



This is a repository copy of *Towards a reliable calculation of relic radiation from primordial gravitational waves*.

White Rose Research Online URL for this paper:

<https://eprints.whiterose.ac.uk/199595/>

Version: Published Version

Article:

Giarè, W. orcid.org/0000-0002-4012-9285, Forconi, M., Di Valentino, E. orcid.org/0000-0001-8408-6961 et al. (1 more author) (2023) Towards a reliable calculation of relic radiation from primordial gravitational waves. *Monthly Notices of the Royal Astronomical Society*, 520 (2). pp. 1757-1773. ISSN 0035-8711

<https://doi.org/10.1093/mnras/stad258>

This article has been accepted for publication in *Monthly Notices of the Royal Astronomical Society* ©: 2023 The Author(s) Published by Oxford University Press on behalf of the Royal Astronomical Society. All rights reserved.

Reuse

Items deposited in White Rose Research Online are protected by copyright, with all rights reserved unless indicated otherwise. They may be downloaded and/or printed for private study, or other acts as permitted by national copyright laws. The publisher or other rights holders may allow further reproduction and re-use of the full text version. This is indicated by the licence information on the White Rose Research Online record for the item.

Takedown

If you consider content in White Rose Research Online to be in breach of UK law, please notify us by emailing eprints@whiterose.ac.uk including the URL of the record and the reason for the withdrawal request.



eprints@whiterose.ac.uk
<https://eprints.whiterose.ac.uk/>

Towards a reliable calculation of relic radiation from primordial gravitational waves

William Giarè^{1,2,3*}, Matteo Forconi^{2,4}, Eleonora Di Valentino³ and Alessandro Melchiorri^{2,4}

¹Galileo Galilei Institute for theoretical physics, Centro Nazionale INFN di studi avanzati, Largo Enrico Fermi 2, I-50125 Firenze, Italy

²Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Roma, P.le A. Moro 2, I-00185 Roma, Italy

³School of Mathematics and Statistics, University of Sheffield, Hounsfield Road, Sheffield S3 7RH, United Kingdom

⁴Physics Department Università di Roma ‘La Sapienza’, P.le Aldo Moro 2, 00185 Rome, Italy

Accepted 2023 January 23. Received 2023 January 18; in original form 2022 November 3

ABSTRACT

Inflationary gravitational waves, behaving as additional radiation in the Early Universe, can increase the effective number of relativistic species (N_{eff}) by a further correction that depends on the integrated energy-density in gravitational waves over all scales. This effect is typically used to constrain (blue-tilted) models of inflation in light of the bounds resulting from the big bang nucleosynthesis. In this paper, we recompute this contribution, discussing some caveats of the state-of-the-art analyses. Through a parametric investigation, we first demonstrate that the calculation is dominated by the ultraviolet frequencies of the integral and therefore by the behaviour of the tensor spectrum on scales corresponding to modes that cross the horizon very close to the end of inflation, when the slow-roll dynamics breaks down and the production of gravitational waves becomes strongly model dependent. Motivated by these results, we realize a theoretical Monte Carlo and, working within the framework of the Effective Field Theory of inflation, we investigate the observable predictions of a very broad class of models. For each model, we solve a system of coupled differential equations whose solution completely specifies the evolution of the spectrum up to the end of inflation. We prove the calculation of $\Delta N_{\text{eff}}^{\text{GW}}$ to be remarkably model dependent and therefore conclude that accurate analyses are needed to infer reliable information on the inflationary Universe.

Key words: gravitational waves – cosmological parameters – early Universe – inflation – primordial nucleosynthesis.

1 INTRODUCTION

According to our current theory of the Early Universe, a phase of almost de-Sitter expansion known as cosmological Inflation (Guth 1981) is expected to drive the Universe towards homogeneity and flatness, setting the appropriate initial conditions for the subsequent hot big bang theory evolution and providing a compelling mechanism to explain the physical origin of the observed anisotropies in the cosmic microwave background (CMB) radiation.

A *unique* prediction of inflation theory is the existence of Primordial Gravitational Waves (PGWs), tensor perturbations on superhorizon scales sourced by a superadiabatic amplification of zero-point quantum fluctuations during inflation (Starobinsky 1980; Linde 1982; Vilenkin 1983; Ungarelli et al. 2005; Guzzetti et al. 2016). Their detection would provide direct evidence for inflation, opening an inestimable observational windows on fundamental physics. For this reason, significant experimental efforts have been devoted to the search for primordial tensor modes, above all by looking for B-modes polarization on large angular scales in the cosmic microwave background angular power spectra (Baumann, Green & Porto 2015; Kamionkowski & Kovetz 2016). Nevertheless, despite the best efforts, a detection of primordial tensor perturbations is still missing and only upper bounds can be inferred by current data (Akrami et al. 2020a; Ade et al. 2021).

More precisely, within the simplest slow-roll scenario (where inflation is achieved by means of a single scalar field minimally coupled to gravity) the power spectrum of primordial tensor perturbations around the CMB scales can be well described by a two-parameter power-law parametrization:

$$\ln \mathcal{P}_T(k) = \ln(r A_s) + n_T \ln(k/k_*). \quad (1)$$

The first parameter, i.e. the tensor amplitude $A_T \doteq r A_s$, is currently constrained to¹ $r < 0.032$ at 95 per cent CL (Tristram et al. 2022) when *Planck* (Akrami et al. 2020c) and BK18 (Ade et al. 2021) data sets are combined together with BAO (Alam et al. 2021) and CMB lensing (Aghanim et al. 2020c). Hopefully, in the upcoming decade, new CMB experiments such as BICEP3 (Grayson et al. 2016), CLASS (Essinger-Hileman et al. 2014), SPT-3G (Benson et al. 2014), Advanced ACTPol (Henderson et al. 2016), LiteBIRD (Suzuki et al. 2018), and CMB-S4 (Abazajian et al. 2016) should reach a better sensitivity $r \sim 0.001$, possibly leading to the first detection of B-mode polarization. As concerns the second parameter, i.e. tensor tilt $n_T \doteq d \ln \mathcal{P}_T / d \ln k$, within the simplest single-field slow-roll framework, its value is fully determined by the slow-roll consistency relation $n_T = -r/8$ that implies an almost scale-invariant slightly red-tilted spectrum. However, this relation can be violated in many non-standard realizations of inflation such as in modified gravity

¹We recall that $A_s \simeq 2.1 \times 10^{-9}$ is the amplitude of primordial scalar perturbations (Akrami et al. 2020b).

* E-mail: william.giare@gmail.com

theories (Baumann, Lee & Pimentel 2016; Odintsov, Oikonomou & Fronimos 2021; Giarè, Renzi & Melchiorri 2021; Oikonomou 2021; Odintsov, Oikonomou & Myrzakulov 2022), in multifields inflationary models (Namba et al. 2016; Peloso, Sorbo & Unal 2016; Pi, Sasaki & Zhang 2019; Özsoy 2021), or from trans-Planckian Physics (Ashoorioon, Hovdebo & Mann 2005; Ashoorioon et al. 2014). Depending on the underlying phenomenology, the tensor tilt can range from being red ($n_T < 0$) to blue ($n_T > 0$) (see e.g. Stewart & Brandenberger 2008; Mukohyama et al. 2014; Giovannini 2016, 2018a, b, 2019; Giarè & Melchiorri 2021; Baumgart, Heckman & Thomas 2022, and the references therein). As a result, constraining the tensor tilt (and in general the shape of the tensor spectrum) without any underlying assumption is crucial for testing new physics and the standard slow-roll scenario (Franciolini et al. 2019; D’Eramo & Schmitz 2019; Giarè, Di Valentino & Melchiorri 2019; Caldwell, Smith & Walker 2019; Clarke, Copeland & Moss 2020).

Relaxing the slow-roll consistency relation, the analysis of the CMB data only weakly constrains the tensor tilt to $-0.55 < n_T < 2.54$ at 95 per cent CL (Akrami et al. 2020a). However, important improvements in the upper limit can be achieved by exploiting other CMB-independent observables. For instance, along with B-modes polarization, primordial tensor fluctuations may contribute also to the stochastic background of gravitational waves (SGWB), the analogous of CMB for gravitational waves (Caprini & Figueroa 2018). Interestingly, if the spectrum is enough blue-tilted, according to equation (1), the inflationary contribution should be much amplified on scales of direct gravitational wave detection so that we can use data from ground-based interferometers, such as LIGO and VIRGO to infer constraints on n_T . These experiments set an upper bound on the fraction of the energy-density of the Universe in gravitational radiation $\Omega_{\text{GW}} \lesssim 10^{-7}$ (Abbott et al. 2017a; Abbott et al. 2019) in the frequency range $f \in (20\text{--}85.8)\text{ Hz}$ (which corresponds to the wave-number range $k_{\text{LV}} \in (1.3\text{--}5.5) \times 10^{16}\text{ Mpc}^{-1}$), leading to a more stringent upper limit $n_T < 0.52$ at 95 per cent CL (Akrami et al. 2020a). While this approach is largely used in the literature, it should be noted that these bounds are obtained by extrapolating the relation (1) on frequencies (those probed by GWs experiments) where it is not granted that the spectrum still follows a power-law behaviour. Indeed, high wave-numbers k correspond to modes that exit the horizon relatively close to the end of inflation where the spectrum may strongly depend on the higher order terms in equation (1) (Giarè & Melchiorri 2021) and therefore on the specific form of the inflationary potential (Kinney 2021), making it extremely difficult to derive reliable model-independent bounds on the tensor-tilt.

Another interesting possibility to gain constraining power on blue-tilted models of inflation is to study the effects induced by PGWs in the early Universe, before the recombination epoch. Behaving as extra radiation, a sizeable amount of tensor perturbations may significantly contribute to the energy budget of the early Universe, increasing the effective number of relativistic species N_{eff} by a further contribution (Maggiore 2000)

$$\Delta N_{\text{eff}}^{\text{GW}} \simeq \frac{h_0^2}{5.6 \times 10^{-6}} \left(\frac{1}{24 z_{\text{eq}}} \right) \int_{f_{\text{min}}}^{f_{\text{max}}} \frac{df}{f} \mathcal{P}_T(f) \quad (2)$$

that depends on the integrated energy density in gravitational waves over all scales and that exponentially grows when $n_T > 0$ (see also Appendix A for a detailed derivation). So, in principle, we can use the big bang nucleosynthesis (BBN) limit on additional radiation ($\Delta N_{\text{eff}} \lesssim 0.4$) to infer constraints on blue-tilted models of inflation. Also, this approach is largely followed in literature, leading to a limit

$n_T \lesssim 0.4$, that is more or less of the same order as those inferred by gravitational wave experiments, (see e.g. Allen & Romano 1999; Smith, Pierpaoli & Kamionkowski 2006; Boyle & Buonanno 2008; Kuroyanagi, Takahashi & Yokoyama 2015; Cabass et al. 2016; Ben-Dayan et al. 2019; Aich et al. 2020).

In this work, we would like to focus a bit closer on this latter scenario. In Section 2, we review the state-of-the-art analyses, outlining some important caveats and showing that the results share the same caveats discussed so far. Also, in this case, the largest inflationary contributions to the effective number of relativistic species come from tensor modes that exit the horizon very close to the end of inflation, precisely when the slow-roll approximation is no longer valid and the power-law parametrization breaks down. Consequently, any calculation becomes model dependent and accurate analyses are needed to correctly estimate the relic radiation resulting from primordial tensor modes. To prove this point further and confer additional physical meaning to our findings, in Section 3, we explicitly compute the energy budget of the Universe in several general effective field theory (EFT) realizations of (blue and red) inflation. By integrating a set of differential equations, we correctly predict the evolution of the spectrum (and all the other dynamical quantities) over the different cosmic epochs and scales. Finally, we present our conclusion in Section 4.

2 PARAMETRIC ANALYSIS

2.1 State-of-the-art analyses

According to equation (2), the contribution of inflationary tensor anisotropies to the effective number of relativistic degrees of freedom in the early Universe will depend on (i) the frequency range $f \in [f_{\text{min}}, f_{\text{max}}]$ over which the integral runs and (ii) the (parametrization of) primordial tensor spectrum.

(i) The choice of the frequency range on which the integral runs is quite debated. In particular, the infrared cutoff can be safely set to $f_{\text{min}} = 10^{-10}\text{ Hz}$ which approximately corresponds to the size of the comoving horizon at the time of BBN (Pritchard & Kamionkowski 2005; Smith et al. 2006; Cabass et al. 2016). Conversely, the ultraviolet cutoff is more arbitrary. Being primordial gravitational waves produced during inflation, we expect an ultraviolet cutoff of the size of the horizon at the end of inflation (Meerburg et al. 2015) (as PGWs with smaller wavelengths cannot be produced). Anyway, the size of the horizon at the end of inflation depends on the reheating temperature T_{RH} at the end of inflation. Assuming an almost GUT-scale inflation and an instant reheating, we can set $T_{\text{RH}} \sim 10^{15}\text{ GeV}$ which corresponds to $k_{\text{end}} \sim 10^{23}\text{ Mpc}^{-1}$, and thus $f_{\text{max}} \simeq 10^8\text{ Hz}$ (Cabass et al. 2016). Nevertheless, inflationary models with (very) lower reheating temperatures $T_{\text{RH}} \sim 10^{10}\text{--}100\text{ GeV}$ have been proposed in the literature (see e.g. Kawasaki, Kohri & Sugiyama 1999, 2000; Giudice, Kolb & Riotto 2001a; Giudice et al. 2001b; Hannestad 2004; Khoury & Steinhardt 2011; Freese et al. 2018; Hasegawa et al. 2019, 2020; Carena et al. 2021; Litsa et al. 2021) and, although such scenarios are typically not easy to realize, in these models, the ultraviolet cutoff may be much smaller, limiting the high-frequency contributions in the integral (2) (see also Vagnozzi 2021; Benetti, Graef & Vagnozzi 2022).

(ii) The main purpose of this section is to study the dependence of the integral (2) from the parametrization used for the primordial tensor spectrum \mathcal{P}_T . The common practice in literature is to assume a power-law tensor spectrum given by equation (1) over the whole range of integration so that the integral (2) can be easily solved

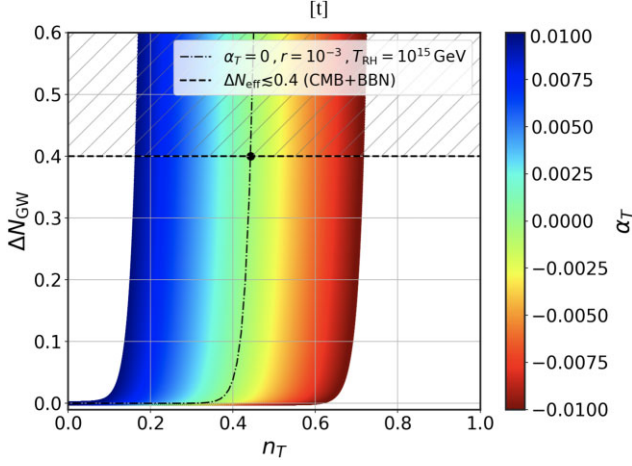


Figure 1. Inflationary tensor mode contribution to the effective number of relativistic degrees of freedom as a function of the tensor tilt and its running α_T . The black dashed line represents the contribution for $\alpha_T = 0$ while the horizontal dashed line represents the limit on additional radiation from the BBN bounds.

analytically:

$$\Delta N_{\text{eff}}^{\text{GW}} \simeq \frac{h_0^2}{5.6 \times 10^{-6}} \left(\frac{r A_s}{24 z_{\text{eq}}} \right) \frac{1}{n_T} \left[\left(\frac{f}{f_*} \right)^{n_T} \right]_{f_{\text{min}}}^{f_{\text{max}}} \quad (3)$$

Interestingly, a blue tensor tilt exponentially amplifies the GWs production on ultraviolet frequencies – that therefore we expect to contribute mostly in equation (2) – possibly leading to a sizable ΔN_{eff} from PGWs. As we already discussed in the introduction, this effect is commonly used in literature to bound blue-tilted models of inflation, with several implications also for gravitational waves observations (Vagnozzi 2021; Benetti et al. 2022; Vagnozzi & Loeb 2022) and fundamental physics (Calcagni & Kuroyanagi 2021). For instance, assuming a GUT scale inflation ($f_{\text{max}} \sim 10^8$ Hz) and a tensor amplitude $r \sim 0.001$, it is easy to see that the BBN limit on the the effective number of relativistic species ($\Delta N_{\text{eff}} \lesssim 0.4$) is naturally translated into a limit $n_T \lesssim 0.4$ by equation (3), see also Fig. 1 and Appendix B, where an updated analysis of the observational constraints resulting from the BBN is carried out. We devote the rest of this section to studying how much robust these bounds are.

2.2 Next-to-leading order parametrization

A first naive consideration is that the above mentioned result is derived assuming the tensor tilt to be exactly constant under the whole range of integration. Typically, in physical models of inflation where the tensor tilt can acquire such large positive values, it may also acquire a non-negligible scale dependence (Giarè & Melchiorri 2021; Giarè et al. 2021). Therefore, a first attempt to question the strength of this result is to study what happens extending the power-law relation (1) to its next-to-leading order generalization

$$\ln \mathcal{P}_T(k) = \ln(r A_s) + n_T \ln(k/k_*) + \alpha_T \ln^2(k/k_*), \quad (4)$$

where, we parametrize the scale dependence of the tensor tilt by including its running $\alpha_T \doteq dn_T/d\ln k$.

In Fig. 1, we show the effect of a relatively small running of the tensor tilt on the calculation of $\Delta N_{\text{eff}}^{\text{GW}}$, finding that it can significantly change the results and so lead to a much tighter (relaxed) constraint on n_T represented by the horizontal dashed line in the figure. We postpone a rigorous analysis of the effects of a running of the tensor tilt on the observational constraints

resulting from the BBN to Appendix B. Here, we point out that a positive (negative) α_T amplifies (suppresses) the power spectrum on high frequency and its contributions in the integral (2), providing another important clue that properly accounting for the ultraviolet behaviour of the tensor spectrum may be crucial in the calculation of $\Delta N_{\text{eff}}^{\text{GW}}$. In this regard, we notice that modes with frequency $f = k/2\pi$ will cross horizon N_k e-folds before the end of inflation, where N_k is given by (Martin, Ringeval & Vennin 2014; Kinney 2021)

$$\begin{aligned} N_k \simeq & -\ln\left(\frac{k}{a_0 H_0}\right) + \ln\left(\frac{H_*}{H_{\text{end}}}\right) - \frac{2}{3} \ln\left(\frac{T_{\text{RH}}}{\Lambda}\right) \\ & + \ln\left(\frac{T_{\text{RH}}}{T_{\text{eq}}}\right) + \frac{1}{3} \ln\left(\frac{g_{*S}(T_{\text{RH}})}{g_{*S}(T_{\text{eq}})}\right) \\ & + \ln\left(\frac{a_{\text{eq}} H_{\text{eq}}}{a_0 H_0}\right). \end{aligned} \quad (5)$$

In the equation above, $a_0 H_0 = 2.248 \times 10^{-4} \text{ Mpc}^{-1}$ is the inverse of the comoving horizon size in the current Universe, H_* is the value of the Hubble parameter at the horizon exit, H_{end} is the Hubble parameter at the end of inflation, Λ is the energy scale of inflation, and the subscript ‘eq’ denotes quantities evaluated at matter–radiation equality. Assuming a standard Λ CDM cosmology, we have $\ln[(a_{\text{eq}} H_{\text{eq}})/(a_0 H_0)] \simeq 3.8$ and $T_{\text{eq}} \simeq 8 \times 10^{-10} \text{ GeV}$ (Martin et al. 2014; Akrami et al. 2020b; Forconi et al. 2021). Approximating $H_{\text{end}} \simeq H_*$ and recalling that the energy scale of inflation can be related to the amplitude of tensor perturbations as $\Lambda \simeq r^{1/4} \times 3.3 \times 10^{16} \text{ GeV}$, we can simplify equation (5) to

$$N_k \simeq 61 - \ln\left(\frac{k}{a_0 H_0}\right) + \frac{1}{3} \ln\left(\frac{T_{\text{RH}}}{10^{15} \text{ GeV}}\right) + \frac{1}{6} \ln(r). \quad (6)$$

Therefore, the ‘high frequencies’ in the integral (2) we are referring to, correspond to tensor modes that exit the horizon extremely close to the end of inflation ($N_k \lesssim 2$ for $k \gtrsim 10^{21} \text{ Mpc}^{-1}$ and $T_{\text{RH}} \sim 10^{15} \text{ GeV}$ and $r \sim 10^{-3}$). This is precisely where, at least in the simplest inflationary scenarios, the potential decreases very rapidly to approach its minimum, and the slow-roll dynamics breaks down. As pointed out in Kinney 2021, Giarè & Melchiorri 2021, it is not sure at all that a power-law parametrization (or even its next-to-leading order generalization) holds – even approximately – on such frequencies because the shape of the tensor spectrum will be strongly related to the shape of the inflationary potential. As a result, we argue the calculation of $\Delta N_{\text{eff}}^{\text{GW}}$ to be largely sensitive to the underlying model.

2.3 Higher order stochastic reconstruction

A more general approach to the problem can be obtained by expanding (the log of) the tensor spectrum as a series of powers

$$\ln \mathcal{P}_T = \sum_{j=0}^{\infty} a_j (x - x_0)^j. \quad (7)$$

If we choose the CMB frequency as the centre of the expansion ($x_0 = \ln f_*$), the coefficients a_j can be trivially related to (the derivatives of) the tensor spectrum evaluated at the CMB scales. In particular, the tensor amplitude and the tensor tilt are simply given by

$$a_0 = \ln(r A_s), \quad a_1 = \frac{d \ln \mathcal{P}_T}{d \ln f} \equiv n_T, \quad (8)$$

while the higher order coefficients are related to the higher order derivatives of the spectrum (or the tensor tilt) as:

$$a_{j>1} = \frac{1}{j!} \frac{d^j \ln \mathcal{P}_T}{d \ln^j f} = \frac{1}{j!} \frac{d^{j-1} n_T}{d \ln^{j-1} f}. \quad (9)$$

Notice that, if we stop the sum expansion at $j = 1$ or $j = 2$, we exactly recover equation (1) or equation (4), respectively. Therefore, including more and more terms in the sum will clearly guarantee a more accurate reconstruction of the tensor spectrum at $x \gg x_0$, since it employs also the other higher order terms in the expansion. However, if we want to adopt this parametrization in the integral (2), we need to make sure that this sum will actually converge on the frequencies over which the integration runs. Although this depends on the specific model of inflation, in most models, the tensor spectrum is a slow-evolving regular function of the frequency, so that it is reasonable to expect a global convergence. For instance, the simplest slow-roll scenario is characterized by a hierarchy of parameters $n_T = \mathcal{O}(\epsilon)$ and $d^j n_T / d \ln^j f \lesssim \mathcal{O}(\epsilon^{j+1})$. Assuming such a scaling, the sum convergence can be easily proved by evaluating the radius of convergence

$$\frac{1}{R} \doteq \lim_{j \rightarrow \infty} \left| \frac{a_{j+1}}{a_j} \right| = \lim_{j \rightarrow \infty} \left| \frac{\mathcal{O}(\epsilon)}{j+1} \right| = 0. \quad (10)$$

So, in principle, we can adopt this parametrization to predict the value of the tensor spectrum at $x \gg x_0$. Anyway, in practice, all the arbitrariness of the method is encapsulated into the coefficients $\{a_j\}$. Ultimately, fixing their values is equivalent to fixing a specific model of inflation. Here, we sample different inflationary models by randomly varying the coefficients $\{a_j\}$ as follows:

(i) We fix the tensor amplitude² on the CMB scales to $r \sim 10^{-3}$ (which is the target of the next CMB experiments) so that a_0 is always fixed by equation (8)

(ii) We let the tensor tilt randomly vary in the range $n_T \in [-0.5, 1]$ thus evaluating a_1 according to equation (8). In this way, we can explore both blue- and red-tilted models³

(iii) We randomly choose the higher order coefficients $\{a_{j>1}\}$ to be extremely small such that $a_1 \gg a_j \gg a_{j+1}$. This is done by assuming the j -order derivative of the tensor spectrum in equation (9) to be a Gaussian distributed with mean $\mu = 0$ and standard deviation $\sigma \simeq 10^{-2j}$. While this is clearly an arbitrary assumption, in this way, we can be sure that the spectrum follows a power law (1) on the CMB scales ($x \simeq x_0$) where such terms remain in fact negligible. In addition, this ensures a fast convergence of the sum expansion on high frequencies ($x \gg x_0$) while granting a certain freedom.

Following this scheme, we simulate 10^6 different shapes of the tensor spectrum as functions of frequency up to the order $j = 10$ in the sum expansion.⁴ Examples of the spectra obtained within this method are provided in Fig. 2, together with a simple leading order power-law approximation (red line).

²Notice that, $\Delta N_{\text{eff}}^{\text{GW}}$ can be easily obtained for any generic r simply rescaling the value obtained for $r = 10^{-3}$ as

$$\Delta N_{\text{eff}}^{\text{GW}}(r) = \left(\frac{r}{10^{-3}} \right) \left[\Delta N_{\text{eff}}^{\text{GW}} \right]_{r=10^{-3}}$$

³Notice that, we are relaxing the slow-roll consistency relation between the tensor tilt and the tensor amplitude ($n_T \neq -r/8$) and considering the two parameters as independent.

⁴We checked that maintaining this scaling for parameters, the 10th order is enough to capture any relevant correction to the tensor spectrum.

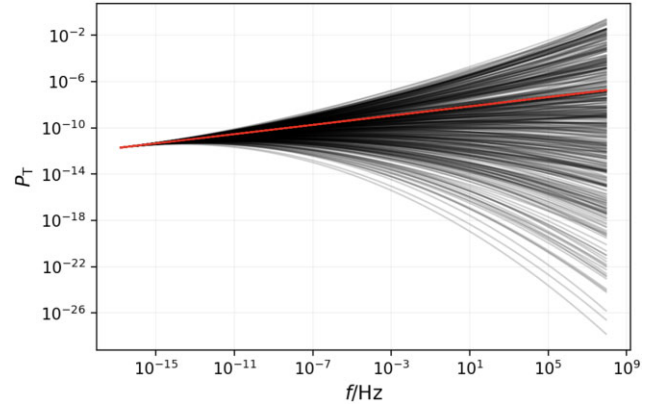


Figure 2. Examples of randomly generated tensor spectra (and the power-law extrapolation, red line) obtained by following the method outlined in Section 2.3.

Notice that, we consider both negative and positive coefficients $\{a_j\}$, so that, on high frequencies, the spectrum can be either suppressed or amplified. Indeed, while in the simplest cases, we expect suppression of power because of the rapid decrease of inflationary potential (see also the subsequent discussion in Section 3), in more elaborated scenarios, it is, in principle, possible to build inflationary models with ultraviolet amplification of tensor perturbations (Barrow, Mimoso & de Garcia Maia 1993; Peng et al. 2021; Ota, Sasaki & Wang 2022; Odintsov & Oikonomou 2022; Baumgart et al. 2022; Oikonomou 2023). As explained in the introduction, in this latter case, we may end up with large amounts of GW on the small scales as those probed by Gravitational Interferometers. Therefore, for all the simulated spectra, we also checked that the amplitude $\mathcal{P}_T(k)$ remains consistent with the LIGO/VIRGO limit, keeping only the models able to satisfy observations. This is the reason why in Fig. 2, we get much more suppressed spectra than amplified ones. From the same figure, we can also appreciate how the usual power-law parametrization is a precise approximation only at frequencies corresponding to the CMB scales (as required by construction) while important deviations are observed at higher frequencies, in spite of our efforts for keeping small the parameters $\{a_j\}$.

Fixing the ultraviolet cutoff to $f_{\text{max}} \simeq 10^8$ Hz, we numerically solve the integral (2) for all the different shapes of $\mathcal{P}_T(f)$, thus computing the corresponding value of $\Delta N_{\text{eff}}^{\text{GW}}$. We ensure the computational relative error due to the numerical integration method to remain smaller than 1 per cent. In Fig. 3, we show the results of our random analysis. Once again the red solid line represents the contribution $\Delta N_{\text{eff}}^{\text{GW}}$ obtained within the power-law parametrization (1). Instead, the grey dots represent the values of $\Delta N_{\text{eff}}^{\text{GW}}$ obtained by the numerical integration method of the randomly obtained tensor spectra.

Despite the intrinsic aleatory nature of this method, we can certainly draw some general conclusions. First of all, as evident from Fig. 2, the high-frequency behaviour of the tensor spectrum may become basically uncorrelated with the value of the tensor tilt on the CMB scales. This goes in the direction of previous analyses already discussed in the literature (see e.g. Giarè & Melchiorri 2021; Kinney 2021). In addition, the results displayed in Fig. 3 lead weight to our previous considerations according to which the value of $\Delta N_{\text{eff}}^{\text{GW}}$ may be strongly sensitive to the high-frequency contributions in the integral (2). Since on such frequencies the spectrum becomes uncorrelated with the behaviour of the tensor tilt on the CMB scale, these findings lead us to believe that the BBN limit on additional

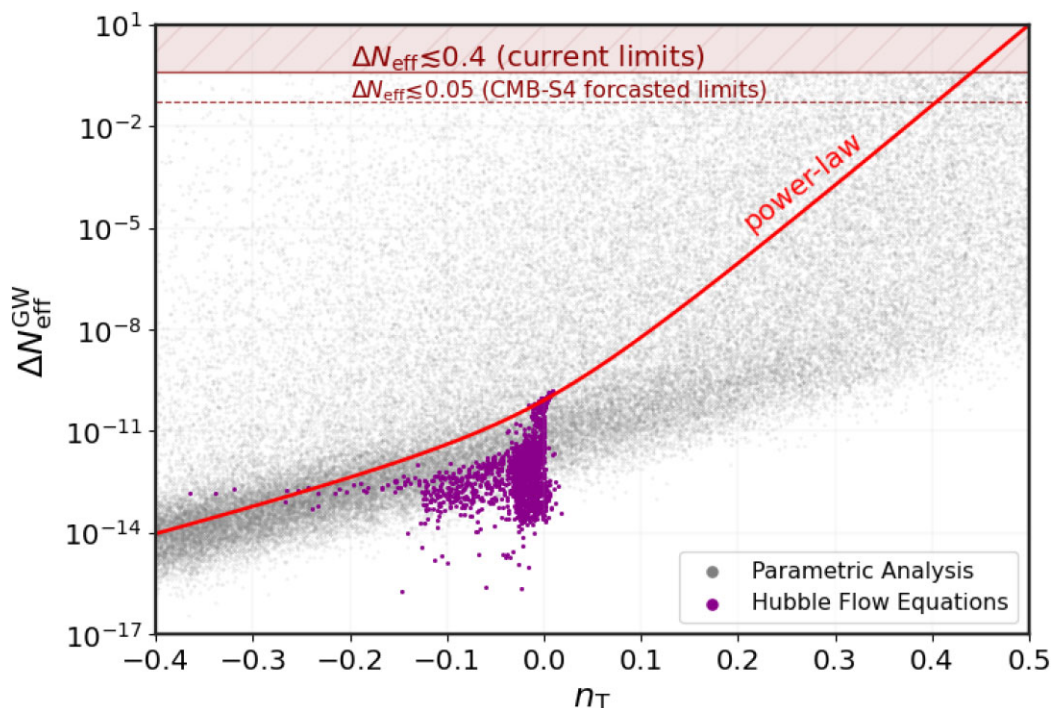


Figure 3. Primordial Gravitational Wave contribution to radiation energy density in the early Universe parametrized as a correction to the effective number of relativistic species ($\Delta N_{\text{eff}}^{\text{GW}}$). All the inflationary models in the figure share the same tensor amplitude ($r \simeq 0.001$) and the same reheating temperature ($T_{\text{RH}} \sim 10^{15}$ GeV) but have different values of tensor tilt (n_{T}) shown in the x -axis. The red thick line represents the predictions for $\Delta N_{\text{eff}}^{\text{GW}}$ inferred by extrapolating a power-law parametrization for the tensor spectrum (equation 1) over all frequencies. The grey dots represent the results of the parametric analysis carried out in Section 2.3 where the spectrum is expanded as a sum of powers up to the 10th order (equation 7) and randomly reconstructed. Finally, the magenta points represent the observable predictions of an ensemble of physical models randomly realized within the framework of the effective field theory of inflation by means of a theoretical Monte Carlo. In this latter case, the spectrum is calculated by integrating a system of coupled differential equations (known as ‘Hubble Flow Equations’), as discussed in Section 3. The horizontal red band (dashed line) represents the current (future forecasted) observational limit on radiation.

radiation can hardly constrain the tensor tilt itself, unless without a full understanding of the underlying model. It is worth noting that the equation equation (7) amplifies higher order terms at ultraviolet frequencies. As a result, increasing the value of the ultraviolet cutoff (f_{max}) will lead to larger contributions from non-linear terms in the integral (2). These larger contributions will cause greater dispersion in the grey points in Fig. 3 and enhance differences in ΔN_{eff} , as noted in Vagnozzi (2021). This indicates that the parametric analysis is dependent, to some degree, on the choice of ultraviolet cutoff (which is fixed to $f_{\text{max}} = 10^8$ Hz in this case). Inflationary models that produce satisfying amounts of gravitational waves typically predict high-scale inflation, so for the values of tensor amplitude of interest to future experiments, a significant reduction in the ultraviolet cutoff is possible only within models with extremely low T_{RH} . While these models are theoretically possible, they are very difficult to realize. In conclusion, this parametric analysis can be useful for pointing out potential limitations and weaknesses in current analyses, but a more reliable investigation of physical models of inflation and their respective contribution to the energy budget of the early Universe is needed. This will be the focus of the next section.

3 PHYSICAL ANALYSIS

The lesson we have learned from the parametric analyses detailed in the previous section is that the calculation of relic radiation from primordial gravitational waves depends crucially on the behaviour of the primordial tensor spectrum at ultraviolet frequencies. Given that assuming a power law continuously on all frequencies is not

reliable (Giarè & Melchiorri 2021; Kinney 2021), the calculation of $\Delta N_{\text{eff}}^{\text{GW}}$ becomes unreliable in turn. Motivated by these results, in this section, we want to provide a definitive evidence that this issue persists in solid theoretical framework of inflation, conferring physical meaning to our findings. In addition, we want to quantify the typical error resulting from extrapolating a power-law parametrization by going through a precise evaluation of the radiation energy density for a reasonable range of different models and possibilities.

In order to investigate the observable predictions of a very broad class of inflationary models in the most general framework, we follow a methodology based on the so-called Hubble Flow Equation (Hoffman & Turner 2001; Kinney 2002; Easther & Kinney 2003; Friedman, Cooray & Melchiorri 2006). The Hubble Flow Equations were first introduced by Hoffman and Turner (Hoffman & Turner 2001) for the simplest single-field slow-roll case where it is straightforward to define an infinite hierarchy of slow-roll parameters that, starting from the Hubble parameter H and its derivatives with respect to the field, completely specify the evolution of the main observable quantities during inflation. Since the integration of the equations yields a trajectory in slow-roll parameter space that can be ultimately interpreted as a model whose dynamics is a solution of the flow equations, solving numerically a truncated system of Hubble Flow Equations for a set of suitably defined initial conditions has been proposed as a sophisticated algorithm for generating large numbers of slow-roll inflationary models, without relying on the explicit form of the action (Kinney 2002).

Recently, in Capurri et al. 2020, the method has been extended to the EFT framework of inflation to include a much broader

class of beyond-standard inflationary models and explore a wide variety of possible high-energy corrections to the simplest slow-roll scenario. In this section, we follow this latter generalized approach to investigate in a more general and reliable way the actual contribution of inflationary tensor perturbations to the energy budget of the early Universe. We start reviewing the Hubble Flow Equations in the EFT of Inflation, strictly following Capurri et al. 2020. Then, we explain how we adapt this method to our investigation. Finally, we discuss the results.

3.1 Hubble Flow Equations and EFT of Inflation

The EFT of inflation (Cheung et al. 2008; Weinberg 2008) is a very general framework for describing fluctuations around a quasi-de Sitter background. The general form of the effective action in the comoving gauge reads

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\text{pl}}^2 R - c(t) g^{00} - \Lambda(t) + \frac{1}{2!} M_2(t)^4 (g^{00} + 1)^2 + \frac{1}{3!} M_3(t)^4 (g^{00} + 1)^3 + \dots - \frac{\bar{M}_1(t)^3}{2} (g^{00} + 1) \delta K_\mu^\mu - \frac{\bar{M}_2(t)^3}{2} \delta K_\mu^{\mu 2} - \frac{\bar{M}_3(t)^3}{2} \delta K_\nu^\mu \delta K_\mu^\nu + \dots \right]. \quad (11)$$

For the following discussion, it is useful to divide this action into two different blocks and analyse them separately.

The first important block is given by the first line of equation (11) that we rewrite below for convenience:

$$S_{\text{bg}} = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\text{pl}}^2 R - c(t) g^{00} - \Lambda(t) \right] \quad (12)$$

It contains the standard Einstein–Hilbert action and terms that are linear perturbations around the background. Therefore, once that the time-dependent coefficients $c(t)$ and $\Lambda(t)$ have been specified, this part of the action completely fixes the background evolution during inflation. Notice also that, the evolution of the parameters $c(t)$ and $\Lambda(t)$ can be related to the evolution of the Hubble parameter by the Friedmann equations

$$H^2 = \frac{1}{3M_{\text{pl}}^2} [c(t) + \Lambda(t)] \quad \text{and} \quad \frac{\ddot{a}}{a} = -\frac{1}{3M_{\text{pl}}^2} [2c(t) - \Lambda(t)], \quad (13)$$

so we need only two independent functions to fully characterize the background evolution that we choose to be $H(t)$ and $c(t)$, fixing $\Lambda(t)$ by equation (13). Starting from equation (12), we take a first step deriving the generalized Hubble Flow Equations for the background parameters. In analogy with the standard case, we take as our fundamental quantity the Hubble parameter as a function of inflaton field, $H(\phi)$. To switch from the time domain to field domain, we can exploit a relation between the time-derivative of the field, $c(\phi)$ and $H(\phi)$ that follows from a combination of the Friedmann equation and the continuity equation, namely

$$\frac{d\phi}{dt} = -\frac{c(\phi)}{M_{\text{pl}}^2 H'(\phi)}, \quad (14)$$

where, from now on, the prime indicates a derivative with respect to the field ($X' = dX/d\phi$). Using the relation above, it is easy to see that the slow roll parameter ϵ becomes

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{c(\phi)}{M_{\text{pl}}^2 H^2(\phi)}. \quad (15)$$

Starting from ϵ , we can define the higher order slow-roll parameters by iterated derivations

$$\begin{aligned} \eta(\phi) &= \frac{c(\phi)}{M_{\text{pl}}^2} \frac{H''(\phi)}{H(\phi)H'(\phi)} \\ &\vdots \\ {}^l \lambda(\phi) &= \left(\frac{c(\phi)}{M_{\text{pl}}^2} \right)^l \left(\frac{1}{H(\phi)} \right)^l \left(\frac{1}{H'(\phi)} \right)^{l+1} \frac{d^{l+1} H(\phi)}{d\phi^{l+1}}, \end{aligned} \quad (16)$$

with $l \geq 2$ and $\eta(\phi) \equiv {}^1 \lambda(\phi)$. Notice however that, in contrast with the standard Hubble flow equations, now the evolution of ϵ and the other higher order parameters will depend also on the additional unknown function $c(\phi)$. Therefore, we need to define other new slow-roll parameters to describe the evolution of $c(\phi)$. Following the notation of Capurri et al. 2020, we introduce the parameter θ

$$\theta \equiv -\frac{\dot{c}}{Hc} = \frac{1}{M_{\text{pl}}^2} \frac{c'(\phi)}{H(\phi)H'(\phi)}, \quad (17)$$

and the other higher-order parameters by taking iterated derivations

$$\begin{aligned} \kappa(\phi) &= \frac{1}{M_{\text{pl}}^2} \frac{c''(\phi)}{H'^2(\phi)} \\ &\vdots \\ {}^l \xi(\phi) &= \left(\frac{c(\phi)}{M_{\text{pl}}^2} \right)^l \left(\frac{1}{H(\phi)} \right)^{l-1} \left(\frac{1}{H'(\phi)} \right)^{l+1} \frac{1}{c(\phi)} \frac{d^{l+1} c(\phi)}{d\phi^{l+1}}, \end{aligned} \quad (18)$$

always with $l \geq 2$ and $\kappa(\phi) \equiv {}^1 \xi(\phi)$. An explicit calculation of the equations above lead to derive the generalized Hubble flow equation for the background parameters (Capurri et al. 2020):

$$\begin{cases} \frac{d\epsilon}{dN} = \epsilon(\theta - 2\epsilon) \\ \frac{d\eta}{dN} = \eta(\theta - \epsilon - 2\eta) + 2\lambda \\ \vdots \\ \frac{d^l \lambda}{dN} = {}^l \lambda [l(\theta - \epsilon) - (l+1)\eta] + {}^{l+1} \lambda \\ \frac{d\kappa}{dN} = \epsilon\kappa - \theta(\epsilon + \eta) \\ \frac{d\xi}{dN} = -2\kappa\eta + 2\xi \\ \vdots \\ \frac{d^l \xi}{dN} = {}^l \xi [(l-1)(\theta - \epsilon) - (l+1)\eta] + {}^{l+1} \xi \end{cases} \quad (19)$$

We stress that the integration of this system of coupled equations completely specifies the dynamics of the background during inflation.

The second block in the action (11), involves the higher order operators that we have organized in powers of the number of perturbations and in terms of the increasing number of derivatives

$$\begin{aligned} \Delta S &= \int d^4x \sqrt{-g} \left[\sum_{n \geq 2} \frac{1}{n!} M_n(t)^4 (g^{00} + 1)^n - \frac{\bar{M}_1(t)^3}{2} (g^{00} + 1) \delta K_\mu^\mu - \frac{\bar{M}_2(t)^3}{2} \delta K_\mu^{\mu 2} - \frac{\bar{M}_3(t)^3}{2} \delta K_\nu^\mu \delta K_\mu^\nu + \dots \right]. \end{aligned} \quad (20)$$

These operators are turned on and off by the M coefficients in the action, whose value will thus weight the relative effects. As we shall see, in their turn, the coefficients M can be related to physical quantities that can be in principle measured and constrained. Therefore, once we have reconstructed the background dynamics

by solving the system (19), it is useful to derive a further system of equations to describe the evolution of the M coefficients in equation (20) over that background. We can do so in a quite general and elegant way by noting that for any quantity described by a generic scalar function $Q(\phi)$, one can always define a slow-roll parameter ϵ_Q as follows:

$$\epsilon_Q = -\frac{\dot{Q}}{H Q} = \frac{1}{M_{\text{pl}}^2} \frac{c(\phi)}{H(\phi) H'(\phi)} \frac{Q(\phi)}{Q'(\phi)} \quad (21)$$

In analogy to the discussion for the background parameters, we define also the higher order parameters for the quantity $Q(\phi)$ by taking its derivatives

$$\begin{aligned} \rho_Q(\phi) &= \frac{1}{M_{\text{pl}}^2} \frac{c(\phi)}{H'^2(\phi)} \frac{Q''(\phi)}{Q(\phi)} \\ &\vdots \\ {}^l \chi_Q(\phi) &= \left(\frac{c(\phi)}{M_{\text{pl}}^2} \right)^l \left(\frac{1}{H(\phi)} \right)^{l-1} \left(\frac{1}{H'(\phi)} \right)^{l+1} \frac{1}{Q} \frac{d^{l+1} Q}{d\phi^{l+1}} \end{aligned} \quad (22)$$

again with $l \geq 2$ and $\rho_Q(\phi) \equiv {}^1 \chi_Q(\phi)$. By explicitly computing these relations, we eventually get the system of Hubble flow equations for $Q(\phi)$:

$$\begin{cases} \frac{d\epsilon_Q}{dN} = \epsilon_Q (\theta - \epsilon - \eta - \epsilon_Q) + \epsilon \rho_Q \\ \frac{d\rho_Q}{dN} = \rho_Q (\theta - 2\eta - \epsilon_Q) + {}^2 \chi_Q \\ \vdots \\ \frac{d{}^l \chi_Q}{dN} = {}^l \chi_Q [l\theta - (l-1)\epsilon - (l+1)\eta - \epsilon_Q] + {}^{l+1} \chi_Q \end{cases} \quad (23)$$

Solving the system, we can predict the evolution of any generic quantity $Q(\phi)$ that will depend also on the background via the slow-roll parameters ϵ , η , and θ , as expected. This means that, in principle, one can evolve all the M coefficients in equation (20) and study different models of inflation in full generality.

3.2 Theoretical Monte Carlo: integration scheme

Our aim is to explore a reasonably large ensemble of physical models of inflation that can lead to a sizeable gravitational wave production and calculate their contribution to the energy density of the early Universe, accurately. In this regard, it is worth noting that taking into account all the operators in the quadratic effective action that induce tensor perturbations, one can derive the following leading order relation for the power spectrum (Creminelli et al. 2014; Noumi & Yamaguchi 2014; Giarè & Renzi 2020)

$$\mathcal{P}_T = \frac{1}{c_T} \left(\frac{H^2}{\pi^2 M_{\text{pl}}^2} \right), \quad (24)$$

where c_T is the propagating speed of tensor modes that can be simply expressed in terms of \bar{M}_3 as $c_T^{-2} = 1 - \bar{M}_3^2/M_{\text{pl}}^2$ where \bar{M}_3 is defined in (11). In this case, it is straightforward to see, from its definition, that the tensor tilt acquires a further correction

$$n_T = -2\epsilon + \epsilon_T, \quad (25)$$

where the evolution of the parameter

$$\epsilon_T = -\frac{\dot{c}_T}{H c_T} \quad (26)$$

is clearly governed by the system (23). It is also worth noting that in this framework the standard relation between the tensor amplitude and the tensor tilt does not hold anymore and more general consistency relations can be derived both in the absence

and in presence of additional EFT operators (see Giarè & Renzi 2020; Capurri et al. 2020 for detailed discussions). Anyway, all the cosmological observables can still be expressed in terms of the slow-roll parameters and in particular, the tensor spectrum and its evolution are fully determined by the evolution of the background and the parameter ϵ_T . This is an important achievement since through the flow equation method, we can actually test the observable predictions of a large number of stochastically generated models, without relying on the specific form of their underlying actions. To optimize our model exploration, we proceed with a theoretical Monte Carlo as follows:

(i) First and foremost, we notice that the hierarchy of flow equations must be truncated at finite order, which we choose to be the 4th order. Then, we draw a suitable set of randomly chosen initial conditions for the background parameters. In particular, we randomly choose the parameters introduced in the Hubble tower (16) within the following ranges

$$\begin{aligned} \epsilon_{\text{in}} &\in [0, 0.8], \\ \eta_{\text{in}} &\in [-0.1, 0.1], \\ {}^2 \lambda_{\text{in}} &\in [-0.05, 0.05], \\ {}^3 \lambda_{\text{in}} &\in [-0.005, 0.005], \end{aligned}$$

while for the $c(\phi)$ tower (18) the initial conditions are taken from the sets

$$\begin{aligned} \theta_{\text{in}} &\in [-0.1, 0.1], \\ \kappa_{\text{in}} &\in [-0.1, 0.1], \\ {}^2 \xi_{\text{in}} &\in [-0.05, 0.05], \\ {}^3 \xi_{\text{in}} &\in [-0.005, 0.005]. \end{aligned}$$

These ranges are very similar to those in Kinney 2002 and Capurri et al. 2020.

(ii) Once the initial conditions are chosen, we solve the Hubble flow equations (19) for the background slow-roll parameters. Specifically, we integrate the equations forward in time for at most $\sim 10^4$ e-folds of expansion. Then, apart from the unfortunate cases where the integration did not survive, we expect two possible outcomes: either we reach a fixed point (that we eliminate) or we manage to get the end of inflation defined by the usual relation $\epsilon = 1$. In this latter case, we store all the background parameters as functions of the number of e-folds N before the end of inflation (*i.e.*, $N = 0$ corresponding to $\epsilon = 1$). Given a large number of repetitions ($\gtrsim 10^4$), approximately 90 per cent of the time, the end of inflation is successfully reached.

(iii) We then check that the models stored in the previous point allow a sufficient long phase of expansion and are able to explain observations. To do so, we use the values reached by parameters at the end of inflation as new initial conditions at $N = 0$ and perform a backward in time integration up to the e-folds when the primordial observables are evaluated ($N = 60$). Once more, we make sure to obtain a successful integration, that is, we do not end up with $\epsilon = 1$ again. For the remaining models, we check whether the spectral index of scalar modes n_s lies within the observed bounds. In particular, we reject all the results outside the range $0.94 < n_s < 0.98$, chosen conservatively around the Planck best-fitting value, ending up with roughly 17 per cent of the total. We store the survived models and proceed to evolve all the other physical quantities involved in our analysis.

(iv) Particularly relevant for the evolution of the tensor spectrum are the quantities related to the propagating speed c_T , see also equation (24). To solve the system of equations (23), we need to specify some initial conditions for ϵ_T and the other high-order tensor

parameters that we randomly choose within the following ranges:

$$\epsilon_T \in [-0.1, 0.1],$$

$$\rho_T \in [-0.01, 0.01],$$

$$^2\chi_T \in [-0.001, 0.001],$$

$$^3\chi_T \in [-0.0001, 0.0001].$$

To optimize the simulations and save computational time, for each realization of the background, our algorithm is able to perform simultaneous evolution of different physical quantities. In particular, starting from some initial conditions, we first perform a forward integration until the end of Inflation. Since we already did such an integration for the background (and given that all the other quantities do not affect the space–time evolution), we can focus exclusively on the stability of the tensor-speed sector. We find that a small part of the total leads to an unsuccessful integration while most models require also a backward integration (for instance, because they reach the controversial value $\epsilon_T = 1$ during the integration or because they show non-physical behaviours for the other parameters). Once that all the consistency checks have been carried out, the model is either accepted or rejected. At the end of the process, only approximately 40 per cent of the attempts resolve in a successful inflation with a non-trivial tensor-speed sector.

(v) As concerns the other physical quantities, we select a subgroup of models that share the same tensor amplitude $r \sim 0.001$ on the CMB scales ($N = 60$) but that differ by the value of the tensor tilt n_T that we estimate at $N = 60$ according to equation (25). Finally, we evolve the tensor spectrum dynamically from $N = 60$ up to the end of inflation by means of the Hubble flow equations. For each spectrum, we calculate the corresponding contribution to the energy density of the early universe parametrized in terms of the effective number of relativistic degrees of freedom $\Delta N_{\text{eff}}^{\text{GW}}$. To do so, we evaluate the corresponding energy density in gravitational waves $\Omega_{\text{GW}}(f)$ by equation (A9) and integrate it over frequency according to equation (2).

3.3 Theoretical Monte Carlo: models

Using this procedure, we are able to collect a sufficiently large ensemble of physical models ($\simeq 10,000$) which spans a reasonable range of possibilities, from realization with a canonical tensor-speed sector (i.e. $c_T = 1$ and $\epsilon_T = 0$) to more general cases with time-dependent tensor parameters. None the less, our integration scheme is focused on a well-defined task, i.e. we are not interested in providing a comprehensive analysis of the model frequency distribution for the different observables as already done in full generality in Capurri et al. 2020, but rather to shed light on the correlation between n_T and the predictions for $\Delta N_{\text{eff}}^{\text{GW}}$. To achieve this task in the most direct and simple way, we necessarily introduce some limitations on the models that we are actually able to explore, that deserve to be further justified and clarified.

A first major restriction comes from limiting our analysis to a small subgroup of models with a fixed tensor amplitude $r \sim 0.001$ on the CMB scales. This clearly introduces a limitation on the number of cases that we are able to reach within our Monte Carlo technique. Notice however that, we do not expect this limitation to introduce a large bias on the frequency distribution of the values obtained for the tensor tilt as the consistency relation between r and n_T does not hold anymore and these two parameters can be regarded as independent. In addition, we are not particularly interested in studying the model frequency distribution, but rather in understanding whether models

sharing similar parameters on the CMB scales may result into a significant different contribution to the energy budget of the early Universe because of their different evolutionary paths. Focusing only on models with the same r at $N = 60$ turns out to be particularly useful for this purpose since it ensures that the predictions for $\Delta N_{\text{eff}}^{\text{GW}}$ do not depend on the value of the tensor amplitude at the CMB scales (which is in fact common to all models). In this way at $N = 60$, all the models will differ only by the value of the tensor tilt and comparing the values of $\Delta N_{\text{eff}}^{\text{GW}}$ predicted by models with a similar n_T one can have an immediate idea of the difference produced by the different evolution of the spectra from $N = 60$ to $N = 0$ and unequivocally understand whether $\Delta N_{\text{eff}}^{\text{GW}}$ and n_T are somehow correlated. Finally, we can directly compare the results obtained within our theoretical Monte Carlo with those derived in Section 2.3 by means of a parametric reconstruction of the spectra (where the tensor amplitude was fixed to $r \sim 0.001$, as well), testing the consistency of these two methods. Last, but not least, $r \sim 0.001$ is the declared target of future CMB-S4-like experiments (Abazajian et al. 2016). Therefore, we believe it should be particularly interesting to understand what kind of physical models future surveys may be able to probe. This is the ultimate reason why we have chosen such a value for the amplitude.

A second minor limitation is introduced by taking only positive initial conditions for the parameter ϵ , without considering models resulting from a background evolution with $\epsilon_{\text{in}} < 0$, like it was done in Capurri et al. 2020. To understand the implications of this limitation, we recall that in the standard single-field models, the null energy condition (NEC) prevents the slow-roll parameter ϵ to be negative. However, this framework is quite general and can be applied also to more complicated scenarios where this possibility is viable, such as superinflation models (Gasperini & Giovannini 1992; Brustein et al. 1995; Baldi, Finelli & Matarrese 2005; Creminelli et al. 2006) (where ϵ can remain always negative) or models with intermittent NEC violation (Cai & Piao 2021, 2022) (where ϵ can be negative for some e-folds and then come back to be positive, restoring the usual end of inflation at $\epsilon = 1$). In this regard, we notice that starting with a positive ϵ as the initial condition does not preclude this parameter to acquire negative values during its evolution. Therefore, the latter intermittent case is included in our Monte Carlo. Conversely, requiring $\epsilon_{\text{in}} > 0$ and the end of inflation to occur at $\epsilon = 1$ exclude the superinflation case. Indeed such models are characterized by a Hubble parameter that increases with time so that the end of inflation is no longer determined by the condition $\epsilon = 1$ but must be forced by external factors, such as an additional field. In the framework of a theoretical Monte Carlo, it becomes very ambiguous to decide when inflation ends since we are not sensitive to the details of the mechanism. For this reason, one needs to choose an arbitrary point during the evolution, as done in Capurri et al. 2020, introducing an element of arbitrariness in the resulting predictions. In addition, this case should be considered separately because one needs to choose the range of integration very carefully in such a way that the energy scale at the end of inflation can lie within the observational upper bound ($H_{\text{fin}} < 2.7 \times 10^{-5} \text{ M}_{\text{pl}}$; Akrami et al. 2020a) and the lower limit (around the MeV scale to guarantee hydrogen and helium production during the BBN; Giudice et al. 2001a). It is important to acknowledge that this limitation can in fact result in a significant reduction in the number of models predicting a blue-shifted tensor tilt that our pipeline is able to investigate (see also Fig. 3). Despite this, our conclusions on $\Delta N_{\text{eff}}^{\text{GW}}$ cannot in any way rely on these exotic scenarios and we can safely exclude such models from the analysis without biasing the results. For a more thorough examination (see Capurri et al. 2020).

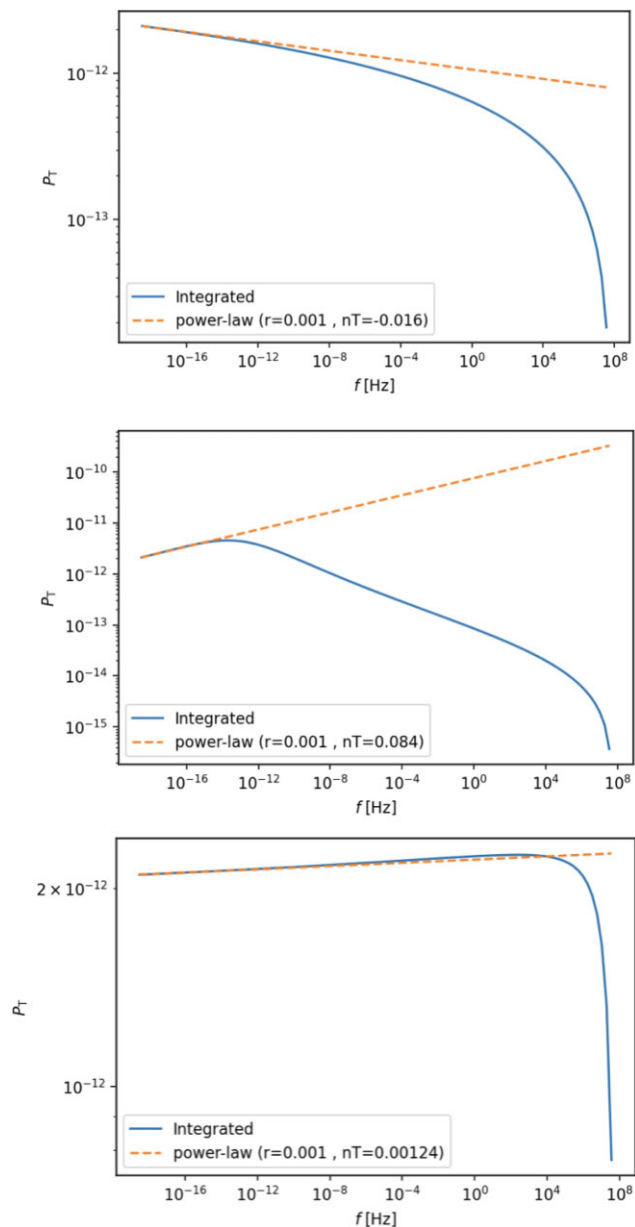


Figure 4. Three physical examples of the primordial tensor spectra (and their power-law extrapolation, orange line) obtained by integrating the Hubble flow equations as discussed in Section 3.

3.4 Observable predictions: primordial tensor spectrum

We start discussing some useful insights about what is obtained following the integration method outlined in the previous subsection. In particular, in Fig. 4, we show three controversial examples that we find particularly enlightening about the very diversified behaviour that the tensor spectrum can acquire in physical models of inflation, validating the need to systematically investigate the observable predictions by means of a theoretical Monte Carlo.

In the top-side panel of the Fig. 4, we display the spectrum predicted by a red-tilted model of inflation where we evolved the background according to equation (19), switching off all the other EFT operators. The shape of this spectrum is not very different from what we got by our previous parametric analysis (see also Fig. 2). Not surprisingly, near the CMB frequencies, the spectrum is well

described by a power law (orange line in the figure) and all the physics of the model is captured only by two parameters, the amplitude and the tilt. Conversely, on frequencies close to the end of inflation, the power in gravitational waves is suddenly dismissed. As soon as the potential starts approaching its minimum (i.e. $\epsilon \rightarrow 1$), the slow-roll dynamics breaks down and both the Hubble parameter and the tensor spectrum ($P \propto H^2$) suddenly decrease. In this frequency range, the behaviour of the spectrum is mostly determined by the shape of the potential (which is no more flat) and consequently, the gravitational wave production becomes strongly model dependent (Kinney 2021). Interestingly, if we compare the power spectrum integrated over the Hubble flow equations with a simple power-law extrapolation, we see that on high frequency there is a difference of almost two orders of magnitude between the two curves. We can easily quantify the impact in terms of $\Delta N_{\text{eff}}^{\text{GW}}$ by integrating both the spectrum and its power-law extrapolation through equation (2). For this particular model (and for models that show a similar behaviour), we estimate a difference of a factor ~ 10 between the contributions obtained by integrating the Hubble flow equation ($\Delta N_{\text{eff}}^{\text{GW}} \simeq 1 \times 10^{-12}$) and the one inferred by a power-law extrapolation ($\Delta N_{\text{eff}}^{\text{GW}} \simeq 1 \times 10^{-11}$). In both cases, however, the contribution is extremely small and well beyond any current or future experimental sensitivity, as expected in red-tilted inflation.

The situation becomes even more intriguing if we turn to the study of blue-tilted models of inflation. Within our framework, such models can be realized either taking $\epsilon < 0$ at $N \sim 60$ or including corrections to the tensor spectrum coming from the extrinsic curvature perturbations in equation (11). In the middle panel of Fig. 4, we plot the tensor spectrum realized in one of the latter cases. In this particular model, the regime $n_T > 0$ is supported only for a few e-folds of inflation, corresponding to the frequency range during which the spectrum follows a blue-tilted power law and the gravitational signal is amplified. After that, because of a combined effect of the background evolution and the evolution of parameter ϵ_T , the spectrum becomes very red-tilted and the power in the gravitational wave is suppressed at high frequencies. Specifically, the more n_T is positive at CMB scales, the greater ϵ_T should be, bringing consequently its derivatives to assume larger values to compensate. Thus, its evolution is accelerated (towards negative values). The blue-tilted regime lasts only a few e-folds and then falls into the red-shifted one. This model is similar to those discussed in Benetti et al. 2022 and in this case assuming a blue-tilted gravitational wave signal by a factor of 10^5 . Repeating the exercise of computing the contribution to the radiation energy density for both the integrated spectrum ($\Delta N_{\text{eff}}^{\text{GW}} \simeq 7 \times 10^{-14}$) and the power-law one ($\Delta N_{\text{eff}}^{\text{GW}} \simeq 2 \times 10^{-9}$), we end up with two completely different results. Therefore, this is the ‘smoking gun’ evidence that leads weight to all the concerns already emerged from our parametric analysis. It makes evident that extrapolating a power-law spectrum over all scales can be an unreliable practice and can lead to strongly overestimating the gravitational wave contribution to the radiation energy density (even by many orders of magnitude, as we have just proved). However, one may ask to what extent such a model can be considered representative of the spectrum’s behaviour in blue-tilted inflation and how easily models like this one can be realized. As a counterexample, in the bottom panel of Fig. 4, we show a blue-tilted spectrum realized within a model where a simple power-law extrapolation still provides a very good approximation of the gravitational wave production even on a frequency very close to the end of inflation, guarantying an accurate estimation of ΔN_{eff} . Notice that, models like that are typically characterized by an extremely

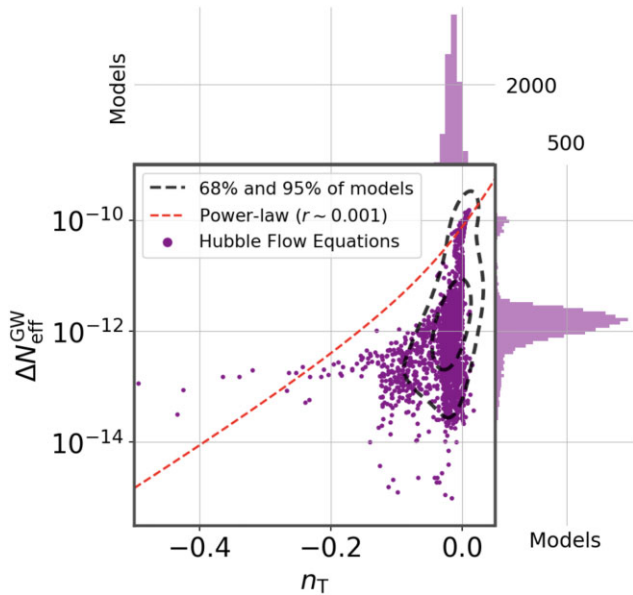


Figure 5. Observable predictions in the plane $(n_T, \Delta N_{\text{eff}}^{\text{GW}})$. The magenta dots represent the models realized within the Hubble flow equation method discussed in Section 3, while the red dashed line represents the power-law prediction. The dashed black lines define the regions of the plane that contain the 68 and 95 per cent of the total models and are calculated by marginalizing over the point frequency distribution (displayed by the two histograms on the axes).

slow evolution of the inflationary parameters and hence by a very flat potential. Therefore, one may argue that they may be not easy to realize, as well.

Clearly to provide a definitive answer and derive reliable results, we need to study the inflationary gravitational wave production for a sufficiently large ensemble of randomly realized physical models where all the possibilities are studied exhaustively in the framework of a theoretical Monte Carlo. In Fig. 3, we compare the results for $\Delta N_{\text{eff}}^{\text{GW}}$ obtained by this latter approach (dark magenta dots in the figure) with those realized in Section 2.3. In Fig. 5, we instead zoom in on the $(n_T, \Delta N_{\text{eff}})$ plane, showing the distribution of the physical models. The dashed black lines in this latter figure define the regions of the plane where the 68 and 95 per cent of the total models lie and are calculated by marginalizing over the point frequency distribution (displayed by the two histograms). An accurate analysis of this figure can reveal several interesting hints about the physics underlying the point distribution that is worthy of being discussed in details.

We start by analysing the observable predictions for the tensor tilt. In particular, we notice that the vast majority of the models are characterized by slightly negative tilt ($n_T \lesssim 0$). As partially explained in the previous subsection, this is due to the fact that, during the integration process, only a few blue-tilted models survive all the physical consistency checks and constraints. In fact, most of the survived models have a canonical tensor speed evolution ($c_T = 1$, $\epsilon_T = 0$) and respect the null energy condition ($\epsilon > 0$) so that their observable predictions follow, or are very close to following, the usual slow-roll consistency relations. This also suggests that realizing well-defined blue-tilted models able to satisfy all the physical requirements (such as stability, causality, and last but not the least the observable constraints) may be a tricky avenue and in general red-tilted models are largely preferred by theoretical Monte Carlo simulations, as already pointed out in Capurri et al.

2020. Focusing on the survived blue-tilted models, it is also evident that only small values of the tensor tilt are realized and we remain far away from the controversial observational upper limit inferred for this parameter. As a matter of fact, the largest n_T we are able to get within our pipeline reads $n_T \simeq 0.08$ (close to the middle panel of Fig. 4). The reasons why the case $n_T > 0$ is generally disfavoured are several. For instance, such values can hardly arise from the extrinsic curvature corrections since this would imply a large $\epsilon_T > 0$ and, by equation (26), a negative time derivative of the tensor speed that would thus be reduced close to the frequencies where it is instead constrained to be unitary by gravitational wave observations (Abbott et al. 2017b, c). As concerns the red-tilted models, most of them show the same preference for very small tilt values ($-0.1 < n_T < 0$ within the 95 per cent region), but a few exceptions with $n_T \lesssim -0.2$ can be observed. While they represent a negligible part of the total points, it is interesting to notice that, in principle, such models do not violate any observable prediction. In our framework, a relatively large negative tilt can be realized by a combined effect of both the background evolution and the tensor speed evolution provided that $\epsilon_T < 0$. In this regard, it is worth pointing out that a negative ϵ_T would imply $\dot{c}_T > 0$ and so a tensor speed that increases over time, around the CMB frequencies. Since the propagating speed of gravitational interactions is not (severely) constrained at those frequencies, the model may remain viable as long as $c_T \in [0, 1]$ (see also Capurri et al. 2020; Giarè & Renzi 2020). On the other hand, a significantly non-unitary c_T may be an element of concern because we are dealing with perturbative departures from General Relativity and so we do not expect large deviations. However, thanks to the narrowed window allowed for the initial conditions of the tensor-speed parameters, these models remain the very minority (we can count only 18 models with $n_T \lesssim -0.2$) and in most of them, the background dynamics importantly contributes to this behaviour. We therefore find this issue to be not particularly relevant to the general aim of this paper and leave it suitable for future investigation. There is yet another interesting aspect that deserves to be remarked: looking at Fig. 5, we can spot a bunch of models that follow a power-law behaviour very closely (dark magenta points that overlap with the red dashed line in the figure). All these models are characterized by a tensor tilt extremely close to zero. Because of equation (25), this means that both ϵ and eventually ϵ_T need to be very close to vanishing, implying an extremely slow-roll dynamics and thus a very flat inflationary potential. They are nothing but models that behave like the one depicted in the bottom panel of Fig. 4. So, looking at Fig. 5, we can finally answer whether such models can be easily realized or not. In particular, we find that they are not in the densest region of the plane. None the less, they still fall within the region containing the 95 per cent of the total models, actually contributing to a second (very) small peak in the histograms of ΔN_{eff} (see also Fig. 5).

3.5 Observable predictions: relic gravitational radiation

We now turn to the study of the observable predictions for ΔN_{eff} that is the point of interest for this analysis. First and foremost, most of the models predict a value very different from what one would expect by extending a power-law parametrization over all scales (see Fig. 3). The same conclusion can be derived from a different perspective by looking at Fig. 5. From the latter figure we can appreciate that, while the histogram of n_T is very sharp and most models share similar values of the tensor tilt, the histogram of $\Delta N_{\text{eff}}^{\text{GW}}$ is instead much broader and the regions containing the 68 and 95 per cent of models are almost vertical, spanning a quite

large range of values $\Delta N_{\text{eff}}^{\text{GW}} \simeq 10^{-10} - \simeq 10^{-14}$. This means that models that share the same inflationary parameters on the CMB scales (i.e. the same amplitude r and the same tilt n_T) can easily result in a completely different contribution to $\Delta N_{\text{eff}}^{\text{GW}}$. As already pointed out in Section 2, this depends on the different evolution of the spectra at high frequencies. More precisely, the results of theoretical Monte Carlo suggest that extrapolating a power-law behaviour over ultraviolet frequencies, in most cases leads to overestimating the gravitational wave contribution to the energy budget of the Universe. As partially explained above, this is due to the fact that, close to the end of inflation, the potential necessarily undergoes a rapid phase of evolution towards its minimum that drives the Hubble parameter (and consequently the power spectrum) to be suddenly dismissed. This is evident in all the three physical spectra shown in Fig. 4. This feature is instead missed within a power-law extrapolation, so that the power in gravitational waves is typically much overestimated on high frequencies, leading to a larger $\Delta N_{\text{eff}}^{\text{GW}}$. None the less, we can observe a few models where the actual contribution to the effective number of relativistic species is larger than predicted by a power law. While such points represent the vast minority of the models, it is still interesting to explain the physical reason underlying this behaviour. In particular, it is evident both from Figs. 3 and 5, that this event is more frequent for those very few points that show a very red tensor tilt $n_T \lesssim -0.2$. As we pointed out in the previous paragraph, in these cases, both the background dynamics (i.e. the value of ϵ at $N = 60$) and the tensor-speed dynamics (i.e. the value of ϵ_T at $N = 60$) should significantly contribute to the final value of the tensor tilt on the CMB scales and to the evolution of the spectrum with frequency. Indeed, in all these models, ϵ must reach the value $\epsilon = 1$ within $\Delta N = 60$ e-folds of evolution (so that inflation can end) while ϵ_T will undergo a similar evolution. If ϵ_T evolves towards less negative values, it can mitigate the loss of power induced by ϵ and the spectrum may remain so much red-tilted only for a few e-folds. Consequently, in this case assuming continuously a power-law parametrization can lead to underestimating $\Delta N_{\text{eff}}^{\text{GW}}$.

We conclude this section with a final remark: all the models obtained with the Hubble flow equation method give an extremely small $\Delta N_{\text{eff}}^{\text{GW}}$. The histogram of this parameter is in fact centred around values $\Delta N_{\text{eff}}^{\text{GW}} \sim 10^{-12}$, with a second small peak of models at $\Delta N_{\text{eff}}^{\text{GW}} \sim 10^{-10}$ (resulting from that class of models with an extremely slow evolution discussed in the previous paragraphs). These values are far away from the total amount of additional radiation allowed by data ($\Delta N_{\text{eff}}^{\text{GW}} \lesssim 0.3 - 0.4$) as well as from any current and future experimental sensitivity. Therefore, one may ask whether this issue is in any way relevant for the purpose of mode building. In this regard, we would like to point out that, while this method is quite general and allows precise calculation without relying on the details of the model, it does not cover all the possibilities proposed in the literature and blue-tilted models with larger gravitational wave production may be obtained by other viable physical mechanisms. On the other hand, the stochastic technique used in Section 2.3 should embrace a much larger class of possibilities since we simply reconstruct the spectrum as a fraction of the frequency. It is also entirely plausible that such spectra can be obtained in well-motivated models, as we may argue by comparing the grey and dark magenta dots in Fig. 3 and noticing that they share similar behaviour. In any case, a detailed study of the observational prospects of the field is beyond the aim of this manuscript where we believe to have already covered a reasonable range of different scenarios and possibilities, consistently getting conclusive evidence that assuming a power-law spectrum over all scales can lead to a wrong estimation of the gravitational wave

contribution in the early Universe. In light of this result, we can definitively conclude that the calculation of $\Delta N_{\text{eff}}^{\text{GW}}$ proves to be remarkably model dependent and more accurate analyses are needed before inferring any reliable conclusion on (blue-titled) inflationary models in light of the BBN bounds on additional radiation.

4 CONCLUSIONS

In this paper, we revisit the calculation of the inflationary gravitational wave contribution to the radiation energy density in the early Universe. Behaving as additional radiation, primordial gravitational waves may in fact increase the effective number of relativistic species (N_{eff}) by a further correction that depends on the integrated energy density in gravitational radiation over all scales (see equation 2). According to the Friedmann equations, extra radiation would imply a faster background expansion and consequently a different thermal evolution of the Universe, with several implications. For instance, a faster expansion would lead to a higher freeze-out temperature of the weak interactions, implying a higher fraction of primordial Helium and Deuterium to be forged during the big bang nucleosynthesis epoch. This effect is particularly relevant, because it is commonly used to infer stringent bounds on the additional radiation energy density and, in its turn, to constrain (blue-titled) models of inflation.

However, the underlying assumption of (most of) the state-of-the-art analyses is that the spectrum of inflationary gravitational waves can be parametrized, *continuously* over all cosmological epochs and scales, by a simple power law with two free parameters: the amplitude r and the tilt n_T . While in most inflationary models such parametrization works very well on the frequencies probed by the CMB experiments (roughly corresponding to $N \sim 60$ e-folds before the end of inflation), as already pointed out in the literature (Giarè & Melchiorri 2021; Kinney 2021) extrapolating a power-law behaviour over all frequencies can be highly non-trivial and risky; above all on the high-frequencies corresponding to tensor modes that cross the horizon very close to the end of inflation, when the slow-roll dynamics breaks down and the gravitational wave production becomes strongly model dependent. Since these frequencies not only contribute to the integral (2), but they are also exponentially amplified within a power-law parametrization ($\mathcal{P}(f) \propto f^{n_T}$), this problem becomes of primary relevance when evaluating the tensor modes contribution in the early Universe because the calculation crucially depends on a parametrization whose validity is anything but reliable.

Driven by this concern, in Section 2, we systematically study how (much) different parametrizations of the tensor spectrum impact on the final predictions of $\Delta N_{\text{eff}}^{\text{GW}}$. In Fig. 1, we show that allowing a \sim few per cent scale variation of the tensor tilt, the resulting $\Delta N_{\text{eff}}^{\text{GW}}$ can be much amplified or suppressed, depending on the sign of the running. In Section 2.3, we perform parametric analysis by expanding the spectrum in full generality as a sum of powers and randomly collecting 10^6 different shapes of the spectrum able to satisfy all the observational constraints, consistently towards all cosmological epochs and scales. The results in Fig. 3 (grey dots) prove that relaxing that assumption of power-law spectrum *on high frequencies*, the value of the tensor tilt becomes basically uncorrelated with $\Delta N_{\text{eff}}^{\text{GW}}$, so that models with the same n_T can contribute very differently to the energy budget of the Universe.

In order to understand to what extent our result can be considered reliable when applied to physical models of Inflation, in Section 3, we investigated the observable predictions of a very broad class of inflationary models. We work within the framework of the effective field theory of inflation and follow a methodology based on the

so-called Hubble flow equation: a system of coupled differential equations whose solution completely specifies the evolution of the main observable quantities during inflation. We solve numerically the truncated system of the Hubble flow equations for a set of suitably defined initial conditions (taking into account also a different combination of additional operators in the EFT of inflation) as a sophisticated algorithm for generating large numbers of slow-roll inflationary models without relying on the explicit form of the action. In this way, we produce an ensemble of very general physical models ($\simeq 10,000$) studying the resulting observable predictions. Examples of the spectra obtained by our method are shown in Fig. 4, while the final results for $\Delta N_{\text{eff}}^{\text{GW}}$ are summarized both in Figs. 3 and 5. Both figures make it evident that in most cases extrapolating a power-law behaviour over 24 orders of magnitude in frequency leads to overestimating the power in gravitational waves, above all on the ultraviolet frequencies that are the most relevant in the calculation. As a result, the predicted relic energy density in gravitational wave can be ultimately incorrect.

We conclude by stressing that this issue seriously calls into question the validity of the observational constraints inferred on the tensor tilt by the indirect effect of additional radiation during the BBN epoch, motivating the need of more accurate calculations before inferring any reliable conclusion on inflation.

ACKNOWLEDGEMENTS

WG thanks Martina Gerbino and Massimiliano Lattanzi for the useful comments at the early stage of this project and Fabrizio Renzi for valid suggestions on the manuscript. WG, MF, and AM are supported by ‘Theoretical Astroparticle Physics’ (TAsP), iniziativa specifica INFN. EDV is supported by a Royal Society Dorothy Hodgkin Research Fellowship.

The data underlying this article will be shared on reasonable request to the corresponding author.

REFERENCES

Abazajian K. N. et al., 2016, CMB-S4 Science Book, 1st edn. preprint ([arXiv:1610.02743](https://arxiv.org/abs/1610.02743))
 Abbott B. P. et al., 2017a, *Phys. Rev. Lett.*, 118, 121101
 Abbott B. et al., 2017b, *ApJ*, 848, L12
 Abbott B. et al., 2017c, *ApJ*, 848, L13
 Abbott B. P. et al., 2019, *Phys. Rev. D*, 100, 061101
 Ade P. A. R. et al., 2021, *Phys. Rev. Lett.*, 127, 151301
 Aghanim N. et al., 2020a, *A&A*, 641, A5
 Aghanim N. et al., 2020b, *A&A*, 641, A6
 Aghanim N. et al., 2020c, *A&A*, 641, A8
 Aich M., Ma Y.-Z., Dai W.-M., Xia J.-Q., 2020, *Phys. Rev. D*, 101, 063536
 Akita K., Yamaguchi M., 2020, *JCAP*, 08, 012
 Akrami Y. et al., 2020a, *A&A*, 641, A10
 Akrami Y. et al., 2020b, *A&A*, 641, A10
 Akrami Y. et al., 2020c, *A&A*, 643, A42
 Alam S. et al., 2021, *Phys. Rev. D*, 103, 083533
 Allen B., Romano J. D., 1999, *Phys. Rev. D*, 59, 102001
 Alpher R. A., Bethe H., Gamow G., 1948, *Phys. Rev.*, 73, 803
 Ashoorioon A., Hovdebo J. L., Mann R. B., 2005, *Nucl. Phys. B*, 727, 63
 Ashoorioon A., Dimopoulos K., Sheikh-Jabbari M. M., Shiu G., 2014, *Phys. Lett. B*, 737, 98
 Aver E., Olive K. A., Skillman E. D., 2015, *JCAP*, 07, 011
 Baldi M., Finelli F., Matarrese S., 2005, *Phys. Rev. D*, 72, 083504
 Barrow J. D., Mimoso J. P., de Garcia Maia M. R., 1993, *Phys. Rev. D*, 48, 3630
 Bartolo N. et al., 2016, *JCAP*, 12, 026
 Baumann D., Green D., Porto R. A., 2015, *JCAP*, 01, 016

Baumann D., Lee H., Pimentel G. L., 2016, *JHEP*, 01, 101
 Baumgart M., Heckman J. J., Thomas L., 2022, *JCAP*, 07, 034
 Ben-Dayan I., Keating B., Leon D., Wolfson I., 2019, *JCAP*, 06, 007
 Benetti M., Graef L. L., Vagnozzi S., 2022, *Phys. Rev. D*, 105, 043520
 Benini R., Lattanzi M., Montani G., 2011, *Gen. Rel. Grav.*, 43, 945
 Bennett J. J., Buldgen G., De Salas P. F., Drewes M., Gariazzo S., Pastor S., Wong Y. Y. Y., 2021, *JCAP*, 04, 073
 Benson B. A. et al., 2014, *Proc. SPIE Int. Soc. Opt. Eng.*, 9153, 91531P
 Boyle L. A., Buonanno A., 2008, *Phys. Rev. D*, 78, 043531
 Boyle L. A., Steinhardt P. J., 2008, *Phys. Rev. D*, 77, 063504
 Brustein R., Gasperini M., Giovannini M., Veneziano G., 1995, *Phys. Lett. B*, 361, 45
 Cabass G., Pagano L., Salvati L., Gerbino M., Giusarma E., Melchiorri A., 2016, *Phys. Rev. D*, 93, 063508
 Cai Y., Piao Y.-S., 2021, *Phys. Rev. D*, 103, 083521
 Cai Y., Piao Y.-S., 2022, *JHEP*, 06, 067
 Calcagni G., Kuroyanagi S., 2021, *JCAP*, 03, 019
 Caldwell R. R., Smith T. L., Walker D. G. E., 2019, *Phys. Rev. D*, 100, 043513
 Caprini C., Figueroa D. G., 2018, *CQG*, 35, 163001
 Capurri G., Bartolo N., Maino D., Matarrese S., 2020, *JCAP*, 11, 037
 Carezza P., Lattanzi M., Mirizzi A., Forastieri F., 2021, *JCAP*, 07, 031
 Cheung C., Creminelli P., Fitzpatrick A. L., Kaplan J., Senatore L., 2008, *JHEP*, 03, 014
 Clarke T. J., Copeland E. J., Moss A., 2020, *JCAP*, 10, 002
 Consiglio R., de Salas P. F., Mangano G., Miele G., Pastor S., Pisanti O., 2018, *Comput. Phys. Commun.*, 233, 237
 Cooke R. J., Pettini M., Steidel C. C., 2018, *ApJ*, 855, 102
 Creminelli P., Luty M. A., Nicolis A., Senatore L., 2006, *JHEP*, 12, 080
 Creminelli P., Gleyzes J., Noreña J., Vernizzi F., 2014, *Phys. Rev. Lett.*, 113, 231301
 Cyburt R. H., Fields B. D., Olive K. A., Skillman E., 2005, *Astropart. Phys.*, 23, 313
 D’Eramo F., Schmitz K., 2019, *Phys. Rev. Research.*, 1, 013010
 D’Eramo F., Di Valentino E., Giarè W., Hajkarim F., Melchiorri A., Mena O., Renzi F., Yun S., 2022, *JCAP*, 09, 022
 de Salas P. F., Pastor S., 2016, *JCAP*, 07, 051
 Dent J. B., Krauss L. M., Sabharwal S., Vachaspati T., 2013, *Phys. Rev. D*, 88, 084008
 Easterher R., Kinney W. H., 2003, *Phys. Rev. D*, 67, 043511
 Essinger-Hileman T. et al., 2014, *Proc. SPIE Int. Soc. Opt. Eng.*, 9153, 91531I
 Forconi M., Giarè W., Di Valentino E., Melchiorri A., 2021, *Phys. Rev. D*, 104, 103528
 Franciolini G., Giudice G. F., Racco D., Riotto A., 2019, *JCAP*, 05, 022
 Freese K., Sfakianakis E. I., Stengel P., Visinelli L., 2018, *JCAP*, 05, 067
 Friedman B. C., Cooray A., Melchiorri A., 2006, *Phys. Rev. D*, 74, 123509
 Froustey J., Pitrou C., Volpe M. C., 2020, *JCAP*, 12, 015
 Galloni G., Bartolo N., Matarrese S., Migliaccio M., Ricciardone A., Vittorio N., 2022, preprint ([arXiv:2208.00188](https://arxiv.org/abs/2208.00188))
 Gariazzo S., de Salas P. F., Pisanti O., Consiglio R., 2021, *Comput. Phys. Commun.*, 271, 108205
 Gasperini M., Giovannini M., 1992, *Phys. Lett. B*, 282, 36
 Giarè W., Melchiorri A., 2021, *Phys. Lett. B*, 815, 136137
 Giarè W., Renzi F., 2020, *Phys. Rev. D*, 102, 083530
 Giarè W., Di Valentino E., Melchiorri A., 2019, *Phys. Rev. D*, 99, 123522
 Giarè W., Renzi F., Melchiorri A., 2021, *Phys. Rev. D*, 103, 043515
 Giarè W., Renzi F., Melchiorri A., Mena O., Di Valentino E., 2022, *MNRAS*, 511, 1373
 Giovannini M., 2016, *Class. Quant. Grav.*, 33, 125002
 Giovannini M., 2018b, *Phys. Rev. D*, 98, 103509
 Giovannini M., 2019, *Phys. Lett. B*, 789, 502
 Giudice G. F., Kolb E. W., Riotto A., 2001a, *Phys. Rev. D*, 64, 023508
 Giudice G. F., Kolb E. W., Riotto A., Semikoz D. V., Tkachev I. I., 2001b, *Phys. Rev. D*, 64, 043512
 Graef L. L., Benetti M., Alcaniz J. S., 2019, *Phys. Rev. D*, 99, 043519
 Grayson J. A. et al., 2016, *Proc. SPIE Int. Soc. Opt. Eng.*, 9914, 99140S
 Guth A. H., 1981, *Phys. Rev. D*, 23, 347
 Guzzetti M. C., Bartolo N., Liguori M., Matarrese S., 2016, *Riv. Nuovo Cim.*, 39, 399

Hannestad S., 2004, *Phys. Rev. D*, 70, 043506
Hasegawa T., Hiroshima N., Kohri K., Hansen R. S. L., Tram T., Hannestad S., 2019, *JCAP*, 12, 012
Hasegawa T., Hiroshima N., Kohri K., Hansen R. S. L., Tram T., Hannestad S., 2020, *JCAP*, 08, 015
Henderson S. W. et al., 2016, *J. Low. Temp. Phys.*, 184, 772
Hoffman M. B., Turner M. S., 2001, *Phys. Rev. D*, 64, 023506
Kamionkowski M., Kovetz E. D., 2016, *Ann. Rev. A&A*, 54, 227
Kawasaki M., Kohri K., Sugiyama N., 1999, *Phys. Rev. Lett.*, 82, 4168
Kawasaki M., Kohri K., Sugiyama N., 2000, *Phys. Rev. D*, 62, 023506
Kawasaki M., Kohri K., Moroi T., 2005, *Phys. Rev. D*, 71, 083502
Khouri J., Steinhardt P. J., 2011, *Phys. Rev. D*, 83, 123502
Kinney W. H., 2002, *Phys. Rev. D*, 66, 083508
Kinney W. H., 2021, *Open J. Astrophys.*, 4, 5
Kuroyanagi S., Takahashi T., Yokoyama S., 2015, *JCAP*, 02, 003
Linde A. D., 1982, *Phys. Lett. B*, 108, 389
Litsa A., Freese K., Sfakianakis E. I., Stengel P., Visinelli L., 2021, *Phys. Rev. D*, 104, 123546
Liu X.-J., Zhao W., Zhang Y., Zhu Z.-H., 2016, *Phys. Rev. D*, 93, 024031
Lyth D. H., Liddle R. A., 2009, *The primordial density perturbation: cosmology, inflation and the origin of structure*. Cambridge Univ. Press, Cambridge
Maggiore M., 2000, *Phys. Rept.*, 331, 283
Ma C.-P., Bertschinger E., 1995, *ApJ*, 455, 7
Mangano G., Miele G., Pastor S., Pinto T., Pisanti O., Serpico P. D., 2005, *Nucl. Phys. B*, 729, 221
Martin J., Ringeval C., Vennin V., 2014, *Phys. Dark Universe*, 5, 75
Meerburg P. D., Hložek R., Hadzhiyska B., Meyers J., 2015, *Phys. Rev. D*, 91, 103505
Mukohyama S., Namba R., Peloso M., Shiu G., 2014, *JCAP*, 08, 036
Namba R., Peloso M., Shiraishi M., Sorbo L., Unal C., 2016, *JCAP*, 01, 041
Noumi T., Yamaguchi M., 2014, preprint (arXiv:1403.6065)
Odintsov S. D., Oikonomou V. K., 2022, *Fortsch. Phys.*, 70, 2100167
Odintsov S. D., Oikonomou V. K., Fronimos F. P., 2021, *Annals Phys.*, 424, 168359
Odintsov S. D., Oikonomou V. K., Myrzakulov R., 2022, *Symmetry*, 14, 729
Oikonomou V. K., 2021, *Class. Quant. Grav.*, 38, 195025
Oikonomou V. K., 2023, *Astropart. Phys.*, 144, 102777
Ota A., Sasaki M., Wang Y., 2022, preprint (arXiv:2209.02272)
Peimbert A., Peimbert M., Luridiana V., 2016, *Rev. Mex. Astron. Astrofis.*, 52, 419
Peloso M., Sorbo L., Unal C., 2016, *JCAP*, 09, 001
Peng Z.-Z., Fu C., Liu J., Guo Z.-K., Cai R.-G., 2021, *JCAP*, 10, 050
Pi S., Sasaki M., Zhang Y.-I., 2019, *JCAP*, 06, 049
Pisanti O., Cirillo A., Esposito S., Iocco F., Mangano G., Miele G., Serpico P. D., 2008, *Comput. Phys. Commun.*, 178, 956
Pitrou C., Coc A., Uzan J.-P., Vangioni E., 2018, *Phys. Rept.*, 754, 1
Pritchard J. R., Kamionkowski M., 2005, *Annals Phys.*, 318, 2
Sabti N., Alvey J., Escudero M., Fairbairn M., Blas D., 2020, *JCAP*, 01, 004
Smith T. L., Pierpaoli E., Kamionkowski M., 2006, *Phys. Rev. Lett.*, 97, 021301
Starobinsky A. A., 1980, *Phys. Lett. B*, 91, 99
Steigman G., 2007, *Ann. Rev. Nucl. Part. Sci.*, 57, 463
Stewart A., Brandenberger R., 2008, *JCAP*, 08, 012
Suzuki A. et al., 2018, *J. Low Temp. Phys.*, 193, 1048
Tristram M. et al., 2022, *Phys. Rev. D*, 105, 083524
Ungarelli C., Corasaniti P., Mercer R. A., Vecchio A., 2005, *Class. Quant. Grav.*, 22, S955
Vagnozzi S., 2021, *MNRAS*, 502, L11
Vagnozzi S., Loeb A., 2022, *ApJ*, 939, L5
Vilenkin A., 1983, *Phys. Rev. D*, 27, 2848
Watanabe Y., Komatsu E., 2006, *Phys. Rev. D*, 73, 123515
Weinberg S., 2004, *Phys. Rev. D*, 69, 023503
Weinberg S., 2008, *Phys. Rev. D*, 77, 123541
Zyla P. A. et al., 2020, *PTEP*, 2020, 083C01
Özsoy O., 2021, *JCAP*, 04, 040

APPENDIX A: GRAVITATIONAL RADIATION IN THE EARLY UNIVERSE

We consider a spatially flat FLRW metric, whose perturbed line element in synchronous gauge reads (Ma & Bertschinger 1995)

$$ds^2 = a^2(\eta) [d\eta^2 - (\delta_{ij} + h_{ij}) dx^i dx^j] \quad (\text{A1})$$

with a and η denoting scale factor and conformal time, respectively. In this picture, generic tensor perturbations are described by the transverse and traceless part of the symmetric 3×3 matrix h_{ij} . In the Fourier space, focusing on one particular polarization state and a given mode k , the gravitational wave field satisfies the usual equation of motion⁵ (Lyth & Liddle 2009)

$$h_k'' + 2\mathcal{H}h_k' + k^2 h_k = 0 \quad (\text{A2})$$

where the prime indicates the derivative with respect to the conformal time and $\mathcal{H} = a'/a$. Since here we are mainly interested in primordial gravitational waves, it is particularly convenient to characterize the gravitational field in terms of its power spectrum

$$\mathcal{P}_T(k) = \frac{2k^3}{\pi} |h_k(\eta_i)|^2 \quad (\text{A3})$$

where, for each mode k , $h_k(\eta_i)$ specifies the value of the field at some initial conformal time η_i . In this way, connecting this picture to inflation simply requires identifying the power spectrum of the gravitational field with the primordial spectrum of inflationary tensor modes.

In the early Universe, a satiable background of gravitational waves will clearly increase the energy budget by providing an additional form of radiation. Here, we parametrize this contribution in terms of corrections to the effective number of relativistic degrees of freedom N_{eff} . Within the Standard Model of particle physics, this parameter acquires the reference value of $N_{\text{eff}} = 3.044$ (Mangano et al. 2005; de Salas & Pastor 2016; Akita & Yamaguchi 2020; Froustey, Pitrou & Volpe 2020; Bennett et al. 2021), counting three different families of relativistic neutrinos plus an additional contribution coming from the non-instantaneous neutrino decoupling. To understand how this reference value is modified in presence of additional gravitational radiation, we focus on temperatures $T \gtrsim \mathcal{O}(1)$ MeV when the relativistic species in the Universe were electrons (and their antiparticles, positrons) e^\pm , neutrinos ν , and photons γ . Including also the contributions of gravitons, the total amount of radiation will read (Maggiore 2000)

$$\rho_{\text{rad}} = \frac{\pi^2}{30} \left[2T_\gamma^4 + \frac{7}{4}T_{e^\pm}^4 + \frac{7}{4}N_{\text{eff}}T_\nu^4 + 2T_{\text{GW}}^4 \right] \quad (\text{A4})$$

where the factor 2 in front of T_{GW} counts the two different polarization states (+, ×) of tensor perturbations. Apart from the gravitons, all the other species were in thermal equilibrium and shared the same temperature: $T_\gamma = T_{e^\pm} = T_\nu$. Therefore, it is straightforward to see that we can describe gravitational radiation as an additional

⁵It should be noted that, as is commonly done in the literature, equation (A2) does not take into account the damping of primordial gravitational waves caused by an anisotropic stress tensor. This is relevant for free-streaming thermal neutrinos and other high-energy particles (see e.g. Weinberg 2004; Watanabe & Komatsu 2006; Benini, Lattanzi & Montani 2011; Dent et al. 2013). Although these corrections are typically small (around 30 per cent), the damping can potentially affect the power-law behaviour of the tensor spectrum. However, incorporating these effects into the current analysis is not straightforward and beyond the scope of this work. This issue is left suitable for future studies.

contribution to the effective number of relativistic species

$$\Delta N_{\text{eff}}^{\text{GW}} = \frac{8}{7} \frac{T_{\text{GW}}^4}{T_\gamma^4} = \frac{8}{7} \frac{\rho_{\text{GW}}}{\rho_\gamma} \Big|_{T_\gamma \gtrsim \mathcal{O}(1) \text{ MeV}} \quad (\text{A5})$$

To rescale this contribution to the present time, we must consider that after $T \gtrsim \mathcal{O}(1) \text{ MeV}$, as the Universe expands, the gravitational wave energy density decays as $\rho_{\text{GW}} \sim 1/a^4$, while, assuming entropy conservation, the CMB photon energy density evolves as $\rho_\gamma \sim 1/(a^4 g_{*s}^{4/3})$ with g_{*s} the number of entropic degrees of freedom. Therefore, the present-day contribution will be given by

$$\Delta N_{\text{eff}}^{\text{GW}} = \frac{8}{7} \left(\frac{g_{*s}(T \gtrsim 1 \text{ MeV})}{g_{*s}(T_0)} \right)^{\frac{4}{3}} \frac{\rho_{\text{GW}}}{\rho_\gamma} \Big|_{\text{Today}} \quad (\text{A6})$$

with $g_{*s}(T_0) \simeq 3.91$, the current number of entropic degrees of freedom.

While the present Cosmic Microwave Background energy density ρ_γ is accurately measured (Aghanim et al. 2020a, b; Akrami et al. 2020b), the present-day fraction of the energy budget of the Universe in gravitational radiation (i.e. the ratio between the present GW energy density ρ_{GW} and the critical density, $\rho_c = 3H^2/8\pi G$), can be easily computed by integrating the spectrum over all scales (Maggiore 2000; Boyle & Steinhardt 2008; Guzzetti et al. 2016)

$$\Omega_{\text{GW}} = \frac{1}{12H_0^2} \int d \ln k \mathcal{P}_T(k) \dot{T}(\eta_0, k)^2 \quad (\text{A7})$$

where the contribution of each mode is weighted by (the time derivative of) the so-called transfer function

$$T(\eta, k) = \frac{h_k(\eta)}{h_k(\eta_i)} \quad (\text{A8})$$

that takes into account the different time evolution of modes with different k according to equation (A2). Assuming that inflation is followed by a standard Hot Big Bang Theory evolution, (i.e. by radiation, matter, and dark energy dominated epochs), the transfer function admits relatively simple semi-analytic solutions and we can estimate the present time contribution at generic frequency $f = k/2\pi$ as (; Bartolo et al. 2016; Cabass et al. 2016; Stewart & Brandenberger 2008; Graef, Benetti & Alcaniz 2019; Liu et al. 2016)

$$\Omega_{\text{GW}}(f) \simeq \frac{\mathcal{P}_T(f)}{24z_{\text{eq}}} \quad (\text{A9})$$

with $z_{\text{eq}} \simeq 3400$, the redshift at equivalence and \mathcal{P}_T the spectrum of primordial tensor modes. By using equation (A9), putting everything together, we finally get (Maggiore 2000)

$$\Delta N_{\text{eff}}^{\text{GW}} \simeq \frac{h_0^2}{5.6 \times 10^{-6}} \left(\frac{1}{24z_{\text{eq}}} \right) \int_{f_{\text{min}}}^{f_{\text{max}}} \frac{df}{f} \mathcal{P}_T(f) \quad (\text{A10})$$

recovering the standard result that Gravitational Waves contribute to the effective number of relativistic species through the logarithmic integral of their power spectrum over frequencies.

APPENDIX B: UPDATED BBN BOUNDS ON INFLATION

To enrich and support the analysis carried out in the manuscript, we devote this appendix to the detailed study of the observational constraints on blue-tilted models inflation resulting from the big bang nucleosynthesis epoch. Our aim is twofold: we first update the state-of-the-art results in light of the most recent cosmological observations. Then, retracing the discussion of Section 2, we quantify

how such results change with the parametrization of the primordial tensor spectrum.

We start recalling that the big bang nucleosynthesis (Alpher, Bethe & Gamow 1948) explains the formation of the first light nuclei heavier than the lightest isotope of hydrogen by a solid understanding of the nuclear interactions involved in their production. It also provides a natural arena to test and constrain extensions to both cosmology and fundamental physics since any proposed model of the early Universe must be able to explain the abundances of light elements inferred by astrophysical and cosmological observations. The reason why the BBN constraining power can be applied to the analysis of blue-tilted models of inflation is quite straightforward: according to the Friedmann equation, additional gravitational radiation (that we parametrized in terms of $\Delta N_{\text{eff}}^{\text{GW}}$) will increase the expansion rate of the Universe $H(z)$. A faster expansion leads to a higher freeze-out temperature of the weak interactions, implying a higher fraction of primordial Helium and Deuterium, as well as a higher fraction of other primordial elements. This makes BBN an extremely powerful and quite general tool for constraining the total amount of relativistic species in the Universe, with several implications for physics beyond the Standard Model (Cyburt et al. 2005; Kawasaki, Kohri & Moroi 2005; Steigman 2007; Sabti et al. 2020; D’Eramo et al. 2022), the Neutrino flavour physics and, in our case, the inflationary cosmology.

It is instructive to start our analysis by assuming a power-law primordial spectrum given by equation (1). This simple parametrization has the benefit that all the models are described only by two quantities: the amplitude r and the tilt n_T . We randomly sample $N = 10^6$ linearly distributed values of the amplitude and the tilt in the ranges $r \in [0, 0.1]$ and $n_T \in [-2, 2]$, respectively. For each of these points, we compute the contribution to the effective number of relativistic species $\Delta N_{\text{eff}}^{\text{GW}}(r, n_T)$ by equation (2). Finally, we randomly sample N values of the baryon energy density in the range $\Omega_b h^2 \in [0.020, 0.025]$ and create a grid in the plane $(\Delta N_{\text{eff}}^{\text{GW}}, \Omega_b h^2)$ similar to those usually obtained within the Monte Carlo methods. Then, we solve numerically the set of differential equations that regulate the BBN nuclear interactions in the primordial plasma (Pisanti et al. 2008; Pitrou et al. 2018; Consiglio et al. 2018; Gariazzo et al. 2021). To do so, we made use of the code PARTHENOPE (Gariazzo et al. 2021). Fixing the values of the neutron lifetime,⁶ for each point in the $(\Delta N_{\text{eff}}, \Omega_b h^2)$ plane the code computes the corresponding value of the primordial Helium fraction Y_p , the Deuterium abundance D/H and all the other light element abundances. In this way, we can directly compare the results with the values inferred by astrophysical and cosmological observations. In this regard, our baseline data set for the BBN analyses consists of:

(i) Two independent measurements of the primordial Helium fraction, $Y_p = 0.2449 \pm 0.0040$ (Aver, Olive & Skillman 2015) and $Y_p = 0.2446 \pm 0.0029$ (Peimbert, Peimbert & Luridiana 2016).

(ii) A per cent determination of the primordial Deuterium abundance $D/H = (2.527 \pm 0.030) \cdot 10^{-5}$ based on six high precision and homogeneously analysed D/H measurements from (Cooke, Pettini & Steidel 2018).

(iii) The value of the baryon energy density parameter $\Omega_b h^2 = 0.0224 \pm 0.0001$ from the final 2018 Planck data release of temperature and polarization CMB angular power spectra (Aghanim et al. 2020b).

⁶The neutron lifetime is fixed to $\tau_n = 879.4 \text{ s}$, corresponding to the latest measurement reported by the Particle Data Group ($\tau_n = 879.4 \pm 0.6 \text{ s}$) (Zyla et al. 2020)

Table B1. Results inferred from BBN primordial abundances. The constraints on $\Omega_b h^2$, Y_p , and $10^5 \cdot (D/H)$ are given at 68 percent CL while the upper bounds on ΔN_{eff} and n_T are given at 95 per cent CL (99 per cent CL). The horizontal lines divide the constraints on the BBN parameters (that are not sensitive to the model of inflation) from those inferred for the inflationary parameters under the two different parametrizations of the spectrum indicated in the table. A BK18 prior ($r < 0.037$ at 95 per cent CL) is assumed on the tensor amplitude.

Parameter	BBN-A ($Y_p + D/H$)	BBN-B ($Y_p + \Omega_b h^2$)	BBN-C ($Y_p + D/H + \Omega_b h^2$)
$\Omega_b h^2$	0.02234 ± 0.00017	0.02240 ± 0.00010	0.022382 ± 0.000086
Y_p	0.24558 ± 0.00010	0.24561 ± 0.00010	$0.245591^{+0.000015}_{-0.000060}$
$(D/H) \cdot 10^{-5}$	2.527 ± 0.030	2.516 ± 0.020	2.519 ± 0.016
ΔN_{eff}	< 0.33 (< 0.40)	< 0.32 (< 0.40)	< 0.16 (< 0.21)
Constraints on Inflation inferred by assuming equation (1)			
n_T	< 0.324 (< 0.376)	< 0.323 (< 0.374)	< 0.32 (0.368)
r	< 0.037	< 0.037	< 0.037
Constraints on Inflation inferred by assuming equation (4)			
n_T	< 1.80 (unc.)	< 1.80 (unc.)	< 1.80 (unc.)
r	< 0.037	< 0.037	< 0.037

(iv) A prior on the tensor amplitude $r < 0.037$ at 95 per cent CL coming from a combination of the final 2018 Planck data release of temperature and polarization CMB angular power spectra (Aghanim et al. 2020b) and the B-modes 2018 likelihood from the Bicep Collaboration (Ade et al. 2021).

We apply these priors on the BBN abundances, reweighting the contributions of the points by means of an ‘importance sampling’ statistical method as done in D’Eramo et al. 2022. Consequently, we obtain informative posterior distributions for the most interesting parameters to be inferred by observations. We summarize the results in Table B1, while Fig. B1 provides the marginalized posterior distributions of parameters.

We start by adopting a prior knowledge of the total amount of the primordial Helium Y_p and Deuterium D/H from direct astrophysical measurements, together with a prior on the tensor amplitude from the BK18 likelihood for B-modes polarization. Therefore, in this case, the free parameters of the sample to be inferred by observations are the baryon energy density and the tensor tilt (the last one to be inferred by the total amount of extra radiation ΔN_{eff}). We refer to this data set as ‘BBN-A’. From it, we derive an upper limit on the additional radiation allowed during BBN epoch of $\Delta N_{\text{eff}} < 0.3$ at 95 per cent CL ($\Delta N_{\text{eff}} < 0.4$ at 99 per cent CL), in perfect agreement with the previous results discussed in the literature (Aver et al. 2015; Peimbert et al. 2016; Cooke et al. 2018; Aich et al. 2020; D’Eramo et al. 2022; Giarè et al. 2022). Assuming all this contribution to be made of primordial gravitational waves, we infer an upper limit on the tensor tilt $n_T < 0.3$ at 95 per cent CL ($n_T < 0.4$ at 99 per cent CL), which is in line with what we argued in Section 2 and with the most recent CMB-analyses (Galloni et al. 2022).

We test the robustness of our result by considering different combinations of data. In particular, we now impose a prior knowledge on the baryon-energy density $\Omega_b h^2$ as inferred by the Planck collaboration analysing the last release of the CMB data (Aghanim et al. 2020b) together with the information on the amount of the primordial Helium Y_p . We label this case ‘BBN-B’. The free parameters to be determined now are D/H and n_T . We find that the constraints on the effective number of relativistic degrees of freedom remain basically unchanged with respect to the previous case and so does the limit on the tensor tilt. Notice that, while in this

case, we are more dependent on the physics at the recombination epoch, we are relaxing the bound on Deuterium. So, we can use the value inferred for this parameter as a consistency check of our analysis, resulting in a great agreement among the different data-combinations.

Finally, for completeness, we combine all these priors together ($Y_p + D/H + \Omega_b h^2 + r$). We refer to this dataset as BBN-C. As already pointed out in Ref. (D’Eramo et al. 2022), assuming all this information leads to an improvement in the constraining power on additional radiation with the limit now reading $\Delta N_{\text{eff}} < 0.16$ at 95 per cent CL ($\Delta N_{\text{eff}} < 0.21$ at 99 per cent CL). Interestingly, this improvement is not transferred into the bound on the tensor tilt which in fact remains basically unchanged with respect to the previous cases. The reason underlying this lack of improvement can be easily understood by looking at the black dashed line in Fig. 1. This line represents the contribution to ΔN_{eff} resulting from a blue-tilted power-law spectrum that exponentially grows for positive n_T . As evident from the figure, when we are close $n_T \sim 0.4$ the line in the plane $(n_T, \Delta N_{\text{eff}})$ becomes almost horizontal. This means that a variation on the y -axis (ΔN_{eff}) does not produce any significant movement in the x -axis direction (n_T), explaining why we do not get a more tight limits on the tensor tilt.

Aiming to quantify the impact on the results from having assumed a vanishing running $\alpha_T = 0$, we repeat the same analysis by allowing α_T to vary in the range $\alpha_T \in [-0.2, 0.2]$. In this case, we parametrize the primordial spectrum through equation (4). We summarize the results inferred for the different data sets in Table B1.

Clearly, both the bounds on the total amount of additional radiation allowed during the BBN epoch and the results on the primordial light element abundance do not change with respect to the previous analysis since they do not depend on the parametrization adopted for the tensor spectrum. Instead, what changes is the limit that we can infer from these limits on the inflationary parameters. In particular, opening to the running completely relaxes the upper limit on positive tensor tilt. This parameter is now constrained to be $n_T < 1.8$ at 95 per cent while it is unbounded at 99 per cent CL. This is due to the strong degeneracy between the tilt and its running, see also their 2D joint marginalized contours shown in Fig. B2. As already discussed in Section 2, a positive running will amplify the power in gravitational waves on small scales, basically miming

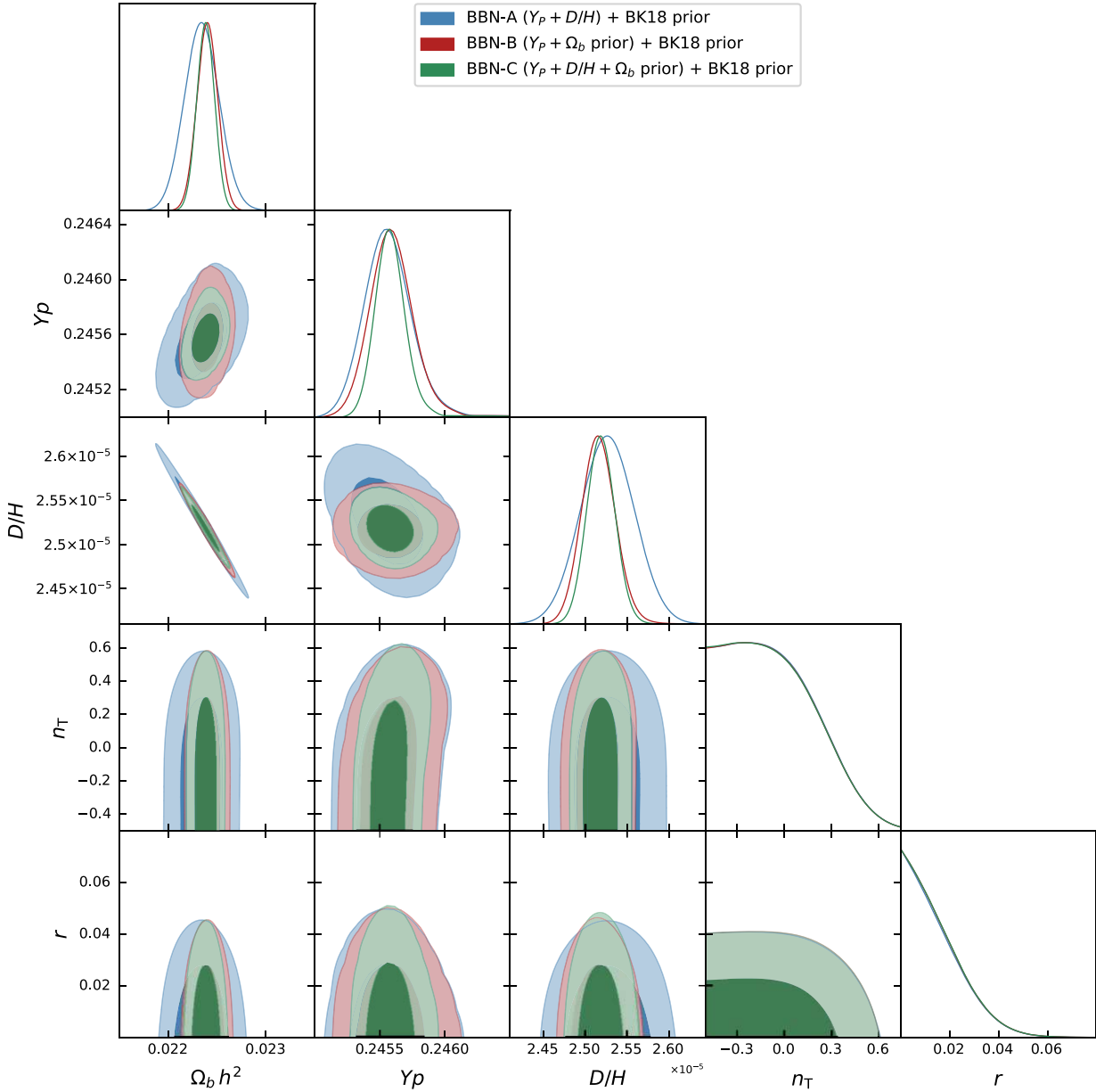


Figure B1. Two-dimensional 68 and 95 per cent CL allowed regions and one-dimensional probability posterior distributions for the most relevant cosmological parameters obtained under the assumption of a power-law spectrum, equation (1). The different colours refer to the different data combinations here considered for BBN analyses, see Table B1.

the effect of a larger scalar tilt. So, when the running becomes positive and relatively large, the tensor tilt is only allowed to be either very close to zero or negative, see Fig. B2. Conversely, when the running acquires negative values it strongly reduces the power in gravitational waves and compensates the effect of a larger n_T . This is why the bounds on the tensor tilt are more relaxed in the region of negative runnings as clearly visible in Fig. B2. These results are in line with what is argued in Section 2 and confirm, one more

time, that the parametrization adopted for the tensor spectrum is in fact crucial when extrapolating constraints on blue-tilted models of inflation.

We conclude by pointing out that if also a running of running is allowed to vary in the sample, the tensor tilt is completely unbounded. By extension, this applies to all the other higher order parametrizations that involve more than two free parameters.

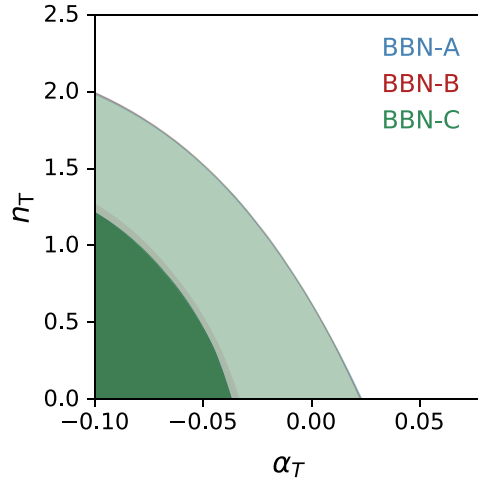


Figure B2. 2D joint marginalized contours in the (α_T, n_T) plane obtained by allowing a non-vanishing running $\alpha_T = dn_T/d\log k$ to vary (see equation 4).

This paper has been typeset from a $\text{\TeX}/\text{\LaTeX}$ file prepared by the author.