Extensive Measurement in Social Choice

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May 23, 2023

Abstract

Extensive measurement is the standard measurement-theoretic approach for constructing a ratio scale. It involves the comparison of objects that can be "concatenated" in an additively representable way. This paper studies the implications of extensively measurable welfare for social choice theory. We do this in two frameworks: an Arrovian framework with a fixed population and no interpersonal comparisons, and a generalized framework with variable populations and full interpersonal comparability. In each framework we use extensive measurement to introduce novel domain restrictions, independence conditions, and constraints on social evaluation. We prove a welfarism theorem for the resulting domains and characterize the social welfare functions that satisfy the axioms of extensive measurement at both the individual and social levels. The main results are simple axiomatizations of strong dictatorship in the Arrovian framework and classical utilitarianism in the generalized framework. We conclude by drawing some lessons regarding the utilitarian significance of Harsanyi's aggregation theorem.

1 Introduction

Kenneth Arrow once called himself "a kind of utilitarian manqué":

> I'd like to be utilitarian but the only problem is I have nowhere those utilities come from. The problem I have with utilitarianism is … that the epistemological foundations are weak. My problem is: What are those objects we are adding

*I am very grateful to Zachary Goodsell, Marcus Pivato, Robert Raschka, and John Weymark for helpful comments and discussion.

up? I have no objection to adding them up if there's something to add. Kelly and Arrow, 1987, p. 59

The content of Arrow's complaint is not entirely transparent. In the orthodox economic sense of "utility," an Arrovian social choice theorist who takes individuals to have numerically representable preferences certainly has somewhere that "utilities come from": a person's utility is just the numerical value of a function that represents her preferences. There is no mystery about how such utilities can be added together: they're just numbers, and we can add whatever numbers we like.

Arrow's complaint cannot be that he lacks a foundation for the numerical representation of preferences. I take his complaint to be this. A utilitarian—more specifically, a *classical* (or "total") utilitarian—believes that we should maximize the sum of *well-being*, where a person's well-being is how good things are for her. (We leave open exactly what makes things good for people.) But what does it mean to "add up" people's well-beings? A person's well-being is not a number, any more than her height or weight is a number. Some properties can, intuitively, be added together: we can add together two heights, or two masses. But we cannot add heights to masses. And it is unclear what it would mean to add up degrees of beauty or of intelligence. Arrow's complaint—or, rather, one complaint inspired by his remarks—is that the classical utilitarian has not shown well-being to be the kind of thing that, like height or mass, can be added up, as opposed to the kind of thing, like beauty or intelligence, that cannot.

Extensive measurement offers a way to precisify this contrast. The idea of extensive measurement is to compare objects that can be "concatenated," or combined, to yield new objects. If the comparison of concatenated objects satisfies certain axioms, it can be represented as maximizing a real-valued function with the concatenation operation represented by the arithmetic operation of addition (Krantz et al., 1971). A classic example is the measurement of length by stacking together rods from end to end, or of mass by stacking together objects in a weightpan.

There are various ways of trying to apply extensive measurement to well-being, which differ based on what kinds of objects are evaluated and how they are concatenated (Nebel, forthcoming). Each of these methods depends on controversial assumptions about what is good for people. It is therefore, in my view, an open question whether or not well-being is susceptible to extensive measurement. In this paper, I want to assume that it is, and thus that well-being can be meaningfully "added up," in order to study the implications of extensive measurement for social choice and welfare theory. In particular, I want to understand what further commitments are necessary and sufficient to characterize classical utilitarianism, once it is granted that well-being is extensively measurable.

We do this by developing a variable-population generalization of Arrow's framework of social welfare functions in which well-being is both interpersonally comparable and extensively measurable. The variable-population aspect of this framework is essential, since what distinguishes classical utilitarianism from other varieties of utilitarianism (e.g., average utilitarianism) is its variable-population commitments. However, we warm up to this complex framework by first applying extensive measurement in a simpler Arrovian setting, in which the population is fixed and well-being is not interpersonally comparable. We provide some limitative results for Arrovian social welfare functions when individual welfare is extensively measurable; these results motivate the use of interpersonal comparisons in the generalized framework.

This project is especially indebted to three others. One is the study of social welfare functionals with ratio-scale measurable utilities (see especially Blackorby & Donaldson, 1982; Tsui & Weymark, 1997). Extensive measurement is the standard measurement-theoretic approach for constructing a ratio scale; indeed, early work on measurement tended to assume that extensive measurement was the *only* fundamental form of measurement (Campbell, 1920; Cohen & Nagel, 1934). It is surprising, then, that while the implications of ratio-scale measurability for social welfare evaluation have been studied, apparently no attention has been paid to extensive structures themselves in social choice theory. A second project is the axiomatization of variable-population ethical principles pioneered by Blackorby et al. (2005). Curiously, however, there doesn't seem to be an axiomatization of classical utilitarianism in this work; they end up, rather, with various generalizations of classical utilitarianism.[1] Third is the broadly "relational" approach to welfare aggregation initiated by Arrow (1951), in which the primitive ingredients of social welfare evaluation are relations on a set of alternatives and axioms are formulated in terms of those relations—in contrast to, for ex-

[1] The closest results are their axiomatizations of "classical means of order r" (discussed in Appendix E below), ex-ante critical-level utilitarian principles (Blackorby et al., 2005, ch. 7), and classical generalized utilitarianism (Blackorby et al., 2005, chs. 6, 9). There are, of course, several axiomatizations of fixed-population utilitarian social welfare functionals (d'Aspremont & Gevers, 1977; Deschamps & Gevers, 1978; Maskin, 1978), but these do not discriminate between classical and other variable-population varieties of utilitarianism. Indeed, they rest on informational invariance conditions that rule out classical utilitarianism (Blackorby et al. (1999), for example, extend Maskin's result to characterize average utilitarianism). Hammond (1988) derives a principle which formally resembles classical utilitarianism, but in later work he is careful to acknowledge the resemblance as "only formal" (Fleurbaey & Hammond, 2004, p. 1268); see also Hammond, 1991. The most direct axiomatization of classical utilitarianism I know of is provided by Xu (1990), which appears never to have been cited.

ample, Sen (1970)'s framework of social welfare functionals (and Blackorby et al.'s variable-population generalization thereof), in which numerical utilities are taken for granted and axioms are formulated in terms of that numerical representation. Other work in the "relational" tradition includes Brandl and Brandt (2020), Dhillon and Mertens (1999), Hammond (1976), Harsanyi (1955), Harvey (1999), Marchant (2019), Pivato (2015), and Raschka (2023), among many others. The use of extensive measurement in this tradition, however, appears to be novel.[2]

The plan is as follows. I introduce the axioms of extensive measurement in section 2. In sections 3 and 4, I consider Arrovian social welfare functions under the assumption that individual well-being is extensively measurable. The main result of section 3 is a welfarism theorem for this setting (Theorem 1). The main results of section 4 are an impossibility theorem for anonymous Paretian evaluation (Theorem 2) and a characterization of strongly dictatorial social welfare functions (Theorem 3). In section 5, I generalize the Arrovian framework to accommodate both interpersonal comparisons and variable populations and prove a welfarism theorem for this setting (Theorem 4). Section 6 provides the promised axiomatization of classical utilitarianism (Theorem 5). Section 7 concludes by drawing some lessons regarding the "social aggregation theorem" of Harsanyi (1955) and its relevance to utilitarianism. All proofs are in appendices.

2 Extensive Measurement

An extensive structure contains three ingredients. There is a set X of objects to be measured: for example, rods of differing lengths. There is a binary relation $\succcurlyeq$ on that set: for example, the *at least as long as* relation. (As usual, $\succ$ denotes the asymmetric part of $\succcurlyeq$, $\sim$ its symmetric part.) And there is a binary *concatenation* operation $\circ : X \times X \to X$ which, in some sense, combines the objects together: for example, by stacking together rods from end to end. Suppose that our set of objects is closed under this operation, so that we can concatenate any two elements of X to form a new element of X. This includes the concatenation of an object with itself, which can be interpreted in the case of length as stacking together perfect copies of that object. For any object $a \in X$, define $1a := a$ and, for any natural number $n > 1$, let $na := (n-1)a \circ a$, so that na is the concatenation of n perfect copies of a.

The triple $(X, \succcurlyeq, \circ)$ is called an *extensive structure* iff the following five axioms are satisfied

[2] As should by now be clear, this paper is unrelated to "extensive social choice" in the sense of Bossert et al. (2013), Ooghe and Lauwers (2005), and Roberts (1995).

for any $a, b, c, d \in X$.

Transitivity If $a \succcurlyeq b$ and $b \succcurlyeq c$, then $a \succcurlyeq c$.

Completeness $a \succcurlyeq b$ or $b \succcurlyeq a$.

Weak Associativity $a \circ (b \circ c) \sim (a \circ b) \circ c$.

Monotonicity $a \succcurlyeq b$ iff $a \circ c \succcurlyeq b \circ c$ iff $c \circ a \succcurlyeq c \circ b$.

Archimedean If $a \succ b$, then for any $c, d \in A$, there is some natural number n such that $na \circ c \succcurlyeq nb \circ d$.

These conditions are necessary and sufficient for a numerical representation of $\succcurlyeq$ that is additive with respect to concatenation—that is:

Proposition 1 (Krantz et al., 1971, ch. 3, Theorem 1). $(X, R, \circ)$ *is an extensive structure iff there is a function* $U : X \to \mathbb{R}$ *such that, for any* $a, b \in X$,

(i) $a \succcurlyeq b$ *iff* $U(a) \geq U(b)$, *and*

(ii) $U(a \circ b) = U(a) + U(b)$.

Another function U' *satisfies (i) and (ii) iff* $U' = kU$ *for some positive real number* k.

We call U an *additive representation* of $\succcurlyeq$. The last line of Proposition 1 says that such a representation is unique *up to similarity transformation*. This is the characteristic uniqueness condition of a ratio scale.

Nebel (forthcoming) explores various possible applications of this sort of structure to the measurement of well-being. For example, X might be a set of experiences. For any experiences $a, b \in X$, their concatenation $a \circ b$ is an experience in which one first undergoes a immediately followed by b. Structures like this have been proposed for the measurement of hedonic well-being by Kahneman et al. (1997) and Skyrms and Narens (2019). A more flexible structure takes the objects to be entire lives, which an impartial spectator might imagine living in sequence (Lewis, 1946; Nagel, 1970). An even more flexible structure takes the objects to be any states of affairs or properties which might be regarded as desirable or undesirable in a nonderivative way. For any such states of affairs a and b, their concatenation $a \circ b$ is the conjunction of two states which are just as good as a and b, and which are "evaluatively independent" in a certain sense (see Danielsson, 1997).

On any of these (or other) interpretations, the appropriateness of the axioms above may of course be questioned. My view is that their acceptability depends on controversial, substantive questions about well-being, so that they should be regarded as open hypotheses worthy of further investigation. In particular, their acceptability should depend on the resulting implications for social welfare evaluation. One motivation for this paper is to draw out some of those implications for assessment. The frameworks developed below, importantly, do not presuppose any particular interpretation of welfare concatenation (or even that welfare is extensively measurable somehow or other—they may have economic applications regardless of our theory of welfare).

An important difference between length and well-being is that, in the case of length, all values of an additive representation are positive. Formally, this is captured by an additional *positivity* axiom, which requires that $a \circ b \succ a$ for all $a, b \in X$. That axiom makes $(X, \succcurlyeq, \circ)$ a *positive* extensive structure. Though this case is of formal interest, we do not impose this restriction here, because we find it hard to think of a conception of welfare on which it seems reasonable: on any plausible theory, not everything is good.

There are many other variations on extensive measurement—for example, without the Archimedean axiom (Carlson, 2007, 2010; Narens, 1974), Completeness (Carlson, 2011), or Transitivity (Krantz, 1967). There are also variations on extensive structures with restricted concatenation operations (Luce & Marley, 1969), those which combine expected utility theory with extensive measurement (Luce, 1972), and a more general class of "concatenation structures" which may be nonassociative (Luce et al., 2014, ch. 19). We leave an exploration of these variations' applications to social choice as a topic for further research.

3 Arrovian Social Welfare Functions

We begin by considering the implications of extensive measurement in a fixed-population setting, without interpersonal comparisons of well-being. This will serve as a warm-up to the more complex setting of section 5.

Let X be a set of alternatives, which is closed under some concatenation operation $\circ : X \times X \to X$. We assume that some alternatives (at least three) are *atomic*—that is, not identical to the concatenation of other alternatives.[3] Let $A \subset X$ be the set of atomic alternatives.

[3] An extensive structure does not need to have atomic elements. They play an important role, however, in the proof of Theorem 1. It may be possible to do without them, if there are instead sufficiently many alternatives that are mutually independent of each other in the sense of having no "parts" in common.

Each alternative $x \in X$ is either atomic or the concatenation of some number of atomic alternatives—that is, $x = a_1 \circ \cdots \circ a_k$ for some $a_1, \ldots, a_k \in A$. For example, the atomic alternatives might be events or histories, and their concatenation would be a sequence of events or series of successive epochs. Or they might be allocations of commodity bundles, or distributions of freely combinable characteristics, with their concatenation simply combining those allocations or distributions.

We assume a fixed population $N = \{1, 2, \ldots, n\}$ of individuals. An *Arrovian profile* $R = (R_1, \ldots, R_n)$ is an n-tuple of orderings on X, one for each individual in N. Our interpretation of these orderings is that xR_iy iff (according to profile R) x is at least as good for i as y; this ordering may but need not be understood in terms of i's actual or enlightened preferences. As usual, I_i denotes the symmetric part of R_i, P_i its symmetric part. $\mathcal{R}$ is the set of all orderings on X. An *Arrovian social welfare function* is a function $f : \mathcal{D} \subseteq \mathcal{R} \to \mathcal{R}$ which assigns an overall betterness or social preference ordering to some set $\mathcal{D}$ of Arrovian profiles. For any profile $R \in \mathcal{D}$, let $\succcurlyeq_R$ denote the ordering $f(R)$.

We adopt the following domain assumption:

Extensive Domain A profile $R \in \mathcal{D}$ iff, for all $i \in N$, $(X, R_i, \circ)$ is an extensive structure.

By Proposition 1, every profile R in an extensive domain can be represented by a *utility profile* $U = (U_1, \ldots, U_n)$, where each U_i additively represents R_i—that is, $U_i(x) \geq U_i(y)$ iff xR_iy and $U_i(x \circ y) = U_i(x) + U_i(y)$—in which case we say that U itself additively represents R. For any Arrovian profile R, let $\mathcal{U}_R$ denote the set of all utility profiles that additively represent R, and let $\mathcal{U}_\mathcal{D} := \bigcup_{R \in \mathcal{D}} \mathcal{U}_R$.

We will be interested in various Pareto principles:

Weak Pareto For any $x, y \in X$ and any Arrovian profile $R \in \mathcal{D}$, if xP_iy for every $i \in N$, then $x \succ_R y$.

Pareto Indifference For any $x, y \in X$ and any Arrovian profile $R \in \mathcal{D}$, if xI_iy for every $i \in N$, then $x \sim_R y$.

Semistrong Pareto For any $x, y \in X$ and any Arrovian profile $R \in \mathcal{D}$, if xR_iy for every $i \in N$, then $x \succcurlyeq_R y$.

Strong Pareto For any $x, y \in X$ and any Arrovian profile $R \in \mathcal{D}$, if xR_iy for every $i \in N$ then xRy; if, in addition, xP_iy for some $i \in N$, then $x \succ_R y$.

Semistrong Pareto and Strong Pareto are, unlike Weak Pareto, strengthenings of Pareto Indifference. (Semistrong Pareto was named and distinguished by Weymark (1991, 1993).)

For any binary relation R on X and any $S \subseteq X$, let $R|_S$ denote the restriction of R to S. Arrow imposed the following "Independence of Irrelevant Alternatives" condition:

Ordinal IIA For any $x, y \in X$ and any Arrovian profiles R and R' in $\mathcal{D}$, if $R_i|_{\{x,y\}} = R'_i|_{\{x,y\}}$ for every $i \in N$, then $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$.

We will instead use a weaker principle, which allows the social comparison of alternatives to depend not just on individuals' rankings of *those* alternatives, but also on their rankings of concatenations involving them. For any $S \subseteq X$, let S° denote the closure of S under $\circ$—that is, the set of all alternatives in S together with those obtainable from repeatedly applying $\circ$ to pairs of alternatives in S (equivalently, the intersection of all subsets of X that contain all elements of S and are closed under $\circ$). According to

Ratio IIA For any $x, y \in X$ and any Arrovian profiles $R, R' \in \mathcal{D}$, if $R_i|_{\{x,y\}^\circ} = R'_i|_{\{x,y\}^\circ}$ for every $i \in N$, then $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$.[4]

The motivation for weakening Ordinal IIA to Ratio IIA (and for its name) is that each $R_i|_{\{x,y\}^\circ}$ fully determines the *ratio* of $U_i(x)$ to $U_i(y)$ for any U_i that additively represents R_i. In a setting where such information is well-defined, there is no reason to exclude it as "irrelevant" to the comparison of alternatives. The naturalness of Ratio IIA is confirmed by the fact that, on our domain, Ratio IIA is equivalent to the following familiar condition:

Utility IIA For any $x, y \in X$, $R, R' \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$, if $U_i(x) = U'_i(x)$ and $U_i(y) = U'_i(y)$ for every $i \in N$, then $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$.

Proposition 2. *If an Arrovian social welfare function f satisfies Extensive Domain, then f satisfies Ratio IIA iff it satisfies Utility IIA.*

The proof of this and other results in this section is in Appendix A. The basic reason why Proposition 2 holds is that the values assigned to particular alternatives by an additive representation of an extensive structure is determined solely by the ordering of concatenations of those particular alternatives. So two utility profiles coincide on a pair of alternatives just in case the orderings represented by those profiles are the same when restricted to concatenations of those alternatives.

[4] A slightly stronger principle would replace the consequent of Ratio IIA with "$\succcurlyeq_R |_{\{x,y\}^\circ} = \succcurlyeq_{R'} |_{\{x,y\}^\circ}$." This strengthening would be harmless for our purposes, but is also unnecessary.

For any utility profile $U \in \mathcal{U}_{\mathcal{D}}$ and alternative $x \in X$, the utility vector assigned by U to x is $U(x) = (U_1(x), \ldots, U_n(x))$. A social welfare function is *welfarist* iff the ordering it assigns to any profile is determined by a single *social welfare ordering* $\succcurlyeq^*$ on the set of attainable utility vectors. According to

Welfarism There is a unique ordering $\succcurlyeq^*$ on $\mathbb{R}^n$ such that, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ iff $U(x) \succcurlyeq^* U(y)$.

When f and $\succcurlyeq^*$ are so related, we say that $\succcurlyeq^*$ is *associated* with f.

The standard "welfarism theorem" in the framework of social welfare functionals appeals to analogues of Pareto Indifference and Utility IIA (Bossert & Weymark, 2004, Theorem 2.2). It assumes, however, an unrestricted domain of utility profiles. It does not apply to the present setting because we have restricted the domain via Extensive Domain. Neither do analogous results for restricted domains due to Mongin (1994) and Weymark (1998). Fortunately, however, we are still able to characterize Welfarism in terms of Pareto Indifference and our IIA condition:

Theorem 1 (Welfarism Theorem). *If an Arrovian social welfare function f satisfies Extensive Domain, then f satisfies Pareto Indifference and Ratio IIA iff it satisfies Welfarism.*

The basic insight behind the proof is that the set of utility vectors attainable by the *atomic* alternatives is unrestricted. We are therefore able to define a social welfare ordering using only atomic alternatives, and then show how this ordering determines the social welfare function's assigned ordering over all alternatives. This strategy makes use of the fact (Lemma 1 in Appendix A) that for any pair of alternatives and any utility vectors which might be assigned to those alternatives, there is some utility profile in which that pair is assigned those same utility vectors and some atomic alternative is assigned one of those vectors as well.

Not just any social welfare ordering is compatible with Extensive Domain, however—only those which are invariant to individual-specific similarity transformations of utilities:

Intrapersonal Ratio-Scale Invariance For any utility vectors $u, v, u', v' \in \mathbb{R}^n$, if for every $i \in N$ there is some $k_i > 0$ such that $u'_i = k_i u_i$ and $v'_i = k_i v_i$, then $u \succcurlyeq^* v$ iff $u' \succcurlyeq^* v'$.

Proposition 3. *If an Arrovian social welfare function f satisfies Extensive Domain and Welfarism, then the social welfare ordering associated with f must satisfy Intrapersonal Ratio-Scale Invariance.*

Intrapersonal Ratio-Scale Invariance plays a key role in the results of section 4.

A social welfare ordering $\succcurlyeq^*$ is *anonymous* iff, for every $u, v \in \mathbb{R}^n$, $u \sim^* v$ whenever there is a permutation $\sigma : N \to N$ such that $u_i = v_{\sigma(i)}$ for every $i \in N$. There are two corresponding properties of social welfare functions:

Anonymity For all profiles $R, R' \in \mathcal{D}$, if there is a permutation $\sigma : N \to N$ such that $R_i = R'_{\sigma(i)}$ for every $i \in N$, then $f(R) = f(R')$.

Utility Anonymity For all $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $U' \in \mathcal{U}_{R'}$, if there is a permutation $\sigma : N \to N$ such that $U_i = U'_{\sigma(i)}$ for every $i \in N$, then $f(R) = f(R')$.

Proposition 4. *If an Arrovian social welfare function f satisfies Extensive Domain, then f satisfies Anonymity iff f satisfies Utility Anonymity. If, in addition, f satisfies Welfarism, then f satisfies Anonymity or Utility Anonymity iff* $\succcurlyeq^*$ *is anonymous.*

The various Pareto principles have obvious analogues in terms of the social welfare ordering as well. We do not state them separately. When we say that a social welfare ordering $\succcurlyeq^*$ violates or satisfies one of the Pareto principles, we mean that it violates or satisfies the obvious translation of that principle for $\succcurlyeq^*$.

4 Possibilities and Impossibilities

Arrow (1951) showed that if a social welfare function defined on an unrestricted domain satisfies Ordinal IIA and Weak Pareto, then it must be *dictatorial*: there must be some $i \in N$ such that, for any profile $R \in \mathcal{D}$ and alternatives $x, y \in X$, $x \succ_R y$ whenever xP_iy. If we weaken Arrow's domain and independence axioms to Extensive Domain and Ratio IIA, this implication is avoided, and even Strong Pareto can be satisfied. These axioms are satisfied, for example, by versions of the "headcount" rule considered by List (2001). A more sophisticated example is the class of social welfare functions associated with the following class of social welfare orderings, characterized by Naumova and Yanovskaya (2001, Theorem 4.1), extending results from Kaneko and Nakamura, 1979:

Example 1. For any $u \in \mathbb{R}^n$, let $O(u) := \{ v \in \mathbb{R}^n \mid \operatorname{sgn} v_i = \operatorname{sgn} u_i \text{ for every } i \in N \}$ denote the *orthant* containing u. Let $\mathcal{O} := \{ U \subset \mathbb{R}^n \mid U = O(u) \text{ for some } u \in \mathbb{R}^n \}$ denote the partition of $\mathbb{R}^n$ into orthants. There is a linear (i.e., antisymmetric) ordering $\geq$ on $\mathcal{O}$ such that, for any $u, v \in \mathbb{R}^n$ where $O(u) \neq O(v)$, if $u_i \geq v_i$ for every $i \in N$, then $O(u) > O(v)$. And, for each

$U \in \mathcal{O}$, there are real numbers $c_1, \ldots, c_n$ such that $\operatorname{sgn} c_i = \operatorname{sgn} u_i$ for every $u \in U$ and $i \in N$. For any $u, v \in \mathbb{R}$:

1. If $O(u) > O(v)$, then $u \succ^* v$.

2. If $O(u) = O(v)$, then $u \succcurlyeq^* v$ iff $\prod_{i \in N} |u_i|^{c_i} \geq \prod_{i \in N} |v_i|^{c_i}$.

It is not difficult to see that a social welfare function which satisfies Extensive Domain and is associated with this ordering satisfies Strong Pareto and Utility IIA and thus Ratio IIA.

The orderings described in Example 1 satisfy a number of further properties which are explored by Naumova and Yanovskaya. For example, they are continuous within each orthant, and they are representable by a real-valued social utility function (Naumova & Yanovskaya, 2001, Corollary 4.1). They can also be anonymous *within* each orthant, by requiring that each $c_i = c_j$ whenever $\operatorname{sgn} u_i = \operatorname{sgn} u_j$. They cannot be made fully anonymous, however, because distinct orthants must be strictly ranked against one another.

Indeed, the failure of anonymity applies more generally:

Theorem 2. *There is no Arrovian social welfare function that satisfies Extensive Domain, Anonymity, Ratio IIA, and Strong Pareto (or, when n is even, Weak Pareto and Pareto Indifference).*

Theorem 2 may suggest that Anonymity is too much to ask of a social welfare function in the present environment. However, the Arrovian axioms can be strengthened in a way that requires the social welfare function to be *strongly* dictatorial: that is, there must be some individual $i \in N$ such that, for any profile $R \in \mathcal{D}$ and alternatives $x, y \in X$, $x \succcurlyeq_R y$ iff xR_iy. One way to do this is to require the social welfare ordering to be continuous in the sense that its upper and lower contour sets are closed in $\mathbb{R}^n$. Tsui and Weymark (1997) show that a continuous social welfare ordering which satisfies Weak Pareto and Intrapersonal Ratio-Scale Invariance must be strongly dictatorial (see also Nebel, 2023, for a simpler proof). In my view, however, the ethical content of and motivation for continuity is not obvious. It is standardly motivated by considerations regarding slight measurement errors (e.g., by d'Aspremont & Gevers, 2002, p. 496). But, while sensitivity to such errors may be unfortunate, it's far from obvious that the ethical ordering of utility vectors *shouldn't* be sensitive to such errors. In order to figure out which alternatives are better or worse, why shouldn't we have to identify the correct profile (as opposed to one that is merely arbitrarily "close" to the correct profile)? Especially given the distinguished role of neutral elements in an extensive structure, discontinuities when some utilities are zero in particular do not seem unreasonable.

We therefore consider a different requirement which does not, by itself, entail continuity:

Extensive Social Preference For each profile $R \in \mathcal{D}$, the triple $(X, \succcurlyeq_R, \circ)$ is an extensive structure.

The Monotonicity and Archimedean axioms are the most questionable of the conditions for extensive measurement in this context. But *if* we assume those axioms for individual welfare, we might reasonably impose them for the social ordering as well. For example, on the successive-epochs interpretation of $\circ$, Monotonicity can be motivated by the idea that a choice between $c \circ a$ and $c \circ b$ is relevantly like choosing between futures a and b after a past epoch c; what happened in previous epochs, we might think, should not matter for future evaluation except insofar as it affects people today or in the future, in which case those effects should be considered in the valuation of a and b. The Archimedean axiom captures the intuition that no alternatives are "infinitely" better or worse than any others. As in the case of individual welfare, my view is that the applicability of these axioms to social evaluation should be regarded as an open question, which depends on the nature of the alternatives, the interpretation of $\circ$, as well as our general ethical commitments.

Our main limitative result for Arrovian social welfare functions is as follows:

Theorem 3. *If a social welfare function f satisfies Extensive Domain, then f satisfies Ratio IIA, Weak Pareto, and Extensive Social Preference iff it is strongly dictatorial.*

The proof goes as follows. First, we show that Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference together entail Semistrong Pareto (Lemma 2). Since Semistrong Pareto entails Pareto Indifference, these axioms together entail Welfarism, by Theorem 1. Next, given Welfarism, Extensive Social Preference is equivalent to $(\mathbb{R}^n, \succcurlyeq^*, +)$ being an extensive structure (Lemma 3). Thus, by Proposition 1, $\succcurlyeq^*$ must be additively representable by a social utility function (or "Bergson–Samuelson social welfare function") $W : \mathbb{R}^n \to \mathbb{R}$. Semistrong Pareto forces this function to be of the weighted utilitarian form—i.e., a linear combination of utilities—with nonnegative weights (Lemma 4). Finally, Weak Pareto and Intrapersonal Ratio-Scale Invariance together require exactly one person's weight to be positive; this proves the theorem. An obvious corollary of this result is that there is no Arrovian social welfare function that satisfies Extensive Domain, Ratio IIA, Strong Pareto, and Extensive Social Preference.

Some might respond to Theorems 2 and 3 by suggesting a restriction of the domain to *positive* extensive structures, in the sense defined on page 6. This might seem promising because anonymous social welfare orderings on $\mathbb{R}^n_{++}$ can satisfy Weak Pareto and Intrapersonal Ratio-Scale Invariance. These conditions uniquely characterize the "symmetric Cobb–Douglas" ordering, which compares vectors by the unweighted product of utilities (Moulin, 1988, p. 38). It is worth pausing to reflect on the qualitative interpretation of this ordering: what does it mean to multiply together people's utilities in the present setting? The answer is easiest to see by reformulating the Cobb-Douglas ordering in terms of products of *ratios* of utilities: $u \succcurlyeq^* v$ iff $\prod_{i \in N} u_i/v_i \geq 1$. For each individual, each alternative stands in a well-defined utility ratio to every other; this ratio is preserved by all admissible transformations of utility functions, and can be understood in terms of the concatenation operation $\circ$. The Cobb-Douglas ordering simply compares alternatives by the product of utility ratios between those alternatives. (A similar interpretation applies to the ordering in Example 1.)

I am not satisfied by this response to our results, for three reasons. First, as I have already said, I find it difficult to imagine a conception of welfare on which everything is good for everyone, in the sense that concatenation always increases everyone's welfare. The restriction to positive extensive structures therefore seems to me unreasonable. Second, if we want social preferences to satisfy the axioms of extensive measurement (positive or not), the symmetric Cobb–Douglas ordering does not meet this desideratum. For example, take the vectors $u = (1, 4)$, $u' = (4, 1)$, $v = (2, 3)$, $v' = (3, 2)$. We have $u + u' = v + v'$, but v and v' are both better than u and u' by the symmetric Cobb–Douglas ordering. That is inconsistent with $(\mathbb{R}^n_{++}, \succcurlyeq^*, +)$ being an extensive structure. Third, we do not have a welfarism theorem for positive extensive structures. Our proof strategy for Theorem 1 would not work for such structures because the analogue of Lemma 1 would not be valid. (For example, if x is the concatenation of all atomic alternatives and all utilities are positive, it's not possible to assign the same utility vector to both x and an atomic alternative.) It would be useful to have a characterization of welfarism on $\mathbb{R}^n_{++}$ for a social welfare function whose domain is restricted to positive extensive structures. I leave that task, however, for future research.

If we want both individual welfare and social evaluation to satisfy the axioms of extensive measurement, a more promising approach is to enrich the informational basis of social evaluation so as to include interpersonal comparisons of welfare levels, rather than merely of welfare ratios. Echoing Sen (1977b, p. 80), "n-tuples of individual orderings"—even when supplemented by an extensive concatenation operation—"are informationally inadequate for representing conflicts of interests."

5 Generalized Variable-Population Social Welfare Functions

We now generalize the framework of section 3 to include both interpersonal comparisons and variable populations. We do this by adapting elements from Hammond (1976) and Blackorby et al. (2005).

Let $\mathbb{N} = \{1, 2, \ldots\}$ represent the set of all possible individuals. Let X once again denote the set of alternatives. For any $x \in X$, $N(x) \subset \mathbb{N}$ is the set of individuals who ever live in x—x's *population*. We assume that $N(x)$ is finite and nonempty for all $x \in X$. $\mathcal{P}$ denotes the set of all finite, nonempty subsets of $\mathbb{N}$. For any $i \in \mathbb{N}$, $X_i \subseteq X$ denotes the set of alternatives in which i ever lives—i.e., $X_i := \{ x \in X : i \in N(x) \}$. For any $N \in \mathcal{P}$, X^N denotes the set of alternatives whose populations are N—i.e., $X^N := \{ x \in X : N(x) = N \}$.

We assume, as before, that X is closed under a concatenation operation $\circ : X \times X \to X$. We assume that, for each $i \in \mathbb{N}$, there are at least three atomic alternatives $a, b, c \in A^{\{i\}}$ in which only i exists. (There may be other atomic alternatives, too.) We assume that all nonatomic alternatives are identical to the concatenation of some atomic alternatives. We require that, for any $x, y \in X$, $N(x \circ y) = N(x) \cup N(y)$.

For any $x \in X$ and $i \in N(x)$, I call the pair (x, i) a *life*. $\mathcal{L}$ denotes the set of all lives—i.e., $\mathcal{L} := \{ (x, i) \in X \times \mathbb{N} \mid i \in N(x) \}$. In order to "add up" the well-beings of different individuals, we will need a way of concatenating these lives. We could take such an operation as primitive, but we can instead define it here in terms of the alternative-concatenation operation $\circ$ which we already have, at the cost of some additional assumptions.

Let $\mathcal{R}_{\mathcal{L}}$ denote the set of all orderings on $\mathcal{L}$, and $\mathcal{R}_X$ the set of all orderings on X. An *interpersonal profile* is an ordering $R \in \mathcal{R}_{\mathcal{L}}$. The interpretation of this ordering is that $(x, i) R (y, j)$ iff x is at least as good for i (according to profile R) as y is for j. Adapting terminology from Hammond (1976), a *generalized social welfare function* is a mapping $f : \mathcal{D} \subseteq \mathcal{R}_{\mathcal{L}} \to \mathcal{R}_X$. For any interpersonal profile $R \in \mathcal{D}$, we write $\succcurlyeq_R$ for $f(R)$.

In order to define a concatenation operation on lives, we impose the following condition:

Matching For any interpersonal profile $R \in \mathcal{D}$ and any $(x, i), (y, j) \in \mathcal{L}$, there is some $k \in \mathbb{N}$ and $x', y' \in X_k$ such that $(x', k) I (x, i)$ and $(y', k) I (y, j)$, and, for any such k, x', y', if $i = j$ then $(x \circ y, i) I (x' \circ y', k)$.

This lets us, for any $R \in \mathcal{D}$, define an operation $\oplus^R : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ as follows: for any $(x, i), (y, j) \in \mathcal{L}$, let $(x, i) \oplus^R (y, j) = (x' \circ y', k)$ for some k, x', y' such that $(x', k) I (x, i)$ and $(y', k) I (y, j)$. When there are multiple such k, x', y', the choice can be arbitrary, since Matching requires all such choices to be equally good according to R.

The axioms of extensive measurement will tell us, for any $x, y \in X$ and $i \in N(x) \cap N(y)$, how to value i's life in $x \circ y$ in terms of her life in x and her life in y: $(x \circ y, i) I (x, i) \oplus (y, i)$. But what if $i \in N(x) \setminus N(y)$? A natural hypothesis is that, since i does not even exist in y, concatenating y to x should not affect i's well-being—that is,

Irrelevance of Nonexistence For any interpersonal profile $R \in \mathcal{D}$, $x, y \in X$, and $i \in N(x) \setminus N(y)$, $(x \circ y, i) I (x, i)$.

There may well be conceptions of well-being, and of alternative-concatenation, on which Irrelevance of Nonexistence fails. But I suspect that those conceptions would either be at odds with the axioms of extensive measurement applied to well-being anyway, or would countenance welfare comparisons to nonexistence, which would require a very different departure from the framework of Blackorby et al. (2005).

We can now state our domain condition:

Interpersonal Extensive Domain A profile $R \in \mathcal{D}$ iff R satisfies the Matching and Irrelevance of Nonexistence conditions, and $(\mathcal{L}, R, \oplus^R)$ is an extensive structure.

Given Interpersonal Extensive Domain, it follows from Proposition 1 that each profile $R \in \mathcal{D}$ can be additively represented by a real-valued utility function. $U : \mathcal{L} \to \mathbb{R}$ additively represents a profile $R \in \mathcal{D}$ iff, for all $(x, i), (y, j) \in \mathcal{L}$, $U(x, i) \geq U(y, j)$ iff $(x, i) R (y, j)$, and $U((x, i) \oplus^R (y, j)) = U(x, i) + U(y, j))$. As before, let $\mathcal{U}_R$ denote the set of all utility functions that additively represent R, and $\mathcal{U}_{\mathcal{D}} := \bigcup_{R \in \mathcal{D}} \mathcal{U}_R$.

The various Pareto conditions have the same interpretation as in section 3; they only apply to fixed-population comparisons:

Weak Pareto For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i) P (y, i)$ for every $i \in N$, then $x \succ_R y$.

Pareto Indifference For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i) I (y, i)$ for every $i \in N$, then $x \sim_R y$.

Semistrong Pareto For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i) R (y, i)$ for every $i \in N$, then $x \succcurlyeq_R y$.

Strong Pareto For any $N \in \mathcal{P}$, $x, y \in X^N$, and $R \in \mathcal{D}$, if $(x, i) R (y, i)$ for every $i \in N$ then $x \succcurlyeq_R y$; if, in addition, $(x, i) P (y, i)$ for some $i \in N$, then $x \succ_R y$.

The reformulation of Ratio IIA in this framework requires some care, because our life-concatenation operation is profile-dependent. For any subset of alternatives $S \subseteq X$, let $L(S) := \bigcup_{x \in S} \{x\} \times N(x)$ denote the set of all lives led among the alternatives in S. For any such S and any profile R, let $L(S)^{\oplus^R}$ denote the closure of $L(S)$ under $\oplus^R$. Given any $S, T \subseteq X$ and any profiles R, R', a *profile isomorphism* is a bijection $\varphi : L(S)^{\oplus^R} \to L(T)^{\oplus^{R'}}$ such that, for all $(x,i), (y,j) \in L(S)$:

(i) $(x,i)R(y,j)$ iff $\varphi(x,i)R'\varphi(y,j)$, and

(ii) $\varphi((x,i) \oplus^R (y,j)) = \varphi(x,i) \oplus^{R'} \varphi(y,j)$.

Our Independence of Irrelevant Alternatives condition will be

Interpersonal Ratio IIA For all $R, R' \in \mathcal{D}$ and $x, y \in X$, if there is a profile isomorphism $\varphi : L(\{x,y\})^{\oplus^R} \to L(\{x,y\})^{\oplus^{R'}}$ such that $\varphi(x,i) = (x,i)$ and $\varphi(y,j) = (y,j)$ for all $i \in N(x)$ and $j \in N(y)$, then $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$.

As with Ratio IIA, this principle is equivalent to a more familiar utility-theoretic condition:

Generalized Utility IIA For any $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ and $x, y \in X$, if for all $i \in N(x), j \in N(y)$, $U(x,i) = U'(x,i)$ and $U(y,j) = U'(y,j)$, then $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$.

Proposition 5. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Interpersonal Ratio IIA iff f satisfies Generalized Utility IIA.*

For any utility profile $U : \mathcal{L} \to \mathbb{R}$, let $U(x,\cdot) : N(x) \to \mathbb{R}$ denote x's *utility distribution* in profile U. For any population $N \in \mathcal{P}$, $\mathbb{R}^N$ denotes the set of all utility distributions with domain N. The set of all utility distributions is $\Omega := \bigcup_{N \in \mathcal{P}} \mathbb{R}^N$. We call these distributions rather than vectors because Ω is not a vector space: we cannot add together utility distributions with different populations. The variable-population analogue of Welfarism is

Variable-Population Welfarism There is a unique social welfare ordering $\succcurlyeq^*$ on Ω such that, for any $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $x, y \in X$, $x \succcurlyeq_R y$ iff $U(x,\cdot) \succcurlyeq^* U(y,\cdot)$.

As in section 3, the key to our welfarism theorem in this setting is that the set of attainable utility distributions for the atomic alternatives is unrestricted. We have not assumed the existence of atomic alternatives for each population, however—only for each *singleton* population. But, for any population, we can find an atomic alternative for each member of the population and concatenate them to form an alternative in which all of those individuals exist. This is the strategy behind the proof of Theorem 4 in Appendix C:

Theorem 4 (Variable-Population Welfarism Theorem). *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Pareto Indifference and Interpersonal Ratio IIA iff it satisfies Variable-Population Welfarism.*

As in the fixed-population setting, Interpersonal Extensive Domain imposes a further constraint on the social welfare ordering—it must be invariant to common similarity transformations of individual utilities:

Interpersonal Ratio-Scale Invariance For every $u, v \in \Omega$ and positive real number k, $u \succcurlyeq^* v$ iff $ku \succcurlyeq^* kv$.

Proposition 6. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain and Variable-Population Welfarism, then the social welfare ordering associated with f must satisfy Interpersonal Ratio-Scale Invariance.*

6 A Qualitative Axiomatization of Classical Utilitarianism

In the present framework, classical utilitarianism can be formulated as follows. For any alternative $x \in X$ and profile $R \in \mathcal{D}$, let $\bigoplus^R_{i \in N(x)}(x, i)$ denote the concatenation of all the individuals' lives in x in arbitrary order.

Classical Utilitarianism For any $x, y \in X$ and $R \in \mathcal{D}$, $x \succcurlyeq_R y$ iff $\bigoplus_{i \in N(x)}(x, i) \geq \bigoplus_{i \in N(y)}(y, i)$.

Given Interpersonal Extensive Domain, Classical Utilitarianism is equivalent to the claim that, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ iff $\sum_{i \in N(x)} U(x, i) \geq \sum_{i \in N(y)} U(y, i)$.

Our axiomatization of Classical Utilitarianism appeals to Weak Pareto, Interpersonal Ratio IIA, and two further conditions. The first is an anonymity condition. It is not obvious how best to generalize Anonymity to the variable-population setting. In the setting of Hammond (1976), with a fixed population $N = \{1, 2, \ldots, n\}$, Anonymity can be reformulated to require that $f(R) = f(R')$ whenever there is a permutation $\sigma : N \to N$ such that, for all $x, y \in X$ and $i, j \in N$, $(x, i)R(y, j)$ iff $(x, \sigma(i))R'(y, \sigma(j))$. The problem is that, in the variable-population framework, there is no nontrivial permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that, for all $x, y \in X$, $i \in N(x)$, and $j \in N(y)$, $(x, i)R(y, j)$ iff $(x, \sigma(i))R'(y, \sigma(j))$. Blackorby et al. (2005) get around this by imposing anonymity as an *intra*profile condition. In the present framework, the natural analogue of their condition would be

Welfare Anonymity For any profile $R \in \mathcal{D}$ and $x, y \in X$, if there is a bijection $\sigma : N(x) \to N(y)$ such that $(x, i)I(y, \sigma(i))$ for all $i \in N(x)$, then $x \sim_R y$

While this has the merit of simplicity and directness, it is much stronger than a mere commitment to impartiality between individuals. For example, it implies Pareto Indifference.[5]

Fortunately, for our purposes, we will not need to assume Welfare Anonymity as a premise. Instead, we simply require the restriction of the social ordering to the alternatives facing a *fixed* population to be invariant to permutations on that fixed set of individuals:

Fixed-Population Anonymity For any $N \in \mathcal{P}$ and $R, R' \in \mathcal{D}$, if there is a permutation $\sigma : N \to N$ and a profile isomorphism $\varphi : L(X^N)^{\oplus^R} \to L(X^N)^{\oplus^{R'}}$ such that $\varphi(x, i) = (x, \sigma(i))$ for all $(x, i) \in L(X^N)$, then for all $x, y \in X^N$, $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$.

In terms of numerical utilities, Fixed-Population Anonymity amounts to the following:

Fixed-Population Utility Anonymity For any $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ and $N \in \mathcal{P}$, if there is a permutation $\sigma : N \to N$ such that $U(x, i) = U'(x, \sigma(i))$ for all $\mathcal{L} \in X^N \times N$, then for all $x, y \in X^N$, $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$.

Proposition 7. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Fixed-Population Anonymity iff f satisfies Fixed-Population Utility Anonymity.*

The final principle we need is

Extensive Social Preference For all $R \in \mathcal{D}$, $(X, \succcurlyeq_R, \circ)$ is an extensive structure.

The Monotonicity axiom of extensive measurement is particularly controversial when applied to variable-population social preference. It rules out views on which the value of additional lives depends on how many other people have ever existed or how well off they were (Asheim & Zuber, 2014; Hurka, 1983; Sider, 1991). However, precisely that feature is what makes such views seem, to many, unattractive (Mulgan, 2001; Nebel, 2022a). It is also unclear whether such views can satisfactorily avoid, in full generality, the "repugnant" conclusions that have been used to motivate them (Spears & Budolfson, 2021).

One especially powerful implication of Extensive Social Preference in the variable-population setting is that, in the presence of Interpersonal Extensive Domain and Pareto Indifference, it implies that the addition of "null" lives to a population is always a matter of social indifference. An alternative $z \in X$ is *null for individual* $i \in N(z)$, relative to a profile R, iff $(z, i) \oplus^R (z, i) I (z, i)$. An alternative z is *universally null*, relative to R, iff z is null for all $i \in N(z)$. According to

[5] Blackorby et al. (2005) avoid this by formally including nonwelfare information as a component of each profile and requiring nonwelfare information to be permuted among individuals in addition to their welfare levels. In addition to the extra complexity introduced by this information, it's not obvious that all "nonwelfare information" can be freely reassigned among individuals in the way required for this to work.

Null Critical Levels For any $R \in \mathcal{D}$ and any universally null $z \in X$, $x \circ z \sim_R x$ for all $x \in X$.

Proposition 8. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, Pareto Indifference, and Extensive Social Preference, then it satisfies Null Critical Levels.*

The intuition behind Proposition 8 is that since concatenating a universally null alternative with itself is a matter of indifference for each individual, by Pareto Indifference its self-concatenation is also a matter of indifference from the social perspective. The axioms of extensive measurement then imply that its concatenation to any other alternative must also be a matter of social indifference.

Null Critical Levels plays a crucial role in the proof of our main result:

Theorem 5 (Characterization of Classical Utilitarianism)**.** *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Interpersonal Ratio IIA, Weak Pareto, Fixed-Population Anonymity, and Extensive Social Preference iff f satisfies Classical Utilitarianism.*

The strategy behind the proof is as follows. Given Null Critical Levels, each utility distribution can be "extended" by adding individuals with zero utility; all such extensions will be equally good. We are therefore able to strengthen Variable-Population Welfarism by constructing an "extended" social welfare ordering on the space $\mathbb{R}^\infty$ of all infinite sequences with finite support (Lemma 7). Fixed-Population Anonymity then requires this extended social welfare ordering to be fully anonymous (Lemma 8). By Extensive Social Preference and Proposition 1, the extended ordering can be additively represented by a real-valued social utility function. The proof of Theorem 5 then amounts to showing that this additive representation is of the weighted utilitarian form and that all weights must be equal.

A great deal of the work in proving Theorem 5 is done by Extensive Social Preference. Clearly this is a very strong condition. Indeed, it is sufficiently strong that we do not even invoke Interpersonal Ratio-Scale Invariance in the proof. We might therefore want to know how Classical Utilitarianism might be derived in this framework without assuming Extensive Social Preference. An answer is provided in Appendix E. Theorem 6 there characterizes Classical Utilitarianism in terms of Interpersonal Extensive Domain, Strong Pareto, Interpersonal Ratio IIA, and five principles imposed directly on the social welfare ordering. This theorem illustrates how the classical utilitarian is committed to a large number of independent principles, some of which lack an obvious ethical motivation or qualitative interpretation. One thing we learn from Theorem 5 is how many of these commitments can be weakened, unified, and subsumed in a simple way via Extensive Social Preference.

7 The Sen–Weymark Critique Revisited

We have seen how extensive measurement can be used to provide simple characterizations of strong dictatorship and classical utilitarianism. As I have emphasized, it is not clear whether the axioms of extensive measurement are satisfied when applied to well-being, so it is (in my view) not clear whether there is even such a thing as the "sum" of people's well-beings, and thus whether classical utilitarianism is even well-defined, let alone true. But we have seen that, if we *can* make sense of adding together individuals' well-being (via Interpersonal Extensive Domain), all we need to obtain classical utilitarianism is Interpersonal Ratio IIA, Weak Pareto, Fixed-Population Anonymity, and, most controversially, Extensive Social Preference. It is noteworthy that analogues of the last two conditions led to impossibilities when interpersonal comparisons were excluded.

The structure of Theorem 5 is reminiscent of Harsanyi (1955)'s "aggregation theorem" (and its multi-profile version in Mongin, 1994). Whereas we impose extensive measurement at the individual and social levels, Harsanyi imposed expected utility theory at both levels. Both results rely on a Pareto principle to connect the individual and social evaluations. Whereas our Pareto principle is applied to concatenations of alternatives, Harsanyi's Pareto principle is applied to lotteries over alternatives—which von Neumann and Morgenstern (1944, p. 24) call a "natural operation" of "combination of two utilities with two given alternative probabilities." Harsanyi's conclusion is that social preferences can be represented as maximizing a weighted sum of von Neumann–Morgenstern utilities.

It is controversial, however, whether Harsanyi's utilities represent an "an attribute of persons which it is meaningful to sum" (Roemer, 1998, p. 30). The key insight behind this critique of Harsanyi is that, as Sen (1977a) and Weymark (1991) remind us, preferences which satisfy the expected utility axioms don't *have* to be represented as maximizing the expectation of von Neumann–Morgenstern utilities (see also Arrow, 1951; Fishburn, 1989; Luce & Raiffa, 1957). If we represent individual preferences using nonexpectational utility functions, the social ordering won't be representable as maximizing a weighted sum of *those* utility functions. In order to get a utilitarian conclusion, then, Harsanyi needs a reason to privilege the expectational rather than nonexpectational representations of individual preferences, and it's not clear what that reason could be.[6]

The connection between this "Sen–Weymark critique" and the present study is antic-

[6] For further discussion of this issue, see Broome (1991), Fleurbaey and Mongin (2016), Grant et al. (2010), Greaves (2017), Nebel (2022b), and Risse (2002).

ipated by Weymark (2005). Drawing on Krantz et al. (1971), Weymark observes that an extensive structure does not *have* to be given an additive representation. Applied to our variable-population framework, the point is as follows. Suppose that a generalized social welfare function satisfies Interpersonal Extensive Domain, so that for each profile $R \in \mathcal{D}$, there is a utility function $U : \mathcal{L} \to \mathbb{R}$ which additively represents R. It is easy to see that the function $V : \mathcal{L} \to \mathbb{R}_{++}$ given by $V = \exp(U)$ *multiplicatively* represents R, in the sense that, for any $(x, i), (y, j) \in \mathcal{L}$, $(x, i)R(y, j)$ iff $V(x, i) \geq V(y, j)$, and $V((x, i) \oplus^R (y, j)) = V(x, i) \times V(y, j)$. Using such a representation, the principle labeled Classical Utilitarianism is equivalent to the principle that, for any alternatives $x, y \in X$, profile $R \in \mathcal{D}$, and $V : \mathcal{L} \to \mathbb{R}_{++}$ which multiplicatively represents R, $x \succcurlyeq_R y$ iff $\prod_{i \in N(x)} V(x, i) \geq \prod_{i \in N(y)} V(y, i)$. What, then, justifies the interpretation of this principle as a utilitarian, rather than "prioritarian," social welfare function?

My answer is that, ultimately, the numerical representation of our social welfare function does not matter; all that matters is the ordering it assigns to each profile. A classical utilitarian believes that alternatives should be compared by their sums of well-being. But since a person's well-being is not a number, this "sum" must be understood in terms of some *qualitative* operation on the objects of individual evaluation, rather than the arithmetic operation of addition. If Interpersonal Extensive Domain is satisfied, then the life-concatenation operation $\oplus$ has as good a claim as anything to determine the semantic value of "sum" as applied to well-being. By way of analogy, in the measurement of length, we could just as well represent the length of a concatenation of rods by the product of numbers assigned to the concatenated rods rather than by the sum of those numbers. But there is no temptation to infer from this that the length of the concatenation is not the sum of the lengths of the rods so concatenated, since we recognize that the "sum" of two lengths refers to the length of the concatenated rod, not to the *number* assigned to that length by some arbitrary scale. This is simply a semantic fact determined by our usage of the word "sum" as applied to lengths. More generally, since alternative numerical representations of relational structures are ubiquitous in the theory of measurement, it seems fetishistic to expect an axiomatization of utilitarianism to deliver a social ordering that can *only* be represented as maximizing an arithmetic sum of numerical utilities. Just as there is nothing wrong with a physicist who uses multiplicative rather than additive representations of extensive physical quantities, there is nothing wrong with a classical utilitarian who represents her social ordering as maximizing the product of numerical utilities, so long as the arithmetic product represents, for her, the same concatenation operation that is conventionally represented by addition.

As Krantz et al. (1971, p. 100) observe, "It would take some getting used to—for example, the multiplicative scales are unique up to positive powers rather than up to multiplication by positive constants—but that is only a matter of familiarity."

A different problem is that there may be multiple concatenation operations for well-being which satisfy the axioms of extensive measurement. For example, in the measurement of length, we could concatenate rods diagonally, by taking $a \circ b$ to be the hypotenuse of a right triangle with sides a and b (Ellis, 1966); this example is also discussed by Weymark, 2005. It's clear, however, that an interpretation of our language which defines the "sum" of the lengths of two rods to be the length of their diagonal concatenation is inconsistent with the facts of usage. In the case of well-being, existing usage might not be sufficiently rich to secure a determinate interpretation of expressions like "sum" (as suggested by Greaves, 2017). Our framework, however, is compatible with many possible interpretations of $\oplus$, and it seems to me that if there is any such thing as the sum of well-being, it must be understood in terms of some extensive concatenation operation or other (Nebel, forthcoming). Given a definition of "sum" as applied to well-being in terms of some interpretation of $\oplus$, the social welfare function characterized uniquely by the axioms of Theorem 5 compares alternatives by their sums of well-being *so defined*. It therefore seems reasonable to call this social welfare function "classical utilitarianism." Indeed, I find it hard to see what else (beyond more compelling axioms) we could reasonably expect from an axiomatization of classical utilitarianism—or, even if we could, how it could possibly matter.

Thus, while a version of the Sen–Weymark critique could be applied to an argument for classical utilitarianism from Theorem 5, so applied it strikes me as uncompelling. This does not mean, however, that the *actual* Sen–Weymark critique, applied to Harsanyi, is similarly uncompelling. It may even be strengthened by the possibility of extensive measurement. For suppose that well-being is susceptible to extensive measurement and that we understand "adding" well-being in terms of concatenation. Then Harsanyi's sum of von Neumann–Morgenstern utilities represents the sum of well-being only if each individual's ranking of lotteries maximizes the expectation of her well-being so understood. This amounts to the requirement that von Neumann–Morgenstern utilities be affine with respect to an additive representation of our extensive structure. But it is not at all obvious why this should be so. And there is some reason to think that it *shouldn't* be: if we can compare arbitrary lotteries with countable support, an expectational utility representation must be bounded; but an additive representation of a closed extensive structure with non-null elements cannot be bounded, so von Neumann–Morgenstern utilities cannot be affine with respect to such a representation. On the other hand, if well-being is not susceptible to extensive measurement, this might strengthen the case for thinking (with Broome, 1991) that quantities of well-being get their meaning from an expectational utility representation (though we would not be forced to that conclusion).

This suggests a sort of middle ground in the debate over the utilitarian relevance of Harsanyi's theorem, between the uniquivocal extremes that "Harsanyi's aggregation theorem is not a theorem about utilitarianism" (Roemer, 1998, p. 143) and that "Harsanyi has gone as far towards defending 'utilitarianism in the original sense' as could coherently be asked" (Greaves, 2017, p. 175). The problem for Harsanyi is not that there are alternative (i.e., nonexpectational) numerical representations of preferences which satisfy the von Neumann–Morgenstern axioms, but rather that there are alternative *qualitative* structures which may be used to measure well-being—other ways of giving "meaning to the utilities to be added" (Arrow, 1973, p. 255).

A Proofs for Section 3

Proof of Proposition 2. Suppose that f satisfies Extensive Domain and Ratio IIA. Take some $x, y \in X$, $R, R' \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ such that $U_i(x) = U'_i(x)$ and $U_i(y) = U'_i(y)$ for every $i \in N$. For each $z \in \{x, y\}^\circ$, there must be n and m such that $U_i(z) = nU_i(x) + mU_i(y)$ and $U'_i(z) = nU'_i(x) + mU'_i(y)$ for every $i \in N$. Thus $U_i(z) = U'_i(z)$ for all $z \in \{x, y\}^\circ$. We must therefore have $R_i|_{\{x,y\}^\circ} = R'_i|_{\{x,y\}^\circ}$ for every $i \in N$, so $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$ by Ratio IIA, and Utility IIA is therefore satisfied.

For the other direction, suppose that f satisfies Extensive Domain and Utility IIA. Take some $x, y \in X$ and $R, R' \in \mathcal{D}$ such that $R_i|_{\{x,y\}^\circ} = R'_i|_{\{x,y\}^\circ}$ for every $i \in N$. Take some $U \in \mathcal{U}_R$ and $V \in \mathcal{U}_{R'}$. For any $w, z \in \{x, y\}^\circ$ and $i \in N$, we have wR_iz iff wR'_iz iff $V_i(w) \geq V_i(z)$, and $V_i(w \circ z) = V_i(w) + V_i(z)$. It follows that each $V_i|_{\{x,y\}^\circ}$ additively represents $R_i|_{\{x,y\}^\circ}$. Since $U \in \mathcal{U}_R$, $U_i|_{\{x,y\}^\circ}$ also additively represents $R_i|_{\{x,y\}^\circ}$. Thus, by the uniqueness component of Proposition 1, for each $i \in N$ there must be some k_i such that $V_i = k_iU_i$. Now let $U'_i = (1/k_i)V_i$ for every $i \in N$, so that $U' = (U'_1, \ldots, U'_n) \in \mathcal{U}_{R'}$ and $U'_i|_{\{x,y\}^\circ} = U_i|_{\{x,y\}^\circ}$. We have $U_i(x) = U'_i(x)$ and $U_i(y) = U'_i(y)$ for every $i \in N$, so $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$ by Utility IIA, and Ratio IIA is therefore satisfied. (Indeed, since $U_i|_{\{x,y\}^\circ} = U'_i|_{\{x,y\}^\circ}$, we also have the stronger consequence that $\succcurlyeq_R|_{\{x,y\}^\circ} = \succcurlyeq_{R'}|_{\{x,y\}^\circ}$.)

□

The following lemma plays a key role in the proof of Theorem 1; it appeals crucially to our assumption that there are at least three atomic alternatives:

Lemma 1. *If an Arrovian social welfare function f satisfies Extensive Domain, then for any alternatives $x, y \in X$, utility profile $U \in \mathcal{U}_{\mathcal{D}}$, and any utility vector $w \in \mathbb{R}^n$, there is an atomic alternative $a \in A \subset X$ and some profile $V \in \mathcal{U}_{\mathcal{D}}$ such that $V(x) = U(x)$, $V(y) = U(y)$, and $V(a) = w$.*

Proof. For any alternatives $x, y \in X$, there are atomic alternatives $a_1, \ldots, a_k \in A$ and non-negative integers $n_1, \ldots, n_k$ (at least one of which is positive) and $m_1, \ldots, m_k$ (at least one of which is positive), where either n_i or m_i is positive for every $i \in \{1, \ldots, k\}$, such that for any profile $V \in \mathcal{U}_{\mathcal{D}}$, $V(x) = \sum_{i=1}^{k} n_i a_i$ and $V(y) = \sum_{i=1}^{k} m_i a_i$. If $k < 3$, the proof is trivial: since there are at least three alternatives in A, simply let $V(a_i) = w$ for some $a_i \in A \setminus \{a_1, a_2\}$ and $V(a_j) = U(a_j)$ for all $j \neq i$. This obviously preserves $V(x) = U(x)$ and $V(y) = U(y)$. Suppose instead, then, that $k \geq 3$. The rest of the proof proceeds by cases. To simplify exposition, let $U(x) = u$ and $U(y) = v$.

Case 1. Assume that, for some real number c, $n_i = cm_i$ for every $i \in \{a_1, \ldots, a_k\}$. Then let $V(a_1) = w$. We can easily preserve $V(x) = u$ by letting $V(a_2) = w(1 - n_1)/n_2$ and $V(a_i) = 0$ for all $i > 2$. (The former is well-defined because at least one of n_2 and m_2 must be positive and $n_2 = cm_2$ for some real number c.) This preserves $V(y) = v$ because $V(y) = cn_1 w + cn_2[w(1 - n_1)/n_2] = cu = v$.

Case 2. Assume there is no real number c such that $n_i = cm_i$ for every $i \in \{a_1, \ldots, a_k\}$. Thus, for some $i, j \in \{1, \ldots, k\}$—say, without loss of generality, $i = 1$ and $j = 2$—$n_i m_j \neq n_j m_i$. (Otherwise the assumption of this case would be contradicted by $c = n_1/m_1$ or $c = m_1/n_1$; at least one of these must be well-defined since at least one of n_1 or m_1 is positive.) Let $V(a_k) = 0$ for all $k > 3$, and $V(a_3) = w$. A bit of algebra shows that, by letting

$$V(a_1) = \frac{m_2(u - n_3 w) + n_2(m_3 w - v)}{m_2 n_1 - m_1 n_2}$$

and

$$V(a_2) = \frac{m_1(u - n_3 w) + n_1(m_3 w - v)}{m_1 n_2 - m_2 n_1},$$

we preserve both $V(x) = u$ and $V(y) = v$.

□

Proof of Theorem 1. Suppose that f satisfies Extensive Domain, Pareto Indifference, and Ratio IIA. By Proposition 2, f also satisfies Utility IIA. Define a social welfare ordering $\succcurlyeq^*$ on $\mathbb{R}^n$ as follows: for any $u, v \in \mathbb{R}^n$, $u \succcurlyeq^* v$ iff for some atomic $a, b \in A$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $U(a) = u$, $U(b) = v$, and $a \succcurlyeq_R b$.

For any $u, v \in \mathbb{R}^n$, there are $a, b \in A$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such that $U(a) = u$ and $U(b) = v$. So, by the completeness of $\succcurlyeq_R$, either $u \succcurlyeq^* v$ or $v \succcurlyeq^* u$.

We then show that, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ if and only if $U(x) \succcurlyeq^* U(y)$:

1. Suppose $x \succcurlyeq_R y$. Using Lemma 1, choose an $R' \in \mathcal{D}$, $U' \in \mathcal{U}_{R'}$ and atomic $a \in A$ such that $U'(a) = U'(x) = U(x)$ and $U'(y) = U(y)$. Utility IIA implies $x \succcurlyeq_{R'} y$, and Pareto Indifference implies $a \sim_{R'} x$, so $a \succcurlyeq_{R'} y$ by the transitivity of $\succcurlyeq_{R'}$. Use Lemma 1 again to find a profile $R'' \in \mathcal{D}$, $U'' \in \mathcal{U}_{R''}$, and atomic $b \in A$ such that $U''(b) = U''(y) = U'(y)$ and $U''(a) = U'(a)$. By Utility IIA, Pareto Indifference, and transitivity again, we have $a \succcurlyeq_{R''} b$, as desired.

2. Suppose $U(x) \succcurlyeq^* U(y)$. Let $U(x) = u$ and $U(y) = v$. Then there must be some $a, b \in A$, $R \in \mathcal{D}$, and $V \in \mathcal{U}_R$ such that $V(a) = u$, $V(b) = v$, and $a \succcurlyeq_R b$. Use Lemma 1 to find an $R' \in \mathcal{D}$, $U' \in \mathcal{U}_{R'}$ and $a' \in A$ such that $U'(a') = U'(x) = u$ and $U'(y) = v$, and then another $R'' \in \mathcal{D}$, $U'' \in \mathcal{U}_{R''}$ and $b' \in A$ such that $U''(b') = U''(y) = v$ and $U''(a') = u$. Since the domain is unrestricted with respect to atomic alternatives and there are at least three of them, there must be some $c \in A \setminus \{b, b'\}$, $R^1, R^2, R^3 \in \mathcal{D}$, and $V^1 \in \mathcal{U}_{R^1}, V^2 \in \mathcal{U}_{R^2}, V^3 \in \mathcal{U}_{R^3}$ such that (i) $V^1(a) = V^1(c) = u$ and $V^1(b) = v$, (ii) $V^2(c) = u$ and $V^2(b) = V^2(b') = v$, and (iii) $V^3(a') = V^3(c) = u$ and $V^3(b') = v$. By Utility IIA, Pareto Indifference, and transitivity again, $a \succcurlyeq_R b$ iff $c \succcurlyeq_{R^1} b$ iff $c \succcurlyeq_{R^2} b'$ iff $a' \succcurlyeq_{R^3} b'$ iff $a' \succcurlyeq_{R''} b'$ iff $a' \succcurlyeq_{R'} y$ iff $x \succcurlyeq_R y$, as desired.

To show that $\succcurlyeq^*$ is transitive, suppose that $u \succcurlyeq^* v$ and $v \succcurlyeq^* w$. There must be some $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $a, b, c \in A$ such that $U(a) = u$, $U(b) = v$, and $U(c) = w$. Given what we just showed above, we must have $a \succcurlyeq_R b \succcurlyeq_R c$, and thus $a \succcurlyeq_R c$ by the transitivity of $\succcurlyeq_R$. Thus $u \succcurlyeq^* w$.

It is easy to see that Welfarism implies Pareto Indifference and Utility IIA and thus, given Extensive Domain and Proposition 2, Ratio IIA. □

Proof of Proposition 3. Suppose that f satisfies Extensive Domain and Welfarism. Take any utility vectors $u, v, u', v' \in \mathbb{R}^n$ for which, for every $i \in N$, there is some $k_i > 0$ such that $u'_i = k_i u_i$ and $v'_i = k_i v_i$. Suppose that $u \succcurlyeq^* v$. Then for any $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $x, y \in X$ such that $U(x) =$

u and $U(y) = v$, $x \succcurlyeq_R y$. For any such R and U, the profile $U' = (k_1 U_1, \ldots, k_n U_n)$ additively represents R as well, by the uniqueness component of Proposition 1. So by Welfarism, $u' \succcurlyeq^* v'$ as well. $\square$

Proof of Proposition 4. Suppose that f satisfies Extensive Domain and Anonymity. Take any $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $U' \in \mathcal{U}_{R'}$, and permutation $\sigma : N \to N$ such that $U_i = U'_{\sigma(i)}$ for every $i \in N$. This is possible only if $R_i = R_{\sigma_i}$. So $f(R) = f(R')$ by Anonymity and Utility Anonymity is satisfied.

Suppose next that f satisfies Extensive Domain and Utility Anonymity. Take any $R, R' \in \mathcal{D}$ and $\sigma : N \to N$ such that $R_i = R'_{\sigma(i)}$ for every $i \in N$. Fix a profile $U \in \mathcal{U}_R$. Let $U' = (U_{\sigma(1)}, \ldots, U_{\sigma(n)})$. Clearly $U' \in \mathcal{U}_{R'}$. So $f(R) = f(R')$ by Utility Anonymity and Anonymity is satisfied.

Now suppose that f satisfies Extensive Domain, Welfarism, and Anonymity and therefore Utility Anonymity. The anonymity of $\succcurlyeq^*$ follows from the proofs of d'Aspremont and Gevers (1977, Lemmas 4 and 5). It is easy to see that if $\succcurlyeq^*$ is anonymous, it must also satisfy Utility Anonymity and therefore Anonymity. $\square$

B Proofs for Section 4

Proof of Theorem 2. Take a social welfare function f that satisfies Extensive Domain, Ratio IIA, and either Strong Pareto or the conjunction of Pareto Indifference and Weak Pareto. By Theorem 1 and Proposition 3, f satisfies Welfarism and its associated social welfare ordering satisfies Intrapersonal Ratio-Scale Invariance. By Proposition 4, f satisfies Anonymity iff its associated social welfare ordering is anonymous. We show that $\succcurlyeq^*$ cannot be anonymous given Strong Pareto or, when n is even, Weak Pareto.

First assume Strong Pareto. Let $a > b > 0$. By Strong Pareto and the anonymity of $\succcurlyeq^*$, $(a, 0, \ldots, 0) \sim (0, \ldots, 0, a) \succ (0, \ldots, 0, b)$, so $(a, 0, \ldots, 0) \succ (0, \ldots, 0, b)$. By the same reasoning, $(0, \ldots, 0, a) \succ (b, 0, \ldots, 0)$. But Intrapersonal Ratio-Scale Invariance implies that $(a, 0, \ldots, 0) \succ (0, \ldots, 0, b)$ iff $(b, 0, \ldots, 0) \succ (0, \ldots, 0, a)$, by multiplying person 1's utilities in both vectors by b/a and person n's by a/b.

Next assume Weak Pareto and suppose that n is even. For any $x, y \in \mathbb{R}$, let $(\mathbf{x}, \mathbf{y})$ denote the vector in $\mathbb{R}^n$ the first half of whose components equal x and whose second half equals y. By Weak Pareto and the anonymity of $\succcurlyeq^*$, $(\mathbf{a}, -\mathbf{b}) \sim (-\mathbf{b}, \mathbf{a}) \succ (-\mathbf{a}, \mathbf{b})$, so $(\mathbf{a}, -\mathbf{b}) \succ (-\mathbf{a}, \mathbf{b})$. By the same reasoning, $(\mathbf{b}, -\mathbf{a}) \sim (-\mathbf{a}, \mathbf{b}) \prec (-\mathbf{b}, \mathbf{a})$, so $(\mathbf{b}, -\mathbf{a}) \prec (-\mathbf{b}, \mathbf{a})$. But these are inconsistent with Intrapersonal Ratio-Scale Invariance, which implies that $(\mathbf{a}, -\mathbf{b}) \succ (-\mathbf{a}, \mathbf{b})$ iff $(\mathbf{b}, -\mathbf{a}) \succ (-\mathbf{b}, \mathbf{a})$.

□

We now lay out three results concerning Extensive Social Preference; these lead to the proof of Theorem 3.

Lemma 2. *If an Arrovian social welfare function f satisfies Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference, then it must also satisfy Semistrong Pareto.*

Proof. Suppose that f satisfies Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference. Suppose for reductio that, for some $x, y \in X$ and $R \in \mathcal{D}$, xR_iy for all $i \in N$ but $y \succ_R x$. Take some $U \in \mathcal{U}_R$ and use Lemma 1 to find an $R' \in \mathcal{D}$, $V \in \mathcal{U}_{R'}$, and $z \in X$ such that $V(x) = U(x)$, $V(y) = U(y)$, $V(z) = U(y) - (1, \ldots, 1)$. By Ratio IIA and Proposition 2, $y \succ_{R'} x$. This implies, by the Archimedean property, that for some natural number n, $ny \circ z \succcurlyeq_{R'} nx \circ x$. By Extensive Domain, $V(ny \circ z) = V(ny) + V(z) = (n+1)V(y) - (1, \ldots, 1)$, and $V(nx \circ x) = V(nx) + V(x) = (n+1)V(x)$. But since $V_i(x) \geq V_i(y)$ for every $i \in N$, $(n+1)V_i(x) > (n+1)V_i(y) - 1$ for every $i \in N$ and natural number n. Thus we cannot have $ny \circ z \succcurlyeq_{R'} nx \circ x$ by Weak Pareto.[7] □

Lemma 3. *If a social welfare function f satisfies Extensive Domain and Welfarism, then f satisfies Extensive Social Preference iff its associated social welfare ordering* $\succcurlyeq^*$ *satisfies Extensive SWO:*

Extensive SWO *The triple* $(\mathbb{R}^n, \succcurlyeq^*, +)$ *is an extensive structure.*

Proof. Suppose that f satisfies Extensive Domain and Welfarism. Transitivity and Completeness are built into the definitions of $\succcurlyeq_R$ and $\succcurlyeq^*$. Vector addition is associative, and Weak Associativity of $\circ$ with respect to $\sim_R$ follows from Extensive Domain and Pareto Indifference, which is implied by Welfarism. So it remains to show that $(X, \succcurlyeq_R, \circ)$ satisfies Monotonicity and Archimedean iff $(\mathbb{R}^n, \succcurlyeq^*, +)$ does.

For Monotonicity, take any $u, v, w \in \mathbb{R}^n$, and any $R \in \mathcal{D}$, $U \in \mathcal{U}_R$, and $x, y, z \in X$ such that $U(x) = u$, $U(y) = v$, and $U(z) = w$. Welfarism implies that $u \succcurlyeq^* v$ iff $x \succcurlyeq_R y$, and $x \circ z \succcurlyeq_R y \circ z$ iff $u + w \succcurlyeq^* v + w$. Extensive Social Preference implies that $x \succcurlyeq_R y$ iff $x \circ z \succcurlyeq_R y \circ z$; Extensive SWO implies $u \succcurlyeq^* v$ iff $u + w \succcurlyeq^* v + w$. Whichever we assume, the other follows. The proof for the Archimedean axiom is analogous. □

[7] I am grateful to Zachary Goodsell for the central insight behind this proof.

Lemma 4. *If a social welfare ordering $\succcurlyeq^*$ satisfies Extensive SWO and Semistrong Pareto, then it is additively represented by a social utility function $W : \mathbb{R}^n \to \mathbb{R}$ of the following form: for some $c_1, \ldots, c_n \geq 0$,*

$$W(u) = \sum_{i \in N} c_i u_i \text{ for all } u \in \mathbb{R}^n. \tag{1}$$

Proof. By Extensive SWO and Proposition 1, $\succcurlyeq^*$ is representable by some $W : \mathbb{R}^n \to \mathbb{R}$ which satisfy's Cauchy's functional equation (2):

$$W(u + v) = W(u) + W(v) \text{ for all } u, v \in \mathbb{R}^n \tag{2}$$

The general solution to such an equation is of the following form (Aczél & Dhombres, 1989, p. 35):

$$W(u) = \sum_{i=1}^{n} W_i(u_i) \tag{3}$$

where each $W_i : \mathbb{R} \to \mathbb{R}$ satisfies equation (4):

$$W_i(x + y) = W_i(x) + W_i(y) \text{ for all } x, y \in \mathbb{R} \tag{4}$$

In order to satisfy Semistrong Pareto, each W_i must be nondecreasing. Thus, by Aczél and Dhombres (1989, Corollary 2.5, p. 15), for each W_i there must be a constant $c_i \geq 0$ such that

$$W_i(x) = c_i x \text{ for all } x \in \mathbb{R} \tag{5}$$

Putting equations (3) and (5) together, we get (1). □

Proof of Theorem 3. Suppose that f satisfies Extensive Domain, Ratio IIA, Weak Pareto, and Extensive Social Preference. By Lemma 2, f also satisfies Semistrong Pareto and thus Pareto Indifference. So, by Theorem 1, Proposition 3, and Lemma 3, f satisfies Welfarism and the associated social welfare ordering $\succcurlyeq^*$ satisfies Intrapersonal Ratio-Scale Invariance and Extensive SWO. Lemma 4 then implies that $\succcurlyeq^*$ must be additively representable by a $W : \mathbb{R}^n \to \mathbb{R}$ which satisfies equation (1) with nonnegative weights.

In order to satisfy Weak Pareto, there must be some $i \in N$ such that $c_i > 0$. We then show that, for any $j \in N \setminus \{i\}$, $c_j = 0$. Suppose for reductio that, for some distinct $i, j \in N$, $c_i > 0$ and $c_j > 0$. Consider the unit vectors $\mathbf{e}_i, \mathbf{e}_j \in \mathbb{R}^n$ with all components equal to 0 except the ith (resp., jth) which equals 1. We have $W(\mathbf{e}_i) = c_i$ and $W(\mathbf{e}_j) = c_j$ by equation (1). If c_i and c_j are both positive, then there must be some natural numbers n and m such that $nc_i > c_j$

and $mc_j > c_i$ by the Archimedean property of the real numbers. Since $W(n\mathbf{e}_i) = nc_i$ and $W(m\mathbf{e}_j) = mc_j$, this implies that $n\mathbf{e}_i \succ^* \mathbf{e}_j$ and $m\mathbf{e}_j \succ^* \mathbf{e}_i$. But, by Intrapersonal Ratio-Scale Invariance, $n\mathbf{e}_i \succ^* \mathbf{e}_j$ implies $\mathbf{e}_i \succ^* m\mathbf{e}_j$.

We have shown there to be exactly one $i \in N$ such that $c_i > 0$; for all other $j \in N$, $c_j = 0$. Thus, $W(u) = c_i u_i$ for all $u \in \mathbb{R}^n$, so the social welfare function must be strongly dictatorial. It is easy to see that if f satisfies Extensive Domain and is strongly dictatorial, it must also satisfy Ratio IIA, Weak Pareto, and Extensive Social Preference. □

C Proofs for Section 5

Proof of Proposition 5. Suppose first that f satisfies Interpersonal Extensive Domain and Interpersonal Ratio IIA, and that for some $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ and $x, y \in X$, $U(x,i) = U'(x,i)$ and $U(y,j) = U'(y,j)$ for all $i \in N(x), j \in N(y)$. Define a bijection $\varphi : L(\{x,y\})^{\oplus R} \to L(\{x,y\})^{\oplus^{R'}}$ as follows. If $s \in L(\{x,y\})$, let $\varphi(s) = s$. If $s \in L(\{x,y\})^{\oplus R} \setminus L(\{x,y\})$, there must be some $s_1, \ldots, s_k \in L(\{x,y\})$ with $k \geq 2$ such that $s = s_1 \oplus^R \cdots \oplus^R s_k$; let $\varphi(s) = s_1 \oplus^{R'} \cdots \oplus^{R'} s_k$. Clearly $U(s) = U'(\varphi(s))$ for all $s \in L(\{x,y\})$; and for all $s = s_1 \oplus^R \cdots \oplus^R s_k \in L(\{x,y\})^{\oplus R} \setminus L(\{x,y\})$, $U(s) = U(s_1 \oplus^R \cdots \oplus^R s_k) = U(s_1) + \cdots + U(s_k) = U'(s_1) + \cdots + U'(s_k) = U'(s_1 \oplus^{R'} \cdots \oplus^{R'} s_k) = U'(\varphi(s))$. Thus, for all $s \in L(\{x,y\})^{\oplus R}$, $U(s) = U'(\varphi(s))$. So for any $s, t \in L(\{x,y\})^{\oplus R}$, $U(s) \geq U(t)$ iff $U'(\varphi(s)) \geq U'(\varphi(t))$, so sRt iff $\varphi(s)R'\varphi(t)$; and, by construction, $\varphi(s \oplus^R t) = \varphi(s) \oplus^{R'} \varphi(t)$. Thus φ is a profile isomorphism, so by Interpersonal Ratio IIA, $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$ and Generalized Utility IIA is satisfied.

Suppose next that f satisfies Interpersonal Extensive Domain and Generalized Utility IIA, and that for some $R, R' \in \mathcal{D}$ and $x, y \in X$, there is a profile isomorphism $\varphi : L(\{x,y\})^{\oplus^R} \to L(\{x,y\})^{\oplus^{R'}}$ such that $\varphi(x,i) = (x,i)$ and $\varphi(y,j) = (y,j)$ for all $i \in N(x)$ and $j \in N(y)$. Pick a $U \in \mathcal{U}_R$ and $U' \in \mathcal{U}_{R'}$. For any $s, t \in L(\{x,y\})^{\oplus^R}$, we have sRt iff $\varphi(s)R'\varphi(t)$ iff $U'(\varphi(s)) \geq U'(\varphi(t))$, and $U'(\varphi(s \oplus^R t)) = U'(\varphi(s) \oplus^{R'} \varphi(t)) = U'(\varphi(s)) + U'(\varphi(t))$. Let $V : L(\{x,y\})^{\oplus^R} \to \mathbb{R}$ denote the composition of $U'|_{L(\{x,y\})^{\oplus^{R'}}}$ with φ. We've just seen that V additively represents $R|_{L(\{x,y\})^{\oplus^R}}$: for any $s, t \in L(\{x,y\})^{\oplus^R}$, sRt iff $V(s) \geq V(t)$ iff $U'(\varphi(s)) \geq U'(\varphi(t))$, and $V(s \oplus^R t) = V(s) + V(t) = U'(\varphi(s)) + U'(\varphi(t))$. Since $U \in \mathcal{U}_R$, $U|_{L(\{x,y\})^{\oplus^R}}$ also additively represents $R|_{L(\{x,y\})^{\oplus^R}}$. Thus, by the uniqueness component of Proposition 1, there must be some $k > 0$ such that $V(s) = kU(s)$ for all $s \in L(\{x,y\})^{\oplus^R}$. Now let $V' = (1/k)U'$, so that $V' \in \mathcal{U}_{R'}$ and $V'(\varphi(s)) = U(s)$ for all $s \in L(\{x,y\})^{\oplus^R}$. Remember that $\varphi(s) = s$ for all $s \in L(\{x,y\})$. So $V'(x,i) = U(x,i)$ and $V'(y,j) = U(y,j)$ for all $i \in N(x)$

and $j \in N(y)$. Therefore, by Generalized Utility IIA, $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$, and Interpersonal Ratio IIA is satisfied.

$\square$

For any population $N \in \mathcal{P}$, let $\{ a_i \}_{i \in N}$ be a set of atomic alternatives with $N(a_i) = \{ i \}$ for each a_i. Let $\bigcirc_{i \in N} a_i$ denote the concatenation of all these alternatives in arbitrary order, so that $N(\bigcirc_{i \in N} a_i) = N$. Let A^N denote the set of all such concatenations of one-person alternatives involving the members of N. For any populations $M, N \in \mathcal{P}$ and $x \in A^M$ and $y \in A^N$, where $x = \bigcirc_{i \in M} a_i$ and $y = \bigcirc_{i \in N} b_i$, say that x and y are *nonoverlapping* iff $\{ a_i \}_{i \in M} \cap \{ b_i \}_{i \in N} = \varnothing$. We have the following lemma:

Lemma 5. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then for any populations $M, N, O \in \mathcal{P}$, there are nonoverlapping alternatives $x \in A^M$, $y \in A^N$, and $z \in A^O$. And, for any such x, y, z, and any utility distributions $u \in \mathbb{R}^M, v \in \mathbb{R}^N, w \in \mathbb{R}^O$, there is a utility profile $U \in \mathcal{U}_{\mathcal{D}}$ such that $U(x, \cdot) = u$, $U(y, \cdot) = v$, $U(z, \cdot) = w$.*

Proof. For each individual, there are at least three atomic alternatives in which only that individual exists. So we can find disjoint sets of atomic alternatives $\{ a_i \}_{i \in M}$, $\{ b_j \}_{j \in N}$, and $\{ c_k \}_{k \in O}$. Let $x = \bigcirc_{i \in M} a_i$, $y = \bigcirc_{j \in N} b_j$, $z = \bigcirc_{k \in O} c_k$, so that $x \in A^M$, $y \in A^N$, and $z \in A^O$ are nonoverlapping. For any $u \in \mathbb{R}^M, v \in \mathbb{R}^N, w \in \mathbb{R}^O$, we can find some $U \in \mathcal{U}_{\mathcal{D}}$ such that $U(a_i, i) = u_i$, $U(b_j, j) = v_j$, and $U(c_k, k) = w_k$ for all $i \in M, j \in N, k \in O$. By the Irrelevance of Nonexistence condition of Interpersonal Extensive Domain, $(x, i) I (a_i, i)$, $(y, j) I (b_j, j)$, and $(z, k) I (c_k, k)$ for all $i \in M, j \in N, k \in O$. So $U(x, i) = u_i$, $U(y, j) = v_j$, and $U(z, k) = w_k$ for every $i \in M, j \in N, k \in O$. Thus $U(x, \cdot) = u$, $U(y, \cdot) = v$, and $U(z, \cdot) = w$, as desired. $\square$

Lemma 5 provides us with a set of *free triples* in the sense of Weymark (1998)—i.e., a set of three alternatives for which the domain of attainable utility distributions is unrestricted. We also have the following analogue of Lemma 1:

Lemma 6. *If f satisfies Interpersonal Extensive Domain, then for any populations $M, N, O \in \mathcal{P}$, alternatives $x \in X^M$ and $y \in X^N$, any utility profile $U \in \mathcal{U}_{\mathcal{D}}$, and utility distribution $w \in \mathbb{R}^O$, there is a $z \in A^O$ and $V \in \mathcal{U}_{\mathcal{D}}$ such that $V(x, \cdot) = U(x, \cdot)$, $V(y, \cdot) = U(y, \cdot)$, and $V(z, \cdot) = w$.*

Proof. The proof is analogous to that of Lemma 1, except that we choose an atomic alternative $a_i \in A^{\{ i \}}$ for each $i \in O$ and let z be the concatenation of all these alternatives. This is trivial for $i \notin O \cap M \cap N$. For $i \in O \cap M \cap N$, we use exactly similar solutions to those in the proof of Lemma 1 to find an $a_i \in A^{\{ i \}}$ and a $V \in \mathcal{U}_{\mathcal{D}}$ such that $V(a_i, i) = w_i$ while preserving $V(x, i) = U(x, i)$ and $V(y, i) = U(y, i)$. We then let z be the concatenation of all these atomic, one-person alternatives, so that $V(z, \cdot) = w_i$ for every $i \in O$ while preserving $V(x, \cdot) = U(x, \cdot)$ and $V(y, \cdot) = U(y, \cdot)$, as desired. $\square$

We can then use Lemmas 5 and 6 to define our social welfare ordering on Ω:

Proof of Theorem 4. Suppose that f satisfies Interpersonal Extensive Domain, Pareto Indifference, and Interpersonal Ratio IIA. Define the social welfare ordering as follows: for any $M, N \in \mathcal{P}$, $u \in \mathbb{R}^M$, and $v \in \mathbb{R}^N$, $u \succcurlyeq^* v$ iff, for some nonoverlapping $a \in A^M$ and $b \in A^N$, $R \in \mathcal{D}$ and $U \in \mathcal{U}_R$ such that $U(a, \cdot) = u$ and $U(b, \cdot) = v$, $a \succcurlyeq_R b$.

By Lemma 5, for any $u \in \mathbb{R}^M$, and $v \in \mathbb{R}^N$, there must be some nonoverlapping $a \in A^M$ and $b \in A^N$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such that $u = U(a, \cdot)$ and $v = U(b, \cdot)$. Since $\succcurlyeq_R$ is complete, we have either $a \succcurlyeq_R b$ or $b \succcurlyeq_R a$, which implies either $u \succcurlyeq^* v$ or $v \succcurlyeq u$ respectively. Thus $\succcurlyeq^*$ is complete.

We then show that, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ if and only if $U(x, \cdot) \succcurlyeq^* U(y, \cdot)$:

1. Suppose $x \succcurlyeq_R y$. Using Lemma 6, choose an $R' \in \mathcal{D}$, $U' \in \mathcal{U}_{R'}$ and $a \in A^{N(x)}$ such that $U'(a, \cdot) = U'(x, \cdot) = U(x, \cdot)$ and $U'(y, \cdot) = U(y, \cdot)$. Generalized Utility IIA implies $x \succcurlyeq_{R'} y$, and Pareto Indifference implies $a \sim_{R'} x$, so $a \succcurlyeq_{R'} y$ by the transitivity of $\succcurlyeq_{R'}$. Use Lemma 6 again to find a profile $R'' \in \mathcal{D}$, $U'' \in \mathcal{U}_{R''}$, and $b \in A^{N(y)}$ such that $U''(b, \cdot) = U''(y, \cdot) = U'(y, \cdot)$ and $U''(a, \cdot) = U'(a, \cdot)$. By Utility IIA, Pareto Indifference, and transitivity again, we have $a \succcurlyeq_{R''} b$, as desired.

2. Suppose $U(x, \cdot) \succcurlyeq^* U(y, \cdot)$. Let $U(x, \cdot) = u$ and $U(y, \cdot) = v$. By the definition of $\succcurlyeq^*$, there must be some nonoverlapping $a \in A^{N(x)}$ and $b \in A^{N(y)}$, $R \in \mathcal{D}$, and $V \in \mathcal{U}_R$ such that $V(a, \cdot) = u$, $V(b, \cdot) = v$, and $a \succcurlyeq_R b$. Use Lemma 6 to find an $R' \in \mathcal{D}$, $U' \in \mathcal{U}_{R'}$ and $a' \in A^{N(x)}$ such that $U'(a', \cdot) = U'(x, \cdot) = u$ and $U'(y, \cdot) = v$, and then another $R'' \in \mathcal{D}$, $U'' \in \mathcal{U}_{R''}$ and $b' \in A^{N(y)}$ such that $U''(a', \cdot) = u$ and $U''(b', \cdot) = U''(y, \cdot) = v$. By Lemma 5, there must be some $c \in A^{N(x)}$ which does not overlap with b or b', and $R^1, R^2, R^3 \in \mathcal{D}$, and $V^1 \in \mathcal{U}_{R^1}$, $V^2 \in \mathcal{U}_{R^2}$, $V^3 \in \mathcal{U}_{R^3}$ such that (i) $V^1(a, \cdot) = V^1(c, \cdot) = u$ and $V^1(b, \cdot) = v$, (ii) $V^2(c, \cdot) = u$ and $V^2(b, \cdot) = V^2(b', \cdot) = v$, and (iii) $V^3(a', \cdot) = V^3(c, \cdot) = u$ and $V^3(b', \cdot) = v$. By Utility IIA, Pareto Indifference, and transitivity again, $a \succcurlyeq_R b$ iff $c \succcurlyeq_{R^1} b$ iff $c \succcurlyeq_{R^2} b'$ iff $a' \succcurlyeq_{R^3} b'$ iff $a' \succcurlyeq_{R''} b'$ iff $a' \succcurlyeq_{R'} y$ iff $x \succcurlyeq_R y$, as desired.

To show that $\succcurlyeq^*$ is transitive, take any $M, N, O \in \mathcal{P}$ and $u \in \mathbb{R}^M, v \in \mathbb{R}^N, w \in \mathbb{R}^O$ such that $u \succcurlyeq^* v \succcurlyeq^* w$. By Lemma 5, there must be some nonoverlapping $a \in A^M, b \in A^N, c \in A^O$,

$R \in \mathcal{D}$, and $U \in \mathcal{U}_R$ such that $U(x,\cdot) = u$, $U(y,\cdot) = v$, $U(z,\cdot) = w$. We have just shown above that $x \succcurlyeq_R y \succcurlyeq_R z$ and thus $x \succcurlyeq_R z$ by the transitivity of $\succcurlyeq_R$, so $u \succcurlyeq^* v$, as required.

It is easy to see that $\succcurlyeq^*$ is unique and that Variable-Population Welfarism implies Pareto Indifference and Generalized Utility IIA and therefore Interpersonal Ratio IIA. □

The proof of Proposition 6 is exactly similar to that of Proposition 3 and is therefore omitted.

D Proofs for Section 6

Proof of Proposition 7. Suppose that f satisfies Interpersonal Extensive Domain and Fixed-Population Anonymity. Take some $R, R' \in \mathcal{D}$, $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$, $N \in \mathcal{P}$, and permutation $\sigma : N \to N$ such that $U(x,i) = U'(x,\sigma(i))$ for all $\mathcal{L} \in X^N \times N$. Define $\varphi : L(X^N)^{\oplus^R} \to L(X^N)^{\oplus^{R'}}$ as follows. For all $(x,i) \in L(X^N)$, let $\varphi(x,i) = (x,\sigma(i))$; if $s \in L(X^N)^{\oplus R} \setminus L(X^N)$, there must be some $s_1,\ldots,s_k \in L(X^N)$ with $k \geq 2$ such that $s = s_1 \oplus^R \cdots \oplus^R s_k$, so let $\varphi(s) = \varphi(s_1) \oplus^{R'} \cdots \oplus^{R'} \varphi(s_k)$. By reasoning analogous to that in the first paragraph of the proof of Proposition 5, φ is a profile isomorphism. Therefore, for all $x, y \in X^N$, $x \succcurlyeq_R y$ iff $x \succcurlyeq_{R'} y$, so Fixed-Population Utility Anonymity is satisfied.

Suppose next that f satisfies Interpersonal Extensive Domain and Fixed-Population Utility Anonymity. Take some $R, R' \in \mathcal{D}$, $N \in \mathcal{P}$, permutation $\sigma : N \to N$, and profile isomorphism $\varphi : L(X^N)^{\oplus^R} \to L(X^N)^{\oplus^{R'}}$ such that $\varphi(x,i) = (x,\sigma(i))$ for all $(x,i) \in X^N \times N$. By reasoning analogous to that in the second paragraph of the proof of Proposition 5, there exist $U \in \mathcal{U}_R$, $U' \in \mathcal{U}_{R'}$ such that $U(x,i) = U'(x,\sigma(i))$ for all $(x,i) \in X^N \times N$. So, by Fixed-Population Utility Anonymity, $\succcurlyeq_R |_{X^N} = \succcurlyeq_{R'} |_{X^N}$, and Fixed-Population Anonymity is satisfied. □

Proof of Proposition 8. Take any profile R and $z \in X$ such that $(z,i) \oplus^R (z,i) I (z,i)$ for all $i \in N(z)$. By the Matching condition of Interpersonal Extensive Domain, $(z,i) \oplus^R (z,i) I (z \circ z, i)$ for all $i \in N(z)$. Thus, by Pareto Indifference, $z \circ z \sim_R z$. So, by the Monotonicity condition of Extensive Social Preference, $x \circ (z \circ z) \sim_R x \circ z$; by Weak Associativity, $x \circ (z \circ z) \sim_R (x \circ z) \circ z$, so $(x \circ z) \circ z \sim_R x \circ z$ by Transitivity; by Monotonicity again, $x \circ z \sim_R x$. □

As mentioned in section 5, the field Ω of the social welfare ordering $\succcurlyeq^*$ is not a vector space: we cannot add together utility distributions with different domains. This can be rectified by strengthening Variable-Population Welfarism in the following way. Let $\mathbb{R}^\infty$ denote the set of all infinite sequences with finite support—i.e., $\mathbb{R}^\infty := \{u : \mathbb{N} \to \mathbb{R} | \, u_i = 0 \text{ for all } i \geq$

n for some $n \in \mathbb{N}\}$. Unlike Ω, $\mathbb{R}^\infty$ is a vector space: for any $u, v \in \mathbb{R}^\infty$, $(u+v)_i = u_i + v_i$ for every $i \in \mathbb{N}$. For any population $N \in \mathcal{P}$, let $\iota_N : \mathbb{R}^N \hookrightarrow \mathbb{R}^\infty$ denote canonical inclusion such that for each $u \in \mathbb{R}^N$, $\iota_N(u)_i = u_i$ for all $i \in N$ and $\iota_N(u)_j = 0$ for all $j \in \mathbb{N} \setminus N$. Let $\iota : \Omega \hookrightarrow \mathbb{R}^\infty$ (no subscript) denote the union of all these inclusions. We call an ordering $\succcurlyeq^\infty$ on $\mathbb{R}^\infty$ an *extended* social welfare ordering. According to

Extended Welfarism There is a unique social welfare ordering $\succcurlyeq^\infty$ on $\mathbb{R}^\infty$ such that, for any profile $R \in \mathcal{D}$, any $U \in \mathcal{U}_R$, and any alternatives $x, y \in X$, $x \succcurlyeq_R y$ iff $\iota(U(x,\cdot)) \succcurlyeq^\infty \iota(U(y,\cdot))$.

Lemma 7. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Welfarism and Null Critical Levels iff it satisfies Extended Welfarism.*

Proof. Take any $M, N \in \mathcal{P}$, $u \in \mathbb{R}^M$, and $v \in \mathbb{R}^N$. Suppose $\iota_M(u) = \iota_N(v)$. We show that $u \sim^* v$. This is obvious if $M = N$, since then $u = v$. So suppose $M \neq N$. Let $u \frown v$ denote the utility distribution in $\mathbb{R}^{M \cup N}$ such that, for all $i \in M \cup N$, $(u \frown v)_i = u_i = v_i$ if $i \in M \cap N$ and $(u \frown v)_i = 0$ otherwise. We show that $u \sim^* (u \frown v) \sim^* v$.

By Lemma 5, there must be some $x \in X^M$, $z \in X^N$, $R \in \mathcal{D}$, and $U : \mathcal{L} \to \mathbb{R}$ which additively represents R such that $U(x,\cdot) = u$ and $U(z,i) = 0$ for all $i \in N$. It follows from Proposition 1 that z is universally null. So by Proposition 8, $x \circ z \sim_R x$. Notice, however, that $U(x \circ z, \cdot) = u \frown v$, so by Welfarism $u \sim^* (u \frown v)$. An exactly similar argument shows $v \sim^* (u \frown v)$. Thus $u \sim^* v$.

We now define $\succcurlyeq^\infty$ as follows: for all $u, v \in \mathbb{R}^\infty$, $u \succcurlyeq^\infty v$ iff, for some $M, N \in \mathcal{P}$ and $u' \in \mathbb{R}^M, v' \in \mathbb{R}^N$ such that $\iota(u') = u$ and $\iota(v') = v$, $u' \succcurlyeq^* v'$. For any such $u, v \in \mathbb{R}^\infty$, there exist $M, N \in \mathcal{P}$ and $u' \in \mathbb{R}^M, v' \in \mathbb{R}^N$ such that $\iota(u') = u$ and $\iota(v') = v$, so $\succcurlyeq^\infty$ inherits completeness from $\succcurlyeq^*$. And we've just seen that for any $M', N' \in \mathcal{P}, u^* \in \mathbb{R}^{M'}, v^* \in \mathbb{R}^{N'}$ such that $\iota(u^*) = \iota(u') = u$ and $\iota(v^*) = \iota(v') = v$, $u' \sim^* u^*$ and $v' \sim^* v^*$, so $u^* \succcurlyeq^* v^*$ iff $u \succcurlyeq^\infty v$. It's easy to see that $\succcurlyeq^\infty$ must also be transitive and is unique.

For the other direction, suppose that f satisfies Extended Welfarism. Then we define the social welfare ordering $\succcurlyeq^*$ as follows: for all $u, v \in \Omega$, $u \succcurlyeq^* v$ iff $\iota(u) \succcurlyeq^\infty \iota(v)$. It's clear that $\succcurlyeq^*$ is an ordering and that, by Extended Welfarism, for any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $x \succcurlyeq_R y$ iff $U(x,\cdot) \succcurlyeq^* U(y,\cdot)$. Finally, to see that Extended Welfarism implies Null Critical Levels, suppose that z is universally null in a profile R. Then for any $U \in \mathcal{U}_R$, $U(z,i) = 0$ for all $i \in N(z)$. For any $x \in X$, $\iota(U(x \circ z, \cdot)) = \iota(U(x,\cdot))$, so by Extended Welfarism, $x \circ z I x$. □

An extended social welfare ordering is *fully anonymous* iff, for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and $u, v \in \mathbb{R}^\infty$ such that $u_i = v_{\sigma(i)}$ for every $i \in \mathbb{N}$, $u \sim^\infty v$.

Lemma 8. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain and Extended Welfarism, then f satisfies Fixed-Population Anonymity iff its associated extended social welfare ordering* $\succcurlyeq^\infty$ *is fully anonymous.*

Proof. Suppose f satisfies Interpersonal Extensive Domain and Extended Welfarism. Clearly if $\succcurlyeq^\infty$ is fully anonymous, then Fixed-Population Anonymity must be satisfied. For the other direction, suppose that f satisfies Fixed-Population Anonymity and thus Fixed-Population Utility Anonymity (by Proposition 7). Take any $u, v \in \mathbb{R}^\infty$ such that, for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$, $u_i = v_{\sigma(i)}$ for every $i \in \mathbb{N}$. Let $N = \{\, i \in \mathbb{N} \mid u_i \neq v_i \,\}$. Since u and v have finite support, N must be finite even if σ itself has infinite support. Consider the distributions $u^*, v^* \in \mathbb{R}^N$ such that $\iota(u^*) = u$ and $\iota(v^*) = v$. There is a permutation $\sigma^* : N \to N$ such that $u^*_i = v^*_{\sigma^*(i)}$ for every $i \in N$. By Fixed-Population Utility Anonymity and Proposition 4, $u^* \sim^* v^*$. Thus, by Extended Welfarism, $u \sim^\infty v$, as desired. □

Proof of Theorem 5. Suppose that f satisfies Interpersonal Extensive Domain, Interpersonal Ratio IIA, Weak Pareto, Fixed-Population Anonymity, and Extensive Social Preference. By Lemma 2, f must also satisfy Semistrong Pareto and thus Pareto Indifference. So by Theorem 4 and Proposition 8, f satisfies Welfarism and Null Critical Levels and thus, by Lemma 7, Extended Welfarism. By Fixed-Population Anonymity and Lemma 8, the extended social welfare ordering $\succcurlyeq^\infty$ is fully anonymous.

The proof of Lemma 3 can be easily adapted to show that $(\mathbb{R}^\infty, \succcurlyeq^\infty, +)$ is an extensive structure. So $\succcurlyeq^\infty$ is additively representable by a social utility function $W : \mathbb{R}^\infty \to \mathbb{R}$ which satisfies Cauchy's functional equation (6):

$$W(u + v) = W(u) + W(v) \text{ for all } u, v \in \mathbb{R}^\infty. \tag{6}$$

For each $i \in \mathbb{N}$, define $W_i : \mathbb{R} \to \mathbb{R}$ so that $W_i(x) = W(\iota_{\{\,i\,\}}(i \mapsto x))$ for all $x \in \mathbb{R}$. For every $u \in \mathbb{R}^\infty$, there is some $k \in \mathbb{N}$ such that

$$u = (u_1, 0, 0, \dots) + (0, u_2, 0, 0, \dots) + \cdots + (0, \dots, 0, u_k, 0, 0, \dots) + (0, 0, \dots) \tag{7}$$

So, by equation (6),

$$W(u) = W(u_1, 0, 0, \dots) + W(0, u_2, 0, 0, \dots) + \cdots + W(0, \dots, 0, u_k, 0, 0, \dots) + W(0, 0, \dots) \tag{8}$$

Since $W(0,0,\dots) = 0$, this simplifies to

$$W(u) = (u_1, 0, 0, \dots) + (0, u_2, 0, 0, \dots) + \cdots + (0, \dots, 0, u_k, 0, 0, \dots) \tag{9}$$

so, by equation (6) and the definition of W_i,

$$W(u) = \sum_{i=1}^{k} W_i(u_i) \tag{10}$$

For each $i \in \mathbb{N}$, we must have:

$$W_i(x+y) = W_i(x) + W_i(y) \text{ for all } x, y \in \mathbb{R}. \tag{11}$$

Thus $W_i(0) = 0$ for all $i \in \mathbb{N}$, so

$$W(u) = \sum_{i=1}^{\infty} W_i(u_i) \text{ for all } u \in \mathbb{R}^\infty \tag{12}$$

Each W_i must be nondecreasing in order to satisfy Semistrong Pareto. So by Aczél and Dhombres (1989, Corollary 2.5, p. 15), for each W_i there must be a constant $c_i \geq 0$ such that

$$W_i(x) = c_i x \text{ for all } x \in \mathbb{R} \tag{13}$$

In order to satisfy Weak Pareto and the full anonymity of $\succcurlyeq_\infty$, there must be some $c > 0$ such that $c_i = c$ for all $i \in \mathbb{N}$. So

$$W(u) = \sum_{i=1}^{\infty} c(u_i) = c \sum_{i=1}^{\infty} u_i \text{ for all } u \in \mathbb{R}^\infty \tag{14}$$

For any such c, and any $x, y \in X$, $R \in \mathcal{D}$, and $U \in \mathcal{U}_R$, $W(\iota(U(x,\cdot))) \geq W(\iota(U(y,\cdot)))$ iff $\sum_{i \in N(x)} U(x,i) \geq \sum_{i \in N(y)} U(y,i)$.

□

E An Alternative Characterization of Classical Utilitarianism

Blackorby et al. (2005, Theorem 6.24) characterize the "classical means of order r." These are social welfare orderings which compare utility distributions as follows: there exist $\beta, r \in \mathbb{R}_{++}$

such that for any $M, N \in \mathcal{P}$, $u \in \mathbb{R}^M$, and $v \in \mathbb{R}^N$, $u \succcurlyeq^* v$ iff

$$\sum_{i \in M: u_i \geq 0} u_i^r - \beta \sum_{i \in M: u_i < 0} (-u_i)^r \geq \sum_{i \in N: v_i \geq 0} v_i^r - \beta \sum_{i \in N: v_i < 0} (-v_i)^r \tag{15}$$

To characterize these orderings we introduce three new conditions:

Variable-Population Continuity For all $N, M \in \mathcal{P}$ and $u \in \mathbb{R}^M$, the sets $\{ v \in \mathbb{R}^N \mid v \succcurlyeq^* u \}$ and $\{ v \in \mathbb{R}^N \mid u \succcurlyeq^* v \}$ are closed in $\mathbb{R}^N$.

Weak Existence of Critical Levels For some $N \in \mathcal{P}$, $u \in \mathbb{R}^N$, $i \in \mathbb{N} \setminus N$, and $v \in \mathbb{R}^{N \cup \{i\}}$ such that $v_j = u_j$ for all $j \in N$, $u \sim^* v$.

Existence Independence For all $u, v, w \in \Omega$ such that $u \cup w, v \cup w \in \Omega$, $u \succcurlyeq^* v$ iff $u \cup w \succcurlyeq^* v \cup w$.

Proposition 9. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f is associated with a classical mean of order r iff f satisfies Interpersonal Extensive Domain, Interpersonal Ratio IIA, Strong Pareto, and Fixed-Population Anonymity and its associated social welfare ordering satisfies Variable-Population Continuity, Weak Existence of Critical Levels, and Existence Independence.*

Proof. Weak Existence of Critical Levels, Existence Independence, and Strong Pareto together imply that there is a single critical level for all alternatives (Blackorby et al., 2005, Theorem 6.9). Interpersonal Ratio-Scale Invariance then implies Null Critical Levels (Blackorby et al., 2005, Theorem 6.23), so the social welfare function satisfies Extended Welfarism (Lemma 7). So, by Fixed-Population Anonymity and Lemma 8, the social welfare ordering must be fully anonymous. The axioms of Blackorby et al. (2005, Theorem 6.24) are therefore satisfied, so $\succcurlyeq^*$ must be a classical mean of order r. □

The classical utilitarian social welfare ordering is the classical mean of order r with $\beta, r = 1$. As Blackorby and Donaldson (1982, Theorem 4) show, we can force $r = 1$ and $\beta \geq 1$ by requiring the social welfare ordering to be weakly averse to inequality, in the following sense. For any $N \in \mathcal{P}$ and distributions $u, v \in \mathbb{R}^N$, u is *unambiguously at least as equal as v* iff either u is a permutation of v or u is obtainable from v via finitely many Pigou-Dalton utility transfers (Blackorby et al., 2005, p. 93).

Weak Inequality Aversion For any $N \in \mathcal{P}$ and $u, v \in \mathbb{R}^N$, if u is unambiguously at least as equal as v, then $u \succcurlyeq *v$.

This still leaves the possibility that $\beta > 1$, in which case negative utilities are weighted more heavily than positive ones. This can be ruled out by imposing

Reflection Anti-Invariance For any $u, v \in \Omega$ and $k < 0$, $u \succcurlyeq^* v$ iff $kv \succcurlyeq^* ku$.[8]

Putting all this together, we have

Theorem 6. *If a generalized social welfare function f satisfies Interpersonal Extensive Domain, then f satisfies Classical Utilitarianism iff f satisfies Strong Pareto and Ratio IIA and its associated social welfare ordering satisfies Variable-Population Continuity, Weak Existence of Critical Levels, Existence Independence, Weak Inequality Aversion, and Reflection Anti-Invariance.*

Proof of Theorem 6. Suppose that f satisfies Interpersonal Extensive Domain, Strong Pareto, and Ratio IIA. By Theorem 4 and Proposition 6, f satisfies Variable-Population Welfarism and the associated social welfare ordering satisfies Interpersonal Ratio-Scale Invariance. Weak Inequality Aversion implies Fixed-Population Anonymity so, by Proposition 9, $\succcurlyeq^*$ is a classical mean of order r. Weak Inequality Aversion then implies (by Blackorby & Donaldson, 1982, Theorem 4) that $r = 1$. It is easy to see that Reflection Anti-Invariance can then be satisfied only if $\beta = 1$ as well. □

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[8] The decision-theoretic analogue of this principle is introduced and defended by Goodsell (2023).

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