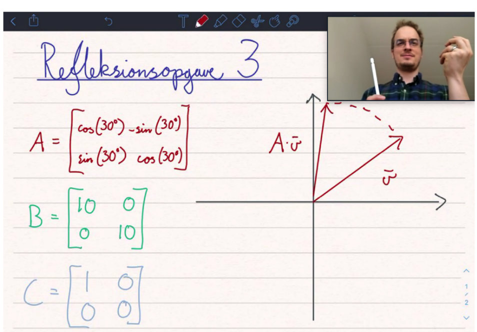


Fig. 3.4. Screenshot of the institutionalisation part of the exercise - using Zoom, an iPad, and the app Notability.

Each classroom session consisted of 2 subsessions of 45 minutes with a 15 minute break in-between (small 5 minute breaks were also held during the 45 minutes). The exercises were presented before the 15 minute break and the institutionalisation phase began 5 to 10 minutes after the break.

Usually, the students stayed in their group rooms for the entire duration of the extended break; though often not discussing the presented problem for the full 25 minutes.

The primary concerns and hopes when designing and formulating the exercises were i) not simply regurgitating traditional calculation-intense linear algebra assignments; the students were supposed to be trained in these during problem solving sessions after the class, ii) encouraging and training students in the habit of presenting solutions and mathematical arguments in concise and understandable writing, and iii) utilize elements from cooperative learning (Johnson & Johnson, 1975) in a class composed of students displaying a considerable span of skills and abilities in mathematics. Specifically, my hope was that the weaker students might benefit from discussion-style exercises in randomized groups.

Note that in some of the exercises, the students were explicitly asked not to do any computations in an attempt to solicit answers based on intuition. At times it felt necessary to emphasize this aspect of the exercise - as well as to remind the students of the intended learning outcomes.

Outcomes and Reception

At the end of the course, the students were asked to fill out an informal course evaluation via Socrative (Showbie Inc., 2020); in which they were asked to express their opinions on the various teaching methods and tools applied during the different parts of the course. A total of 8 questions were posed to the 23 attendees.

The responses to the question about the reflection exercises are plotted in Figure 3.5; and as shown the polled students were generally appreciative of the teaching activity. *En passant*, it is worth noting that this survey is likely biased towards students with positive opinions of the classroom sessions and teaching activities as they were more likely to be present during the survey. Conversely, students with negative opinions might be more likely to express them in this setting.

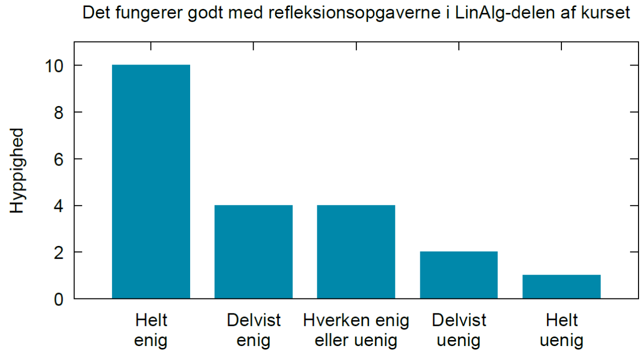


Fig. 3.5. Student responses to "*The reflection exercises in linear algebra work well*" in the unofficial course evaluation. The responses range from "*completely agree*" (Left, in Danish: "*Helt enig*") to "*completely disagree*" (Right, in Danish: "*Helt uenig*").

The students were reasonably active during the reflection exercises, and most groups usually made quite an effort as evidenced by the amount and overall quality of the postings on the associated Padlet website. The subsequent institutionalisation sessions were definitely the parts of the lecture sessions with the most questions and student engagement. Perhaps the students found it easier to engage in discussions or fomulate questions on problems, over which they had deliberated with their peers.

From the teacher's perspective, the exercises offered good opportunity to gauge the students' proficiency in and understanding of some of the more subtle points in basic linear algebra. As the exercises were placed in the middle of a classroom session, they allowed for correcting apparent misunderstandings on the spot and for further elaboration on essential points relevant to the exercises. In the context of the flow of the lecture sessions, they offered convenient segues from a longer break to institutionalisation and onwards to the traditional lecture format and the day's material to be covered.

The exercises offer some interesting opportunities for some just-in-time learning (Novak et al., 1999): one can design the exercise to emphasize exactly the point, theme, or algorithm needed for lecture session directly following the exercise. Or sometimes to highlight pitfalls in previously covered material. Again, this also adds to the flow of the different sessions and the overall congruence (Hounsell & Hounsell, 2007) of the different elements of the course.

A few students mentioned and appreciated the exercises in their official course evaluation:

"*It worked well being sent to break-out rooms to do a few exercises together...*"

Anonymous student, *MatN* 2020 Course evaluation

whereas others questioned the efficiency of the format:

"*... if you could not answer, you were just sort of sitting there.*"

Anonymous student, *MatN* 2020 Course evaluation

As the student implies, if one's randomly assigned group does not make progress with (or even manage to properly approach) the assigned problem, the exercise is lengthy and feels somewhat pointless and unrewarding. And while the aforementioned quality of the responses was decent, each group did not always produce an answer on Padlet; supporting the issue raised by the student. A second iteration of these exercise should facilitate a system of hints or additional guidance for groups that are "stuck".

Anectodally, during the classes, some students mentioned during the breaks that they appreciated the "forced" interactions with the other students, as student life could get slightly lonesome during the Covid-19 quarantine. One student even brought it up in the official course evaluation:

"*... made the class more interactive, which is nice at a time where one is always just sitting at home.*"

Anonymous student, *MatN* 2020 Course evaluation

Perhaps, this was the greater success and benefit of adding this this type exercises to this year's edition of the course. And perhaps the social aspect of this exercise does to some extent represent the manner, in which the students will need their mathematical skills in future research projects: in discussions with other students and/or supervisors.

Discussion

While the first iteration of this type of exercise was somewhat successful, there are a few changes to be made, should they become an integrated, standardized part of the course.

First of all, the issue raised by the student with the exercise being wasted time is concerning and must be remedied. Though I believe this year's online format exacerbated the issue; in a traditional classroom setting, the students would likely have had an easier time consulting the teacher (or other groups of students). That being said, there should - or even must - be a clear route to take for student groups struggling with “getting into” the problem at hand.

Secondly, as written, student responses were collected using Padlet. After a couple of sessions, it became evident that Padlet is somewhat of a double-edged sword when used for student responses as it is here. Padlet is practical, accessible, and easily archived for later reviewing. But due to the nature of mathematics; replies are usually easily and unambiguously labelled as right or wrong. And hence, writing a potentially wrong answer to a presented problem publicly for everyone to see likely constricted some groups of weaker students from presenting their thoughts and considerations.

I believe it is worth reconsidering, whether or not Padlet should really be the weapon of choice for these assignments. That being said, it would have been interesting to ask the groups for a more rigorous evaluation of the other groups’ responses and point out flaws in argumentation, counter-examples etc. to emphasize the validation stage of the exercise a bit more; and to perhaps train and test the students in reading and expressing themselves in terms of mathematical rigour.

Furthermore, I am left feeling that there is room for improvement in the institutionalisation part of the exercise as well. Ultimately, the responses provided by the students play too small a role in the section of the exercise - the good responses are quickly reviewed; the mistakes in the less good ones are very briefly visited, but in the end the right answer and arguments are eventually presented regardless of the student responses. One could do another iteration of the breakout room sessions on the same topic as suggested by e.g. Mazur (Mazur, 1997). Though in the interest of time, I have a hard time envisioning a better - or more efficient, I suppose - structure.

Another counterpoint to making this type of exercises a recurring teaching activity in this course is worth noting: the reflection exercises are not very well-aligned (J. Biggs, 2003) or perhaps rather incongruent (Hounsell & Hounsell, 2007) with the exam format in this course, which is a classical written exam based on standard exercises in linear algebra. The skills trained in these exercises are not always directly applicable in the following exam.

In hindsight, there is plenty of room for improvement in the problem descriptions (i.e. the slides in the appendix). Some of the problems need to be reworked as students sometimes got stuck on misperceptions of the problem at hand, or sometimes even misunderstanding the problem entirely (and thus providing a rather convoluted and mysterious answer on Padlet).

As mentioned in the introduction, at times the students question the importance of linear algebra in the nanoscience. At the end of the course, the students were polled on their view on several aspects of the course; among others the pertinence of a linear algebra course in their study programme. As shown in Figure 3.6, the students replied that the necessity of linear algebra in their study programme is quite apparent to them.

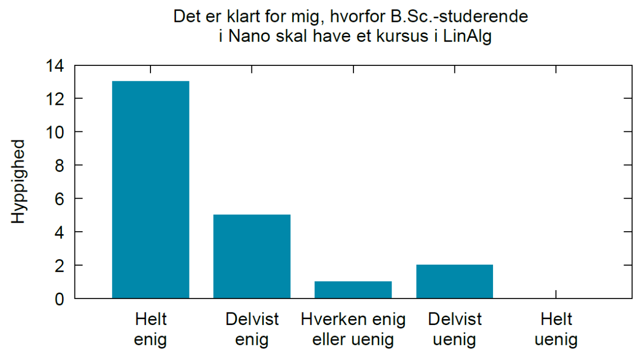


Fig. 3.6. Student responses to "*It is clear to me, why a course in linear algebra is mandatory for nanoscience students*" in the unofficial course evaluation. The response range is identical to the one in Figure 3.5.

It would have been intersting to compare these answers to similar surveys from previous editions of the course. Unfortunately, these data do not exist. But it is my hope - and belief - that emphasizing the intuitive aspect of linear algebra this year might have contributed to this.

Regardless, I believe future editions of the course should continue stressing the general applicability of linear algebra along with the importance of a multistructural understanding of the field.

As a final observation, I believe these exercises would scale reasonably well in courses with considerably more students. Though it would be even harder to involve the individual student responses into the institutionalisation phase.

Conclusions and Outlook

The addition of the reflection exercises was reasonably succesful; possibly moreso due to the restrictions placed on the course by the Covid-19 quarantine measures. The student responses to the two presented surveys accompanied by my own impressions and the students' general commitment during the exercises seem to support this conclusion.

Thus, I have every intention of keeping the reflection exercises around for the 2021 edition of the course in one form or another; though as discussed there is plenty of room for improvements in the various aspects of the exercises. The time-wise efficiency of this style of exercises needs to be (re-)evaluated, as they are very consuming and might not benefit the weaker students nearly as much as initially envisioned. It will be interesting to see if the format works equally well off-line. Obviously, the break-out room functionality will have to be replaced; and perhaps an off-line implementation of this style of exercises should look to a medium other than Padlet as communication/reporting tool.

Similarly, the actual problems posed to the students could likely use a rework in light of the lessons learned during the first exposition. Gathering more data on recurring mistakes and misconceptions should aid in this process as well; needless to say, this basis for an element of this type in a course should be an ever-evolving product.

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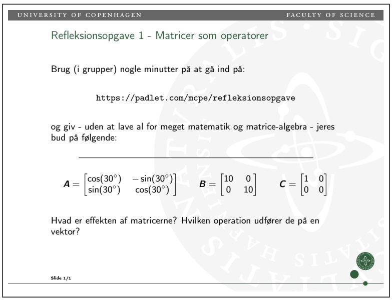
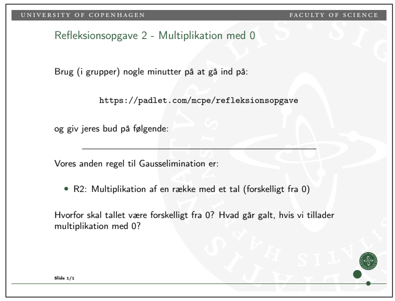
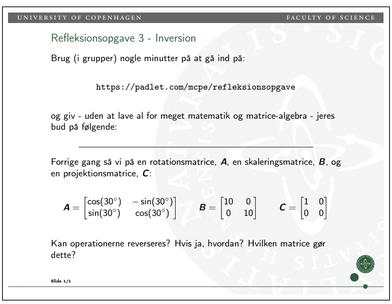
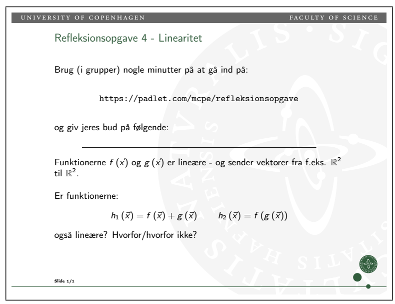
A Slides with reflection exercise problems

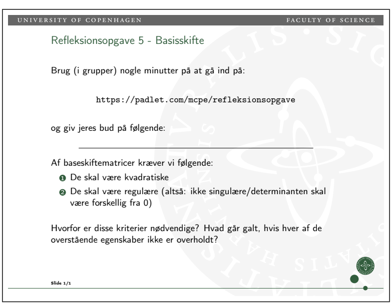
All 10 slides introducing the students to the various exercises can be found below.

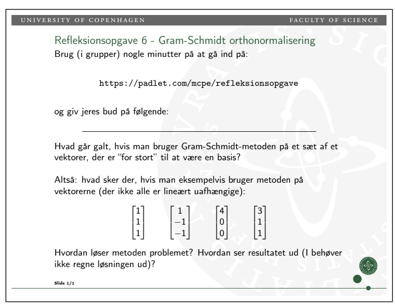












Refleksionsopgave 7 - Egenvektorer og egenværdier

Brug (i grupper) nogle minutter på at gå ind på:

`https://padlet.com/mcpe/refleksionsopgave`

og giv - uden at lave al for meget matematik og matrice-algebra - jeres bud på følgende:

Tidligere i kurset har vi haft kig på rotationsmatrice, $\boldsymbol{A}$, en skaleringsmatrice, $\boldsymbol{B}$, og en projektionsmatrice, $\boldsymbol{C}$:

$$\boldsymbol{A} = \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Hvad er egenvektorerne til de forskellige matricer? Og hvad er de tilhørende egenværdier?

Refleksionsopgave 8 - Egenrum

Brug (i grupper) nogle minutter på at gå ind på:

`https://padlet.com/mcpe/refleksionsopgave`

og giv jeres bud på følgende:

For at finde egenvektorerne til en 2×2-matrice, $\boldsymbol{A}$, tilknyttet en egenværdi, λ, løser vi ligningssystemet:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\,\vec{v} = \vec{0} \quad \rightarrow \quad \left[\begin{array}{cc|c} A_{11} - \lambda & A_{12} & 0 \\ A_{21} & A_{22} - \lambda & 0 \end{array}\right]$$

Hvad gør vi, hvis ligningssystemet kun har en løsning og ikke giver os en familie af egenvektorer? Kan dette ske? Hvorfor/hvorfor ikke?

Refleksionsopgave 9 - Diagonalisering

Brug (i grupper) nogle minutter på at gå ind på:

`https://padlet.com/mcpe/refleksionsopgave`

og giv jeres bud på følgende:

Matricer kan kun være similære med andre matricer med samme determinant. Altså:

$$\det(\boldsymbol{A}) = \det(\boldsymbol{B})$$

hvis $\boldsymbol{A}$ og $\boldsymbol{B}$ er similære.

Kan I komme med:

- Et algebraisk argument for, hvorfor det skal være sådan (hint: tænk på determinanter og baseskift)
- Et geometrisk argument for, hvorfor det skal være sådan (hint: tænk på hvordan $\boldsymbol{A}$ og $\boldsymbol{B}$ transformerer $\hat{i}$ og $\hat{j}$)

Refleksionsopgave 10 - Differentialligninger

Brug (i grupper) nogle minutter på at gå ind på:

`https://padlet.com/mcpe/refleksionsopgave`

og giv jeres bud på følgende:

Når man beskriver kemiske reaktioner eller ligevægte med et system af differentialligninger som dette:

$$\frac{d\vec{x}}{dt} = \boldsymbol{A}\vec{x}$$

finder man som regel kun egenværdier lig med eller mindre end nul.

Hvorfor? Hvad sker der, hvis egenværdierne er positive, og hvad er den fysiske/kemiske fortolkning af dette?