Mathematical Modelling of Drug Abuse Reduction Strategies taking into account the Treatment Type and Risks Level

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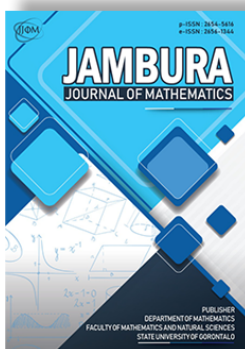
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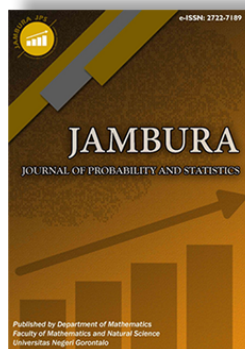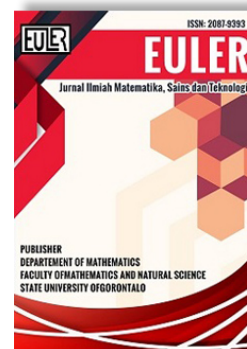
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ABSTRACT. Drug abuse is one of the global issues and has spread among teenagers. Drugs may lead to subordination, health problems and even death. There are several policies made in each country related to the problem of drug abuse, both punishment and treatment. In this paper, we discuss the treatment and strategy to reduce the number of drug users. Drug users can recover themselves by undergoing rehabilitation in the form of inpatient or outpatient care. We first conduct qualitative analyses including stability analysis of equilibrium points of the model, the basic reproduction number and parameter sensitivity analysis. Mathematical model of drug abuse reduction by concerning type of treatment along with risk level without control has two equilibrium points, namely non-endemic or drug-free equilibrium and endemic equilibrium. Sensitivity analysis is provided to investigate which parameter that most affects the dynamical behaviour of the drug abuse model in terms of stability of the non-endemic and endemic equilibrium point. Then we impose an anti-drug campaign on the model as strategy control to reduce the number of drug abusers. Simulation results show that the anti-drug campaign has a significant effect in reducing both the number of drug abusers who received any treatment and do not get any treatment.

1. Introduction

Juvenile delinquency is a problem of society. Juvenile delinquency always leads to criminal acts so that people become anxious. A number of studies demonstrated that violent crime is one cause of delinquency [1]. One form of juvenile delinquency is drug abuse [2, 3]. Drug abuse has become popular among teenagers even adults [1, 4]. John et al.[5] demonstrated a compartmental model for general substance abuse. They showed both drug kings and person-to-person contact have an important role in the prevalence of substance abuse. Drug abuse is a crucial global problem. The development of technology and information in this globalization era make drug distribution throughout the world become faster [4]. Therefore, drug abuse is a problem that deserves the attention of the public.

Narcotics do not include hazardous substances when they are used in accordance with medical instructions. The medicines classified as ilicit drugs are originally used to treat diseases and relieve pain because they generally make a relaxed feeling [6]. Besides that drugs can cause dependency behaviour which leads to abuse. Drug dependence is considered a health disorder that is often followed by relapses and chronic diseases [7].

Drug use can cause several symptoms including dehydration, hallucinations, seizure, decreased levels of consciousness and impaired body health. Excessive use can be fatal, such as death. Apart from having a negative impact on health, drug abuse might also cause crime [8]. The legal form for drug abusers varies around the world. Criminal punishment is an effort to reduce the

spread of drug abuse [7]. It can be seen that law enforcement is able to reduce the supply of drugs so reducing access for potential users [5]. Drug report 2021 [9] presents that the population growth projection for 2030 translates into a potential increase of 11 per cent in the global population who use drugs, with a much greater impact in low-income than in high-income countries.

The mathematical model plays an important role in understanding the behaviour of epidemics and biology. There are several mathematicians who have developed mathematical models related to the general drugs [10, 11] or mathematical models of the spread of drug abuse [12–17]. It has identical characteristics with conventional epidemics. Bhunu et al. [12] demonstrated the relation between homelessness and drug abuse. Mushanyu et al. [13] constructed and analysed mathematical models of the spread of drug abuse by paying attention to the limitations of rehabilitation capacity. Mushanyu et al. [14] constructed and analysed models of methamphetamine abuse by dividing the type of rehabilitation treatment into two, namely outpatient and inpatient care. Liu et al. [16] formulated the SQIR (Susceptible- Susceptible with history of drug abuse-Users not in treatment-Treatment) model to explain the transmission of the spread of synthetic drugs with regard to relapse and treatment. Furthermore, Mushanyu and Nyabadza [17] formulated the SUTR (Susceptible-Users-Treatment-Recovered) model to explain the transmission of drug abuse. The Susceptible and Recovered populations are divided according to the risk level, namely high risk and low risk of drug abuse. The spread of drug abuse occurs when interacting with drug abusers who are not in a rehabilitation center.

Table 1. Parameters. Description of parameters in the model of drug abuse reduction.

Parameter	Description
Λ	The rate of recruitment
p	The proportion of individuals who entered the high risk susceptible
q	The proportion of drug abusers who are hospitalized in a rehabilitation center
β_1	The rate of transmission due to interaction between susceptible individuals and drug abusers
β_2	The rate of transmission due to interaction between recovery individuals and drug abusers
β_3	The rate of transmission due to interaction between outpatients and drug abusers
η	Parameter modification to minimize the chances of vulnerable individuals with low risk becoming drug abusers
ω_1	The rate of transition from high-risk susceptibility to low-risk susceptibility
ω_2	The rate of transition from low-risk susceptibility to high-risk susceptibility
α_1	The rate of transition from abusers who get outpatient care treatment to abusers who get inpatient care treatment
α_2	The rate of transition from abusers who get inpatient care treatment to abusers who get outpatient care treatment
ρ	The natural recovery rate
γ_1	Recovery rate of drug abusers under treatment into outpatient care
γ_2	Recovery rate of drug abusers under treatment into inpatient care
σ	The rate of transition from drug users become rehabilitated patients
δ	The death rate of drug abusers
μ	The natural death rate

According to a journal written by Mushanyu and Nyabadza [17], they do not pay attention to outpatient care so the spread of drugs caused by outpatient abusers who are not in the rehabilitation center is ignored. In addition, there is no optimal control in the journal to control the spread of drug abuse. This paper presents modification of the model by considering outpatient care as one kind of treatment. In addition we add optimal control in the form of anti-drug campaign efforts on a model that has been modified.

The paper is organized as follows: the formulation and stability analysis of the drug abuse model is presented in Section 2. In Section 3, We provide parameter sensitivity analysis to examine which parameter has significant effect on the system. Next, in Section 4 we employ anti-drug campaigns as control variables. We then conduct a numerical exploration of the drug abuse model with control in Section 5. We conclude by discussing our findings and suggesting future work in Section 6.

2. Formulation of Drug Abuse Model

In this section, a mathematical model of drug abuse by taking into account the type of treatment along with the level of risk is formulated. The assumptions used for the model construction are as follows:

1. Drug abusers who have recovered may relapse.
2. Drug abusers who receive treatment as an outpatient can recur when interacting with abusers who do not receive treatment.
3. Only drug abusers who do not get treatment that may be distributing drugs.
4. Abusers who receive inpatient treatment form cannot interact with drug abusers who do not receive treatment.
5. Human population becomes vulnerable when it reaches the age of 15 years.
6. Vulnerable individuals with low risk have less chance to become drug abusers than susceptible individuals with a high risk.

The human population is divided into six compartments of human populations which are the population at high risk of initiating drug abuse (S_H), the population at low risk of initiating drug abuse (S_L), the population of drug abusers who do not receive treatment (I), the population of drug abusers who receive treatment as an outpatient care (T_j), the population of drug abusers who received treatment in the form of hospitalization (T_r), and human populations are recovering from drug abuse (R). We divide susceptible populations into low and high risk susceptible populations. Risk structure is very important in this behaviour. Many factors affect the behaviour of society. It depends on community environment, life principle, ambitions, supporting system in their life. Defining parameters can be seen in Table 1.

Based on the assumptions, we can set the transmission diagram that is shown in Figure 1. From the diagram in Figure 1 transmission models can be formulated as follows:

$$\frac{dS_H}{dt} = p\Lambda - \frac{(\beta_1 I S_H)}{N} - (\mu + \omega_1) S_H + \omega_2 S_L, \tag{1}$$

$$\frac{dS_L}{dt} = (1-p)\Lambda - \frac{\eta\beta_1 I S_L}{N} - (\mu + \omega_2) S_L + \omega_1 S_H, \tag{2}$$

$$\frac{dI}{dt} = \frac{\beta_1 I(S_H + \eta S_L)}{N} + \frac{\beta_2 I R}{N} + \frac{\beta_3 I T_j}{N} - (\mu + \sigma + \delta + \rho) I, \tag{3}$$

$$\frac{dT_j}{dt} = (1-q)\sigma I - \frac{\beta_3 I T_j}{N} - (\mu + \alpha_1 + \gamma_1) T_j + \alpha_2 T_r, \tag{4}$$

$$\frac{dT_r}{dt} = q\sigma I - (\mu + \alpha_2 + \gamma_2) T_r + \alpha_1 T_j, \tag{5}$$

$$\frac{dR}{dt} = \gamma_1 T_j + \gamma_2 T_r + \rho I - \frac{\beta_2 I R}{N} - \mu R. \tag{6}$$

Furthermore, for reason of simplicity, $S_H(t)$, $S_L(t)$, $I(t)$, $T_j(t)$, $T_r(t)$, $R(t)$ written into S_H, S_L, I, T_j, T_r, R with $\{S_H, S_L, I, T_j, T_r, R\} \geq 0$. Then defined $N(t)$ as total population at t, with $N = S_H + S_L + I + T_j + T_r + R \geq 0$. Afterwards, all parameters that have been defined are positive, with $\Lambda > 0$ and $0 < \eta, \omega_1, \omega_2, \alpha_1, \alpha_2, \rho, \gamma_1, \gamma_2, \sigma, \delta, \mu < 1$.

The drug-free equilibrium of drugs abuse is a condition where no spread of drugs. This equilibrium attainable when no human who become drug abuser $(I = 0)$. In addition, due to the absence of the population of drug abusers, there are also no human being receiving treatment as an outpatient and inpatient care $T_r = 0, T_j = 0$. Thus a drug-free equilibrium is obtained by
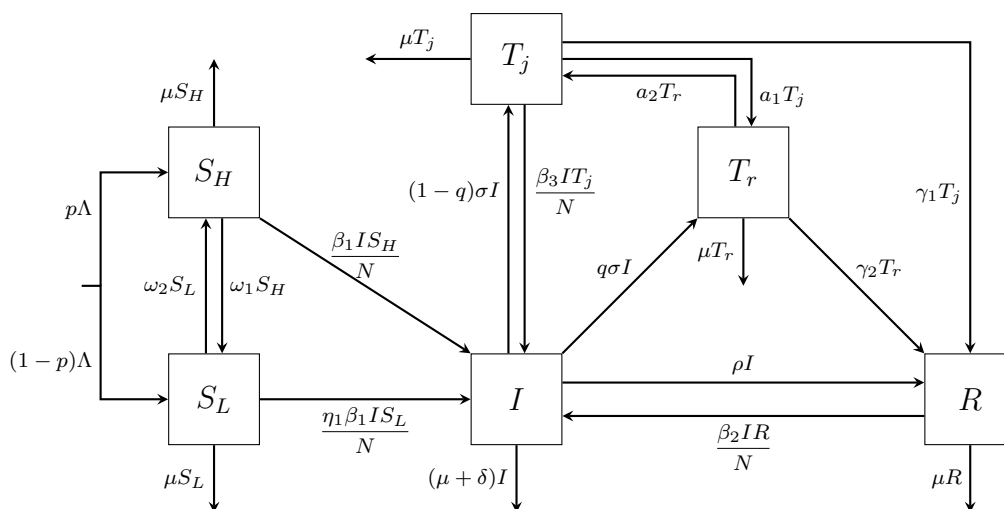
Figure 1. Transmission Diagram. Mathematical model diagram of drug abuse by taking into account the type of treatment along with risks level.

$$E_0 = \left(S_H{}^0, S_L{}^0, I^0, T_j{}^0, T_r{}^0, R^0\right)$$
$$= \left(\frac{\Lambda\left(\mu p + \omega_2\right)}{\mu\left(\mu + \omega_1 + \omega_2\right)}, \frac{\Lambda\left(\mu\left(1-p\right) + \omega_1\right)}{\mu\left(\mu + \omega_1 + \omega_2\right)}, 0, 0, 0, 0\right).$$

Then the basic reproduction number (R_0) will be determined, which represents the expectation of the average number of new drug abusers due to contacted between drug abuser patients and vulnerable individuals. In this study, we applied the Next Generation Matrix (NGM) method to get R_0 which has been developed by [18] to obtain

$$R_0 = \frac{\beta_1\left(\mu p + \omega_2 + \eta\left(\mu\left(1-p\right) + \omega_1\right)\right)}{\left(\mu + \sigma + \rho\right)\left(\mu + \omega_1 + \omega_2\right)}.$$

The drug-free equilibrium E^0 will be locally asymptotically stable if $R_0 < 1$ and will be unstable when $R_0 > 1$.

The endemic equilibrium is the condition that there is a drug abuser patient, as well as the spread of that behaviour. Endemic equilibrium $E^* = (S_H{}^*, S_L{}^*, I^*, T_j{}^*, T_r{}^*, R^*)$ is obtained when $S_H \neq 0, S_L \neq 0, I \neq 0, T_j \neq 0, T_r \neq 0, R \neq 0$

Setting the right-hand sides of the model (1)-(6) equals to zero. The endemic equilibrium point of the model is

$$S_H^* = \frac{\Lambda^2(\mu p(\eta\beta_1 I^* + \Lambda) + \Lambda\omega_2)}{\mu((\eta\beta_1 I^* + \Lambda)(\mu\beta_1 I^* + \Lambda(\mu + \omega_1)) + \Lambda\omega_2(\beta_1 I^* + \Lambda))},$$

$$S_L^* = \frac{\Lambda^2(\mu(1-p)(\beta_1 I^* + \Lambda) + \omega_1\Lambda)}{\mu((\eta\beta_1 I^* + \Lambda)(\mu\beta_1 I^* + \Lambda(\mu + \omega_1)) + \Lambda\omega_2(\beta_1 I^* + \Lambda))},$$

$$T_j^* = \frac{\Lambda\sigma I^*((\mu + \gamma_2)(1-q) + \alpha_2)}{(\mu\beta_3 I^* + \Lambda(\mu + \gamma_1))(\mu + \alpha_2 + \gamma_2) + \Lambda\alpha_1(\mu + \gamma_2)},$$

$$T_r^* = \frac{\sigma I^*[q(\mu\beta_3 I^* + \Lambda(\mu + \gamma_1)) + \Lambda\alpha_1]}{(\mu\beta_3 I^* + \Lambda(\mu + \gamma_1))(\mu + \alpha_2 + \gamma_2) + \Lambda\alpha_1(\mu + \gamma_2)},$$

$$R^* = \frac{\Lambda(\gamma_1 T_j{}^*(\mu + \alpha_2 + \gamma_2) + \gamma_2(q\sigma I^* + \alpha_1 T_j{}^*) + \rho I^*(\mu + \alpha_2 + \gamma_2))}{\mu(\mu + \alpha_2 + \gamma_2)(\beta_2 I^* + \Lambda)}.$$

while I^* shows that are the roots of a degree four polynomial as follows:

$$B_1 I^{*4} + B_2 I^{*3} + B_3 I^{*2} + B_4 I^* + B_5 = 0, \qquad (7)$$

where

$$B_1 = -\Lambda\beta_1{}^2\beta_2\beta_3\eta n_3\mu^5\left(q\sigma + \mu + \alpha_2 + \gamma_2\right),$$

$$B_2 = \Lambda^2\sigma n_1\mu^3\beta_1\beta_3 n_3\left(\mu(\beta_2\eta + \beta_1\eta + \beta_2) + \beta_2(\omega_1\eta + \omega_2)\right)$$
$$+ \Lambda^2\beta_1{}^2\beta_2\mu^3\eta\left(\sigma n_1 n_2 + \beta_3 n_3\mu\right) + \Lambda^2\beta_1\beta_2\mu^3$$
$$\left(\gamma_2 q\sigma + n_3\left(\rho - s\right)\right)\left(\beta_1\eta(\alpha_1\gamma_2 + \alpha_1\mu + \gamma_1 n_3 + n_3\mu)\right.$$
$$\left.+ \beta_3 n_3(\eta\mu + \eta\omega_1 + \mu + \omega_2)\right) - \Lambda^2\beta_1{}^2\beta_3\eta n_3{}^2\mu^4 s,$$

$$B_3 = \Lambda^3\beta_1\mu^2\left(\sigma\beta_2 n_1 n_2 + \beta_3 n_3\mu\left(\sigma n_1 - n_3 s\right)\right)$$
$$\left(\eta\left(\mu + \omega_1\right) + \mu + \omega_2\right) + \Lambda^3\beta_2\mu^3\beta_3 n_3$$
$$\left(q\sigma\gamma_2 + n_3\left(\rho - s\right) + \sigma n_1\right)\left(\mu + \omega_1 + \omega_2\right)$$
$$+ \Lambda^3\beta_2\mu^2\left(q\sigma\gamma_2 + n_3\left(\rho - s\right)\right)\left(\alpha_1\beta_1\left(\eta\gamma_2\mu + \eta\gamma_2\omega_1\right.\right.$$
$$\left.\left. + \eta\mu^2 + \eta\mu\omega_1 + \gamma_2\mu + \gamma_2\omega_2 + \mu^2 + \mu\omega_2\right)\right)$$
$$+ \Lambda^3\beta_2\mu^2\left(q\sigma\gamma_2 + n_3\left(\rho - s\right)\right)\left(\beta_1 n_3\left(\eta\gamma_1\mu\right.\right.$$
$$\left.\left. + \eta\gamma_1\omega_1 + \eta\mu^2 + \eta\mu\omega_1 + \gamma_1\mu + \gamma_1\omega_2 + \mu^2 + \mu\omega_2\right)\right)$$
$$+ \Lambda^3\beta_1{}^2 n_3\mu^3\eta\left(\beta_2 - s\right)\left(\alpha_1\left(\gamma_2 + \mu\right) + n_3\left(\gamma_1 + \mu\right)\right)$$
$$+ n_3{}^2\beta_3\mu(\Lambda^3\beta_1{}^2\mu^3\eta + \beta_2 v),$$

$$B_4 = \mu^2\Lambda^4\sigma n_1\left(\beta_2 n_2 + \beta_3 n_3\mu\right)\left(\mu + \omega_1 + \omega_2\right)$$
$$+ \mu^2\Lambda^4\beta_2\left(\left(\gamma_2 q\sigma + n_3\left(\rho - s\right)\right)\left(\alpha_1\left(\gamma_2 + \mu\right)\right.\right.$$
$$\left.\left. + n_3\left(\gamma_1 + \mu\right)\right) - \beta_3 n_3{}^2\mu s\right)\left(\mu + \omega_1 + \omega_2\right)$$
$$+ \mu^2\Lambda n_3\left(\left(\Lambda^3\beta_1{}^2\mu\eta + \beta_2 v - \Lambda^3\beta_1 s\left(\mu + \omega_2\right.\right.\right.$$
$$\left.\left. + \eta\left(\mu + \omega_1\right)\right)\right)\left(\alpha_1\left(\gamma_2 + \mu\right) + n_3\left(\gamma_1 + \mu\right)\right)$$
$$\left. + \Lambda n_3{}^2\beta_3\mu^2 v\right),$$

$$B_5 = \Lambda^2 n_3\mu^2\left(\alpha_1\gamma_2 + \alpha_1\mu + \gamma_1 n_3 + n_3\mu\right)$$
$$\left(v - \Lambda^3 s\left(\mu + \omega_1 + \omega_2\right)\right),$$

$$n_1 = \left(\mu + \gamma_2\right)\left(1 - q\right) + \alpha_2,$$

$$n_2 = \gamma_1\mu + \gamma_1\alpha_2 + \gamma_1\gamma_2 + \alpha_1\gamma_2,$$

$$n_3 = \mu + \alpha_2 + \gamma_2,$$

$$v = \beta_1\Lambda^2\left(\Lambda\mu\left(p + \eta\left(1 - p\right)\right) + \Lambda\left(\eta\omega_1 + \omega_2\right)\right),$$

$$s = \mu + \sigma + \rho.$$

It should be noted that the coefficient B_1 in eq. (7) is always negative. Then, we can clearly note that $B_5 < 0 \Leftrightarrow R_0 < 1$ and

Table 2. Number of positive roots possibility

$B_1 < 0$								
$B_2 > 0$								
$B_3 > 0$				$B_3 < 0$				
$B_4 > 0$		$B_4 < 0$		$B_4 > 0$		$B_4 < 0$		
$B_5 > 0$	$B_5 < 0$	$B_5 > 0$	$B_5 < 0$	$B_5 > 0$	$B_5 < 0$	$B_5 > 0$	$B_5 < 0$	
I^*	1	2	3	2	3	4	3	2
$B_1 < 0$								
$B_2 < 0$								
$B_3 > 0$				$B_3 < 0$				
$B_4 > 0$		$B_4 < 0$		$B_4 > 0$		$B_4 < 0$		
$B_5 > 0$	$B_5 < 0$	$B_5 > 0$	$B_5 < 0$	$B_5 > 0$	$B_5 < 0$	$B_5 > 0$	$B_5 < 0$	
I^*	1	2	3	2	1	2	1	0

Table 3. Parameter Value of Mathematical Model of Drug Abuse by Concerning Type of Treatment along with Level Risks

Parameter	Value	Unit	Source
Λ	5000	person/year	Assumed
p	0.352	−	[17]
q	0.352	−	[13]
β_1	0.3	1/year	[17]
β_2	0.15	1/year	[17]
β_3	0.1	1/year	[13]
η	0.09	−	[17]
ω_1	0.2	1/year	[17]
ω_2	0.65	1/year	[17]
α_1	0.02961	1/year	[13]
α_2	0.003	1/year	[13]
μ	0.02	1/year	[17]
ρ	0.0082	1/year	[17]
γ_1	0.01	1/year	[13]
γ_2	0.3142	1/year	[13]
σ	0.02827	1/year	[17]

Table 4. Index of Parameter Sensitivity

Parameter	Sensitivity
β_1	1
σ	-0.50
μ	-0.37
ω_2	0.21
ω_1	-0.20
ρ	-0.15
η	0.03
p	0.01

$B_5 > 0 \Leftrightarrow R_0 > 1$ based on B_5 formula below:

$$v - \Lambda^3 s \left(\mu + \omega_1 + \omega_2\right) \qquad (8)$$

The eq. (8) can be written as follows

$$\Lambda^3 \beta_1 \ \left(\mu \left(p + \eta \left(1 - p\right)\right) + \eta \omega_1 + \omega_2\right)$$
$$- \Lambda^3 \left(\mu + \sigma + \rho\right) \left(\mu + \omega_1 + \omega_2\right)$$
$$\Leftrightarrow \Lambda^3 \left(\mu + \sigma + \rho\right) \left(\mu + \omega_1 + \omega_2\right)$$
$$\left(\frac{\beta_1 \left(\mu\left(p+\eta\left(1-p\right)\right)+\eta\omega_1+\omega_2\right)}{\left(\mu+\sigma+\rho\right)\left(\mu+\omega_1+\omega_2\right)} - 1\right)$$
$$\Leftrightarrow \Lambda^3 \left(\mu + \sigma + \rho\right) \left(\mu + \omega_1 + \omega_2\right) \left(R_0 - 1\right).$$

We now determine the number of possible positive real roots of the polynomial (7) using the Descartes Rule of Signs. The number of positive roots are at most four. The possibilities can be tabulated as shown in Table 2.

3. Parameter Sensitivity Analysis

We analyse parameter sensitivity to investigate the role of each parameter in terms of stability of the non-endemic and endemic equilibrium point through sensitivity index (e_m) of each parameter. The parameter values used to calculate the sensitivity index refer to the Table 3. Then, the parameter sensitivity index is formulated as follows and the calculation results can be seen in Table 4,

$$e_m = \left(\frac{\partial R_0}{\partial m}\right) \frac{m}{R_0},$$

m : Parameter to be analysed
e_m : Sensitivity index of parameter m.

Table 4 shows positive or negative value for sensitivity index. The positive value of sensitivity index indicates the R_0 value increases after the parameter value is increased. On the other hand, the negative value of sensitivity index indicates the R_0 value decreases after the parameter value is increased. For example, the sensitivity index of β_1 is 1 which means that if the transmission rate of drug abusers increased by 10%, the value of R_0 increases up to 10% and vice versa if the value of β_1 decreased by

10% then the value of R_0 will also decrease 10%. In similar way, if the value of ω_2 increased by 10%, the value of R_0 increases up to 2.1%. However, for the rate of drug abusers who become rehabilitated patients (σ) increased by 10%, then the value of R_0 goes down to 0.5%. The analysis also applies to other parameters.

Based on the explanation above, it can be concluded that the β_1 and σ parameters have an important role on the mathematical model of strategies of drug abuse reduction because the absolute value of the sensitivity index β_1 and σ are the biggest among the other parameters. The simulation result in the form of β_1 and σ toward R_0 can be seen in Figure 2. The values are $\sigma = 0,002827, 0,02827$, and $0,2827$, where β_1 is in the interval $0,1 \leq \beta_1 \leq 0,8$.
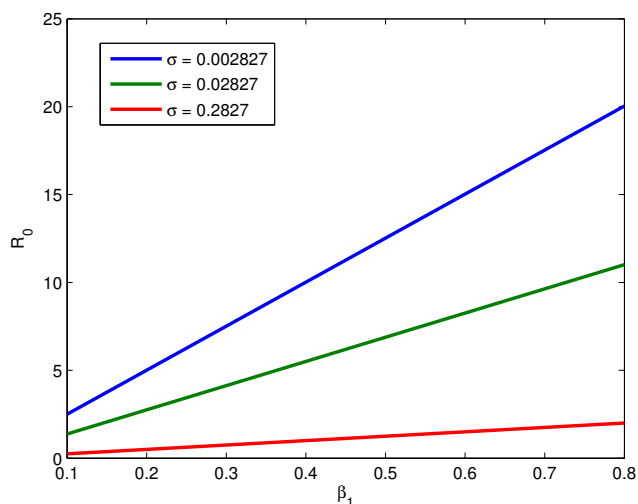

Figure 2. The sensitivity β_1 to R_0 values with three different σ values

Based on Figure 2, it can be concluded that if the transmission rate of drug abusers (β_1) increases then R_0 value also goes up which means that drug abuse is getting more widespread. This happens because the value of (β_1) index is positive. Then the smaller value of σ resulting in a greater value of R_0, this is because the value of (σ) index is negative.

4. Application of Optimal Control

In this study, an optimal control analysis will be conducted for a mathematical model of drug abuse by concerning type of treatment along with level of risks. To determine the optimal control, the construction of the model is carried out with the addition of the control variable. The mathematical model with the following control is

$$\frac{dS_H}{dt} = p\Lambda - (1-u)\frac{\mu\beta_1 I S_H}{\Lambda} - (\mu+\omega_1)S_H + \omega_2 S_L, \quad (9)$$

$$\frac{dS_L}{dt} = (1-p)\Lambda - (1-u)\frac{\mu\eta\beta_1 I S_L}{\Lambda}$$
$$- (\mu+\omega_2)S_L + \omega_1 S_H \quad (10)$$

$$\frac{dI}{dt} = (1-u)\frac{\mu\beta_1 I(S_H + \eta S_L)}{\Lambda} + \frac{\mu\beta_2 IR}{\Lambda}$$
$$+ \frac{\mu\beta_3 I T_j}{\Lambda} - (\mu+\sigma+\rho)I, \quad (11)$$

$$\frac{dT_j}{dt} = (1-q)\sigma I - \frac{\mu\beta_3 I T_j}{\Lambda}$$
$$- (\mu+\alpha_1+\gamma_1)T_j + \alpha_2 T_r, \quad (12)$$

$$\frac{dT_r}{dt} = q\sigma I - (\mu+\alpha_2+\gamma_2)T_r + \alpha_1 T_j, \quad (13)$$

$$\frac{dR}{dt} = \gamma_1 T_j + \gamma_2 T_r + \rho I - \frac{\mu\beta_2 IR}{\Lambda} - \mu R. \quad (14)$$

where $u(t)$ is defined by control input variable such as the anti-drug campaign at time t and imposed in eqs. (9) to (11). The application of this control variable aims to reduce the number of drug abusers who do not receive treatment by reducing the rate of spread of drug abuse and maximizing anti-drug campaign with minimum cost. Pontryagin's Maximum Principle method is used to achieve this objective.

The performance index that can be formed based on the above explanation is as follows:

$$\min J(u) = \int_0^{t_f} \left(I + \frac{K}{2}u^2\right)dt,$$

with coefficient K is a weighting constant in the form of costs that should be used for anti-drug campaigns. The interval value of optimal control is $0 \leq u(t) \leq 1, 0 \leq t \leq t_f$ where t_f is the end time of observation. The quadratic function of the control cost is adopted, as stated in [19–21].

Based on Pontryagin's Maximum Principle [22], the first step carried out in the analysis of the optimal control problem is to form a Hamiltonian (H) function, that is:

$$H = I + \frac{K}{2}u^2 + \psi_1\left[p\Lambda - (1-u)\frac{\mu\beta_1 I S_H}{\Lambda} - (\mu+\omega_1)S_H + \omega_2 S_L\right]$$
$$+ \psi_2\left[(1-p)\Lambda - (1-u)\frac{\mu\eta\beta_1 I S_L}{\Lambda} - (\mu+\omega_2)S_L + \omega_1 S_H\right]$$
$$+ \psi_3\left[(1-u)\frac{\mu\beta_1 I(S_H+\eta S_L)}{\Lambda} + \frac{\mu\beta_2 IR}{\Lambda} + \frac{\mu\beta_3 I T_j}{\Lambda} - (\mu+\sigma+\rho)I\right]$$
$$+ \psi_4\left[(1-q)\sigma I - \frac{\mu\beta_3 I T_j}{\Lambda} - (\mu+\alpha_1+\gamma_1)T_j + \alpha_2 T_r\right]$$
$$+ \psi_5\left[q\sigma I - (\mu+\alpha_2+\gamma_2)T_r + \alpha_1 T_j\right]$$
$$+ \psi_6\left[\gamma_1 T_j + \gamma_2 T_r + \rho I - \frac{\mu\beta_2 IR}{\Lambda} - \mu R\right].$$

where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$ and ψ_6 are adjoint variables or co-state variables.

Furthermore, in order to obtain optimal conditions, the Hamiltonian function above must meet stationary conditions, namely $\frac{\partial H}{\partial u} = 0$. So that the optimal controller u is obtained

$$u^* = \min\left(1, \max\left(0, \frac{\mu\beta_1 I(-\psi_1 S_H - \psi_2\eta S_L + \psi_3(S_H+\eta S_L))}{K\Lambda}\right)\right) \quad (15)$$

The controller form of u^* depends on state and co-state variable. The state equations are as follows:

$$\dot{S_H} = \frac{\partial H}{\partial \psi_1}$$
$$= p\Lambda - (1-u)\frac{\mu\beta_1 I S_H}{\Lambda} - (\mu+\omega_1)S_H + \omega_2 S_L$$

$$\dot{S_L} = \frac{\partial H}{\partial \psi_2}$$
$$= (1-p)\Lambda - (1-u)\frac{\mu\eta\beta_1 I S_L}{\Lambda} - (\mu+\omega_2)S_L + \omega_1 S_H$$

$$\dot{I} = \frac{\partial H}{\partial \psi_3}$$

$$= (1-u) \frac{\mu \beta_1 I (S_H + \eta S_L)}{\Lambda} + \frac{\mu \beta_2 IR}{\Lambda}$$

$$+ \frac{\mu \beta_3 IT_j}{\Lambda} - (\mu + \sigma + \rho) I \qquad (16)$$

$$\dot{T}_j = \frac{\partial H}{\partial \psi_4}$$

$$= (1-q) \sigma I - \frac{\mu \beta_3 IT_j}{\Lambda} - (\mu + \alpha_1 + \gamma_1) T_j + \alpha_2 T_r$$

$$\dot{T}_r = \frac{\partial H}{\partial \psi_5}$$

$$= q \sigma I - (\mu + \alpha_2 + \gamma_2) T_r + \alpha_1 T_j$$

$$\dot{R} = \frac{\partial H}{\partial \psi_6}$$

$$= \gamma_1 T_j + \gamma_2 T_r + \rho I - \frac{\mu \beta_2 IR}{\Lambda} - \mu R.$$

Meanwhile, the co-state equations are as follows :

$$\dot{\psi}_1 = -\frac{\partial H}{\partial S_H}$$

$$= - \left[-\psi_1 (1-u) \frac{\mu \beta_1 I}{\Lambda} - \psi_1 (\mu + \omega_1) + \psi_2 \omega_1 \right.$$

$$\left. + \psi_3 (1-u) \frac{\mu \beta_1 I}{\Lambda} \right]$$

$$\dot{\psi}_2 = -\frac{\partial H}{\partial S_L}$$

$$= - \left[\psi_1 \omega_2 - \psi_2 (1-u) \frac{\mu \eta \beta_1 I}{\Lambda} - \psi_2 (\mu + \omega_2) \right.$$

$$\left. + \psi_3 (1-u) \frac{\mu \beta_1 I \eta}{\Lambda} \right]$$

$$\dot{\psi}_3 = -\frac{\partial H}{\partial I}$$

$$= - \left[1 - \psi_1 (1-u) \frac{\mu \beta_1 S_H}{\Lambda} - \varnothing_2 (1-u) \frac{\mu \beta_1 \eta S_L}{\Lambda} \right.$$

$$\left. + \psi_3 (1-u) \frac{\mu \beta_1 (S_H + \eta S_L)}{\Lambda} + \frac{\psi_3 \mu \beta_2 R}{\Lambda} \right] \qquad (17)$$

$$- \left[\frac{\psi_3 \mu \beta_3 T_j}{\Lambda} - \psi_3 (\mu + \sigma + \rho) + (1-q) \sigma \psi_4 \right.$$

$$\left. - \frac{\mu \beta_3 \psi_4 T_j}{\Lambda} + q \sigma \psi_5 + \rho \psi_6 - \frac{\mu \beta_2 \psi_6 R}{\Lambda} \right]$$

$$\dot{\psi}_4 = -\frac{\partial H}{\partial T_j}$$

$$= - \left[\frac{\psi_3 \mu \beta_3 I}{\Lambda} - \frac{\mu \beta_3 \psi_4 I}{\Lambda} - \psi_4 (\mu + \alpha_1 + \gamma_1) \right.$$

$$\left. + \psi_5 \alpha_1 + \psi_6 \gamma_1 \right]$$

$$\dot{\psi}_5 = -\frac{\partial H}{\partial T_r}$$

$$= - \left[\psi_4 \alpha_2 - \psi_5 (\mu + \alpha_2 + \gamma_2) + \psi_6 \gamma_2 \right]$$

$$\dot{\psi}_6 = -\frac{\partial H}{\partial R}$$

$$= - \left[\frac{\psi_3 \mu \beta_2 I}{\Lambda} - \frac{\psi_6 \mu \beta_2 I}{\Lambda} - \psi_6 \mu \right].$$

Based on the description above, to get the value of S_H, S_L, I, T_j, T_r and R from the optimal form u^* then it is nec-essary to solve the non-linear state and co-state equations. The non-linear equation system is hard to be solved analytically, so it will be solved numerically.

5. Numerical Results

The numerical simulation is carried out by comparing a mathematical model of drug abuse spread without control variable to one with control variable. Its goal is to determine the effectiveness of the control effort in order to meet the goal of the cost function presented. To solve the optimal control strategy, we use the fourth order Runge-Kutta (RK4) scheme. To solve the state system, we first implement the forward RK4 technique. We employ the backward RK4 scheme to unravel the co-state system from then on.

The initial value for all the population in this simulation are $S_H(0) = 20000$, $S_L(0) = 25000$, $I(0) = 15000, T_j(0) = 5000$, $T_r(0) = 10000, R(0) = 3000$ and performed at $t = 0$ to $t = 100$. The parameter values on this numerical simulation refer to the Table 3 and the weighting constant for the cost of campaign is $K = 10$.

The profile of optimal control u is plotted in Figure 3. The anti-drug campaign work intensively. Furthermore, the dynamics of drug abusers who do not receive treatment (I) are given in Figure 4. The dynamics of drug abusers who receive treatment as outpatient care (T_j) and inpatient care (T_r) are given in Figures 5 and 6.
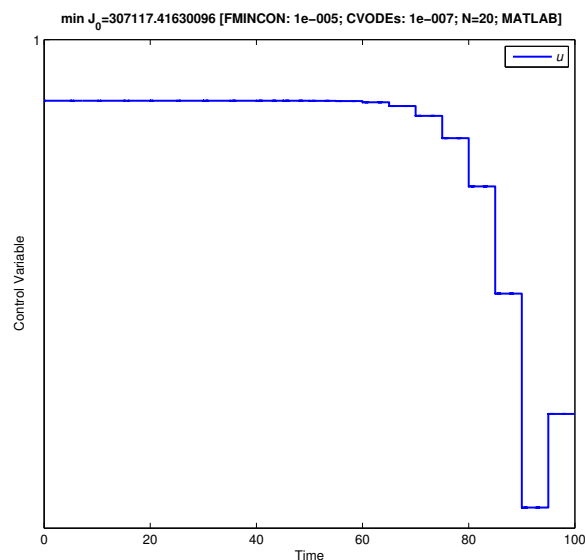

Figure 3. Control profile of anti-drug campaign efforts

Figure 4 shows that campaign of drug abuse provide a significant reduction in drug abusers who do not get any treatment (I) compared to having no control. The total population without control decreases until year-7 then goes up steadily until the end of the observation. But when the anti-drug campaign is applied, the population of untreated drug-abusers decreases continuously from the beginning to the end of the observation. Furthermore, it shows that the number of these population is towards zero.

Moreover, we can see similar pattern in Figures 5 and 6, the population number of drug abusers who receive outpatient and inpatient care without any control go down until certain time

Table 5. Comparison of the number of Abusers I, T_j, T_r with and without control

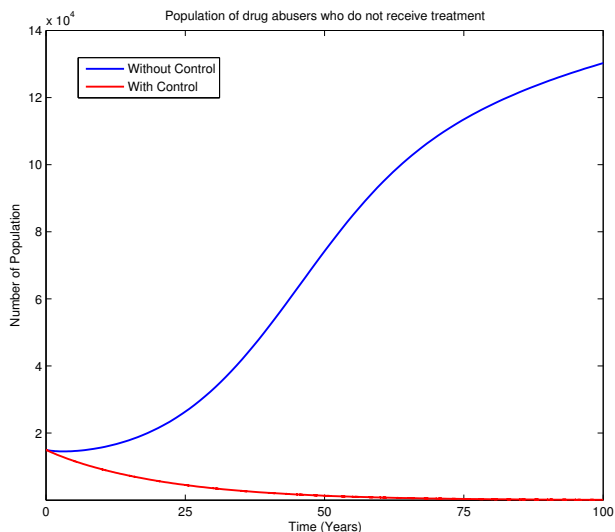| Scenario	Population of I	Population of T_j	Population of T_r
without Control	130.300	20.950	5.620
with Control	97	135	17

Figure 4. Comparison between the number population drug abusers who do not receive treatment (I) without and with control
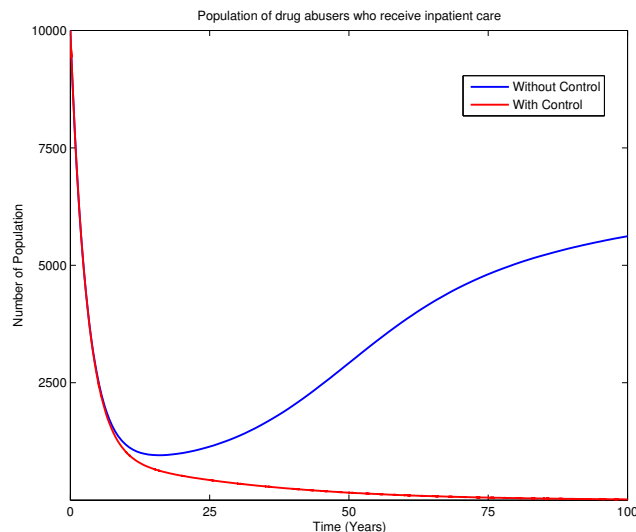

Figure 6. Comparison between the number population drug abusers who receive treatment as inpatient care (T_r) without and with control
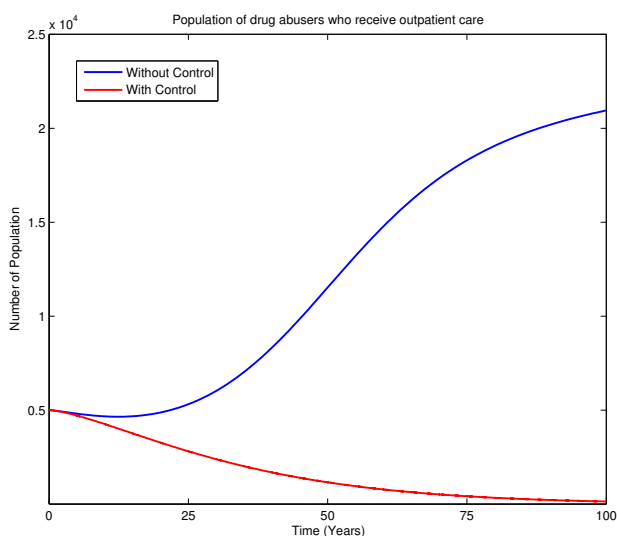

Figure 5. Comparison between the number population drug abusers who receive treatment as outpatient care (T_j) without and with control

and then up steadily until the end of the observation. But when the anti-drug campaign was given to those populations, they decreased from the initial observation until the end of the observation. This shows that the anti-drug campaign has a significant effect to reduce the population number of drug abusers who receive treatment either as outpatient or inpatient care.

Table 5 shows that the anti-drugs Campaign provides a great effect to minimize the number of drug abusers who do not receive treatment, receive treatment as outpatient care, or receive treatment as inpatient care.

6. Conclusion

In this paper, we have analysed the model of drug abuse reduction by concerning type of treatment along with risks level by applying the optimal control problem. The model has two equilibria, namely the drug-free equilibrium and the endemic equilibrium. The drug-free equilibrium will be locally asymptotically stable when the basic reproduction number less than one. We also analysed our model by evaluating the parameter sensitivity index to determine the most influential parameters on the spread of drug abuse.

The optimal control is then applied to the drug abuse model in the form of anti-drug campaign. Based on the results of numerical simulations before and after being given control shows that anti-drug campaign has a significant effect to reduce the number of drug abusers who do not receive any treatment (I), and also reduce the number of drug abusers who receive treatment as outpatient care (T_j) or inpatient care (T_r).

In a subsequent study can be developed a mathematical model of drug abuse by concerning the population of drug distributors. Drug distributors have a huge effect on the dynamics of the spread of drugs so that there is a greater potential for drug abuse to be endemic.

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