Non-unitary Trace Formulae

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Même si nos paroles sont justes,
même si nos pensées sont exactes, Cela
n'est pas conforme à la verité.

Verse 15 of the "Shin jin Mei" by Kanchi Sōsan
(translation by Taisen Deshimaru)

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Last but not least, I thank Frank Loose for accepting to review my thesis.

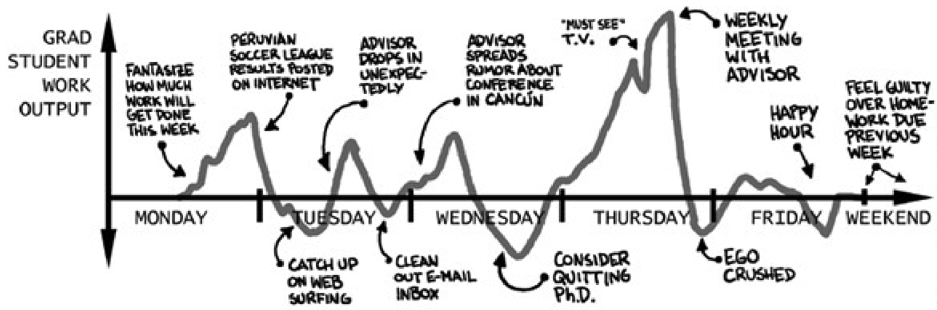


Figure 1: With permission from "Piled Higher and Deeper" by Jorge Cham, `www.phdcomics.com`

Contents

Introduction

In his paper [Mül11], Werner Müller started to investigate a non unitary trace formula. In this thesis, we would like to give several generalizations of his results. Firstly, let us summarize the unitary case, which goes back to Selberg [Sel56] and is known as the Selberg trace formula.

Let G be a first countable, locally compact group and $\Gamma \subset G$ a uniform lattice, or in other words, a discrete subgroup $\Gamma \subset G$, such that the quotient $\Gamma\backslash G$ carries an invariant Radon-measure μ and $\Gamma\backslash G$ is compact. The right regular representation R of G on $L^2(\Gamma\backslash G)$ is defined as

$$R(g)\varphi(h) = \varphi(hg),$$

for $\varphi \in L^2(\Gamma\backslash G)$. By the invariance of the measure μ, the right regular representation is unitary:

$$\|R(g)\varphi\|_2^2 = \int_{\Gamma\backslash G} |\varphi(hg)|^2 \, d\mu(h) = \int_{\Gamma\backslash G} |\varphi(h)|^2 \, d\mu(h) = \|\varphi\|_2^2.$$

Because of the unitarity and the compactness of the quotient $\Gamma\backslash G$, one can prove (see for ex. [DE09, Theorem 9.2.2]), that the right regular representation decomposes as a direct sum of unitary irreducible representations of G, with finite multiplicity:

$$L^2(\Gamma\backslash G) \cong \widehat{\bigoplus}_{\pi\in\widehat{G}} N_\Gamma(\pi)\pi. \tag{1}$$

The sum ranges over the unitary dual $\widehat{G}$, and the natural number $N_\Gamma(\pi)$ counts the multiplicity of the representation $\pi \in \widehat{G}$ in $L^2(\Gamma\backslash G)$.

For a function $f \in L^1(G)$ and a unitary representation (π, V_π) of G, we can define a bounded operator on the representation space V_π as the integral

$$\pi(f) = \int_G f(g)\pi(g) \, dg. \tag{2}$$

Let us for example consider the right regular representation R. Then $R(f)$ applied to a function $\varphi \in L^2(\Gamma\backslash G)$ gives

$$R(f)\varphi(h) = \int_G f(g)\varphi(hg) \, dg.$$

If the function $f \in C_c(G) * C_c(G)$ is the convolution of two continuous compactly supported functions on G, the operator $R(f)$ can be shown to be a trace class operator. Using (1), we compute the trace of the operator $R(f)$ as

$$\operatorname{tr} R(f) = \sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \operatorname{tr} \pi(f). \tag{3}$$

On the other hand, we can show, that the operator is an integral operator, with integral kernel

$$k_f(x,y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y).$$

The trace of such an operator can be computed by integrating the kernel along the diagonal

$$\operatorname{tr} R(f) = \int_G \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x)\, dx. \tag{4}$$

Instead of summing over the whole group Γ, we can sum over the conjugacy classes $[\gamma]$ of the group Γ, which of course needs to be compensated for, such that we obtain

$$\int_G \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x)\, dx = \int_G \sum_{[\gamma]} \sum_{\sigma \in \Gamma_\Gamma \backslash \Gamma} f((\sigma x)^{-1}\gamma \sigma x)\, dx,$$

where Γ_γ is the centralizer of γ in Γ.

After some further, rather straightforward transformations, it turns out, that

$$\operatorname{tr} R(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, dx, \tag{5}$$

where G_γ, Γ_γ is the centralizer of γ in G and Γ, respectively. Thus, by equating (3) and (5) we obtain the celebrated Selberg trace formula

$$\sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \operatorname{tr} \pi(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, dx. \tag{6}$$

We can generalize the situation as follows. Instead of considering the right regular representation on $L^2(\Gamma\backslash G)$, we additionally twist with a finite dimensional complex representation $\chi : \Gamma \to V$. More precisely, we regard functions $f : G \to V$, which are Γ-equivariant with respect to χ:

$$f(\gamma g) = \chi(\gamma) f(g), \tag{7}$$

for all $g \in G$ and every $\gamma \in \Gamma$. By introducing an appropriate hermitian form on this function space, we obtain a Hilbert space $L^2(\Gamma\backslash G, \chi)$ in the usual manner: We consider measurable functions with property (7), which have finite norm and we take the quotient with respect to almost everywhere equality. On $L^2(\Gamma\backslash G, \chi)$ we can define the right regular representation as before, which we will again denote by R. Since the representation χ is allowed to be any complex finite dimensional representation, it is in general not true, that R is unitary. Hence we have to introduce a different class of representations, which is suitable in our context and we will make further assumptions on the group G. We let G be a semisimple Lie group with finite center and maximal compact subgroup K. Instead of unitary representations, we will work with *admissible* representations. These are Hilbert space representations ρ, such that the restriction of ρ to the maximal compact subgroup K is unitary, and each K-isotype is finite dimensional. But even now, R does in general not decompose directly as in (1).

The main result we will prove in this context, is Theorem 9.20. It assures the existence of an increasing and exhaustive filtration of subspaces for the right regular representation R on $L^2(\Gamma\backslash G, \chi)$,

$$0 = V_0 \subset V_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} V_i = L^2(\Gamma\backslash G, \chi),$$

and the induced representations on the quotients V_i/V_{i-1} are admissible and irreducible. Furthermore it is true, that the graduated G-module

$$\bigoplus_{i=0}^{\infty} V_{i+1}/V_i, \tag{8}$$

is independent of the chosen filtration. Accordingly, we can associate to each admissible irreducible representation π of G a natural number $N_{\Gamma,\chi}(\pi)$, giving the multiplicity of π in the graduated G-module (8), such that we can write

$$\bigoplus_{i=0}^{\infty} V_{i+1}/V_i = \bigoplus_{\pi\in\widehat{G}_{\mathrm{adm}}} N_{\Gamma,\chi}(\pi)\,\pi. \tag{9}$$

As in (2), we can define for admissible π and $f \in C_c(G)$ the operator $\pi(f)$, acting on the representation space of π. The operator $R(f)$ is again a trace class operator, and according to our above considerations, the trace is given by

$$\operatorname{tr} R(f) = \sum_{\pi \in \widehat{G}_{\mathrm{adm}}} N_{\Gamma,\chi}(\pi) \operatorname{tr} \pi. \tag{10}$$

On the other hand, $R(f)$ is an integral operator, with integral kernel

$$k_f(x, y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)\chi(\gamma).$$

As in (4), by integrating along the diagonal, we obtain

$$\operatorname{tr} R(f) = \int_G \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x) \operatorname{tr} \chi(\gamma)\, dx.$$

We can reorder the above sum, according to the conjugacy classes $[\gamma]$ of Γ and we will obtain

$$\sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, dx \operatorname{tr} \chi(\gamma). \tag{11}$$

Since we computed $\operatorname{tr} R(f)$ via spectral data in (10) and via geometric data in (11) we get the trace formula

$$\sum_{\pi \in \widehat{G}_{\mathrm{adm}}} N_{\Gamma,\chi}(\pi) \operatorname{tr} \pi = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, dx \operatorname{tr} \chi(\gamma).$$

For $\chi = 1$ the trivial representation, we again get the Selberg trace formula from (6).

Although it seems natural, to present the compact case before the non compact case, we decided to push it to the very end. The methods are mainly representation theoretic and the requirements on the group G are much less restrictive, than in the non compact case. Because of this higher degree of abstraction, we felt it is appropriate to present the compact case in the final chapter. At the very end we will present a trace formula for compact quotients $\Gamma \backslash G$, where now, G is a totally disconnected group.

The non compact case

Up to now, we only discussed the case, that the quotient $\Gamma\backslash G$ is compact. We can also consider the case of a quotient $\Gamma\backslash G$ with finite area, but not necessarily compact. We do not know how to deal with this situation in full generality. There are several obstacles, the main problem being the continuous spectrum. If χ is unitary, it is known, that $L^2(\Gamma\backslash G,\chi)$ decomposes into a discrete part, and a continuous part:

$$L^2(\Gamma\backslash G,\chi) = L^2_{\mathrm{disc}}(\Gamma\backslash G,\chi) \oplus L^2_{\mathrm{cont}}(\Gamma\backslash G,\chi). \tag{12}$$

The discrete part $L^2_{\mathrm{disc}}(\Gamma\backslash G,\chi)$ is a direct sum of unitary irreducible representations, while the continuous part $L^2_{\mathrm{cont}}(\Gamma\backslash G,\chi)$ cannot be described as a direct sum anymore, but as a direct integral of unitary irreducible representations. The main tool, to derive such a decomposition is the existence of the Eisenstein series. In our situation, where we additionally twist with a non-unitary χ we do not know how to replace these Eisenstein series. Therefore, we restrict ourselves to $G = \mathrm{PSL}(2,\mathbb{R})$ and we take the quotient G/K with respect to the maximal compact subgroup $K = \mathrm{PSO}(2,\mathbb{R})$. In other words, we only consider the K-invariants $L^2(\Gamma\backslash G,\chi)^K$ in the representation space $L^2(\Gamma\backslash G,\chi)$. The upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is a homogeneous space for G, i.e. G acts (by Moebius transformations) transitively on $\mathbb{H}$. The fix group of i is exactly $\mathrm{PSO}(2,\mathbb{R})$, and thus we find

$$G/K \cong \mathbb{H}.$$

The double quotient $\Gamma\backslash G/K \cong \Gamma\backslash\mathbb{H}$ is a non compact hyperbolic surface. An important datum of the surface $\Gamma\backslash\mathbb{H}$ is the number of cusps. Geometrically, this gives the number of cylindrical ends of the surface. To each cusp $\mathfrak{a}$, we can associate its fix group $\Gamma_{\mathfrak{a}}$. The only restriction we have to put on the representation χ, is *unitarity at cusps*. This means, that $\chi(\gamma)$ is required to be unitary for each $\gamma \in \Gamma_{\mathfrak{a}}$. These assumptions allow us, to establish the existence of the appropriate Eisenstein series. For simplicity, assume that we have a cusp at infinity. The prototype for a classical Eisenstein series is given by

$$E(z,s) = \sum_{\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s, \tag{13}$$

where $z \in \mathbb{H}$ and $s \in \mathbb{C}$. This series converges absolutely for $\mathrm{Re}(s) > 1$ and can be meromorphically continued to the complex plane. For our scenario, it seems very plausible to mimic the definition in (13) with an additional twist by χ:

$$E(z,s,\chi) = \sum_{\Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s \chi(\gamma)^{-1}. \tag{14}$$

In the above definition, for the sake of simplicity, we implicitly assumed that χ is trivial on cusps, i.e. $\chi(\gamma) = \mathrm{Id}$ for all $\gamma \in \Gamma_{\mathfrak{a}}$, such that the summand is invariant under Γ_∞ and the above series (14) is well-defined. Looking at the definition (13), it is easy to formally compute the χ-equivariance of $E(z,s,\chi)$:

$$E(\gamma z, s, \chi) = \chi(\gamma) E(z,s,\chi).$$

Having found a reasonable candidate, it is by no means clear that (14) converges anywhere. This is the main reason, why we restricted ourselves to

$$\mathrm{PSL}(2,\mathbb{R}) / \mathrm{PSO}(2,\mathbb{R}) \cong \mathbb{H},$$

because at this point we use very geometric considerations to prove estimates for the representation χ (Proposition 2.10), which enable us to show convergence for (14) for $\mathrm{Re}(s)$ large enough. This will help us, to understand the spectral decomposition of the Hilbert space $L^2(\Gamma\backslash\mathbb{H},\chi)$ with respect to the hyperbolic laplacian. It turns out, that again we have a decomposition into a discrete and a continuous part

$$L^2(\Gamma\backslash\mathbb{H},\chi) = L^2_{\mathrm{disc}}(\Gamma\backslash\mathbb{H},\chi) \oplus L^2_{\mathrm{cont}}(\Gamma\backslash\mathbb{H},\chi),$$

the main difference being, that $L^2_{\mathrm{disc}}(\Gamma\backslash\mathbb{H},\chi)$ is spanned by generalized eigenfunctions of the hyperbolic Laplacian Δ (meaning $(\Delta - \lambda)^N f = 0$, for some $\lambda \in \mathbb{C}$ and some sufficiently large N). In contrast, in the unitary case, $L^2_{\mathrm{disc}}(\Gamma\backslash\mathbb{H},\chi)$ is spanned by proper eigenfuctions of the Laplacian. Having understood the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H},\chi)$, we can deduce a trace formula in Theorem 8.1.

Applications

Our results can be used to refine the prime geodesic theorem. We would like to sketch the idea. The prime geodesic theorem for a hyperbolic surface $\Gamma\backslash\mathbb{H}$ is analogous to the classical prime number theorem. A prime geodesic is a closed curve, which is a geodesic and traces out its image exactly once. We let $\pi(x)$ be the counting function for prime geodesics of length less, or equal to x. A hyperbolic conjugacy class $[P]$ is said to be primitive, if P cannot be written as Q^j for some $j > 1$. There is a one-to-one correspondence between primitive hyperbolic conjugacy classes and prime geodesics of $\Gamma\backslash\mathbb{H}$. We define the norm NP as the logarithm of the length of the geodesic, which is associated to the class $[P]$. Introducing the Selberg zeta function as the infinite product,

$$Z_\Gamma(s) = \prod_{[P]}\prod_{k=0}^{\infty}(1 - NP^{-s-k}),$$

where $[P]$ runs through all primitive hyperbolic conjugacy classes, one can show with the help of the Selberg trace formula, that $Z_\Gamma(s)$ converges absolutely for $\mathrm{Re}(s) > 1$ and extends to a meromorphic function on the whole complex plane. This functions satisfies a weak form of the Riemann hypothesis, in the sense that all its non trivial zeros lie on $\mathrm{Re}(s) = 1/2$, with finitely many exceptional zeros, which all lie in $[0,1]$. These zeros are linked to the unitary representations π, that are subrepresentations of the right regular representation R of $L^2(\Gamma\backslash G)$ and have a K-fixed vector. Furthermore, $Z_\Gamma(s)$ has a simple zero in 1, which is linked to the trivial representation. This allows one to use a tauberian theorem and one can prove the asymptotic formula

$$\pi(x) \sim x/\log(x),$$

in complete analogy to the prime number theorem. Thus the number of prime geodesics, with length smaller than x is asymptotically equivalent to $x/\log(x)$.

If we have a representation $\chi : \Gamma \to \mathrm{GL}(V)$, which is unitary at cusps, we can similarly define a zeta function

$$Z_{\Gamma,\chi}(s) = \prod_{[P]}\prod_{k=0}^{\infty}(1 - NP^{-s-k}\det\chi(P)),$$

and show with the help of the trace formula, that this defines a meromorphic function on the complex plane. Now, we would like to explain how this zeta function can be made useful. It is a classical result of Fricke and Klein [FK65], that a Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is generated by elements

$$A_1, \ldots, A_g, B_1, \ldots, B_g, E_1, \ldots, E_l, P_1, \ldots, P_h,$$

satisfying the only relations

$$[A_1, B_1] \cdots [A_g, B_g] E_1 \cdots E_l P_1 \cdots P_h = 1 \tag{15}$$

and

$$E_j^{m_j} = 1, \tag{16}$$

where the A_i, B_i are hyperbolic, the E_i elliptic and the P_i parabolic elements. Furthermore, $[A_i, B_i]$ is the commutator of A_i and B_i and g denotes the genus of the surface $\Gamma\backslash\mathbb{H}$ and h the number of cusps. We can now easily construct one dimensional representations: Assigning to the hyperbolic generators A_i, B_i arbitrary non zero complex numbers, and 1 to the other generators, is seen to define a representation, from the relations (15) and (16). Thus it is possible to put arbitrary weights on the hyperbolic elements A_i, B_i, and this will define a one dimensional χ representation, unitary at cusps. Now, we form the according zeta function $Z_{\Gamma,\chi}$. Applying a tauberian theorem will give us a prime geodesic theorem.

1 Preliminaries

In this chapter we will recall some definitions and important results, which we will need throughout the rest of this work. In section 1.1 we start by introducing analysis on the upper half-plane $\mathbb{H}$, which will furnish the objects we will work with: We let discrete subgroups $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ act on $\mathbb{H}$ via Moebius-transformations. Taking the quotient $\Gamma\backslash\mathbb{H}$ with respect to this action, yields a hyperbolic surface. A very important notion, while working with a quotient of the form $\Gamma\backslash\mathbb{H}$ is that of a fundamental domain, which will be discussed in the consecutive section 1.2.

The group Γ will, in this work, also come with a finite-dimensional complex representation $\chi : \Gamma \to \mathrm{GL}(V)$. We will be interested in functions $f : \mathbb{H} \to V$, which are Γ-equivariant with respect to the representation χ, more precisely

$$f(\gamma z) = \chi(\gamma) f(z) \text{ for all } \gamma \in \Gamma, z \in \mathbb{H}.$$

These functions can also be understood as sections of a complex vector bundle E over $\Gamma\backslash\mathbb{H}$. By introducing a suitable norm on this function space, we can define a Hilbert-space $L^2(\Gamma\backslash\mathbb{H},\chi)$, which will be fundamental in our study. The hyperbolic Laplacian furnishes an unbounded operator on $L^2(\Gamma\backslash\mathbb{H},\chi)$ and one of our goals will be to understand its spectral properties. Details will be explained in section 1.3.

1.1 The upper half plane

We write $e(z) = e^{2\pi i z}$ for the exponential. Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the upper half-plane. The manifold $\mathbb{H}$ is equipped with the Riemannian metric

$$ds^2 = \frac{(dx^2 + dy^2)}{y^2}, \tag{17}$$

which turns $\mathbb{H}$ into a Riemannian manifold. The Laplacian corresponding to this metric is given by

$$\Delta = y^2 \left(\frac{\partial^2}{(\partial x)^2} + \frac{\partial^2}{(\partial y)^2} \right)$$

and is usually called the *hyperbolic Laplacian*. The Riemannian measure derived from (17) is given by

$$d\mu(z) = y^{-2}\, dx\, dy,$$

where $dx\, dy$ is the Lebesgue measure. The distance function on $\mathbb{H}$ is given explicitly by

$$\rho(z, w) = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}. \tag{18}$$

For $z \in \mathbb{H}, r > 0$ we let

$$B_r(z) = \{w \in \mathbb{H} : \rho(w, z) < r\}$$

the hyperbolic ball of radius r with center z. We will write $B_r^e(z)$ for the euclidian ball of radius r of center z to distinguish it from the hyperbolic ball. Instead of working with the distance function ρ we consider the function

$$u(z, w) = \frac{|z - w|^2}{\operatorname{Im} z \operatorname{Im} w} \tag{19}$$

which satisfies

$$\cosh \rho(z, w) = 1 + \frac{1}{2} u(z, w). \tag{20}$$

We let

$$\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) / \pm 1.$$

By abuse of notation, we will write frequently drop the $\pm$ and just write

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

for elements of Γ. The group $\mathrm{PSL}(2, \mathbb{R})$ acts on the upper half plane $\mathbb{H}$ via fractional linear transformations:

$$\gamma z = \frac{az + b}{cz + d}, \qquad \gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In the same manner $\mathrm{PSL}(2, \mathbb{R})$ acts on $\mathbb{R} \cup \{\infty\}$. Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a lattice, i.e. Γ is a discrete subgroup, such that the quotient $\Gamma \backslash \mathbb{H}$ has finite measure.

The action of Γ on $\mathbb{H}$ is properly discontinuous, meaning that for each compact subset $K \subset \mathbb{H}$ there exist only finitely many $\gamma \in \Gamma$ such that

$$\gamma K \cap K \neq \varnothing.$$

The measure $d\mu$ induces a measure on the quotient space $\Gamma\backslash\mathbb{H}$ which we denote for simplicity again by $d\mu$. There are two cases which we will distinguish.

The compact case: The first, and more simple one, is the case that the quotient space $\Gamma\backslash\mathbb{H}$ is compact, or equivalently, that Γ has no parabolic elements (see [Kat92, Corollary 4.2.7]).

The non compact case: The second case is the quotient $\Gamma\backslash\mathbb{H}$ being non compact. A *cusp* for the group Γ is an element of $\mathbb{R} \cup \{\infty\}$ which is stabilized by some non-trivial element of Γ. The cusps will be denoted by german letters $\mathfrak{a}, \mathfrak{b}, \dots$. The stabilizer group

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}$$

for the cusp $\mathfrak{a}$ can be shown to be isomorphic to $\mathbb{Z}$, so we can choose a generating element $\gamma_{\mathfrak{a}}$ for $\Gamma_{\mathfrak{a}}$. For each cusp $\mathfrak{a}$ there exists an element $\sigma_{\mathfrak{a}} \in \mathrm{PSL}(2,\mathbb{R})$ which is unique up to a translation from the right by upper triangular matrices with ones on the diagonal in $\mathrm{PSL}(2,\mathbb{R})$, with the property that

$$\sigma_{\mathfrak{a}}\infty = \mathfrak{a}, \qquad \sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$\sigma_{\mathfrak{a}}$ is called the scaling matrix for the cusp $\mathfrak{a}$.

1.2 Fundamental domains

We say that two points $z, w \in \mathbb{H} \cup \hat{\mathbb{R}}$, where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, are equivalent if $w \in \Gamma z$. We will also write $w \equiv z \pmod{\Gamma}$. A domain $F \subset \mathbb{H}$ is called *fundamental domain for* Γ, if

1. distinct points in F are not equivalent,

2. any orbit Γz has non-trivial intersection with $\overline{F}$, by which we mean the closure of F in the $\mathbb{H}$-topology,

3. the boundary of F has measure zero: $\mu(\partial F) = 0$.

We can use the following construction to obtain a fundamental domain: Take any $z \in \mathbb{H}$ fixed only by the trivial element of Γ. Then the set

$$D(z) = \{w \in \mathbb{H} : \rho(z,w) < \rho(\gamma z, w) \text{ for all } \gamma \in \Gamma, \gamma \neq e\} \tag{21}$$

can be shown to be a fundamental domain for Γ (see [Bea95, p. 226 ff.]). It is called *Dirichlet domain* and has only finitely many faces. We would like to consider cuspidal parts and a compact subset of $\Gamma\backslash\mathbb{H}$ seperately to simplify work. It is useful to introduce the invariant height:

$$y_\Gamma(z) = \max_{\mathfrak{a}} \max_{\gamma\in\Gamma} \{\operatorname{Im} \sigma_{\mathfrak{a}}^{-1} \gamma z\}.$$

We say $z \in \mathbb{H}$ approaches the cusp $\mathfrak{a}$ if $\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} z) \to \infty$. Assuming that z approaches the cusp $\mathfrak{b}$ we obtain $y_\Gamma(\sigma_{\mathfrak{a}} z) = \operatorname{Im}(\sigma_{\mathfrak{b}}^{-1} z)$, for $\operatorname{Im}(\sigma_{\mathfrak{b}}^{-1} z)$ large enough. For $Y > 0$, we introduce the strip

$$P(Y) = \{z = x + iy : 0 < x < 1, y \geq Y\}.$$

If $\mathfrak{a}$ is a cuspidal vertex of the fundamental domain F, then for large enough Y this vertical strip is mapped via the scaling matrix $\sigma_{\mathfrak{a}}$ into F with image

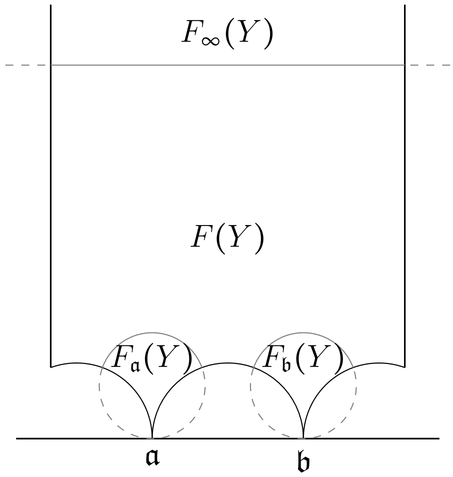
$$F_{\mathfrak{a}}(Y) = \sigma_{\mathfrak{a}} P(Y).$$

For different cusps these cuspidal zones are pairwise disjoint for Y large enough and we set

$$F(Y) = F \smallsetminus \left(\bigcup_{\mathfrak{a}} F_{\mathfrak{a}}(Y)\right).$$

$F(Y)$ is relatively compact and we have divided the fundamental domain F into the central part $F(Y)$ and finitely many cuspidal zones $F_{\mathfrak{a}}(Y)$

$$F = F(Y) \cup \bigcup_{\mathfrak{a}} F_{\mathfrak{a}}(Y).$$



1.3 Representations

Let $(V, \langle\cdot,\cdot\rangle)$ be a finite dimensional unitary vector space. The vector space of endomorphisms of V, $\mathrm{End}(V)$ will be equipped with the Frobenius norm $\|\cdot\|_F$:

$$\|S\|_F^2 = \mathrm{tr}(SS^*),$$

where S^* is the adjoint operator to S. We omit the subscript F and just write $\|S\|$, when it is clear that $S \in \mathrm{End}(V)$. It is an exercise to prove, that

1. $\|\cdot\|_F$ is submultiplicative:

$$\|ST\| \leq \|S\|\|T\| \text{ for every } S, T \in \mathrm{End}(V),$$

2. for arbitrary $S \in \mathrm{End}(V)$ and U unitary

$$\|US\| = \|SU\| = \|S\|.$$

By χ we will always denote a representation

$$\chi : \Gamma \to \mathrm{GL}(V).$$

In the case that Γ has parabolic elements we make the additional assumption that χ is *unitary at cusps* meaning that for every cusp $\mathfrak{a}$ the automorphism $\chi(\gamma_{\mathfrak{a}})$ is unitary. When talking about $\chi(\gamma_{\mathfrak{a}})$ we will consider the orthogonal projection $P_{\mathfrak{a}}$ onto the eigenspace for the eigenvalue 1 of $\chi(\gamma_{\mathfrak{a}})$, denoted by $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)$. We will say that χ is *singular* at the cusp $\mathfrak{a}$ if $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}), 1) \neq \{0\}$. In case $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}), 1) = \{0\}$ for every cusp $\mathfrak{a}$ we will say that χ is *non singular*.

We would like to define a Hilbert space of square integrable functions. Our definition will depend on the chosen fundamental domain F. We consider the set of all measurable functions $f : \mathbb{H} \to V$ with the property

$$f(\gamma z) = \chi(\gamma) f(z), \qquad \text{almost everywhere in } z \in \mathbb{H} \text{ for all } \gamma \in \Gamma.$$

On this set we install an inner product via

$$(f, g) = \int_F \langle f(z), g(z) \rangle \, d\mu(z), \tag{22}$$

and set

$$L^2(F, \chi) = \left\{ f : \|f\|^2 = \int_F \langle f(z), f(z) \rangle \, d\mu(z) < \infty \right\} / \sim,$$

where the equivalence relation $\sim$ is given by $f \sim g$ if, and only if, $f - g = 0$ almost everywhere.

Fortunately, this definition depends only mildly on the chosen fundamental domain F as we will show now.

Definition 1.1. We install an equivalence relation on the set of all fundamental domains F for the group Γ. We say that two fundamental domains F_1, F_2 are equivalent, if F_1 can be covered by finitely many Γ-translates of F_2 and vice versa.

Proposition 1.2. *The identity map provides a topological isomorphism*

$$L^2(F_1, \chi) \to L^2(F_2, \chi),$$

if F_1 and F_2 are equivalent fundamental domains.

Proof. Let $\gamma_1, \ldots, \gamma_l$ be finitely many elements of Γ, such that

$$F_2 \subset \bigcup_{n=1}^{l} \gamma_n F_1.$$

Then we get an estimate of the integral over F_2:

$$\int_{F_2} \|f(z)\|\, d\mu(z) \leq \sum_{n=1}^{l} \|\chi(\gamma_n)\| \int_{F_1} \|f(z)\|\, d\mu(z).$$

Bounding the finitely many $\chi(\gamma_n)$ by a constant we obtain

$$\int_{F_2} \|f(z)\|\, d\mu(z) \leq c \int_{F_1} \|f(z)\|\, d\mu(z)$$

with some constant c independent of f. Hence the identity map $L^2(F_1, \chi) \to L^2(F_2, \chi)$ is continuous. □

Definition 1.3. A fundamental domain F is called *geometrically finite*, if F is a polygon (connected and convex) with a finite number of faces.

Proposition 1.4. *Two geometrically finite fundamental domains F_1, F_2 are equivalent. In particular, in view of Proposition 1.2 the L^2-spaces $L^2(F_1, \chi)$ and $L^2(F_2, \chi)$ are topologically isomorphic.*

Proof. We will introduce the Borel-Serre compactification $\mathbb{H}_\Gamma$: As a set we take

$$\mathbb{H}_\Gamma = \mathbb{H} \cup \bigcup_{\mathfrak{a}} \partial\mathbb{H} \setminus \{\mathfrak{a}\}.$$

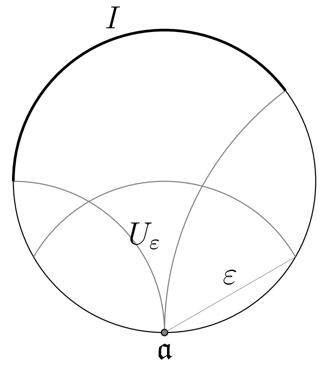
Hence, for each cusp $\mathfrak{a}$ we add a copy of $\partial\mathbb{H} \cong S^1$ and delete the cusp $\mathfrak{a}$. For $\partial\mathbb{H} \setminus \{\mathfrak{a}\}$ we will also write $B_\mathfrak{a}$. For points of $B_\mathfrak{a}$ we will write $b_\mathfrak{a}(x)$, where $x \in \partial\mathbb{H} \setminus \{\mathfrak{a}\}$, to distinguish the points from the different copies of $\mathbb{R}$. To install a topology on $\mathbb{H}_\Gamma$, it is sufficient to give for each point $x \in \mathbb{H}_\Gamma$ a neighbourhood basis. We distinguish two cases

1. Let $x \in \mathbb{H}$. In this case we choose the common neighbourhood basis of open hyperbolic balls $B_r(x) \subset \mathbb{H}$.

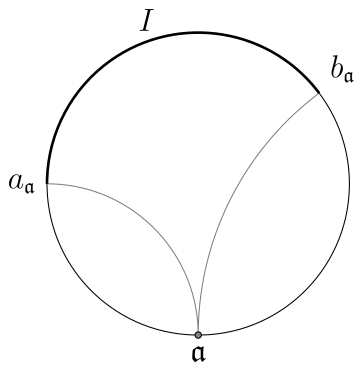
2. Let $x \in \partial\mathbb{H} \setminus \{\mathfrak{a}\}$. We switch to the disc model of hyperbolic space to give a neighbourhood basis of x. On the disk, $d(x, y)$ shall denote the euclidian distance of x and y. Then a neighbourhood basis will be given

by $\mathcal{U}(x) = \{U_{I,\varepsilon}\}$, where $\varepsilon > 0$ and $I \subset B_{\mathfrak{a}}$ is open:

$$\begin{aligned} U_{I,\varepsilon} =& \{y \in I : I \subset B_{\mathfrak{a}}, x \in I\} \cup \{z \in B_1^e(0) : d(z, \mathfrak{a}) \leq \varepsilon \\ & \quad \text{and } z \text{ is an element of a geodesic with endpoints in } I \text{ and } \mathfrak{a}\} \\ =:& I \cup U_\varepsilon \end{aligned}$$



Now we will explain how Γ acts on $\mathbb{H}_\Gamma$. For points $z \in \mathbb{H}$ the operation is the given one by fractional linear transformations. For a point $b_{\mathfrak{a}}(x)$ we set $\gamma \cdot b_{\mathfrak{a}}(x) = b_{\gamma\mathfrak{a}}(\gamma x)$. We can now describe a fundamental domain for the action of Γ on $\mathbb{H}_\Gamma$. Let F a geometrically finite fundamental domain for the action of Γ on $\mathbb{H}$. It has finitely many inequivalent cusps. For each cusp $\mathfrak{a}$ of F we choose $a_{\mathfrak{a}}, b_{\mathfrak{a}} \in B_{\mathfrak{a}} \cong \mathbb{R}$ such that the unique geodesics $\alpha_{\mathfrak{a}}, \beta_{\mathfrak{a}}$ which join $\mathfrak{a}$ to $a_{\mathfrak{a}}$ respectively $b_{\mathfrak{a}}$ contain the two faces of F which meet at $\mathfrak{a}$.



Then $F_\Gamma := F \cup \bigcup_{\mathfrak{a}} (a_{\mathfrak{a}}, b_{\mathfrak{a}})$ can be shown to be a fundamental domain of the action of Γ on $\mathbb{H}_\Gamma$. The closure of F_Γ in $\mathbb{H}_\Gamma$ is given by

$$\overline{F_\Gamma}^{\mathbb{H}_\Gamma} = \overline{F}^{\mathbb{H}} \cup \bigcup_{\mathfrak{a}} [a_{\mathfrak{a}}, b_{\mathfrak{a}}].$$

From the definition of the topology on $\mathbb{H}_\Gamma$, it can be seen from the fact that F has only finitely many sides, that the set $\overline{F_\Gamma}^{\mathbb{H}_\Gamma}$ is compact.
Now, take two geometrically finite fundamental domains $F, F' \subset \mathbb{H}$ for the action of Γ on $\mathbb{H}$ and consider the fundamental domains F_Γ, F'_Γ for the action of Γ on $\mathbb{H}_\Gamma$. Since both are relatively compact we can cover F_Γ by finitely many translates of F'_Γ and vice versa. But then we can cover F by finitely many translates of F' and vice versa. To conclude the proof, apply Proposition 1.2.

□

Our canonical choice for a fundamental domain will be a geometrically finite one. So let's fix once and for all a geometrically finite fundamental domain F (for example a Dirichlet domain). We will from now on write $L^2(\Gamma\backslash\mathbb{H}, \chi)$ instead of $L^2(F, \chi)$ since by choosing another geomtrically finite fundamental domain, we will obtain topologically the same Hilbert space.

We can yet introduce an L^2-space in a different manner, such that we get again a topological isomorphism between the L^2-space in the former definition. To this end consider the action of Γ on $\mathbb{H} \times V$ via

$$\gamma(z, v) = (\gamma z, \chi(\gamma)v).$$

The quotient space

$$\mathbb{H} \times_\Gamma V := \Gamma\backslash(\mathbb{H} \times V)$$

yields a vector bundle over $\Gamma\backslash\mathbb{H}$ with each fibre isomorphic to V. We choose a smooth Hermitian fibre metric $\langle\cdot,\cdot\rangle_s$ in such a manner that it coincides near the cusps with the given inner product on V. This means that for $z \in \mathbb{H}$ whose invariant height $y_\Gamma(z)$ is larger than some constant, say $y_\Gamma(z) > c$, the inner product on the fibre above z coincides with the inner product on the given representation space V. It is possible to find such a smooth Hermitian fibre metric, since the representation χ is unitary at cusps. Thus only for a compact subset of $\Gamma\backslash\mathbb{H}$ the fibre metric does not come from the representation space $(V, \langle\cdot,\cdot\rangle)$, while in cuspidal areas it coincides with the metric induced by $(V, \langle\cdot,\cdot\rangle)$. Now we define the Hilbert space $L^2(\Gamma\backslash\mathbb{H}, \chi)_s$ as the set of all measureable sections

$$f : \Gamma\backslash\mathbb{H} \to \mathbb{H} \times_\Gamma V$$

such that

$$\int_{\Gamma\backslash\mathbb{H}} \langle f(z), f(z)\rangle_{s,z} \, d\mu(z) < \infty,$$

and the inner product for $f, g \in L^2(\Gamma\backslash\mathbb{H}, \chi)$ is given by

$$\int_{\Gamma\backslash\mathbb{H}} \langle f(z), g(z)\rangle_{s,z} \, d\mu(z).$$

A section $f : \Gamma\backslash\mathbb{H} \to \mathbb{H} \times_\Gamma V$ can also be considered as a function $f : \mathbb{H} \to V$ such that $f(\gamma z) = \chi(\gamma) f(z)$ for every $z \in \mathbb{H}$ and every $\gamma \in \Gamma$. Thus we can compare the integral

$$\int_{\Gamma\backslash\mathbb{H}} \langle f(z), f(z)\rangle_{s,z} \, d\mu(z) = \int_F \langle f(z), f(z)\rangle_{s,z} \, d\mu(z)$$

with the integral we defined earlier (22)

$$\int_F \langle f(z), f(z)\rangle \, d\mu(z).$$

Proposition 1.5. *The identity map yields a bicontinuous map*

$$L^2(\Gamma\backslash\mathbb{H}, \chi) \to L^2(\Gamma\backslash\mathbb{H}, \chi)_s.$$

Proof. We pull back the smooth fibre metric $\langle\cdot,\cdot\rangle_s$ onto the constant bundle $\mathbb{H} \times V$. This pull back metric we denote by $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. Since any two norms on a finite dimensional vector space are equivalent, and since $\langle\cdot,\cdot\rangle$ and $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ coincide on cuspidal areas $F_{\mathfrak{a}}(Y)$, for Y large enough, there exist constants $m, M > 0$, such that

$$m\langle\!\langle(z,v),(z,v)\rangle\!\rangle \le \langle v, v\rangle = M\langle\!\langle(z,v),(z,v)\rangle\!\rangle,$$

for all $(z, v) \in \mathbb{H} \times V$, since F(Y) is relatively compact. Hence we get

$$m \int_F \langle\!\langle f(z), f(z)\rangle\!\rangle \, d\mu(z) \le \int_F \langle f(z), f(z)\rangle \, d\mu(z) \le M \int_F \langle\!\langle f(z), f(z)\rangle\!\rangle \, d\mu(z).$$

This yields the bicontinuity of the identity map

$$L^2(\Gamma\backslash\mathbb{H},\chi) \to L^2(\Gamma\backslash\mathbb{H},\chi)_s. \qquad \square$$

For this given smooth fibre metric there exists a corresponding Laplacian, which we denote by Δ_s to distinguish it from the hyperbolic Laplacian

$$\Delta = y^2\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right).$$

Let $(\cdot,\cdot)_s$ be the inner product in $L^2(\Gamma\backslash\mathbb{H},\chi)_s$. With the help of Δ_s we can introduce the Sobolev spaces $H^i(\Gamma\backslash\mathbb{H},\chi)_s$ where $i = 1, 2$:

$$H^1(\Gamma\backslash\mathbb{H},\chi)_s = \{f \in L^2(\Gamma\backslash\mathbb{H},\chi)_s : (f,f)_s + (\Delta_s f, f)_s < \infty\},$$
$$H^2(\Gamma\backslash\mathbb{H},\chi)_s = \{f \in L^2(\Gamma\backslash\mathbb{H},\chi)_s : (f,f)_s + (\Delta_s f, \Delta_s f)_s < \infty\},$$

where of course $\Delta_s f$ is understood in the distributional sense.

Equivalently, H^1 respectively H^2 can be regarded as the set of functions in $L^2(\Gamma\backslash\mathbb{H},\chi)_s$ with the extra property that their distributional derivates up to first, respectively second order be again square integrable.

2 Estimating Representations

In order to understand the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H},\chi)$ it is essential to find an equivalent for the classical Eisenstein series. Firstly, let us recall the definition of the classical Eisenstein series $E(z,s)$. When $\mathfrak{a}$ is a cusp with corresponding scaling matrix $\sigma_{\mathfrak{a}}$, then

$$E_{\mathfrak{a}}(z,s) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s,$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$. This series converges locally uniformly absolutely for $\operatorname{Re}(s) > 1$. We will later see, that it is most reasonable to define the Eisenstein series in our situation as

$$E_{\mathfrak{a}}(z,s,\chi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \chi(\gamma^{-1}) P_{\mathfrak{a}},$$

where $P_{\mathfrak{a}}$ is the orthogonal projection onto $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}),1)$. Since in this definition the representation χ is involved, we need to study the growth of χ to show convergence of $E_{\mathfrak{a}}(z,s,\chi)$. We will prove, that there exists an $\alpha > 0$, such that $\|\chi(\gamma)\| \le O\left(\mathrm{Im}(\gamma z)^{\alpha}\right)$, where the norm is the Frobenius norm $\|\cdot\| = \|\cdot\|_F$. This will give the convergence of the Eisenstein series, since we can now estimate them against the classical Eisenstein series.

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \|\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \chi(\gamma^{-1})P_{\mathfrak{a}}\| \le \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{s+\alpha}$$

Thus we get convergence of $E_{\mathfrak{a}}(z,s,\chi)$ for $\mathrm{Re}(s) > 1+\alpha$.

Developing the trace formula we will also encounter the invariant kernel

$$K(z,w,\chi) = \sum_{\gamma\in\Gamma} k(z,\gamma w)\chi(\gamma)$$

where k is a function on $\mathbb{H}\times\mathbb{H}$, which depends only on the hyperbolic distance of its two arguments. To show convergence and continuity of the invariant kernel K an estimate for χ will also be required.

The idea to show an estimate for χ is as follows: Let us assume, that Γ has parabolic elements. Since the representation $\chi:\Gamma\to V$ is unitary at cusps we have

$$\|\chi(\gamma_{\mathfrak{a}}\gamma)\| = \|\chi(\gamma)\| \tag{23}$$

for every cusp $\mathfrak{a}$. This behaviour is very similar to the imaginary part of an element $z\in\mathbb{H}$. If we take an element of $\gamma\in\mathrm{SL}(2,\mathbb{R})$, which fixes the cusp at infinity, and hence when γ is of the form

$$\gamma = \pm\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

with $t\in\mathbb{R}$, we have

$$\mathrm{Im}(\gamma z) = \mathrm{Im}(z).$$

We will show, that a similar result is true for $\chi(\gamma)$ namely

$$\|\chi(\gamma)\| = O(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)), \tag{24}$$

since we have similarly as in (23) $\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\gamma z) = \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)$. Recall that the fundamental domain F is of the form $F = D(z_0)$ where $z_0 \in \mathbb{H}$ is not an elliptic fixed point, see (21). The fundamental domain F has a finite number of neighbouring fundamental domains of the form $\gamma_i F, \gamma_i \in \Gamma, i = 1, \ldots, l$. The finite number of elements $\gamma_1, \ldots, \gamma_l$ generate the group Γ.
We will start, proving some technical results (Propositions 2.2, 2.3, 2.5), which will culminate in Theorem 2.6. Basically, this theorem gives an estimate of the number of generators we will need to express an element $\gamma \in \Gamma$ as a product of these. These results go back to Eichler and are part of his work in [Eic65].

Definition 2.1 (Normal representation of elements in Γ)**.** A representation $\gamma = \eta_1\eta_2\cdots\eta_r$ of $\gamma \in \Gamma$ with $\eta_i \in \{\gamma_j\}_{j=1}^{l}$ is called *normal* if the η_i are chosen in the following manner: Besides z_0 we fix another point $z_1 \in F$ different from z_0. We assume $\eta_1, \ldots, \eta_{j-1}$ to be already found. Now join $\eta_1\cdots\eta_{j-1}z_0$ with γz_1 by the unique geodesic which contains both of these elements. Following this geodesic from the point $\eta_1\cdots\eta_{j-1}z_0$ in the direction of γz_1, after leaving the fundamental domain $\eta_1\cdots\eta_{j-1}F$ the geodesic enters a fundamental domain of the form $\eta_1\cdots\eta_{j-1}\gamma_k F, k \in \{1, \ldots, l\}$.
In the following we will set

$$\zeta_j := \eta_1\cdots\eta_j z_0.$$

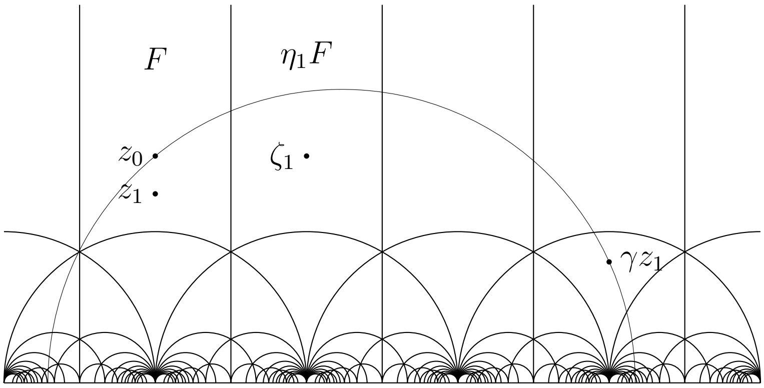
Let $g(\zeta_{i-1}, \gamma z_1)$ be the unique geodesic which joins ζ_{i-1} and γz_1. Let $\eta \in g(\zeta_{i-1}, \gamma z_1)$ be the unique point inbetween ζ_{i-1} and γz_1 such that $\eta \in \partial\eta_1\cdots\eta_{i-1}F$. Then by definition of the fundamental domain F we have

$$\rho(\zeta_{j-1}, \gamma z_0) = \rho(\zeta_j, \gamma z_0),$$

and hence

$$\begin{aligned} \rho(\zeta_j, \gamma z_0) &\leq \rho(\zeta_j, \eta) + \rho(\eta, \gamma z_0) \\ &= \rho(\zeta_{j-1}, \eta) + \rho(\eta, \gamma z_0) \\ &= \rho(\zeta_{j-1}, \gamma z_0). \end{aligned}$$

Furthermore ζ_j is different from $z_0, \zeta_1, \ldots, \zeta_{j-1}$. Then, because there are only finitely many translates of z_0 with bounded distance from γz_0 there exists j_0 with $\zeta_{j_0} \in \gamma F$ and the processus terminates at j_0.



The reason why we have chosen besides z_0 a different point z_1 is that under these circumstances the following proposition holds:

Proposition 2.2. *There exists a positive constant c_1 independent of γ and z_1, such that*

$$\rho(\zeta_i, \gamma z_1) \leq \rho(\zeta_{i-1}, \gamma z_1) - c_1$$

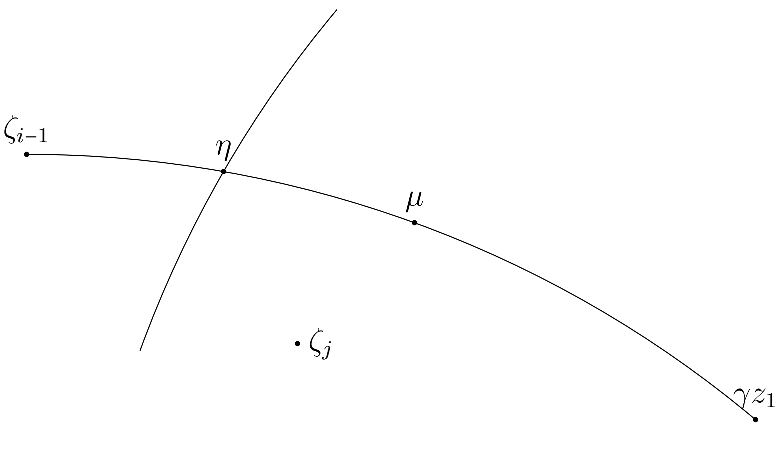
if η_i is not parabolic.

Proof. We recall that the fundamental domain F is of the form

$$F = \{z \in \mathbb{H} : \rho(z, z_0) < \rho(z, \gamma z_0) \text{ for all } \gamma \in \Gamma\},$$

where z_0 is no fix point. Let $g(\zeta_{i-1}, \gamma z_1)$ be the unique geodesic which joins ζ_{i-1} and γz_1. Let $\eta \in g(\zeta_{i-1}, \gamma z_1)$ be the unique point inbetween ζ_{i-1} and γz_1 such that $\eta \in \partial \eta_1 \cdots \eta_{i-1} F$. Finally choose $\mu \in g(\zeta_i, \gamma z_1)$ to be the unique point $\neq \zeta_{i-1}$ such that

$$\rho(\eta, \mu) = \rho(\zeta_{i-1}, \eta). \tag{25}$$



By the triangle inequality we get

$$\rho(\zeta_i, \gamma z_1) < \rho(\zeta_i, \mu) + \rho(\mu, \gamma z_1). \tag{26}$$

The fact that ζ_i does not belong to the geodesic joining $g(\zeta_{i-1}, \gamma z_1)$ excludes equality in (26). Because of inequality (26) and $\rho(\mu, \gamma z_1) = \rho(\zeta_{i-1}, \gamma z_1) - \rho(\zeta_{i-1}, \mu)$ we get

$$\rho(\zeta_i, \gamma z_1) - \rho(\zeta_{i-1}, \gamma z_1) < \rho(\zeta_i, \mu) - \rho(\zeta_{i-1}, \mu),$$

or equivalently

$$\rho(\zeta_{i-1}, \mu) - \rho(\zeta_i, \mu) < \rho(\zeta_{i-1}, \gamma z_1) - \rho(\zeta_i, \gamma z_1). \tag{27}$$

The left hand side of (27) can be estimated from below by using (25) and the triangle inequality

$$0 = \rho(\zeta_{i-1}, \mu) - \rho(\zeta_i, \eta) - \rho(\eta, \mu) \leq \rho(\zeta_{i-1}, \mu) - \rho(\zeta_i, \mu), \tag{28}$$

since $\rho(\zeta_i, \eta) = \rho(\eta, \mu) = 1/2\rho(\zeta_{i-1}, \mu)$. Thus we get by (27) and (28)

$$\rho(\zeta_{i-1}, \mu) - \rho(\zeta_i, \mu) = 2\rho(\eta, \mu) - \rho(\zeta_i, \mu) \geq c_1, \tag{29}$$

with some positive constant $c_1 > 0$ depending on γz_1. The left hand side of the equality in (29) can also be regarded as a continuous function of η. But if η_i is not parabolic, then η varies only on the compact boundary where the two fundamental domains $\eta_1 \cdots \eta_{i-1} F$ and $\eta_1 \cdots \eta_i F$ meet. Thus we see that the constant c_1 can be chosen independent of γz_1. Hence the assertion is proved. □

Now we will go one step further and prove the following proposition:

Proposition 2.3. *There exists a positive constant c_2 independent of γ and z_1, such that*

$$\rho(\zeta_{i+1}, \gamma z_1) \leq \rho(\zeta_{i-1}, \gamma z_1) - c_2 \tag{30}$$

as long as η_i, η_{i+1} are not parabolic elements belonging to the same conjugacy class.

Proof. In the case that either η_i or η_{i+1} is not parabolic, (30) is handled by using the previous Proposition 2.2.
Now assume η_i and η_{i+1} are two parabolic elements belonging to distinct parabolic conjugacy classes. They fix two distinct cusps $\mathfrak{a}, \mathfrak{b}$ of F respectively. $\mathfrak{a}$ can also be described as the cusp which F and $\eta_i F$ have in commen. Similar for $\mathfrak{b}$. If we vary γz_1 then η varies on the boundary which F and $\eta_i F$ share. Analogously as η and μ are defined for ζ_{i-1} in the proof of Proposition 2.2, one defines η' and μ' for ζ_i.
Now let a positive constant $C > 0$ be given. First we consider the case that

$$\rho(\zeta_{i-1}, \eta) \leq C \text{ or } , \rho(\zeta_i, \eta') \leq C.$$

This means that either η or η' are restricted to a compact subset and the same argument as in the proof of Proposition 2.2 applies. So only the case

$$\rho(\zeta_{i-1}, \eta), \rho(\zeta_i, \eta') \geq C$$

remains. We use the inequality

$$\rho(\zeta_{i-1}, \eta) < \rho(\zeta_{i-1}, \gamma z_1) \tag{31}$$

which is clear by the definition of η. The same of course holds for ζ_i. We will show that $\rho(\zeta_{i-1}, \eta)$ and $\rho(\zeta_i, \eta')$ can not both at the same time tend to infinity, which means that either η or η' is restricted to a compact subset and the argument from Proposition 2.2 applies. If $\rho(\zeta_{i-1}, \eta) \to \infty$, then by (31) also $\rho(\zeta_{i-1}, \gamma z_1) \to \infty$. Same for ζ_{i-1} replaced by ζ_i. But this means that γz_1 moves into the cuspidal area of $\mathfrak{a}$. Assuming that $\rho(\zeta_i, \eta) \to \infty$ too we get by the same reasoning that γz_1 moves into the cuspidal area of $\mathfrak{b}$. But since $\mathfrak{a} \neq \mathfrak{b}$, we get a contradiction. This finishes the proof. □

Now we want to give an upper bound for the number of factors l in the normal representation of γ. More precisely we regroup consecutive elements belonging to the same parabolic class and count them only once. This is made precise as follows:

Definition 2.4. Let $\gamma = \eta_1 \cdots \eta_r$ be given in its normal representation. If η_1 is not parabolic set $\nu_1 = \eta_1$. If η_1 is parabolic then set $\nu_1 = \eta_1 \cdots \eta_{r_1}$ where r_1 is the biggest number, such that $\eta_1, \eta_2, \ldots, \eta_{r_1}$ all belong to the same parabolic conjugacy class. Continuing in the same way for $\nu_2, \nu_3, \ldots$ we obtain a representation

$$\gamma = \nu_1 \cdots \nu_k. \tag{32}$$

The occuring ν_i are called the *tranches* of γ. We furthermore define for $\gamma \in \Gamma$

$$\mu(\gamma) := a^2 + b^2 + c^2 + d^2, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$\mu(\gamma)$ is intimately connected to the hyperbolic distance $\rho(i, \gamma i)$ as the following proposition shows.

Proposition 2.5. *We have*

$$\frac{1}{2}\mu(\gamma) \leq \exp(\rho(i, \gamma i)) \leq \mu(\gamma).$$

By the singular value decomposition, one can find for each $\gamma \in \mathrm{SL}(2, \mathbb{R})$ two ortogonal matrices A_1, A_2 such that $A_1 \gamma A_2$ is diagonal. Since neither $\mu(\gamma)$ nor $\rho(i, \gamma i)$ change under multiplication of γ by an orthogonal matrix from the left, or the right, we can hence assume γ to be diagonal

$$\gamma = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}.$$

Then $\rho(i, \gamma i) = |\log a^2|$. Now

$$\frac{1}{2}(a^2 + a^{-2}) \leq \max(a^2, a^{-2}) \leq a^2 + a^{-2}$$

and the proof is finished.

Theorem 2.6. *Let $\gamma \in \Gamma$ an element and*

$$\gamma = \eta_1 \ldots \eta_r \tag{33}$$

its normal representation. The number of tranches k in the representation (32) can be estimated by

$$k \leq O(\log(\mu(\gamma)) + 1) \tag{34}$$

where the implied constant is independent of γ.

Proof. Let $\gamma = \nu_1 \cdots \nu_k$ be the representation of γ in tranches. Using Proposition 2.3 we obtain

$$\rho(z_0, \gamma z_1) \geq \rho(\nu_2 \nu_1 z_0, \gamma z_1) + c_2 \geq \cdots \geq \frac{c_2 k}{2} + \rho(\gamma z_0, \gamma z_1) = \rho(z_0, z_1).$$

The triangle inequality yields

$$\rho(z_0, \gamma z_1) \leq \rho(z_0, i) + \rho(i, \gamma i) + \rho(\gamma i, \gamma z_1),$$

and thus

$$k \leq O(\rho(i, \gamma i) + \rho(i, z_0) + \rho(i, z_1) - \rho(z_0, z_1)).$$

Using Proposition 2.5 concludes the proof. □

2.1 The general case

Now we can formulate a first result concerning the growth of representations $\chi : \Gamma \to V$. This result is valid in the compact as well as in the non compact case.

Proposition 2.7. *Let $\chi : \Gamma \to V$ be a finite dimensional representation (unitary at cusps if Γ contains parabolic elements). Then there exists an $\alpha > 0$ such that for $z, w \in \mathbb{H}$*

$$\|\chi(\gamma)\| = O(1 + u(\gamma z, w)^{\alpha}), \tag{35}$$

where the implied constant depends on z and w but is independent of γ. Furthermore the implied constant can be chosen independently for z and w in a compact subset of $\mathbb{H} \times \mathbb{H}$.

Proof. Take $\gamma \in \Gamma$ and let $\gamma = \nu_1 \cdots \nu_k$ be its representation in tranches. Note that in the compact case it coincides with its normal representation, since there are no parabolic elements. Then we estimate

$$\|\chi(\gamma)\| \leq \prod_{i=1}^{k} \|\chi(\nu_i)\|.$$

So if we bound the finitely many $\|\chi(\gamma_i)\|$ (where $\gamma_1, \ldots, \gamma_l$ is a generating set of Γ as introduced in the definition 2.1) by a constant K we obtain $\|\chi(\gamma)\| \leq K^k$. Now by theorem 2.6 we have $k \leq C(\log(\mu(\gamma)) + 1)$ with some positive constant $C > 0$ and we thus obtain the estimate

$$\|\chi(\gamma)\| \leq K^{C \log \mu(\gamma) + C} \leq \mu(\gamma)^{\alpha} K^C \ll \mu(\gamma)^{\alpha} \tag{36}$$

where $\alpha = C \log(K)$. We further analyze $\mu(\gamma)$. As in Proposition 2.5, we find that there exists a constant c, which can be chosen independent of w in a compact subset such that

$$\mu(\gamma) \leq c \exp(\rho(w, \gamma w)). \tag{37}$$

We continue by using the triangle inequality to compute

$$\begin{aligned} \exp(\rho(w, \gamma w)) &\leq \exp(\rho(w, \gamma z) + \rho(\gamma z, \gamma w)) \\ &\leq \exp(\rho(z, w)) \exp(\rho(w, \gamma z)). \end{aligned} \tag{38}$$

Remember that the relation between the hyperbolic distance and u is given by

$$\cosh \rho(z, w) = 1 + \frac{1}{2} u(z, w).$$

Of course we have $\exp/2 \leq \cosh$ and hence we get

$$\mu(\gamma) = O(1 + u(\gamma z, w))$$

where the implied constant can be chosen independently on z and w varying in a compact subset of $\mathbb{H} \times \mathbb{H}$. Combining this with (37) and (38) yields the result. $\square$

2.2 The non compact case

After having shown a growth estimate for the case of general Γ, we now specialize to the case of Γ having parabolic elements. As said before, we wish to deduce a bound of the form

$$\|\chi(\gamma)\| = O(\operatorname{Im}(\gamma z)^{\alpha}) \tag{39}$$

for some $\alpha > 0$, as to be able to show convergence of the Eisenstein series, which we will introduce in chapter 3.
In (36) we have shown, that we can estimate $\|\chi(\gamma)\|$ against the entries of γ namely

$$\|\chi(\gamma)\| \ll \mu(\gamma)^\alpha.$$

Recall that $\mu(\gamma) = a^2 + b^2 + c^2 + d^2$. We want to make the estimate (39) independent of the matrix entries of γ in the first row. This will be achieved by the following proposition.

Proposition 2.8. *Let Γ have a cusp at ∞ of width one. For given*

$$\begin{pmatrix} a & b \\ m & n \end{pmatrix} \in \Gamma$$

there exists $\gamma \in \Gamma_\infty$ such that

$$\gamma \begin{pmatrix} a & b \\ m & n \end{pmatrix} = \begin{pmatrix} a' & b' \\ m & n \end{pmatrix}$$

and $a'^2 + b'^2 \ll m^2 + n^2$, the implied constant depending only on the group Γ.

Proof. Let $\left(\begin{smallmatrix} a & b \\ m & n \end{smallmatrix}\right) \in \Gamma$ be given. We can assume $n > 0$ and $m \neq 0$, otherwise the statement is clear. Then we can find $q \in \mathbb{Z}$ such that $a = mq + l, 0 \leq l < |m|$. Then set $\gamma = \left(\begin{smallmatrix} 1 & -q \\ 0 & 1 \end{smallmatrix}\right)$. Clearly $a'^2 = l^2 \leq m^2$. Furthermore $b' = b - nq = (ln - 1)/m$ by the condition $an - bm = 1$. Now

$$\left(\frac{ln-1}{m}\right)^2 \leq \frac{(l+1)^2 n^2}{m^2} \leq \frac{(m+1)^2}{m^2} n^2 \ll n^2$$

since $|m| > c > 0$ for some constant c depending only on the group Γ. $\square$

Remark 2.9. For a matrix $\eta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2, \mathbb{R})$ we already defined $\mu(\eta) = a^2 + b^2 + c^2 + d^2$. Now let Γ have a cusp at ∞ of width one. By the previous proposition there exists a constant $C > 0$ such that for any $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma$ there exists an element $\gamma' \in \Gamma$ such that γ and γ' differ only by multiplication by an element in Γ_∞ from the left and $\mu(\gamma') \leq C(c^2 + d^2)$. This we will now use to estimate $\chi(\gamma)$.

Proposition 2.10. *Let Γ have a cusp at ∞ of width one. Then there exists a constant $\alpha > 0$ such that for $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma$ we can estimate the operator norm by $\|\chi(\gamma)\| \ll (c^2 + d^2)^\alpha$. The implied constant depends on the representation χ only.*

Proof. Since the representation is unitary at cusps we can assume by remark 2.9 that $\mu(\gamma) \leq C(c^2 + d^2)$ for some fixed constant C independent of γ. Let $\gamma = \nu_1 \cdots \nu_k$ be its representation in tranches (see Definition 2.4). Then we estimate

$$\|\chi(\gamma)\| \leq \prod_{i=1}^{k} \|\chi(\nu_i)\|.$$

So if we bound the finitely many $\|\chi(\gamma_i)\|$ (where $\gamma_1, \ldots, \gamma_l$ is a generating set of Γ as introduced in the Definition 2.1) by a constant K we obtain $\|\chi(\gamma)\| \leq K^k$. Now by Theorem 2.6 we have $k \leq C(\log(\mu(\gamma)) + 1)$ with some positive constant $C > 0$ and thus obtain the estimate

$$\|\chi(\gamma)\| \leq K^{C \log(\mu(\gamma)) + C} \leq \mu(\gamma)^\alpha K^C \ll (c^2 + d^2)^\alpha$$

where $\alpha = C \log(K)$. By using Remark 2.9 this proves $\|\chi(\gamma)\| \ll (c^2 + d^2)^\alpha$. □

To make the above estimate useful to show the convergence of Eisenstein series, we need to bound $c^2 + d^2$ by some term involving a power of $\operatorname{Im}(\gamma z)$. This is the content of the next proposition.

Proposition 2.11. *For $c, d \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$ we have*

$$\left(\frac{y^2}{1 + |z|^2}\right)(c^2 + d^2) \leq |cz + d|^2.$$

In particular, combining this result with Proposition 2.10 yields

$$\|\chi(\gamma)\| \leq O\left(\left(\frac{1 + |z|^2}{y^2}\right)^\alpha |cz + d|^{2\alpha}\right), \tag{40}$$

where the implied constant depends on the representation χ only. Here, as above, we write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Proof. To obtain the estimate, observe that $|cz+d|^2 \geq y^2c^2$ and $|\overline{z}|^2|cz+d|^2 = |c|z|^2 + dz|^2 \geq d^2y^2$, so that $|cz+d|^2 \geq \frac{y^2(c^2+d^2)}{1+|z|^2}$. □

3 Eisenstein series

3.1 Incomplete Eisenstein series

To obtain smooth functions $f : \mathbb{H} \to V$ with $f(\gamma z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma$ and all $z \in \mathbb{H}$, the most natural procedure is to take a smooth compactly supported function ψ on the positive real axis and average it over the group Γ:

$$f(z) = \sum_{\gamma\in\Gamma} \psi(\mathrm{Im}(\gamma z))\chi(\gamma^{-1}).$$

We formally compute that the above series has the invariance condition, but nevertheless this naïve approach does not work, since convergence will fail at the cusps. Hence, instead of summing over Γ it is more reasonable to sum over the quotient $\Gamma_\mathfrak{a}\backslash\Gamma$. But then we need a summand which is invariant under $\Gamma_\mathfrak{a}$. Thus we modify the summand to be

$$\psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}z))\chi(\gamma^{-1})P_\mathfrak{a}.$$

Definition 3.1. Let $\psi \in C_c^\infty(\mathbb{R}^+)$ be a smooth, compactly supported function on the positive real axis. For a cusp $\mathfrak{a}$ we define the *incomplete Eisenstein series* to ψ as

$$E_\mathfrak{a}(z|\psi) = \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_\mathfrak{a}^{-1}\gamma z))\chi(\gamma^{-1})P_\mathfrak{a}.$$

For $v \in V$ consider the function $E_\mathfrak{a}(\,\cdot\,|\psi)v : \mathbb{H} \to V$. Since ψ has compact support the function $E_\mathfrak{a}(\,\cdot\,|\psi)v$ is bounded on the fundamental domain F, hence an element of $L^2(\Gamma\backslash\mathbb{H},\chi)$.

We let $\mathcal{E}_\mathfrak{a}(\Gamma\backslash\mathbb{H},\chi)$ be the closure of the set of all incomplete Eisenstein series $E_\mathfrak{a}(z|\psi)v$ in $L^2(\Gamma\backslash\mathbb{H},\chi)$, where ψ varies in $C_c^\infty(\mathbb{R}^+)$ and v ranges over all elements of V. Then we set

$$\mathcal{E}(\Gamma\backslash\mathbb{H},\chi) = \bigcup_\mathfrak{a} \mathcal{E}_\mathfrak{a}(\Gamma\backslash\mathbb{H},\chi).$$

Note that $\mathcal{E}_{\mathfrak{a}}(\Gamma\backslash\mathbb{H},\chi) = \{0\}$ if $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}),1) = 0$. More generally the size of the eigenspaces $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}),1)$ will determine the size of the continuous spectrum, as we will see later. Hence, there will be no continuous spectrum when $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}),1) = \{0\}$ for all cusps $\mathfrak{a}$.

3.2 Eisenstein series

Definition 3.2. We now come to a slightly different case, than in Definition 3.1. Instead of taking $\psi \in C_c^\infty(\mathbb{R}^+)$, we now choose $\psi(z) = \mathrm{Im}(z)^s$ for some $s \in \mathbb{C}$. For the cusp $\mathfrak{a}$ we then define the *Eisenstein series*

$$E_{\mathfrak{a}}(z,s,\chi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \chi(\gamma^{-1}) P_{\mathfrak{a}}.$$

In comparison to the incomplete Eisenstein series, the functions $E_{\mathfrak{a}}(z,s,\chi)v$ where $v \in V$, will not be elements of $L^2(\Gamma\backslash\mathbb{H},\chi)$. But at first we shall show that the series actually converges in some half plane $\mathrm{Re}(s) > \sigma_0$.

Proposition 3.3. *There exists $\sigma_0 > 0$, such that $E_{\mathfrak{a}}(z,s,\chi)$ converges locally uniformely absolutely (l.u.a.) in the half plane* $\mathrm{Re}(z) > \sigma_0$ *for any cusp $\mathfrak{a}$. In particular, $E_{\mathfrak{a}}(z,s,\chi)$ represents a holomorphic function in s on the half plane* $\mathrm{Re}(s) > \sigma_0$.

Proof. We use the inequality (40) which gives

$$\|\chi(\gamma)\| \leq O\left(\left(\frac{1+|\sigma_{\mathfrak{a}}^{-1}z|^2}{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}z)^2}\right)^{\alpha} |c\sigma_{\mathfrak{a}}^{-1}z+d|^{2\alpha}\right),$$

for some $\alpha > 0$ and the implied constant depends on χ only. As always we write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Keeping in mind, that

$$\mathrm{Im}(\gamma z) = \frac{\mathrm{Im} z}{|cz+d|^2}$$

we get

$$\begin{aligned}\|E_{\mathfrak{a}}(z,s,\chi)\| &\leq \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \|(\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}z))^{s-\alpha}\chi(\gamma^{-1})P_{\mathfrak{a}}\| \\ &\ll \left(\frac{1+|\sigma_{\mathfrak{a}}^{-1}z|^2}{\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}z)}\right)^{\alpha} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} (\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}z))^{s-\alpha}.\end{aligned}$$

It is well-known that the series

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}z)^s$$

converges l.u.a. in the half-plane $\operatorname{Re}(s) > 1$. Hence it follows, that $E_{\mathfrak{a}}(z,s,\chi)$ converges l.u.a. in the half-plane $\operatorname{Re}(s) > 1+\alpha$. Now α may vary if we choose a different cusp $\mathfrak{b}$. But since there are only finitely many inequivalent cusps we will find a uniform bound for α. Hence we also find σ_0 such that $E_{\mathfrak{a}}(z,s,\chi)$ converges l.u.a. in $\operatorname{Re}(s) > \sigma_0$ for every cusp $\mathfrak{a}$. □

3.3 Fourier expansion

To estimate the growth of $E_{\mathfrak{a}}(z,s,\chi)$ when z approaches some cusp $\mathfrak{b}$ we need to study its Fourier expansion. We will study the more general case $E_{\mathfrak{a}}(z|\psi)$ and later we will set $\psi(z) = \operatorname{Im}(z)^s$. To fix notation, we set

$$b(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

and we let B be the subgroup in $\operatorname{PSL}(2,\mathbb{R})$ generated by $b(1)$. By definition of the incomplete Eisenstein series we find

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}(z+1)|\psi) = \chi(\gamma_{\mathfrak{b}})E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z|\psi).$$

Thus we expect a Fourier expansion of the incomplete Eisenstein series. But first we would like to introduce some necessary tools. We start with the *K-Besselfunction*.

Definition 3.4. Let $\lambda \in \mathbb{C}$ and consider the second-order differential equation

$$F''(y) + (\lambda y^{-2} - 1)F(y) = 0$$

on $\mathbb{R}^+$. We have two lineary independent solutions F_1, F_2 which solve this differential equation, where F_1 decays exponentially and F_2 grows exponentially if $y \to \infty$. We define the *K-Besselfunction* to be the function which satisfies

$$(2\pi^{-1}y)^{1/2}K_{s-1/2}(y) = F_1(y),$$

and $s \in \mathbb{C}$ is chosen such that $s(1-s) = \lambda$.

The second tool we need to introduce is the double coset decomposition of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ with respect to the group B. This is given as follows:

Theorem 3.5. *Let $\mathfrak{a}, \mathfrak{b}$ be cusps for Γ. We then have a disjoint union*

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a}\mathfrak{b}}\Omega_\infty \cup \bigcup_{c>0} \bigcup_{d \bmod c} \Omega_{d/c},$$

where $\Omega_\infty = B\omega_\infty B$, $\Omega_{d/c} = B\omega_{d/c}B$ with

$$\omega_\infty = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}, \quad \omega_{d/c} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}},$$

respectively.

Proof. For a proof we refer the reader to Theorem 2.7. in [Iwa02]. □

Now it is possible to deduce the Fourier expansion of the Eisenstein series. Let $\eta(\gamma) := \chi(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1})$. Then we use the double coset decomposition to write

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z|\psi) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)\chi(\gamma)^{-1}P_{\mathfrak{a}} = \sum_{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} \psi(\gamma z)\eta(\gamma)^{-1}P_{\mathfrak{a}} \tag{41}$$

$$= \delta_{\mathfrak{a}\mathfrak{b}}\psi(y)P_{\mathfrak{a}} + \sum_{c>0}\sum_{d(\text{mod } c)}\sum_{n\in\mathbb{Z}} \psi(\omega_{cd}(z+n))\chi(\gamma_{\mathfrak{b}})^{-n}\eta(\omega_{cd}b(n))^{-1}P_{\mathfrak{a}}. \tag{42}$$

$\chi(\gamma_{\mathfrak{b}})$ is a unitary automorphism, so if $e(\nu_1) = 1, \ldots, e(\nu_{j(\mathfrak{b})})$ are the eigenvalues where $\nu_k \in [0, 2\pi)$ and $P_1 = P_{\mathfrak{b}}, \ldots, P_{j(\mathfrak{b})}$ the corresponding orthogonal projections onto the eigenspaces, we have

$$\chi(\gamma_{\mathfrak{b}}) = \sum_{k=1}^{j(\mathfrak{b})} e(\nu_k)P_k.$$

Hence we continue to compute

$$= \delta_{\mathfrak{a}\mathfrak{b}}\psi(y)P_{\mathfrak{a}} + \sum_{c>0}\sum_{d(\bmod c)}\sum_{k=1}^{j(\mathfrak{b})} P_k\eta(\omega_{cd})^{-1}\sum_{n\in\mathbb{Z}} e(-n\nu_k)\psi(\omega_{cd}(z+n))P_{\mathfrak{a}}.$$

We keep looking just at the summand for a fixed k

$$\sum_{n\in\mathbb{Z}} e(-n\nu_k)\psi(\omega_{cd}(z+n))P_{\mathfrak{a}},$$

and apply Poisson summation to obtain

$$\sum_{n\in\mathbb{Z}} e(-n\nu_k)\psi(\omega_{cd}(z+n))P_{\mathfrak{a}} = \sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} e(-\nu_k t)\psi(\omega_{cd}(z+t))e(-nt)P_{\mathfrak{a}}\,dt.$$

A small computation shows

$$\omega_{cd}(z+t) = a/c - c^{-2}(t+x+d/c+iy)^{-1},$$

such that by performing the change of variables $t \mapsto t - x - d/c$ we arrive at

$$\sum_{n\in\mathbb{Z}} e((n+\nu_k)(x+d/c))\int_{-\infty}^{\infty}\psi\left(\frac{yc^{-2}}{t^2+y^2}\right)e(-(\nu_k+n)t)\,dt.$$

Hence

$$\begin{aligned} &E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z|\psi) = \delta_{\mathfrak{a}\mathfrak{b}}\psi(y)P_{\mathfrak{a}} \\ &+ \sum_{k=1}^{j(\mathfrak{b})} P_k\sum_{n\in\mathbb{Z}} e((n+\nu_k)x)\sum_{c>0}\mathcal{S}_{\mathfrak{a}\mathfrak{b}}(n+\nu_k,c,\chi)\int_{-\infty}^{\infty}\psi\left(\frac{yc^{-2}}{t^2+y^2}\right)e(-(\nu_k+n)t)\,dt \end{aligned} \tag{43}$$

where $\mathcal{S}_{\mathfrak{a}\mathfrak{b}}(r,c,\chi)$ is the *Kloosterman sum*

$$\mathcal{S}_{\mathfrak{a}\mathfrak{b}}(r,c,\chi) = \sum_{d(\bmod c)} e\left(r\frac{d}{c}\right)\eta(\omega_{cd})^{-1}P_{\mathfrak{a}}.$$

We now specialize to the Eisenstein series, that is in the above we consider the case $\psi(y) = y^s$. Then the integral in (43) can be computed explicitely using

[Iwa02, p. 205]. This yields

$$\int_{-\infty}^{\infty} (t^2 + y^2)^{-s}\, dt = \pi^{1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} y^{1-2s} \tag{44}$$

and

$$\int_{-\infty}^{\infty} (t^2 + y^2)^{-s} e(-rt)\, dt = 2\pi^s \Gamma(s)^{-1} |r|^{s-1/2} y^{-s+1/2} K_{s-1/2}(2\pi|r|y) \tag{45}$$

where $r \in \mathbb{R}, r \neq 0$ and K_s is the K-Besselfunction.

Theorem 3.6. *Let $\mathfrak{a}, \mathfrak{b}$ be cusps for Γ and let s be in the domain of absolute convergence for the Eisenstein series. We have the Fourier expansion*

$$\begin{aligned} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) =& (\delta_{\mathfrak{ab}} y^s + \varphi_{\mathfrak{ab}}(s) y^{1-s} \\ &+ \sum_{n \neq 0} \varphi_{\mathfrak{ab}}(n, s) W_s(nz) + \sum_{k=2}^{j(\mathfrak{b})} \sum_{n \in \mathbb{Z}} \varphi_{\mathfrak{ab}}(n + \nu_k, s) W_s((n + \nu_k) z)) P_{\mathfrak{a}} \end{aligned}$$

where

$$\varphi_{\mathfrak{ab}}(s) = \pi^{1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} P_{\mathfrak{a}} \sum_{c>0} c^{-2s} \mathcal{S}_{\mathfrak{ab}}(0, c, \chi),$$

$$\varphi_{\mathfrak{ab}}(n + \nu_k, s) = \pi^s \Gamma(s)^{-1} |n + \nu_k|^{s-1} P_k \sum_{c>0} c^{-2s} \mathcal{S}_{\mathfrak{ab}}(n + \nu_k, c, \chi),$$

and $W_s(z)$ is the Whittaker function

$$W_s(z) = 2y^{1/2} K_{s-1/2}(2\pi y) e(x).$$

The above theorem allows us to give an estimate about the growth of $E_{\mathfrak{a}}(z, s, \chi)$ when z approaches the cusp $\mathfrak{b}$.

Proposition 3.7. *For s in the domain of absolute convergence of $E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}(z, s, \chi))$ we have*

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) = (\delta_{\mathfrak{ab}} y^s + \varphi_{\mathfrak{ab}} y^{1-s}) P_{\mathfrak{a}} + O(e^{-\beta y}),$$

as $\mathrm{Im}(z) \to \infty$ where $0 < \beta < \min_k(v_k)$ is arbitrary. The implied constant depends on the group Γ, χ and $s \in \mathbb{C}$ only.

Proof. Use the asymptotics $W_s(z) \sim e^{-2\pi y}$. Since for $y \geq \varepsilon > 0$ we have

$$\sum_{n=0}^{\infty} (n + \nu_k)^{s-1} e^{-2\pi y(\nu_k + n)} = O(e^{-\beta y})$$

for $0 < \beta < \nu_k$, the proposition is proved. □

Proposition 3.8. *Let $s \in \mathbb{C}$ be in the domain of absolut convergence of $E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi)$ with $\mathrm{Re}(s) = \sigma$. Then*

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) \ll \frac{1}{y^{\sigma}} + y^{\sigma}$$

where the implied constant depends on Γ, χ and σ only.

Proof. First we consider the case $y \geq 1$. Since

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) = \delta_{\mathfrak{a}\mathfrak{b}} y^s P_{\mathfrak{a}} + O(y^{1-\sigma}),$$

it follows

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) \ll y^{\sigma}.$$

On the other hand for $z = x + iy$ and $\tilde{z} = x + i$ the inequality

$$\mathrm{Im}(z)\mathrm{Im}(\gamma z) \leq \mathrm{Im}(\gamma \tilde{z})$$

holds for $y \leq 1$ and arbitrary $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. In consequence this yields for $y \leq 1$

$$\| E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) y^{\sigma} \| \leq \| E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} \tilde{z}, \sigma) \| \ll 1.$$ □

4 The discrete spectrum

4.1 Cusp forms

We would like to find a subspace of $W \subset L^2(\Gamma \backslash \mathbb{H}, \chi)$, such that we obtain a direct sum decomposition

$$L^2(\Gamma \backslash \mathbb{H}, \chi) = W \oplus \mathcal{E}(\Gamma \backslash \mathbb{H}, \chi).$$

In the classical theory for unitary χ, the choice for W is the orthogonal complement to $\mathcal{E}(\Gamma \backslash \mathbb{H}, \chi)$, which is equal to the space of cusp forms. Hence, a reasonable choice in our case will also be to choose W equal to the space of cusp forms.

Definition 4.1. $f \in L^2(\Gamma\backslash\mathbb{H},\chi)$ is said to be a *cusp form*, if for each cusp $\mathfrak{a}$

$$\int_0^1 P_{\mathfrak{a}} f(\sigma_{\mathfrak{a}} z)\, dx = 0,$$

where $z = x + iy$. The space of cusp forms will be denoted by $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$. Later (Proposition 6.1), we will show the direct sum decomposition

$$L^2(\Gamma\backslash\mathbb{H},\chi) = \mathcal{C}(\Gamma\backslash\mathbb{H},\chi) \oplus \mathcal{E}(\Gamma\backslash\mathbb{H},\chi).$$

Remark 4.2. The function $P_{\mathfrak{a}} f(\sigma_{\mathfrak{a}} z)$ is invariant under $z \mapsto z+1$. Hence, the above condition is equivalent to saying, that the zero-th Fourier coefficient is zero.

Similar as in the classical case, we will show that the space of cusp forms decomposes discretely. But since the hyperbolic Laplacian

$$\Delta = y^2\left(\frac{\partial^2}{(\partial x)^2} + \frac{\partial^2}{(\partial y)^2}\right)$$

is in general not self-adjoint with respect to the inner product on $L^2(\Gamma\backslash\mathbb{H},\chi)$, we can not expect to obtain a decomposition of $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ into a direct sum of eigenspaces for Δ. But fortunately, Δ is not far from being self-adjoint, since for Δ the obstacle of being self-adjoint arises only from a compact subset of $\Gamma\backslash\mathbb{H}$. More precisely, the operators Δ and Δ_s coincide outside of a compact subset of $\Gamma\backslash\mathbb{H}$. What we will obtain, is, that the space of cusp forms decomposes into a direct sum of root spaces, also known as generalized eigenspaces (Proposition 4.12). To prove this result we will use the theory of Eisenstein series, as developed in chapter 3 . The most involved part will be, to show that all root vectors actually span the space of cusp forms.

4.2 Invariant integral operators

To begin with let $k \in C^\infty(\mathbb{R}_+)$ be a smooth function on the positive real axis. For k we require $k(u), k'(u) \ll (u+2)^{-\sigma}$ where $\sigma > \sigma_0$ and σ_0 is as in Proposition 3.3. For $z, w \in \mathbb{H}$ we recall the distance function

$$u(z,w) = \frac{|z-w|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}.$$

By abuse of notation, we then obtain an integral kernel $k(z,w) = k(u(z,w))$. As usual we can define an integral operator $L = L_k$, by

$$(Lf)(z) = \int_{\mathbb{H}} k(z,w) f(w)\, d\mu w,$$

for $f : \mathbb{H} \to V$. If we restrict ourselves to functions $f \in L^2(\Gamma\backslash\mathbb{H}, \chi)$ (so in particular $f(\gamma z) = \chi(\gamma) f(z)$), we obtain

$$(Lf)(z) = \int_{\Gamma\backslash\mathbb{H}} K(z,w) f(w)\, d\mu(w),$$

where $K(z,w) = K(z,w,\chi) = \sum_{\gamma\in\Gamma} k(z,\gamma w)\chi(\gamma)$. If σ is large enough, then one can show that this sum converges absolutely locally uniformly.

Proposition 4.3. *Let the kernel k satisfy*

$$k(z,w) \ll u(z,w)^{-s}. \tag{46}$$

Then the invariant kernel $K(z,w)$ represents a continuous function on the domain $\{(z,w) : z \not\equiv w \pmod{\Gamma}\} \subset \mathbb{H}\times\mathbb{H}$ for $\mathrm{Re}(s)$ large enough. In the case that the kernel satisfies the stronger condition

$$k(z,w) \ll (u(z,w)+2)^{-s}, \tag{47}$$

we even get continuity of the kernel K on the whole of $\mathbb{H}\times\mathbb{H}$.

Proof. From (35) we know, that there exists $\alpha > 0$ such that

$$\|\chi(\gamma)\| = O(1 + u(\gamma z, w)^{\alpha}),$$

where the implied constant is independent of z, w in a compact subset of $\mathbb{H}\times\mathbb{H}$. We estimate as follows, using (35):

$$\begin{aligned}\sum_{\gamma\in\Gamma} \|k(z,\gamma w)\chi(\gamma)\| &\ll \sum_{\gamma\in\Gamma} u(z,\gamma w)^{-\sigma}\|\chi(\gamma)\| \\ &\ll \sum_{\gamma\in\Gamma} u(z,\gamma w)^{\alpha-\sigma}.\end{aligned}$$

But Lemma 2.11 in [Iwa02] yields absolute convergence of

$$\sum_{\gamma\in\Gamma} u(z,\gamma w)^{-s},$$

for $\mathrm{Re}(s) > 1$ and $z \not\equiv w \pmod{\Gamma}$. Now, if we fix z_0, w_0 such that $z_0 \neq \gamma w_0$ for every $\gamma \in \Gamma$, we choose $\varepsilon > 0$ so small, that this is true for all $z \in B_{\varepsilon/2}(z_0)$ and $w \in B_{\varepsilon/2}(w_0)$ and $u(\gamma z_0, w_0) \geq 2\varepsilon$ for all $\gamma \in \Gamma$. Then we get

$$\sum_{\gamma\in\Gamma} u(\gamma z, w)^{-s} \leq \sum_{\gamma\in\Gamma} (u(\gamma z_0, w_0) - \varepsilon)^{-s} \leq 2^s \sum_{\gamma\in\Gamma} u(\gamma z_0, w_0)^{-s},$$

and we see that the sum

$$\sum_{\gamma\in\Gamma} k(z,\gamma w)\chi(\gamma)$$

converges locally uniformely absolutely in the domain $\{(z,w) : z \not\equiv w \pmod{\Gamma}\}$ of $\mathbb{H}\times\mathbb{H}$ and hence is a contionuous function on this domain.
If the kernel k satisfies (47), it is by the same procedure clear, that K is continuous on $\mathbb{H}\times\mathbb{H}$. Condition (47) just excludes a singularity at 0 of k. □

We also take notice of the following proposition:

Proposition 4.4. *The integral operator L maps $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ into itself.*

Proof. Let $f \in \mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$, $g = Lf$ and $n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. For the constant Fourier coefficient $g_{\mathfrak{a}}$ of g, we compute

$$\begin{aligned} g_{\mathfrak{a}}(y) &= \int_0^1 P_{\mathfrak{a}} g(\sigma_{\mathfrak{a}} n(t) z)\, dt = \int_0^1 \left(\int_{\mathbb{H}} k(\sigma_{\mathfrak{a}} n(t) z, w) P_{\mathfrak{a}} f(w)\, d\mu w \right) dt \\ &= \int_{\mathbb{H}} k(z,w) \left(\int_0^1 P_{\mathfrak{a}} f(\sigma_{\mathfrak{a}} n(t) w)\, dt \right) d\mu w = \int_{\mathbb{H}} k(z,w) f_{\mathfrak{a}}(\mathrm{Im}(w))\, d\mu w = 0. \end{aligned}$$

□

The kernel $K(z,w)$ is not square-integrable on $\Gamma\backslash\mathbb{H}\times\Gamma\backslash\mathbb{H}$, so we can not prove easily, that it induces a compact operator. Similar as in the classical case we have to subtract for each cusp the prinicpal part

$$H_{\mathfrak{a}}(z,w) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{-\infty}^{\infty} k(z, \sigma_{\mathfrak{a}} n(t) \sigma_{\mathfrak{a}}^{-1} \gamma w)\, dt\, P_{\mathfrak{a}}\chi(\gamma).$$

The resulting operator will be shown to have the same effect on cusp forms (Proposition 4.5), and the kernel will be square-integrable, and thus will induce a compact operator. This will yield the desired decomposition of $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$. For $f \in L^2(\Gamma\backslash\mathbb{H},\chi)$ the function $w \mapsto H_{\mathfrak{a}}(z,w)f(w)$ is easily seen to be invariant under Γ. This makes the integral

$$\int_{\Gamma\backslash\mathbb{H}} H_{\mathfrak{a}}(z,w)f(w)\,d\mu w$$

meaningful. We claim, that the integral operator induced by $H_{\mathfrak{a}}(z,w)$ annihilates cusp forms.

Proposition 4.5. *We have* $\int_{\Gamma\backslash\mathbb{H}} H_{\mathfrak{a}}(z,w)f(w)\,d\mu(w) = 0$ *for every* $f \in \mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$.

Proof. By unfolding the integral we get

$$\begin{aligned}&\int_{\Gamma\backslash\mathbb{H}} H_{\mathfrak{a}}(\sigma_{\mathfrak{a}}z,w)f(w)\,d\mu w\\ &\quad= \int_0^\infty\int_0^1\int_{-\infty}^\infty k(\sigma_{\mathfrak{a}}z,\sigma_{\mathfrak{a}}n(t)\sigma_{\mathfrak{a}}^{-1}\gamma w)\,dt\,P_{\mathfrak{a}}\chi(\gamma)f(w)\,d\mu w\\ &\quad= \int_0^\infty\left(\int_{-\infty}^\infty k(z,t+iv)\,dt\right)(P_{\mathfrak{a}}f(\sigma_{\mathfrak{a}}(u+iv))\,du)\,v^{-2}\,dv = 0.\end{aligned}$$ □

We add all principal parts over inequivalent cusps

$$H(z,w) = \sum_{\mathfrak{a}} H_{\mathfrak{a}}(z,w).$$

The kernel function

$$\widehat{K}(z,w) = K(z,w) - H(z,w) \tag{48}$$

is called the compact part of $K(z,w)$. If L is the integral operator with integral kernel $K(z,w)$, we let $\widehat{L}$ be the integral operator with integral kernel $\widehat{K}(z,w)$. We will show, that $\widehat{L}$ is a Hilbert-Schmidt operator. To show this, it is sufficient to bound its L^2-norm.

Proposition 4.6. *Assume that the kernel satisfies the stronger growth condition (2)*

$$k(u) \ll (2+u)^{-s}$$

with $\mathrm{Re}(s) > \sigma_0$ *and* σ_0 *as in proposition 3.3. Then we have* $K(z,w) = \sum_{\mathfrak{a}}\sum_{\gamma\in\Gamma_{\mathfrak{a}}} k(z,\gamma w)\chi(\gamma) + O(1)$.

Proof. Fix a cusp $\mathfrak{a}$. By $\sum'_{\gamma\in\Gamma}$ we indicate that γ is not equal to a power of $\gamma_{\mathfrak{a}}$. Hence, we can write

$$K(z,w) = \sum_{\mathfrak{a}} \sum_{\gamma\in\Gamma_{\mathfrak{a}}} k(z,\gamma w)\chi(\gamma) + \sum_{\gamma\in\Gamma}{}' k(z,\gamma w)\chi(\gamma).$$

We define a representation $\tilde{\chi} : \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} \to V$ by $\tilde{\chi}(\gamma) = \chi(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{a}}^{-1})$. Then we arrive at

$$\sum_{\gamma\in\Gamma}{}' k(\sigma_{\mathfrak{a}}z,\gamma\sigma_{\mathfrak{a}}w)\chi(\gamma) = \sum_{\gamma\in B\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}}{}' \sum_{n\in\mathbb{Z}} k(z,\gamma w+n)\tilde{\chi}(\gamma+b(n)).$$

Now,

$$\begin{aligned} k(z,\gamma w+n) &\ll (2+u(z,\gamma w+n))^{-s} \\ &= \left(2+\frac{(\mathrm{Re} z-\mathrm{Re}(\gamma w)-n)^2+(\mathrm{Im}\, z-\mathrm{Im}(\gamma w))^2}{\mathrm{Im}\, z\,\mathrm{Im}(\gamma w)}\right)^{-s} \\ &\ll \frac{\mathrm{Im}(\gamma w)^s}{\left(\mathrm{Im}\, z+\frac{(\mathrm{Re}\, z-\mathrm{Re}(\gamma w)-n)^2}{\mathrm{Im}\, z}\right)^s}, \end{aligned}$$

which yields

$$\begin{aligned} \sum_{n\in\mathbb{Z}} k(z,\gamma w+n) &\ll \mathrm{Im}(\gamma w)^s \sum_{n\in\mathbb{Z}} \left(\mathrm{Im}\, z+\frac{(\mathrm{Re}\, z-\mathrm{Re}(\gamma w)-n)^2}{\mathrm{Im}\, z}\right)^{-s} \\ &\ll \mathrm{Im}(\gamma w)^s \sum_{n=0}^{\infty} \left(\mathrm{Im}\, z+\frac{n^2}{\mathrm{Im}\, z}\right)^{-s}. \end{aligned}$$

We are assuming, that $\mathrm{Im}\, z \geq A$ for some positive constant $A > 0$, which gives

$$\sum_{n=0}^{\infty} \left(\mathrm{Im}\, z+\frac{n^2}{\mathrm{Im}\, z}\right)^{-s} < \mathrm{Im}(z)^{-s} + \int_0^{\infty} \left(\mathrm{Im}\, z+\frac{u^2}{\mathrm{Im}\, z}\right)^{-s} du \ll \mathrm{Im}(z)^{-s+1}$$

and thus,

$$\sum_{n\in\mathbb{Z}} k(z,\gamma w+n) \ll \mathrm{Im}(z)^{-s+1}\mathrm{Im}(\gamma w)^s.$$

Finally, we get an estimate from above, by the Eisenstein series:

$$\sum_{\gamma\in\Gamma}{}' k(\sigma_{\mathfrak{a}}z,\gamma\sigma_{\mathfrak{a}}w)\chi(\gamma) \ll \mathrm{Im}(z)^{-s+1} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}{}' \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma w)^s \|\chi(\gamma)\|.$$

But, we know from Proposition 3.7 that for each cusp $\mathfrak{b}$ we have

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma}{}' \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}w)^s\|\chi(\gamma)\| = O(y^{1-s}) \tag{49}$$

for $y \to \infty$ and hence, the sum (49) is uniformly bounded in F. Thus we find, that $\sum'_{\gamma\in\Gamma} k(\sigma_{\mathfrak{a}}z, \gamma\sigma_{\mathfrak{a}}w)\chi(\gamma)$ is uniformely bounded for z and w in F with $\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}z) \geq A$.

From the above results we see, that

$$\sum_{\gamma \text{ not parabolic}} k(z, \gamma w)\chi(\gamma)$$

is uniformely bounded for z and w in F and $\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}z) \geq A$. Since this is true for all cusps $\mathfrak{a}$, we find that this sum is actually uniformely bounded for all z and w in F. □

Proposition 4.7. *We have*

$$H_{\mathfrak{a}}(z, w) = \int_{-\infty}^{\infty} k(z, \sigma_{\mathfrak{a}}n(t)\sigma_{\mathfrak{a}}^{-1}w)\, dt P_\alpha + H'_{\mathfrak{a}}(z, w)$$

where $H'_{\mathfrak{a}}(z, w)$ has bounded L^2-norm.

Proof. This proof is similar to the previous. Speaking loosely

$$H'_{\mathfrak{a}}(z, w) = H_{\mathfrak{a}}(z, w) - \int k(z, \sigma_{\mathfrak{a}}n(t)\sigma_{\mathfrak{a}}^{-1}w)\, dt$$

is just the "continuous analogue" of $\sum'_{\gamma\in\Gamma} k(z, \sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{a}}^{-1}w)\chi(\gamma)$. □

Now it remains to show, that

$$J_{\mathfrak{a}}(z, w) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}} k(z, \gamma w)\chi(\gamma) - \int_{-\infty}^{\infty} k(z, \sigma_{\mathfrak{a}}n(t)\sigma_{\mathfrak{a}}^{-1}w)\, dt\, P_{\mathfrak{a}}$$

is bounded on $\mathbb{H} \times \mathbb{H}$. We first consider

$$J_{\mathfrak{a}}(z, w)P_{\mathfrak{a}} = \sum_{\gamma\in\Gamma_{\mathfrak{a}}} k(z, \gamma w)P_{\mathfrak{a}} - \int_{-\infty}^{\infty} k(z, \sigma_{\mathfrak{a}}n(t)\sigma_{\mathfrak{a}}^{-1}w)\, dt\, P_{\mathfrak{a}}.$$

If we set $\psi(t) = t - [t] - 1/2$, we apply the Euler-MacLaurin formula and obtain

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{-\infty}^{\infty} f(t)\,dt + \int_{-\infty}^{\infty} \psi(t)\,df(t).$$

Applying this in our case, yields

$$\begin{aligned} J_{\mathfrak{a}}(\sigma_{\mathfrak{a}} z, \sigma_{\mathfrak{a}} w)P_{\mathfrak{a}} &= \sum_{n\in\mathbb{Z}} k(z, w+n)P_{\mathfrak{a}} - \int_{-\infty}^{\infty} k(z, n(t)w)\,dt\,P_{\mathfrak{a}} \\ &= \int_{-\infty}^{\infty} \psi(t)\,dk(z, w+t)\,P_{\mathfrak{a}} \ll \int_0^1 |k'(u)|\,du \ll 1. \end{aligned}$$

Now we consider the contribution of $J_{\mathfrak{a}}(z,w)$ for the orthogonal complement of the eigenspace for 1, $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}),1)^{\perp}$. Here, we get

$$\begin{aligned} (\mathrm{Id} - \chi(\gamma_{\mathfrak{a}})) \sum_{\gamma\in\Gamma_{\mathfrak{a}}} k(\sigma_{\mathfrak{a}} z, \gamma\sigma_{\mathfrak{a}} w)\chi(\gamma) &= \sum_{n\in\mathbb{Z}} (k(w, z+n) - k(w, z+n-1))\chi^n(\gamma_{\mathfrak{a}}) \\ &\ll \int_0^{\infty} dk(w, z+t) \ll \int_0^{\infty} |k'(u)|\,du \ll 1. \end{aligned}$$

Since the restriction of $\mathrm{Id} - \chi(\gamma_{\mathfrak{a}})$ to the orthogonal complement $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}),1)^{\perp}$ of the eigenspace for the eigenvalue 1 is invertible, we get

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}} k(z,w)\chi(\gamma)Q_{\mathfrak{a}} \ll 1,$$

where $Q_{\mathfrak{a}}$ is the orthogonal projection on the orthocomplement $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}),1)^{\perp}$. Hence, we see that $\widehat{K}$ defines an integral operator with L^2-kernel and is thus Hilbert-Schmidt.

4.3 The resolvent kernel

In this section we want to construct a resolvent for the Laplacian Δ on $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ by an integral kernel. This will give us the means to show that $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ is spanned by generalized eigenfunctions of the Laplacian (Proposition 4.12). We call a function, a *generalized eigenfunction* of the Laplacian, with generalized eigenvalue λ, if it is annihilated by some power of $\Delta - \lambda$.
We start by giving a resolvent for Δ on $\mathbb{H}$. Let $G_s(u)$ be the integral

$$G_s(u) = \frac{1}{4\pi}\int_0^1 (\xi(1-\xi))^{s-1}(\xi+u)^{-s}\,d\xi. \tag{50}$$

The next proposition resumes the most prominent properties of this integral.

Proposition 4.8. *The integral (50) converges absolutely for* $\mathrm{Re}(s) = \sigma > 0$. *The function* $G_s(u)$ *on* $\mathbb{R}^+$ *satisfies the differential equation*

$$(\Delta + s(1-s))F = 0, \tag{51}$$

and the following bounds:

$$G_s(u) = -\frac{1}{4\pi}\log(u) + O(1), \quad u \to 0, \tag{52}$$

$$G_s'(u) = -(4\pi u)^{-1} + O(1), \quad u \to 0, \tag{53}$$

$$G_s(u) \ll u^{-\sigma}, \quad u \to \infty. \tag{54}$$

Proof. For a proof, see [Iwa02, Lemma 1.7]. □

Theorem 4.9. *Let* $s \in \mathbb{C}$ *with* $\mathrm{Re}(s) > 0$ *and let* $-R_s$ *be the integral operator on* $\mathbb{H}$ *given by*

$$-(R_s f)(z) = \int_{\mathbb{H}} G_s(u(z,w)) f(w)\, d\mu w.$$

If $f : \mathbb{H} \to V$ *is smooth and satisfies the growth condition* $\|f(z)\| \ll y^\sigma + y^{-\sigma}$ *for* $\sigma > 0$ *then, if* $\mathrm{Re}(s) > \sigma + 1$ *we get*

$$(\Delta + s(1-s))R_s f = f.$$

In other words, R_s *is the right inverse to* $\Delta + s(1-s)$.

Proof. [Iwa02, Theorem 1.17]. □

Let us assume we have a function $f : \mathbb{H} \to V$ satisfying the invariance property

$$f(\gamma z) = \chi(\gamma) f(z) \tag{55}$$

for all $z \in \mathbb{H}$ and every $\gamma \in \Gamma$. Under the assumption that the integral

$$\int_{\mathbb{H}} G_s(u(z,w)) f(w)\, d\mu w$$

converges absolutely we get

$$\int_{\mathbb{H}} G_s(u(z,w)) f(w)\, d\mu w = \int_F \sum_{\gamma \in \Gamma} G_s(u(z,\gamma w)) \chi(\gamma) f(w)\, d\mu w,$$

so that we will consider the integral kernel

$$G_s(z/w,\chi) = \sum_{\gamma\in\Gamma} G_s(u(z,\gamma w))\chi(\gamma).$$

By formal computation, we easily check, that it satisfies the invariance properties

$$\begin{aligned} G_s(\gamma z/w,\chi) &= \chi(\gamma)G_s(\gamma z/w,\chi), \\ G_s(z/\gamma w,\chi) &= G_s(z/w,\chi)\chi(\gamma^{-1}), \end{aligned}$$

for arbitrary $\gamma\in\Gamma$ and $z,w\in\mathbb{H}$. Using Proposition 4.3 and (54) we find, that the sum $\sum G_s(u(z,\gamma w))\chi(\gamma)$ represents a continuous function on the domain $\{(z,w) : z \not\equiv w \pmod{\Gamma}\} \subset \mathbb{H}\times\mathbb{H}$ for $\mathrm{Re}(s)$ large enough. Furthermore, (52) yields

$$G_s(z/w,\chi) = -\frac{1}{2\pi}\log|z-\gamma_0 w|\chi(\gamma_0)\sum_{\gamma\in\Gamma_w}\chi(\gamma) + O(1). \tag{56}$$

Now take $f : \mathbb{H}\to V$ as in (55) with the extra property that f is smooth and its restriction to F has compact support. Because $G_s(z/w,\chi)$ is continuous in $F\times F$ outside the diagonal and has a logarithmic singularity on the diagonal only, the integral

$$\int_F G_s(z/w,\chi)f(w)\,d\mu w$$

exists.

We want to modify this operator to obtain a bounded integral operator, which will then be compact. Therefore, we define

$$L = R_s - R_a.$$

If $a > s > \sigma$ and σ is sufficiently large, this operator is given by an integral kernel k, which satisfies the growth estimate

$$k(u), k'(u) \ll (u+2)^{-\sigma-1}.$$

Now, we can use some general theory, to show that the generalized eigenvectors of the Laplacian span $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$. Firstly, we use a classical result of Carleman on the growth of resolvents of Hilbert-Schmidt operators.

Theorem 4.10. *[Car21] Let T be a Hilbert-Schmidt operator on a Hilbert space $\mathcal{H}$. Let $\{\lambda_i\}$ be the sequence of non-zero eigenvalues of T, each repeated according to its multiplicity and let $R_\lambda(T)$ be the resolvent operator. Then we have the following:*

1. *The series $\sum_i |\lambda_i|^2$ converges absolutely.*

2. *Put*
$$C(\lambda) = \prod_i \left(1 - \frac{\lambda_i}{\lambda}\right) \exp\left(\frac{\lambda_i}{\lambda} + \frac{1}{2}\left(\frac{\lambda_i}{\lambda}\right)^2\right).$$
Then $C(\lambda)$ is a well-defined entire function of $1/\lambda$ vanishing at the points λ_i. Furthermore we have a growth estimate
$$\|C(\lambda)R_\lambda(T)\| \leq \exp(c|\lambda|^{-2}),$$
where c is some constant depending on T only.

This result we use to prove the following:

Theorem 4.11. *Same assumptions as before. For $\varepsilon > 0$ there exists a sequence of positive numbers $\rho_i \to 0$, such that R_λ exists everywhere on $|\lambda| = \rho_i$ and*
$$\|R_\lambda(T)\| \leq \exp(c|\lambda|^{-2-\varepsilon}) \text{ for } |\lambda| = \rho_i.$$

Proof. $C(1/\lambda)$ is an entire function of order 2. It follows from the minimum modulus theorem for entire functions (c.f. [Lan77] Chapter X, Theorem 3.3), that there exists a sequence of positive numbers $\rho_i \to 0$, such that
$$|C(\lambda)| \geq \exp(-|\lambda|^{-2-\varepsilon}) \text{ for } |\lambda| = \rho_i.$$
On the other hand, we know from the previous theorem, that
$$\|R_\lambda(T)\| \leq |C(\lambda)|^{-1} \exp(c|\lambda|^{-2}) \text{ for } |\lambda| = \rho_i.$$
This concludes the proof. □

The above Theorem 4.11 we want to apply to the resolvent of the Laplacian. More precisely, we consider the operator $L = R_s - R_a$, where R_t is the Resolvent of Δ with parameter $t(1-t)$. For $a > s > \sigma$ and σ sufficiently large, this operator exists. This operator has dense range in $L^2(\Gamma\backslash\mathbb{H},\chi)$. The modified operator $\widehat{L}$ is bounded on $L^2(\Gamma\backslash\mathbb{H},\chi)$. The range of $\widehat{L}$ is dense in the subspace $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$.

Proposition 4.12. *There exists a sequence $|\lambda_i| \to \infty$, such that the space $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ decomposes as a direct sum of generalized eigenspaces*

$$\mathcal{C}(\Gamma\backslash\mathbb{H},\chi) = \bigoplus_{i=1}^{\infty} R(\Delta,\lambda_i)$$

where $R(\Delta,\lambda_i) = \{f \in \mathcal{C}(\Gamma\backslash\mathbb{H},\chi) : (\Delta-\lambda_i)^n f = 0$ for some $n \in \mathbb{N}\}$. Each generalized eigenspace is finite dimensional.

Proof. For $a > s > \sigma$ and σ sufficiently large, we consider the operator

$$L = R_s - R_a,$$

where R_s is the resolvent of Δ with parameter $s(1-s)$. We know that this operator is an integral operator with kernel k, which satisfies the growth estimate

$$k(u), k'(u) \ll u^{\sigma+1}.$$

By the resolvent formula

$$L = R_s - R_a = (s(1-s) - a(1-a))R_s R_a$$

we see, that L has dense range in $L^2(\Gamma\backslash\mathbb{H},\chi)$. As in (48) we modify this operator to an operator $\widehat{L}$, which has the same effect on cusp forms as the original L according to Proposition 4.7. But, since $\widehat{L}$ has an L^2-kernel, the operator $\widehat{L} : \mathcal{C}(\Gamma\backslash\mathbb{H},\chi) \to \mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ is Hilbert-Schmidt.
An element $\phi \in L_0^2$ is said to be a generalized eigenfuction of L, corresponding to the eigenvalue μ, if $(L-\mu)^j\phi = 0$ for some $j \geq 1$. Since L is compact, it is well known, that the space of generalized eigenfunctions corresponding to an eigenvalue $\mu \neq 0$ is finite dimensional and its dimension is called the multiplicity of the eigenvalue μ. In general, the linear span of all generalized eigenfuctions does not span the whole space. But in our case, we will prove that it does.
The Resolvent $R_\lambda(L)$ is a meromorphic operator valued function of $1/\lambda$. Its poles are exactly the eigenvalues of L. If μ is an eigenvalue of L, then in a

sufficiently small neigbourhood of μ we have a Laurent series expansion

$$R_\lambda(L) = \sum_{n=-N}^{\infty} A_n(\lambda-\mu)^n.$$

Applying $L-\lambda$ we find the relations

$$(L-\mu)A_{-N} = 0, \quad (L-\mu)A_{-N+1} = A_{-N}, \ldots (L-\mu)A_{-1} = A_{-2}.$$

Thus we see, that the operator A_{-1} maps $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ into the generalized eigenspace of the corresponding eigenvalue μ.

We let W be the span of all generalized eigenfunctions in $\mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ for the operator L. To show, that $W = \mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$, we take $f \in W^\perp$ and show, that $f = 0$. Consider the function

$$F(\lambda) = (f, R_{1/\lambda}(L)\, g)_{L^2},$$

where $g \in L_0^2$ is arbitrary. $F(\lambda)$ is holomorphic outside of those λ for which λ^{-1} is in the spectrum of L. But, for $\lambda_j^{-1} \in \sigma(L)$ we find as above, that $R_{1/\lambda_j}(L)\, g$ is an element of the generalized eigenspace for λ_j^{-1}. By the choice of f, we find, that F is also regular in λ_j^{-1}. Thus F is entire.

Applying Theorem 4.11, it follows, that for every $\varepsilon > 0$ there exists a sequence of positive numbers $r_i \to \infty$, such that

$$|F(\lambda)| \le \exp(c|\lambda|^{2+\varepsilon}) \text{ for } |\lambda| = r_i. \tag{57}$$

The spectrum of Δ is contained in the convex hull of a parabola of the form $a+biy+cy^2$ with $a,b,c \in \mathbb{R}$. So we obtain a growth estimate $\|R_\lambda(\Delta)\| = O(|\lambda^{-1}|)$ as $|\lambda| \to \infty$ on any ray $\arg\lambda \ne 0$. Hence,

$$|F(\lambda)| = O(|\lambda|) \text{ as } \lambda \to \infty \text{ along any ray } \arg\lambda \ne 0. \tag{58}$$

We take finitely many rays $\arg\lambda = \theta_j \ne 0$, which divide the complex plane into angles of size $< \pi/2$. On the sides of the angles (58) holds and on a sequence of circles with radii tending to infinity (57) holds. Thus we are in a position to apply the Phragmen-Lindelöf principle in each angle. This gives the uniform bound $|F(\lambda)| = O(|\lambda|)$, as $\lambda \to \infty$, in each angle and hence, in the whole plane.

Since F is entire, it must be a linear function $F(z) = c_0 + c_1 z$. On the other hand, we have in a neighbourhood of the origin the expansion

$$F(z) = z(f, g)_{L^2} + z^2(f, Lg)_{L^2} + \dots.$$

This leads to $(f, Lg)_{L^2} = 0$. Since $f \in W^\perp$ was arbitrary and the range of L is dense in $\mathcal{C}(\Gamma\backslash\mathbb{H}, \chi)$ we obtain $W^\perp = \{0\}$. Thus, we obtain a decomposition of $\mathcal{C}(\Gamma\backslash\mathbb{H}, \chi)$ into the generalized eigenspaces of the operator L

$$\mathcal{C}(\Gamma\backslash\mathbb{H}, \chi) = \bigoplus_{i=1}^{\infty} R(L, \eta_i),$$

where $\eta_i \to 0$. Since the Laplacian Δ commutes with the operator L, it leaves the root spaces $R(L, \eta_i)$ invariant. To conclude, we have a decomposition

$$\mathcal{C}(\Gamma\backslash\mathbb{H}, \chi) = \bigoplus_{i=1}^{\infty} R(\Delta, \lambda_i) \qquad \square$$

5 Meromorphic continuation of Eisenstein series

As the title already points out, we will in this section prove the meromorphic continuation of the Eisenstein series. Our main tool will be Fredholm theory, which is classical a way to show meromorphic continuation of Eisenstein series. We will adapt the theory for our needs below. In Theorem 5.3 we will show a functional equation, which will connect the values of the Eisenstein series at s and $1 - s$. Theorem 5.3 will later be used, while computing the spectral side of the trace formula.

5.1 Fredholm theory

Let $\lambda \in \mathbb{C}$ and $F \subset \mathbb{H}$ the given fundamental domain. $K : F \times F \to \mathrm{End}(V)$ shall denote a given kernel function and $f : F \to V$ a function. The *Fredholm equation* is given by

$$g(x) - \lambda \int_F K(x, y) g(y) \, d\mu(y) = f(x). \tag{59}$$

We are looking for solutions $g \in L^2(\Gamma\backslash\mathbb{H}, \chi)$. We assume $f \in L^2(\Gamma\backslash\mathbb{H}, \chi)$ and $K \in L^2(F \times F, \mathrm{End}(V))$. We let

$$\|K\|^2 = \iint_{F\times F} |K(x,y)|^2 \, dx \, dy,$$

be the norm of the kernel K. By abuse of notation, we also just write K for the operator on $L^2(\Gamma\backslash\mathbb{H}, \chi)$ induced by the kernel K. We say λ is a *characteristic number* of the kernel $K(x,y)$ if the homogeneous equation

$$(I - \lambda K)g = 0, \tag{60}$$

has a nonzero solution $g \in L^2(\Gamma\backslash\mathbb{H}, \chi)$. Of course, this entails $\lambda \neq 0$ and then λ^{-1} is just an eigenvalue of the operator K with eigenfunction g. For small λ it is possible to construct a solution of (60) by succesive approximation. We set $g_0 = f$ and define inductively $g_p = \lambda K g_{p-1} + f$. Written differently this means

$$g_p = \sum_{j=0}^{p} \lambda^j K^j f.$$

We introduce the *Neumann series*

$$g = \sum_{j=0}^{\infty} \lambda^j K^j f.$$

This series converges absolutely in the disc $|\lambda| < \|K\|^{-1}$, and yields a solution for (59), as is clearly seen by integrating term by term. Furthermore this solution is unique up to a function vanishing almost everywhere. Assuming that we have two solutions g_1, g_2, then (60) implies

$$\|g_1 - g_2\| \leq |\lambda| \|K\| \|g_1 - g_2\|.$$

Since $|\lambda|\|K\| < 1$ whe get $\|g_1 - g_2\| = 0$ and hence $g_1 = g_2$ almost everywhere. Hence the operator $(I - \lambda K)^{-1}$ exists in the disk $|\lambda| < \|K\|^{-1}$ and we get the estimate

$$\|(I - \lambda K)^{-1}\| \leq (1 - |\lambda|\|K\|)^{-1}.$$

We define inductively new operators K^j via the kernels

$$K_j(x,y) = \int_F K(x,z)K_{j-1}(z,y)\,d\mu(z), \quad j = 2,3,\dots,$$

and $K_1 = K$. Supposing that

$$A(x)^2 = \int_F \|K(x,y)\|^2\,d\mu(y) < \infty,$$
$$B(x)^2 = \int_F \|K(x,y)\|^2\,d\mu(x) < \infty,$$

and applying the Cauchy-Schwarz inequality, we find by induction that

$$\|K_j(x,y)\| \le A(x)B(x)\|K\|^{j-2}, \quad j = 2,3,\dots.$$

Hence, we can estimate the series

$$R_\lambda(x,y) = \sum_{j=1}^{\infty} \lambda^{j-1} K_j(x,y)$$

by

$$\|K(x,y)\| + |\lambda| A(x)B(y) \sum_{j=0}^{\infty} |\lambda\|^j \|K\|^j.$$

Consequently, this series converges locally uniformely absolutely in the disc $|\lambda| < \|K\|^{-1}$ and yields a function $R_\lambda(x,y)$ in $L^2(F \times F, \mathrm{End}(V))$ which is holomorphic in λ.
Integrating term by term we get

$$g(x) = f(x) + \lambda \int_F R_\lambda(x,y) f(y)\,d\mu(y). \tag{61}$$

If we let R be the integral operator with kernel function $R_\lambda(x,y)$, then (61) asserts, that

$$(I - \lambda K)^{-1} = I + \lambda R.$$

R is called the *resolvent* of K and it can be easily seen to satisfy the equation

$$R_\lambda(x,y) = K(x,y) + \lambda \int_F K(x,z) R_\lambda(z,y)\,d\mu(z). \tag{62}$$

Since R_λ is holomorphic in the disc $|\lambda| < \|K\|^{-1}$, it follows by analytic continuation, that the solution g is unique for all λ to which R_λ has analytic continuation and is in $L^2(F \times F)$.

5.2 An explicit resolvent kernel

We will give an explicit construction of the resolvent kernel $R_\lambda(x,y)$ in the case that $K(x,y)$ is bounded on $F \times F$, say by M, which follows the construction by Fredholm.

We will show the existence of two entire functions $\mathcal{D}(\lambda) \not\equiv 0$ and $\mathcal{D}_\lambda(x,y)$ with values in $\mathrm{End}(V)$ such that

$$R_\lambda(x,y) = \mathcal{D}_\lambda(x,y)\mathcal{D}(\lambda)^{-1},$$

for all $\lambda \in \mathbb{C}$ where $\mathcal{D}(\lambda) \in \mathrm{Aut}(V)$.

We put

$$K\left(\begin{smallmatrix} \xi_1,\dots,\xi_m \\ \eta_1,\dots,\eta_m \end{smallmatrix}\right) = \sum_{\sigma \in \mathrm{Per}_m} \mathrm{sign}(\sigma) \prod_{i=1}^{m} K(\xi_i, \eta_{\sigma(i)}).$$

$K\left(\begin{smallmatrix} \xi_1,\dots,\xi_m \\ \eta_1,\dots,\eta_m \end{smallmatrix}\right)$ can be seen as the determinant of a matrix with entries in $\mathrm{End}(V)$, namely $K(\xi_i,\eta_j)$. By Hadamard's inequality we have for real numbers a_{ij}

$$|\det(a_{ij})|^2 \le \prod_j \left(\sum_i |a_{ij}|^2 \right),$$

so that

$$\left\| K\left(\begin{smallmatrix} \xi_1,\dots,\xi_m \\ \eta_1,\dots,\eta_m \end{smallmatrix}\right) \right\| \le (\sqrt{m} M)^m. \tag{63}$$

We write

$$C_m = \int \cdots \int K\left(\begin{smallmatrix} \xi_1,\dots,\xi_m \\ \eta_1,\dots,\eta_m \end{smallmatrix}\right) \, d\xi_1 \cdots d\xi_m,$$
$$C_m(x,y) = \int \cdots \int K\left(\begin{smallmatrix} x,\xi_1,\dots,\xi_m \\ y,\eta_1,\dots,\eta_m \end{smallmatrix}\right) \, d\xi_1 \cdots d\xi_m,$$

so that by (63) and by letting $V = \mathrm{vol}(F)$ we get

$$\|C_m\| \le (\sqrt{m} M V)^m,$$
$$\|C_m(x,y)\| \le (\sqrt{m+1} M)^{m+1} V^m.$$

Thus, we find that

$$\mathcal{D}(\lambda) = 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m$$

converges absolutely, since

$$\left\| 1 + \sum_{m=1}^{\infty} \frac{(-\lambda)^m}{m!} C_m \right\| \leq 1 + \sum_{m=1}^{\infty} \frac{(\sqrt{m}|\lambda| KV)^m}{m!} < \infty.$$

Therefore, $\mathcal{D}(\lambda)$ represents an entire function in λ. Furthermore, we have $\mathcal{D}(0) = \mathrm{Id}_V$. By an anologous reasoning we get

$$\|\mathcal{D}_\lambda(x,y)\| = \left\| \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} C_m(x,y) \right\| \leq eK \sum_{m=0}^{\infty} \frac{(\sqrt{m+1}|\lambda| KV)^m}{m!} < \infty.$$

where $C_0(x,y) = K(x,y)$. Thus, $\mathcal{D}_\lambda(x,y)$ is also an entire function in λ. Developing the determinant $K\left(\begin{smallmatrix} x,\xi_1,\dots,\xi_m \\ y,\eta_1,\dots,\eta_m \end{smallmatrix}\right)$ by the first row yields

$$\begin{aligned}
&C_m(x,y) \\
&\quad = \int \cdots \int \Big(K(x,y) K\left(\begin{smallmatrix} \xi_1,\dots,\xi_m \\ \xi_1,\dots,\xi_m \end{smallmatrix}\right) \\
&\qquad + \sum_{l=1}^{m} (-1)^l K(x,\xi_l) K\left(\begin{smallmatrix} \xi_1,\xi_2,\dots\dots\dots,\xi_m \\ y,\xi_1,\dots,\hat{\xi}_l,\dots\xi_m \end{smallmatrix}\right) \Big) d\xi_1 \dots d\xi_m \\
&\quad = K(x,y) C_m - \sum_{l=1}^{m} \int K(x,\xi_l) \int \cdots \int K\left(\begin{smallmatrix} \xi_l,\xi_1,\dots\xi_l\dots,\xi_m \\ y,\xi_1,\dots,\hat{\xi}_l,\dots\xi_m \end{smallmatrix}\right) d\xi_1 \dots d\xi_m \\
&\quad = K(x,y) C_m - \sum_{l=1}^{m} \int K(x,\xi_l) C_{m-1}(\xi_l, y)\, d\xi_l.
\end{aligned}$$

Hence we obtain for $C_m(x,y)$:

$$C_m(x,y) = K(x,y) C_m - m \int K(x,z) C_{m-1}(z,y)\, dz,$$

for $m = 1, 2, \dots$. By the definition of $\mathcal{D}_\lambda(x,y)$ as

$$\sum_{m=0}^{\infty} \frac{(-\lambda)^m}{m!} C_m(x,y)$$

we get

$$\mathcal{D}_\lambda(x,y) = K(x,y)\mathcal{D}(\lambda) + \lambda \int K(x,z) \mathcal{D}_\lambda(z,y)\, dz. \tag{64}$$

Mutliplying (64) from the right by $\mathcal{D}(\lambda)^{-1}$, we find, that $\mathcal{D}_\lambda(x,y)\mathcal{D}(\lambda)^{-1}$ statisfies the Fredholm equation (62) as does $R_\lambda(x,y)$. But a solution for (62) is unique for $|\lambda| < \|K\|^{-1}$, as can be seen easily. Consequently, we have the identity

$$R_\lambda(x,y) = \mathcal{D}_\lambda(x,y)\mathcal{D}(\lambda)^{-1}$$

for $|\lambda| < \|K\|^{-1}$, and by analytic continuation, for all λ where $\mathcal{D}(\lambda)$ is invertible.

5.3 Applying Fredholm theory

Now, we can proceed to obtain the meromorphic continuation of the Eisenstein series $E_\mathfrak{a}(z,s,\chi)$. We recall their definition:

$$E_\mathfrak{a}(z,s,\chi) = \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma} \operatorname{Im}(\gamma z)^s \chi(\gamma)^{-1} P_\mathfrak{a},$$

and remind the reader of the existence of some $\sigma_0 > 0$, such that the above series converges locally uniformally absolutely in $\operatorname{Re}(s) > \sigma_0$. The starting point to apply Fredholm theory is the following equation:

$$-E_\mathfrak{a}(z,s,\chi) = (a(1-a) - s(1-s)) \int_F G_a(z/w,\chi) E_\mathfrak{a}(w,s)\, d\mu(w),$$

where $a > \operatorname{Re}(s) > \sigma_0$. This is the homogeneous Fredholm equation, but we can not yet employ the theory that we have developed above. The first problem are the singularities of $G_a(z/w,\chi)$ on the diagonal $z = w$. This is only a minor problem, which can be easily dealt with by taking the difference

$$G_{ab}(z/w,\chi) = G_a(z/w,\chi) - G_b(z/w,\chi)$$

for fixed b with $a > b$. Then we obtain the homogeneous Fredholm equation

$$-\nu_{ab}(s)E_\mathfrak{a}(z,s) = \int_F G_{ab}(z/w,\chi) E_\mathfrak{a}(w,s,\chi)\, d\mu(w), \tag{65}$$

with

$$\nu_{ab}(s) = (a(1-a) - s(1-s))^{-1} - (b(1-b) - s(1-s))^{-1}.$$

We let $\lambda_{ab}(s) = -\nu_{ab}(s)^{-1}$ and note that λ_{ab} is a polynomial of degree four in s. The new kernel $G_{ab}(z/w,\chi)$ is now continuous, since the leading and singular term of $G_s(z/w,\chi)$ does not depend on s.

In our previous considerations on the explicit construction of the Fredholm resolvent, we required that the kernel $K(z, w)$ be bounded on F. For $G_{ab}(z/w, \chi)$ this is not the case, so further modification is required. First, we look at the w variable. Here we subtract some particular contributions when w is in the cuspidal zones of F. We remind the reader of the partition of F into a relatively compact part $F(Y)$, plus cuspidal zones $F_{\mathfrak{a}}(Y)$ depending on $Y > 0$:

$$F = F(Y) \cup \bigcup_{\mathfrak{a}} F_{\mathfrak{a}}(Y).$$

We define the truncated kernel $G^Y_{ab}(z/w, \chi)$ on $F \times \mathbb{H}$ by setting $G^Y_{ab}(z/w, \chi) = G_{ab}(z/w, \chi)$ for $w \in F(Y)$ and

$$\begin{aligned} G^Y_{ab}(z/w, \chi) = G_{ab}(z/w, \chi) &- (2a-1)^{-1}(\mathrm{Im}\sigma_{\mathfrak{b}}^{-1}w)^{1-a} E_{\mathfrak{b}}(z, a, \chi) \\ &+ (2b-1)^{-1}(\mathrm{Im}\sigma_{\mathfrak{b}}^{-1}w)^{1-b} E_{\mathfrak{b}}(z, b, \chi), \end{aligned}$$

for $w \in F_{\mathfrak{b}}(Y)$. The new kernel $G^Y_{ab}(z/w, \chi)$ is continuous in w, except for jumps on the horocycles $L_{\mathfrak{b}}(Y)$. When w approaches a cusp, $G^Y_{ab}(z/w, \chi)$ decays exponentially, but still in the z variable the kernel is not bounded. The Eisenstein series grows polynomially in z. More precisely, we have

$$\|G^Y_{ab}(\sigma_{\mathfrak{a}} z/\sigma_{\mathfrak{b}} w, \chi)\| \ll y^a e^{-2\pi \max(y'-y, 0)}, \qquad \text{if } y, y' > 0.$$

If we replace in (65) the kernel $G_{ab}(z/w, \chi)$ by the kernel $G^Y_{ab}(z/w, \chi)$ we have to compute the integrals of the subtracted terms, namely:

$$\begin{aligned} &\int_{F_{\mathfrak{b}}(Y)} (\mathrm{Im}\sigma_{\mathfrak{b}}^{-1}w)^{1-a} E_{\mathfrak{a}}(w, s, \chi)\, d\mu w \\ &\quad = \int_0^1 \int_Y^\infty y^{-1-a}(\delta_{\mathfrak{ab}} y^s + \varphi(s) y^{1-s} + \cdots) P_{\mathfrak{a}}\, dx\, dy \\ &\quad = \delta_{\mathfrak{ab}} \frac{Y^{s-a}}{a-s} + \varphi_{\mathfrak{ab}}(s) \frac{Y^{1-a-s}}{a+s-1}, \end{aligned}$$

and similarly for the second term, when we replace all occurrences of b with a.

Thus we obtain the inhomogenous Fredholm equation

$$
\begin{aligned}
-\nu_{ab}(s)E_{\mathfrak{a}}(z,s,\chi) = &\int_F G^Y_{ab}(z/w,\chi)E_{\mathfrak{a}}(w,s,\chi)\, d\mu w \\
&+ \frac{Y^{s-a}}{(2a-1)(a-s)}E_{\mathfrak{a}}(z,a,\chi) \\
&- \frac{Y^{s-b}}{(2b-1)(b-s)}E_{\mathfrak{b}}(z,b,\chi) \\
&+ \frac{Y^{1-a-s}}{(2a-1)(a+s-1)}\sum_{\mathfrak{b}}\varphi_{\mathfrak{ab}}(s)E_{\mathfrak{b}}(z,a,\chi) \\
&- \frac{Y^{1-b-s}}{(2b-1)(b+s-1)}\sum_{\mathfrak{b}}\varphi_{\mathfrak{ab}}(s)E_{\mathfrak{b}}(z,b,\chi).
\end{aligned} \tag{66}
$$

To kill all terms involving the scattering matrix $\varphi_{\mathfrak{ab}}$, whose meromorphic continuation is not yet established, we add a suitable linear combination of (66) for the values of $Y, 2Y$ and $4Y$. We denote the right hand side of (66) by $R(Y)$ and obtain

$$
\begin{aligned}
&(1-2^{2s-1})^{-1}(R(Y)-2^{s-1}(2^a+2^b)R(2Y)+2^{2s-2+a+b}R(4Y)) \\
&=(2^{2s-1}-1)^{-1}(2^{s-1+a}-1)(2^{s-1+b}-1)\nu_{ab}(s)E_{\mathfrak{a}}(z,s,\chi) \\
&=\frac{2^{2s-1+a-b-1}}{(2b-1)(b-s)}Y^{s-b}E_{\mathfrak{a}}(z,b,\chi)-\frac{2^{2s-1-a+b-1}}{(2a-1)(a-s)}Y^{a-b}E_{\mathfrak{a}}(z,a,\chi) \\
&\quad+(1-2^{2s-1})^{-1}\int_F(G^Y_{ab}-2^{s-1}(2^a+2^b)G^{2Y}_{ab}+2^{2s-2+a+b}G^{4Y}_{ab})(z/w,\chi) \\
&\quad\times E_{\mathfrak{a}}(w,s,\chi)\ d\mu w.
\end{aligned} \tag{67}
$$

If we let

$$
h(z) = (2^{2s-1}-1)^{-1}(2^{s-1+a}-1)(2^{s-1+b}-1)\nu_{ab}(s)E_{\mathfrak{a}}(z,s,\chi), \tag{68}
$$

$$
f(z) = \frac{2^{2s-1+a-b-1}}{(2b-1)(b-s)}Y^{s-b}E_{\mathfrak{a}}(z,b,\chi)-\frac{2^{2s-1-a+b-1}}{(2a-1)(a-s)}Y^{a-b}E_{\mathfrak{a}}(z,a,\chi)Y^{s-a},
$$

and

$$
\begin{aligned}
H(z,w) =&(2^{s-1+a}-1)^{-1}(2^{s-1+b}-1)^{-1} \\
&\times(G^Y_{ab}-2^{s-1}(2^a+2^b)G^{2Y}_{ab}+2^{2s-2+a+b}G^{4Y}_{ab})(z/w,\chi)E_{\mathfrak{a}}(w,s,\chi),
\end{aligned}
$$

then we can rewrite (67) as

$$h(z) = f(z) + \lambda \int_F H(z,w)h(w)\, d\mu(w), \tag{69}$$

where $\lambda = \lambda_{ab}(s) = -\nu_{ab}(s)^{-1}$.

We are almost prepared to apply Fredholm theory. The only remaining problem is the polynomial growth of $f(z)$ and $H(z,w)$ in z. Fredholm theory requires these functions to be bounded. We have

$$\|f(\sigma_{\mathfrak{a}}z)\| \ll y^a$$

and

$$H(\sigma_{\mathfrak{a}}z, \sigma_{\mathfrak{b}}w) \ll y^a, e^{-2\pi \max\{y'-y,0\}}$$

for $y, y' \geq 4Y$. This obstacle will be removed, after multiplying (69) by $\eta(z) = e^{-\eta y(z)}$, where η is a constant $0 < \eta < 2\pi$. We obtain

$$\eta(z)h(z) = \eta(z)f(z) + \lambda \int_F \eta(z)\eta(w)^{-1}H(z,w)\eta(w)h(w)\, d\mu w.$$

Now we have that $\eta(z)f(z)$ is bounded in F and $\eta(z)\eta(w)^{-1}H(z,w)$ is bounded in $F \times F$.

We can now apply Fredholm theory, which tells us that the kernel $\eta(z)\eta(w)^{-1}H(z,w)$ has a resolvent of the form

$$R_\lambda(z,w) = \mathcal{D}_\lambda(z,w)\mathcal{D}(\lambda)^{-1},$$

where $\mathcal{D}(\lambda) \not\equiv 0$ and $\mathcal{D}_\lambda(z,w)$ are holomorphic in λ. For any λ, where $\mathcal{D}(\lambda)$ is invertible, we have a unique solution

$$\eta(z)h(z) = \eta(z)f(z) + \lambda \int_F R_\lambda(z,w)\eta(w)f(w)\, d\mu w.$$

Multiplying by $\eta(z)^{-1}$, we get

$$h(z) = f(z) + \lambda \int_F \eta(z)^{-1}\eta(w)\mathcal{D}_\lambda(z,w)\mathcal{D}(\lambda)^{-1}f(w)\, d\mu w. \tag{70}$$

From this equation we derive the meromorphic continuation of the Eisenstein series: The function $\mathcal{D}_\lambda(z,w)$ has a power series expansion in λ, whose coefficients are bounded in $F\times F$. Hence, the integral in (70) is again a meromorphic function in λ with values in $\mathrm{End}(V)$. Consequently h is a meromorphic function with values in $\mathrm{End}(V)$ and by (68) we see that $E(z,s,\chi)$ is meromorphic. We will collect these results in the next proposition. Firstly, we let

$$\begin{aligned} A_{\mathfrak{a}}(s) &= (2^{2+a-1}-1)(2^{s+b-1})\mathcal{D}(\lambda), \\ A_{\mathfrak{a}}(z,s) &= (2^{2s-1}-1)\lambda\mathcal{D}(\lambda)h(z), \end{aligned}$$

where $\lambda = \lambda_{ab}(s)$ and $h(z)$ are as in (67) and (68) respectively.

Proposition 5.1. *Let $c > \sigma_0$. Denote $\mathcal{S} = \{s \in \mathbb{C} : 1-c \le \mathrm{Re}(s) \le c\}$. There are functions $A_{\mathfrak{a}}(s) \not\equiv 0$ on $\mathcal{S}$ and $A_{\mathfrak{a}}(z,s)$ on $\mathbb{H}\times\mathcal{S}$ with values in* $\mathrm{End}(V)$*, such that*

1. *$A_{\mathfrak{a}}(s)$ is holomorphic in s,*
2. *$A_{\mathfrak{a}}(z,s)$ is holomorphic in s,*
3. *$A_{\mathfrak{a}}(z,s)$ is real-analytic in (z,s),*
4. *$A_{\mathfrak{a}}(\gamma z,s) = \chi(\gamma)A_{\mathfrak{a}}(z,s)$, for all $\gamma\in\Gamma$ and $z\in\mathbb{H}$,*
5. *$A_{\mathfrak{a}}(z,s) = A_{\mathfrak{a}}(s)E_{\mathfrak{a}}(z,x,\chi)$ if $\sigma_0 < \mathrm{Re}(s) \le c$.*

5.4 The functional equation

To prove a functional equation for the Eisenstein series, we will use the fact, that for the eigenvalues $s(1-s)$ of the Laplacian Δ ,considered as an operator on $L^2(\Gamma\backslash\mathbb{H},\chi)$, the real part $\mathrm{Re}(a)$ is bounded from above by σ_0.

Proposition 5.2. *Let $f:\mathbb{H}\to\mathrm{End}(V)$ be a smooth function with $\Delta f = s(1-s)f$ with $\mathrm{Re}(s)>\sigma_0$. We require, that f satisfies the usual invariance condition $f(\gamma z) = \chi(\gamma)f(z)$ for all $z\in\mathbb{H},\gamma\in\Gamma$. Furthermore, we assume that f does not grow to fast, more precisely*

$$\|f(z)\| \ll e^{\varepsilon y(z)} \tag{71}$$

with $0<\varepsilon<2\pi$, and that f is periodic at all cusps:

$$f(\gamma_{\mathfrak{a}}z) = f(z).$$

Then f can be written in terms of the Eisenstein series:

$$f(z) = \sum_{\mathfrak{a}} \alpha_{\mathfrak{a}} E_{\mathfrak{a}}(z, s, \chi),$$

where $\alpha_{\mathfrak{a}} \in \mathrm{End}(V)$.

Proof. Since f is periodic at every cusp we obtain a Fourier expansion

$$f(\sigma_{\mathfrak{a}} z) = \alpha_{\mathfrak{a}} P_{\mathfrak{a}} y^s + \beta_{\mathfrak{a}} P_{\mathfrak{a}} y^{1-s} + O(1),$$

where $\alpha_{\mathfrak{a}}, \beta_{\mathfrak{a}} \in \mathrm{End}(V)$. That the error term is bounded is a consequence of (71). We form a new function

$$g(z) = f(z) - \sum_{\mathfrak{a}} \alpha_{\mathfrak{a}} E_{\mathfrak{a}}(z, s, \chi).$$

Since we killed the leading terms $\alpha_{\mathfrak{a}} y^s$ of f the function g is an element of $L^2(\Gamma\backslash\mathbb{H}, \chi)$ and $\Delta g = s(1-s)g$. Hence $g \equiv 0$. □

The functional equation of the Eisenstein series is an easy consequence of the previous proposition. We let $\mathcal{E}(z, s, \chi) = (E_{\mathfrak{a}}(z, s, \chi))_{\mathfrak{a}}$ be the coloumn vector of the Eisenstein series $E_{\mathfrak{a}}(z, s, \chi)$, where $\mathfrak{a}$ ranges over all inequivalent cusps $\mathfrak{a}$. We let

$$\Phi(s) = (\varphi_{\mathfrak{a}\mathfrak{b}}(s)), \tag{72}$$

where $\mathfrak{a}, \mathfrak{b}$ range through all inequivalent cusps and $\varphi_{\mathfrak{a}\mathfrak{b}}$ is given by the constant term of the Fourier expansion of the Eisenstein series:

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) = \delta_{\mathfrak{a}\mathfrak{b}} y^s P_{\mathfrak{a}} + \varphi_{\mathfrak{a}\mathfrak{b}}(s) y^{1-s} + O(1).$$

We let $V_{\mathfrak{a}} = \mathrm{Im} P_{\mathfrak{a}}$ and $W_{\mathfrak{a}} = \mathrm{Im} Q_{\mathfrak{a}}$. We recall, that $Q_{\mathfrak{a}}$ is the orthogonal projection onto the orthocomplement $\mathrm{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)^{\perp}$. We furthermore put $\kappa_{\mathfrak{a}} = \dim V_{\mathfrak{a}}$ and $\kappa = \sum_{\mathfrak{a}} \kappa_{\mathfrak{a}}$. Then, $\Phi(s)$ yields an endomorphism of $\bigoplus_{\mathfrak{a}} V_{\mathfrak{a}}$, which is a vector space of dimension κ. If we reduce to the case, that χ is the trivial representation we find the well-known scattering matrix from the classical theory.

Theorem 5.3. *The vector $\mathcal{E}(z, s, \chi)$ satisfies the functional equation*

$$\mathcal{E}(z, s, \chi) = \Phi(s) \mathcal{E}(z, 1-s, \chi).$$

Proof. We take s with $\mathrm{Re}(s) > \sigma_0$ and $A_{\mathfrak{a}}(1-s)^{-1}$ exists. Then the Eisenstein series $E_{\mathfrak{a}}(z, 1-s, \chi)$ is defined by meromorphic continuation and satisfies

$$\Delta E_{\mathfrak{a}}(z, 1-s, \chi) = s(1-s) E_{\mathfrak{a}}(z, 1-s, \chi).$$

Using proposition 5.2 we find

$$E_{\mathfrak{a}}(z, 1-s, \chi) = \sum_{\mathfrak{b}} \varphi_{\mathfrak{a}\mathfrak{b}}(1-s) E_{\mathfrak{b}}(z, s).$$

By analytic continuation, this is true for all $s \in \mathbb{C}$ and the theorem is proved. □

6 The continuous spectrum

6.1 The orthogonal complement of $\mathcal{E}(\Gamma\backslash\mathbb{H}, \chi)$

Consider the following pullback diagram:

$$\begin{tikzcd} \mathbb{H} \times V \arrow[r] \arrow[d] & \mathbb{H} \times_\Gamma V \arrow[d] \\ \mathbb{H} \arrow[r, "\pi"] & \Gamma\backslash\mathbb{H} \end{tikzcd}.$$

The pullback of the bundle $\mathbb{H} \times_\Gamma V$ over $\Gamma\backslash\mathbb{H}$ along the projection $\pi : \mathbb{H} \to \Gamma\backslash\mathbb{H}$ is just the constant bundle $\mathbb{H} \times V$. We can pull back sections $\Gamma(X, E)$ along π and the set $\pi^*(\Gamma(X, E))$ is exactly the set of maps

$$f : \mathbb{H} \to V,$$

with the property, that $f(\gamma z) = \chi(\gamma) f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathbb{H}$.
The pull back of the smooth fibre metric on $\mathbb{H} \times_\Gamma V$ yields a metric on $\mathbb{H} \times V$ with the property, that

$$\langle \chi(\gamma) v, \chi(\gamma) w \rangle_{\gamma z} = \langle v, w \rangle_z,$$

for arbitrary $v, w \in V$, $\gamma \in \Gamma$ and $z \in \mathbb{H}$. Since χ is unitary at cusps we can choose the fibre metric in such a way, that the pull back metric on the constant bundle $\mathbb{H}\times V$ is constant if we move along horocycles. We will explain this in more detail:

The horocycles for a cusp $\mathfrak{a}$ are given by

$$H_{\mathfrak{a}}(y) = \{\sigma_{\mathfrak{a}}(x+iy) : x \in \mathbb{R}\},$$

where $y > 0$. Since $\chi(\gamma_{\mathfrak{a}})$ is unitary we can choose the metric, such that

$$\langle v,w\rangle_{\sigma_{\mathfrak{a}}(iy)} = \langle v,w\rangle_{\sigma_{\mathfrak{a}}(iy+x)}$$

for arbitrary $x \in \mathbb{R}$ and $v,w \in V$. Thus, the metric is constant along the horocycle $H_{\mathfrak{a}}(y)$.

Now, choose a cusp form $f \in \mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ and an incomplete Eisenstein series $\psi(\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))\chi(\gamma^{-1})v$. We compute the inner product as

$$\begin{aligned}
&\int_F \langle f(z), \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))\chi(\gamma^{-1})v\rangle_z \, d\mu(z)\\
&= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma F} \langle f(\sigma_{\mathfrak{a}} z), \psi(y)v\rangle_{\sigma_{\mathfrak{a}} z} \, d\mu(z)\\
&= \int_P \langle f(\sigma_{\mathfrak{a}} z), \psi(y)v\rangle_{\sigma_{\mathfrak{a}} z} \, d\mu(z)\\
&= \int_0^\infty \left(\int_0^1 \langle f(\sigma_{\mathfrak{a}} z), \psi(y)v\rangle_{\sigma_{\mathfrak{a}} z} \right) dx \, \frac{dy}{y^2}\\
&= \int_0^\infty \langle \int_0^1 f(\sigma_{\mathfrak{a}} z)\, dx, \psi(y)v\rangle_{\sigma_{\mathfrak{a}} z} \frac{dy}{y^2},
\end{aligned}$$

where $P = \{x+iy : x \in (0,1), y > 0\}$. The last step in the computation is justified by the above argument, that the metric is constant along the horocycles $H_{\mathfrak{a}}(y)$. We find that the above expression is equal to 0, since $f \in \mathcal{C}(\Gamma\backslash\mathbb{H},\chi)$ and hence

$$\int_0^1 f(\sigma_{\mathfrak{a}} z)\, dx.$$

Thus, we have:

Proposition 6.1. *The Hilbert space* $L^2(\Gamma\backslash\mathbb{H},\chi)$ *splits orthogonally into the space of cusp forms and the space of incomplete Eisenstein series*

$$L^2(\Gamma\backslash\mathbb{H},\chi) = \mathcal{C}(\Gamma\backslash\mathbb{H}) \oplus \mathcal{E}(\Gamma\backslash\mathbb{H},\chi).$$

6.2 The spectral decomposition of $\mathcal{E}(\Gamma\backslash\mathbb{H},\chi)$

The spectral decomposition of the space $\mathcal{E}(\Gamma\backslash\mathbb{H},\chi)$ of incomplete Eisenstein series is very similar to the case for χ unitary, due to the existence of Eisenstein series in both cases. The Mellin transform, the meromorphic continuation of the Eisenstein series and a contour integration yield the decomposition of $\mathcal{E}(\Gamma\backslash\mathbb{H},\chi)$ into a finite dimensional part, spanned by the residues of Eisenstein series and an infinite dimensional part, where Δ has absolutely continuous spectrum. Since $\mathcal{E}(\Gamma\backslash\mathbb{H},\chi)$ is spanned by the functions

$$E(z|\psi)v = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \psi(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))\chi(\gamma)^{-1}P_{\mathfrak{a}}v,$$

for $\psi\in C_c^\infty(\mathbb{R}_{>0})$ and $v\in V$, it will be enough to decompose $E(z|\psi)v$.

According to the Mellin inversion we find

$$E(z|\psi) = \frac{1}{2\pi i}\int_\sigma \hat{\psi}(s)E_{\mathfrak{a}}(z,s,\chi)\,ds$$

where $\sigma > 0$ is any real number, right to the axis of convergence $\mathrm{Re}(s) = \sigma_0$ of the Eisenstein series $E_{\mathfrak{a}}(z,s,\chi)$.

By Phragmen-Lindelöf we can move the integration to the axis $\mathrm{Re}(s) = 1/2$ but we have to account for the poles of the Eisenstein series $E_{\mathfrak{a}}(z,s,\chi)$ at values s_j with $1/2 < \mathrm{Re}(s_j) < \sigma$ with respective residues $\theta_{\mathfrak{a}j}$ and obtain

$$E(z|\psi) = \sum_{1/2<\mathrm{Re}(s_j)<\sigma} \hat{\psi}(s_j)\theta_{\mathfrak{a}j}(s_j) + \frac{1}{2\pi i}\int_{(1/2)} \hat{\psi}(s)E_{\mathfrak{a}}(z,s,\chi)\,ds$$

Now, computing the inner product of an incomplete Eisenstein series $E_{\mathfrak{a}}(\cdot|\psi)v$ and an Eisenstein series $E_{\mathfrak{b}}(z,s,\chi)w$, where $v,w\in V$ we obtain

$$
\begin{aligned}
&\int_F \langle E_{\mathfrak{a}}(\cdot|\psi)v, E_{\mathfrak{b}}(z,s,\chi)w\rangle_z \, d\mu(z) \\
&\quad= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \int_{\sigma_{\mathfrak{a}}^{-1}\gamma F} \langle \psi(\mathrm{Im}(z))\chi(\gamma)^{-1}P_{\mathfrak{a}}v, \chi(\gamma)^{-1}E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,s,\chi)w\rangle_{\gamma\sigma_{\mathfrak{a}}z}\, d\mu(z) \\
&\quad= \int_0^\infty \left(\int_0^1 \langle \psi(y)P_{\mathfrak{a}}v, E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,s,\chi)w\rangle_{x+iy}\right) dx\, \frac{dy}{y^2} \\
&\quad= \int_0^\infty \langle \psi(y)P_{\mathfrak{a}}v,\ \int_0^1 E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,s,\chi)w\, dx\rangle_{iy}\, \frac{dy}{y^2} \\
&\quad= \int_0^\infty \langle \psi(y)(P_{\mathfrak{a}}v, \delta_{\mathfrak{a}\mathfrak{b}}y^s + \varphi_{\mathfrak{b}\mathfrak{a}}(s)y^{1-s}P_{\mathfrak{b}}w)\rangle_{iy}\, \frac{dy}{y^2}
\end{aligned}
$$

For each cusp $\mathfrak{b}$ we choose an orthonormal basis $B_{\mathfrak{b}}$ of the subspace $P_{\mathfrak{b}}V$ of V. Multiplying the last term by $E_{\mathfrak{b}}(z,s,\chi)v$ and summing over all $\mathfrak{b}$ and all $v\in B_{\mathfrak{b}}$ we get by the functional equation of the Eisenstein series (5.3)

$$
\sum_{\mathfrak{b}}\sum_{v\in B_{\mathfrak{b}}} \langle E_{\mathfrak{a}}(\cdot|\psi)w, E_{\mathfrak{b}}(\cdot,s)v\rangle E_{\mathfrak{b}}(z,s,\chi)v = \hat{\psi}(s)E_{\mathfrak{a}}(z,s,\chi)w + \hat{\psi}(1-s)E_{\mathfrak{a}}(z,1-s)w
$$

Integrating this expression in s along the line $\mathrm{Re}(s) = 1/2$ we obtain:

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{(1/2)} \hat{\psi}(s)E_{\mathfrak{a}}(z,s,\chi)w\, ds \\
&\qquad\qquad= \sum_{\mathfrak{b}}\sum_{w\in B_{\mathfrak{b}}} \frac{1}{4\pi i}\int_{(1/2)} \langle E_{\mathfrak{a}}(\cdot|\psi)v, E_{\mathfrak{b}}(\cdot,s)w\rangle E_{\mathfrak{b}}(z,s,\chi)v\, ds.
\end{aligned}
$$

We sum up the spectral decomposition of $\mathcal{E}(\Gamma\backslash\mathbb{H},\chi)$ in the following theorem:

Theorem 6.2. *The space $\mathcal{E}(\Gamma\backslash\mathbb{H},\chi)$ of incomplete Eisenstein series has a direct sum decomposition*

$$
\mathcal{E}(\Gamma\backslash\mathbb{H},\chi) = \mathcal{R}(\Gamma\backslash\mathbb{H},\chi) \oplus_{\mathfrak{a}} \mathcal{E}_{\mathfrak{a}}(\Gamma\backslash\mathbb{H},\chi).
$$

The space $\mathcal{R}(\Gamma\backslash\mathbb{H},\chi)$ of residues of Eisenstein series is finite dimensional and the spectrum of the hyperbolic Laplacian Δ on $\mathcal{R}(\Gamma\backslash\mathbb{H},\chi)$ consists of a finite number of points. The spectrum of the hyperbolic Laplacian Δ is continuous in each $\mathcal{E}_{\mathfrak{a}}(\Gamma\backslash\mathbb{H},\chi)$ and covers the segment $[1/4,\infty)$ uniformly with multiplicity *equal to the dimension of* $P_{\mathfrak{a}}V$*. In particular* $\mathcal{E}_{\mathfrak{a}}(\Gamma\backslash\mathbb{H}) = \varnothing$*, if* χ *is non-singular at the cusp* $\mathfrak{a}$*. Every* $f \in \mathcal{E}(\Gamma\backslash\mathbb{H},\chi)$ *has the expansion*

$$\begin{aligned} f(z) &= \sum_{\mathfrak{a}} \sum_{v\in B_{\mathfrak{a}}} \sum_j \langle f, \theta_{\mathfrak{a}j}(z)v\rangle \theta_{\mathfrak{a}j}(z)v \\ &+ \frac{1}{4\pi}\sum_{\mathfrak{a}} \sum_{v\in B_{\mathfrak{a}}} \int_{-\infty}^{\infty} \langle f, E_{\mathfrak{a}}(\cdot, 1/2+ir,\chi)v\rangle E_{\mathfrak{a}}(z,1/2+ir,\chi)v\, dr, \end{aligned}$$

where the above equality is understood in the L^2*-sense.*

7 Spectral expansion of the automorphic kernel

So far we considered the bundle $E := \mathbb{H}\times_\Gamma V$ with base space X, equipped with a smooth metric, which induces the Hilbert space $L^2(\Gamma\backslash\mathbb{H},\chi)$. For the bundle E we now consider the dual bundle E^*. The fibre of $x \in \Gamma\backslash\mathbb{H}$ in E^* is the dual space of the fibre of x in E.

In our case we have a canonical description of E^*. Consider the dual space V^* of the original representation space V. The group Γ acts on V^* via the contragredient representation. This is given as follows: For $\varphi \in V^*, v \in V$ and $\gamma\in\Gamma$ we let $\chi^*(\gamma)\varphi(v) = \varphi(\chi(\gamma^{-1})v)$. The inner product on V gives a canonical identification $V \cong V^*$ and with respect to this identification the contragredient representation is given by $\gamma \mapsto \chi(\gamma^{-1})^t$, where t denotes the transpose with respect to the inner product. For any $k(u) \in C_0^\infty(\mathbb{R}_{>0})$ the automorphic kernel

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w)\chi(\gamma)$$

can be identified with an element of $C^\infty(\Gamma\backslash\mathbb{H}, E\boxtimes E^*)$. We can use the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H},\chi)$ from Proposition 4.12 and Theorem 6.2 to deduce a spectral expansion of the kernel $K(z,w)$.

Firstly, we choose an orthonormal basis (u_i) for the discrete spectrum such that each u_i is either an element of a generalized eigenspace $R(\Delta,\lambda_i)$, or a residue of an Eisenstein series. We let u_i^* be the corresponding dual element.

For a cusp $\mathfrak{a}$ we have the Eisenstein series

$$E_{\mathfrak{a}}(z,s,\chi^{-t}) \in C^{\infty}(\Gamma\backslash\mathbb{H},E^*).$$

We introduce the Selberg transform h for $k \in C_0^{\infty}(\mathbb{R}_{>0})$ in three steps:

$$q(v) = \int_v^{\infty} k(u)(u-v)^{-1/2}\,du, \tag{73}$$

$$g(r) = 2q\left((\sinh r/2)^2\right), \tag{74}$$

$$h(t) = \int_{-\infty}^{\infty} e^{irt} g(r)\,dr. \tag{75}$$

Then, we have by Theorem 1.16 in [Iwa02], that

$$\langle K(\cdot,w),u_j\rangle = h(\lambda_j)u_j^*(w),$$
$$\langle K(\cdot,w),E_{\mathfrak{a}}(\cdot,1/2+ir,\chi)v\rangle = h(r)E_{\mathfrak{a}}(w,1/2+ir,\chi^{-t})v.$$

Thus from the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H},\chi)$ we obtain the following theorem:

Theorem 7.1. *Let*

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w)\chi(\gamma)$$

be an automorphic kernel with selberg transform $h(r)$. Then it has a spectral expansion

$$\begin{aligned} K(z,w) &= \sum_j h(t_j)u_j(z) \otimes u_j^*(w) \\ &\quad + \sum_{\mathfrak{a}} \sum_{v\in B_{\mathfrak{a}}} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r)E_{\mathfrak{a}}(z,1/2+ir,\chi)v \otimes E_{\mathfrak{a}}(w,1/2+ir,\chi^{-t})v\,dr \end{aligned}$$

8 Selberg trace formula for hyperbolic surfaces

Having deduced the spectral expansion of the automorphic kernel $K(z,w)$, we are now ready to deduce the Selberg trace formula.
We start by inserting the spectral decomposition of the integral kernel $K(z,w)$ to compute the integral and to obtain the spectral side of the trace formula.

The computation will be done asymptotically, by computing it on the relatively compact set $F(Y) \subset F$ and letting Y tend to ∞. The asymptotic computation of $\mathrm{tr}^Y K$ becomes necessary, since the integral

$$\int_F \mathrm{tr}\, K(z,z)\, d\mu(z)$$

will not converge, due to the existence of the continuous spectrum. We will write

$$\mathrm{tr}^Y K = \int_{F(Y)} \mathrm{tr}\, K(z,z)\, d\mu(z), \tag{76}$$

for the truncated trace. In a second step, we will use the formula

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w)\chi(\gamma) \tag{77}$$

to compute the integral (76) and obtain the geometric side of the trace formula. In the above sum (77), we reorder according to the conjugacy classes of Γ. We thus write

$$\mathrm{tr}\, K(z,z) = \sum_{\mathcal{C}} \sum_{\gamma\in\mathcal{C}} k(z,\gamma z)\, \mathrm{tr}\, \chi(\gamma).$$

The sum $\sum_{\mathcal{C}}$ ranges over all conjugacy classes of Γ. We distinguish the trivial conjugacy class, hyperbolic, elliptic and parabolic conjugacy classes and accordingly we will compute for all conjugacy classes $\mathcal{C}$, except the parabolic ones, the partial trace

$$\int_F \sum_{\gamma\in\mathcal{C}} k(z,\gamma z)\, \mathrm{tr}\, \chi(\gamma)\, d\mu(z). \tag{78}$$

Since

$$\eta\gamma\eta^{-1} = \nu\gamma\nu^{-1}$$

if, and only if $\eta\nu^{-1}$ is an element of the centralizer $Z(\gamma)$ of γ we may write (78) also as

$$\sum_{\tau\in Z(\gamma)\backslash\Gamma} \int_F k(z,\tau^{-1}\gamma\tau z)\, \mathrm{tr}\, \chi(\gamma)\, d\mu(z) = \int_{Z(\gamma)\backslash\mathbb{H}} k(z,\gamma z)\, \mathrm{tr}\, \chi(\gamma)\, d\mu(z),$$

where the equality is true, by the unfolding trick. For the parabolic conjugacy classes, the integral

$$\int_F \sum_{\gamma \in \mathcal{C}} k(z, \gamma z) \operatorname{tr} \chi(\gamma)\, d\mu(z)$$

does not converge. Instead we will, once again as for the spectral side, compute a truncated trace

$$\int_{F(Y)} \sum_{\gamma \in \mathcal{C}} k(z, \gamma z) \operatorname{tr} \chi(\gamma)\, d\mu(z)$$

for parabolic classes.

By computing $\operatorname{tr}^Y K$ spectrally and geometrically, as sketched above, we can compare both results. It will turn out, that on both sides a term $\log Y$ will appear, which is responsible for the divergence. Cancelling this term, both, the spectral and the geometric side will converge, as Y tends to infinity and this will yield the trace formula (Theorem 8.6).

We recall the following definitions: The map $P_{\mathfrak{a}}$ is the orthogonal projection onto $V_{\mathfrak{a}} = \operatorname{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)$ and $Q_{\mathfrak{a}}$ the orthogonal projection onto the orthogonal complement $W_{\mathfrak{a}} = \operatorname{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)^{\perp}$. We furthermore let $\kappa_{\mathfrak{a}} = \dim \operatorname{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)$ and $\kappa = \sum_{\mathfrak{a}} \kappa_{\mathfrak{a}}$.

8.1 The spectral contribution

By the spectral decomposition of the integral kernel we obtain

$$\begin{aligned} \operatorname{tr}^Y K = & \sum_j h(t_j) \int_{F(Y)} |u_j(z)|^2 \, d\mu(z) \\ & + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \sum_{\mathfrak{a}} \sum_{v \in B_{\mathfrak{a}}} \int_{F(Y)} |E_{\mathfrak{a}}^Y(z, 1/2 + ir, \chi) v|^2 \, d\mu(z)\, dr. \end{aligned} \tag{79}$$

Since we integrate over $F(Y)$ only, we can replace the Eisenstein series by their truncated equivalents. These are defined as follows:

$$E_{\mathfrak{a}}^Y(z, s, \chi) = \begin{cases} E_{\mathfrak{a}}(z, s, \chi) - \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} z)^s P_{\mathfrak{a}} + \varphi(s) \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{1-s} \text{ if } z \in F_{\mathfrak{a}}(Y), \\ E_{\mathfrak{a}}(z, s, \chi) \text{ if } z \in F(Y). \end{cases}$$

Hence, $E_{\mathfrak{a}}^Y(z, s, \chi)$ is equal to the original Eisenstein series $E_{\mathfrak{a}}(z, s, \chi)$ in the relatively compact part $F(Y)$, and in the cuspidal area $F_{\mathfrak{a}}(Y)$ it is the original Eisenstein series with the constant term in its Fourier expansion eliminated. The reason to do this, is that exactly the constant term of the Eisenstein series is the obstacle to L^2-integrability.

Now, we can compute the following integral in the same manner as in [Iwa02, 6.35]:

$$\begin{aligned}\sum_{\mathfrak{a}}\sum_{v\in B_{\mathfrak{a}}}\int_F |E^Y_{\mathfrak{a}}(z,1/2+ir,\chi)v|^2\,d\mu(z) &= \operatorname{tr}\left\langle \mathcal{E}^Y(\cdot,1/2+ir,\chi),\mathcal{E}^Y(\cdot,1/2+ir,\chi)\right\rangle\\ &= \frac{1}{2ir}\operatorname{tr}\left(\Phi(1/2-ir)Y^{2ir}-\Phi(1/2+ir)Y^{-2ir}\right)\\ &\quad + 2\kappa h\log Y - \operatorname{tr}\Phi'(s)\Phi^{-1}(s).\end{aligned}$$

Here, Φ is the scattering matrix, as introduced in (72). Recall, that Φ defines an endomorphism of $\bigoplus_{\mathfrak{a}} V_{\mathfrak{a}}$. We need to compute the integral

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{h(r)}{2ir}\left(\Phi(1/2-ir)Y^{2ir}-\Phi(1/2+ir)Y^{-2ir}\right)dr. \tag{80}$$

By adding $0=\Phi(1/2)-\Phi(1/2)$ and using the symmetry $h(r)=h(-r)$ we get

$$\begin{aligned}&\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{h(r)}{2ir}\left(\Phi(1/2-ir)Y^{2ir}-\Phi(1/2)-\Phi(1/2+ir)Y^{-2ir}\right)+\Phi(1/2)\,dr\\ &= \frac{1}{4\pi i}\int_{-\infty}^{\infty}r^{-1}h(r)\left(\Phi(1/2-ir)Y^{2ir}-\Phi(1/2)\right)dr.\end{aligned}$$

We regard the above integral as a complex line integral and move the integration to $\operatorname{Im} r=\varepsilon$:

$$\begin{aligned}&\frac{1}{4\pi i}\int_{\operatorname{Im} r=\varepsilon}r^{-1}h(r)\left(\Phi(1/2-ir)Y^{2ir}-\Phi(1/2)\right)dr\\ &\qquad= -\Phi(1/2)\frac{1}{4\pi i}\int_{\operatorname{Im} r=\varepsilon}r^{-1}h(r)\;+O(Y^{-2\varepsilon}),\end{aligned}$$

where the equality is true, since Φ is bounded in a neighborhood of the critical axis $\operatorname{Re} s=1/2$. Now, by the residue theorem and the equality

$$\int_{\operatorname{Im} r=\varepsilon}r^{-1}h(r)\,dr=\int_{\operatorname{Im} r=-\varepsilon}r^{-1}h(r)\,dr,$$

since $h(r)=h(-r)$, we have

$$\frac{1}{2\pi i}\int_{\operatorname{Im} r=\varepsilon}r^{-1}h(r)\,dr=-\frac{1}{2}h(0).$$

Thus we find, that (80) is equal to

$$\frac{1}{4}\Phi(1/2)h(0) + O(Y^{-\varepsilon}).$$

Now we come back to the truncated trace. Recall that in its definition (79) the domain of integration was limited to the relatively compact set $F(Y)$. We will let Y tend to infinity, such that $F(Y)$ exhausts the whole fundamental domain F. If we replace the domain of integration $F(Y)$ in (79) by F, we get by the above

$$\begin{aligned}\operatorname{tr}^Y K \le \sum_j h(t_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \operatorname{tr}\Phi'(s)\Phi^{-1}(s)h(r)\,dr \\ + \frac{1}{4}h(0)\operatorname{tr}\Phi(1/2) + \kappa g(0)h\log Y + O(Y^{-\varepsilon}).\end{aligned} \tag{81}$$

But the contribution of the integral, if we integrate over the cuspidal area $F_{\mathfrak{a}}(Y)$ is $O(Y^{-1})$ as $Y \to \infty$. This contribution gets absorbed by $O(Y^{-\varepsilon})$ in (81). Thus the above inequality becomes an equality and we find the spectral contribution

$$\begin{aligned}\operatorname{tr}^Y K = \sum_j h(t_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} \operatorname{tr}\Phi'(s)\Phi^{-1}(s)h(r)\,dr \\ + \frac{1}{4}h(0)\operatorname{tr}\Phi(1/2) + \kappa g(0)h\log Y + O(Y^{-\varepsilon}).\end{aligned}$$

8.2 The hyperbolic terms

Following the tradition, we will denote primitive hyperbolic conjugacy classes by P, to emphasize their resemblance to prime ideals in number fields. Recall, that every hyperbolic conjugacy class is of the form P^l for some primitive hyperbolic conjugacy class P and some non-zero integer l. Furthermore $P^l = P^{-l}$. If $\gamma_P \in P$ is a primitive hyperbolic element and $\gamma = \gamma_P^l$ some integer power of γ_P, then the centralizers $Z(\gamma_P) = Z(\gamma)$ coincide. We will compute the trace of the restricted kernel K_{P^l}. By unfolding we get

$$\operatorname{tr} K_{P^l} = \int_{\Gamma\backslash\mathbb{H}} \sum_{\gamma\in P^l} k(z,\gamma z)\operatorname{tr}\chi(\gamma)\,d\mu(z) = \int_{Z(\gamma_P)\backslash\mathbb{H}} k(z,\gamma z)\operatorname{tr}\chi(\gamma)\,d\mu(z). \tag{82}$$

After conjugation, if necessary, we can assume, that γ_P acts on $\mathbb{H}$ as $z \mapsto pz$. We can furthermore assume, that $p > 1$, otherwise we substitute P by P^{-1}. A fundamental domain for the centralizer can be chosen as the strip $1 < y < p$, where y is the imaginary part of $z \in \mathbb{H}$. Thus, continuing with (82) we obtain

$$\operatorname{tr} K_{P^l} = \int_1^p \int_{-\infty}^{\infty} k(z, p^l z) \operatorname{tr} \chi(\gamma_P)\, d\mu(z).$$

We put $2d = |p^{l/2} - p^{-l/2}|$ and continue the computation:

$$\begin{aligned}
\operatorname{tr} K_{P^l} &= \int_1^p \int_{-\infty}^{\infty} k\left((d|z|/y)^2\right) y^{-2} \operatorname{tr} \chi(\gamma)\, dx\, dy \\
&= \left(\int_1^p y^{-1}\, dy\right) \int_{-\infty}^{\infty} k(d^2(x^2+1))\, dx \operatorname{tr} \chi(\gamma) \\
&= \frac{\log p}{d} \int_{d^2}^{\infty} \frac{k(u)}{\sqrt{u - d^2}}\, du \operatorname{tr} \chi(\gamma) \\
&= \frac{\log p}{d} q(d^2) \operatorname{tr} \chi(\gamma) \\
&= \frac{\log p}{2d} g(2 \log(\sqrt{d^2+1} + d)) \operatorname{tr} \chi(\gamma) \\
&= \frac{\log p}{2d} g(l \log p) \operatorname{tr} \chi(\gamma) \\
&= |p^{l/2} - p^{-l/2}|^{-1} g(l \log p) \log p \operatorname{tr} \chi(\gamma_P)^l,
\end{aligned}$$

where the functions q and g are as in (73).

8.3 The identity term

We compute the trace for the trivial conjugacy class $\mathcal{C} = \{1\}$.

$$\operatorname{tr} K_{\mathcal{C}} = \int_F k(z,z) \operatorname{tr} \chi(1)\, d\mu(z) = k(0) \operatorname{vol}(\Gamma \backslash \mathbb{H}) \dim V.$$

Furthermore

$$k(0) = \frac{1}{4\pi} \int_{-\infty}^{\infty} r \tanh(\pi r) h(r)\, dr,$$

see (1.64') in [Iwa02].

8.4 The parabolic terms

There are as many primitive parabolic conjugacy classes, as there are inequivalent cusps. Let $\mathcal{C}_{\mathfrak{a}}$ be the primitive parabolic conjugacy classes belonging to a cusp $\mathfrak{a}$. Every other parabolic conjugacy class $\mathcal{C}$ is of the form $\mathcal{C} = \mathcal{C}_{\mathfrak{a}}^l$ for some cusp $\mathfrak{a}$ and with some non-zero integer l. If $\gamma_{\mathfrak{a}}$ is the generator of the fix group $\Gamma_{\mathfrak{a}}$ of $\mathfrak{a}$, we let $\gamma = \gamma_{\mathfrak{a}}^l$. Then for the centralizers we have the following equality:

$$Z(\gamma) = Z(\gamma_{\mathfrak{a}}) = \Gamma_{\mathfrak{a}}.$$

To compute the trace of the partial kernel $K_{\mathcal{C}}$, we will split the computation into two parts. Here again, $P_{\mathfrak{a}}$, the orthogonal projection onto $\operatorname{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)$ and $Q_{\mathfrak{a}}$, the orthogonal projection onto the orthogonal complement $\operatorname{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)^{\perp}$, and the numbers $\kappa_{\mathfrak{a}} = \dim \operatorname{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)$ and $\kappa = \sum_{\mathfrak{a}} \kappa_{\mathfrak{a}}$, will be involved. Then we can write

$$K_{\mathcal{C}} = K_{\mathcal{C}} P_{\mathfrak{a}} + K_{\mathcal{C}} Q_{\mathfrak{a}}.$$

We firstly compute the contribution of the first term $K_{\mathcal{C}} P_{\mathfrak{a}}$.

8.4.1 The singular contribution

The unfolding trick yields

$$\begin{aligned} \operatorname{tr}^Y K_{\mathcal{C}} P_{\mathfrak{a}} &= \int_{Z(\gamma)\backslash\mathbb{H}(Y)} k(z, \gamma z) \operatorname{tr} \chi(\gamma) P_{\mathfrak{a}} \, d\mu(z) \\ &= \kappa_{\mathfrak{a}} \int_{Z(\gamma)\backslash\mathbb{H}(Y)} k(z, \gamma z) \, d\mu(z). \end{aligned}$$

The set $\mathbb{H}(Y)$ is the upper half-plane where the cuspidal zones are removed at height Y. After the change of variables $z \mapsto \sigma_{\mathfrak{a}} z \sigma_{\mathfrak{a}}^{-1}$, this equals

$$\operatorname{tr}^Y K_{\mathcal{C}} P_{\mathfrak{a}} = \kappa_{\mathfrak{a}} \int_{B\backslash\sigma_{\mathfrak{a}}\mathbb{H}(Y)} k(z, z + l) \, d\mu(z).$$

To estimate the above integral from below and above, we firstly make the domain of integration smaller and in a second step we will enlarge the domain of integration. A fundamental domain for $B\backslash\sigma_{\mathfrak{a}}\mathbb{H}(Y)$ is contained in the rectangle

$$\{z \in \mathbb{H} : 0 < x \le 1, Y' < y \le Y, Y'Y = c_{\mathfrak{a}}^{-2}\}.$$

On the other hand this fundamental domain contains the rectangle

$$\{z \in \mathbb{H} : 0 < x \le 1, 0 < y \le Y\}.$$

Thus, the following chain of inequalities holds true:

$$\kappa_{\mathfrak{a}} \int_0^1 \int_{Y'}^Y k(z, z+l)\, d\mu(z) \le \operatorname{tr}^Y K_{\mathcal{C}} P_{\mathfrak{a}} \le \kappa_{\mathfrak{a}} \int_0^1 \int_0^Y k(z, z+l)\, d\mu(z). \quad (83)$$

We continue the computation of the right-hand side of (83)

$$\int_0^1 \int_0^Y k(z, z+l)\, d\mu(z) = \int_0^Y k\left(\left(\frac{l}{2y}\right)^2\right) y^{-2}\, dy = |l|^{-1} \int_{(l/2Y)^2}^{\infty} k(u) u^{-1/2}\, du,$$

by the change of variables $u = (l/2y)^2$. Now, summing the above expression over l and interchanging the summation and integration, we get

$$\begin{aligned} &2 \int_{(2Y)^{-2}}^{\infty} k(u) u^{-1/2} \left(\sum_{1 \le l < 2Y\sqrt{u}} l^{-1} \right) du \\ &\qquad = 2 \int_{(2Y)^{-2}}^{\infty} k(u) u^{-1/2} (\log 2Y\sqrt{u} + \gamma + O(u^{-1/2} Y^{-1}))\, du \\ &\qquad = L(Y) + O(Y^{-1} \log Y), \end{aligned}$$

where γ here denotes the Euler-Mascheroni constant and $L(Y)$ stands for the term

$$L(Y) = 2 \int_0^{\infty} k(u) u^{-1/2} (\log 2Y\sqrt{u} + \gamma).$$

For the left-hand term of (83), we do a similar computation and replace Y by Y' to obtain

$$\begin{aligned} \int_0^1 \int_0^{Y'} k(z, z+l)\, d\mu(z) &= 2 \int_{(2Y')^{-2}}^{\infty} k(u) u^{-1/2} \left(\sum_{1 \le l < 2Y'\sqrt{u}} l^{-1} \right) du \\ &= \int_{(2Y')^{-2}}^{\infty} k(u) u^{-1/2} \log(u+2)\, du \ll O(Y'). \end{aligned}$$

Hence, we find

$$\begin{aligned} L(Y) + O(Y^{-1}\log Y) &= \sum_{l\neq 0} \int_0^1 \int_{Y'}^{Y} k(z, z+l)\, d\mu(z) \\ &\le \sum_{\mathcal{C}=\mathcal{C}_{\mathfrak{a}}^l} \operatorname{tr}^Y K_{\mathcal{C}} \\ &\le \sum_{l\neq 0} \int_0^1 \int_{Y'}^{Y} k(z, z+l)\, d\mu(z) = L(Y) + O(Y^{-1}\log Y), \end{aligned}$$

and consequently, we also have

$$\sum_{\mathcal{C}=\mathcal{C}_{\mathfrak{a}}^l} \operatorname{tr}^Y K_{\mathcal{C}} P_{\mathfrak{a}} = \kappa_{\mathfrak{a}} L(Y) + O(Y^{-1}\log Y).$$

To express $L(Y)$ in terms of g and h, we proceed as follows: We write

$$L(Y) = g(0)(\log 2Y + \gamma) + \int_0^\infty k(u) u^{-1/2} \log u\, du. \tag{84}$$

In the above formula we made $g(0)$ appear, via

$$g(0) = 2q(0) = 2\int_0^\infty k(u) u^{-1/2}\, du.$$

The second term can be transformed as

$$\begin{aligned} \int_0^\infty k(u) u^{-1/2} \log u\, du &= \frac{-1}{\pi} \int_0^\infty \left(\int_0^v \frac{\log u}{\sqrt{u(v-u)}} \right) dq(v) \\ &= \frac{-1}{\pi} \int_0^\infty \left(\int_0^1 \frac{\log uv}{\sqrt{u(1-u)}}\, du \right) dq(v) \\ &= \frac{1}{\pi} q(0) \int_0^1 \frac{\log u}{\sqrt{u(1-u)}}\, du \\ &\quad - \frac{1}{\pi} \int_0^1 \frac{du}{\sqrt{u(1-u)}} \int_0^\infty \log v\, dq(v). \end{aligned}$$

The integrals after the last equality can be evaluated to $-2\pi \log 2$, π and $\int_0^\infty \log(\sinh r/2)\, dg(r)$, respectively. If we note that $q(0) = \frac{1}{2} g(0)$, we obtain

$$\int_0^\infty k(u) u^{-1/2} \log u\, du = -\log 2\, g(0) - \int_0^\infty \log(\sinh r/2)\, dg(r). \tag{85}$$

We want to further modify the integral

$$\int_0^\infty \log(\sinh r/2)\, dg(r).$$

We use the formula

$$g'(r) = -\frac{1}{2\pi i}\int_{\operatorname{Im} t=\varepsilon} e^{irt} h(t) t\, dt$$

and the Laplace transform

$$\int_0^\infty -\log(\sinh r/2)\nu e^{-\nu r}\, dr = \gamma + \log 2 - \frac{1}{2\nu} + \psi(1+\nu),$$

where

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n+1}\right).$$

Now we can write

$$\begin{aligned}
\int_0^\infty \log(\sinh r/2)\, dg(r) &= \int_0^\infty \log(\sinh r/2)\frac{-1}{2\pi i}\int_{\operatorname{Im} t=\varepsilon} e^{irt} h(t) t\, dt\, dr \\
&= \int_{\operatorname{Im} t=\varepsilon} h(t)\frac{-1}{2\pi i}\int_0^\infty \log(\sinh r/2) e^{irt} t\, dr\, dt \\
&= \frac{1}{2\pi}\int_{\operatorname{Im} t=0}\left(\gamma + \log 2 + \frac{1}{2it} + \psi(1-it)\right) h(t)\, dt \\
&= g(0)(\gamma + \log 2) - 1/4h(0) + \frac{1}{2\pi}\int_{-\infty}^\infty h(t)\psi(1+it)\, dt,
\end{aligned}$$

since

$$\frac{1}{2\pi i}\int_{\operatorname{Im} t=\varepsilon} h(t)/t\, dt = -1/2h(0),$$

and h is even, so

$$\int_{\operatorname{Im} t=\varepsilon} h(t)\psi(1-it)\, dt = \int_{\operatorname{Im} t=\varepsilon} h(t)\psi(1+it)\, dt.$$

Collecting the above computations we find

$$\int_0^\infty k(u)u^{-1/2}\log u\, du = -g(0)(\gamma+\log 4) + \frac{1}{4}h(0) - \frac{1}{2\pi}\int_{-\infty}^\infty h(t)\psi(1+it)\, dt. \quad (86)$$

Thus, if we plug (86) into (84) we find

$$L(Y) = g(0)\log\frac{Y}{2} + \frac{1}{4}h(0) - \frac{1}{2\pi}\int_{-\infty}^\infty h(t)\psi(1+it)\, dt.$$

In total, we get that the singular contribution of all parabolic classes, with fixed point the cusp $\mathfrak{a}$, is equal to

$$\mathrm{tr}^Y \sum_{l\neq 0} K_{\mathcal{C}_{\mathfrak{a}}^l} P_{\mathfrak{a}} = \kappa_{\mathfrak{a}} \left(g(0) \log \frac{Y}{2} + \frac{1}{4} h(0) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)\psi(1+it)\, dt \right).$$

8.4.2 The non-singular contribution

Now we will take care of the contribution of $K_{\mathcal{C}} Q_{\mathfrak{a}}$. Without loss of generality, we can assume, that $\dim \mathrm{Im}\, Q_{\mathfrak{a}} = 1$ and that $\chi(\gamma_{\mathfrak{a}})$ acts on the image by multiplying each vector with $e^{i\alpha}$, where $\alpha \in (0, 2\pi)$. By the unfolding trick and the change of variables $z \mapsto \sigma_{\mathfrak{a}} z \sigma_{\mathfrak{a}}^{-1}$, we find

$$\begin{aligned} \mathrm{tr} \sum_{l\neq 0} K_{\mathcal{C}_{\mathfrak{a}}^l} &= \int_{Z(\gamma_{\mathfrak{a}})\backslash\mathbb{H}} \sum_{l\neq 0} k(z, \gamma^l z)\, \mathrm{tr}\, \chi(\gamma_{\mathfrak{a}}^l)\, d\mu(z) \\ &= \int_{B\backslash\mathbb{H}} \sum_{l\neq 0} k(z, z+l) e^{il\alpha}\, d\mu(z). \end{aligned} \tag{87}$$

If we group the terms for l and $-l$ together, we obtain that (87) equals

$$\int_0^\infty \sum_{l=1}^\infty \cos(l\alpha) k(l^2/y^2) \frac{dy}{y^2} = \int_0^\infty \sum_{l=1}^\infty \cos(l\alpha) k(l^2 u^2)\, du.$$

To be able to interchange summation and integration, we write this as

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\infty \sum_{l=1}^\infty \cos(l\alpha) k(l^2 u^2)\, du,$$

and continue

$$\begin{aligned} \int_\varepsilon^\infty \sum_{l=1}^\infty \cos(l\alpha) k(l^2 u^2)\, du &= \sum_{l=1}^\infty \cos(l\alpha) \int_\varepsilon^\infty k(l^2 u^2)\, du \\ &= \sum_{l=1}^\infty \frac{\cos(l\alpha)}{l} \int_{l\varepsilon}^\infty k(u^2)\, du \\ &= \int_0^\infty k(u^2) \left(\sum_{1/leql\leq u/\varepsilon} \frac{\cos(l\alpha)}{l} \right) du. \end{aligned}$$

Since $\alpha \in (0, 2\pi)$, we have

$$\sum_{1\leq l\leq u/\varepsilon} \frac{\cos(l\alpha)}{l} = \log \frac{1}{|1 - e^{i\alpha}|} + O(\sqrt{\varepsilon/u}).$$

Substituting this estimate, we can continue our above computation:

$$\begin{aligned}\int_{\varepsilon}^{\infty}\sum_{l=1}^{\infty}\cos(l\alpha)k(l^2u^2)\,du &= \log\frac{1}{|1-e^{i\alpha}|}\int_0^{\infty}k(u^2)\,du + O\left(\sqrt{\varepsilon}\int_0^{\infty}k(u^2)\frac{du}{\sqrt{u}}\right)\\ &= \frac{1}{2}g(0)\log\frac{1}{|1-e^{i\alpha}|}+O(\sqrt{\varepsilon}).\end{aligned}$$

Now, inserting this into (87), and letting $\varepsilon \to 0$ we get, that (87) is equal to

$$g(0)\log\frac{1}{|1-e^{i\alpha}|} = g(0)\log\frac{1}{|1-\chi(\gamma_{\mathfrak{a}})|} = -g(0)\log|1-\chi(\gamma_{\mathfrak{a}})|.$$

In the general case, that $\dim\operatorname{Im}Q_{\mathfrak{a}} > 1$, each eigenvalue $e^{i\alpha} \neq 1$ of $\chi(\gamma_{\mathfrak{a}})$ contributes the above expression with the respective multiplicity. Adding these up, we obtain the contribution

$$\sum_{l\neq 0}\operatorname{tr}K_{\mathcal{C}_{\mathfrak{a}}^l}Q_{\mathfrak{a}} = -g(0)\log|\det(\mathrm{Id}-\chi(\gamma_\alpha))|_{W_\alpha}|.$$

Here, $(\mathrm{Id}-\chi(\gamma_\alpha))|_{W_\alpha}$ is the restriction of the endomorphism $\mathrm{Id}-\chi(\gamma_\alpha)$ to $W_\alpha = \operatorname{Im}Q_{\mathfrak{a}}$.

8.5 The elliptic terms

Let R be a primitive elliptic conjugacy class. An elliptic conjugacy class has only fixed points in $\mathbb{H}$ and each elliptic class $\mathcal{C}$ with the same fixed points is a power of R: $\mathcal{C} = R^l$ with some $0 < l < m$, where m is the order of R. After conjugation, we can assume, that the matrix

$$\gamma_R = \begin{pmatrix}\cos(\pi m^{-1}) & \sin(\pi m^{-1})\\ -\sin(\pi m^{-1}) & \cos(\pi m^{-1})\end{pmatrix}$$

is a representative for R.

The matrix γ_R acts as a rotation of angle $2\pi m^{-1}$ at $i \in \mathbb{H}$. As a fundamental domain for that centralizer, a hyperbolic sector S at i of angle $2\pi m^{-1}$ may be chosen. Similar as for the hyperbolic situation, unfolding yields

$$\operatorname{tr}K_{R^l} = \int_S k(z,\gamma_R^l z)\operatorname{tr}\chi(\gamma_R)^l\,d\mu(z) = \frac{1}{m}\int_{\mathbb{H}}k(z,\gamma_R^l z)\operatorname{tr}\chi(\gamma_R)^l\,d\mu(z).$$

Applying geodesic polar coordinates

$$z = \theta(\varphi)e^{-r}i,$$

where

$$\theta(\varphi) = \begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

and φ ranges in $[0,\pi)$ and r in $[0,\infty)$, we get

$$\operatorname{tr} K_{R^l} = \frac{\pi}{m}\int_0^\infty k(e^{-r}i, \theta(\varphi l)e^{-r}i)(2\sinh r)\operatorname{tr}\chi(\gamma_R)^l.$$

Computing this term gives

$$\operatorname{tr} K_{R^l} = (2m\sin(\pi l/m))^{-1}\operatorname{tr}\chi(R)^l\int_{-\infty}^\infty h(r)\frac{\cosh\pi(1-2l/m)r}{\cosh\pi r}\,dr.$$

8.6 The trace formula

If we equate the above computations for the spectral contribution on the one side and for the hyperbolic, parabolic, elliptic and the contribution from the identity on the other side we obtain the following formula:

$$\begin{aligned}
\sum_j h(t_j) &- \frac{1}{4\pi}\int_{-\infty}^\infty \operatorname{tr}\Phi'(s)\Phi^{-1}(s)h(r)\,dr + \frac{1}{4}h(0)\operatorname{tr}\Phi(1/2) + \kappa g(0)h\log Y + O(Y^{-\varepsilon}) \\
&= \dim(V_\chi)\frac{\operatorname{vol}(F)}{4\pi}\int_{\mathbb{R}} rh(r)\tanh(\pi r)\,dr \\
&\quad + \sum_P\sum_{l=1}^\infty (p^{l/2}-p^{-l/2})^{-1}g(l\log p)\log p\operatorname{tr}\chi(P)^l \\
&\quad + \sum_R\sum_{0<l<m}(2m\sin(\pi l/m))^{-1}\operatorname{tr}\chi(R)^l\int_{\mathbb{R}} h(r)\frac{\cosh\pi(1-2l/m)r}{\cosh\pi r} \\
&\quad - g(0)\sum_{\mathfrak{a}}\log|\det(\mathrm{Id}-\chi(\gamma_\alpha))|_{W_\alpha}| \\
&\quad + \kappa\left(g(0)\log\frac{Y}{2} + \frac{1}{4}h(0) - \frac{1}{2\pi}\int_{-\infty}^\infty h(t)\psi(1+it)\,dt\right)
\end{aligned}$$

We see, that the $\log Y$ terms on both sides of the equation cancel each other. Finally, we let Y tend to infinity, which makes the error term $O(Y^{-\varepsilon})$ vanish, and we have proven:

Theorem 8.1. *If* $\kappa = \sum_{\mathfrak{a}} \kappa_{\mathfrak{a}}$, *where* $\kappa_{\mathfrak{a}} = \dim \operatorname{Eig}(\chi(\gamma_{\mathfrak{a}}), 1)$ *the trace formula*

$$\begin{aligned}
\sum_j h(t_j) = \dim(V_\chi) \frac{\operatorname{vol}(F)}{4\pi} \int_{\mathbb{R}} r h(r) \tanh(\pi r)\, dr \\
+ \sum_P \sum_{l=1}^{\infty} (p^{l/2} - p^{-l/2})^{-1} g(l \log p) \log p \operatorname{tr} \chi(P)^l \\
+ \sum_R \sum_{0<l<m} (2m \sin(\pi l/m))^{-1} \operatorname{tr} \chi(R)^l \int_{\mathbb{R}} h(r) \frac{\cosh \pi(1-2l/m) r}{\cosh \pi r} \\
+ \kappa \left(\frac{h(0)}{4} - \log 2 g(0) - \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \psi(1+ir)\, dr \right) \\
- g(0) \sum_{\mathfrak{a}} \log |\det(\mathrm{Id} - \chi(\gamma_{\mathfrak{a}}))|_{W_{\mathfrak{a}}} \\
+ \frac{1}{4}(\kappa - \operatorname{tr} \Phi(1/2)) h(0) + \frac{1}{4\pi} \int_{\mathbb{R}} h(r) \operatorname{tr} \Phi'(s) \Phi^{-1}(s)\, dr,
\end{aligned}$$

holds. The sum $\sum_P$ *ranges over all primitive hyperbolic conjugacy classes, and the sum* $\sum_R$ *ranges over all primitive hyperbolic conjugacy classes. The Hilbert space* $L^2(\Gamma\backslash\mathbb{H}, \chi)$ *decomposes into a discrete part, which is spanned by generalized eigenfunctions, and a continuous part, spanned by the Eisenstein series* $E_{\mathfrak{a}}(z, s, \chi)$. *The spectrum of the Laplacian* Δ *in the continuous part covers the segment* $[1/4, \infty)$ *uniformly with multiplicity* κ.

In particular, in the non-singular case $\kappa = 0$, *we obtain*

$$\begin{aligned}
\sum_j h(t_j) = \dim(V_\chi) \frac{\operatorname{vol}(F)}{4\pi} \int_{\mathbb{R}} r h(r) \tanh(\pi r)\, dr \\
+ \sum_P \sum_{l=1}^{\infty} (p^{l/2} - p^{-l/2})^{-1} g(l \log p) \log p \operatorname{tr} \chi(P)^l \\
+ \sum_R \sum_{0<l<m} (2m \sin(\pi l/m))^{-1} \operatorname{tr} \chi(R)^l \int_{\mathbb{R}} h(r) \frac{\cosh \pi(1-2l/m) r}{\cosh \pi r} \\
- g(0) \sum_{\mathfrak{a}} \log |\det(\mathrm{Id} - \chi(\gamma_{\mathfrak{a}}))|_{W_{\mathfrak{a}}}
\end{aligned}$$

and $L^2(\Gamma\backslash\mathbb{H}, \chi)$ *decomposes discretely.*

9 Selberg trace formula for compact quotients

9.1 The Lie group case

In this chapter we will only consider the compact case, so we do not need to consider the continuous spectrum and the Eisenstein series. This makes the development of a trace formula considerably easy. We also will not restrict ourselves to the case of $\mathbb{H}$ and a Fuchsian group Γ. Instead, we will consider the quotient $\Gamma\backslash G$, with G a Lie-group and Γ a lattice. The exact assumptions are as follows:

Let G be a connected, semisimple Lie-group with finite center and K a maximal compact subgroup of G and we denote a Haar measure by dg. Let $\Gamma \subset G$ be a torsion free, uniform lattice. We let $X := \Gamma\backslash G$ be the compact quotient space with G-invariant measure dx, such that for $f \in C_c(G)$ the integral formula

$$\int_G f(g)\, dg = \sum_{\gamma\in\Gamma} \int_X f(\gamma x)\, dx$$

holds.

9.1.1 Representation theory

Definition 9.1. For a complex vector space V we let $\mathrm{GL}(V)$ be the group of all automorphisms of V. A *representation* (π, V) of G is a group homomorphism $\pi : G \to \mathrm{GL}(V)$.

For a complex Hilbert space we let $\mathrm{GL}(H)$ be the group of all bijective and bounded endomorphisms of H. A *(continuous) representation* (π, H) of G is a group homomorphism $\pi : G \to \mathrm{GL}(H)$, such that the map

$$\begin{aligned} G \times H &\to H, \\ (g, v) &\mapsto \pi(g)v, \end{aligned}$$

is continuous. A continuous representation π is said to be *admissible*, if π restricted to K is unitary and each $\tau \in \widehat{K}$ occurs with finite multiplicity only. When the underlying representation space is a Hilbert space, we will always mean a continuous representation, without mentioning it anymore.

A *(Lie algebra) representation* representation!Lie algebra (π, V) of the Lie algebra $\mathfrak{g}$ is a complex vector space V, together with a Lie algebra homomorphism

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{gl}(V), \\ X &\mapsto \pi(X). \end{aligned}$$

A $(\mathfrak{g}, K)$-module is a vector space V, which is both a Lie algebra representation of $\mathfrak{g}$ and a group representation of K, such that the representations are compatible in the following way:

1. for any $v \in V$, $k \in K$ and $X \in \mathfrak{g}$

$$k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v),$$

2. for any $v \in V$ and $Y \in \mathfrak{k}$

$$\left(\frac{d}{dt} \exp(tY) \cdot v\right)\bigg|_{t=0} = Y \cdot v.$$

The third condition gives the K-finiteness:

3. for any $v \in V$ the set Kv spans a finite-dimensional subspace of V.

Recall that the K-finite vectors of an admissible representation (π, V) give rise to a $(\mathfrak{g}, K)$-module. Two admissible representations π and η are called *equivalent*, if the associated $(\mathfrak{g}, K)$-modules are isomorphic.
The *unitary dual* $\widehat{G}$ of the group G is the set of all irreducible unitary representations modulo unitary equivalence.
The *admissible dual* $\widehat{G}_{\mathrm{adm}}$ of G is the set of all irreducible admissible representations module admissible equivalence.

As always χ is a finite dimensional complex representation of Γ, not necessarily unitary, with representation space $V = V_\chi$. Let $E = E_\chi$ be the associated vector bundle over $\Gamma\backslash G$. More precisely, we consider the bundle $\Gamma\backslash(G \times V)$, where Γ acts via $\gamma \cdot (g, v) = (\gamma g, \chi(\gamma))v$ on $G \times V$. The image of (g, v) under the canonical projection $G \times V \to E$ will be written as $\Gamma(g, v)$. Note, that $\Gamma(\gamma g, v) = \Gamma(g, \chi(\gamma^{-1})v)$. Furthermore the group G acts on E via

$$g \cdot \Gamma(h, v) = \Gamma(hg^{-1}, v).$$

There is a canonical identification of the smooth sections $\Gamma^\infty(X,E)$ of the bundle E with the set of functions

$$C^\infty(G,V_\chi)^\Gamma = \{f \in C^\infty(G,V_\chi) : f(\gamma g) = \chi(\gamma)f(g) \text{ for all } \gamma \in \Gamma, g \in G\},$$

and we will freely switch between these two interpretations.

We choose any smooth hermitian metric $\langle\cdot,\cdot\rangle$ on E. If we choose a Haar-measure dk on K we can form the integral

$$\int_K \langle\Gamma(gk,v),\Gamma(gk,w)\rangle_{\Gamma gk}\ dk,$$

where x is the image of g under the canonical projection $G \to \Gamma\backslash G$. This gives again a smooth hermitian fiber metric on E, which is K-equivariant and by replacing $\langle\cdot,\cdot\rangle$ with this one, we can assume, that

$$\langle\Gamma(gk,v),\Gamma(gk,w)\rangle_{\Gamma gk} = \langle\Gamma(g,v),\Gamma(g,w)\rangle_{\Gamma g},$$

for arbitrary $g \in G, k \in K$ and $v,w \in V$.

Together with this smooth K-equivariant metric we obtain a pre-Hilbert space structure on the set of smooth sections $\Gamma^\infty(X,E)$, via

$$(f,g) := \int_{\Gamma\backslash G} \langle f(x),g(x)\rangle_x\ dx.$$

We complete $\Gamma^\infty(X,E)$ with respect to the induced norm and obtain the Hilbert space of square integrable sections $L^2(X,E)$. Because of the compactness of $\Gamma\backslash G$ the definition of $L^2(X,E)$ is independent of the chosen smooth fibre metric on E, since by compactness of $\Gamma\backslash G$ and the finite-dimensionality of the fibres, another smooth metric induces an equivalent norm on $\Gamma^\infty(X,E)$.

Definition 9.2. On $\Gamma^\infty(X,E)$ we define the *right regular representation* R of G as

$$R(g)f(x) := g\cdot f(xg),$$

where $f \in \Gamma^\infty(X,E), x \in X$ and $g \in G$.

If we use the identification with $C^\infty(G,V_\chi)^\Gamma$, the right regular representation for elements $f \in C^\infty(G, V_\chi)^\Gamma$ is just given by

$$R(g)f(x) = f(xg).$$

In the following paragraph we will show, that the right regular representation R is continuous. We start out with the following proposition.

Proposition 9.3. *There exists a continuous function ψ on G, such that*

$$\langle \Gamma(gh, v), \Gamma(gh, v)\rangle_{\Gamma gh} \le \psi(h) \langle \Gamma(g, v), \Gamma(g, v)\rangle_{\Gamma g},$$

for arbitrary $g, h \in G, x \in X$ and $v \in V$. In particular, there exists for each compact set $C \subset G$ a constant M depending only on C, such that

$$\langle \Gamma(gh, v), \Gamma(gh, v)\rangle_{\Gamma gh} \le M \langle \Gamma(g, v), \Gamma(g, v)\rangle_{\Gamma g},$$

for all $h \in C$ and arbitrary $g \in G, v \in V$.

Proof. For every $h \in G$, there exists a contionuous section $A_h \in \Gamma(X, \mathrm{Hom}(E, E))$, such that for $v \in V$ we have

$$\langle \Gamma(gh, v), \Gamma(gh, v)\rangle_{\Gamma gh} = \langle \Gamma(g, A_h(\Gamma g)v), \Gamma(g, A_h(\Gamma g)v)\rangle_{\Gamma g}.$$

As the metric is smooth, the dependence of A_h on h is smooth, in particular continuous. We let

$$\|A_g(x)\|^2_{\Gamma g} = \sup_{v \neq 0} \frac{\langle \Gamma(g, A_h(x)v), \Gamma(g, A_h(x)v)\rangle_{\Gamma g}}{\langle \Gamma(g, v), \Gamma(g, v)\rangle_{\Gamma g}}.$$

Then we define $\psi(h) = \max_{\Gamma g \in \Gamma\backslash G} \|A_h(\Gamma g)\|^2_{\Gamma g}$, which satisfies the conditions of the proposition. □

We choose once and for all a representative (τ, V_τ) for each class in $\widehat{K}$, the unitary dual of K. For a unitary representation (π, V) of K we let $V(\tau)$ be the τ-isotype. Recall the following theorem:

Theorem 9.4. [DE09, Theorem 7.3.2.] *For (π, V) a unitary representation of K, the representation space is the direct Hilbert space sum of all K-isotypes:*

$$V = \widehat{\bigoplus}_{\tau \in \widehat{K}} V(\tau).$$

Proposition 9.5. *The right regular representation R on $\Gamma^\infty(X,E)$ is continuous with respect to the L^2-topology. In particular, it extends to a continuous representation of $L^2(X,E)$. The restriction of R to the maximal compact subgroup K is unitary and hence, we get a K-isotypical decomposition*

$$L^2(X,E) = \widehat{\bigoplus}_{\tau\in\widehat{K}} L^2(X,E)(\tau),$$

where $\widehat{K}$ is the unitary dual of K.

Proof. Let $C \subset G$ be a compact subset and $h \in C$. Let ψ and M be as in Proposition 9.3. Then for $f \in \Gamma^\infty(X,E)$ we estimate

$$\begin{aligned}
(R(h)f, R(h)f) &= \int_X \langle h\cdot f(xh), h\cdot f(xh)\rangle_x \, dx \\
&\leq \int_X \psi(h^{-1}, x)\, \langle f(xh), f(xh)\rangle_{xh} \, dx \\
&= \int_X \psi(h^{-1}, xh^{-1})\, \langle f(x), f(x)\rangle_x \, dx \\
&\leq M(f,f),
\end{aligned}$$

independent of $h \in C$. Hence, the operator norm of $R(h)$ is uniformally bounded on each compact subset $C \subset G$. Since for fixed $f \in \Gamma^\infty(X,E)$ the map

$$\begin{aligned}
G &\to L^2(X,E) \\
g &\mapsto R(g)f
\end{aligned}$$

is continous, the continuity of the representation R follows. The unitarity as a representation of K is clear from the K-equivariance of the fibre metric. If we let $k \in K$, then we get

$$\begin{aligned}
(R(k)f, R(k)f) &= \int_X \langle k\cdot f(xk), k\cdot f(xk)\rangle_x \, dx \\
&= \int_X \langle f(xk), f(xk)\rangle_{xk} \, dx \\
&= \int_X \langle f(x), f(x)\rangle_x \, dx \\
&= (f,f).
\end{aligned}$$

□

9.1.2 The spectral decomposition of the Casimir operator

Definition 9.6. We let $\mathfrak{g}, \mathfrak{g}_{\mathbb{C}}, U(\mathfrak{g}_{\mathbb{C}}), Z(\mathfrak{g}_{\mathbb{C}})$ be the Lie-algebra of G, its complexification, the universal envelopping algebra, as well as its center, respectively. We endow G/K with the G-invariant metric induced by the Killing form $\langle\cdot,\cdot\rangle$. The Killing form is non-degenerate, and hence it gives an identification of $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$. If $X_1,\dots,X_n$ is a basis of $\mathfrak{g}$, then we define the *Casimir element* as

$$\Omega = Y_1X_1 + \dots Y_nX_n \in U(\mathfrak{g}_{\mathbb{C}}),$$

where $Y_1,\dots,Y_n$ is a dual basis of $X_1,\dots,X_n$ with respect to the Killing form. The definition of Ω is independent of the chosen orthonormal basis and $\Omega \in Z(\mathfrak{g}_{\mathbb{C}})$ [Kna01, Proposition 8.6.].
If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ and θ is the Cartan involution on $\mathfrak{g}$ the map

$$(X,Y) \mapsto -\langle X, \theta(Y)\rangle,$$

is a positive definite bilinear form. If $X_1,\dots,X_l$ is an orthonormal basis of $\mathfrak{k}$ and $Y_1,\dots,Y_k$ an orthonormal basis of $\mathfrak{p}$ we find

$$\begin{aligned}\Omega &= -X_1^2 - \dots - X_l^2 + Y_1^2 + \dots + Y_k^2\\ &= \Omega_K + Y_1^2 + \dots + Y_k^2,\end{aligned}$$

where Ω_K is the Casimir element of $U(\mathfrak{k}_{\mathbb{C}})$.

We will need the following results, to deduce a nice spectral decomposition of the Casimir operator on $L^2(X,E)$.

Theorem 9.7. [Shu01, Theorem 8.4.] *Let M be a closed manifold and D an elliptic differential operator on a metric bundle $\mathcal{E}$. If the resolvent set $\rho(D) \neq \varnothing$ is not empty, the spectrum $\sigma(D)$ is discrete and for each $\lambda \in \sigma(D)$ there exists a decomposition $L^2(M,\mathcal{E}) = E_\lambda \oplus E'_\lambda$, such that*

1. *$E_\lambda \subset \Gamma^\infty(M,\mathcal{E}), \dim E_\lambda < \infty$, and E_λ is invariant under D and there exists some $n > 0$, such that $(D-\lambda)^n E_\lambda = 0$,*
2. *E'_λ is a closed subspace of $L^2(M,\mathcal{E})$ invariant under D. If we denote by A_λ the restriction of A to E'_λ, then $\lambda \notin \sigma(A_\lambda)$.*

Theorem 9.8. [Shu01, Theorem 8.4.] *Let M be a closed manifold and D an elliptic differential operator on a metric bundle $\mathcal{E}$. For an interval I we define the cone*

$$\Lambda_I = \{re^{i\theta} : 0 \le r < \infty, \theta \in I\}.$$

For $\varepsilon > 0$ there exists an $R > 0$ such $\sigma(D)$ is contained in the set $B_R(0) \cup \Lambda_{[-\varepsilon,\varepsilon]}$.

Definition 9.9. Let $\mathcal{E}$ be a vector bundle over a Riemannian manifold M. A second order differential operator D is a *Laplace type operator* if for the principal symbol

$$\sigma_2(D)(x,\xi) = \|\xi\|^2,$$

for arbitrary $x \in M$ and $\xi \in T^*M$. In particular each Laplace type operator is elliptic.

The reason why we introduce the notion of Laplace type operators, is that a reasonable spectral theory can be developed for those.

Proposition 9.10. *The Casimir element Ω induces on each K-isotype $L^2(X,E)(\tau)$ a Laplace type operator, having discrete spectrum. We will denote the induced operator by Ω_τ. Let*

$$V_{\tau,\lambda} := \{f \in L^2(X,E)(\tau) : (\Omega_\tau - \lambda)^n f = 0 \text{ for some } n \in \mathbb{N}\},$$

the generalized eigenspace belonging to $\lambda \in \operatorname{spec}(\Omega_\tau)$. Then $V_{\tau,\lambda} \subset \Gamma^\infty(X,E)(\tau)$, $\dim V_{\tau,\lambda} < \infty$ and $V_{\tau,\infty}$ is stable under K as well as $Z(\mathfrak{g}_\mathbb{C})$.

Proof. We let $E_{\chi,\tau}$ be the vector bundle $G \times_{\Gamma\times K} V_\chi \otimes V_\tau$ on $\Gamma\backslash G/K$ where $\Gamma \times K$ acts on $G \times V_\chi \otimes V_\tau$ as

$$(\gamma,k)\cdot(g, v\times w) = (\gamma g k^{-1}, \chi(\gamma)v \times \tau(k)w).$$

Let $X \in \mathfrak{g}$ and $f \in C^\infty(G, V_\chi \times V_\tau)$. X induces a differential operator via

$$Xf(g) = \frac{d}{dt} f(g\exp(tX))\Big|_{t=0},$$

and this map from the Lie algebra to the algebra of differential operators extends to the universal envelopping algebra $U(\mathfrak{g}_\mathbb{C})$. From the definition it is easily seen, that the induced differential operator is left invariant, meaning that

$$X(L_h(f))(g) = (Xf)(hg),$$

where L_h is the left translation by the element h. On the other hand, we find for $X \in \mathfrak{g}$ and the right translation R_h, for an element $h \in G$, that

$$\begin{aligned} X(R_h(f))(g) &= \frac{d}{dt} f(g\exp(tX)h)\Big|_{t=0} \\ &= \frac{d}{dt} f(gh\exp(\mathrm{Ad}(h)tX))\Big|_{t=0} \\ &= (\mathrm{Ad}(h)X)(f)(gh). \end{aligned}$$

Thus we get $X(R_h(f))(g) = (\mathrm{Ad}(h)X)(f)(gh)$ for all $X \in U(\mathfrak{g}_\mathbb{C})$. Since $\Omega \in Z(\mathfrak{g}_\mathbb{C})$ this yields

$$\Omega(R_h(f))(g) = (\Omega f)(gh),$$

for arbitrary $g, h \in G$ and consequently we have for $f \in C^\infty(G, V_\chi \otimes V_\tau)^{\Gamma\times K}$, that

$$(\Omega f)(\gamma x k^{-1}) = \chi(\gamma) \otimes \tau(k) \Omega f(x)$$

and thus, $C^\infty(G, V_\chi \otimes V_\tau)^{\Gamma\times K}$ is stable under Ω.

Furthermore

$$\Gamma^\infty(X, E)(\tau) \cong V_\tau \otimes \mathrm{Hom}_K(V_\tau, \Gamma^\infty(X, E)),$$

and

$$\begin{aligned} \mathrm{Hom}_K(V_\tau, \Gamma^\infty(X, E)) &\cong (\Gamma^\infty(X, E) \otimes V_\tau)^K \\ &\cong (C^\infty(G) \otimes V_\chi \otimes V_\tau)^{\Gamma\times K} \\ &\cong \Gamma^\infty(\Gamma\backslash G/K, E_{\chi,\tau}). \end{aligned}$$

Hence $\mathrm{Id} \otimes \Omega$ induces an operator Ω_τ on $L^2(E)(\tau)$.

Next we will show, that it is a Laplace type operator.

Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and basises $X_1, \dots X_l$ and $Y_1, \dots Y_k$ as before, such that

$$\Omega = \Omega_K + Y_1^2 + \cdots + Y_k^2.$$

For $\Gamma gK \in \Gamma\backslash G/K$ there exists a neighbourhood U, such that we can choose the map

$$g\exp(y_1Y_1 + \cdots + y_kY_n) \mapsto (y_1, \ldots, y_k)$$

as a local coordinate map on U. Inside U, with these coordinates we find

$$(\Omega f)(g) = \sum \frac{\partial^2}{\partial y_i^2} f(g) + \mathrm{Id} \otimes \sum (d\tau)(\Omega_K) f(g).$$

But, since the representation V_τ is irreducible and $\Omega_K \in Z(\mathfrak{k}_\mathbb{C})$, the operator $(d\tau)(\Omega_K)$ acts as a scalar, according to the Lemma of Schur. Consequently, Ω induces a second order differential operator with principal symbol $(\Omega_\tau)(x,\xi) = \|\xi\|^2$.

From Theorem 9.8 and 9.7 it is now clear, that the spectrum is discrete. By Theorem 9.7 it follows also that the generalized eigenspace $V_{\tau,\lambda}$ is finite dimensional and $V_{\tau,\lambda} \subset \Gamma^\infty(X,E)(\tau)$.

$V_{\tau,\lambda}$ is stable under K, since $K \cdot L^2(X,E)(\tau) \subset L^2(X,E)(\tau)$ and $\mathrm{Ad}(k)\Omega = \Omega$. Since $Z(\mathfrak{g}_\mathbb{C}) \cdot \Gamma^\infty(X,E)(\tau) \subset \Gamma^\infty(X,E)(\tau)$ it is also clear, that $Z(\mathfrak{g}_\mathbb{C}) \cdot V_{\tau,\lambda} \subset V_{\tau,\lambda}$. □

Proposition 9.11. *The space $L^2(X,E)(\tau)$ is the closure of the algebraic direct sum of all generalized eigenspaces:*

$$L^2(X,E)(\tau) = \overline{\bigoplus_{\lambda\in\sigma(\Omega_\tau)} V_{\tau,\lambda}}.$$

To prepare the proof of Proposition 9.11 we need:

Definition 9.12. Let H be a Hilbert space and G a linear operator with non-empty resolvent set $\rho(G) \neq \varnothing$. An operator B is said to be compact relative to G if $D(G) \subset D(B)$ and the operator $BR_\lambda(G)$ is compact, where $R_\lambda(G) = (G-\lambda)^{-1}$ is the resolvent of G.

Theorem 9.13. [Mar88, Theorem 4.3.] *Let H be a Hilbert space and G a self-adjoint operator. The resolvent $R_\lambda(G)$ is assumed to be a Schatten class operator and B an operator relatively compact to G. Then the operator $C = G + B$ has a compact resolvent and H is the closure of the generalized eigenspaces of C:*

$$H = \overline{\bigoplus_{\lambda\in\sigma(C)} V_\lambda}$$

Proof of Proposition 9.11. According to Proposition 9.10 we have $\sigma(\Omega_\tau)(x,\xi) = \|\xi\|^2$. Hence we get $\Omega_\tau = \Delta + B$, where Δ is the Bochner-Laplace operator, which is self-adjoint, and B is a first order differential operator. The resolvent $R_\lambda(\Delta)$ is of order -2, hence compact and a Schatten class operator. Similarly $BR_\lambda(\Delta)$ is of order -1 and also compact. The statement follows now by applying Theorem 9.13. □

Definition 9.14. Let $V_{\text{fin}} \subset \Gamma^\infty(X,E)$ be the set of all smooth sections, which are K- as well as $Z(\mathfrak{g}_\mathbb{C})$-finite.

Proposition 9.15. *V_{fin} is the algebraic direct sum of all generalized eigenspaces for the operators Ω_τ:*

$$V_{fin} = \bigoplus_{\tau\in\widehat{K}} \bigoplus_{\lambda\in\sigma(\Omega_\tau)} V_{\tau,\lambda}. \tag{88}$$

In particular, V_{fin} is dense in $L^2(X,E)$ and consequently, because $V_{fin} \subset \Gamma^\infty(X,E)$, it is dense in $\Gamma^\infty(X,E)$.

Proof. If f is an element in the above direct sum, it is clear that $f \in V_{\text{fin}}$, since each generalized eigenspace $V_{\tau,\lambda}$ is finite dimensional and K- and $Z(\mathfrak{g}_\mathbb{C})$-invariant.

If we now take $f \in V_{\text{fin}}$ we obtain

$$f \in \bigoplus_{\tau\in\widehat{K}} L^2(X,E)(\tau),$$

because of the K-finiteness of f. Hence, we can assume, that $f \in L^2(X,E)(\tau)$ for some $\tau \in \widehat{K}$. Let $W \subset L^2(X,E)(\tau)$ be the finite-dimensional $Z(\mathfrak{g}_\mathbb{C})$- and K-invariant vectorspace, generated by f. W is stable under Ω_τ, since this operator is induced by $\Omega \in Z(\mathfrak{g}_\mathbb{C})$. Consider the operator $\Omega_\tau|_W$. By the theorem about the Jordan normal form we have a direct sum decomposition

$$W = \bigoplus_{\lambda\in\sigma(\Omega_\tau)} V_{\tau,\lambda} \cap W.$$

This proves, that f lies in the above direct sum. □

We will now cite two propositions we will need to infer a filtration of the $(\mathfrak{g},K)$-module V_{fin}.

Proposition 9.16. [Wal88, Corollary 3.4.7.] *Let V a finitely generated $(\mathfrak{g}, K)$-module, such that* $\dim Z(\mathfrak{g}_{\mathbb{C}})v < \infty$ *for all $v \in V$. Then V is admissible.*

Proposition 9.17. [Kna01, Corollary 10.42.] *Each Harish-Chandra-module V (in other words: a finitely generated, admissible $(\mathfrak{g}, K)$-module) has a finite composition series*

$$V = W_k \supset W_{k-1} \supset \cdots \supset W_0 = 0$$

with irreducible quotients W_j/W_{j-1}. The multiplicities of the irreducible subquotients are independent of the chosen composition series.

Proposition 9.18. *The exists a seperated, exhaustive and increasing filtration $Fil_i V$, where i ranges over all nonnegative integers, of V_{fin} as a $(\mathfrak{g}, K)$-module, such that each quotient $Fil_i V/Fil_{i-1} V$ is admissible and irreducible. The graduated module*

$$Gr V_{fin} = \bigoplus_{i=0}^{\infty} Fil_{i+1} V/Fil_i V$$

is independent of the chosen filtration.

Proof. We choose a generalized eigenspace $V_{\tau,\lambda}$ from the direct sum in (88). Since it is K- and $Z(\mathfrak{g}_{\mathbb{C}})$-stable, we find $U(\mathfrak{g})V_{\tau,\lambda} \subset V_{\mathrm{fin}}$. According to Proposition [Wal88] the $(\mathfrak{g}, K)$-module $U(\mathfrak{g})V_{\tau,\lambda}$ is admissible and according to Proposition [Kna01] there exists a finite composition series

$$U(\mathfrak{g})V_{\tau,\lambda} = \mathrm{Fil}_k V \supset \mathrm{Fil}_{k-1} V \supset \cdots \supset \mathrm{Fil}_0 V = 0$$

such that $\bigoplus_{i=1}^{k} \mathrm{Fil}_i V/\mathrm{Fil}_{i-1} V$ is independent of the chosen composition series. Now proceed in the same manner with $V_{\mathrm{fin}}/U(\mathfrak{g})V_{\tau,\lambda}$ to obtain the filtration. □

Proposition 9.19. *Each element $f \in V_{\tau,\lambda}$ is real-analytic.*

Proof. If $f \in V_{\tau,\lambda}$, there exists some $N \in \mathbb{N}$, such that $(\Omega_\tau - \lambda)^N f = 0$. Hence, f is annihilated by the elliptic differential operator $(\Omega_\tau - \lambda)^N$, whence it is real-analytic. □

The filtration of V_{fin} furnishes a filtration of the right regular representation, as we will show now.

Theorem 9.20. *The representation* $(R, L^2(X,E))$ *admits a seperated, exhaustive and increasing filtration of subrepresentations*

$$0 = V_0 \subset V_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} V_i = L^2(X,E),$$

induced by the filtration of the $(\mathfrak{g},K)$*-module* V_{fin} *from Proposition 9.18.*

Proof. We let $V_i = \overline{\mathrm{Fil}_i V}$ the closure of $\mathrm{Fil}_i V$ in $L^2(X,E)$. It is easy to see, that V_i is stable under G. To show this, it is enough to prove $G\mathrm{Fil}_i V \subset V_i$, since the representation is continuous. Let $h \in V_i^{\perp}$. For $f \in \mathrm{Fil}_i V \subset V_{\mathrm{fin}}$ the function

$$g \mapsto (R(g)h, f)$$

is real-analytic according to Proposition 9.1.2. For $X \in \mathfrak{g}$ on a sufficient small neighbourhood of 0, we have a Taylor development:

$$\begin{aligned}\langle R(\exp X)h, f\rangle &= \sum_{n=0}^{\infty} \frac{1}{n!} X^n \langle R(g)h, f\rangle \bigg|_{g=1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \langle X^n h, f\rangle .\end{aligned}$$

Since $h \in \mathrm{Fil}_i V$, hence also $X^n h \in \mathrm{Fil}_i V$ we get $\langle R(\exp X)h, f\rangle = 0$. But then $\langle R(g)h, f\rangle = 0$ in a neighbourhood of 1 and because of analycity, on the whole of G. Since $h \in \mathrm{Fil}_i V$ and $f \in V_i^{\perp}$ were arbitrary, we get $G \cdot \mathrm{Fil}_i V \subset V_i$. □

For each $i \in \mathbb{N}$ we obtain the quotient representation on V_i/V_{i-1} which we will denote by R_i.

Proposition 9.21. *For the* K*-finite vectors of* R_i *we obtain*

$$(V_i/V_{i-1})_{R_i,K} \cong Fil_i V/Fil_{i-1}V.$$

Proof. This is is clear, because R_i is admissible, and so $(V_i/V_{i-1})(\tau)$ is finite dimensional, but

$$(\mathrm{Fil}_i V/\mathrm{Fil}_{i-1}V)(\tau) \subset (V_i/V_{i-1})(\tau)$$

is dense, and thus the both must be equal. □

Together with the following theorem, it follows that the representations R_i are irreducible.

Theorem 9.22. [Wal88, Theorem 3.4.12.] *Let (π, H) be an admissible Hilbert-space representation of G. Then (π, H) is irreducible, iff the associated $(\mathfrak{g}, K)$-module $H_{\pi,K}$ is irreducible.*

9.1.3 The trace formula

Definition 9.23. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert-space. A compact operator T is of *trace class*, if $\sum_i s_i(T) < \infty$, where $s_i(T)$ denote the singular values of the operator T. Let $(e_i)_{i\in I}$ be an orthonormal basis of H. The *trace* of a trace class operator T is defined as

$$\mathrm{tr}(T) = \sum_{i\in I} \langle Te_i, e_i\rangle.$$

One can show that this sum converges absolutely and is independent of the chosen orthonormal basis $(e_i)_{i\in I}$.

Now let (π, H) be an admissible representation of G. The representation is said to be of *trace class*, if for each $f \in C_c^\infty(G)$ the operator

$$\pi(f) = \int_G f(g)\pi(g)\, dg$$

is of trace class.

Theorem 9.24. *Let $f \in C_c^\infty(G)$. The operator $R(f)$ on $L^2(X, E)$ is an integral operator with integral kernel*

$$k_f(x, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\chi(\gamma).$$

Thus $R(f)$ is of trace class and the following trace formula holds

$$\sum_{[\gamma]} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f)\mathrm{tr}\chi(\gamma) = \sum_{\pi\in\widehat{G}_{adm}} N_{\Gamma,\chi}(\pi)\mathrm{tr}\,\pi(f),$$

where

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, dx,$$

and $N_{\Gamma,\chi}(\pi)$ is the multiplicity of the representation π in $\widehat{\bigoplus}_{i=0}^\infty V_i/V_{i-1}$.

Proof. Let $h \in C_c(G)$, such that $\sum_{\gamma\in\Gamma} h(\gamma x) = 1$. For $\varphi \in L^2(X,E)$ we compute

$$\begin{aligned}
R(f)\varphi(x) &= \int_G f(y)\varphi(xy)\,dy \\
&= \int_G f(x^{-1}y)\varphi(y)\,dy \\
&= \int_G \sum_{\gamma\in\Gamma} h(\gamma^{-1}y) f(x^{-1}y)\varphi(y)\,dy \\
&= \sum_{\gamma\in\Gamma} \int h(y) f(x^{-1}\gamma y)\varphi(\gamma y)\,dy \\
&= \int_G h(y) \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\chi(\gamma)\varphi(y)\,dy \\
&= \int_{\Gamma\backslash G} \sum_{\gamma'\in\Gamma} h(\gamma' y) \sum_{\gamma\in\Gamma} f(x^{-1}\gamma\gamma' y)\chi(\gamma)\varphi(\gamma' y)\,dy \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\chi(\gamma)\varphi(y)\,dy.
\end{aligned}$$

This computation shows that $R(f)$ is an integral operator with integral kernel

$$k_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\chi(\gamma).$$

Then we can compute the trace of $R(f)$ by integrating the kernel along the diagonal:

$$\operatorname{tr} R(f) = \int_X \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x) \operatorname{tr}\chi(\gamma)\,dx.$$

Breaking the integration up into the different conjugacy classes of G, we get

$$\begin{aligned}
\operatorname{tr} R(f) &= \int_{\Gamma\backslash G} \sum_{\gamma'\in\Gamma} g(\gamma' x) \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\operatorname{tr}\chi(\gamma)\, dx \\
&= g(x) \int_G \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\operatorname{tr}\chi(\gamma)\, dx \\
&= \sum_{\gamma\in\Gamma} \int_G g(x) f(x^{-1}\gamma x)\operatorname{tr}\chi(\gamma)\, dx \\
&= \sum_{[\gamma]} \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} \int_G g(\sigma^{-1}x) f(x^{-1}\gamma x)\operatorname{tr}\chi(\gamma)\, dx \\
&= \sum_{[\gamma]} \int_{\Gamma_\gamma\backslash G} \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} \sum_{\eta\in\Gamma_\gamma} g(\sigma^{-1}\eta x) f(x^{-1}\gamma x)\operatorname{tr}\chi(\gamma)\, dx \\
&= \sum_{[\gamma]} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\operatorname{tr}\chi(\gamma)\, dx \\
&= \sum_{[\gamma]} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f((\sigma x)^{-1}\gamma\sigma x)\operatorname{tr}\chi(\gamma)\, d\sigma\, dx \\
&= \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f)\operatorname{tr}\chi(\gamma).
\end{aligned}$$

On the other hand, according to Proposition 9.20, we have a filtration of the representation space $L^2(X,E)$

$$0 = V_0 \subset V_1 \subset \cdots \subset \bigcup_{i=0}^{\infty} V_i = L^2(X,E).$$

Thus, when R_i is the representation induced by R on V_i/V_{i-1} we get

$$\operatorname{tr} R(f) = \sum_{i=0}^{\infty} \operatorname{tr} R_i(f),$$

but the right-hand term is obviously equal to

$$\sum_{\pi\in\widehat{G}_{\mathrm{adm}}} N_{\Gamma,\chi}(\pi)\operatorname{tr}\pi. \qquad \square$$

9.2 The totally disconnected case

From now on, we let G be a locally compact and totally disconnected topological group. We can equally say, that G is a topological group, which has a neighborhood basis $(K_i)_{i\in I}$ of 1 consisting of compact open subgroups. We

additionally assume, that G is first countable, in particular we can find a countable neighborhood basis $(K_n)_{n\in\mathbb{N}}$ of 1, consisting of compact open subgroups. $\Gamma \subset G$ again denotes a uniform lattice.
The Hilbert space $L^2(\Gamma\backslash G, \chi)$ is defined similarly as in section 1.3. The only difference is the choice of a hermitian metric on the bundle $E = \Gamma\backslash(G \times V)$, which we require to be continuous, whereas in the Lie case we required it to be smooth.
We define $\mathcal{H} = C_c^\infty(G)$ as the set of all compactly supported and locally constant functions on G. Recall, that $\mathcal{H}$ becomes an algebra under convolution.

Proposition 9.25. *If $f \in C_c^\infty(G)$, then there exists a compact open subgroup $K \subset G$, such that f is invariant under K from left and right.*

Proof. The function f is locally constant and hence, $f^{-1}(z) \subset G$ is open for $z \in \mathbb{C}$. Consequently $\operatorname{supp}(f) = G \smallsetminus f^{-1}(0)$ is closed and thus compact. The open sets $f^{-1}(z), z \neq 0$ cover $\operatorname{supp}(f)$, and because of compactness we see, that finitely many will suffice to cover $\operatorname{supp}(f)$. In particular, f takes only finitely many values. Since $f^{-1}(z)$ is compact, we can cover it by finitely many open sets $g_1U_{i_1}, \dots, g_nU_{i_{n(z)}}$. Thus we see, that f is a finite linear combination of characteristic functions $\mathbb{1}_{gU_i}$. If we take the intersection of the finitely many occurring subgroups U_i, we obtain an open subgroup U', such that f is invariant under U' from the right. Proceeding in the same manner, we get an open subgroup U'', such that f is invariant under U'' from the left. Taking the intersection $K = U' \cap U''$ we get the required subgroup of the proposition. □

Definition 9.26. Define $\mathcal{H}_K$ as the set of all $f \in C_c^\infty(G)$, which are invariant under K from the left and right. According to Proposition 9.25 we have

$$\mathcal{H} = \bigcup_K \mathcal{H}_K = \bigcup_{n\in\mathbb{N}} \mathcal{H}_{K_n}.$$

The union in the middle ranges over all compact open subgroups $K \subset G$. It equals the union over the neighborhood basis $(K_n)_{n\in\mathbb{N}}$ of 1 on the right-hand side, since for $K \subset K'$ we have $\mathcal{H}_{K'} \subset \mathcal{H}_K$.

Definition 9.27. Let $K \subset G$ be a compact open subgroup. We define $F(\Gamma\backslash G, \chi)^K$ to be the set of functions $f : G \to V$, which satisfy

$$f(\gamma g k) = \chi(\gamma) f(g),$$

for arbitrary $\gamma \in \Gamma$, $g \in G$ and $k \in K$.

Proposition 9.28. *The double quotient $\Gamma\backslash G/K$ for a compact open subgroup $K \subset G$ is finite. In particular the complex vector space $F(\Gamma\backslash G,\chi)^K$ is finite dimensional.*

Proof. Firstly, the quotient $\Gamma\backslash G$ is compact and furthermore the sets ΓgK for arbitrary $g \in G$ are open in $\Gamma\backslash G$. Thus, there are finitely many $g_1,\dots,g_n \in G$, such that the $\Gamma g_i K$ cover the quotient $\Gamma\backslash G$. But this means, that the g_i contain a full set of representatives for the double quotient $\Gamma\backslash G/K$. Consequently, $F(\Gamma\backslash G,\chi)^K$ is a finite dimensional complex vector space. □

Proposition 9.29. *Let $f \in C_c^\infty(G)$. The operator $R(f)$ on $L^2(\Gamma\backslash G,\chi)$ has finite dimensional image. More precisely, let K be a compact open subgroup, under which the function f is invariant. The image of $R(f)$ is contained in $F(\Gamma\backslash G,\chi)^K$.*

Proof. Considering the integral

$$R(f)\varphi(y) = \int_G f(x)\varphi(yx)\,d\mu(x),$$

we easily find, that $R(f)\varphi(y)$ is invariant under an element $u \in K$ from the right:

$$\begin{aligned} R(f)\varphi(yu) &= \int_G f(x)\varphi(yux)\,d\mu(x) \\ &= \int_G f(u^{-1}x)\varphi(yx)\,d\mu(x) \\ &= \int_G f(x)\varphi(yx)\,d\mu(x) = R(f)\varphi(y). \end{aligned}$$

Thus we see, that $R(f)$ has finite dimensional image, contained in $F(\Gamma\backslash G,\chi)^K$. □

Definition 9.30. Let (π, V) be a continuous Banach space representation of G. A vector $v \in V$ is *smooth*, if the stabilizer $\mathrm{Stab}_G(v)$ of v is open in G. We let V^∞ be the set of all smooth vectors.

Proposition 9.31. *The set of smooth vectors V^∞ is dense in V. Furthermore we have*

$$V^\infty = \pi(\mathcal{H})V.$$

Proof. Let $f \in \mathcal{H}$. Then there exists a compact open subgroup $K \subset G$, such that $f \in C_c^\infty(K\backslash G/K)$. The stabilizer of $\pi(f)v$ contains K, as can be seen from the following computation:

$$\pi(k)\pi(f)v = \int_G f(g)\pi(kg)v\,dg = \int_G f(k^{-1}g)\pi(g)v\,dg = \pi(f)v.$$

Thus $\mathrm{Stab}_G(v)$ is open and we get $\pi(\mathcal{H})V \subset V^\infty$. If on the other hand $v \in V^\infty$ is given, then by definition $\mathrm{Stab}_G(v)$ is open. Consequently, there exists a compact open subgroup $K \subset \mathrm{Stab}_G(v)$. The function $f = \frac{\mathbb{1}_K}{|K|} \in C_c^\infty(K\backslash G/K)$ leaves v invariant:

$$\pi(f)v = |K|^{-1}\int_G \mathbb{1}_K(g)\pi(g)v\,dg = |K|^{-1}\int_K v\,dg = v.$$

Hence $V^\infty \subset \pi(\mathcal{H})V$. To show the density, we take a decreasing family of compact open subgroups $(K_n)_{n\in\mathbb{N}}$, with

$$\bigcap K_n = \{1\}$$

and consider the weighted characteristic functions

$$f_n = |K_n|^{-1}\mathbb{1}_{K_n}.$$

Now let $v \in V$, and compute:

$$\begin{aligned}
\|\pi(f_n)v - v\| &= \|\int_G f_n(g)\pi(g)v\,dg - v\| \\
&\leq |K_n|^{-1}\int_{K_n} \|\pi(g)v - v\|\,dg. \\
&\leq |K_n|^{-1}\int_{K_n} \|\pi(g) - \mathrm{Id}\|_{\mathrm{op}}\|v\|\,dg.
\end{aligned}$$

The representation π is continuous and consequently $\|\pi(g) - \mathrm{Id}\|_{\mathrm{op}} < \varepsilon$ for all $g \in K_n$, if n is large enough. This shows

$$\pi(f_n)v \to v,$$

and V^∞ is dense in V. $\square$

Theorem 9.32. *V^∞ has a filtration*

$$0 = F_0 \subset F_1 \subset \cdots \subset \bigcup_{j=0}^{\infty} F_j = V^\infty,$$

such that the respective quotients are simple $\mathcal{H}$-modules. In addition, for every $f \in \mathcal{H}$ there exists a natural number n_0, such that

$$R(f)(F_{n+1}/F_n) = 0, \tag{89}$$

for $n \geq n_0$.

Proof. It suffices to show, that there exists an increasing filtration $(F_n)_{n\in\mathbb{N}}$, such that F_{n+1}/F_n has finite length and satisfies (89), since we then can refine the filtration, such that respective quotients are simple $\mathcal{H}$-modules. Choose a countable family $(K_n)_{n\in\mathbb{N}}$ of neighborhoods of the neutral element, consisting of compact open subgroups with

$$K_{n+1} \subset K_n \text{ and } \bigcap_{n\in\mathbb{N}} K_n = \{1\}.$$

We let $F_n = R(\mathcal{H})F(\Gamma\backslash G, \chi)^{K_n}$. Each vector space $F(\Gamma\backslash G, \chi)^{K_n}$ is finite dimensional and hence the quotients F_{n+1}/F_n must have finite length. If $f \in \mathcal{H}$, say $f \in C_c^\infty(K\backslash G/K)$, then we have shown in proposition 9.29, that $R(f)F_n \subset F(\Gamma\backslash G, \chi)^K$. If we choose $n_0 \in \mathbb{N}$, such that $K_{n_0} \subset K$, we thus have $R(f)F_n \subset F_{n_0}$ for all $n \in \mathbb{N}_0$. Consequently

$$R(f)(F_{n+1}/F_n) = 0,$$

for all $n \geq n_0$. □

Definition 9.33. Let $\widehat{\mathcal{H}}$ be the set of equivalence classes of simple $\mathcal{H}$-modules. For $M \in \mathcal{H}$, we let $N_\chi(M) \in \mathbb{N}_0$ be the multiplicity of M in $\bigoplus_{n=1}^\infty F_n/F_{n-1}$, such that

$$\bigoplus_{n=1}^{\infty} F_n/F_{n-1} \cong \bigoplus_{M\in\widehat{\mathcal{H}}} N_\chi(M)M.$$

Theorem 9.34. *For $M \in \widehat{\mathcal{H}}$ and $f \in \mathcal{H}$ we let $\mathrm{tr}_M(f)$ be the trace of the linear operator*

$$m \mapsto f \cdot m$$

on M. Then the trace formula

$$\sum_{M\in\widehat{\mathcal{H}}} N_\chi(M)\mathrm{tr}_M(f) = \sum_{[\gamma]} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\, dx \, \mathrm{tr}(\chi(\gamma))$$

holds.

Proof. We firstly compute the trace of the operator $R(f)$ via the filtration of Theorem 9.32. By

$$\bigoplus_{n=1}^{\infty} F_n/F_{n-1} \cong \bigoplus_{M\in\widehat{\mathcal{H}}} N_\chi(M)M$$

we get

$$\mathrm{tr}\, R(f) = \sum_{M\in\widehat{\mathcal{H}}} N_\chi(M)\, \mathrm{tr}_M(f).$$

Note, that by (89) this sum is finite. On the other hand, the geometric side can be computed in exactly the same fashion as in the proof of Theorem 9.24. □

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Appendix A - Zusammenfassung in deutscher Sprache

In dieser Arbeit stellen wir nicht unitäre Spurformeln vor. Sei G eine lokalkompakte topologische Gruppe und $\Gamma \subset G$ eine diskrete Untergruppe zusammen mit einer endlichdimensionalen komplexen Darstellung

$$\chi : \Gamma \to \mathrm{GL}(V).$$

In der klassischen Selbergschen Spurformel fordert man zusätzlich, dass χ eine unitäre Darstellung ist, worauf wir verzichten. Wir betrachten Funktionen $f : G \to V$ mit der Eigenschaft

$$f(\gamma g) = \chi(\gamma) f(g),$$

für alle $g \in G$ und $\gamma \in \Gamma$. Nach Einührung eines geeigneten Skalarprodukts auf dem Raum dieser Funktionen, erhalten wir einen Hilbertraum $L^2(\Gamma\backslash G, \chi)$. Auf diesem Raum betrachten wir die Rechstdarstellung R, gegeben durch

$$R(g)\varphi(x) = \varphi(xg),$$

wobei $\varphi \in L^2(\Gamma\backslash G, \chi)$. Ist $f : G \to \mathbb{C}$ eine Funktion mit hinreichend guten Eigenschaften, so erhält man einen Operator $R(f)$ auf $L^2(\Gamma\backslash G, \chi)$, der für $\varphi \in L^2(\Gamma\backslash G, \chi)$ durch

$$R(f)\varphi(x) = \int_G f(g)\varphi(xg)\, dg$$

gegeben ist. Ist der Quotient $\Gamma\backslash G$ kompakt, so ist $R(f)$ ein Spurklasseoperator. Ist $\Gamma\backslash G$ nicht kompakt, so definiert man den Unterraum $L^2_{\text{cusp}}(\Gamma\backslash G, \chi) \subset L^2(\Gamma\backslash G, \chi)$ der Spitzenformen. Es stellt sich heraus, dass $R(f)$ den Raum der Spitzenformen stabil lässt und die Einschränkung einen Spurklassoperator liefert. Zur Herleitung der Spurformel zerlegt man einerseits die Darstellung R und berechnet mit Hilfe dieser Zerlegung die *spektrale Seite* der Spurformel. Andererseits zeigt man, dass $R(f)$ ein Integraloperator ist. Die Spur von $R(f)$ berechnet man dann durch Integration des Integralkerns auf der Diagonalen und die Zerlegung von G in seine Konjugationsklassen. Dies liefert die *geome-*

trische Seite der Spurformel.
Während die geometrische Seite im nicht unitären Fall keine größeren Schwierigkeiten bereitet, ist die Zerlegung der Rechtsdarstellung a priori nicht klar.
In dieser Arbeit stellen wir Lösungen für die folgenden 3 Fälle vor:

1. Als einfachsten Fall betrachten wir die Gruppe $G = \mathrm{PSL}(2, \mathbb{R})$.
2. Im Anschluss stellen wir den Fall einer halbeinfachen Lie-Gruppe genauer dar. Zusätzlich verlangen wir die Kompaktheit des Quotienten $\Gamma \backslash G$.
3. Zuletzt untersuchen wir kompakte Quotienten $\Gamma \backslash G$, wobei G eine total unzusammenhängende Gruppe ist.

Appendix B - Lebenslauf

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