Global Formalism of Loop Quantum Gravity

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Contents

0. Overview (in German)

Laut Definition ist die *Quantengravitation* eine Quantenfeldtheorie der Allgemeinen Relativitätstheorie von Albert Einstein [69]. Sie ist somit eine Theorie, welche die beiden fundamentalen Bausteine der modernen Physik, (1.) die allgemeine Kovarianz der *Allgemeinen Relativitätstheorie* und (2.) die Unschärferelation der *Quantenmechanik*, verbindet.
Ein möglicher Kandidat einer solchen Theorie der Quantengravitation ist die *Schleifenquantengravitation (***L***oop* **Q***uantum* **G***ravity)*. Sie ist demnach ein Versuch eine nicht perturbative, Hintergrund unabhängige – wie es für Gravitationstheorien natürlich scheint – Quantenfeldtheorie der Gravitation zu konstruieren. Unter dem Begriff *Hintergrundunabhängigkeit* versteht man vereinfacht gesprochen die Annahme, dass die Gesetze der Physik, welche mathematisch durch die klassischen Einstein Gleichungen ausgedrückt werden, allgemein kovariant sind.

Die Schleifenquantengravitation wurde in den neunziger Jahren des letzten Jahrhunderts von Ashtekar, Lewandowski, Rovelli, Smolin, Thiemann und weiteren entwickelt [69, 62]. Ausgangspunkt der Schleifenquantengravitation ist eine Hamilton'sche Formulierung der Allgemeinen Relativitätstheorie. Im Rahmen dieser Formulierung wird zunächst eine sogenannte (3+1)-Zerlegung durchgeführt, wodurch die vierdimensionale Raumzeit, modelliert durch eine Loretz-Mannigfaltigkeit $(\mathcal{M}, g)$, als eine Blätterung aus dreidimensionalen raumartigen Cauchy-Hyperflächen dargestellt ist. Hierbei sind die Hyperflächen isomorph zu einer Riemannschen Mannigfaltigkeit (Σ, q), das heißt es gilt $\mathcal{M} \cong (\mathbb{R} \times \Sigma, -N^2\,\mathrm{d}t^2 + q_t)$, wobei N die Lapse-Funktion bezeichnet. Den Ansatz für die Entwicklung der Schleifenquantengravitation lieferte Ashtekar in den Jahren 1986 und 1987 mit der Einführung der sogenannten *Ashtekar-Variablen* [5, 6]. Das Besondere an diesen Variablen ist, dass sie eine Hamilton'sche Formulierung der klassischen Gravitationstheorie ermöglichen, welche vergleichsweise gut quantisierbar zu sein scheint. Die Ashtekar-Variablen bilden somit die Basis der Schleifenquantengravitation. Sie bestehen aus den kanonischen Variablen (A, E), wobei A als Ashtekar-Zusammenhang und E als gewichtetes Dreibein bezeichnet wird. Die Rolle der Koordinaten in dieser Theorie übernimmt A auf $T\Sigma$ und die zugehörenden konjugierten Impulse sind durch ein gewichtetes Dreibein (orthogonaler Rahmen) E auf der Cauchy-Hyperfläche Σ gegeben. Mit Hilfe dieser Variablen erhält man eine klassische Gravitationstheorie in Hamilton'scher Formulierung, deren Zwangsbedingungen (die Gauß-, Diffeomorphismen- und Hamilton-Zwangsbedingung) polynomial in diesen Variablen sind [6].

Eines der zentralen Ergebnisse dieser Quantentheorie der Gravitation ist die Vorhersage einer diskreten Struktur der Raumzeit, anhand welcher neue physkalische Vorhersagen möglich sind. Im Einzelnen können einige langjährige Probleme wie die Beschaffenheit des Big Bangs, welcher durch einen sogenanntem *Big Bounce* ersetzt wird [24, 29], oder die Physik des frühen Universums (*Inflation*) [25, 1] und die Eigenschaften von quantisierten Schwarzen Löchern [7] mit Methoden der Theorie der Schleifenquantengravitation gelöst werden. Die Kinematik der Theorie, welche in den Gauss- und Diffeomorphismus-Zwangsbedingungen kodiert ist, ist wohlverstanden und dessen Lösungsraum wird durch die sogenannte *Spin-Netzwerk-Basis* aufgespannt [31]. Jedoch ist keine vollständige allgemeine Lösungstheorie bezüglich der Hamilton-Zwangsbedingung, sprich der Dynamik bekannt. Mit dem Lösungsansatz von Thiemann [69] erhält man einerseits zwar eine wohldefinierte Hamilton-Zwangsbedingung, deren Wirkung explizit bekannt und endlich ist. Und darüber hinaus kann gezeigt werden, dass Thiemanns Hamilton-Zwangsbedingung frei von Anomalien ist, das heißt, dass keine weiteren Zwangsbedingung notwendig sind, um den semiklassischen Limes rückzugewinnen. Dennoch kann man die Theorie nicht als vollständig bezeichnen, denn weder das volle Spektrum der Hamilton-Zwangsbedingung noch die physikalische Charakterisierung des Hilbert-Raums ist vollkommen verstanden. Genauer gesagt liegt das Problem darin, dass die Hamilton-Zwangsbedingung nicht frei von Mehrdeutigkeiten ist, und somit die physikalische Interpretation der Lösungen unklar ist. Um diese Mehrdeutigkeiten zu beseitigen, werden Auswahlkriterien eingeführt, deren physi kalische Bedeutung jedoch noch nicht geklärt ist [11]. Somit besteht die Herausforderung darin, in der Quantendynamik der Theorie Lösungen aller quantisierten Zwangsbedingungen zu finden und diese physkalischen Zustände mit der Struktur eines geeigneten Hilbert-Raumes zu versehen.

Diese Problematik der Schleifenquantengravitation und deren mathematische Struktur soll in der vorliegenden Arbeit mit Hilfe eines differentialgeometrischen Zugangs diskutiert werden. Insbesondere wird untersucht, inwieweit die Variablen und die Zwangsbedingungen der Theorie auch in einer global geltenden Form dargestellt werden können und ob dadurch ein besseres Verständnis der Theorie ermöglicht wird. Diese Arbeit ist im Wesentlichen in zwei Teile gegliedert.

Konstruktion und Eigenschaften des Ashtekar-Zusammenhangs

Im ersten Teil wird die Konstruktion des Ashtekar-Zusammenhangs studiert. Hierbei wird die Diskussion von [33] aufgegriffen und fortgeführt und inbesondere werden die Beweise aus [33] mathematisch detailliert ausgearbeitet. Dabei konstruiert man den Ashtekar-Zusammenhang mit Hilfe der Theorie der Faserbündel als ein global definiertes Objekt. Das übergeordnete Ziel dabei ist die Klassifizierung der Menge aller Zusammenhänge $\mathcal{C}(\mathrm{O}^+(\Sigma, q))$ auf dem SO(3)-Hauptfaserbündel der orthonormalen, geordneten und orientierten Rahmen $\mathrm{O}^+(\Sigma, q)$ über Σ. In diesem Zusammenhang wird $\mathcal{C}(\mathrm{O}^+(\Sigma, q))$ mit der Menge der $(1,1)$-Tensorfelder auf Σ identifiziert. Die Konstruktion ist in drei Schritte aufgeteilt.

i.) Zunächst wird gezeigt, dass der Raum aller Zusammenhänge auf $\mathrm{O}^+(\Sigma, q)$ ein affiner Raum mit zugrundeliegendem Vektorraum $\Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, q), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$ der horizontalen 1-Formen vom Typ ad auf $\mathrm{O}^+(\Sigma, q)$ mit Werten in der Lie-Algebra $\mathfrak{so}(3)$ ist. Dieser Raum kann mit $\Omega^1(\Sigma, E^{\mathrm{ad}})$, dem Raum der 1-Formen auf Σ mit Werten im assoziierten Bündel $E^{\mathrm{ad}} = \mathrm{O}^+(\Sigma, q) \times_{(\mathrm{SO}(3),\mathrm{ad})} \mathfrak{so}(3)$, durch einen

Isomorphismus $\mathfrak{X}$ identifiziert werden.

ii.) Anschließend benutzt man die Äquivalenz der adjungierten Darstellung ad und definierenden Darstellung ρ von SO(3). Der Isomorphismus $\mathfrak{f}: \mathfrak{so}(3) \to \mathbb{R}^3$ induziert einen Isomorphismus $\mathfrak{F}$ zwischen E^{ad} und $E^\rho := \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3),\rho)} \mathbb{R}^3$. Hierbei soll darauf hingewiesen werden, dass anhand des Isomorphismus' $\mathfrak{f}$ die Wahl der Standardbasis des $\mathbb{R}^3$ beziehungsweise die Basis der $\mathfrak{so}(3)$ explizit in die Konstruktion des Ashtekar-Zusammenhangs eingeht. Diese Wahl ist fundamental für die Konstruktion des Ashtekar-Zusammenhangs und es ist somit ersichtlich, dass die Konstruktion ausschließlich auf vierdimensionalen Raumzeiten, und somit dreidimensionalen Cauchy-Hyperflächen möglich ist.

iii.) Zu guter Letzt nutzt man den Isomorphismus $\mathfrak{V}$ zwischen dem Vektorbündel E^ρ und dem Tangentialbündel $T\Sigma$.

Somit gibt es, wie in Chapter 4, Theorem 4.1.1 gezeigt wird, eine Eins-zu-eins-Beziehung $\mathfrak{I}$ zwischen der Menge der Zusammenhangsformen $\mathcal{C}(\mathrm{O}^+(\Sigma, q))$ auf $\mathrm{O}^+(\Sigma, q)$ und der Menge der $(1,1)$-Tensorfelder $\mathrm{T}^{(1,1)}(\Sigma)$ auf Σ. Der zugehörige Isomorphismus, der diese Identifizierung ermöglicht ist mit

$$\mathfrak{I} : \Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, q), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})} \xrightarrow{\cong} \Omega^1(\Sigma, \mathrm{T}\Sigma), \quad \mathfrak{I} := \mathfrak{V} \circ \mathfrak{F} \circ \mathfrak{X} \tag{0.1}$$

bezeichnet.

Daraus lässt sich direkt folgendes zentrale Resultat folgern, siehe Chapter 4,Theorem 4.1.2.

Theorem 0.0.1. *(Siehe [33]) Die Menge aller Zusammenhänge auf* $\mathrm{O}^+(\Sigma, q)$ *über* Σ *ist durch*

$$\mathcal{C}(\mathrm{O}^+(\Sigma, q)) \cong \{\omega^{\mathrm{LC}} + \mathfrak{I}^{-1}(S) | S \in \Omega^1(\Sigma, T\Sigma) = \mathrm{T}^{(1,1)}(\Sigma)\}$$

gegeben. Hierbei bezeichnet ω^{LC} *den Levi-Civita-Zusammenhang und* $\mathfrak{I}$ *die durch Eq.* (0.1) *definierte Abbildung.*

Die Konstruktion, die zur Identifizierung von $\mathcal{C}(\mathrm{O}^+(\Sigma, q))$ mit der Menge der $(1,1)$-Tensorfelder $\mathrm{T}^{(1,1)}(\Sigma)$ führt, ist die Verallgemeinerung der Konstruktion des Ashtekar-Zusammenhangs. Man erhält den Ashtekar-Zusammenhang, indem man die spezielle Wahl $S = \beta \mathrm{Wein}$ trifft, wobei Wein die Weingarten-Abbildung der Cauchy-Hyperfläche $\Sigma \subset \mathcal{M}$ und $\beta \in \mathbb{R}^*$ den Barbero-Immirzi-Parameter [16, 17, 44] bezeichnet. Aus diesen Vorbereitungen kann folgende geometrische Definition des Ashtekar-Zusammenhangs gewonnen werden:

Theorem/Definition 0.0.2. *(Siehe [33]) Der Ashtekar Zusammenhang bezüglich β ist definiert durch*

$$A := \omega^{\mathrm{LC}} + \beta \mathfrak{I}^{-1}(\mathrm{Wein}) \in \Omega^1(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}. \tag{0.2}$$

Um globale Ausdrücke für die kovariante Ableitung, die Torsion und die Krümmung des Ashtekar-Zusammenhangs zu erhalten, wird auf $T\Sigma$ ein Produkt eingeführt, welches das Kreuzprodukt auf $\mathbb{R}^3$ verallgemeinert:

Definition 0.0.3. *(Siehe [33]) Es sei $e \in \mathrm{O}^+(\Sigma, q)$ ein orientierter, orthonormaler Rahmen und $X = \sum_i X^i e_i$, $Y = \sum_j Y^j e_j \in T\Sigma$, mit $X^i, Y^j \in \mathbb{R}$. Für jeden orientierten, orthonormalen Rahmen e ist die Produktstruktur auf $T\Sigma$ durch*

$$\bowtie : T\Sigma \times T\Sigma \longrightarrow T\Sigma, \quad X \bowtie Y := \sum_{ijk} \epsilon_{ijk} X^i Y^j e_k$$

definiert. Diese Produktstruktur lässt sich durch faserweise Konstruktion auf Schnitte in $T\Sigma$, $\Gamma(T\Sigma)$, übertragen.

Theorem 0.0.4. *(Siehe [33]) Seien $X, Y \in \Gamma(T\Sigma)$. Die zum Ashtekar-Zusammenhang Eq. (0.2) gehörende kovariante Ableitung $\nabla^{\mathrm{A}} : \Gamma(T\Sigma) \longrightarrow \Gamma(T^*\Sigma \otimes T\Sigma)$ ist durch*

$$\nabla^{\mathrm{A}}_X := \nabla^{\mathrm{LC}}_X Y + \beta \mathrm{Wein}(X) \bowtie Y$$

gegeben.

Proposition 0.0.5. *i.) ∇^{A} ist metrisch mit nichtverschwindender Torsion, siehe [33]. Die Torsion T^{A} kann wie folgt ausgedrückt werden:*

$$T^{\mathrm{A}}(X,Y) = \beta[\mathrm{Wein}(X) \bowtie Y - \mathrm{Wein}(Y) \bowtie X]$$

ii.) Die Krümmung des Ashtekar-Zusammenhangs ist gegeben durch, siehe [33]

$$\begin{aligned} R^{\mathrm{A}}(X,Y)Z =& R^{\mathrm{LC}}(X,Y)Z \\ &+ \beta[(\nabla^{\mathrm{LC}}_X \mathrm{Wein})(Y) - (\nabla^{\mathrm{LC}}_Y \mathrm{Wein})(X)] \bowtie Z \\ &+ \beta^2[\mathrm{Wein}(X) \bowtie \mathrm{Wein}(Y)] \bowtie Z, \end{aligned}$$

wobei $X, Y, Z \in \Gamma(T\Sigma)$ und R^{LC} den Krümmungstensor bezüglich des Levi-Civita-Zusammenhangs darstellt.

iii.) Für die Skalarkrümmung des Ashtekar-Zusammenhang erhalt man den Ausdruck

$$\mathcal{R}^{\mathrm{A}} = \mathcal{R}^{\mathrm{LC}} + \beta^2[\mathrm{tr}(\mathrm{Wein})^2 - \mathrm{tr}(\mathrm{Wein}^2)],$$

wobei $\mathcal{R}V$ die Skalarkrümmung bezüglich des Levi-Civita-Zusammenhangs ist.

Des Weiteren lassen sich die Bianchi-Identitäten verallgemeinern.

Theorem 0.0.6. *Der Krümmungstensor $R^{\mathrm{A}} \in \Gamma(\Lambda^2 T^*\Sigma \otimes \mathrm{End}(T\Sigma))$ des Ashtekar-Zusammenhangs erfüllt für alle Vektorfelder $X, Y, Z \in \Gamma(T\Sigma)$ die folgenden verallgemeinerten Bianchi-Identitäten*

1. Bianchi-Identität:

$$\begin{aligned} \mathfrak{S}\{R^{\mathrm{A}}(X,Y)Z\} = \mathfrak{S}\{&(\mathrm{Wein}(X) \bowtie \mathrm{Wein}(Y)) \bowtie Z \\ &+ \nabla^{\mathrm{LC}}_X T^{\mathrm{A}}(Y,Z) + T^{\mathrm{A}}(X,[Y,Z])\}; \end{aligned}$$

2. Bianchi-Identität:

$$\mathfrak{S}\{(\nabla^{\mathrm{A}}_Z R^{\mathrm{A}})(X,Y)\} = \mathfrak{S}\{R^{\mathrm{A}}(T^{\mathrm{A}}(X,Y),Z)\}$$
$$= - \mathfrak{S}\{R^{\mathrm{A}}(\beta[\mathrm{Wein}(X) \bowtie Y - \mathrm{Wein}(Y) \bowtie X], Z)\},$$

wobei $\mathfrak{S}$ die zyklische Summe bezüglich X, Y, Z ist.

Nach der Einführung des Ashtekar-Zusammenhangs und dessen Eigenschaften wird die Spinstruktur des Ashtekar-Zusammenhangs diskutiert. Hierbei wird folgendes Resultat bewiesen:

Theorem 0.0.7. *Sei $\omega \in \Omega^1(\mathrm{O}^+(\mathcal{M},q),\mathfrak{so}(3))$ eine Zusammenhangsform und $\tilde{\omega} \in \Omega^1(S(\Sigma),\mathfrak{su}(2))$ die zugehörige Zusammenhangsform im Spinbündel. Diese induzieren die gleichen kovarianten Ableitungen auf dem Tangentialbündel $T\mathcal{M}$.*

Dieses Theorem rechtfertigt also die Konstruktion des Ashtekar-Zusammenhangs als SO(3)-Zusammenhang, im Gegenstatz zu dem in der Literatur verwendeten Ausdrucks als SU(2)-Zusammenhang, da die Wirkung des Ashtekar-Zusammenhangs auf dem Tangentialbündel $T\Sigma$ unabhängig davon ist, ob der Ashtekar-Zusammenhang als SO(3)-Zusammenhang oder durch Liftung in das Spinbündel $S(\Sigma)$ als SU(2)-Zusammenhang betrachtet wird, siehe dazu auch [33].

Gleichung (0.2) erlaubt es zudem die Hamilton'sche Formulierung der Gravitation und die zugehörigen Zwangsbedingungen in diesem neuen globalen Formalismus darzustellen. Man erhält beispielsweise folgenden modifizierten Ausdruck für die Einstein-Hilbert-Wirkung:

Theorem 0.0.8. *Hinsichtlich des Ashtekar Zusammenhangs ist die Einstein-Hilbert-Wirkung durch*

$$S_{\mathrm{EH}} = \int_{\mathcal{M}} \big(\mathcal{R}^{\mathrm{A}} + (1+\beta^2)[\mathrm{tr}(\mathrm{Wein}^2) - \mathrm{tr}(\mathrm{Wein})^2]\big)\, \mathrm{dvol}[g],$$

gegeben. Wählt man – wie in den ursprünglichen Arbeiten Ashtekars' – $\beta = \mathrm{i}$, so erhält man den folgenden sehr eleganten Ausdruck:

$$S_{\mathrm{EH}} = \int_{\mathcal{M}} \mathcal{R}^{\mathrm{A}}\, \mathrm{dvol}[g].$$

Als Fortführung der Arbeit werden erste Schritte unternommen die Zwangsbedingungen in dem hier entwickelten Formalismus zu übersetzen. Dies führt für die Wahl $\beta = \mathrm{i}$ zu folgendem ästhetischen Ausdruck für die Hamilton-Zwangsbedingung

$$H = \mathcal{R}^{\mathrm{A}}. \tag{0.3}$$

Quantisierung des Hamilton-Zwangsbedingung

Der zweite Teil der Arbeit befasst sich mit der Quantisierung der globalen Hamilton-Zwangsbedingung Eq. (0.3), indem ein Krümmungsskalar-Operator $\widehat{\mathcal{R}^{\mathrm{A}}}$ definiert wird. Hierfür wird die *diskrete Quantengeometrie* des dualen Bildes der Schleifenquantengravitation herangezogen. Die Konstruktion des dualen Bildes ist mit der Konstruktion der *Wigner-Seitz-Zelle* in der Festkörperphysik vergleichbar. Der Quantenzustand der 3-Geometrie Σ wird durch eine Linearkombination sogenannter *Spin-Netzwerk-Zustände* $\Psi(\Gamma)$ dargestellt. Ein Spin-Netzwerk-Zustand $\Psi(\Gamma; e, n)$ ist durch einen Graphen $\Gamma \subset \Sigma$, bestehend aus Kanten e und Knoten n, definiert. Somit entspricht die Spin-Netzwerk-Struktur von Σ der Struktur eines Kristallgitters, wobei die Gitterpunkte den Knoten des Graphen gleichkommen. Dadurch liegt das folgende duale Bild der Quantengeometrie eines Spin-Netzwerk-Zustandes vor, siehe auch Abbildung 3.8(b): jeder Knoten $n \in \Gamma$ des Spin-Netzwerks entspricht einer Region R_n mit bestimmetem Volumen Vol, den sogenannten *chunks of space*. Die Kanten, welche zwei Knoten verbinden, entsprechen der Fläche S_i mit bestimmtem Flächeninhalt Ar. Diese Flächen sind der Abschluss der *chunks of sapce*. Des Weiteren idenfizieren die beiden Flächen S_i, S_j in ihrem Schnittpunkt eine Kurve c mit bestimmter Länge L . Das bedeutet also, dass ein Spin-Network-Zustand im dualen Bild zu einer Triangulierung $\triangle$ der Cauchy-Fläche Σ führt und dass sich Größen wie Volumen, Flächeninhalt und Länge quantifizieren lassen.

Diese Diskretisierung von Σ ist somit in natürlicher Art und Weise mit dem *Regge-Kalkül* der Allgemeinen Relativitätstheorie verbunden [60] . Im Regge-Kalkül werden Mannigfaltigkeiten durch einen Simplizialkomplex $\triangle$, der aus Simplizes $\sigma \in \triangle$ besteht, trianguliert. Die sogenannte Regge-Wirkung der Allgemeinen Relativitätstheorie ist durch

$$S_{\mathrm{Regge}}(L_h^\sigma, \epsilon_h) = \sum_{\sigma \in \triangle} \sum_{h \in \sigma} L_h^\sigma \epsilon_h$$

definiert, wobei L_h^σ die Kantenlänge eines *hinges* h von σ und $\epsilon_h = 2\pi - \sum_{\sigma \ni h} \mathrm{Ang}_h^{\sigma,\alpha_h}$ den *Defizitwinkel* bezeichnet. Hier beschreibt $\mathrm{Ang}(\sigma, h)$ den Öffnungswinkel zwischen dem ausgezeichneten Symplex σ und dem benachbarten Symplex σ', welche sich in h schneiden. Der Index α_h verdeutlicht die Abhängigkeit des Öffnungswinkels von den angrenzenden Simplizes. Somit sind $(L_h^\sigma, \epsilon_h(\mathrm{Ang}))$ die dynamischen Variablen in diesem Zugang. Es zeigt sich, dass für immer feinere Triangulierungen die Riemann-Summe der Regge-Wirkung in den Integralausdruck der Einstein-Hilbert-Wirkung übergeht, das heißt

$$\lim_{\triangle \to 0} S_{\mathrm{Regge}}(L_h^\sigma, \epsilon_h) = \frac{1}{2} S_{\mathrm{EH}} = \frac{1}{2} \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathcal{R}^{\mathrm{A}} \, \mathrm{dvol}[q].$$

Aufgrund des Theorems von Gauß-Bonnet kann die Skalarkrümmung $\mathcal{R}^{\mathrm{A}}$ demnach mit der Summe der Defizitwinkel über alle hinges $h \in \sigma$ mit Kantenlänge L, d.h. $\sum_{h \in \sigma} L_h^\sigma \epsilon_h$ identifiziert werden [49, 55]. Daher ist es möglich, einen Krümmungsskalar-Operator $\widehat{\mathcal{R}^A}$ durch einen Längenoperator $\widehat{\mathrm{L}}$ und Winkeloperator $\widehat{\mathrm{Ang}}$ zu definieren und dadurch die Hamilton-Zwangsbedingung Eq. (0.3) zu quantisieren.

THEOREM 0.0.9. *Auf dem kinematischen Hilbert-Raum* $\mathcal{H}_{\mathrm{kin}} = \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma$, *d.h. dem Lösungsraum der Gauß- und Diffeomorphismen-Zwangsbedingung der Schleifenquantengravitation, wobei* $\mathcal{H}_\Gamma$ *den Hilbertraum entsprechend einem gegebenen Graphen* Γ *darstellt, ist der Längenoperator bezüglich einer Kurve* c *durch*

$$\widehat{\mathrm{L}}(\mathrm{c}_\omega) = \sqrt{\delta_{ij} \widehat{G}^{i\dagger}(\mathrm{c}_\omega) \widehat{G}^j(\mathrm{c}_\omega)} \tag{0.4}$$

gegeben, wobei $\widehat{G}^j$ ein beschränkter Operator auf $\mathcal{H}_{\text{kin}}$ ist. $\widehat{L}(\mathrm{c}_\omega)$ ist selbstadjungiert. Die Kurve c_ω ist dual zum Tripel $\omega := \{n, e_1, e_2\}$, dem sogenannten wedge, *bestehend aus einem Knoten n und zwei Kanten e_1, e_2 des Graphs Γ, siehe [21].*

Bei der Konstruktion des Winkeloperators werden ähnliche Regularisierungstechniken wie bei der Konstruktion des Längenoperators angwandt, und man erhält den folgenden Aussdruck für den Winkeloperator zwischen den beiden Flächen $S_1 \in \sigma_1$ (ausgezeichnet durch e_1) und $S_2 \in \sigma_2$ (ausgezeichnet durch e_2) auf $\mathcal{H}_{\text{kin}}$ durch

$$\widehat{\text{Ang}}(\mathrm{c}_\omega) = \arccos\left(\frac{\widehat{Y}^i(\mathrm{c}_\omega)}{\widehat{\text{Ar}}(S_1)\widehat{\text{Ar}}(S_2)}\right). \tag{0.5}$$

Hierbei bezeichnen $\widehat{\text{Ar}}(S_i)$ beziehungsweise $\widehat{Y}^i(c_\omega)$, die in der Schleifenquantengravitation bekannten Operatoren, den *Flächen-Operator* beziehungsweise den sogenannten *two-hand-Operator* [62]. $\widehat{\text{Ang}(\mathrm{c}_\omega)}$ ist selbstadjungiert. Die Winkel zwischen σ_1 und σ_2 ist also durch das Tripel $\omega := \{n, e_1, e_2\}$, bestehend aus einem Knoten n und zwei Kanten e_1, e_2 des Graphs Γ, eindeutig festgelegt. Daraus folgt das zentrale Resultat. Der Krümmungsskalar-Operator bezüglich des Ashtekar-Zusammenhangs $\widehat{\mathcal{R}^A_\sigma}$ ist anhand von Eq. (0.4) und Eq. (0.5) durch

$$\widehat{\mathcal{R}^A_\sigma} := \sum_{h\in\sigma}\left[\widehat{\mathrm{L}^\sigma_h}\left(2\pi - \sum_{\sigma\ni h}\widehat{\text{Ang}^{\sigma,\alpha_h}_h}\right)\right] \tag{0.6}$$

gegeben, hierbei gilt $\widehat{L^\sigma_h} := \widehat{L}(\mathrm{c}_\omega)$ und $\widehat{\text{Ang}^\sigma_h} := \widehat{\text{Ang}}(\mathrm{c}_\omega)$. Der in Eq. (0.6) definierte Krümmungsskalar-Operator $\widehat{\mathcal{R}^A_\sigma}$ hat folgende Eigenschaften:

i.) $\widehat{\mathcal{R}^A_\sigma}$ ist selbstadjungiert,

ii.) $\widehat{\mathcal{R}^A_\sigma}$ ist von der Wahl der Triangulierung $\triangle$ abhängig, denn

  a.) Die Wirkung von $\widehat{\mathcal{R}^A_\sigma}$ auf ein Spin-Netzwerk-Zustand $\Psi(\Gamma') \in \mathcal{H}_\Gamma$, mit $\Gamma' \neq \Gamma$, ergibt null, außer ein Knoten $n \in \Gamma'$ liegt im Symplex σ, b.) Die Wirkung von $\widehat{\mathcal{R}^A_\sigma}$ ist abhängig von dem Symplex σ, welches die Knoten n von Γ einschließt (Auswahl der wedges) und von den Symplizes, welche an σ angrenzen (bestimmt α_h).

Zum Abschluss der Arbeit wird die Quanten-Hamilton-Zwangsbedingung, ausgedrückt in der urprünglichen Darstellung, d.h. mit einer komplexen Zusammenhangsform $\beta = \mathrm{i}$, im dem hier vorgestelltem Formalismus angegeben. Die Quanten-Hamilton-Zwangsbedingung ist in der globalen, hier diskutierten Form, explizit duch

$$\widehat{\mathcal{R}^A_\sigma}|\psi(\gamma)\rangle = \sum_{h\in\sigma}\left[\widehat{\mathrm{L}^\sigma_h}\left(2\pi - \sum_{\sigma\ni h}\widehat{\mathrm{Ang}^{\sigma,\alpha_h}_h}\right)\right]|\psi(\gamma)\rangle \equiv 0, \quad \forall\sigma\in\triangle$$

gegeben.

Fazit

In der vorliegenden Arbeit ist die geometrische Struktur des Ashtekar-Zusammenhangs hergeleitet worden. In diesem Kontext konnte die Schleifenquantengravitation in diesem mathematischen Rahmen global formuliert werden. Darüber hinaus konnte gezeigt werden, dass die Hamilton-Zwangsbedingung äquivalent der Forderung ist, dass die Skalarkrümmung des Ashtekar-Zusammenhangs identisch verschwindet. Diese Sichtweise ermöglichte es schließlich, eine quantisierte Version dieser Zwangsbedingung anzugeben.

1. Introduction

The text in hand is build up by two main parts and is concerned with a mathematical investigation of canonical quantum gravity. The first part contains the mathematical construction of the so-called Ashtekar connection within the theory of fibre bundles. The second part includes by using the Regge calculus the implementation of the Hamiltonian constraint in the presented global formalism of Loop Quantum Gravity (LQG), which by itself is a candidate for a Quantum Field Theory in four dimensions which achieves to unify the principles of Quantum Theory and General Relativity, see e.g. for reviews [11] and [62, 69] for books.

In 1987, Abhay Ashtekar reformulated Einstein's field equations of general relativity using what have come to be known as Ashtekar variables [5, 6]. Around 1990, Rovelli and Smolin obtained an explicit basis of states of quantum geometry which illustrated the quantization of geometry, that is, the (non-gauge-invariant) quantum operators representing the discreetness of the spectrum of area and volume [63, 61] which is one of the main predictions of LQG. Thereforet LQG implements the fundamental feature of general relativity which is its non-perturbative background independence [30], in a quantum setting. The main advantage of the Ashtekar variables has been that they drastically simplified the constraints of gravity, which become polynomial. This enables a completely new way to approach the quantization of gravity, ultimately leading to Loop Quantum Gravity.

In Loop Quantum Gravity, the main message of general relativity is taken seriously: in general relativity the metric itself is considered as a dynamical object in other words this means gravity is geometry. For this reason in a fundamental quantum gravity theory, there should be no background metric. Therefore geometry and matter should both arise quantum mechanically at once. Thus in contrast to approaches according to particle physicists one does not start with quantum matter on a background geometry and use perturbation theory to implement quantum effects of gravity. Briefly spoken there is a manifold but no metric, or indeed any other physical fields, in the background. This point mentioned above explains why a non-perturbative and thus background independent quantization is chosen in the LQG framework.

In classical gravity the appropriate mathematical language to formulate the physical, kinematical notions as well as the final dynamical equations is provided by Riemannian geometry. Now in quantum general relativity, *quantum* Riemannian geometry adopts this rôle. In the classical domain, the best available theory of gravity is represented by Einstein's general relativity, whose predictions have been examined to an amazing degree of precision. Hence, a natural question arises: exists a quantum general relativity as a consistent theory non-perturbatively? But at this point we want to mention that there is no consequence that such a theory would be the unique, final and complete description of Nature. Nonetheless, in its own right this is a really exciting and important open question.

Over the last quarter of a century, there has been only a single, but significant extension of Ashtekar's variables. In the mid-90s, Barbero [16, 17] and Immirzi [44] added a new parameter $\beta \in \mathbb{R}$ and $\beta \in \mathbb{C}$, respectively. Where the choice $\beta = i$ giving the original variables. Now the great benefit of real β is that the structure group is no longer SlC(2), but SU(2).

For the integration theory of Loop Quantum Gravity this fact has been crucial. Before the Barbero-Immirzi idea has been introduced, in order to implement the real structure of the theory complicated reality condition had been necessary. However, the new formulation has the disadvantage that the Hamiltonian constraint is no longer polynomial. Only by the so-called *Thiemann's trick* [67] this have been mitigated. Thereby the term with prefactor $1 + \beta^2$ has been rewritten by means of certain Poisson brackets. This fact unites the advantages of functional analysis of polynomial constraints and the integration theory on compact structure groups. But we want to emphasize that the original choice, i.e. $\beta = i$, has significant advantage over the real β, as we will show in this thesis.

But in the full theory the challenge of quantum dynamics is to find solutions to the quantum constraint equations and equip these physical states with the structure of an appropriate Hilbert space. In the community of LQG the general consensus is that while on the one hand the situation for the Gauss and diffeomorphism constraints is well-understood, but on the other hand it is far from being definite for the Hamiltonian constraint. In 1996 non-trivial development due to Thiemann is that well-defined candidate operators representing the Hamiltonian constraint exist on the space of solutions to the Gauss and diffeomorphism constraints [69]. However there are several ambiguities [11], which must be fixed in order to make progress but, unfortunately, we have no understanding for the physical meaning of choices made to resolve them.

In the reduced context of Loop Quantum Cosmology detailed analysis has shown that mathematically natural choices can nonetheless lead to intolerable physical consequences. For example departures from general relativity in completely less exciting situations with low curvature [13]. Thus the Hamiltonian constraint remains the major unsolved problem in Loop Quantum Gravity and therefore, much more work must be done in the full theory.

The present status can be summarized as follows. After my opinion two main ways have been proceeded to construct and to solve the quantum Hamiltonian constraint.

i.) The first one is the so-called *Master constraint program*, which was introduced by Thiemann [69, 68] in 2003. The key idea of this method is to avoid using an infinite number of Hamiltonian constraints $H(N) = \int H(x)N(x)\,\mathrm{d}^3x$, each integrated against a so-called *lapse function* N. Instead, the integrand $H(x)$ is squared itself in a suitable sense and then it is integrated on the 3-manifold $\mathcal{M}$. Thus one gets finally a single constraint. This method leads in simple examples to physically feasible quantum theories. However, in the definition of the Hamiltonian constraint of LQG the method does not remove any of the ambiguities. Rather, the principal strength of the method lies in its potential to complete the final step in quantum dynamics if the ambiguities are resolved, i.e. finding the physically suitable scalar product on physical states.

ii.) The second strategy comes from *spin-foam models* [62, 58]. Spin-foam models provide a path integral approach to quantum gravity. Over the last four years, there has been extensive work in this field, discussed in the articles by Rovelli, Speziale, Perez, Freidel, Alexandrov, Bianchi and others [22, 59, 34, 2]. Transition amplitudes from path integrals can be applied to limit the choice of the Hamiltonian constraint operator in the canonical theory. This is a very promising direction and Freidel, Rovelli, Perez, Noui and others have analyzed this issue especially in $2 + 1$ dimensions.

But to the best of my knowledge there is no unique way out to resolve the problem in the quantum dynamics of the full theory down to the present day. Additionally the precise mathematical structure underlying Loop Quantum Cosmology [26, 12, 13, 76] and the sense in which it implements the full quantization method of LQG in a symmetry reduced model has not been made explicit. Therefore it seems useful to obtain a better understanding of the theory and thus a detailed studying of the fundamental principles is necessary. Despite the fundamental rôle of Ashtekar's variables, their geometric origin have remained open. As far as we know, only local versions using sophisticated index notations have been available so far. But there exists a obviously way out. The modern approach to *differential geometry* is the fact that although coordinate systems have an important rôle to play, the key concepts are developed in a manner which is explicitly independent of any specific reference to coordinates. Thus in the thesis in hand we want to elaborate and to complete the discussion of [33] in the construction of the Ashtekar variables in a *differential geometrical* manner and to rewrite the classical domain of LQG using mathematically global defined objects in order to gain new insights into the fundamental level of the theory of Loop Quantum Gravity. Furthermore we want to make a proposal how to turn the classical expression of the Hamiltonian as derived in that differential geometrical manner into a well defined quantum operator.

Outline of the thesis The present thesis aims at a first glimpse of the differential geometry underlying Ashtekar's variables. The road map of the thesis is as follows: In Chapter 2 we give an overview of the differential geometry underlying the characterization of the Ashtekar variables. In order to get a feel of the variables it is also mandatory to give a short introduction of the Hamilton formulation of general relativity (GR), i.e. the physical origin of the Ashtekar variables in Chapter 3. The variables are a connection in some principal fibre bundle to be determined and a densitized dreibein field. The latter one is rather easy to state and is discussed in Section 3.1.2, whence we will focus on the connection variables which form the configuration space of the theory (up to gauge transformations). More precisely in Chapter 4, we will describe the principal fibre bundle the connection lives in, and then discuss Ashtekar-type connections. This Chapter is based on [33]. The Ashtekar-type connections are compared with physical notation in Section 4.2. In Section 4.3 we want to clarify the spin structure of the Ashtekar connection. The reformulation of the constraints in the new global formalism is outlined in Chapter 5. In Chapter 6 we orient our interest towards the strategy of implementation of the Hamiltonian constraint in the new framework by using the Regge calculus.

2. Mathematical Prologue

This first chapter is intended to develop the necessary mathematical tools and techniques for the construction of the Ashtekar connection which is the central object of this work. We will start with the basic theory of space-times spaces and their geometric properties. We will proceed to discuss some aspects of the theory of fibre bundles. The next step will be to study connections on fibre bundles. We close the chapter with a discussion about the covariant differentiation and 2nd fundamental form, respectively.

2.1. Space-times

In this Section we want to introduce some central statements about space times, which are fundamental for the construction of the *Ashtekar connection*. In this Section we will follow [14]. The starting point should be a Lorentzian manifold denoted by $(\mathcal{M}, g)$.

Notation 2.1.1. *(See also [14].) A time cone* τ *is defined by* $\tau := \{x \in$

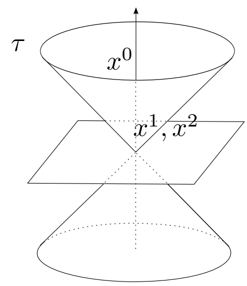


Figure 2.1.: Illustration of the time cone τ, see Notation 2.1.1.

$\mathbb{R}^{n+1} | \langle x, x\rangle = 0$. *Furthermore, we define*

$$\begin{aligned}
I &:= \{x \in \mathbb{R}^{n+1} | \langle x, x\rangle < 0\}, \\
I_\pm &:= \{x \in I | \pm x^0 > 0\}, \\
\tau_\pm &:= \{x \in \tau | \pm x^0 \geq 0\}, \\
J &:= \tau \cup I, \\
J_\pm &:= \tau_\pm \cup I_\pm, \\
\{\textit{light-like vectors}\} &:= \tau \setminus \{0\}, \\
\{\textit{time-like vectors}\} &:= I, \\
\{\textit{space-like vectors}\} &:= (\mathbb{R}^{n+1} \setminus J) \cup \{0\}, \\
\{\textit{causal vectors}\} &:= J \setminus \{0\}, \\
\{\textit{future-directed light-like vectors}\} &:= \tau_+ \setminus \{0\}, \\
\{\textit{past-directed light-like vectors}\} &:= \tau_- \setminus \{0\}, \\
\{\textit{future-directed time-like vectors}\} &:= I_+, \\
\{\textit{past-directed time-like vectors}\} &:= I_-.
\end{aligned}$$

Definition 2.1.2. *Let ζ be a function on $\mathcal{M}$ that assigns to each point p a time cone τ_p in $T_p(\mathcal{M})$. ζ is smooth if for each $p \in \mathcal{M}$ there is a (smooth) vector field V on some neighborhood $\mathcal{U}$ of p such that $V \in \tau_q, \forall q \in \mathcal{U}$. Such a smooth function is called a* **time-orientation** *of $\mathcal{M}$. If $\mathcal{M}$ admits a* *time-orientation, then $\mathcal{M}$ is said to be time-orientable. Then to choose a specific time-orientation on $\mathcal{M}$ is to time-orient $\mathcal{M}$.*

In the following we consider invariably time-oriented Lorentzian manifolds as space-times.

THEOREM 2.1.3. *A Lorentzian manifold $(\mathcal{M}, g)$ is time-orientable if and only if there exists a time-like vector field $X \in \mathfrak{X}(\mathcal{M})$, where $\mathfrak{X}$ denotes the set of all differentiable vector fields on $\mathcal{M}$.*

Proof. See [57]. QED.

DEFINITION 2.1.4. *A manifold $\mathcal{M}$ is orientable provided there exists a collection $\mathcal{O}$ of coordinate systems in $\mathcal{M}$ whose domains cover $\mathcal{M}$ and such that for each $\xi, \eta \in \mathcal{O}$ the Jacobian determinant function $\mathfrak{J}(\xi, \eta) = \det(\mathrm{d}y^i/\mathrm{d}x^j)$ is positive. ($\mathcal{O}$ is called an orientation atlas for $\mathcal{M}$.)*

In the following let $(\mathcal{M}, g, \zeta)$ a connected, time-oriented Lorentzian manifold. We define:

DEFINITION 2.1.5. *In respect of a point $m \in \mathcal{M}$*

$$I_+(m) := \{q \in \mathcal{M} | \exists \quad \textit{future-directed, time-like curve from } m \textit{ to } q\}$$

denotes the chronological future of m and

$$J_+(m) := \{q \in \mathcal{M} | \exists \quad \textit{future-directed, causal curve from } m \textit{ to } q\}$$

the causal future of m, respectively. Analogously, the chronological past $I_-(m)$ of m respectively the causal past $J_-(m)$ of m is defined.

REMARK 2.1.6. *Let $\mathcal{A} \subset \mathcal{M}$. We have $I_\pm(\mathcal{A}) = \cup_{m \in \mathcal{A}} I_+(m)$ and $J_\pm(\mathcal{A}) = \cup_{m \in \mathcal{A}} J_+(m)$.*

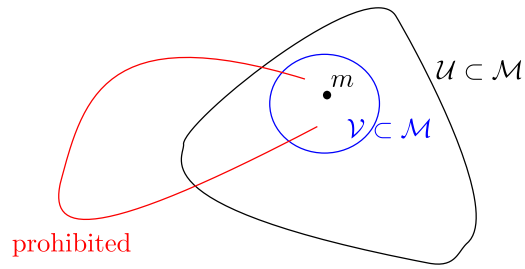


Figure 2.2.: Illustration of the strong causality condition, see Definition 2.1.9.

Corollary 2.1.7. *Let $\mathcal{A} \subset \mathcal{M}$ arbitrary. Then $I_\pm(\mathcal{A}) \subset \mathcal{M}$ is open.*

Proof. See [14]. QED.

Proposition 2.1.8. *Let $(\mathcal{M}, g)$ a compact, time-oriented Lorentzian manifold. Then there exists at least one closed time-like curve.*

Proof. See [14]. QED.

Definition 2.1.9. *A connected, time-oriented Lorentzian manifold $\mathcal{M}$ fulfills*

i.) *the* **chronology condition**, *if no closed time-like curve in $\mathcal{M}$ exists;*

ii.) *the* **causality condition**, *if no closed causal curve in $\mathcal{M}$ exists;*

iii.) *the* **strong causality condition**, *if for every $m \in \mathcal{M}$ and every neighborhood $\mathcal{U} \subset \mathcal{M}$ of m a neighborhood $\mathcal{V} \subset \mathcal{M}$ of m exists, such that every causal curve, which starts and ends in $\mathcal{V}$ is completely in $\mathcal{U}$, see Figure 2.1.*

Remark 2.1.10. *We have the following implications:*

strong causality condition $\Longrightarrow$ causality condition $\Longrightarrow$ chronology condition

In general the converse is not valid.

Definition 2.1.11. *A connected, time-oriented Lorentzian manifold $\mathcal{M}$ is called* **global-hyperbolic** *if*

i.) $\mathcal{M}$ fulfills the strong causality condition;

ii.) for all $p, q \in \mathcal{M}$ the set $J(p,q) := J_+(p) \cap J_-(q) \subset \mathcal{M}$ is compact.

Definition 2.1.12. *A* **Cauchy hypersurface** *of a time-oriented Lorentzian manifold $\mathcal{M}$ is a hyper-surface $\Sigma \subset \mathcal{M}$ which is met exactly once by every inextendible time-like curve in $\mathcal{M}$.*

Theorem 2.1.13. *(Geroch, 1969.) If $\mathcal{M}$ is a globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$, then $\mathcal{M}$ has a Cauchy hypersurface (Σ, q) and there exists a homeomorphism*

$$\mathbb{R} \times \Sigma \longrightarrow \mathcal{M}$$

on which each $\{t\} \times \Sigma$ is mapped onto a Cauchy hypersurface.

Proof. See [38, 20]. QED.

Theorem 2.1.14. *(Bernal-Sánchez, 2004.) If $\mathcal{M}$ is global-hyperbolic, then $(\mathcal{M}, g)$ is isometric to*

$$(\mathbb{R} \times \Sigma, -f \, \mathrm{d}t \otimes \mathrm{d}t + q_t),$$

where $f : \mathbb{R} \times \Sigma \longrightarrow \mathbb{R}$ is a positive smooth function and $q_t|_{t \in \mathbb{R}}$ is a smooth family of Riemannian metrics on Σ. Moreover each $\{t\} \times \Sigma$ is a Cauchy hypersurface.

Proof. See [20]. QED.

Definition 2.1.15. *A* **space-time** *is a 4-dimensional, connected, time- and space-oriented Lorentz manifold, which is globally hyperbolic.*

2.2. Elements of differential geometry

In this Section we want briefly illustrate the theory of fibre bundles and connections. The discussion will be oriented by [19]. Theorems and examples of particular interest for the construction of the *Ashtekar connection* will be presented in detail.

2.2.1. Local trivial Fibre bundles

Definition 2.2.1. *A* **fibre bundle** $(\mathcal{E}, \pi, \mathcal{M}; \mathcal{F})$ *consists of manifolds* $\mathcal{E}, \mathcal{M}, \mathcal{F}$ *and a smooth mapping* $\pi : \mathcal{E} \longrightarrow \mathcal{M}$ *; furthermore it is required that each* $m \in \mathcal{M}$ *has an open neighborhood* $\mathcal{U} \subset \mathcal{M}$ *such that* $\mathcal{E}|_{\mathcal{U}} := \pi^{-1}(\mathcal{U})$ *is diffeomorphic to* $\mathcal{U} \times \mathcal{F}$ *via a fiber respecting diffeomorphism:*

$$\begin{tikzcd} \mathcal{E}|_U \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \mathcal{U} \times \mathcal{F} \arrow[dl, "pr_1"] \\ & \mathcal{U} & \end{tikzcd}$$

$\mathcal{E}$ *is called the total space,* $\mathcal{M}$ *is called the base space,* π *is called the projection , and* $\mathcal{F}$ *is called standard fiber.* $(\mathcal{U}, \varphi)$ *as above is called a fiber chart or a local trivialization of* $\mathcal{E}$*.*

A collection of fiber charts $\{\mathcal{U}_\alpha, \varphi_\alpha\}$, such that $\{\mathcal{U}_\alpha\}$ is an open cover of $\mathcal{M}$, is called a (fiber) bundle atlas . If we fix such an atlas, then the transition mapping is given by $(\varphi_\alpha \circ \varphi_\beta^{-1})(m, f) = (m, \phi_{\alpha\beta}(m)(f))$, where $\phi_{\alpha\beta} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \mathcal{F} \longrightarrow \mathcal{F}$ is smooth and $\phi_{\alpha\beta}(m, f)$ is a diffeomorphism of $\mathcal{F}$ for each $m \in \mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$. We may thus consider the mappings $\phi_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow \mathrm{Diff}(\mathcal{F})$ with values in the group $\mathrm{Diff}(\mathcal{F})$ of all diffeomorphisms of $\mathcal{F}$. In either form these mappings $\phi_{\alpha\beta}$ are called the transition functions of the bundle. They satisfy the cocycle condition : $\phi_{\alpha\beta}(m) \circ \phi_{\beta\gamma}(m) = \phi_{\alpha\gamma}(m)$ for $m \in \mathcal{U}_{\alpha\beta\gamma}$ and $\phi_{\alpha\alpha}(m) = \mathrm{Id}_{\mathcal{F}}$ for $m \in \mathcal{U}_\alpha$. Therefore the collection $(\phi_{\alpha\beta})$ is called a cocycle of transition functions.

PROPOSITION 2.2.2. *Two local-trivial fibre bundles* $(\mathcal{E}, \pi, \mathcal{M}; \mathcal{F})$ *and* $(\widetilde{\mathcal{E}}, \widetilde{\pi}, \mathcal{M}; \widetilde{\mathcal{F}})$ *are called isomorphic, if there exists a fibre-preserving diffeomorphism* $f : \mathcal{E} \longrightarrow \widetilde{\mathcal{E}}$, *i.e.* $\widetilde{\pi} \circ f = \pi$.

DEFINITION 2.2.3.

i.) *A* **smooth section** s *in a local-trivial fibre bundle* $(\mathcal{E}, \pi, \mathcal{M}; \mathcal{F})$ *is a smooth function* $s : \mathcal{M} \longrightarrow \mathcal{E}$ *such that* $\pi \circ s = \mathrm{Id}_{\mathcal{M}}$, *where* π *is the projection* $\mathcal{E} \longrightarrow \mathcal{M}$. $\Gamma(\mathcal{E})$ *denotes the set of all smooth sections in* $\mathcal{E}$.

ii.) *A smooth mapping* $s : \mathcal{M} \longrightarrow \mathcal{E}|_{\mathcal{U}}$ *such that* $\pi \circ s = \mathrm{Id}_{\mathcal{U}}$ *is called* **local section** *in* $\mathcal{E}$ *over* $\mathcal{U}$. $\Gamma(\mathcal{U}, \mathcal{E}) = \Gamma(\mathcal{E}_{\mathcal{U}})$ *denotes the set of all local, smooth sections in* $\mathcal{E}$ *over* $\mathcal{U}$.

Principle fibre bundles

DEFINITION 2.2.4. *Let* G *be a Lie group and* $\pi : \mathcal{P} \longrightarrow \mathcal{M}$ *a smooth mapping. The tuple* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *is called* **(differentiable) principle fibre bundle** *over* $\mathcal{M}$ *with group* G, *if it is satisfying the following conditions:*

i.) G *acts freely on* $\mathcal{P}$ *on the right, i. e.* $\Psi : \mathcal{P} \times \mathrm{G} \longrightarrow \mathcal{P}$. *The action is fibre preserving and transitive on the fibres;*

ii.) $\mathcal{M}$ *is the quotient space of* $\mathcal{P}$ *by the equivalence relation induced by* G, $\mathcal{M} = \mathcal{P}/\mathrm{G}$, *and the canonical projection* $\pi : \mathcal{P} \longrightarrow \mathcal{M}$ *is differentiable;*

iii.) $\mathcal{P}$ *is locally trivial, that is, every point* $m \in \mathcal{M}$ *has a neighborhood* $\mathcal{U}$ *such that* $\pi^{-1}(\mathcal{U})$ *is isomorphic to* $\mathcal{U} \times \mathrm{G}$ *in the sense that there is a diffeomorphism* $\chi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \longrightarrow \mathcal{U}_\alpha \times \mathrm{G}$ *such that* $\chi_\alpha(e \circ g) = \chi_\alpha(e) \circ g$ *for all* $e \in \mathcal{P}$ *and* $g \in \mathrm{G}$ *(*G*-equivariant) , where the action of* G *on* $\mathcal{U}_\alpha \times \mathrm{G}$ *is given by* $(m, a) \circ g = (m, ag)$ *and* $pr_1 \circ \chi_\alpha = \pi$.

G acts freely on $\mathcal{P}$ on the right: $\Psi : \mathcal{P} \times \mathrm{G} \longrightarrow \mathcal{P}$. The action of G induces the following mapppings

$$\begin{aligned} \Psi_p : \mathrm{G} &\longrightarrow \mathcal{P} \\ g &\longmapsto \Psi(p, g) \quad \forall p \in \mathcal{P}, \end{aligned}$$

respectively

$$\begin{aligned} \Psi_g : \mathcal{P} &\longrightarrow \mathcal{P} \\ \mathcal{P} &\longmapsto \Psi(p, g) \quad \forall g \in \mathrm{G}. \end{aligned}$$

The mapping $\chi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{G}$, called transition function, is defined by $\chi_{\alpha\beta}(m) = \chi_\alpha(p) \circ \chi_\beta(p)^{-1}, \forall p \in \mathcal{P}_m$ is satisfying the cocycle condition. In the other hand we have

Proposition 2.2.5. *Let $\mathcal{M}$ be a manifold, $\{\mathcal{U}_\alpha\}$ an open covering of $\mathcal{M}$ and* G *a Lie group. Given a mapping $\chi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{G}$ for every $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, in such a way, that the cocycle condition is satisfied, then we can construct a (differentiable) principle fiber bundle $\mathcal{P}(\mathcal{M}; \mathrm{G})$ with transition functions $\chi_{\alpha\beta}$.*

Proof. See [47]. QED.

Proposition 2.2.6. *Let $\mathcal{M}$ be a manifold, $\pi : \mathcal{P} \longrightarrow \mathcal{M}$ a smooth mapping and* G *a Lie group. Then the tuple $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ is a principle fibre bundle, if and only if there exists an open covering $\{\mathcal{U}_\alpha, \chi_\alpha\}$ and a family of smooth mappings $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{G}, \alpha, \beta \in A$, in such a way, that the cocycle condition is satisfied, such that the transition functions $\chi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{Diff}\, \mathrm{G}$ are given by the translation on the left with cocycles $\chi_{\alpha\beta}(m) = L_{g_{\alpha\beta}(m)} : \mathrm{G} \longrightarrow \mathrm{G}$.*

Proof. See [19]. QED.

Definition 2.2.7. *Two* G*-principle fibre bundles* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *and* $(\widetilde{\mathcal{P}}, \widetilde{\pi}, \mathcal{M}; \mathrm{G})$ *are called isomorphic, if there exists a* G*-equivariant, fibre-preserving diffeomorphism* $f : \mathcal{P} \longrightarrow \widetilde{\mathcal{P}}$.

At this point we want to determine bundles of orthonormal frames in some detail, because in Chapter 4 the *Ashtekar connection* will be constructed on the frame bundle of a Cauchy hypersurface.

Example 2.2.8. **Bundles of orthonormal frames over $\mathcal{M}$**

Let $\mathcal{M}$ *be a n-dimensional manifold. A frame* e_m *over* $m \in \mathcal{M}$ *is a ordered base* $e = (e_1, \ldots, e_n)$ *of* $T_m\mathcal{M}$. *Let*

$$\mathrm{GL}_m(\mathcal{M}) := \{e_m = (e_1, \ldots, e_n)\}$$

the collection of all frames at points of $m \in \mathcal{M}$ *and*

$$\mathrm{GL}(\mathcal{M}) = \dot{\bigcup}_{m\in\mathcal{M}} \mathrm{GL}_m(\mathcal{M}).$$

The projection $\pi : \mathrm{GL}(\mathcal{M}) \longrightarrow \mathcal{M}$ *assigns to each fibre* $e \in \mathrm{GL}_m(\mathcal{M})$ *the point* p *of* $\mathcal{M}$ *at which the frame is located. The group* $\mathrm{GL}(n, \mathbb{R})$ *acts freely on* $\mathrm{GL}(\mathcal{M})$ *on the right by*

$$\begin{aligned} \Psi : \mathrm{GL}(\mathcal{M}) \times \mathrm{GL}(n, \mathbb{R}) &\longrightarrow \mathrm{GL}(\mathcal{M}) \\ (e = (e_1, \ldots, e_n), A = (A_{ij})_{1\leq i,j\leq n}) &\longmapsto (\sum_{j=1}^{n} e_j A_{j1}, \ldots, \sum_{j=1}^{n} e_j A_{jn}). \end{aligned}$$

For short we write $\Psi(a, A) = e \cdot A$, *where '$\cdot$' denotes the matrix multiplication. The action is fibre preserving and transitive on the fibres. The induced mappings*

$$\begin{aligned} \Psi_A : \mathrm{GL}(\mathcal{M}) &\longrightarrow \mathrm{GL}(\mathcal{M}) \\ e &\longmapsto e \cdot A, \quad A \in \mathrm{GL}(n, \mathbb{R}) \end{aligned}$$

fulfill $\Psi_A \circ \Psi_B = \Psi_{BA}, \forall A, B \in \mathrm{GL}(n, \mathbb{R})$. *The corresponding* $\mathrm{GL}(n, \mathbb{R})$*-principle fibre bundle* $(\mathrm{GL}(\mathcal{M}), \pi, \mathcal{M}; \mathrm{GL}(n, \mathbb{R})$ *is called* **frame bundle** *over* $\mathcal{M}$.

Remark 2.2.9. *i.) Let* $(\mathcal{M}, \mathrm{O}_{\mathcal{M}})$ *be a oriented manifold of dimension* n*, then we are able to consider the* $\mathrm{GL}(n,\mathbb{R})^{+}$*-principle fibre bundle* $(\mathrm{GL}(\mathcal{M})^{+}, \pi, \mathcal{M}; \mathrm{GL}(n,\mathbb{R})^{+})$ *of all positive-oriented frames. Then the fibres* $\mathrm{GL}_m(\mathcal{M})^{+}$ *consists of all positive-oriented bases of* $T_m\mathcal{M}, m \in \mathcal{M}$*.*

ii.) Let $(\mathcal{M}, g)$ *a semi-Riemannian manifold with signature* (k, l)*. Then we are able to consider the set of orthonormal bases* $\mathrm{O}_m(\mathcal{M}, g) = \{e \in \mathrm{GL}_m(\mathcal{M}) | (e_1, \ldots, e_{k+l})$ *is* g_m*-orthonormal*$\}$ *over each point* $m \in \mathcal{M}$*. And we obtain then the* $\mathrm{O}(k,l)$*-principle fibre bundle* $(\mathrm{O}(\mathcal{M}, g), \pi, \mathcal{M}; \mathrm{O}(k,l))$ *of all orthonormal frames.*

Associated fibre bundles

In the following let $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ a G-principle fibre bundle over $\mathcal{M}$ with right G-action $\Psi : \mathcal{P} \times \mathrm{G} \longrightarrow \mathcal{P}$ and $\mathcal{F}$ a manifold on which G acts on the left $\sigma : \mathrm{G} \times \mathcal{F} \longrightarrow \mathcal{F}$. On the product manifold $\mathcal{P} \times \mathcal{F}$ we let G act on the right as follows:

$$\begin{aligned} (\mathcal{P} \times \mathcal{F}) \times \mathrm{G} &\longrightarrow \mathcal{P} \times \mathcal{F} \\ ((p, f), g) &\longmapsto (\Psi(p, g), \sigma(g^{-1}, f)) =: (p \circ g, g^{-1} \circ f). \end{aligned}$$

The quotient space of $\mathcal{P} \times \mathcal{F}$ by this group action is denoted by $\mathcal{E} := \mathcal{P} \times_{\mathrm{G}} \mathcal{F} := (\mathcal{P} \times \mathcal{F})/\mathrm{G}$ and $[p, f] \in \mathcal{E}$ is the equivalence class of $(p, f) \in \mathcal{P} \times \mathcal{F}$. We define a mapping $\pi_{\mathcal{E}}$, called projection, from $\mathcal{E}$ onto $\mathcal{M}$ by

$$\begin{aligned} \pi_{\mathcal{E}} : \mathcal{E} &\longrightarrow \mathcal{M} \\ [p, f] &\longmapsto \pi(p). \end{aligned}$$

Theorem/Definition 2.2.10. *The tuple* $(\mathcal{E}, \pi_{\mathcal{E}}, \mathcal{M}; \mathcal{F})$ *is a local-trivial fibre bundle over the base* $\mathcal{M}$ *with (standard) fibre* $\mathcal{F}$ *and (structure) group* G*, which is* **associated** *with the principle fibre bundle* $\mathcal{P}$*.*

Proof. See [19]. QED.

THEOREM 2.2.11. *Let* G *a Lie-Group and* $\mathcal{M}$ *and* $\mathcal{F}$ *manifolds, whereas G acts on* $\mathcal{F}$ *smooth on the left. Let* $(\mathcal{U}_{alpha})_{\alpha\in A}$, *where* $A \in \mathrm{GL}(n,\mathbb{R})$, *a open covering of* $\mathcal{M}$ *and* $g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{G}$ *a family of cocycles. Then there is, up to isomorphisms, exact one local-trivial fibre bundle* $\mathcal{E}, \pi_{\mathcal{E}}, \mathcal{M}; \mathcal{F}$ *over* $\mathcal{M}$ *of fibre* $\mathcal{F}$*, which transition function is given by the left-translation with* $g_{\alpha\beta}(m)$ *on* $\mathcal{F}$*. This fibre bundle is associated to the uniquely determined* G*-principle fibre bundle, which transition function is given by the left-translation with* $g_{\alpha\beta}(m)$ *on* G*.*

Proof. See [19]. QED.

REMARK 2.2.12. *On the basis of the mapping*

$$\begin{aligned} \iota_p : \mathcal{F} &\longrightarrow \mathcal{E}_m \\ f &\longmapsto [p, f] \end{aligned}$$

each $p \in \mathcal{P}$ *gives a diffeomorphism from* $\mathcal{F}$ *onto the fibre of* $\mathcal{E}$ *with* $\pi(p) = m$*. The mapping* ι_p *is called* **defined fibre diffeomorphism** *by* p*. For the fibre diffeomorphisms defined by* $p \circ q, q \in \mathrm{G}$ *we have* $\iota_{p\circ q}(f) = [p \circ q, f] = [p, g \circ f] = \iota_p(\sigma(g, f))$*, i.e.* $\iota_{p\circ q} = \iota_p \rho(g)$*. Here* $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(\mathcal{F})$ *denotes the mapping given by* $\rho(g)(f) := \sigma(g, f)$ *for* $g \in \mathrm{G}, f \in \mathcal{F}$*.*

In the following $\mathcal{C}^\infty(\mathcal{P}, \mathcal{F})^{(\rho,\mathrm{G})}$ denotes the set of smooth, G-equivariant mappings from $\mathcal{P}$ onto $\mathcal{F}$

$$\mathcal{C}^\infty(\mathcal{P}, \mathcal{F})^{(\rho,\mathrm{G})} := \{\bar{s} \in \mathcal{C}^\infty(\mathcal{P}, \mathcal{F}) | \bar{s}(p \circ q) = \rho(g^{-1})\bar{s}(p) \quad \forall p \in \mathcal{P}, g \in \mathrm{G}\}.$$

Thus we have the following isomorphism

THEOREM 2.2.13. *(See [19]) Let* $\mathcal{E} : \mathcal{P} \times_{\mathrm{G}} \mathcal{F}$ *the associated fibre bundle with reference to the* G*-principle fibre bundle* $\mathcal{P}$ *over* $\mathcal{M}$ *and the left* G*-manifold* $\mathcal{F}$*. Then we can identify the set of smooth sections in* $\mathcal{E}$ *with the set of the* G*-equivariant mappings:*

$$\Gamma(\mathcal{E}) \cong \mathcal{C}^\infty(\mathcal{P}, \mathcal{F})^{(\rho,\mathrm{G})}.$$

Proof. See A.1.1. QED.

Vector bundles

Definition 2.2.14. *A* $\mathbb{K}$**-vector bundle** $(\mathcal{E}, \pi, \mathcal{M}; V)$ *over a manifold* $\mathcal{M}$ *consists of a manifold* $\mathcal{E}$ *and a smooth map (projection)* $\pi : \mathcal{E} \longrightarrow \mathcal{M}$ *such that*

i.) each $\mathcal{E}_p := \pi^{-1}(m), m \in \mathcal{M}$ *is a* $\mathbb{K}$*-dimensional vector space;*

ii.) there exists a collection of trivial trivializations $(\mathcal{U}_\alpha, \Phi_\alpha)_{\alpha \in A}$ *of* $\mathcal{E}$ *and diffeomorphisms*

$$\Phi_{\alpha,m} : \mathcal{E}_m \longrightarrow V$$

such that for each $m \in \mathcal{M}$ *and* $\alpha \in A$ *the map* $\Phi_{\alpha,m}$ *is a linear vector space isomorphism.*

An example of a vector bundle is the tangent bundle $T\mathcal{M}$ of a differentiable manifold $\mathcal{M}$.

Definition 2.2.15. *Let* $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ *a vector bundle over* $\mathcal{M}$. *A linear mapping*

$$\nabla : \Gamma(\mathcal{E}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes \mathcal{E})$$

is called **covariant derivative** *if it satisfies the following product rule*

$$\nabla(fe) = \mathrm{d}f \otimes e + f\nabla e$$

for all $f \in \mathcal{C}^\infty(\mathcal{M})$ *and* $e \in \Gamma(\mathcal{E})$.

Most operations on vector spaces can be extended to vector bundles by performing the vector space operation fibrewise. For example:

- If $(\mathcal{E}, \pi, \mathcal{M}; V)$ is a vector bundle, then there is a bundle $(\mathcal{E}^*, \pi^*, \mathcal{M}; V)$, with $\pi^* : \mathcal{E}^* \longrightarrow \mathcal{M}$, called the *dual bundle*;
- The vector bundle $(\mathcal{E} \oplus \widetilde{\mathcal{E}}, \pi_\oplus, \mathcal{M}; V \oplus \widetilde{V})$ is called *Whitney sum* of $(\mathcal{E}, \pi, \mathcal{M}; V)$ and $(\widetilde{\mathcal{E}}, \pi, \mathcal{M}; V)$. The projection $\pi_\oplus : \mathcal{E} \oplus \widetilde{\mathcal{E}} \longrightarrow \mathcal{M}$ is given by $\pi_\oplus(e \oplus \widetilde{e}) = m$ for $e \oplus \widetilde{e} \in \mathcal{E}_m \oplus \widetilde{\mathcal{E}}_m$;

- The *tensor product* bundle $(\mathcal{E} \otimes \widetilde{\mathcal{E}}, \pi_{\otimes}, \mathcal{M}; V \otimes_{\mathbb{K}} \widetilde{V})$ is defined in a similar way, using fibrewise tensor product of vector spaces.

Remark 2.2.16. *Every vector bundle is associated to a principle fibre bundle with linear structure group.*

i.) *Let $(\mathcal{E}, \pi, \mathcal{M}; V)$ be a vector bundle over $\mathcal{M}$ with fibre type V and let $\mathrm{GL}(V)$ the linear Lie-Group of all isomorphisms of V. Furthermore let $\{\mathcal{U}_\alpha, \varphi_\alpha\}_{\alpha \in A}$ be a bundle atlas of $\mathcal{E}$. Since the transition functions $\varphi_{\alpha\beta}(m) : V \xrightarrow{\varphi_{\beta,m}^{-1}} \mathcal{E}_m \xrightarrow{\varphi_{\alpha,m}} V$ are linear isomorphisms, they are $\mathrm{GL}(V)$ valued, i.e $g_{\alpha\beta} := \varphi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{GL}(V)$ in such a way that the cocycle condition is satisfied. Due to Proposition 2.2.5 and Proposition 2.2.6 there exists a uniquely determined principle fibre bundle $\mathcal{P}$ over $\mathcal{M}$ with structure group $\mathrm{GL}(V)$, whose transition function is given by left translation with $g_{\alpha\beta}$. According to Theorem 2.2.11 $\mathcal{E}$ is up to isomorphms the uniquely determined associated fibre bundle to $\mathcal{P}$ with typical fibre V.*

ii.) *Let $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ a principle fibre bundle and $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ a representation, which characterizes the G-action on V. Then $E := \mathcal{P} \times_{(\mathrm{G},\rho)} V$ is a vector bundle. The vector space structure on the fibres $E_m = \mathcal{P}_m \times_{(\mathrm{G},\rho)} V$ is given by the vector space structure in V:*

$$\lambda[p, v] + \mu[p, \widetilde{v}] = [p, \lambda v + \mu \widetilde{v}], \forall p \in \mathcal{P}; v, \widetilde{v} \in V; \lambda, \mu \in \mathbb{K}.$$

E is called the **associated vector bundle** *to the principle fibre bundle $\mathcal{P}$ with G-representation (ρ, V).*

Proposition/Example 2.2.17. *Let $\mathcal{M}$ be a n-dimensional manifold. We consider the bundle $\mathrm{T}^{r,s}\mathcal{M}$ of (r,s)-tensor fields on $\mathcal{M}$. $\mathrm{GL}(\mathcal{M})$ denotes the $\mathrm{GL}(n, \mathbb{R})$-principle fibre bundle of all frames on $\mathcal{M}$ and $\rho : \mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathrm{GL}(\mathbb{R}^n)$ the representation, given by $\rho(A)(x) := A \cdot x, \forall A \in \mathrm{GL}(n, \mathbb{R}), x \in \mathbb{R}^n$. Let $(\rho^*, \mathbb{R}^{n*})$ the dual representation to ρ on the dual space of $\mathbb{R}^n$ given by $(\rho^*(A)\varphi)(x) := \varphi(\rho(A^{-1})x), \forall A \in \mathrm{GL}(n, \mathbb{R}), \varphi \in \mathbb{R}^{n*}, x \in \mathbb{R}^n$. Then we* *have the following isomorphisms*

$$T\mathcal{M} \cong \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n$$
$$T^*\mathcal{M} \cong \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho^*)} \mathbb{R}^{n*}.$$

Then the isomorphism between the tangent bundle $T\mathcal{M}$ *and the bundle* $\mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n$ *associated with the frame bundle is given by*

$$\begin{aligned}\Phi : \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n &\longrightarrow T\mathcal{M} \\ [e, x] &\longmapsto \sum_{i=1}^{n} e_i x_i =: e \cdot x,\end{aligned}$$

where $e = (e_1, \ldots, e_n) \in \mathrm{GL}(\mathcal{M})$ *and* $x = (x_1, \ldots, x_n) \in \mathbb{R}$.

Proof. See [47]. QED.

The following theorem is essential for the construction of the *Ashtekar connection*:

THEOREM 2.2.18. *Let* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *a principle fiber bundle and let* (ρ, V) *respectively* $(\widetilde{\rho}, \widetilde{V})$ *two linear, equivalent representations of* G. *Then the vector bundles* $\mathcal{E} := \mathcal{P} \times_{(\mathrm{G},\rho)} V$ *and* $\widetilde{\mathcal{E}} := \mathcal{P} \times_{(\mathrm{G},\widetilde{\rho})} \widetilde{V}$ *are isomorphic.*

Proof. Since (ρ, V) and $(\widetilde{\rho}, \widetilde{V})$ are equivalent representations, there exists a vector space isomorphism $f : V \longrightarrow \widetilde{V}$ given by

$$f(\rho(g), v) = \widetilde{\rho}(g) f(v), \quad \forall g \in \mathrm{G}, v \in V.$$

Defining the mapping

$$\begin{aligned}F : \mathcal{E} &\longrightarrow \widetilde{\mathcal{E}} \\ [p, v &\longmapsto [p, f(v)]],\end{aligned}$$

which is well defined due to $F([p \circ g, \rho(g^{-1})v]) = [p \circ g, f(\rho(g^{-1})v)] = [p \circ g, \widetilde{\rho}(g^{-1})f(v)] = [p, f(v)] = F([p, v])$. In addition F is fibre preserving and bijective due to construction. The inverse function is given by

$$\begin{aligned} F^{-1} : \widetilde{\mathcal{E}} &\longrightarrow \mathcal{E} \\ [p, \widetilde{v} &\longmapsto [p, f^{-1}(\widetilde{v})]], \end{aligned}$$

where $f^{-1} : \widetilde{V} \longrightarrow V$ denotes the inverse isomorphism. It remains to show, that F is smooth. For this purpose we regard the collection of local trivializations $(\mathcal{U}_\alpha, \chi_\alpha)_{\alpha \in A}$ of $\mathcal{P}$ and the corresponding fibre charts $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha \in A}$ of $\mathcal{E}$ resp. $(\mathcal{U}_\alpha, \widetilde{\varphi}_\alpha)_{\alpha \in A}$ of $\widetilde{\mathcal{E}}$

$$\begin{aligned} \varphi_\alpha : \pi_{\mathcal{E}}^{-1}(\mathcal{U}_\alpha) &\longrightarrow \mathcal{U}_\alpha \times V \\ [p, v] &\longmapsto (\pi(p), \rho(\kappa_\alpha(p))v); \\ \widetilde{\varphi}_\alpha : \pi_{\widetilde{\mathcal{E}}}^{-1}(\mathcal{U}_\alpha) &\longrightarrow \mathcal{U}_\alpha \times \widetilde{V} \\ [p, \widetilde{v}] &\longmapsto (\pi(p), \widetilde{\rho}(\kappa_\alpha(p))\widetilde{v}), \end{aligned}$$

where $\kappa_\alpha := \mathrm{pr}_2 \circ \chi_\alpha$. The mapping $\widetilde{\varphi}_\alpha \circ F \circ \varphi_\beta^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times V \longrightarrow (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \widetilde{V}, \quad \alpha, \beta \in A$ yields

$$\begin{aligned} \widetilde{\varphi}_\alpha \circ F \circ \varphi_\beta^{-1} &= \widetilde{\varphi}_\alpha \circ F([p, \rho(\kappa_\beta(p)^{-1})v]) \\ &= \widetilde{\varphi}_\alpha([p, f(\rho(\kappa_\beta(p)^{-1})v]) = \widetilde{\varphi}_\alpha([p, \widetilde{\rho}(\kappa_\beta(p)^{-1})f(v)]) \\ &= (\pi(p), \widetilde{\rho}(\kappa_\alpha(p)) \circ \widetilde{\rho}(\kappa_\beta(p)^{-1})f(v)) \\ &= (\pi(p), \widetilde{\rho}(\kappa_\alpha(p)\kappa_\beta(p)^{-1})f(v)) \\ &= (m, \widetilde{\rho}(\kappa_{\alpha\beta}(m))f(v)). \end{aligned}$$

Since $\kappa_{\alpha,\beta}$ and $\widetilde{\rho}$ are smooth, F itself is smooth. Analogously we obtain:

$$\begin{aligned} \varphi_\beta \circ F^{-1} \circ \widetilde{\varphi}_\alpha^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times \widetilde{V} &\longrightarrow (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times V \\ (m, \widetilde{v}) &\longmapsto (m, \rho(\kappa_{\beta\alpha}(m))f^{-1}(\widetilde{v})). \end{aligned}$$

and therefore the smoothness of F^{-1}. QED.

Example 2.2.19. *The adjoint representation , i.e* $\mathrm{Ad} : \mathrm{SO}(3) \longrightarrow \mathrm{GL}(\mathfrak{so}(3))$ *and defining representation , i.e* $\rho : \mathrm{SO}(3) \longrightarrow \mathrm{GL}(\mathbb{R}^3)$ *of* $\mathrm{SO}(3)$.

LEMMA 2.2.20. *Let* $(\mathcal{M}, g)$ *be a 3-dimensional Riemannian manifold and* $\mathrm{O}^+(\mathcal{M}, g)$ *denotes the* $\mathrm{SO}(3)$*-principle fibre bundle of the oriented, orthonormal triads over* $\mathcal{M}$*. Using Theorem 2.2.18, we obtain the following identification*

$$\mathrm{O}^+(\mathcal{M}, g) \times_{(\mathrm{SO}(3),\rho)} \mathcal{E} \cong \mathrm{O}^+(\mathcal{M}, g) \times_{(\mathrm{SO}(3),\mathrm{Ad})} \mathfrak{so}(3)$$
$$\mathrm{O}^+(\mathcal{M}, g) \times_{(\mathrm{SO}(3),\rho)} \mathcal{E} \ni [e, u_i] \stackrel{\varphi}{\longleftrightarrow} [e, M_i] \in \mathrm{O}^+(\mathcal{M}, g) \times_{(\mathrm{SO}(3),\mathrm{Ad})} \mathfrak{so}(3),$$

where the isomorphism φ *is given in the proof of Theorem 2.2.18.*

Reduction of the structure group

At this point we want to show, how the structure group of a principle fibre bundle can be varied.

DEFINITION 2.2.21. *Let* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *and* $(\mathcal{Q}, \tilde{\pi}, \mathcal{M}, \mathrm{H})$ *two principle fibre bundles,* $\lambda : \mathrm{H} \longrightarrow \mathrm{G}$ *a homomorphism of Lie-groups and* $f : \mathcal{Q} \longrightarrow \mathcal{P}$ *a smooth mapping. Then the pair* $(\mathcal{Q}, f)$ *is called* **λ-reduction** *of the principle fibre bundle* $\mathcal{P}$*, if*

i.) $\pi \circ f = \widetilde{\pi}$,

ii.) $f(q \circ h) = f(q) \circ \lambda(h) \quad \forall q \in \mathcal{Q}, h \in \mathrm{H}$

hold.

In the case that $\mathrm{H} \subset \mathrm{G}$ is a Lie subgroup of G and $\lambda = \iota$ the inclusion mapping, then a λ-reduction $(\mathcal{Q}, f)$ is also called reduction from $\mathcal{P}$ to H. The mapping $f : \mathcal{Q} \longrightarrow \mathcal{P}$ is an embedding.

EXAMPLE 2.2.22. **Reduction of the frame bundle.** *Let* $\mathcal{M}$ *a n-dimensional, smooth manifold and* $(\mathrm{GL}(\mathcal{M}), \pi, \mathcal{M}; \mathrm{GL}(n, .\mathbb{R}))$ *the* $\mathrm{GL}(n, \mathbb{R})$*-principle fibre bundle of all franes on* $\mathcal{M}$*. Then every additional geometrical structure on* $\mathcal{M}$ *provides a reduction of the frame bundle* $GL(\mathcal{M})$ *on a subgroup of* $\mathrm{GL}(n, \mathbb{R})$*. For a detailled discussion see [19].*

THEOREM 2.2.23. *Let* $(\mathcal{P}, \pi, \mathcal{M}, \mathrm{G})$ *a principle fibre bundle with continuous, non-compact structure group. Then* $\mathcal{P}$ *is reducible to every maximal compact subgroup* $K \subset \mathrm{G}$.

Proof. See [19]. QED.

THEOREM 2.2.24. *Let* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *and* $(\mathcal{Q}, \tilde{\pi}, \mathcal{M}, \mathrm{H})$ *two principle fibre bundles,* $\lambda : \mathrm{H} \longrightarrow \mathrm{G}$ *a homomorphism of Lie-groups and* $f : \mathcal{Q} \longrightarrow \mathcal{P}$ *a smooth mapping and* $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ *a finite dimensional representation on a real valued vector space* V. *If* $(\mathcal{Q}, f)$ *is a* λ*-reduction of* $\mathcal{P}$, *then the associated vector bundles* $\mathcal{P} \times_{(\mathrm{G},\rho)} V$ *and* $\mathcal{Q} \times_{(\mathrm{H},\rho\circ\lambda)} V$ *are isomorphic.*

Proof. See A.1.2. QED.

EXAMPLE 2.2.25. *As seen in Proposition/Example 2.2.17 the tangent bundle* $T\mathcal{M}$ *of a* n*-dimensional manifold* $\mathcal{M}$ *is represented as an associated vector bundle w.r.t. the frame bundle* $\mathrm{GL}(\mathcal{M})$

$$T\mathcal{M} \overset{\mathfrak{V}}{\cong} \mathrm{GL}(M) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n.$$

By inclusion a pseudo-Riemannian metric g *with signature* (k,l) *on* $\mathcal{M}$ *provides a reduction of the frame bundle to the principle* $\mathrm{O}(k,l)$*-bundle* $\mathrm{O}(\mathcal{M}, g)$ *of the orthonormal frames. By theorem Theorem 2.2.24 we get the following identification*

$$\mathrm{O}(\mathcal{M}, g) \times_{(\mathrm{O}(k,l),\rho)} \mathbb{R}^n \cong \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n,$$

in the case of a given orientation of $\mathcal{M}$ *we obtain respectively*

$$\mathrm{O}^+(\mathcal{M}, g) \times_{(\mathrm{SO}(k,l),\rho)} \mathbb{R}^n \cong \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n$$

In summary, we obtain the following identifications:

$$\begin{aligned} T\mathcal{M} &\overset{\mathfrak{V}}{\cong} \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n \\ &\cong \mathrm{O}(\mathcal{M}, g) \times_{(\mathrm{O}(k,l),\rho)} \mathbb{R}^n \\ &\cong \mathrm{O}^+(\mathcal{M}, g) \times_{(\mathrm{SO}(k,l),\rho)} \mathbb{R}^n. \end{aligned}$$

2.2.2. Connections in principle fibre bundles

In the following let be $(\mathcal{P}, \pi, \mathcal{M}; G)$ a smooth G-principle fibre bundle over a manifold $\mathcal{M}$ and $\mathfrak{g}$ denotes the Lie-algebra of G. Since $\pi : \mathcal{P} \longrightarrow \mathcal{M}$ is a submersion, the fibres $\mathcal{P}_m = \pi^{-1}(m)$, $m \in \mathcal{M}$ are smooth submanifolds of $\mathcal{P}$. The tangent space $T_p\mathcal{P}_m$ is called *vertical tangent space* of $\mathcal{P}$ in p $(\pi(p) = m)$ and is denoted by $V_p := T_p\mathcal{P}_m$. On $\mathcal{P}$ we let act G on the right as $\Psi : \mathcal{P} \times \mathrm{G} \longrightarrow \mathcal{P}$. By

$$\begin{aligned} \Psi_p : \mathrm{G} &\longrightarrow \mathcal{P} \\ g &\longmapsto \Psi(p, g) \end{aligned}$$

the orbit is given, which provides the identification of the fibres $P_m = \mathrm{Image}(\Psi_p)$ with G. For each $A \in \mathfrak{g}, \widetilde{A} \in \Gamma(T\mathcal{P})$ is called the *fundamental vector field* corresponding to A given by $\mathcal{P} \ni p \longmapsto \widetilde{A} := \mathrm{d}\Psi_p(A) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, p \circ \exp(tA) \in V_p$. We have a unique characterization of the vertical spaces by the fundamental vector fields:

LEMMA 2.2.26. *For all $p \in \mathcal{P}$ the vertical tangent space V_p is isomorphic to the Lie algebra $\mathfrak{g}$*

$$\mathfrak{g} \ni A \mapsto \mathrm{d}\psi_p(A) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, p \circ \exp(tA) \in V_p,$$

where $\mathrm{d}\psi$ is the tangent mapping.

Proof. See [47]. QED.

REMARK 2.2.27. *One has $\widetilde{[A, B]} = [\widetilde{A}, \widetilde{B}]$ for all $A, B \in \mathfrak{g}$. The mapping $g \ni A \mapsto \widetilde{A} \in X(\mathcal{P})$ is then a Lie algebra-homomorphism.*

LEMMA 2.2.28. *Let $\widetilde{A}$ the fundamental vector field corresponding to $A \in \mathfrak{g}$. For each $g \in \mathrm{G}$, $(\Psi_g)_*\widetilde{A}$ is the fundamental vector field corresponding to $(\mathrm{Ad}(g^{-1}))A \in \mathfrak{g}$.*

Proof. See [47]. QED.

The complementary vector space to $V_p \subset T_p\mathcal{P}$ is called *horizontal tangent space* of $\mathcal{P}$ in $p \in \mathcal{P}$, denoted by H_p.

DEFINITION 2.2.29. *A* **connection form** Γ *on a principle fibre bundle* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *is given by a mapping*

$$\mathcal{P} \ni p \mapsto H_p \subset T_p\mathcal{P}$$

for which one has:

i.) H_p *is complementary to* $V_p : T_p\mathcal{P} = H_p \oplus V_p \quad \forall p \in \mathcal{P}$*;*

ii.) H_p *is compatible with the* G*-action on* $\mathcal{P} : \mathrm{d}\Psi_g H_p = H_{\Psi_g(p)} \quad \forall g \in \mathrm{G}, p \in \mathcal{P}$*;*

iii.) Γ *is a smooth distribution, i.e.:* $\forall p \in \mathcal{P}$ *there exists a neighborhood* $\mathcal{U} \subset \mathcal{P}$ *and smooth vector fields* $X_1, \ldots, X_m$*, such that* $H_q = \mathrm{span}(X_1(q), \ldots, X_m(q)), \quad \forall q \in \mathcal{U}$*.*

In the following hor resp. ver denote the projection on the horizontal resp. vertical subspaces $\mathrm{hor} : T_p\mathcal{P} \longrightarrow H_p$ resp. $\mathrm{ver} : T_p\mathcal{P} \longrightarrow V_p$, for arbitrary $p \in \mathcal{P}$. Hence every vector $X \in T_p\mathcal{P}$ can be written as $X = \mathrm{hor}\, X + \mathrm{ver}\, X$.

LEMMA 2.2.30. $V_p = \ker \pi_p$ *for all* $p \in \mathcal{P}$*.*

Proof. See A.1.3. QED.

Due to *Lemma* 2.2.30 the differential of the projection π, $\mathrm{d}\pi_p|_{H_p} \longrightarrow T_{\pi(p)}\mathcal{M}$, restricted to the horizontal spaces H_p, is a linear isomorphism. Hence we can lift vector fields on $\mathcal{M}$ uniquely to P.

DEFINITION 2.2.31. *A vector field* $X^* \in \Gamma(T\mathcal{P})$ *on* $\mathcal{P}$ *is called the* **horizontal lift** *of a vector field* $X \in \Gamma(T\mathcal{M})$ *on* $\mathcal{M}$*, if* *i.)* X^* *is horizontal, i.e.* $X^*_p \in H_p \quad \forall p \in \mathcal{P}$*;*

ii.) $\pi_* X^* = X$

hold.

THEOREM 2.2.32. *i.) For all vector field* $X \in \Gamma(T\mathcal{M})$ *there exists exactly one horizontal lift* $X^* \in \Gamma(T\mathcal{P})$*.* X^* *is* G*-invariant, i.e.* $X^* \circ \Psi_g = \mathrm{d}\Psi_g X^* \quad \forall g \in \mathrm{G}$.

ii.) Every horizontal, G*-invariant vector field* $Y \in \Gamma(T\mathcal{P})$ *is the horizontal lift of a vector field* $X \in \Gamma(T\mathcal{M})$*.*

Proof. See [19]. QED.

We have the following properties:

LEMMA 2.2.33. *Let* $X, Y \in \Gamma(T\mathcal{M})$*,* $f \in \mathcal{C}^\infty(\mathcal{M})$*,*$Z \in \Gamma(T\mathcal{P})$ *and* $A \in \mathfrak{g}$ *with fundamental vector field* $\widetilde{A} \in \Gamma(T\mathcal{P})$*. Then we have*

i.) $(X+Y)^* = X^* + Y^*$,

ii.) $(fX)^* = f^* X^*$ *with* $f^* := f \circ \pi$,

iii.) $[X, Y]^* = \mathrm{hor}\,[X^*, Y^*]$,

iv.) $[\widetilde{A}, \mathrm{hor}\, Z]$ *is horizontal,*

v.) $[\widetilde{A}, X^*] = 0$.

Proof. See [19]. QED.

Now we want to illustrate, how connections on principle fibre bundles can be characterized by specific 1-forms.

DEFINITION 2.2.34. *Let* $\widetilde{A} \in \Gamma(T\mathcal{P})$ *fundamental vector field of* A*. A connection form on a principle fibre bundle* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *is a Lie-algebra valued* 1*-form* $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *with the following properties:*

i.) $\omega(\widetilde{A}) = A \quad \forall A \in \mathfrak{g}$;

ii.) $(\Psi_g^* \omega)(Y) = \mathrm{Ad}(g^{-1})\omega(Y) \quad \forall g \in \mathrm{G}, Y \in X(\mathcal{P})$.

THEOREM 2.2.35. *On a principle fibre bundle* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$, *there is a one-to-one correspondence between the connections and the connection forms.*

Proof. See A.1.4. QED.

REMARK 2.2.36. *A connection form* $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *is able to identify the vertical part of a vector by the corresponding Lie-algebra element. Let* $A \in \mathfrak{g}$ *with fundamental vector field* $\widetilde{A} = \mathrm{ver}\, X$. *Due to* $\omega \circ \mathrm{hor} = 0$ *and the isomorphism* $\mathrm{d}\Psi_p : g \longrightarrow V_p$ *we obtain* $\omega_p(X) = \omega_p(\mathrm{ver}\, X) = \omega_p(\mathrm{d}\Psi_p A) = A = \mathrm{d}\Psi_p^{-1}(\mathrm{ver}\, X)$. *Thus we have* $\omega_p(X) = \mathrm{d}\Psi^{-1}(\mathrm{ver}\, X), \forall p \in \mathcal{P}, X \in T_p\mathcal{P}$.

A further identification of connections is given by *local connection forms*:

DEFINITION 2.2.37. *Let* $s : \mathcal{U} \longrightarrow \mathcal{P}$ *a local section from the subset* $\mathcal{U} \subset \mathcal{M}$ *to* $\mathcal{P}$ *and* $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *a connection form. Then the pull back*

$$\omega^s := s^* \omega \in \Omega^1(\mathcal{U}, \mathfrak{g})$$

from ω *to* $\mathcal{U}$ *is called* **local connection form** *or* **gauge-potential**.

Let $s_\alpha : \mathcal{U}_\alpha \longrightarrow \mathcal{P}$ and $s_\beta : \mathcal{U}_\beta \longrightarrow \mathcal{P}$ two overlapping sections to $\mathcal{P}$ $(\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset)$. Then there exists a smooth transition function $\kappa_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathrm{G}$, such that $s_\beta = s_\alpha \circ \kappa_{\alpha\beta}$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$. Using this relation, we are able to compare local connection forms ω^{s_α} and ω^{s_β}. One obtains

$$\omega^{s_\beta} = \mathrm{Ad}(\kappa_{\alpha\beta}^{-1})\omega^{s_\alpha} + \kappa_{\alpha\beta}\Theta, \tag{2.1}$$

where $\Theta \in \Omega^1(\mathrm{G}, \mathfrak{g})$ denotes the *Maurer-Cartan-form* on G, which is the uniquely determined $\mathfrak{g}$-valued 1-form on G given by $\Theta(X) = X$, $\forall X \in \mathfrak{g}$. Otherwise Eq. (2.1) gives us a unique characterization of connections on $\mathcal{P}$. In summary, we have

THEOREM 2.2.38.

i.) *Let be* $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *a connection form on* $\mathcal{P}$ *and let be* $(\mathcal{U}_\alpha, s_\alpha)$ *and* $(\mathcal{U}_\beta, s_\beta)$ *local sections in* $\mathcal{P}$ *with* $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq 0$. *Then we have*

$$\omega^{s_\beta} = \mathrm{Ad}(\kappa_{\alpha\beta}^{-1})\omega^{s_\alpha} + \kappa_{\alpha\beta}^*\Theta; \tag{2.2}$$

ii.) *Vice versa given a cover of the bundle* $\mathcal{P}$ *by local sections* $\{(\mathcal{U}_\alpha, s_\alpha)\}_{\alpha\in A}$ *and a family of local 1-forms* $\{\omega_\alpha \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})\}_{\alpha\in A}$, *such that for* $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq 0$ *we have*

$$\omega_\beta = \mathrm{Ad}(\kappa_{\alpha\beta}^{-1})\omega_\alpha + \kappa_{\alpha\beta}^*\Theta,$$

then there exists one connection form ω *on* $\mathcal{P}$, *which is given by* ω_α, *i.e.* $s_\alpha^*\omega = \omega_\alpha, \quad \forall\alpha \in A$.

Proof. See [19]. QED.

REMARK 2.2.39.

i.) *For the Maurer-Cartan-form* $\Theta \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *we have* $\Theta_g(Y) = \mathrm{d}L_{g^{-1}}(Y), \forall Y \in T_g\mathrm{G}, g \in \mathrm{G}$, *where* $L_g : \mathrm{G} \longrightarrow \mathrm{G}$ *denotes the action on the left of* $g \in \mathrm{G}$, *given by* $L_g : \mathrm{G} \ni h \longmapsto gh \in \mathrm{G}$. *Then the Maurer-Cartan-form* $\kappa_{\alpha\beta}^*\Theta$ *on* $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ *is given by* $(\kappa_{\alpha\beta}^*\Theta)(X) = \mathrm{d}L_{\kappa_{\alpha\beta}(m)^{-1}}(\mathrm{d}\kappa_{\alpha\beta}|_m X)$ *for all* $X \in T_m(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$.

ii.) *For a matrix group* $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ *we have due to the linearity of the action* $\mathrm{d}L_g(X) = gX$ *(matrix multiplication) and hence* $\mathrm{Ad}(g)X = \mathrm{d}L_g \circ \mathrm{d}R_{g^{-1}}X = gXg^{-1}$ *for all* $g \in \mathrm{G}$ *and* $X \in \mathfrak{g}$. *Then Eq.* (2.1) *yields*

$$\omega^{s_\beta} = \kappa_{\alpha\beta}^{-1}\omega^{s_\alpha}\kappa_{\alpha\beta} + \kappa_{\alpha\beta}^{-1}\,\mathrm{d}\kappa_{\alpha\beta}. \tag{2.3}$$

Linear connections

Throughout this Section, we shell denote a n-dimensional, smooth manifold by $\mathcal{M}$ and the $\mathrm{GL}(n, \mathbb{R})$-principle fibre bundle of linear frames over $\mathcal{M}$ by

$\mathrm{GL}(\mathcal{M})$.

Definition 2.2.40. *Connections in the frame-bundle* $\mathrm{GL}(\mathcal{M})$ *are called* **linear connections** .

We have the following relation between the covariant derivative on $T\mathcal{M}$ and the linear connetions.

Theorem 2.2.41. *The set of covariant derivatives on the tangent bundle* $T\mathcal{M}$ *is bijective to the set of connections on the* $\mathrm{GL}(n,\mathbb{R})$ *- principle fibre bundle* $\mathrm{GL}(\mathcal{M})$ *of all frames of* $\mathcal{M}$.

$$\{\textit{covariant derivative on } T\mathcal{M}\} \overset{1:1}{\longleftrightarrow} \{\textit{connection form on } \mathrm{GL}(\mathcal{M})\}.$$

Proof. i.) On the one hand let $\omega \in \Omega^1(\mathrm{GL}(\mathcal{M}), \mathfrak{gl}(n,\mathbb{R}))$ a connection form on the frame bundle $\mathrm{GL}(\mathcal{M})$ and $(E_{ij})_{1\leq i,j\leq n}$ denotes the standard basis of $\mathfrak{gl}(n,\mathbb{R})$, which is given by the $n\times n$ matrices E_{ij}. In that basis ω can be rewritten as $\omega = \sum_{1\leq i,j\leq n} \omega_{ij} E_{ij}$ with real valued 1-forms $\omega_{ij} \in \Omega^1(\mathrm{GL}(\mathcal{M}))$. Let be $e = (e_1, \dots, e_n) : \mathcal{U} \longrightarrow \mathrm{GL}(\mathcal{M})$ a local section. Then we can consider local connections forms $\omega^e_{ij} := e^*\omega_{ij}$, which transforms by Eq. (2.1) when changing basis. The covariant derivative associated to ω, $\nabla : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes T\mathcal{M})$ on $T\mathcal{M}$ is then given by $\nabla e_j := \sum_{i=1}^n \omega^e_{ij}(X) \otimes e_i$ and the Leibniz-rule $\nabla fY := \mathrm{d}f \otimes Y + f\nabla Y \,\forall f \in \mathcal{C}^\infty, Y \in \Gamma(T\mathcal{M})$, together with the requirement of $\mathbb{R}$-linearity. Due to Eq. (2.1) the expression of the covariant derivative is well defined under change of basis $e \longrightarrow f \in \Gamma(\mathcal{U}, \mathrm{GL}(\mathcal{M}))$. Let $f : \mathcal{U} \longrightarrow \mathrm{GL}(\mathcal{M})$ an additional basis section and let be $A : \mathcal{U} \longrightarrow \mathrm{GL}(n,\mathbb{R})$ the associated transition function to e and f, given by $f = e \cdot A$. According to Eq. (2.3) we have for matrix valued connections forms

$$\omega^f = \mathrm{Ad}(A^{-1})\omega^e + A^{-1}\,\mathrm{d}A. \tag{2.4}$$

Writing ω^f resp. ω^e in the basis $(E_{ij})_{1\leq i,j\leq n}$ of $\mathfrak{gl}(n,\mathbb{R})$, i.e. $\omega^e =$

$\sum_{1\leq i,j\leq n}\omega^e_{ij}E_{ij}$ resp. $\omega^f = \sum_{1\leq i,j\leq n}\omega^f_{ij}E_{ij}$, the transformation formula yields

$$\omega^f_{ij} = \sum_{1\leq i,j\leq n}(A^{-1})_{ik}\omega^e_{ij}A_{lj} + (A^{-1}\,\mathrm{d}A)_{ij}, \tag{2.5}$$

where A_{ij} denote the components of A. The latter equation is equivalent to

$$\sum_{1\leq i,j\leq n} A_{lk}\omega^f_{kj} = \mathrm{d}A_{lj} + \sum_{1\leq i,j\leq n}\omega^e_{lk}A_{kj}. \tag{2.6}$$

In order to validate if $\nabla : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T^*\mathcal{M}\otimes T\mathcal{M})$ is well defined, we have to proof

$$\nabla(e\cdot A)_j = \nabla\sum_{1\leq i,j\leq n}A_{kj}e_k \stackrel{!}{=} \nabla f_j. \tag{2.7}$$

By using Eq. (2.6) the left hand side of Eq. (2.7) yields

$$\begin{aligned}
\nabla\sum_{1\leq i,j\leq n}A_{kj}e_k &= \sum_{1\leq i,j\leq n}\nabla A_{kj}e_k = \sum_k(\mathrm{d}A_{kj}\otimes e_k + A_{kj}\otimes\nabla e_k)\\
&= \sum_k(\mathrm{d}A_{kj}\otimes e_k + A_{kj}\sum_l \omega^e_{lk}\otimes\nabla e_l)\\
&= \sum_l(\mathrm{d}A_{lj} + \sum_k\omega^e_{lk}\,A_{kj})\otimes e_l = \sum_{l,k}A_{lk}\,\omega^f_{kj}\otimes e_l\\
&= \sum_k\omega^f_{kj}\otimes\sum_l e_lA_{lk} = \sum_k\omega^f_{kj}\otimes f_k = \nabla f_j,
\end{aligned} \tag{2.8}$$

where in the fifth step we have used Eq. (2.6).

ii.) On the other hand let be $\nabla : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T^*\mathcal{M}\otimes T\mathcal{M})$ a covariant dervative on $T\mathcal{M}$ and $e = (e_1,\dots,e_n) \in \Gamma(\mathcal{U},\mathrm{GL}(\mathcal{M}))$ a local section in the frame bundle. Then there exists real-valued 1-forms $\omega^e_{ij} \in \Omega^1(\mathcal{U})$, such that $\nabla e_j = \sum_{i=1}^n\omega^e_{ij}\otimes e_i$. For each additional basis section $f : \mathcal{U} \longrightarrow \mathrm{GL}(\mathcal{M})$ with transition function $A : \mathcal{U} \longrightarrow \mathrm{GL}(n,\mathbb{R})$, such that $f = e\cdot A$, we have

$$\nabla(e\cdot A)_j = \nabla\sum_{1\leq i,j\leq n}A_{kj}e_k = \nabla f_j. \tag{2.9}$$

Analogous to Eq. (2.8), we obtain that the real-valued 1-forms $\omega^e_{ij} \in \Omega^1(\mathcal{U})$ and $\omega^f_{ij} \in \Omega^1(\mathcal{U})$ are linked by Eq. (2.6)

$$\omega^f_{ij} = \sum_{1\leq i,j\leq n} (A^{-1})_{ik}\omega^e_{ij}A_{lj} + (A^{-1}\,\mathrm{d}A)_{ij}. \tag{2.10}$$

Defining as stated above $\mathfrak{gl}(n,\mathbb{R})$-valued, local 1-forms on $\mathcal{U}$ by $\omega^e := \sum_{1\leq i,j\leq n}\omega^e_{ij}E_{ij}$ resp. $\omega^f := \sum_{1\leq i,j\leq n}\omega^f_{ij}E_{ij}$, such that the transformation formula Eq. (2.5) is fulfilled. It seems prudent to construct a family of local, consistent connection forms with the help of the real-valued 1-forms $\{\omega^e_{ij}|_e$ is a local section in $\mathrm{GL}(\mathcal{M})\}$. For that reason let $\{\mathcal{U}_\alpha, e_\alpha\}_{\alpha\in A}$ a collection of local sections of $\mathrm{GL}(\mathcal{M})$. Defining now for all $\alpha \in A$ local $\mathfrak{gl}(n,\mathbb{R})$-valued 1-forms $\Omega^1(\mathcal{U}_\alpha, \mathfrak{gl}(n,\mathbb{R}))$ by $\omega_\alpha = \sum_{1\leq i,j\leq n}\omega^{e_\alpha}_{ij}E_{ij}$, whereas $\omega^{e_\alpha}_{ij} \in \mathcal{U}_\alpha$ is given as above. As already seen $\omega^{e_\alpha}_{ij} \in \Omega(\mathcal{U}_\alpha)$ transforms under change of basis $e_\alpha \longrightarrow e_\beta = e_\alpha \circ \kappa_{\alpha\beta}$, such that the family $\{\omega_\alpha\}_{\alpha\in A}$ of $\mathfrak{gl}(n,\mathbb{R})$-valued 1-forms fulfills the transformation rule Eq. (2.2). Hence there exists exactly one connecion form $\omega \in \Omega^1(\mathrm{GL}(\mathcal{M})\mathfrak{gl}(n,\mathbb{R}))$ with $e^*_\alpha\omega = \omega_\alpha$ for all $\alpha \in A$.

QED.

Metric connections of a pseudo-riemannian manifold

Let $(\mathcal{M}, g)$ be a n-dimensional, smooth pseudo-riemannian manifold with metric $g \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$.

Definition 2.2.42. *i.) A linear connection on $(\mathcal{M}, g)$ is called metric , if g is parallel to the covariant derivative ∇ given by the linear connection on $T\mathcal{M}$, i.e. $\nabla g = 0$.*

ii.) If $\nabla g = 0$ for covariant derivative on $T\mathcal{M}$, then ∇ is called metric.

Theorem 2.2.43. *The set of covariant, metric derivatives on the tangent bundle $T\mathcal{M}$ is bijective to the set of connections on the $\mathrm{O}(\mathcal{M}, g)$ - principle fibre bundle $\mathrm{O}(\mathcal{M}, g)$ of all orthonormal, ordered frames of $\mathcal{M}$.*

$$\{\textit{covariant, metric derivative on } T\mathcal{M}\} \overset{1:1}{\longleftrightarrow} \{\textit{connection form on } \mathrm{O}(\mathcal{M}, g)\}.$$

Proof. i.) Let $\nabla : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes T\mathcal{M})$ a covariant, metric derivation on $T\mathcal{M}$ and let be $e : \mathcal{U} \longrightarrow \mathrm{O}(\mathcal{M}, g)$ a local, orthonormal basis section. Then for real-valued, local 1-forms $\omega^e_{ij} \in \Omega^1(\mathcal{U})$ we have

$$\nabla e_j = \sum_i^n \omega^e_{ij} \otimes e_i \tag{2.11}$$

For each $e \in \Gamma(\mathcal{U}, \mathrm{O}(\mathcal{M}, g))$ we construct a $\mathfrak{gl}(n, \mathbb{R}^n)$-valued 1-form $\omega^e := \sum_{i<j} \omega^e_{ij} E_{ij}$. As seen this gives a family of local 1-forms, which satisfies Eq. (2.4). In order to construct a connection form on $\mathrm{O}(\mathcal{M}, g)$ analogously to Theorem 2.2.41, it suffices to show, that ω^e is valued in the Lie-algebra $\mathfrak{o}(k, l)$ of $\mathrm{O}(k, l)$. For this purpose we use that ∇ is metric. Since $e : \mathcal{U} \longrightarrow \mathrm{O}(\mathcal{M}, g)$ is an orthonormal, local basis section, from Eq. (2.11) we obtain

$$g(\nabla e_j, e_k) = \epsilon_k \omega^e_{kj}, \tag{2.12}$$

whereas ϵ_k is given by

$$\epsilon_k := g(e_k, e_k) = \begin{cases} -1 & \text{if} \quad i = 1, \ldots, k; \\ 1 & \text{if} \quad i = k+1, \ldots, n. \end{cases}$$

Since ∇ is metric (i.e. $\nabla g = 0$), we get

$$\begin{aligned} 0 =& (\nabla_X g)(e_j, e_k) = X(g(e_j, e_k)) - g(\nabla_X e_j, e_k) - g(e_j, \nabla_X e_k) \\ &\Longleftrightarrow g(\nabla e_j, e_k) = -g(e_j, \nabla e_k) \end{aligned}$$

for all $X \in \Gamma(T\mathcal{M})$. Inserting in Eq. (2.12) we obtain the following symmetry

$$\epsilon_k \omega^e_{kj} = g(\nabla e_j, e_k) = -g(e_j, \nabla e_k) = \epsilon_j \omega^e_{jk}. \tag{2.13}$$

In particular we have $\omega^e_{ii} = 0$ for all $i = 1, \ldots, n$. Using Eq. (2.13) local 1-forms ω^e yields

$$\begin{aligned}
\omega^e &= \sum_{1\le i,j\le n} \omega^e_{ij} E_{ij} = \sum_{i\neq j} \omega^e_{ij} E_{ij} = \sum_{i<j} \omega^e_{ij} E_{ij} + \sum_{j<i} \omega^e_{ij} E_{ij} \\
&= \sum_{i<j} \omega^e_{ij} E_{ij} + \sum_{j<i} \epsilon_i^2 \omega^e_{ij} E_{ij} = \sum_{i<j} \omega^e_{ij} E_{ij} - \sum_{j<i} \epsilon_i \epsilon_j \omega^e_{ij} E_{ij} \\
&= \sum_{i<j} \epsilon_j^2 \omega^e_{ij} E_{ij} - \sum_{i<j} \epsilon_j \epsilon_i \omega^e_{ij} E_{ji} = \sum_{i<j} \epsilon_j \omega^e_{ij} (\epsilon_j E_{ij} - \epsilon_i E_{ji}) \\
&= -\sum_{i<j} \epsilon_j \omega^e_{ij} (\epsilon_i E_{ji} - \epsilon_j E_{ij}) = -\sum_{i<j} \epsilon_j \omega^e_{ij} O_{ij},
\end{aligned} \tag{2.14}$$

with $n \times n$ matrices $O_{ij} := \epsilon_i E_{ji} - \epsilon_j E_{ij}\ \forall\, 1 \le i < j \le n$. $(O_{ij})_{1\le i<j\le n}$ is a basis of $\mathfrak{o}(k,l)$. Thus ω^e are $\mathfrak{o}(k,l)$-valued and according to Theorem 2.2.38 ω^e are the connections forms associated to a uniquely determined connection ω on $\mathrm{O}(\mathcal{M}, g)$.

ii.) On the other hand let $\omega \in \Omega^1(\mathrm{O}(\mathcal{M}, g), \mathfrak{o}(k,l))$ a connection form on the bundle of the orthonormal frames $\mathrm{O}(\mathcal{M}, g)$. Then ω is given in the basis $(E_{ij})_{1\le i \le j\le n}$ of $\mathfrak{gl}(n,\mathbb{R})$ by $\omega = \sum_{i,j} \omega_{ij} E_{ij}$, where $\omega_{ij} \in \Omega^1(\mathrm{O}(\mathcal{M}, g))$. Associated to a local section $e : \mathcal{U} \longrightarrow \mathrm{O}(\mathcal{M}, g)$ we have $\omega^e := e^*\omega = \sum_{i\le j} \omega^e_{ij} E_{ij}$, where $\omega^e_{ij} := e^*\omega_{ij} \in \Omega^1(\mathcal{U})$. Defining now the corresponding covariant derivation on $T\mathcal{M}$ by $\nabla e_j := \sum_i \omega^e_{ij} \otimes e_i$, such that it is $\mathbb{R}$-linea and ∇ fulfills the Leibnizrule. Acoording to Theorem 2.2.41 ∇ is well defined. Since ω is $\mathfrak{o}(k,l)$-valued, we obtain

$$\omega^T = -\eta\omega\eta, \quad \text{with} \quad \eta = \begin{pmatrix} -1_{k\times k} & 0 \\ 0 & 1_{(n-k)\times(n-k)} \end{pmatrix}$$

and hence

$$\omega^e = -\sum_{k,l} \eta_{jk} \omega_{kl} \eta_{li} = -\sum_{k,l} \epsilon_j \delta_{jk} \omega_{kl} \epsilon_l \delta_{li} = -\epsilon_i \epsilon_j \omega_{ij}. \tag{2.15}$$

Analogously to Eq. (2.14) we get $\omega^e = -\sum_{i<j} \epsilon_j \omega^e_{ij} O_{ij}$. And due to Eq. (2.15) we finally obtain $g(\nabla e_j, e_i) = \epsilon_i \omega^e_{ij} = -\epsilon_j \omega^e_{ji} = -g(\nabla e_i, e_j)$. Thus ∇ is metric.

QED.

Example 2.2.44. **Levi-Civita-connection of a pseudo-riemannian manifold**

Let $(\mathcal{M}, g)$ *a* n*-dimensional, smooth pseudo-riemannian manifold with signature* (k, l) *of dimension* $n = k + l$*.*

Theorem 2.2.45. *On* $T\mathcal{M}$ *there exists a unique metric and torsion free covariant derivative*

$$\nabla^{\mathrm{LC}} : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes T\mathcal{M}), \tag{2.16}$$

which is given by the Koszul-formula

$$\begin{aligned} 2g(\nabla^{\mathrm{LC}}_X Y, Z) =& X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) \\ &+ g(Y, [Z, X]) + g(Z, [X, Y]). \end{aligned}$$

Proof. See [19]. QED.

The corresponding connection form in the principle fibre bundle of all orthonormal frames $(\mathrm{O}(\mathcal{M}, g), \pi, \mathcal{M}; \mathrm{O}(k, l))$ *is denoted by* $\omega^{\mathrm{LC}} \in \Omega^1(\mathrm{O}(\mathcal{M}, g), \mathfrak{o}(k, l))$*. The appertaining connection is called Levi-Civita-connection . Given a local field of orthonormal basis vectors* $e = (e_1, \ldots, e_n) : \mathcal{U} \longrightarrow \mathrm{O}(\mathcal{M}, g)$ *and using Eq.* (2.14)*,* ω^{LC} *can locally be rewritten as*

$$\omega^{\mathrm{LC},e} := \sum_{i<j} \epsilon_i \epsilon_j g(\nabla^{\mathrm{LC}} e_i, e_j) O_{ij} \in \Omega^1(\mathcal{U}, \mathfrak{o}(k, l)). \tag{2.17}$$

Reduction of connections

Now we want to illustrate the behavior of a connection, when reducing the structure group.

THEOREM/DEFINITION 2.2.46. *Let* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *and* $(\mathcal{Q}, \tilde{\pi}, \mathcal{M}, H)$ *two principle fibre bundles,* $\lambda : H \longrightarrow \mathrm{G}$ *a homomorphism of Lie-groups and* $f : \mathcal{Q} \longrightarrow \mathcal{P}$ *a smooth mapping, such that* $(\mathcal{Q}, \mathcal{P})$ *is a* λ*-reduction of* $\mathcal{P}$*. Furthermore let* $\omega \in \Omega^1(\mathcal{Q}, \mathfrak{h})$ *a connection form on* $\mathcal{Q}$*. Then there exists exactly one connection form* $\tilde{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{q})$*, such that the horizontal spaces* $H^{\mathcal{Q}} = \ker \omega$ *and* $H^{\mathcal{P}} = \ker \tilde{\omega}$ *are connected as follows:*

$$\mathrm{d}f_q H_q^{\mathcal{Q}} = H_{f(q)}^{\mathcal{P}}.$$

In addition we have

$$f^*\tilde{\omega} = \lambda_* \circ \omega$$
$$f^*\tilde{\Omega} = \lambda_* \circ \Omega,$$

where $\tilde{\Omega} := \mathcal{D}^{\tilde{\omega}}\tilde{\omega}$ *and* $\Omega := \mathcal{D}_\omega \omega$ *denotes the corresponding curvature forms.*[1] *The connection* $\tilde{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *is called the* λ**-extension** *of* $\omega \in \Omega(\mathcal{Q}, \mathfrak{h})$*. The connection* $\omega \in \Omega^1(\mathcal{Q}, \mathfrak{h})$ *is called the* λ**-reduction** *of* $\tilde{\omega} \in \Omega(\mathcal{P}, \mathfrak{g})$*.*

Proof. See [19]. QED.

Thus there exists always an extension of a connection. In general a λ-reduction does not exist conversely. Therefor a criteria is given by

THEOREM 2.2.47. *Let* $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ *a* G*-principle fibre bundle with connection form* $\tilde{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *and* $\mathrm{H} \subset \mathrm{G}$ *a closed subgroup with Lie-algebra* $\mathfrak{h}$*. Furthermore let* $\mathcal{Q} \subset \mathcal{P}$ *a* H*-reduction of* $\mathcal{P}$*. If there exists a vector-space-decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *of the Lie-algebra of* G*, such that*

$$\mathrm{Ad}(\mathrm{H})\mathfrak{m} \subset \mathfrak{m},$$

then

$$\omega := \mathrm{pr}_{\mathfrak{h}} \circ \tilde{\omega}|_{T\mathcal{Q}} : T\mathcal{Q} \longrightarrow \mathfrak{h}$$

is a connection form on $\mathcal{Q}$*. In particular if* $\tilde{\omega} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *is valued in the Lie-algebra* $\mathfrak{h} \subset \mathfrak{g}$*, then* $\omega := \tilde{\omega}|_{T\mathcal{Q}}$ *is a connection form on* Ω*.*

Proof. See A.1.5. QED.

Since we will consider the bundle $\mathrm{O}^+(\mathcal{M}, g)$ of all orthonormal, oriented frames during the construction of the Ashtekar connection, we give the following example:

EXAMPLE 2.2.48. *Let* $(\mathcal{M}, g)$ *a* n*-dimensional, pseudo-Riemannian manifold with signature* (k, l) *of* g*. In addition let be* $\nabla : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes T\mathcal{M})$ *a covariant derivation of a metric connection on* $T\mathcal{M}$ *and let be* $\omega \in \Omega^1(\mathrm{GL}(\mathcal{M}), \mathfrak{gl}(n, \mathbb{R}))$ *the corresponding connection form on the bundle of all frames* $\mathrm{GL}(\mathcal{M})$*. The pseudo-Riemannian metric induces a* $\mathrm{O}(k, l)$*-reduction of* $\mathrm{GL}(\mathcal{M})$ *on the subbundle* $\mathrm{O}(\mathcal{M}, g)$ *of all orthonormal frames. As seen in Theorem 2.2.43 the connection* ω *is valued in* $\mathfrak{o}(k, l) \subset \mathfrak{gl}(n, \mathbb{R})$*. By restriction on* $T\mathrm{O}(\mathcal{M}, g)$*, we obtain a connection form on* $\mathrm{O}(\mathcal{M}, g)$*, as illustrated in Theorem 2.2.47. Thus we have:*

LEMMA 2.2.49. *A linear connection on a pseudo-Riemannian manifold is metric, if and only if it is reducible on* $\mathrm{O}(\mathcal{M}, g)$*.*

If orientation is imposed, $\mathrm{O}(\mathcal{M}, g)$ *can be further reduced to the* $\mathrm{SO}(k, l)$*-principle fibre bundle* $\mathrm{O}^+(\mathcal{M}, g)$ *of all orthonormal, oriented frames. Due to* $\mathfrak{o}(k, l) = \mathfrak{so}(k, l)$ *and Theorem 2.2.47 the metric connection* ω *can be reduced to a connection on* $\mathrm{O}^+(\mathcal{M}, g)$ *by restriction on* $T\mathrm{O}^+(\mathcal{M}, g)$*.*

Covariant differentiation in associated vector bundles

Hereafter let $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ a G-principle fibre bundle over a smooth manifold $\mathcal{M}$ and let be $\rho : \mathrm{G} \to \mathrm{GL}(V)$ a finite dimensional G-representation, which provides a smooth left action on V. $\mathcal{E} := \mathcal{P} \times_{(\mathrm{G},\rho)} V$ denotes the corresponding associated vector bundle. Furthermore we denote by $\Omega^k(\mathcal{M}, \mathcal{E}) := \Gamma(\Lambda^k T^*\mathcal{M} \otimes \mathcal{E})$ the set of the $\mathcal{E}$-valued k-forms on $\mathcal{M}$. In addition the k-forms on $\mathcal{P}$ valued in the vector space V are denoted by $\Omega^k(\mathcal{M}, V) := \Gamma(\Lambda^k T^*\mathcal{M} \otimes \underline{V})$, whereas $\underline{V}$ is the trivial bundle over $\mathcal{M}$ with fibre V.

Definition 2.2.50. *A k-form $\varsigma \in \Omega^k(\mathcal{P}, V)$ on $\mathcal{P}$ with values in V is called*

i.) **horizontal** *if $\varsigma(X_1, \cdots, X_k) = 0$ holds, in the case of at least one of the vectors $X_i \in T_p\mathcal{P}$ is vertical.*

ii.) **of type** ρ, *if $\Psi_g^*\varsigma = \rho(g^{-1})\varsigma \quad \forall g \in \mathrm{G}$ holds.*

We shell denote the set of all horizontal k-forms of type ρ on $\mathcal{P}$ valued in V by $\Omega^k_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}$.

Remark 2.2.51. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra and in addition let be ω_1 and ω_2 two connections forms on $\mathcal{P}$. Then their difference is given by a horizontal 1-form of type (ad). Consequently, for every $\mathfrak{g}$-valued horizontal 1-form η of type (ad) on $\mathcal{P}$, $\omega_1 + \eta$ is a connection form on $\mathcal{P}$. The set of all connections is an affine space in subject to the vector space $\Omega^1_{\text{hor}}(\mathcal{P}, \mathfrak{g})^{(\mathrm{G},\text{ad})}$.*

No we give a generalization of Theorem 2.2.13

Theorem 2.2.52. *The V-valued vector space $\Omega^k_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}$ of horizontal 1-forms of type (ρ) is isomorphic to the vector space $\Omega^k(\mathcal{M}, \mathcal{E})$ of k-forms on $\mathcal{M}$ with values in the vector bundle $\mathcal{E} := \mathcal{P} \times_{(\mathrm{G},\rho)} V$.*

$$\begin{array}{ccc} \Omega^k(\mathcal{M}, \mathcal{E}) & \cong & \Omega^k_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)} \\ \varsigma & \longleftrightarrow & \bar{\varsigma} \end{array}$$

Proof. (outline) Let $p \in \mathcal{P}_m$ be an arbitrary point in the fibre over $m \in \mathcal{M}$. The vector space isomorphism $\Omega^k(\mathcal{M}, \mathcal{E}) \cong \Omega^k_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}$ is obtained as follows

- Let $\bar{\varsigma} \in \Omega^k_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}$. We define $\varsigma \in \Omega^k(\mathcal{M}, \mathcal{E})$ by

$$\begin{aligned} \varsigma_m(X_1, \ldots, X_k) &:= [p, \bar{\varsigma}_p(\widetilde{X}_1, \ldots, \widetilde{X}_k)] \\ &= \iota_p(\bar{\varsigma}_p(\widetilde{X}_1, \ldots, \widetilde{X}_k)), \end{aligned}$$

  where $\widetilde{X}_i \in T_p\mathcal{P}$ denotes an arbitrary lift of $X_i \in T_m\mathcal{M}$. Since $\bar{\varsigma}$ of type ρ is horizontal, ς is well defined.

- We have $\varsigma \in \Omega^k(\mathcal{M}, \mathcal{E})$. Then we obtain the corresponding k-form $\bar{\varsigma} \in \Omega^k_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}$ by

$$\begin{aligned}\bar{\varsigma}_p(Y_1, \dots, Y_k) &:= \iota_p^{-1}(\varsigma_{\pi(p)}(\mathrm{d}\pi_p Y_1, \dots, \mathrm{d}\pi_p Y_k)) \\ &= \iota_p^{-1} \circ (\pi^*\varsigma)_p(Y_1, \dots, Y_k), \quad \forall Y_1, \dots, Y_k \in T_p\mathcal{P}.\end{aligned}$$

QED.

Let $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ a G-principle fibre bundle with fixed connection Γ, $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ a representation of G and $\mathcal{E} := \mathcal{P} \times_{\mathrm{G}} V$ be the associated vector bundle over $\mathcal{M}$. Analogously to the covariant derivatives on the tangent bundle $T\mathcal{M}$, we are able to define covariant derivatives on arbitrary vector bundles. With Theorem 2.2.52 we define a covariant derivative $\nabla : \Gamma(\mathcal{E}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes \mathcal{E})$ on $\mathcal{E}$ associated to a connection form Γ on $\mathcal{P}$ by:

Definition 2.2.53. *The linear mapping $\mathcal{D}_\omega : \Omega^k(\mathcal{P}, V) \longrightarrow \Omega^{k+1}(\mathcal{P}, V)$ given by*

$$(\mathcal{D}_\omega\varsigma)_p(X_1, \cdots X_{k+1}) := \mathrm{d}\varsigma(\mathrm{hor}X_1, \dots, \mathrm{hor}X_{k+1}) \quad \textit{for} \quad X_1, \dots X_{k+1} \in T_p\mathcal{P} \tag{2.18}$$

is called the **absolute differential** *on $\mathcal{P}$ given by the connection form Γ.*

The following Theorem shows, that the modified derivative $\mathcal{D}_\omega$ is preserving the invariance of k-forms and it provides a formula in order to compare $\mathcal{D}_\omega$ with the usual differential d.

Theorem 2.2.54. *The absolute differential given by ω maps horizontal differential forms of type ρ to horizontal differential forms of type ρ, i.e.*

$$\mathcal{D}_\omega : \Omega^k_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)} \longrightarrow \Omega^{k+1}_{\text{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}.$$

For each horizontal k-form $\varsigma \in \Omega^k_{\text{hor}}(\mathcal{P}, V)^{\mathrm{G},\rho}$ we have

$$\mathcal{D}_\omega\varsigma : \mathrm{d}\varsigma + \rho_*(\omega) \wedge \varsigma, \tag{2.19}$$

where the second addend is given by

$$\rho_*(\omega) \wedge \varsigma(X_1 \dots X_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i-1} \rho_*(\omega(X_i))\varsigma(X_1, \dots, \hat{X}_i, \dots, X_{k+1}).$$

Proof. See [19]. QED.

By means of Theorem 2.2.52 the absolute differential $\mathcal{D}_\omega$ induces a linear mapping d_ω on k-forms on $\mathcal{M}$ valued in $\mathcal{E}$:

Definition 2.2.55.

i.) The **absolute differential**

$$\begin{aligned} \mathrm{d}_\omega : \Omega^k(\mathcal{M}, \mathcal{E}) &\longrightarrow \Omega^{k+1}(\mathcal{M}, \mathcal{E}) \\ \varsigma &\longmapsto \mathrm{d}_\omega \varsigma \end{aligned}$$

of forms on $\mathcal{M}$ with values in $\mathcal{E}$ is given by the mapping

$$\overline{\mathrm{d}_\omega \varsigma} := \mathcal{D}_\omega \overline{\varsigma},$$

where $\overline{\varsigma} \in \Omega^k_{\mathrm{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}$ denotes the k-form of type ρ associated to $\varsigma \in \Omega^k(\mathcal{M}, \mathcal{E})$.

ii.) The covariant derivative on $\mathcal{E}$ induced by ω is then given by

$$\nabla := \mathrm{d}_\omega|_{\Omega^0(\mathcal{M},\mathcal{E})} : \Gamma(\mathcal{E}) = \Omega^0(\mathcal{M}, \mathcal{E}) \longrightarrow \Omega^1(\mathcal{M}, \mathcal{E}) = \Gamma(T^*\mathcal{M} \otimes \mathcal{E}).$$

Thus we obtain for $\varsigma \in \Omega^k(\mathcal{M}, \mathcal{E})$:

$$(\mathrm{d}_\omega \varsigma)_m(X_1, \ldots, X_{k+1}) = [p, (\mathcal{D}_\omega \overline{\varsigma})_p(\widetilde{X}_1, \ldots, \widetilde{X}_{k+1})] = [p, \mathrm{d}\, \overline{\varsigma}_p(X^*_1, \ldots, X^*_{k+1})], \tag{2.20}$$

where $m \in \mathcal{M}$, $X_1, \ldots X_{k+1} \in T_p\mathcal{M}$, $p \in \mathcal{P}_m$ are arbitrary elements of the fibre over m, $\widetilde{X}_1, \ldots \widetilde{X}_{k+1} \in T_p\mathcal{P}$ are arbitrary lifts and $X^*_1, \ldots X^*_{k+1} \in T_p\mathcal{P}$ are horizontal lifts of $X_1, \ldots X_{k+1}$ ($\mathrm{d}\pi_p \widetilde{X}_i = X_i = \mathrm{d}\pi_p X^*_i$ and $\mathrm{hor} X^*_i = X^*_i$).

Using a local section $e : \mathcal{U} \longrightarrow \mathcal{P}$ in the neighborhood of $m \in \mathcal{U} \subset \mathcal{M}$ one can choose $p = e(m)$ and $\widetilde{X}_i = \mathrm{d}e_m X_i$, such that (2.20) yields

$$(\mathrm{d}_\omega \varsigma)_m(X_1, \ldots, X_{k+1}) = [e(m), (\mathcal{D}_\omega \overline{\varsigma})_{e(m)}(\mathrm{d}e_m X_1, \ldots, \mathrm{d}e_m X_{k+1})]. \tag{2.21}$$

In particular one gets for 0-forms, i.e. sections $s \in \Gamma(\mathcal{E})$, the following expression:

$$(\nabla_X s)_m = (\mathrm{d}_\omega s)_m(X) = [p, \mathrm{d}\overline{s}_p(X^*)] = [p, X^*(\overline{s}(p)] \tag{2.22}$$

with the horizontal lift $X^* \in T_p\mathcal{P}$ of $X \in T_m\mathcal{M}$.

THEOREM 2.2.56.

i.) $\nabla : \Gamma(\mathcal{E}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes \mathcal{E})$ *is a covariant derivative in* $\mathcal{E}$.

ii.) Let $s \in \Gamma(\mathcal{U}, \mathcal{E})$ *a local section in the vector bundle* $\mathcal{E}$, *which is represented by a local section* $e : \mathcal{U} \longrightarrow \mathcal{P}$ *and a smooth function* $v \in C^\infty(\mathcal{U}, V)$ $(v \equiv \overline{s} \circ e)$, *i.e.*

$$s(m) = [e(m), v(m)] \quad \forall m \in \mathcal{U}.$$

Then we have

$$\nabla s = [e, \mathrm{d}v + \rho_*(\omega^e)v],$$

where $\omega^e := e^*(\omega) \in \Omega^1(\mathcal{U}, \mathfrak{g})$ *is the pullback of* ω *to* $\mathcal{U}$.

Proof. See A.1.6 and [19], respectively. QED.

Covariant differentiation induced by linear connections

Let $\mathcal{M}$ a n-dimensional smooth manifold and $\nabla^{T\mathcal{M}} : \Gamma(T\mathcal{M}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes T\mathcal{M})$ a covariant derivation on the tangent bundle. Furthermore $\mathrm{GL}(\mathcal{M})$ denotes the $\mathrm{GL}(n, \mathbb{R})$-principle bundle of all frames on $\mathcal{M}$ and let $\rho : \mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathrm{GL}(\mathbb{R}^n)$ be the natural representation of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$. According to Proposition/Example 2.2.17 the tangent bundle $T\mathcal{M}$ and the associated vector bundle $\mathcal{E} := \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n$ are isomorphic. The isomorphism is explicitly given by

$$\begin{aligned} \Phi : \mathcal{E} = \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n &\longrightarrow T\mathcal{M} \\ [e, u_i] &\longmapsto e_i, \end{aligned}$$

where $(u_i)_{1\leq i\leq n}$ denotes the standard basis of $\mathbb{R}^n$ and $e = (e_1, \ldots, e_n)$ a element of $\mathrm{GL}(\mathcal{M})$. The covariant derivative $\nabla^{T\mathcal{M}}$ provides a connection form $\omega \in \Omega^1(\mathrm{GL}(\mathcal{M}), \mathrm{GL}(n, \mathbb{R}))$ on $\mathrm{GL}(\mathcal{M})$ as illustrated in Theorem 2.2.41, which in turn induces a covariant derivation $\nabla^{\mathcal{E}} : \Gamma(\mathcal{E}) \longrightarrow \Gamma(T^*\mathcal{M} \otimes \mathcal{E})$ on the associated bundle $\mathcal{E} = \mathrm{GL}(\mathcal{M}) \times_{(\mathrm{GL}(n,\mathbb{R}),\rho)} \mathbb{R}^n$.

In the following we want to compare the derivations $\nabla^{T\mathcal{M}}$ and $\nabla^{\mathcal{E}}$ by reference of the isomorphism Φ. To this end let $X \in \Gamma(\mathcal{U}, T\mathcal{M})$ a local vector field and $e = (e_1, \ldots, e_n) : \mathcal{U} \longmapsto \mathrm{GL}(\mathcal{M})$ a local basis field. Then we are able to rewrite X as $X = \sum_{i=1}^n X^i e_i$, with real valued functions $X^i \in \mathcal{C}^\infty(\mathcal{U})$. Then the associated local section in $\mathcal{E}$ is given by

$$\Phi^{-1}(X) = \sum_i X\Phi^{-1}(e_i) = \sum_i X^i[e, u_i] = [e, \sum_i X^i u_i] =: [e, v].$$

Id est $\Phi^{-1}(X) \in \Gamma(\mathcal{U}, \mathcal{E})$ has the form of a local section as in Theorem 2.2.56 with $v := \sum_i X^i u_i \in \mathcal{C}(\mathcal{U}, \mathbb{R}^n)$. Hence the covariant derivative in $\mathcal{E}$ along the vector field $Y \in \Gamma(T\mathcal{M})$ yields

$$\nabla^{\mathcal{E}}_Y \Phi^{-1}(X) = [e, \mathrm{d}v(Y) + \rho_*(\omega^e(Y))v].$$

Since $\rho : \mathrm{GL}(n, \mathbb{R}) \longrightarrow \mathrm{GL}(\mathbb{R}^n)$ is given by matrix multiplication, this applies equally to $\rho_* : \mathfrak{gl}(n, \mathbb{R}^n) \longrightarrow \mathfrak{gl}(\mathbb{R}^n)$. In addition according to Theorem 2.2.41 $\omega^e \in \Omega^1(\mathcal{U}, \mathfrak{gl}(\mathbb{R}^n))$ is given by $\omega^e = \sum_{i,j} \omega^e_{ij} E_{ij}$, where $\omega^e_{ij} \in \Omega^1(\mathcal{U})$ is defined by

$$\nabla^{T\mathcal{M}} e_j = \sum_i \omega^e_{ij} \otimes e_i \quad \forall \quad i, j = 1, \ldots, n. \tag{2.23}$$

Hence we obtain

$$\rho_*(\omega^e(Y))v = \sum_{i,j,k} X^k \omega^e_{ij}(Y) E_{ij} u_k = \sum_{i,k} X^k \omega^e_{ik}(Y) u_i.$$

Using $\mathrm{d}v(Y) = \sum_k \mathrm{d}X^k(Y) u_k$, one gets

$$\begin{aligned}\nabla^{\mathcal{E}}_Y \Phi^{-1}(X) &= [e, \sum_k \mathrm{d}X^k(Y) u_k + \sum_{i,k} X^k \omega^e_{ij}(Y) u_i] \\ &= \sum_k \mathrm{d}X^k(Y)[e, u_k] + \sum_{i,k} X^k \omega^e_{ik}(Y)[e, u_i]\end{aligned}$$

and therefore by using Eq. (2.23)

$$\begin{aligned}
\Phi(\nabla^{\mathcal{E}}_Y \Phi^{-1}(X)) &= \sum_k \mathrm{d}X^k(Y)e_k + \sum_{i,k} X^k \omega^e_{ij}(Y)e_i \\
&= \sum_k \mathrm{d}X^k(Y)e_k + \sum_k X^k \nabla^{T\mathcal{M}}_Y e_k \\
&= \sum_k (\mathrm{d}X^k \otimes e_k + X^k \nabla^{T\mathcal{M}} e_k)(Y) = \sum_k \nabla^{T\mathcal{M}} X^k e_k \\
&= \nabla^{T\mathcal{M}} X(Y) = \nabla^{T\mathcal{M}}_Y X.
\end{aligned}$$

Finally, Φ transfers the constructed derivatives in each other. We obtain

$$\Phi(\nabla^{\mathcal{E}}_Y s) = \nabla^{T\mathcal{M}}_Y \Phi(s) \tag{2.24}$$

for all vector fields $Y \in \Gamma(T^*\mathcal{M})$ and sections $s \in \Gamma(\mathcal{E})$. Thereby the isomorphism Φ is distinguished.

Remark 2.2.57. *As seen in Eq.* (2.2.48) *metric connections are reducible on the bundle* $\mathrm{O}(\mathcal{M}, g)$ *of orthonormal frames. According to Theorem 2.2.43 the metric covariant derivatives on* $T\mathcal{M}$ *are bijective to the set of the connections forms on* $\mathrm{O}(\mathcal{M}, g)$. *Using Example 2.2.25 the tangent bundle* $T\mathcal{M}$ *is equal to the associated bundle of* $\mathrm{O}(\mathcal{M}, g)$, *i.e.* $T\mathcal{M} \overset{\Phi}{\cong} \mathrm{O}(\mathcal{M}, g) \times_{(\mathrm{O}(k,l),\rho)} \mathbb{R}^n$. *The isomorphism* Φ *transfers the covariant derivative, induced by the reduced connection* $\mathrm{O}(\mathcal{M}, g) \times_{(\mathrm{O}(k,l),\rho)} \mathbb{R}^n$ *into the associated covariant derivative on* $T\mathcal{M}$.

In the oriented case, Φ *transfers analogously the covariant derivative, induced by the reduced connection* $\mathrm{O}^+(\mathcal{M}, g) \times_{(\mathrm{SO}(k,l),\rho)} \mathbb{R}^n$ *on* $\mathrm{O}^+(\mathcal{M}, g)$ *into the associated covariant derivative on* $T\mathcal{M}$.

Curvature of connections

In this Section we want to define the curvature form of connections. The curvature form is a 2-form associated to the connection. In the complete Section let $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ a G-principle fibre bundle with fixed connection Γ and associated connection form ω, $\rho : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ a representation of G and $\mathcal{E} := \mathcal{P} \times_{\mathrm{G}} V$ be the associated vector bundle over $\mathcal{M}$.

Definition 2.2.58. *The 2-form associated to the connection form* ω

$$F^{\omega} := \mathcal{D}_{\omega}\omega \in \Omega^2(\mathcal{P},\mathfrak{g})$$

is called **curvature form** *of* ω*, resp.* **curvature form** *of the connection* Γ*.*

Remark 2.2.59. *Since* $\omega \in \Omega^1(\mathcal{P},\mathfrak{g})^{(\mathrm{G},\mathrm{Ad})}$ *is of type* Ad *and the absolute differential* $\mathcal{D}_{\omega}$ *is type preserving,* F^{ω} *is also of type* Ad*. In addition* F^{ω} *is horizontal according to the definition of* $\mathcal{D}_{\omega}$*:*

$$F^{\omega} \in \Omega^2_{\mathrm{hor}}(\mathcal{P},\mathfrak{g})^{(\mathrm{G},\mathrm{Ad})}.$$

With the intend to simplify, we introduce the following commutator of Lie-algebra-valued differential forms

$$[,]^{\wedge} : \Omega^k(\mathcal{M},\mathfrak{g}) \otimes \Omega^l(\mathcal{M},\mathfrak{g}) \longrightarrow \Omega^{k+l}(\mathcal{M},\mathfrak{g}).$$

Let $\varsigma \in \Omega^k(\mathcal{M},\mathfrak{g}), \varrho \in \Omega^l(\mathcal{M},\mathfrak{g})$ and $(\tau_1,\dots,\tau_n)$ be a basis of $\mathfrak{g}$. Then we have $\varsigma = \sum_{i=1}^n \varsigma^i\tau_i$ resp. $\varrho = \sum_{i=1}^n \varrho^i\tau_i$, where $\varsigma^i \in \Omega^k(\mathcal{M})$ resp. $\varrho^i \in \Omega^l(\mathcal{M})$. We define

$$[\varsigma,\varrho]^{\wedge} := \sum_{i,j=1}^{n} (\varsigma^i \wedge \varrho^j) \otimes [a_i,a_j] \in \Omega^{k+l}(\mathcal{M},\mathfrak{g}).$$

Then we obtain for 1-forms $\varsigma,\varrho \in \Omega^1(\mathcal{M},\mathfrak{g})$

$$[\varsigma,\varrho]^{\wedge}(X,Y) = \sum_{i,j=1}^{n} (\varsigma^i \wedge \varrho^j)(X,Y) \otimes [a_i,a_j] = [\varsigma(X),\varrho(Y)] - [\varsigma(Y),\varrho(X)]$$

and hence

$$[\varsigma,\varsigma]^{\wedge}(X,Y) = 2[\varsigma(X),\varsigma(Y)].$$

Next, we proof basic identities of the curvature form of a connection.

Theorem 2.2.60. *The curvature form* $F^{\omega} \in \Omega^2(\mathcal{P},\mathfrak{g})$ *of the connection* $\omega \in \Omega^1(\mathcal{P},\mathfrak{g})$ *fulfills the following identities:*

i.) *Structure equation:* $F^\omega = \mathrm{d}\omega + \frac{1}{2}[\omega,\omega]^\wedge$*;*

ii.) *Bianchi - identity:* $\mathcal{D}_\omega F^\omega = 0$*;*

iii.) *for horizontal k-forms* $\varsigma \in \Omega^k_{\mathrm{hor}}(\mathcal{P}, V)^{(\mathrm{G},\rho)}$ *of type* ρ *holds:* $\mathcal{D}_\omega\mathcal{D}_\omega\varsigma = \rho_*(F^\omega)\wedge\varsigma$*;*

iv.) *moreover for horizontal 1-forms* $\varsigma \in \Omega^1_{\mathrm{hor}}(\mathcal{P}, \mathfrak{g})^{(\mathrm{G},\mathrm{Ad})}$ *of type* Ad *with with values in the Lie-algebra* $\mathfrak{g}$ *holds* $\mathcal{D}_\omega\varsigma = \mathrm{d}\varsigma + [\omega,\varsigma]^\wedge$.

Proof. See A.1.7. QED.

As for covariant derivatives on manifolds $\mathcal{M}$ resp. its tangent bundle $T\mathcal{M}$, we can assign to every covariant derivative $\nabla : \Gamma(\mathcal{E}) \longrightarrow \Gamma(T^*\mathcal{M}\otimes\mathcal{E})$ on the vector bundle $\mathcal{E}$ over $\mathcal{M}$ a curvature endomorphism.

DEFINITION 2.2.61. *Let* $\nabla : \Gamma(\mathcal{E}) \longrightarrow \Gamma(T^*\mathcal{M}\otimes\mathcal{E})$ *a covariant derivation on the vector bundle* $\mathcal{E}$*. The 2-form* $R^\nabla \in \Gamma(\Lambda^2 T^*\mathcal{M}\otimes \mathrm{End}(\mathcal{E}))$ *is defined by*

$$R^\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$$

for $X, Y \in \Gamma(T\mathcal{M})$ *and* $s \in \Gamma(\mathcal{E})$ *is called* **curvature endomorphism** *of* ∇.

We have the following relation between the curvature form $F^\omega \in \Omega^2(\mathcal{P}, \mathrm{g})$ on $\mathcal{P}$ and the curvature tensor $R^\nabla \in \Gamma(\Lambda^2 T^*\mathcal{M}\otimes\mathrm{End}(\mathcal{E}))$ on $\mathcal{E}$.

THEOREM 2.2.62. *Let* $p \in \mathcal{P}_m$ *a point in the fibre over* $m \in \mathcal{M}$ *and* $\iota_p : V \longrightarrow \mathcal{E}_m$ *the fibre diffeomorphism given by* p*. Then we have*

$$R^\nabla_m(X,Y) = \iota_p \circ \rho_*(F^\omega_p(\tilde{X},\tilde{Y}))\circ\iota_p^{-1} : \mathcal{E}_m \longrightarrow \mathcal{E}_m$$

where $\tilde{X},\tilde{Y} \in T_p\mathcal{P}$ *are arbitrary lifts of* $X, Y \in T_m\mathcal{M}$*. I.e. we have*

$$(R^\nabla(X,Y)s)_m = [p, \rho_*(F^\omega_p(\tilde{X},\tilde{Y}))\bar{s}(p)]$$

for sections $s \in \Gamma(\mathcal{E})$ *and vector fields* $\tilde{X},\tilde{Y} \in \Gamma(T\mathcal{P}), X, Y \in \Gamma(T\mathcal{M})$ *with* $\pi_*\tilde{X} = X$ *and* $\pi_*\tilde{Y} = Y$.

Proof. See [19], where the following Lemma was used:

LEMMA 2.2.63. *Let* $\tilde{A} \in \Gamma(T\mathcal{P})$ *the fundamental vector field of* $A \in \mathfrak{g}$ *and* $f \in C^\infty(\mathcal{P}, W)^{(\mathrm{G},\rho)}$ *a function of type* σ*, where* $\sigma : \mathrm{G} \longrightarrow \mathrm{GL}(W)$ *is a representation in a finite dimensional vector space* W*. Then the function* $\tilde{A}(f) \in C^\infty(\mathcal{P}, V)$ *is given by*

$$\tilde{A}(f)(p) = -\sigma_*(A)f(p) \quad \forall p \in \mathcal{P}.$$

QED.

The local expression of Theorem 2.2.62 is given by the following corollary.

COROLLARY 2.2.64. *Let* $s \in \Gamma(\mathcal{U}, \mathcal{E})$ *a local section in the vector bundle* $\mathcal{E}$*, which is represented by a local section* $e : \mathcal{U} \longrightarrow \mathcal{P}$ *and a smooth function* $v \in C^\infty$ $(v \equiv \bar{s} \circ e)$*, i.e.*

$$s(m) = [e(m), v(m)] \quad \forall m \in \mathcal{U}.$$

Then we have

$$(R^\nabla(X,Y)s)_m = [e(m), \rho_*(F^{\omega,e}_m(X,Y))v(m)] \quad \forall m \in \mathcal{U},$$

where $\Omega^e := e^*\Omega \in \Omega^2(\mathcal{U}, \mathfrak{g})$ *is the pullback of* $\Omega \in \Omega^2_{\mathrm{hor}}(p, \mathfrak{g})^{(\mathrm{G},\mathrm{Ad})}$ *to* $\mathcal{U}$*.*

Proof. See A.1.8. QED.

According to Remark 2.2.51 the space of all connection forms on a G-principle fibre bundle $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ is a affine space which underlying vector space is given by the set $\Omega^1_{\mathrm{hor}}(\mathcal{P}, \mathfrak{g})^{(G,\mathrm{Ad})}$ of horizontal, Lie-algebra valued 1-forms of type Ad. Modifying a connection form with a horizontal 1-form of type Ad, then we obtain the following curvature form.

Theorem 2.2.65. *Let $(\mathcal{P}, \pi, \mathcal{M}; \mathrm{G})$ a* G*-principle fibre bundle, $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ a connection form on $\mathcal{P}$ and $\sigma \in \Omega^1_{\mathrm{hor}}(\mathcal{P}, \mathfrak{g})^{(\mathrm{G},\mathrm{Ad})}$ a horizontal lie algebra valued 1-form on $\mathcal{P}$ of type* Ad*. The curvature form of $A := \omega + \sigma$ is given by*

$$F^A = F^\omega + \mathcal{D}_\omega \sigma + \frac{1}{2}[\sigma, \sigma]^\wedge, \tag{2.25}$$

where F^ω denotes the curvature form of ω and $\mathcal{D}_\omega$ the differential relative to ω.

Proof. We have

$$\begin{aligned} F^A &= \mathcal{D}_A A = \mathrm{d}A + \frac{1}{2}[A, A]^\wedge = \mathrm{d}(\omega + \sigma) + \frac{1}{2}[\omega + \sigma, \omega + \sigma]^\wedge \\ &= \mathrm{d}\omega + \mathrm{d}\sigma + \frac{1}{2}([\omega, \omega]^\wedge + [\omega, \sigma]^\wedge + [\sigma, \omega]^\wedge + [\sigma, \sigma]^\wedge) \\ &= \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]^\wedge + \mathrm{d}\sigma + [\omega, \sigma]^\wedge + \frac{1}{2}[\sigma, \sigma]^\wedge = F^\omega + \mathcal{D}^\omega \sigma + \frac{1}{2}[\sigma, \sigma]^\wedge, \end{aligned}$$

where we used $[\sigma, \omega]^\wedge = (-1)^{(1 \cdot 1 + 1)}[\omega, \sigma]^\wedge$. QED.

2.2.3. Covariant Differentiation and 2nd fundamental form

In the following we consider $\mathcal{M}$-vector fields on Σ. These are vector fields along the inclusion mapping $i : \Sigma \hookrightarrow \mathcal{M}$. Additionally we denote by $\Gamma(T\mathcal{M})|_\Sigma$ the set of all differentiable vector fields; it is a real vector space and a module over the algebra $\mathfrak{C}(\Sigma)$ of differentiable functions on Σ. For every $Y \in \Gamma(T\mathcal{M})$, the restriction $Y|_\Sigma$ lies in $\Gamma(T\mathcal{M})|_\Sigma$. $\Gamma(T\mathcal{M})$ is a submodule of $\Gamma(T\mathcal{M})|_\Sigma$. Since $(\Sigma, g|_\Sigma)$ is a Riemannian submanifold of $\mathcal{M}$, every tangent space $T_p\Sigma$ is a non-degenerate subspace of $T_p\mathcal{M}$ and we get a decomposition $T_p\mathcal{M} = T_p\Sigma \oplus T_p\Sigma^\perp$, where $T_p\Sigma^\perp$ is also non-degenerate. The corresponding projections are $\mathbb{R}$-linear and provide a unique decomposition $v = \tan v + \mathrm{nor} v$ for all vectors $v \in T_p\mathcal{M}$

$$\tan : T_p\mathcal{M} \longrightarrow T_p\Sigma \tag{2.26}$$

$$\mathrm{nor} : T_p\mathcal{M} \longrightarrow T_p\Sigma^\perp. \tag{2.27}$$

A vector field $X \in \Gamma(T\mathcal{M})|_\Sigma$ is said to be normal to Σ, if $X_p \in T_p\Sigma^\perp$ for all $p \in \Sigma$. The set of all normal vector fields $\Gamma(T\Sigma)^\perp$ forms a submodule of $\Gamma(T\mathcal{M})|_\Sigma$. Applying Eq. (2.26) and Eq. (2.27) for each $p \in \Sigma$ to a vector field $X \in \Gamma(T\mathcal{M})|_\Sigma$, we obtain vector fields $\tan X \in \Gamma(T\Sigma)$ and $\mathrm{nor}X \in \Gamma(T\Sigma)^\perp$. The resulting projections

$$\tan \ : \ \Gamma(T\mathcal{M})|_\Sigma \longrightarrow \Gamma(T\Sigma) \tag{2.28}$$

$$\mathrm{nor} \ : \ \Gamma(T\mathcal{M})|_\Sigma \longrightarrow \Gamma(T\Sigma)^\perp \tag{2.29}$$

are $\mathfrak{C}(\Sigma)$-linear.

Let $X, Y \in \Gamma(T\Sigma)$ and $p \in \Sigma$. Since $(\nabla^{\mathcal{M}}_X Y)_p$ is defined for each $p \in \Sigma$, we shall denote by $\tan(\nabla^{\mathcal{M}}_X Y)_p$ its tangential component and by $\mathrm{nor}(\nabla^{\mathcal{M}}_X Y)_p$ its normal component so that

$$(\nabla^{\mathcal{M}}_X Y)_p = \tan(\nabla^{\mathcal{M}}_X Y)_p + \mathrm{nor}(\nabla^{\mathcal{M}}_X Y)_p, \tag{2.30}$$

where

$$\tan(\nabla^{\mathcal{M}}_X Y)_p \in T_p\Sigma \qquad \text{and} \qquad \mathrm{nor}(\nabla^{\mathcal{M}}_X Y)_p \in T_p\Sigma^\perp.$$

In Eq. (2.30) $\tan(\nabla^{\mathcal{M}}_X Y)_p$ is introduced just as a symbol for the tangential component; now we want to show that it is in fact the covariant differentiation for the Levi-Civita connection of Σ.

Proposition 2.2.66. *(See [48].) The tangential component of $\nabla^{\mathcal{M}}$ is the covariant differentiation for the Levi-Civita connection of Σ. We have*

$$\tan\nabla^{\mathcal{M}}_X Y = \nabla^{\mathrm{LC}}_X Y \tag{2.31}$$

for all $X, Y \in \Gamma(T\Sigma)$.

Proof. See A.1.9. QED.

In a next step we want to prove the basic properties concerning the normal component $\mathrm{nor}(\nabla^{\mathcal{M}}_X Y)_p$. In what follows we denote the normal component with K.

$$K(X,Y) := \mathrm{nor}\nabla^{\mathcal{M}}_X Y \qquad \textit{for} \quad X, Y \in \Gamma(T\Sigma). \tag{2.32}$$

Thus the composition of $(\nabla^{\mathcal{M}}_X Y)$, that is Eq. (2.30), yields

$$\nabla^{\mathcal{M}}_X Y = \nabla^{\mathrm{LC}}_X Y + K(X,Y) \qquad for \quad X,Y \in \Gamma(T\Sigma). \tag{2.33}$$

PROPOSITION 2.2.67. *(See [48]) The mapping $K : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \longrightarrow \Gamma(T\Sigma)^{\perp}$ is symmetric (i.e., $K(X,Y) = K(Y,X)$) and bilinear over $\mathfrak{C}(\Sigma)$. Furthermore we have $[K(X,Y)]_p =: K_p(X,Y) = K_p(V,W)$, where the vector fields $X = V$ and $Y = W$ in the neighborhood $\mathcal{U}$ of $p \in \Sigma$. Consequently, $K_p(X,Y)$ depends only on X_p and Y_p, and there is induced a symmetric bilinear mapping $K_p : T\Sigma \times T\Sigma \longrightarrow T\Sigma^{\perp}$.*

Proof. see [48]. QED.

DEFINITION 2.2.68. *We define $K : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \longrightarrow \Gamma(T\Sigma)^{\perp}$ as the* **second fundamental form** *of Σ for the given immersion in $\mathcal{M}$. For each $p \in \Sigma$, $K_p : T_p\Sigma \times T_p\Sigma \longrightarrow T_p\Sigma^{\perp}$ is called second fundamental form of Σ at p.*

In the case where Σ is a hypersurface immersed in $\mathcal{M}$ (see Section 2.1), choosing a unit normal vector field n in a neighborhood $\mathcal{U}$ of a point p in Σ, and we get

$$K(X,Y) = k(X,Y)n \qquad \forall X,Y \in \Gamma(T\Sigma),$$

where $k : \Gamma(T\Sigma) \times \Gamma(T\Sigma) \longrightarrow \mathfrak{C}(\Sigma)$ is symmetric and bilinear over $\mathfrak{C}(\Sigma)$. k_p is a symmetric bilinear function on $T_p\Sigma \times T_p\Sigma$. In classical literature, k is called the second fundamental form of $\mathcal{M}$. Due to $g(n,n) = -1$, we obtain

$$k(X,Y) = -g(K(X,Y),n). \tag{2.34}$$

REMARK 2.2.69. *Next we want to explain, how K describes the 'extrinsic curvature' of Σ in $\mathcal{M}$. Let c be a geodesic in Σ, which satisfies $c(0) = p \in \Sigma$, and $\frac{\mathrm{d}}{\mathrm{d}t}c|_p = v \in T_p\Sigma$. Since $\nabla_{\dot{c}}\dot{c} = 0$ the acceleration acting on c in $\mathcal{M}$ originates from the curvature of Σ in $\mathcal{M}$; the extrinsic curvature. But that acceleration is just given by the second fundamental form:*

$$\nabla^{\mathcal{M}}_{\dot{c}}\dot{c} = \nabla^{\mathrm{LC}}_{\dot{c}}\dot{c} + K(\dot{c},\dot{c}) = K(\dot{c},\dot{c}).$$

Weingarten mapping

Next, let be $\xi \in \Gamma(T\Sigma)^\perp$ respectively $X \in \Gamma(T\Sigma)$ and write

$$\nabla^{\mathcal{M}}_X \xi = \tan \nabla^{\mathcal{M}}_X \xi + \mathrm{nor} \nabla^{\mathcal{M}}_X \xi, \tag{2.35}$$

where, for the moment, $\tan \nabla^{\mathcal{M}}_X \xi \equiv A_\xi(X)$ and $\mathrm{nor} \nabla^{\mathcal{M}}_X \xi$ are just symbols for the tangential and normal components depending on X and ξ. About A_ξ we prove

PROPOSITION 2.2.70. *(See [48])*

i.) The mapping

$$\begin{aligned} A : \Gamma(T\Sigma) \times \Gamma(T\Sigma)^\perp &\longrightarrow \Gamma(T\Sigma) \\ (X, \xi) &\longmapsto \tan \nabla^{\mathcal{M}}_X \xi \end{aligned}$$

is bilinear over $\mathfrak{C}(\Sigma)$; consequently, $(A_\xi(X))_p$ depends only on X_p and ξ_p, and there is induced a bilinear mapping of

$$A_p : T_p\Sigma \times T_p\Sigma^\perp \longrightarrow T_p\Sigma,$$

where p is an arbitrary point of $\mathcal{M}$.

ii.) For each $\xi \in T_p\Sigma^\perp$, we have

$$g(A_\xi(X, Y)) = -g(K(X, Y), \xi)$$

for all $X, Y \in T_p\Sigma$; consequently, A_ξ is a symmetric linear transformation of $T_p\Sigma$ with respect to g_p.

Proof. See A.1.10. QED.

This shows that $A_\xi : T_p\Sigma \longrightarrow T_p\Sigma$ is the linear transformation which corresponds to the symmetric bilinear function $K_p : T_p\Sigma \times T_p\Sigma \longrightarrow T_p\Sigma^\perp$. Thus A_ξ is symmetric w.r.t. g_p: $g_p(A_\xi(X), Y) = -g_p(K_p(X, Y), \xi) = -g_p(K_p(Y, X), \xi) = g_p(A_\xi(Y), X)$.

Now we will consider the case of a hypersurface Σ. On Σ there exists a uniquely determined – up to sign – unit normal vector field n. Then differentiating $g(n,n) = -1$ covariant in the direction of $X \in \Gamma(T\Sigma)$, we obtain $0 = g(\nabla^{\mathcal{M}}_X n, n) = g(\mathrm{nor}\nabla^{\mathcal{M}}_X n, n)$. Since $\mathrm{nor}\nabla^{\mathcal{M}}_X n$ is normal and therefore a scalar multiple of n we must have $\mathrm{nor}\nabla^{\mathcal{M}}_X n = 0$ for all $p \in \Sigma$.
Every $\xi \in \Gamma(T\Sigma^{\perp})$ can be rewritten as $\xi = f \cdot n$ for $f \in \mathfrak{C}(\Sigma)$. Then we obtain for the covariant differentiation in the direction of $X \in \Gamma(T\Sigma)$ $\mathrm{nor}\nabla^{\mathcal{M}}_X \xi = \mathrm{nor}(X(f)n + f\nabla^{\mathcal{M}}_X n) = X(f)n$.

Thus we have developed for all $X, Y \in \Gamma(T\Sigma)$ and $\xi = fn \in T\Sigma^{\perp}$ the first set of basic formulas for submanifolds, namely,

$$\nabla^{\mathcal{M}}_X Y = \nabla^{\mathrm{LC}}_X Y + K(X,Y) \tag{2.36}$$

$$\nabla^{\mathcal{M}}_X \xi = A_\xi(X) + X(f)n \tag{2.37}$$

respectively

$$\nabla^{\mathcal{M}}_X n = A_n(X), \tag{2.38}$$

where Eq. (2.36) is called *Gauss's formula* and Eq. (2.37) *Weingarten's formula*, respectively. This leads us to the following definition:

DEFINITION 2.2.71. *The Weingarten mapping with respect to n is given by*

$$\begin{aligned} \mathrm{Wein} : \Gamma(T\Sigma) &\longrightarrow \Gamma(T\Sigma) \\ X &\longmapsto \nabla^{\mathcal{M}}_X n = A_n(X). \end{aligned}$$

REMARK 2.2.72. *Generally*

$$\begin{aligned} A_\xi : \Gamma(T\Sigma) &\longrightarrow \Gamma(T\Sigma) \\ X &\longmapsto A_\xi(X) \end{aligned}$$

is called the Weingarten mapping with respect to $\xi \in \Gamma(T\Sigma)^{\perp}$.

In the following, we want to specify some properties of the Weingarten mapping.

Remark 2.2.73. *i.) We have*

$$g(\mathrm{Wein}(X), Y) = g(A_n(X), Y) = -g(K(X,Y), n) = k(X,Y), \quad (2.39)$$

for all $X, Y \in \Gamma(T\Sigma)$. *Consequently* $\mathrm{Wein} : \Gamma(T\Sigma) \longrightarrow \Gamma(T\Sigma)$ *is the linear transformation which corresponds to the symmetric tensor field* k.

ii.) Since k *is symmetric, the Weingarten mapping is also symmetric.*

$$g(\mathrm{Wein}(X), Y) = k(X,Y) = k(Y,X) = g(\mathrm{Wein}(Y), X).$$

Proposition 2.2.74. *Let* X *and* $Y \in \Gamma(T\Sigma)$ *vector fields on* Σ. *Then we have*

$$g(K(X,Y), n) = -\frac{1}{2}(\mathcal{L}_n g)(X,Y).$$

Proof. See A.1.11. QED.

Equations of Gauss and Codazzi

In this Section we shall find a relationship between the curvature tensor fields of Σ and $\mathcal{M}$, denoted by R respectively $R^{\mathcal{M}}$, see [48]. Using the formula of Weingarten (2.37) and Gauss (2.36), we obtain for any vector fields X, Y and Z tangent to Σ

$$\begin{aligned} \nabla^{\mathcal{M}}_X(\nabla^{\mathcal{M}}_Y Z) =& \nabla^{\mathcal{M}}_X(\nabla^{\mathrm{LC}}_Y Z + K(Y,Z)) \\ =& \nabla^{\mathrm{LC}}_X(\nabla^{\mathrm{LC}}_Y Z) + K(X,Z) + A_{K(Y,Z)}(X) + X(k(Y,Z))n \\ =& \nabla^{\mathrm{LC}}_X(\nabla^{\mathrm{LC}}_Y Z) + k(Y,Z) \underbrace{A_n(X)}_{=\mathrm{Wein}(X)} \\ &+ [k(X, \nabla^{\mathrm{LC}}_Y Z) + X(k(Y,Z))]n. \end{aligned} \quad (2.40)$$

For $\nabla^{\mathcal{M}}_Y(\nabla^{\mathcal{M}}_X Z)$ we may simply interchange X and Y in Eq. (2.40) and the same calculation then reveals

$$\begin{aligned} \nabla^{\mathcal{M}}_Y(\nabla^{\mathcal{M}}_X Z) =& \nabla^{\mathrm{LC}}_Y(\nabla^{\mathrm{LC}}_X Z) + k(X,Z)\mathrm{Wein}(Y) \\ &+ [k(Y, \nabla^{\mathrm{LC}}_X Z) + Y(k(X,Z))]n. \end{aligned} \quad (2.41)$$

Also we get

$$\begin{aligned}\nabla^{\mathcal{M}}_{[X,Y]}Z &= \nabla_{[X,Y]Z} + K([X,Y],Z)\\ &= \nabla_{[X,Y]Z} + [k(\nabla^{\mathrm{LC}}_X Y, Z) - k(\nabla^{\mathrm{LC}}_Y X, Z)]n\end{aligned}$$

by virtue of $[X,Y] = \nabla^{\mathrm{LC}}_X Y - \nabla^{\mathrm{LC}}_Y X$ on Σ. Collecting terms we get

$$\begin{aligned}R^{\mathcal{M}}(X,Y)Z =&\nabla^{\mathcal{M}}_X(\nabla^{\mathcal{M}}_Y Z) - \nabla^{\mathcal{M}}_Y(\nabla^{\mathcal{M}}_X Z) - \nabla^{\mathcal{M}}_{[X,Y]}Z\\ =&\nabla^{\mathrm{LC}}_X(\nabla^{\mathrm{LC}}_Y Z) - \nabla^{\mathrm{LC}}_Y(\nabla^{\mathrm{LC}}_X Z) - \nabla^{\mathcal{M}}_{[X,Y]}Z + k(Y,Z)\mathrm{Wein}(X)\\ &- k(X,Z)\mathrm{Wein}(Y) + [k(X,\nabla^{\mathrm{LC}}_Y Z) + X(k(Y,Z))\\ &- k(Y,\nabla^{\mathrm{LC}}_X Z) - Y(k(X,Z)) - k(\nabla^{\mathrm{LC}}_X Y, Z) - k(\nabla^{\mathrm{LC}}_Y X, Z)]n\end{aligned} \tag{2.42}$$

Thus the relationship between the Riemannian curvature tensors of $\mathcal{M}$ and Σ is given by

Proposition 2.2.75. *(Equation of Gauss, see [48]) Regarding Eq.* (2.42), *we find that the tangential component of* $R^{\mathcal{M}}(X,Y)Z$ *is given by*

$$\begin{aligned}R^{\mathcal{M}}(X,Y,Z,V) :=&g(R^{\mathcal{M}}(X,Y)Z,V)\\ =&g(R(X,Y)Z,V) - k(X,Z)k(Y,V) + k(Y,Z)k(X,V)\\ =&R(X,Y,Z,V) - g(K(X,Z),K(Y,V))\\ &+ g(K(Y,Z),K(X,V))\\ =&R(X,Y,Z,V) + g(\mathrm{Wein}(Y),Z)g(\mathrm{Wein}(X),V)\\ &- g(\mathrm{Wein}(X),Z)g(\mathrm{Wein}(Y),V),\end{aligned} \tag{2.43}$$

where we have defined $g(R(X,Y)Z,V) =: R(X,Y,Z,V)$ *and we used Eq.* (2.39) *in the fourth step. Furthermore* X,Y,Z *and* $V \in \Gamma(T\Sigma)$.

Proposition 2.2.76. *(Equation of Codazzi, see [48]) For all* X,Y *and* $Z \in \Gamma(T\Sigma)$ *the normal component of* $R^{\mathcal{M}}(X,Y)Z$ *is given by*

$$\begin{aligned}\mathrm{nor}R^{\mathcal{M}}(X,Y)Z &= [(\tilde{\nabla}_X k)(Y,Z) - (\tilde{\nabla}_Y k)(X,Z)]n\\ &= g((\nabla^{\mathrm{LC}}_X \mathrm{Wein})(Y) - (\nabla^{\mathrm{LC}}_Y \mathrm{Wein})(X), Z)n,\end{aligned} \tag{2.44}$$

where we define the covariant derivative, denoted by $\tilde{\nabla}_X k$ for the second fundamental form k, to be

$$(\tilde{\nabla}_X k)(Y, Z) = X(k(Y, Z)) - k(\nabla_X^{\mathrm{LC}} Y, Z) - k(Y, \nabla_X^{\mathrm{LC}} Z).$$

Proof. See A.1.12. QED.

Corollary 2.2.77. *If $\mathcal{M}$ is of constant sectional curvature, then we have*

$$(\nabla_X^{\mathrm{LC}} \mathrm{Wein})(Y) = (\nabla_Y^{\mathrm{LC}} \mathrm{Wein})(X)$$

for all $X, Y \in \Gamma(T\Sigma)$.

Proof. See A.1.13. QED.

2.3. Spin Structure

With the intension of studying the spin structure of the *Ashtekar connection* in Section 4.3, we want to introduce the spin structure on a space-time $\mathcal{M}$.

Definition 2.3.1. *(See [35]) A spin structure* Spin *on a space-time $\mathcal{M}$ is a pair $(S(\mathcal{M}), \Lambda)$ consisting of*

i.) a $\mathrm{SL}(2, \mathbb{C})$ principle fibre bundle $(S(\mathcal{M}), \tilde{\pi}, \mathcal{M}); \mathrm{SL}(2, \mathbb{C}))$ over $\mathcal{M}$,

ii.) a double cover $\Lambda : S(\mathcal{M}) \to \mathrm{O}^+(\mathcal{M}, g)$ such that the following diagram commutes:

$$\begin{tikzcd}
S(\mathcal{M}) \times \mathrm{SL}(2,\mathbb{C}) \arrow[r, "\tilde{\mu}"] \arrow[dd, "\Lambda\times\lambda"] & S(\mathcal{M}) \arrow[dd, "\Lambda"] \arrow[dr, "\tilde{\pi}"] & \\
 & & \mathcal{M} \\
\mathrm{O}^+(\mathcal{M}, g) \times \mathrm{SO}_0(1,3) \arrow[r, "\mu"] & \mathrm{O}^+(\mathcal{M}, g) \arrow[ur, "\pi"] &
\end{tikzcd}$$

where $\lambda : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}_0(1,3)$ *denotes the universal cover of* $\mathrm{SO}_0(1,3)$*. In the rows the respective group action of the principle bundles* $S(\mathcal{M})$ *and* $\mathrm{O}^+(\mathcal{M},g)$ *is indicated. A manifold with a fixed* Spin *is called a* spin manifold.

THEOREM 2.3.2. *(Bichteler, 1967.) Let* $(\mathcal{M},g)$ *a 4-dimensional, connected, time- and space-oriented Lorentzian manifold. Then* $\mathcal{M}$ *has spin structure* Spin*, if and only if the second Stiefel-Whitney-class* $\mathfrak{w}_2(\mathcal{M})$ *on* $\mathcal{M}$ *is zero, i.e.*

$$\mathcal{M} \quad is \quad \mathrm{Spin} \Longleftrightarrow 0 = \mathfrak{w}_2(\mathcal{M}) \in H^2(\mathcal{M};\mathbb{Z}_2),$$

where H^2 *denotes the 2nd homology group of* $\mathcal{M}$*. For a definition of Stiefel-Whitney-classes see [54].*

Proof. See [23]. QED.

THEOREM 2.3.3. *Let* Σ *a orientable 3-dimensional manifold. Then we have* $\mathfrak{w}_2(\Sigma) = 0$*.*

Proof. See [46]. QED.

THEOREM 2.3.4. *(Geroch, 1968.) Let* $\mathcal{M}$ *a space-time. Then* $\mathcal{M}$ *can be parallelized* Par *this means* $\mathcal{M}$ *can be covered by a single distinguished frame* $e = (e_1, \ldots, e_4)$*, if and only if* $\mathcal{M}$ *is* Spin*, i.e.* $\mathcal{M} \quad is \quad \mathrm{Spin} \Longleftrightarrow \mathcal{M} \quad is \quad \mathrm{Par}$.

Proof. See [37]. QED.

3. Physical Prologue

In the chapter in hand we want to introduce the bacis tools of loop quantum gravity with the intension the explain the origin of the Ashtekar variables. Furthermore the understandig of the pysical background is essential for the construction of the Riemannian scalar curvature operator in Chapter 6. Since there are many good books and reviews on both general relativity [71, 65] and loop quantum gravity [69, 62, 11] we will only give a short introduction.

3.1. Hamiltonian formulation of General Relativity (GR)

In this Section we provide a self-contained exposition of the classical *Hamiltonian formulation* of General Relativity. It is mandatory to know all the details of this classical work as it lays the ground for the interpretation of the theory. It also defines the platform on which the quantum theory is based. A Hamiltonian (canonical) formulation of a field theory requires a breakup of space time into space and time. This split is necessary in a canonical approach, as otherwise we cannot define velocities and hence momenta conjugate to the configuration variables. The $(d+1)$ split seems to break diffeomeorphism invariance. But this is not the case because we do not fix the split in space and time, rather we keep it arbitrary, this means we do *not* fix a coordinate system. Indeed, the first step in producing a Hamiltonian formulation of a field theory consists of choosing a time function t and a vector field t^a on a space time such that the surfaces Σ_t of constant t are space-like Cauchy surfaces – for a mathematical definition see Definition 2.1.12 – and such that $t^a \nabla_a t = 1$.The vector fields t_a may be interpreted as describing the *flow of time* in space time and can be used to identify each Σ_t with initial Σ_0. This selection of a particular time direction seems to break the space-time covariance, bit in the end the formalism itself will tell us that it really did not matter which direction of time we took to begin with. In the sixties of the past century this approach was applied to GR by Arnowitt, Deser and Misner (ADM), Dirac, Wheeler and De Witt, among many others.

3.1.1. Arnowitt-Deser-Misner (ADM) formalism

The ADM action

The standard Hamilton formulation for general relativity was developed by Arnowitt, Deser and Misner [3, 4]. A modern treatment can be found in [71].
Let us consider a four-dimensional, Lorentzian manifold $(\mathcal{M}, g_{\mu\nu})$, compare Section 2.1,Definition 2.1.11. The space time metric will be denoted by $g_{\mu\nu}$ and will have the Lorentzian signature $(-,+,+,+)$. Here Greek indices from the middle of the alphabet $\mu, \nu, \rho, ... = 0, 1, 2, 3$ are indices for the component of 4 dimensional space time tensors and we denote in the following X^μ as coordinates of $\mathcal{M}$ in local trivializations. The object of interest is the *Einstein-Hilbert action* for the metric tensor field $g_{\mu\nu}$ which evolves – as a dynamical object – on a manifold $\mathcal{M}$,

$$S_{\mathrm{EH}}[g] = \frac{1}{\kappa} \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{|g(x)|} \mathcal{R}^{\mathcal{M}}, \tag{3.1}$$

where $\mathcal{R}^{\mathcal{M}}$ is the Riemannian curvature scalar associated with $g_{\mu\nu}$ and $\kappa = 16\pi G/c^3$ where G is Newton's constant.

We make the assumption that $\mathcal{M}$ has the topology $\mathcal{M} \cong \mathbb{R} \times \Sigma$, where Σ is a fixed three-dimensional manifold of arbitrary topology with metric q_{ab} in order to derive a canonical form of the action (3.1). By a theorem due to Geroch (Section 2.1, Theorem 2.1.13, [38]) and improved by Bernal and Sanchez [20], any globally hyperbolic space time is necessarily of this

kind of topology. Thus, $\mathcal{M}$ admits a foliation into a one-parameter family of hypersurfaces $\Sigma_t = X_t(\Sigma)$, that is, we have for all $t \in \mathbb{R}$ an embedding of $X_t = \Sigma \to \mathcal{M}$ defined by $X_t(x) := X(t, x)$, where x^a are local coordinates of Σ. Latin indices from the beginning of the alphabet $a, b, c... = 1, 2, 3$ are indices for three dimensional manifold coordinates. The foliation allows us to identify the coordinate t as a time parameter. Notice however that this *time* should not be regarded as an absolute quantity, because of the diffoemorphism invariance of the action. Any diffeomorphism $\varphi \in \mathrm{Diff}(\mathcal{M})$ of $\mathcal{M}$ is of the form $\varphi = X' \circ X^{-1}$, where X, X' are two different foliations differ on a new time parameter t'. Any two foliations are related by φ via $X' = \varphi \circ X$. Therefore, we can work with a chosen foliation, but the diffeomorphism invariance of the theory guarantee that the physical quantities are independent of this choice.
Summarizing the freedom of the choice of the foliation is equivalent to $\mathrm{Diff}(\mathcal{M})$ and since the action Eq. (3.1) is invariant under all diffeomorphisms of $\mathcal{M}$ the foliations X are not specified by it and we must allow them to be completely arbitrary.

Given a foliation X_t and corresponding ADM coordinates (t, x). A useful parametrization of the embedding can be given through its deformation vector field

$$T^\mu(X) \equiv \frac{\partial X^\mu(t, x)}{\partial t}_{|X=X(x,t)} = (1, 0, 0, 0) =: N(X)n^\mu(X) + N^\mu(X),$$

here x are local coordinates of Σ and n^μ is a unit vector normal to Σ_t, that is, $g_{\mu\nu}n^\mu n^\nu = -1$ and N^μ is tangential, $g_{\mu\nu}n^\nu X^\nu_{,a} = 0$. It is convenient to parametrize $n^\mu = (1/N, -N^a/N)$, so that $N^\mu = (0, N^a)$. T^μ should not be confused with the unit normal vector n^μ. They are both timelike $(g_{\mu\nu}T^\mu T^\nu = g_{00})$ but they are not parallel in general. The coefficients N and N^μ respectively are called *lapse function* and *shift vector* respectively. In terms of lapse and shift, we obtain

$$\begin{aligned}
&g_{\mu\nu}T^\mu T^\nu = g_{00} = -N^2 + g_{ab}N^a N^b, \\
&g_{\mu\nu}T^\mu N^\nu = g_{0b}N^b = g_{\mu\nu}\left(Nn^\mu + N^\mu\right) = g_{ab}N^a N^b \to g_{0a} = g_{ab}N^b \equiv N_a.
\end{aligned}$$

Therefore one can explicitly recast the metric into the form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (-N^2 + N_a N^a)\mathrm{d}t^2 + 2N_a \mathrm{d}t\mathrm{d}x^a + g_{ab}\mathrm{d}x^a \mathrm{d}x^b.$$

Notice that the spatial part g_{ab} is *not* in general the intrinsic metric on Σ_t. Rather in combination with a unit vector field n^μ normal to the foliation the space-time metric $g_{\mu\nu}$ defines a unique metric on Σ_t, given by

$$q_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu \tag{3.2}$$

and is called the *first fundamental form* of Σ_t. The quantity $q^\mu_\nu = g^{\mu\rho} q_{\rho\nu}$ acts as a projector on Σ_t, offering us to define the tensorial calculus on Σ_t from the one on $\mathcal{M}$. As an important quantity in the canonical description we now consider the following tensor field, the extrinsic curvature $K_{\mu\nu}$ of Σ_t, also called the *second fundamental form* of Σ, compare Section 2.2.3, Definition 2.2.68 respectively Remark 2.2.69. This is defined by

$$K_{\mu\nu} := q^\nu_\mu q^\sigma_\nu \nabla_\rho n_\sigma, \tag{3.3}$$

where all indices are moved with respect to $g_{\mu\nu}$ and ∇ is the torsion-free derivative compatible with $g_{\mu\nu}$. Notice that both tensors (3.2) and (3.3) are spatial, that means, they vanish when either of their indices is contracted with n^μ. An important property of $K_{\mu\nu}$ is its symmetry. Because of this fact one derives another useful identity connecting it to the Lie derivative of the intrinsic metric, compare Section 2.2.3,Proposition 2.2.74:

$$2K_{\mu\nu} = (\mathcal{L}_n q)_{\mu\nu}.$$

And also we obtain

$$\dot{q}_{\mu\nu} := (\mathcal{L}_t q)_{\mu\nu} = 2NK_{\mu\nu} + (\mathcal{L}_{\vec{N}} q)_{\mu\nu}.$$

That is, the extrinsic curvature $K_{\mu\nu}$ allows us to give a measure of the variation of the three-dimensional metric with respect to the fiducial time introduced by the foliation, that is, $K_{\mu\nu}$ essentially contains the information about the *time derivative* of $q_{\mu\nu}$. Up to now we have defined quantities defined on Σ_t in terms of which we can reconstruct the space-time metric and its time derivatives. Now, we proceed to rewrite the Einstein-Hilbert action Eq. (3.1) in terms of these variables.

The extrinsic curvature $K_{\mu\nu}$ enters the *Gauss equation* (see Section 2.2.3, Eq. (2.43)), which provides a relation between the Riemann curvature tensor of Σ, denoted by R^{Σ} and that of $\mathcal{M}$, denoted by $R^{\mathcal{M}}$, namely

$$R^{\Sigma;\sigma}_{\rho\mu\nu} = q^{\rho'}_{\rho} q^{\mu'}_{\mu} q^{\nu'}_{\nu} q^{\sigma}_{\sigma'} R^{\mathcal{M};\sigma'}_{\rho'\mu'\nu'} + 2K^{\sigma}_{[\nu} K_{\mu]\rho}.$$

With this formula we can concentrate on the Riemann curvature scalar $\mathcal{R}^{\mathcal{M}}$ of the *Einstein-Hilbert action.* Employing the abbreviations $K := K_{\mu\nu} q^{\mu\nu}$ and $K^{\mu\nu} = q^{\mu\rho} q^{\nu\sigma} K_{\rho\sigma}$, we obtain the following expression for the Riemannian curvature scalar associated with Σ

$$\mathcal{R}^{\Sigma} = -K^2 + K_{\mu\nu} K^{\mu\nu} + q^{\mu\rho} q^{\nu\sigma} R^{\mathcal{M}}_{\mu\nu\rho\sigma}. \tag{3.4}$$

Next we want to eliminate the last term in (3.4) by using $g = q - n \otimes n$ and the definition of curvature $R^{\mathcal{M}}_{\mu\nu\rho\sigma} n^{\sigma} = 2\nabla_{[\mu}\nabla_{\nu]} n_{\rho}$ in order to express the latter equation purely in terms of $\mathcal{R}^{\mathcal{M}}$ alone. We get

$$\begin{aligned} \mathcal{R}^{\mathcal{M}} = R^{\mathcal{M}}_{\mu\nu\rho\sigma} g^{\mu\rho} g^{\nu\sigma} &= q^{\mu\rho} q^{\nu\sigma} R^{\mathcal{M}}_{\mu\nu\rho\sigma} - 2q^{\rho\mu} n^{\nu} [\nabla_{\mu}, \nabla_{\nu}] n_{\rho} \\ &= q^{\mu\rho} q^{\nu\sigma} R^{\mathcal{M}}_{\mu\nu\rho\sigma} - 2n^{\nu} [\nabla_{\mu}, \nabla_{\nu}] n^{\nu}, \end{aligned} \tag{3.5}$$

where in the first step we used the antisymmetry of the Riemann tensor to eliminate the quadratic term in n and in the second step we used again $g = q - n \otimes n$ and additionally the antisymmetry in the $\mu\nu$ indices. Next, we have

$$n^{\nu}([\nabla_{\mu}, \nabla_{\nu}] n^{\nu}) = -(\nabla_{\mu} n^{\nu})(\nabla_{\nu} n^{\mu}) + (\nabla_{\mu} n^{\mu})(\nabla_{\nu} n^{\nu}) + \nabla_{\mu}(n^{\nu} \nabla_{\nu} n^{\mu} - n^{\mu} \nabla_{\nu} n^{\nu}).$$

In particular the addends yields:

$$\nabla_{\mu} n^{\mu} = g^{\mu\nu} \nabla_{\nu} n_{\mu} = q^{\mu\nu} \nabla_{\nu} n^{\mu} = K \tag{3.6}$$

and

$$(\nabla_{\mu} n^{\nu})(\nabla_{\nu} n^{\mu}) = g^{\nu\sigma} g^{\rho\mu} (\nabla_{\mu} n_{\sigma})(\nabla_{\nu} n_{\rho}) = q^{\nu\sigma} q^{\rho\mu} (\nabla_{\mu} n_{\sigma})(\nabla_{\nu} n_{\rho}) = K_{\mu\nu} K^{\mu\nu}. \tag{3.7}$$

Combining Eq. (3.4), Eq. (3.5) and Eq. (3.6) respectively Eq. (3.7), we obtain the Codacci equation

$$\mathcal{R}^{\mathcal{M}} = \mathcal{R}^{\Sigma} - K^2 + K_{\mu\nu}K^{\mu\nu} - 2\nabla_\mu(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu). \tag{3.8}$$

At this point it is useful to pull back various quantities to Σ, see [69]. We define $N(x,t) := N(X(x,t)), \vec{N}^a(x,t) := q^{ab}(x,t)(X^\mu_b g_{\mu\nu}N^\nu)(X(x,t))$. Then we get

$$K_{ab}(x,t) = \frac{1}{2N}\left(\dot{q}_{ab} - (\mathcal{L}_{\vec{N}}q)_{ab}\right)(x,t). \tag{3.9}$$

After pulling back the quantities appearing in Eq. (3.8) such as the extrinsic curvature $K_{\mu\nu}$ and after dropping the total differential in Eq. (3.8) as a result the *Einstein-Hilbert action* (3.1) yields the *Arnowitt-Deser-Misner action*

$$S_{\mathrm{ADM}}[q] = \frac{1}{\kappa}\int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x\sqrt{|q(x)|}|N|\left[\mathcal{R}^\Sigma + K_{ab}K^{ab} - (K^a_a)^2\right]. \tag{3.10}$$

Now we want to cast this action into canonical form, that is, we would like to perform a Legendre transform from the Lagrangian density in (3.10) to the corresponding Hamiltonian density.

Legendre transform and Dirac analysis of constraints

Before we move on, we will give some useful mathematical definitions.

Definition 3.1.1. *Let $T_*(\mathcal{C})$ be the tangent bundle over the m-dimensional configuration manifold $\mathcal{C}$, where $v := \dot{q}$ defines the corresponding action principle and consider a Lagrangean function $L : T_*(\mathcal{C}) \to \mathbb{C}; (q^a, v^a) \mapsto L(q,v)$. Then the map*

$$\rho_L : T_*(\mathcal{C}) \to T_*(\mathcal{C}); (q,v) \mapsto \left(q, p(q,v) := \frac{\partial L}{\partial v}(q,v)\right)$$

is called Legendre transformation. A Lagrangean is called singular provided that ρ_L is not surjective, that is,

$$\det\left(\left(\frac{\partial^2 L}{\partial v^a \partial v^b}\right)^m_{a,b=1}\right) = 0. \tag{3.11}$$

The rank of the matrix in Eq. (3.11) is $m-r$, with $0<r\leq m$. Using the inverse function theorem, we are able to solve (at least locally) $m-r$ velocities for $m-r$ momenta and the remaining velocities, that is w.l.g.

$$p_A = \frac{\partial L}{\partial v^A}(q,v) \longrightarrow v^A = u^A(q^A, p_A, v^j).$$

where $a,b,\dots = 1,\dots,m$, $A,B,\dots = 1,\dots,m-r$ and $j,k,\dots = m-r+1,\dots,m$. Inserting the latter equation into the remaining equations it follows that $p_j = \partial L/\partial v_j$ cannot depend on the v_j any more as otherwise the rank would exceed. Therefore we get equations of the form

$$p_j = \left(\frac{\partial L}{\partial v^j}(q,v)\right)_{v^A=u^A(q^A,p_A,v^k)} =: \pi_j(q^a,p_A).$$

The latter equation shows that the p_a are not independent of each other.

Definition 3.1.2. *The functions*

$$\phi_j(q^a,p_a) := p_j - \pi_j(q^a,p_A)$$

are called primary constraints. Furthermore the function

$$\mathcal{H}'(q^a,p_a,v^j) := [p_a v^a - L(q^a,p_a)]_{v^A=u^A(q^A,p_A,v^k)}$$

is called the primary Hamiltonian corresponding to L.

At this point we want to continue with the transformation of the Lagrangian density appearing in the *ADM action* to the corresponding Hamilton density. Eq. (3.10) do not depend on the velocities of N and N^a, which implies that N and N^a are Lagrange multipliers but the action depends – using Eq. (3.9) – on the velocities $\dot{q}_{ab}$ of q_{ab}. Therefore we obtain for the conjugate momenta (use the fact that ${}^{\Sigma_t}R$ does not contain time derivatives)

$$\begin{aligned}
\Pi(t,x) &:= \frac{\delta L}{\delta \dot{N}(t,x)} = 0,\\
\Pi_a(t,x) &:= \frac{\delta L}{\delta \dot{N}^a(t,x)} = 0,\\
P^{ab} &:= \frac{\delta L}{\delta \dot{q}_{ab}(t,x)} = \frac{\sqrt{\det(q)}}{\kappa}\left(K^{ab} - K^c_c q^{ab}\right).
\end{aligned} \tag{3.12}$$

Since one canot solve all velocities for momenta, the Lagrangian in Eq. (3.10) is said to be a *singular* Lagrangian, see Definition 3.1.1. In particular Eq. (3.12) shows, that it is not possible to solve $\dot{N}$, $\dot{N}^a$ respectively in terns of q_{ab},N,N^a and P^{ab}, rather we obtain the so-called *primary constraints* $C := \Pi(t,x) = 0$ and $C_a := \Pi^a(t,x) = 0$, for which we introduce the Lagrange multiplier fields $\lambda(t,x)$ and $\lambda_a(t,x)$. After performing the Legendre transform and a spatial integration by parts one can cast (3.10) into the following compact form:

$$
\begin{aligned}
S_{\mathrm{ADM}} = \frac{1}{\kappa}\int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \mathrm{d}^3x[\dot{q}_{ab}P^{ab} + \dot{N}\Pi + \dot{N}^a\Pi_a \\
- (\lambda C + \lambda^a C_a + N^a H_a + |N|H)],
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
H_a &:= -2\sqrt{q}\mathcal{D}_b\left(\frac{P_a^b}{\sqrt{q}}\right), \\
H &:= \frac{1}{\sqrt{q}} G_{abcd}P^{ab}P^{cd} - \sqrt{q}\mathcal{R}^{\Sigma}, \quad G_{abcd} = q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd},
\end{aligned}
\tag{3.14}
$$

where G_{abcd} is called the *super-* or *DeWitt-metric* and $\mathcal{D}_a$ is the spatial covariant derivative. H_a are called the *(spatial) Diffeomorphism or vector constraint* and H is called *Hamiltonian or scalar constraint*, for reasons we will see below.

Definition 3.1.3. *A symplectic structure for a differential manifold $\mathcal{M}$ is a non-degenerate, closed two-form Ω. The pair $(\mathcal{M}, \Omega)$ is called a symplectic manifold.*

Theorem 3.1.4. *Let $(\mathcal{M}, \Omega)$ be a symplectic manifold. Then for a neighborhood Z of each point p one can choose so-called canonical coordinates $(x^\mu)_{\mu=1}^{2m} = (q^a, P_a)_{a=1}^m$ such that $\Omega = \mathrm{d}P_a \wedge \mathrm{d}q^a$, where $2m = dim(\mathcal{M})$. The coordinates (q, P) are called configuration and momentum variables respectively.*

Proof. See [69]. QED.

Equipped with the Hamiltonian form of the *Einstein-Hilbert* action, cf. (3.13), we can evaluate the phase space of GR. The phase space is parametrized by the pair (q_{ab}, P^{ab}), with the symplectic structure Ω or Poisson bracket is given by

$$\{P^{ab}(t,x), q_{cd}(t,x')\} = \kappa\delta^a_{(c}\delta^b_{d)}\delta^{(3)}(x,x') \tag{3.15}$$

In the language of symplectic geometry, the first term in (3.13) is a symplectic potential for the symplectic structure (3.15).

At this point we want to turn to the meaning of the term in brackets in Eq. (3.13), the so called *Hamiltomian* of the action

$$\begin{aligned}\mathcal{H} &:= \frac{1}{\kappa}\int_\Sigma \mathrm{d}^3x(\lambda C + \lambda^a C_a + N^a H_a + |N|H) \\ &=: C(\lambda) + \vec{C}(\vec{\lambda}) + \vec{H}(\vec{N}) + H(|N|).\end{aligned} \tag{3.16}$$

The variation of the action (3.13) with respect to the Langrange multipliers gives the equations

$$H_\mu = (H(q,P), H_a(q,P)) = 0.$$

Physical configurations, also called *on-shell* configurations, i.e. $G_{\mu\nu} = 0$, must satisfy these four constraints. Now we obtain the *Dirac algebra* $\mathfrak{D}$

$$\begin{aligned}\{H_a(x), H_b(y)\} &= H_a(y)\partial_b\delta(x-y) - H_b(x)\partial'_a\delta(x-y), \\ \{H_a(x), H(y)\} &= H(x)\partial_a\delta(x-y), \\ \{H(x), H(y)\} &= H^a(y)\partial_a\delta(x-y) - H^a(y)\partial'_a\delta(x-y).\end{aligned}$$

From the above equations we recognize that the constraint surface $\overline{\mathcal{M}}$ of $\mathcal{M}$, the submanifold of $\mathcal{M}$, where the constraints hold, is preserved under the motions generated by the constraints, see Figure 3.1.. In the terminology of Dirac, all constraints are first class. Following [42] first class constraints generates gauge transformations.

Geometrical interpretation of the gauge transformations

We obtain the reduced action, the so-called *canonical ADM action*

$$S_{\mathrm{cADM}}(q_{ab}, P^{ab}, N, N^a) = \frac{1}{\kappa}\int_{\mathbb{R}}\mathrm{d}t\int_\Sigma \mathrm{d}^3x[P^{ab}\dot{q}_{ab} - (N^a H_a + |N|H)], \tag{3.17}$$

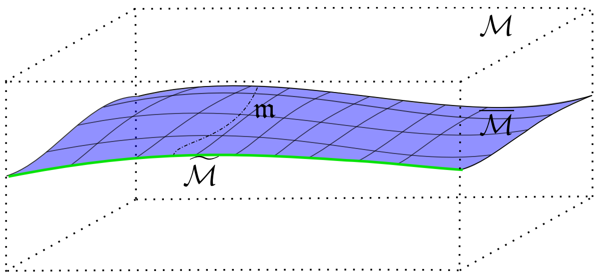


Figure 3.1.: Unconstrained phase space, constraint surface $\overline{\mathcal{M}}$, gauge orbit $\mathfrak{m}$ and physical phase space $\widetilde{\mathcal{M}}$.

From Eq. (3.17) we obtain the *Hamiltonian* of the action and the associated equation of motions:

$$\mathcal{H} := \frac{1}{\kappa} \int_{\Sigma} \mathrm{d}^3x [N^a H_a + |N| H]. \tag{3.18}$$

Since it is proportional to the Lagrange multipliers, the Hamiltonian has the characteristic, that it *vanishes* on-shell. Thus GR is an example of a so-called constrained Hamiltonian system with no *true* Hamiltonian.

Now, we want to interpret the motions that the constraints generate on $\mathcal{M}$ geometrically. Since Eq. (3.18) is a linear combination of constraints, we get the equation of motion once we know the Hamilton flow of the functions $H(N)$ and $H(\vec{N})$ for any $N, \vec{N}$ seperarately. To see what the gauge transformations look like we integrate (3.14) against suitable *test* functions, so that both constraint functions are simple polynominals in P_{ab}. Hence we obtain the smeared constraints $H(N) := \int_\Sigma H(x)N(x)\mathrm{d}^3x$ respectively $\vec{H}(\vec{N}) := \int_\Sigma H^a(x)N_a(x)\mathrm{d}^3x$. After a lengthly calculation, see [69], we obtain

$$\left\{\vec{H}(\vec{N}), q_{ab}\right\} = \mathcal{L}_{\vec{N}} q_{ab}, \tag{3.19}$$

$$\left\{\vec{H}(\vec{N}), P_{ab}\right\} = \mathcal{L}_{\vec{N}} P_{ab}, \tag{3.20}$$

$$\{H(N), q_{ab}\} = \mathcal{L}_{\vec{n}N} q_{ab}, \tag{3.21}$$

$$\{H(N), P_{\mu\nu}\} = \mathcal{L}_{\vec{n}N} P_{\mu\nu} + \frac{1}{2} q^{\mu\nu} N H - 2N\sqrt{q} q^{\mu[\rho} q^{\nu]\sigma} R^{\mathcal{M}}_{\rho\sigma}, \tag{3.22}$$

where in the latter equation we used again $g = q - n \otimes n$. Equations (3.19) and (3.20) shows, that the *Diffeomorphism or vector constraint* is the generator of space-diffeomorphisms on Σ. For the *Hamiltonian or scalar constraint* we get following meaning. Equation (3.21) respectively (3.22) give the action of time diffeomorphisms on q_{ab} respectively P_{ab}. But notice that (3.22) contains also two extra addends, which only vanish iff $H = 0$ and $R^{\mathcal{M}}_{\mu\nu} = 0$. This means only on the constraint surface $\overline{\mathcal{M}}$ and only when the (vacuum) equation of motion holds – i.e. *on shell*, $G_{\mu\nu} = 0$ – the Hamilton flow of $P_{\mu\nu}$ with respect to $H(N)$ can be interpreted as the action of a diffeomorphism in the direction perpendicular to Σ_t. Thus we can conclude, that the constraints H^μ are the generators of the space time diffeomorphism group $\mathrm{Diff}(\mathcal{M})$ on physical configurations.

Fully constrained system and physical degrees of freedom

The canonical formalism has the advantage that it allows us the counting of the number of degrees of freedom in a robust way. Recall in fact that in classical physics a physical trajectory is characterized by each point in phase space, i.e. initial position and momentum, and the number of degrees of freedom is defined to be half the dimensionality of the phase space. In GR but also gauge theories as examples of constrained theories, one has to be careful with the constraints. For this purpose, it is ordinary to distinguish a notion of *kinematical phase space*, and *physical phase space* .

The Poisson structure of the theory defines the kinematical phase space. In our case, the space (q_{ab}, P^{ab}) , with Poisson brackets (3.15). Its dimensionality is $(6+6)\cdot\infty^3 = 12\cdot\infty^3$. On this space, the constraints $H_\mu = 0$

define a hypersurface, the so-called constraint surface $\overline{\mathcal{M}}$ within the the full phase space $\mathcal{M}$ where they are satisfied, i.e. the space of (q_{ab}, P^{ab}) such that the constraints Eq. (3.1.1) are satisfied. The dimension of this space is $(12-4)\cdot\infty^3 = 8\cdot\infty^3$. The gauge motions are defined on all of $\mathcal{M}$ but the fact that the algebra of constraints is first class guarantees that the gauge transformations generated by the constraints leave the constraint hypersurface invariant. Thus the orbit of a point m in the hypersurface under gauge transformations will be a curve or gauge orbit $\mathfrak{m}$ entirely within it. The set of these curves defines the so-called physical phase space and Dirac observables restricted to $\overline{\mathcal{M}}$ depend only on these orbits. Points along one orbit correspond to the same physical configuration, only described in different coordinate systems. In order to select the physical degrees of freedom, we have to divide by the gauge orbits in such a way identical to what happens in gauge theories. Since the orbits span a manifold of dimension four at each space point, dividing by the orbits gives $(8-4)\cdot\infty^3 = 4\cdot\infty^3$. This is the physical phase space $\widetilde{\mathcal{M}}$. It has four dimensions per space point, therefore the theory has precisely the two physical degrees of freedom per space point of general relativity. See e.g. [42] for more details.

This has far as the counting goes. However, in the case of the linearized analysis we are also able to identify the 2 degrees of freedom as the two helicities, and associate a physical trajectory to each point in phase space, thanks to the fact that we are able to solve the dyna- mics. Therefore, if we want to know what the two physical degrees of freedom of general relativity are, we need to control the general solution of the theory. This is a formidable task due to the high non-linearity of the equations, and in spite of the effort in this direction, still little is known. See [45] for a review of some attempts.

Remark on the ADM-formalism

We want to close this section with a remark. After lifting the theory onto the quantum level, trouble appears when we want the wave functions to be annihilated by the Hamiltonian constraint. We have to promote the constraint to a wave equation, use some factor ordering, pick some regularization and try to solve the resulting equation, the so-called Wheeler-DeWitt equation

. But unfortunately it turns out, that this aim has never accomplished in general. One of the main difficulties encountered in that formulation is the fact that the Hamiltonian constraint (3.14) is a non-polynomial function of the basic variables.

3.1.2. Ashtekar formalism

As seen the traditional canonical approach to quantum general relativity faces serious obstructions at a very early stage. Thus in this section we will introduce the shift from the ADM variables q_{ab}, P^{ab} to the connection variables also called *Ashtekar variables* introduced first by Ashtekar [5, 6] and later somewhat generalized by Immirzi [43] and Barbero [16]. The construction actually consists of two steps: first an extension of the ADM phase space and second a canonical transformation on the extended phase space. In a third step we will rewrite the constraints in terms of the new variables.

ADM phase space extension

We will extend the phase space described in Section 3.1.1 to a larger symplectic co-isotropic constraint surface. We define a so-called co-3-Bein field e_a^i on Σ. Here the indices $i, j, k...$ take values $1, 2, 3$. The 3-Bein is defined by the relations

$$e_j^a e_a^k := \delta_j^k, \quad e_j^a e_b^j := \delta_b^a, \quad \{e_i^a\} \in \mathrm{GL}\,(3, \mathbb{R}) \quad \text{and} \quad \det e_i^a > 0 \ .$$

These triads contain all the spatial information and thus the 3-metric q_{ab} is defined in terms of e_a^i as

$$q_{ab} := \delta_{jk} e_a^j e_b^k. \tag{3.23}$$

Thus the first part of the variables used in the framework of LQG, called Ashtekar variables, is formed by a densitized dreibein , cf. Eq. (3.25). If the manifold Σ is three dimensional, an orthonormal frame on Σ is called triad or dreibein, depending on whether one prefers Greek or German.

At this point we want to give a mathematical description of the triads and we want to emphasize that the definition of frames works in any dimension n in contrast to the Ashtekar connection to be defined in Chapter 4.

DEFINITION 3.1.5. *A* frame *at $m \in \Sigma$ is a vector space isomorphism*

$$e : \mathbb{R}^n \longrightarrow T_m\Sigma. \tag{3.24}$$

Another possibility to specify a frame is to select a basis of $T_m\Sigma$, see [47]. Recall that the frame bundle on Σ is given by the disjoint union, indexed by $m \in \Sigma$, of all frames at m. The differentiable structure on it is naturally induced from that on Σ by decomposing each frame with respect to some appropriate local coordinate system on $T\Sigma$. In the case of choosing some local basis for $T_m\Sigma$ and the canonical basis of $\mathbb{R}^n$, then – as being a vector space isomorphism – any frame e at m is characterized by some matrix. Its determinant is called $(\det e)$ of e. We might get an additional factor, if we choose a different basis on $T_m\Sigma$. In fact, the transformation matrix intertwining two bases is some $\mathrm{Gl}(n)$ element, whose determinant is precisely that factor. But note if we consider general local frames that this prefactor may change from point to point. However, in the case at hand the tangent bundle will be globally trivial, such that we may assume that each frame is globally defined and such a change of bases corresponds to multiplication by some function on full Σ.

DEFINITION 3.1.6. *A frame is called*

i.) orthonormal *w.r.t the metic q on Σ iff it is an isometry, where we have the standard Euclidean metric on $\mathbb{R}^n$;*

ii.) oriented *if and only if it preserve the orientation.*

Again, one may specify an orthonormal frame by an orthonormal basis of $T_m\Sigma$. On the one hand any frame defines a metric such that the frame is orthonormal with repsect to that metric. In particular,

$$q(X,Y) := \langle e^{-1}(X), e^{-1}(Y)\rangle_{\mathrm{Eucl}} \quad \text{for} \quad X, Y \in T_m\Sigma$$

defines a metric q on Σ, such that e is an isometry, if $\langle\cdot,\cdot\rangle_{\mathrm{Eucl}}$ denotes the Euclidean scalar product. While frames determine a metric uniquely, on the other hand a metric does not state the orthonormal frame. Particularly e and e' are isometries for q if and only if $e' = e \circ L_g$ for some $g \in \mathrm{O}(n)$. Here L_g denotes the left translation by g.
By replacing $\mathrm{Gl}(n)$ by $\mathrm{O}(n)$, the bundle $\mathrm{O}_q(\Sigma)$ of orthonormal frames can be defined completely analogously to that of general frames. Thereby, $\mathrm{O}_q(\Sigma)$ is the reduction of the structure group $\mathrm{Gl}(n)$ of the frame bundle to the structure group $\mathrm{O}(n)$. Analogous arguments apply to the bundle $\mathrm{O}_q^+(n)$ of oriented orthonormal frames with structure group $\mathrm{SO}(n)$.

Now we want to continue with the physical introduction of the Ashtekar formalism. Remark that Eq. (3.23) is invariant under local SO(3) rotations i.e.

$$e_a^i \mapsto e_a^{\prime i} = O_i^a e_a^j, \quad (O)_{ij} \in \mathrm{SO}(3)$$

leaves the metric invariant. Therefore we can view e_a^i as an $\mathfrak{su}(2)$-valued one form. Our conventions are such that the generators of Lie algebra $\mathfrak{su}(2)$ in the adjoint - or equivalently, of $\mathfrak{so}(3)$ in the defining representation are given by $[\tau_i, \tau_j] = \epsilon_{ijk}\tau^k$, i.e., $2i\tau_i = \sigma_i$, where σ_i are the Pauli matrices. Recall that the adjoint representation of SU(2) on its Lie algebra is isomorphic with the defining representation of SO(3) on $\mathbb{R}^3$ under the isomorphism $\mathbb{R}^3 \to \mathfrak{su}(2); v^i \to v^i\tau_i$, where τ_i is a basis of $\mathfrak{su}(2)$. This observation makes it obvious that we have to get rid of the $3(3-1)/2 = 3$ rotational degrees of freedom sitting in e_a^i but not in q_{ab}. This extra degrees of freedom justifies the naming *phase space extension*.

The consequence of these novelties is a much more complicated structure than in the metric case. In particular, the constraint algebra is second class. However, there is a particular choice of variables which simplifies the analysis, making it possible to implement a part of the constraint and reducing the remaining ones to first class again. These are the famous Ashtekar variables, which we now introduce.

The Ashtekar representation is a triad formulation, but uses the triad in a denstized form. The denstized triad E_j^a , with density weight $+1$, is then related to the triad by

$$E_j^a := \frac{1}{2}\epsilon_{jj_1j_2}\epsilon^{aa_1a_2}e_{a_1}^{j_1}e_{a_2}^{j_2} = \sqrt{\det(q)}e_j^a, \tag{3.25}$$

where $\sqrt{\det(q)} := |\det(E_j^a)|^{1/(\dim(\Sigma)-1)} = |\det(E_j^a)|^{1/2}$ and has the same properties concerning gauge rotations and its orientation as the triad e_i^a. Mathematically spoken the multiplication of any tensorial object with $(\det e)^{-k}$ gives the coresponding tensor density of weight k. Hence we arrive at the mathematical definition of the Ashtekar fields, i.e. of Eq. (3.25).

Definition 3.1.7. *(See [33]) The* Ashtekar field *E to a frame e is the densitized frame field*

$$E := (\det e)^{-1}e \tag{3.26}$$

of weight 1.

The latter definition depends on the choice of the basis on each $T_m\Sigma$. If that basis is given by the imagine of the canonical basis on $\mathbb{R}^n$, then $(\det e)$ is 1. In the case $n \neq 1$, then the frame can reconstructed from the Ashtekar field by using

$$(\det E) = (\det((\det e)^{-1}e)) = (\det e)^{-n}(\det e) = (\det e)^{1-n}$$

and we obtain

$$e = (\det e)E = ((\det E)^{-1+n})E.$$

Spatial geometry is obtained directly from the densitized triad, which is related to the spatial metric by

$$q^{ab} = \frac{E_j^a E_j^b}{\det(q)} \tag{3.27}$$

by which $R = R(q)$ is considered as a function of E_j^a.

Next we introduce another independent one-form K_a^i on Σ, which we also consider as $\mathfrak{su}(2)$-valued, from which the extrinsic curvature – for a mathematical treatment see Section 2.2.3, Remark 2.2.69 – is derived by

$$K_{ab} := K_{(a}^i e_{b)}^i. \tag{3.28}$$

Since K_{ab} was a symmetric tensor field, we recognize with equation (3.25) that K_a^i must satisfy the constraint

$$G_{jk} := K_{a[j}E^a_{k]} = 0. \tag{3.29}$$

The square brackets denote anti-symmetrization defined as an idempotent operation $x^{[a}x^{b]} := 1/2\left(x^a x^b - x^b x^a\right)$. Now we consider the following functions on the extended phase space

$$q_{ab} := E_a^j E_b^j |\det(E_l^c)|, \qquad P^{ab} := 2|\det(E_l^c)|^{-1} E_k^a E_k^d K^j_{[d}\delta^b_{c]} E_j^c, \tag{3.30}$$

where E_a^j is the inverse of E_j^a. It is easy to see that when $G_{jk} = 0$, the functions (3.30) precisely reduce to the ADM coordinates. Inserting (3.30) in (3.14) we obtain the diffoemorphism and Hamiltonian constraint as functions on the extended phase space, which one can check to be explicitly given by

$$H_a := -2\mathcal{D}_b\left[K_a^j E_j^b - \delta_a^b K_c^j E_j^c\right] \tag{3.31}$$

$$H := \frac{1}{\det(q)}\left(K_a^l K_b^j - K_a^j K_b^l\right) E_j^a E_l^b - \det(q)\mathcal{R}^\Sigma. \tag{3.32}$$

Here $\mathcal{R}^\Sigma = \mathcal{R}^\Sigma(q)$ is considered as a function of E_j^a by $\sqrt{\det(q)} := |\det(E_j^a)|^{1/2}$ and Eq. (3.27). Notice that, using Eq. (3.28), Eq. (3.29), expressions (3.31) indeed reduce to Eq. (3.14) up to terms proportional to G_{jk}.

In the next step, we equip the extended phase space coordinatized by the pair (K_a^i, E_i^a) with the symplectic structure defined by

$$\{E_j^a(x), E_k^b(y)\} = \{K_a^j(x), E_b^k(y)\} = 0, \tag{3.33}$$

$$\{E_i^a(x), K_b^j(y)\} = \frac{\kappa}{2}\delta_b^a\delta_i^j\delta(x,y). \tag{3.34}$$

In order to prove that the symplectic reduction with respect to the constraint G_{ij} of the constrained Hamiltonian system subject to the constraints Eq. (3.31) results the ADM phase space of Section 3.1.1 with the original diffeomeorphism and Hamilton constraint.

First by using Eq. (3.33) we compute the Poisson algebra of the smeared rotation constraints $G(\Lambda) := \int_\Sigma \mathrm{d}^3x \Lambda^{ik} K_{aj} E^a_k$, where $\Lambda^T = -\Lambda$ is an arbitrary antisymmetric matrix, that is, an $\mathfrak{so}(3)$-valued scalar on Σ. We get

$$\{G(\Lambda), G(\Lambda')\} = \frac{\kappa}{2} G([\Lambda, \Lambda']),$$

i.e. $G(\Lambda)$ generates infinitesimal SO(3) rotations as expected. Since the functions Eq. (3.30) are SO(3)-invariant by construction, the Poisson commute with $G(\Lambda)$ and as the constraints Eq. (3.31) are functions of these, $G(\Lambda)$ also Poisson commutes with Eq. (3.31).
Second we compute the Poisson brackets among $q_a b, P^{cd}$, given by Eq. (3.30) on the extended phase space with symplectic structure Eq. (3.33). We obtain

$$\{q_{ab}(x), q_{cd}(y)\} = 0, \tag{3.35}$$

$$\left\{P^{ab}(x), q_{cd}(y)\right\} = \kappa \delta^a_{(c} \delta^b_{d)} \delta(x, y), \tag{3.36}$$

$$\begin{aligned}\left\{P^{ab}(x), q_{cd}(y)\right\} = & -\kappa[\frac{\det(e)}{4}(q^{bc}G^{ad} + q^{bd}G^{ac} \\ & + q^{ac}G^{bd} + q^{ad}G^{bc})](x)\delta(x, y),\end{aligned} \tag{3.37}$$

where (3.37) only vanishes at $G_{ab} := G_{jk} e^j_a e^k_b = 0$, the so called *rotation constraints*. Thus the functions (3.30), their Poisson brackets among each other and the diffeomorphism respectively Hamiltonian constraint reduce at $G_{jk} = 0$ to those of the ADM phase space. Therefore the ADM system and the extended one are completely equivalent and we are able to work with the latter. Thus we can summarize: the symplectic reduction with respect to G_{jk} of the constrained Hamiltonian system described by the action

$$S := \frac{1}{\kappa} \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}x^3 \left[2\dot{K}^j_a E^a_j - (-\Lambda^{jk} G_{jk} + N^a H_a + NH)\right] \tag{3.38}$$

is given by the system described by the ADM action in Section 3.1.1. In equation (3.38) Λ acts as Lagrange multiplier.

Canonical transformation on the extended phase space

At the beginning of this section we want to give the following definition:

Definition 3.1.8. *The spin connection is defined as an extension of the spatial covariant derivative $\mathcal{D}_a$ from tensors to generalized tensors with $\mathfrak{so}(D)$ indices. One defines*

$$\mathcal{D}_a u_{b\ldots} v_j := (\mathcal{D}_a u_b)_{\ldots} v_j + \ldots + u_{b\ldots}(\mathcal{D}_a v_j), \text{ where } \quad \mathcal{D}_a v_j := \partial_a v_j + \Gamma_{ajk} v^k$$

extends by linearity, Leibniz rule and imposes that $\mathcal{D}_a$ commutes with contractions, see [69].

As explicitly shown in e.g. [69], the motivation of introducing the above expression and its derivation starts with the extension of the metric compatibility condition $\mathcal{D}_a q_{bc} = 0$ to e_a^j that is

$$\mathcal{D}_a e_b^j = 0 \Rightarrow \Gamma_{ajk} = -e_k^b \left[\partial_a e_b^j - \Gamma_{ab}^c e_c^j\right]. \tag{3.39}$$

Now our goal is to rewrite the constraint G_{jk} in such a form equal to the Gauss constraint of an SO(3) gauge theory, i.e. in the form $G_{jk} = (\partial_a E^a + [A_a, E^a])_{jk}$ for some $\mathfrak{so}(3)$ connection. In order to achieve this goal, we have to make a canonical transformation, which consists of a constant Weyl (rescaling) transformation and an affine transformation.

We start with the constant Weyl transformation. The rescaling $(K_a^j, E_j^a) \mapsto ({}^{(\beta)}K_a^j := \beta K_a^j, {}^{(\beta)}E_j^a := K_j^a/\beta)$ is a canonical transformation, since the Poisson brackets (3.33) are obviously invariant under this map. Now we can rewrite the rotational constraint as follows

$$G_j = \epsilon_{jkl} K_a^k E_l^a = \epsilon_{jkl}({}^{(\beta)}K_a^k)({}^{(\beta)}E_l^a) \tag{3.40}$$

which is invariant under this rescaling.

Now we will continue with the affine transformation. Using equation (3.39) we get $\mathcal{D}_a E_j^a = 0$ and particularly we have

$$\mathcal{D}_a E_j^a = [\mathcal{D}_a E^a]_j + \Gamma_{aj}^k E_k^a = \partial E_j^a + \epsilon_{jkl} \Gamma_a^k E_l^a = 0.$$

The square bracket means that D acts only on tensorial indices. Thus we are able to make an affine transformation by replacing D by ∂ as E_j^a

is an SU(2)-valued vector with density of weight one. In order to define $\Gamma_a := \Gamma^l_a \tau_l, (\tau_l)_{jk} = \epsilon_{jkl}$, we used the isomorphism between antisymmetric tensors of second rank and vectors in Euclidean space. Another important tool is the notion of the *spin connection.*

Solving the spin connection in terms of E^a_j from equation (3.40) we find

$$\begin{aligned}\Gamma^i_a =& \frac{1}{2}\epsilon^{ijk} e^b_k (e^j_{a,b} - e^j_{b,a} + e^c_j e^l_a e^l_{c,b}) \\ =& -\frac{1}{2}\epsilon^{ijk} E^b_k (E^j_{a,b} - E^j_{b,a} + E^c_j E^l_a E^l_{c,b}) \\ &+ \frac{1}{4}\epsilon^{ijk} E^b_k \left(2E^j_a \frac{(\det(E))_{,b}}{\det(E)} - E^j_b \frac{(\det(E))_{,a}}{\det(E)} \right).\end{aligned}$$

From the last equation we get the important conclusion, that

$$({}^{(\beta)}\Gamma^j_a) := \Gamma^j_a({}^{(\beta)}E) = \Gamma^j_a = \Gamma^j_a(E).$$

is itself invariant under the rescaling transformation. Therefore the we obtain $\mathcal{D}_a({}^{(\beta)}E^a_j) = 0$, since the derivative $\mathcal{D}_a$ is independent of the *Immirzi parameter* β. Finally we can rewrite the rotational constraint as

$$\begin{aligned}G_j &= 0 + \epsilon_{jkl}({}^{(\beta)}K^k_a)({}^{(\beta)}E^a_l) = \partial_a({}^{(\beta)}E^a_j) + \epsilon_{jkl}[\Gamma^k_a + ({}^{(\beta)}K^k_a)]({}^{(\beta)}E^a_l) \\ &:=^{(\beta)} \mathcal{D}_a \; {}^{(\beta)}E^a_j\end{aligned} \tag{3.41}$$

Notice that this equation has *exactly* the structure of a Gauss law constraint for an SU(2) gauge theory. Hence we will call G_j the *Gauss constraint.* Eq. (3.41) suggests introducing the new connection, the so called *Ashtekar-Immirzi-Barbero* connection

$${}^{(\beta)}A^j_a := \Gamma^j_a + \beta K^j_a, \tag{3.42}$$

where for $\beta \in \mathbb{R}^*$ the Barbero connection [16], for complex β the Immirzi connection [44] and for $\beta = \pm \mathrm{i}$ the original Ashtekar connection [5, 6] arises. For short, we will refer to it as the Ashtekar connection, since we will make the choice $\beta = \mathrm{i}$ in Chapter 5. The exact mathematical structure of Eq. (3.42) and their geometric origin will be discussed in detail in Chapter 4.

Loosely speaking the Ashtekar connection can be seen as the pull-back to Σ by local sections of a connection on an SU(2) fiber bundle. As such it transforms under a local gauge transformation $g : \Sigma \to \mathrm{SU}(2)$ (i.e. transformations between two sections of the principal bundle $P(\mathcal{M}, \mathrm{SU}(2))$ in the following way:

$$A \mapsto A^g = gAg^{-1} - \mathrm{d}gg^{-1},$$

where $\mathrm{d} : \Lambda^p(\mathcal{M}) \to \Lambda^{p+1}(\mathcal{M})$ is the exterior derivative whose action on a p-form is defined by

$$\mathrm{d}\omega = \mathrm{d}\left(\frac{1}{p!}\omega_{\mu_1\ldots\mu_p}\mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}\right) = \left(\partial_\nu \omega_{\mu_1\ldots\mu_p}\right)\mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}$$

with the property $\mathrm{d}^2 = 0$. On the other hand the densitized triad transforms according to

$$E \mapsto E^g = g^{-1}Eg.$$

The new phase space, which is similar to that of a Yang-Mills theory with SU(2) as structure group, is spanned by the variables (A^j_a, E^a_j) and its symplectic structure is given by

$$\begin{aligned} \{{}^{(\beta)}A^j_a(x), {}^{(\beta)}A^k_b(y)\} &= \{{}^{(\beta)}E^a_j(x), {}^{(\beta)}E^b_k(y)\} = 0, \\ \{{}^{(\beta)}E^a_j(x), {}^{(\beta)}A^k_b(y)\} &= \frac{\kappa}{2}\delta^a_b\delta^k_j\delta(x,y). \end{aligned} \tag{3.43}$$

Yang-Mills theory is a theory defined on a background space time geometry. Dynamics in such a theory is described by a non vanishing Hamiltonian. We can regard general relativity in the new Ashtekar variables as a background independent relative of SU(2) Yang-Mills theory. Without these simple bracket structure classically it would be very hard to find Hilbert space representations that turn these Poisson bracket relations into canonical commutation relations.

To accomplish the Legendre transformation of the Einstein-Hilbert action, the Ashtekar representation can be used. This lengthly calculation can be found in [69] and results finally in a fully constrained system, which is given by

$$S = \frac{1}{2\kappa\beta}\int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left(2\,{}^{(\beta)}\dot{A}^i_a\,{}^{(\beta)}E^a_i - [\Lambda^j G_j + N^a H_a + NH]\right),$$

where G_j is the Gauss constraint, H_a the (spatial) diffeomorphism (or vector) constraint, $\mathcal{H}$ the Hamiltonian and Λ^j, N^a, N are Lagrange multipliers. The geometrical meaning of these quantities is as follows: At fixed t the fields $N^a(t,x)$, $N(t,x)$ label points in an phase space $\mathcal{M}$.

Summarizing, in the Ashtekar formulation of General Relativity the theory is described by an extended phase space of dimension $18 \cdot \infty^3$ with the fundamental Poisson bracket given in Eq. (3.43). In order to recover the $12 \cdot \infty^3$ dimensional phase space of the ADM formulation, we have to regard the hypersurface where the Gauss constraint is satisfied and have to divide by the gauge orbits generated by G_j.

Constraints in terms of the new variables

It remains to write the constraints in terms of the variables ${}^{(\beta)}A_a^j, K_a^j, E_j^a$, for a detailed calculation see [69]. For this purpose we introduce the curvature of the connection A on Σ

$$F_{ab}^j = 2\partial_{[a}\,{}^{(\beta)}A_{b]}^j + \epsilon_{jkl}\;{}^{(\beta)}A_a^k\,{}^{(\beta)}A_b^l.$$

The *Gauss constraint* given by the covariant derivative of E_j^a w.r.t. the connection A_a^j, i.e.

$$G_j = \frac{1}{\beta}\,{}^{(\beta)}\mathcal{D}_a E_j^a = \frac{1}{\beta}\left[\partial_a E_j^a + \epsilon_{ijk}\,{}^{(\beta)}A_a^j E_k^a\right] \tag{3.44}$$

stems from the fact that gravity has to be invariant under SO(3)-rotations of the triad $E_j^a \to O_i^j E_j^a$, where $O_i^j \in \mathrm{SO}(3)$. The *diffeomorphism constraint* originates from the requirement of independence from any spatial coordinate system or background and is given by

$$H_a = \frac{1}{\beta}\,{}^{(\beta)}F_{ab}^j E_j^b. \tag{3.45}$$

Finally the *Hamiltonian constraint* tells us that gravity must be invariant under a reparametrization of the coordinate time and is given by

$$H = \left[{}^{(\beta)}F_{ab}^j - (1+\beta^2)\epsilon_{jmn}K_a^m K_b^n\right]\frac{\epsilon_{jkl}E_k^a E_l^b}{\sqrt{\det(q)}}, \tag{3.46}$$

The components of the extrinsic curvature in Eq. (3.46) are functions of the Ashtekar connection A_a^j and the densitized triad E_j^a because of the dependence of the spin connection Γ_a^j on the triad e_j^a, see Eq. (3.39). In Section 5.2 we translate the constraints as given in Eq. (3.44), Eq. (3.45) and Eq. (3.46) into our preceding differential geometrical framework of Chapter 4.
From now on we will only consider in view of several considerations, given in [69], positive β. In order to simplify our notation we will drop the label β in what comes, but mean by the fields E, A the fields ${}^{(\beta)}E$, ${}^{(\beta)}A$ for $\beta = 1$ respectively for arbitrary β. In summary general relativity can be written in terms of connections with a compact structure group resembling a Yang-Mills theory, where E_j^a respectively F_{ab}^j plays the role of the electric respectively magnetic field and the Gauss law $G_j = 0$ for gravity in the new variables format is identical of that for Yang-Mills equations. But we want to point out the appearance of the Hamiltonian- and diffeomorphism constraint, which generates time evolution.

Holonomy-flux Poisson algebra $\mathfrak{A}$

With the intension to quantize gravity according to the algorithm for the quantization of constrained systems devised by Dirac (for the original account, see his Lectures on Quantum Mechanics, for a modern treatment, see [69]), we have to proceed in two steps.

i.) Quantization of the canonical variables (the so-called *kinematic quantization*);

ii.) Impose the constraints as operator equations on states, and solve these equations to obtain physical states.

The first step is what we will discuss in the present section. What we want is a representation of the canonical commutation relations, see Eq. (3.43),

$$\{{}^{(\beta)}E_j^a(x), {}^{(\beta)}A_b^k(y)\} = \frac{\kappa}{2}\delta_b^a\delta_j^k\delta(x,y)$$

on a Hilbert space. Fields evaluated at points are usually too singular to give good operators in the quantum theory. Thus one has to form suitably integrated *smeared* quantities, similar we did for the fields q_{ab}, P^{ab} in Section 3.1.1, that correspond to well defined operators in the quantum theory. Poisson brackets then suggest commutation relations for these quantities, and one obtains an abstract algebra of operators in order to proceed with the quantization.

But before introducing the holonomy-flux Poisson algebra $\mathfrak{A}$ we will give the definitions of *curves, edges* and *graphs*:

DEFINITION 3.1.9. *(See [69])*

i.) *By a curve c we mean a map $c: [0,1] \to \Sigma; t \mapsto c(t)$ which is continuous, oriented, piecewise semianalytic, parametrized, compactly supported and embedded in Σ. The set of curves is denoted $\mathcal{C}$ in what follows.*

ii.) *The beginning point, final point and the range of a curve is defined, respectively, by*

$$b(c) \coloneqq c(0), \quad f(c) \coloneqq c(1), \quad r(c) \coloneqq c([0,1]).$$

iii.) *On $\mathcal{C}$ we define maps $\circ$, $(.)^{-1}$ called composition and inversion respectively by*

$$[c_1 \circ c_2](t) \coloneqq \begin{cases} c_1(2t) & t \in [0, \frac{1}{2}] \\ c_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

if $f(c_1) = b(c_2)$ and

$$c^{-1}(t) \coloneqq c(1-t).$$

iv.) *An edge e is an equivalence class of a curve $c_e \in \mathcal{C}$ which is semianalytic in all of $[0,1]$. In this case $r(e) \coloneqq r(c_e)$.*

v.) *An independent set of edges $\{e_1, \ldots, e_N\}$ defines an oriented graph γ by $\gamma \coloneqq \bigcup_{k=1}^N r(e_k)$, where $r(e_k) \subset \gamma$ carries the arrow induced by e_k. We denote by $E(\gamma)$ the edge set of γ. From γ we can recover its set of edges $E(\gamma) = \{e_1, \ldots, e_N\}$ as the maximal semianalytic segments of γ together with their orientations as well as the set of vertices of γ as $V(\gamma) = \{b(e), f(e); e \in E(\gamma)\}$. The set of graphs is denoted by Γ.*

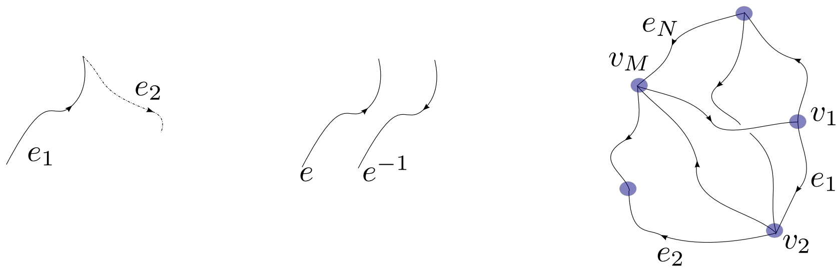


Figure 3.2.: Edges and its collection $\gamma = \{e_1, \ldots, e_N\}$.

These objects are depicted in figure 3.2.

Now we can go over to the implementation of the *Holonomy-flux Poisson algebra* $\mathfrak{A}$. Here the different tensorial nature of A^j_a and E^a_j plays a central role. The connection A^j_a is a 1-form, so it is natural to smear it along a 1-dimensional graph. The topic we just described is not unique to gravity but appears in a non-Abelian Yang-Mills theory. The only known solution is to work with so-called *Wilson loops.* Before introducing *Wilson loops* we shall first give a geometrical definition of a holonomy along the lines of [15, 56].

Definition 3.1.10. *Let $A = A^j_a \tau_i \mathrm{d}x^a \in \mathfrak{su}(2)$ be a connection. Given a curve $\gamma : [0,1] \to \Sigma$ in Σ we define by the holonomy $h_\gamma(A) \in SU(2)$ of the connection A along γ the unique solution to the following ordinary differential equation*

$$\frac{d}{ds} h_{\gamma_s}(A) = h_{\gamma_s}(A) A(\gamma(s)), \qquad h_{\gamma_0} = 1_2, \quad h_\gamma(A) := h_{\gamma_1}(A),$$

where $\gamma_s(t) := \gamma(st), s \in [0,1]$ and $A(\gamma(s)) := A^j_a(\gamma(s))\tau_j/2\dot{\gamma}^a(s)$. The solution to this equation is explicitly given by the holonomy

$$h_\gamma(A) = \mathcal{P} \exp\left[\int_\gamma \mathrm{d}s\, A\left(\dot{\gamma}(s)\right)\right], \tag{3.47}$$

where $\mathcal{P}$ denotes the path ordering symbol which orders the curve parameters from left to right according to their value beginning with the smallest one.

Now we want to list some basic properties of the holonomy which are explicitly given by:

i.) The definition of $h_\gamma[A]$ is independent of the parametrization of the path γ.

ii.) The holonomy of a path given by a single point is the identity, given two oriented paths γ_1 and γ_2 such that the end point of γ_1 coincides with the starting point of γ_2 so that we can define $\gamma = \gamma_1 c_2$ in the standard fashion, then we have

$$h_\gamma[A] = h_{\gamma_1}[A] h_{\gamma_2}[A],$$

where the multiplication on the right is the SU(2) multiplication. We also have that

$$h_{\gamma^{-1}}[A] = h_\gamma^{-1}[A].$$

iii.) The holonomy transforms in a very simple way under the action of diffeomorphisms (transformations generated by the vector constraint). Given $\varphi \in \mathrm{Diff}(\Sigma)$ we have

$$h_\gamma[\varphi^* A] = h_{\varphi^{-1}(\gamma)}[A],$$

where $\varphi^* A$ denotes the action of φ on the connection. In other words, transforming the connection with a diffeomorphism is equivalent to simply *moving* the path with φ^{-1}.

iv.) Under a local gauge transformation $g(x) \in \mathrm{SU}(2)$ the holonomy transforms according to

$$h_\gamma[A^g] = g(\gamma(0)) h_\gamma[A] g(\gamma(1))^{-1},$$

where $\gamma(0)$ and $\gamma(1)$ are respectively the source and target points of the line γ.

v.) Suppose $\gamma : [0,1] \to \Sigma$ is a loop, i.e. $\gamma(0) = \gamma(1)$. A consequence of the above transformation rule and the invariance of the trace is that the so-called *Wilson loop*

$$W_\gamma[A] = \mathrm{tr}(h_\gamma[A]) = \mathrm{tr}\left(\mathcal{P} \exp \oint_\gamma A\right)$$

is gauge invariant.

vi.) The functional derivative with respect to the connection gives

$$\frac{\delta h_\gamma[A]}{\delta A_a^j(x)} = \begin{cases} \frac{1}{2}\dot{x}^a \delta^{(3)}(\gamma(s), x)\tau_i h_\gamma & \text{if } x \text{ is the source of } \gamma, \\ \frac{1}{2}\dot{x}^a \delta^{(3)}(\gamma(s), x) h_\gamma \tau_i & \text{if } x \text{ is the target of } \gamma, \\ \dot{x}^a \delta^{(3)}(\gamma(s), x)[h_\gamma(0, s)\tau_i h_\gamma(s, 1)] & \text{if } x \text{ is inside } \gamma. \end{cases} \tag{3.48}$$

Next we turn to the conjugate electric field E. The vector density E_j^a, defined in (3.25), is dual to the two-form $(*E)^j_{a1,a2}$ defined by

$$(*E)^j_{a1,a2} := \epsilon_{aa1a2} E_j^a, \tag{3.49}$$

where ϵ_{aa1a2} is the tensor density with weight -1 that is equal to the totally anti-symmetric symbol in any coordinate system. We note that E^a has density weight $+1$ whereas ϵ_{aa1a2} has weight -1, so the quantity $(*E)^j_{a1,a2}$ is in fact a two-form.

Since a two-form is naturally integrated in two dimensions we are led to the following quantity. We integrate Eq. (3.49) on a two - dimensional oriented surface S, and therefore we obtain the so-called (electric) *fluxes*

$$E_n(S) := \int_S n^j (*E)_j^\sigma \, \mathrm{d}\sigma_1 \wedge \mathrm{d}\sigma_2,$$

where $n = n^j$ is a Lie algebra valued smearing field on S and σ_1 and σ_2 are local coordinates on S. The quantity $E_n(S)$ is the *flux* of E across S. Since the integrand is a two-form the integral, using the orientation of S, is hence coordinate independent. The densitized triad or (electric) *flux* $E_n(S)$ has simple geometrical interpretation. E_j^a encodes the full background independent Riemannian geometry of Σ. Therefore any geometrical quantity in space can be written as a functional of E_j^a. One of the simplest is the area-functional $\mathrm{Ar}_S[E_j^a]$ for a parameterized surface $X_S : S \to \Sigma, S \subset \mathbb{R}^2$, which is in terms of E_j^a explicitly given by

$$\mathrm{Ar}_S[E_j^a] = \int_S \mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \sqrt{E_j^a E_k^b \delta^{jk} n_a n_b},$$

where $n_a = \epsilon_{abc}\frac{\partial x^b}{\partial \sigma_1}\frac{\partial x^c}{\sigma_2}$ is the normal to the surface S and σ_1 and σ_2 are local coordinates on S. This expression for the area of a surface is a well-defined gauge-invariant quantity and will be a cornerstone in the quantum theory.

Now we have regularized the resulting Poisson algebra using paths and surfaces, instead of the all of space in traditional smearings such as done in the ADM formulation. The resulting smeared algebra of $h_\gamma[A]$ and $E_n(S)$ is called *holonomy-flux Poisson algebra* $\mathfrak{A}$. All requirements of the program of canonical quantization with constraints for it to be a classical starting point of quantization are satisfied by the algebra $\mathfrak{A}$ generated from fluxes and holonmies.

Cylindrical functions and the kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$

Shifting the focus from connections to holonomies results in the idea of *generalized connections.*

DEFINITION 3.1.11. *A generalized connection is an assignment of* $h_\gamma \in SU(2)$ *to any graph* $\gamma \subset \Sigma$. *The space of generalized connections is denoted by* $\overline{\mathcal{A}}$.

In short the fundamental observable in LQG is taken to be the holonomy itself and not its relationship (see Eq. (3.47)) to a smooth connection. The algebra of kinematical observables is defined to be the algebra of the so - called *cylindrical functions* of generalized connections denoted by Cyl. Thus in order to define the integration measure on the space of connections without relying on a fixed background metric we use Cyl, which we introduce next.

The algebra of the cylindrical functions

More or less, a *cylindrical function* is a functional of a field that depends only on some subset of components of the field itself. In the present case, the field is the connection, and the cylindrical functions are functionals that depend on the connection only through the holonomies $h_e[A] = \mathcal{P}\exp\left(\int_e A\right)$ along some finite set of edges e. Accordingly a couple (γ, f) of a graph and a smooth function $f : \mathrm{SU}(2)^{\mathrm{N_e}} \longrightarrow \mathbb{C}$, defines a functional of the connection A, called *cylindrical function* which is explicitly given by a functional of the connection defined as

$$\langle A \mid \gamma, f\rangle = \psi(\gamma, f)[A] := f(h_{e_1}[A], \ldots, h_{e_{N_e}}[A]) \in \mathrm{Cyl}_\gamma \tag{3.50}$$

where e_i with $i = 1, \ldots, N_e$ are the edges of the corresponding graph γ. The algebra of kinematical observables is defined to be the algebra of *cylindrical functions* denoted by Cyl. We can depict the latter algebra as the union of the set of *cylindrical functions* defined on graphs $\gamma \subset \Sigma$, namely

$$\mathrm{Cyl} = \bigcup_\gamma \mathrm{Cyl}_\gamma,$$

where $\bigcup_\gamma$ denotes the union of Cyl_γ for all graphs in Σ. On this algebra we will base the definition of the *kinematical Hilbert space* $\mathcal{H}_{\mathrm{kin}}$.

But before turning towards the construction of the representation of Cyl that defines $\mathcal{H}_{\mathrm{kin}}$ we will give the definintion of a *spin network* and *spin-network function*:

Definition 3.1.12. *i.) Given a graph γ, label each edge $e \in E(\gamma)$ with a triple of numbers (j_e, m_e, n_e) where $j_e \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, ..\}$ is a half-integral spin quantum number and $m_e, n_e \in \{-j_e, -j_e + 1, .., j_e\}$ are magnetic quantum numbers. A quadruple*

$$s := \left(\gamma, \vec{j} := \{j_e\}_{e\in E(\gamma)}, \vec{m} := \{m_e\}_{e\in E(\gamma)}, \vec{n} := \{n_e\}_{e\in E(\gamma)}\right)$$

is called a spin network (SNW). We also write $\gamma(s)$ etc. for the entries of a SNW.

ii.) Choose once and for all one representative ρ_j, $j > 0$ half integral, from each equivalence class of irreducible representations of SU(2)*. Then*

$$T_s : \overline{\mathcal{A}} \to \mathbb{C};\ A \mapsto \prod_{e\in E(\gamma)} \left\{\sqrt{2j_e + 1}\, [\rho_{j_e}(A(e))]_{m_e n_e}\right\}$$

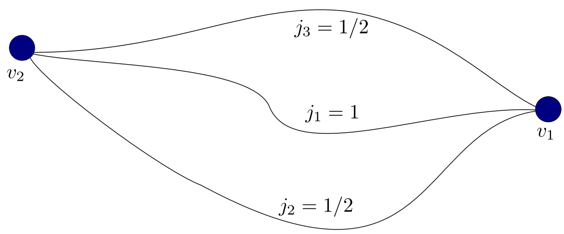


Figure 3.3.: A generic spin-network with two trivalent nodes.

is called the spin-network function (SNWF) of s. Here $[\rho_j(.)]_{mn}$ *denotes the matrix elements of the matrix valued function* $\rho_j(.)$.

A generic spin-network is depicted in Figure 3.3.

So far we have introduced the algebra of functionals of generalized connections Cyl. SNWs are special examples of Cyl, which in addition are SU(2) gauge invariant. In [75] it is shown how SNWFs define a complete basis of $\mathcal{H}_{\text{kin}}$.

The Ashtekar-Lewandowski representation of Cyl

In this section we want to turn the space of functionals into an Hilbert space $\mathcal{H}_{\text{kin}}$, i.e. we have to equip it with a scalar product. For this purpose basically we need the notion of a measure in the space of generalized connections in order to obtain a definition of the kinematical scalar product.

In this respect, the modification from the connection to the holonomy is the crucial factor, because the holonomy is a SU(2) element, and the integration over SU(2) is well-defined. In particular, there is the so-called *Haar measure* $\mathrm{d}\mu_{\mathrm{H}}$ of SU(2), a unique gauge-invariant and normalized measure. Given a cylindrical function $\psi(\gamma, f)[A] \in \text{Cyl}$ as in Eq. (3.50), the

Ashtekar-Lewandowski measure $\mu_{\mathrm{AL}}(\psi(\Gamma, f))$ is explicitly defined as

$$\mu_{\mathrm{AL}}(\psi_{(\gamma,f)}) := \int \prod_{e\subset\gamma} \mathrm{d}\mu_{\mathrm{H},e} f(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_M}[A]),$$

where $\mathrm{d}\mu_{\mathrm{H}}$ is the Haar measure of SU(2) and $h_e \in \mathrm{SU}(2)$. Using M copies of the Haar measure, and the properties of μ_{AL} we define on Cyl_γ the following scalar product,

$$\begin{aligned}\left\langle \psi(\gamma, f) \mid \psi_{(\Gamma,g)} \right\rangle :=& \mu_{\mathrm{AL}}(\overline{\psi(\gamma, f)}, \psi(\gamma, g)) \\ =& \int \prod_{e\subset\gamma} \mathrm{d}\mu_{\mathrm{H},e} \overline{f(h_{e_1}[A], \ldots, h_{e_M}[A])} g(h_{e_1}[A], \ldots, h_{e_M}[A]).\end{aligned} \tag{3.51}$$

This shifts Cyl_γ into a *Hilbert space* $\mathcal{H}_\gamma$ associated to a given graph γ.

Definition 3.1.13. *The Hilbert space $\mathcal{H}_\gamma$ is defined as the space of square integrable functions over $\overline{\mathcal{A}}$ with respect to the Ashtekar-Lewandowski measure μ_{AL}, that is*

$$\mathcal{H}_\gamma = L_2[\overline{\mathcal{A}}, \mathrm{d}\mu_{\mathrm{AL}}]. \tag{3.52}$$

Additionally, it can be shown, that all proper subspaces $\mathcal{H}_\gamma$ are orthogonal to each other, and they span $\mathcal{H}_{\mathrm{kin}}$. This justifies the definition of the Hilbert space of all cylindrical functions for all graphs as the direct sum of Hilbert spaces on a given graph.

Definition 3.1.14. *The Hilbert space $\mathcal{H}_{\mathrm{kin}}$ of all all cylindrical functions for all graphs is defined as*

$$\mathcal{H}_{\mathrm{kin}} = \bigoplus_{\gamma\subset\Sigma} \mathcal{H}_\gamma. \tag{3.53}$$

The scalar product on $\mathcal{H}_{\mathrm{kin}}$ is easily induced from (3.51) in the following manner: if ψ and ψ' share the same graph, then (3.51) immediately applies. In the case they have different graphs, such as γ_1 and γ_2, we consider a further graph $\gamma_3 \equiv \gamma_1 \cup \gamma_2$, we extend f_1 and f_2 trivially on γ_3, and define the scalar product as (3.51) on γ_3:

$$\langle \psi(\gamma_1, f_1) \mid \psi(\gamma_2, f_2) \rangle \equiv \langle \psi(\gamma_1 \cup \Gamma_2, f_1) \mid \psi(\gamma_1 \cup \gamma_2, f_2) \rangle . \tag{3.54}$$

Ashtekar and Lewandowski showed that (3.53) defines an Hilbert space over the space of generalized connections $\overline{\mathcal{A}}$ on Σ with respect to the Ashtekar-Lewandowski measure (see [9] for details).

Eq. (3.52) implies that (3.54) can be seen as a scalar product between cylindrical functionals of the connection with respect to the Ashtekar-Lewandowski measure:

$$\int \mathrm{d}\mu_{\mathrm{AL}}\, \overline{\psi(\gamma_1, f_1)}(A)\psi(\gamma_2, f_2)(A) \equiv \langle \psi(\gamma_1, f_1) \mid \psi(\gamma_2, f_2)\rangle \,.$$

The latter equation is the rigorous definition of the kinematical scalar product claimed in step one of the quantization program of LQG. $\mathcal{H}_{\mathrm{kin}}$ is the Cauchy completion of the space of cylindrical functions Cyl in the Ashtekar-Lewandowski measure. In other words we add to $\mathcal{H}_{\mathrm{kin}}$ the limits of all Cauchy convergent sequences in the μ_{AL} norm in addition to the cylindrical functions. Now we have a candidate kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}}$ which does not require a fixed background metric.

An orthonormal basis of $\mathcal{H}_{\mathrm{kin}}$

Until now we have a definition of the kinematical Hilbert space. In a next step we want to search for a representation of the holonomy-flux algebra on it. To that end it is convenient to introtroduce an orthogonal basis in the space. The main tool for introducing an orthogonal basis is the *Peter-Weyl* theorem, The Peter-Weyl theorem states that a basis on the Hilbert space $L_2(G, \mathrm{d}\mu_{\mathrm{H}})$ of functions on a compact group G is given by the matrix elements of the unitary irreducible representation (irreps) of the group, namely

$$f(g) = \sum_j f_j^{mn} \mathcal{D}^{(j)}_{mn}(g), \qquad j = 0, \frac{1}{2}, 1, \ldots, \qquad m, n = -j, \ldots, j,$$

where $f_j^{mn} = \int_{\mathrm{SU}(2)} \mathrm{d}\mu_{\mathrm{H}} \mathcal{D}^{(j)}_{nm}(g^{-1}) f(g)$, where for the case of SU(2) irreps are labeled by half-integer spin j. The Wigner matrices $\mathcal{D}^{(j)}_{mn}(g)$ give the spin-j irreducible matrix representation of the group element g.

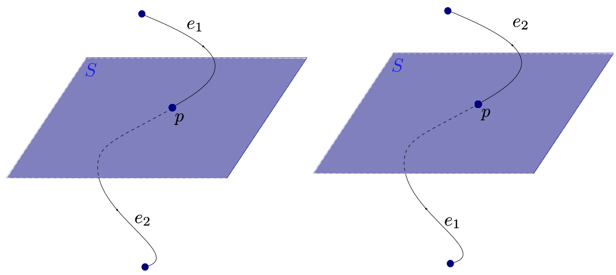


Figure 3.4.: An edge that intersects the surface at an individual point p. Two different relative orientations of e and S are depicted.

The *Peter-Weyl* theorem immediately applies to $\mathcal{H}_\gamma$, since the latter is just a tensor product of $L_2(\mathrm{SU}(2), \mathrm{d}\mu_{\mathrm{H}})$. Therefore, the basis elements are

$$\langle A \mid \gamma; j_e, m_e, n_e \rangle \equiv \mathcal{D}^{(j_1)}_{m_1 n_1}(h_{e_1}) \ldots \mathcal{D}^{(j_n)}_{m_n n_n}(h_{e_n}),$$

and an arbitrary cylindrical function $\psi(\gamma, f)[A] \in \mathcal{H}_\gamma$ can be decomposed as

$$\begin{aligned} \psi(\gamma, f)[A] &= f(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_M}[A]) \\ &= \sum_{j_e, m_e, n_e} f^{j_1, \ldots, j_n}_{m_1, \ldots, m_n, n_1, \ldots, n_n} \mathcal{D}^{(j_1)}_{m_1 n_1}(h_{e_1}[A]) \ldots \mathcal{D}^{(j_n)}_{m_n n_n}(h_{e_n}[A]). \end{aligned}$$

Hence for all values of the spin j and any graph γ the product of components of irreps $\prod_{i=1}^n \mathcal{D}^{j_i}_{m_i n_i}[h_{e_i}]$ associated with the n edges $e \subset \gamma$ is a complete orthonormal basis of $\mathcal{H}_{\mathrm{kin}}$.

Representation of the holonomy-flux algebra $\mathfrak{A}$ on $\mathcal{H}_{\mathrm{kin}}$

On the basis introduced in the previous Section, we can give a Schrödinger representation for the regularized holonomy-flux version of the algebra. Let $\vec{\sigma} = (\sigma_1, \sigma_2)$ be coordinates on the surface S, whereas the surface is defined by $S : (\sigma_1, \sigma_2) \mapsto x^a(\sigma_1, \sigma_2)$. Consider for simplicity the fundamental representation, $h_e \equiv \mathcal{D}^{(\frac{1}{2})}(h_e)$. The holonomy - operator acts by multiplication on the holonomy $h_e[A]$, explicitly given by

$$\hat{h}_\gamma[A]h_e[A] = h_\gamma[A]h_e[A]. \tag{3.55}$$

Next we want to compute the action of the flux operator $\hat{E}_i[S] = -i\hbar\beta \int_S \mathrm{d}^2\sigma n_a \frac{\delta}{\delta A_a^i(x(\sigma))}$ on the holonomy $h_e[A]$. For simplicity, let us assume that the edge e crosses the surface S at most once, and denote the intersection point (if any) by p. Here the edge e is separated into two parts $e = e_1 \cup e_2$ by the point p at which the triad acts and the sign depends on the relative orientation of e and S, see Figure 3.4. Applying Eq. (3.48) we obtain

$$\hat{E}_i[S]h_e[A] \;=\; -i\hbar\beta \int_S \mathrm{d}^2\sigma n_a \frac{\delta h_e[A]}{\delta A_a^i(x(\sigma))} = \pm i\hbar\beta h_{e_1}[A]\tau_i h_{e_2}[A] \tag{3.56}$$

The action vanishes, $\hat{E}[S]h_{e_1}[A] = 0$, when e is tangential to S or $e \cap S = 0$. Hence we obtain the simple result. The action of the operator $\hat{E}_i[S]$ on holonomies consists of just inserting the matrix $\pm i\hbar\tau_i$ at the point of intersection, we say that the operator $\hat{E}_i[S]$ *grasps* γ. Eq. (3.1.2) plays a key role in the construction of gemoetrical operators, see Section 3.1.2 and Section 6.1.2. Diagrammatically it is illustrated in the following way, see [21]:

The link represented with a dashed line denotes a link in the adjoint representation $j = 1$.

The generalization to multiple intersections is immediate. We have

$$\hat{E}_i[S]h_e[A] = \sum_{p\in(S\cap\gamma)} \pm i\hbar\beta h^p_{e_1}[A]\tau_i h^p_{e_2}[A],$$

where p labels different intersections points. Thus $\hat{E}_i[S]$ is a well defined operator in $\mathcal{H}_{\mathrm{kin}}$.

In a next step we consider the action of the scalar product of two fluxes acting inside the link,

$$\hat{E}_i[S]\hat{E}^i[S]h_e[A] = -\hbar^2\beta^2\, h_{e_1}[A]\tau^i\tau_i h_{e_2}[A]. \tag{3.57}$$

On the right hand side, we see the appearance of the scalar contraction of algebra generators, $\tau^i\tau_i \equiv C^2$. This scalar product is known as the *Casimir operator* of the algebra. In the fundamental representation considered here, $C^2 = -\frac{3}{4}\mathbb{1}_2$. The Casimir clearly commutes with all group elements, thus (3.57) can be written as

$$\hat{E}_i[S]\hat{E}^i[S]h_e[A] = -\hbar^2 C^2\beta^2\, h_{e_1}[A]h_{e_2}[A] = -\hbar^2 C^2\beta h_e[A]. \tag{3.58}$$

This expression will be useful below. On the other hand, if two consecutive fluxes act on one endpoint, say the target, we get

$$\hat{E}_i[S]\hat{E}_j[S]h_e[A] = -\hbar^2\beta^2\, h_e[A]\tau_i\tau_j.$$

From this result we immediately find that two flux operators *do not commute*,

$$\left[\hat{E}_i[S], \hat{E}_j[S]\right] h_e[A] = -\hbar^2\beta^2\, h_e[A][\tau_i,\tau_j] = -\hbar^2\beta^2\epsilon_{ij}{}^k\, h_e[A]\tau_k.$$

The actions of the holonomy-flux algebra, given by Eq. (3.55) resp. Eq. (3.1.2), trivially extends to a generic basis element $\mathcal{D}^{(j)}(h)$. The action of the holonomy - operator (3.55) is unchanged, and in the case of the flux - operator one simply has to replace in the right hand side of (3.1.2) τ_i by the generator J_i in the arbitrary irreducible j. Consequently, in (3.58) we have the Casimir $C_j^2 = -j(j+1)\mathbb{1}_{2j+1}$ on a generic irreducible representation,

$$\hat{E}_i[S]\hat{E}^i[S]\mathcal{D}^{(j)}(h_e) = \hbar^2\beta^2 j(j+1)\,\mathcal{D}^{(j)}(h_e). \tag{3.59}$$

Finally, the action is extended by linearity over the whole $\mathcal{H}_{\text{kin}}$. The remarkable fact is that this representation of the holonomy-flux algebra on $\mathcal{H}_{\text{kin}}$ is *unique*, as proved by Fleischhack [32] and Lewandowski, Okolow, Sahlmann, Thiemann [51]. This uniqueness result can be compared to the

Von Neumann theorem in quantum mechanics on the uniqueness of the Schrödinger representation. It is well-known that the uniqueness does not extend to interacting field theories on flat spacetime. Remarkably, insisting on background-independence reintroduces such uniqueness also for a field theory. But we want to remark that Eq. (3.59) is unfeasible if γ intersects S more than once, due to the fact, that in this case the τ_i matrices at different points get contracted and thus we do not obtain a gauge-invariant state.

Summarizing with this construction we have accomplished the definition of a well-behaved kinematical Hilbert space for GR. It carries a representation of the canonical Poisson algebra, and in addition, this representation is unique. Following Dirac, we now have a well-posed problem of reduction by the constraints:

$$\mathcal{H}_{\mathrm{kin}} \xrightarrow{\hat{G}_j = 0} \mathcal{H}^{G}_{\mathrm{kin}} \xrightarrow{\hat{H}^a = 0} \mathcal{H}^{\mathrm{Diff}}_{\mathrm{kin}} \xrightarrow{\hat{H} = 0} \mathcal{H}_{\mathrm{phys}}. \tag{3.60}$$

By kinematical we mean here the *Gauss* and *spatial diffeomorphism constraint* which will be the same for any background-independent gauge field theory. On the other hand, the *Hamiltonian constraint* is the the only which depends on the Lagrangian of the classical Hamiltonian. Thus the *Hamiltonian constraint* distinguish the different background-independent gauge field theories and we we denote the physical Hilbert space by $\mathcal{H}_{\mathrm{phys}}$.

Gauge-invariant Hilbert space $\mathcal{H}^{G}_{\mathrm{kin}}$

It is not really necessary to implement the *Gauss constraint* since we can work directly with gauge invariant functions, that is we solves the constraint classically and quantizes only the phase space reduced with respect to the *Gauss constraint*. Thus as a first step to obtain a gauge invariant Hilbert space we want to find the states in $\mathcal{H}_{\mathrm{kin}}$ that are SU(2) gauge invariant. Therefore these solutions define a new Hilbert space $\mathcal{H}^{G}_{\mathrm{kin}}$. The subindex indicates that there are still other constraints to be solved before arriving $\mathcal{H}_{\mathrm{phys}}$. In previous sections we already introduced SNWFs as natural SU(2) gauge invariant functionals. In this section we will show how these are effectively a complete set of orthogonal solutions of the Gauss constraint, that is a basis of $\mathcal{H}^G_{\text{kin}}$.

The action of the *Gauss constraint* is easily represented in $\mathcal{H}^G_{\text{kin}}$. In fact, recall that under gauge transformations

$$h_e \longrightarrow h'_e = \hat{U}_G[g] h_e = g_{s(e)} h_e g^{-1}_{t(e)}, \tag{3.61}$$

where $\hat{U}_G[g]$ denotes the operator generating a local $g(x) \in \mathrm{SU}(2)$ transformation and $g_{s(e)}$ is the value of $g(x)$ at the source of the edge e and $g_{t(e)}$ the value of $g(x)$ at the target. Similarly, in a generic irrep j we have

$$\begin{aligned} \mathcal{D}^{(j)}(h_e) \longrightarrow \mathcal{D}^{(j)}(h'_e) =& \hat{U}_G[g] \mathcal{D}^{(j)}(h_e) = \mathcal{D}^{(j)}(g_{s(e)} h_e g^{-1}_{t(e)}) \\ =& \mathcal{D}^{(j)}(g_{s(e)}) \mathcal{D}^{(j)}(h_e) \mathcal{D}^{(j)}(g^{-1}_{t(e)}). \end{aligned} \tag{3.62}$$

From this it follows that gauge transformations act on the source and targets of the edge, namely on the vertices of a graph. Imposing gauge-invariance then means requiring the cylindrical function to be invariant under action of the group at the vertices:

$$f_0(h_1, \ldots, h_{N_e}) \equiv f_0(g_{s_1} h_1 g_{t_1}{}^{-1}, \ldots, g_{s_{N_e}} h_{N_e} g_{t_{N_e}}{}^{-1}) \tag{3.63}$$

This property can be easily implemented via *group averaging*: given an arbitrary $f \in \mathrm{Cyl}_\gamma$, the function

$$f_0(h_1, \ldots, h_{N_e}) \equiv \int \prod_n \mathrm{d}g_n \; f(g_{s_1} h_1 g_{t_1}{}^{-1}, \ldots, g_{s_{N_e}} h_{N_e} g_{t_{N_e}}{}^{-1}) \tag{3.64}$$

clearly satisfies (3.63).

The group averaging amounts to inserting on each vertex v the following projector,

$$\mathcal{P} = \int \mathrm{d}g \prod_{e \in v} \mathcal{D}^{(j_e)}(g). \tag{3.65}$$

The integrand of Eq. (3.65) is an element in the tensor product of SU(2) irreducible representations,

$$\prod_e \mathcal{D}^{(j_e)}_{m_e n_e}(h_e) \in \bigotimes_e V^{(j_e)}. \tag{3.66}$$

As such, it transforms non-trivially under gauge transformation and is in general reducible,

$$\bigotimes_e V^{(j_e)} = \bigoplus_i V^{(j_i)}. \tag{3.67}$$

Then, the integration in Eq. (3.65) selects the gauge invariant part of $\bigotimes_e V^{(j_e)}$, namely the *singlet* space $V^{(0)}$, if the latter exists. Since $\mathcal{P}$ is a projector, we can decompose it in terms of a basis of $V^{(0)}$. Denoting i_α a vector (*ket*) in this basis, $\alpha = 1, \ldots, \dim V^{(0)}$, and i^*_α the dual (*bra*),

$$\mathcal{P} = \sum_{\alpha=1}^{\dim V^{(0)}} i_\alpha \, i^*_\alpha. \tag{3.68}$$

These invariants are called *intertwiners*. For the case of a 3-valent vertex as in the above example, $\dim V^{(0)} = 1$ and the unique intertwiner i is given by Wigner's 3j-m symbols. More precisely in the case of a three-valent vertex the space

$$\left[V^{(j_1)} \otimes V^{(j_2)} \otimes V^{(j_3)}\right]_{\text{inv}} \tag{3.69}$$

is non-empty only when the following Clebsch-Gordan conditions hold,

$$|j_2 - j_3| \leq j_1 \leq j_2 + j_3. \tag{3.70}$$

For an n-valent vertex, the space $V^{(0)}$ can have a larger dimension. To visualize the intertwiners, it is convenient to add first two irreps only, then the third, and so on. This gives rise to a decomposition over virtual links, which for $n = 4$ and $n = 5$ is depicted in Fig. 3.5, where the virtual spins k_i label the intertwiners.

The facts that $\mathcal{P}$ acts only on the nodes of the graph that label the basis of $\mathcal{H}_{kin}$ and equation (3.68) implies that the result of the action of $\mathcal{P}$ on elements of $\mathcal{H}_{kin}$ can be written as a linear combination of products of representation matrices $\mathcal{D}^{(j)}(h_e)$ contracted with intertwiners.

The states labeled with a graph γ, with an irreducible representation $\mathcal{D}^{(j)}(h)$ of spin-j of the holonomy h along each link, and with an element i of the intertwiner space $\mathcal{H}_v \equiv \mathrm{Inv}[\bigotimes_{e\in v} V^{(j_e)}]$ in each node, are called spin

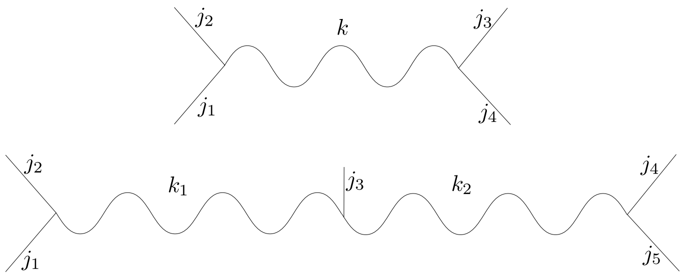
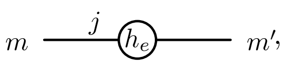


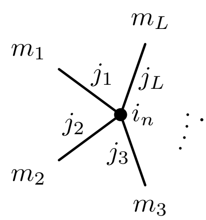
Figure 3.5.: The virtual spins k_i label the intertwiners.

network states, and are given by

$$\psi_{(\gamma,j_e,i_v)}[h_e] = \bigotimes_e \mathcal{D}^{(j_e)}(h_e) \bigotimes_v i_v. \tag{3.71}$$

Here the indices of the matrices and of the interwiners are hidden for simplicity of notation. Their contraction pattern can be easily reconstructed from the connectivity of the graph. Before summarizing let us introduce some graphical notation. We represent the holonomy $h_e[A]$ in representation j along a curve e embedded in Σ by a labeled edge and an intertwining tensor v_{i_n}, that is the tensor associated to an invariant vector i_n in the tensor product of L irreducible $SU(2)$ representations, by a L-valent node. A spin network state can be represented diagrammatically using these building blocks, see [21].

$\mathcal{D}^{(j_e)}[h_e]_m{}^{m'} =$ 

$i_v{}^{(j_1 \ldots j_L)}_{m_1 \ldots m_L} =$ 

(3.72)

Summarizing, spin network states (3.71) form a complete basis of the Hilbert space of solutions of the quantum Gauss law, $\mathcal{H}^G_{\text{kin}}$. The structure of this space is nicely organized by the spin networks basis. As before, different graphs γ select different orthogonal subspaces, thus $\mathcal{H}^G_{\text{kin}}$ decomposes as a direct sum over spaces on a fixed graph,

$$\mathcal{H}^{\mathrm{G}}_{\text{kin}} = \bigoplus_{\gamma \in \Sigma} \mathcal{H}^{\mathrm{G}}_{\gamma} \tag{3.73}$$

Furthermore, the Hilbert space on a fixed graph decomposes as a sum over intertwiner spaces,

$$\mathcal{H}^{G}_{\gamma} = L_2[\mathrm{SU}(2)^{\mathrm{N_e}}/\mathrm{SU}(2)^{\mathrm{N_v}}, \mathrm{d}\mu_{\mathrm{Haar}}] = \oplus_{\mathrm{j_l}} \left(\otimes_{\mathrm{v}} \mathcal{H}_{\mathrm{v}}\right), \tag{3.74}$$

where N_v are the vertices to the corresponding graph. Equations (3.73) and (3.74) are the analogue in loop gravity of the Fock decomposition of the Hilbert space of a free field in Minkowski spacetime into a direct sum of v-particle states, and play an equally important fundamental role.

Kinematical geometrical operators

In this Section we will describe the kinematical geometrical operators of Loop Quantum Gravity. To be well defined on $\mathcal{H}^G_{\text{kin}}$, the space of the sates invariant under local SU(2), an operator has to be invariant under internal gauge transformations. Although the operator $\hat{E}^a_j$ cannot be gauge invariant, as the index j transforms under internal gauges, in Eq. (3.1.2) we have introduced the action of the triad field E^a_j (3.25) on a spin network state ψ in order to obtain the quantization of the Gauss constraint and area- respectively the volume-operator, that lead to one of the main physical prediction of LQG: discreteness of geometry eingenvalues. Let us assume that there is just one intersection P between the surface S and the graph γ of the spin network ψ. Let be j the spin of the link at the intersection. We obtain from Eq. (3.59)

$$\widehat{E}^a_j(S_n)\widehat{E}^b_j(S_n)|\psi(\gamma)\rangle = \hbar^2 j(j+1)|\psi(\gamma)\rangle. \tag{3.75}$$

i.) **Area operator**

Now we will introduce the Area operator of Loop Quantum Gravity [10]. Let us define a gauge-invariant operator Ar$[S]$, see (3.76), associated to the two-dimensional surface $S \subset \Sigma$, where S is an oriented, open, embedded, completely supported, semi-analytical surface. In order to generalize (3.75) to a generic graph, with arbitrary amount of intersections. To circumvent the difficulties already described at the end of Section 3.1.2 we will regularize the expression for the area in the following way. For any N, we will decompose the surface S in small surfaces S_n (two-cells), which are shrinking as $N \to \infty$, and such $\forall N, \bigcup_n S_n = S$. With this decomposition of S, we can write the integral defining the area as the limit of a *Riemann sum*, in particular

$$\mathrm{Ar}[S] = \lim_{N \to \infty} \mathrm{Ar}_S^N[S],$$

where the *Riemann sum* can explicity written as

$$\mathrm{Ar}^N[S] = \sum_{n=1}^{N} \sqrt{E_j^a(S_n) E_j^b(S_n)}. \tag{3.76}$$

Here $E_j^a(S_n)$ denotes the flux of E_j^a through the n-th two cell.

The strategy of quantization of the area is to plug into (3.76) the quantization of $E_j(S)$,to apply cylindrical function and to hope that in the limit $N \to \infty$ we obtain a consistently defined family of positive semidefinite operators. In particular the area operator is defined as

$$\widehat{\mathrm{Ar}}[S] = \lim_{N \to \infty} \widehat{\mathrm{Ar}}_S^N[S],$$

In a next step we will compute the action of the area operator on a generic spin-network state $\psi(\gamma)$, where the graph is generic and can cross S many times. Using equation (3.75), we obtain immediately

$$\begin{aligned} \widehat{\mathrm{Ar}}^N[S]|\psi(\gamma)\rangle &= \sum_{n=1}^{N} \sqrt{\widehat{E_j^a}(S_n)\widehat{E_j^b}(S_n)}|\psi(\gamma)\rangle \\ &= \sum_{p \in S \cup \gamma} \hbar \sqrt{\beta^2 j_p(j_p+1)}|\psi(\gamma)\rangle. \end{aligned} \tag{3.77}$$

Next we list the main properties of the Area operator, namely:

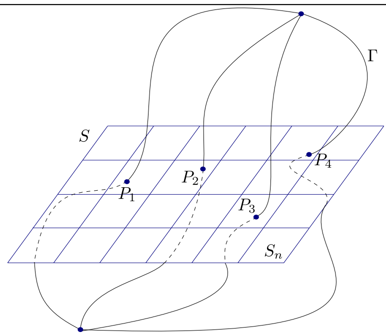


Figure 3.6.: A simple spin network (SNW) γ intersecting the surface S.

THEOREM 3.1.15. *The area functional admits a well-defined quantization* $\widehat{\mathrm{Ar}}(S)$ *on* $\mathcal{H}_{kin}$ *with the following properties:*

a.) $\widehat{\mathrm{Ar}}(S)$ *is positive semidefinite, (essentially) self-adjoint with* $\mathrm{Cyl}^2(\overline{\mathcal{A}})$ *as domain of (essential) self-adjointness.*

b.) The spectrum Spec$(\widehat{\mathrm{Ar}}(S))$ *is pure point (discrete) with eigenvectors being given by finite linear combinations of spin network functions.*

c.) In physical units the eigenvalues are given explicitly by $\lambda_{j_p} = \frac{\beta\ell_P^2}{4}\sqrt{j_p(j_p+1)}$, *where* $\ell_P^2 = \hbar\kappa$ *is the Planck area. The spectrum has an area gap (smallest non-vanishing eigenvalue) given by* $\lambda_0 = \beta\ell_P^2\frac{\sqrt{3}}{8}$.

d.) Spec$(\widehat{\mathrm{Ar}}(S))$ *contains information about the topology of* Σ, *for instance it matters whether* $\partial S = \emptyset$ *or not.*

The area operator $\widehat{\mathrm{Ar}}_S^N[S]$ has contribution from each edge of ψ that crosses S. For a detailed discussion of the area operator see [10].

ii.) **Volume operator**

Here we describe in some detail the construction of the volume operator, we quote [21]. We will follow similar steps when introducing the length operator in Section 6.1.2. In order to introduce in the quantum theory an operator corresponding to the volume of a region, the starting point is the classical expression, given by

$$\mathrm{Vol}[R] := \int_R \mathrm{d}^3x \sqrt{\det(q)} = \sqrt{\frac{1}{3!}\left|\epsilon_{ijk}\epsilon^{abc}E_i^a E_j^b E_k^c\right|}. \tag{3.78}$$

Applying the canonical quantization procedure, however, is not straightforward: at the quantum level the well-defined operator representing the geometry of space is not $E_i^a(x)$ but its flux through a surface given by $F_i(S) = \int_S n_a E_i^a(x)\,\mathrm{d}^2\sigma$ and the holonomy of the connection A along a curve c namely $h_{\mathrm{c}}[A]$. Therefore the quantization strategy is to find a regularized expression for the classical volume in terms of fluxes, to promote this expression to an operator and then analyze the existence of the limit in the Hilbert space topology. If the limit exists, then we can say that we have a *candidate* for the volume operator. At this point one can forget the construction, study the properties of this operator both in the deep quantum regime and in the semi-classical regime and understand if it actually has the meaning of *volume of a region* at both levels.

Given the number of choices to be made, it is not surprising that two distinct mathematically well-defined volume operators exist in the literature, one due to Rovelli and Smolin [63] the other to Ashtekar and Lewandowski [10]. Both of them act non-trivially only at the nodes of a spin network state. In this sense, both of them fit into the picture 3.8(b). For a discussion of the relation between the two operators see [10, 31, 40, 41]. Here we describe in detail some aspects of the Rovelli-Smolin construction [63] of the volume operator as it will play a role in the following.

a.) *External regularization of the volume*

The construction of the regularized expression for the volume to be used as starting point for quantization goes through the following steps. We want to mention that the construction we discuss at hand is based on [63], [28], [62] but does not completely

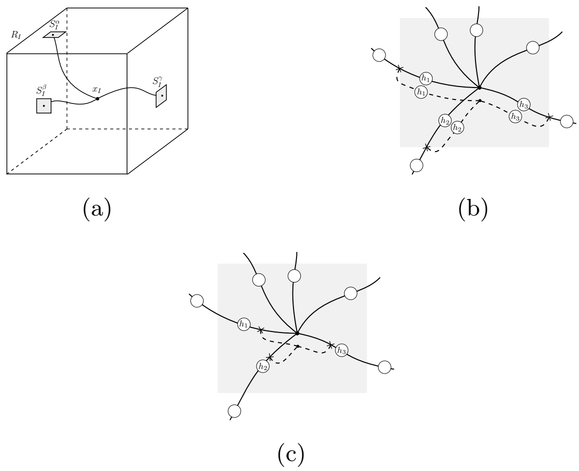


Figure 3.7.: 3.7(a) Cubic cell with regularized quantity (6.4) shown. Figure 3.7(b) Action of the three-hand operator. The cubic cell is shown in gray. Figure 3.7(c) Shrinking property of the three-hand operator [21].

coincide with it. See also [50] and [10] for a comprehensive discussion of the many subtleties involved and a comparison with the Ashtekar-Lewandowski construction.

i. The integral over R is replaced by the limit of a Riemann sum. More specifically, we choose coordinates x^a in a neighborhood in Σ containing R and consider a partition of the neighborhood in cubic cells R_I of coordinate side Δx. Therefore the region R is contained in the union of a number of cells, $R \subseteq \cup_N R_N$, and the integral $\int_R d^3x$ can be approximated from above by the sum $\sum_N (\Delta x)^3$ with N running on the cells containing points of R.

ii. The argument of the square root in (3.78) in a point contained in the cell R_N is written in terms of the limit of a quantity $W_{\Delta x}(x_N)$ given by a triple surface integral over the boundary of the cell:

$$W_{\Delta x}(x_N) = \frac{1}{8\times 3!}\frac{1}{(\Delta x)^6}\iiint_{\partial R_N} [|T^{ijk}_{x_N}(\sigma,\sigma',\sigma'')\times E^a_i(\sigma)n_a(\sigma)\, E^b_j(\sigma')n_b(\sigma')\, E^c_k(\sigma'')n_c(\sigma'')|] \tag{3.79}$$

where $\iiint_{\partial R_N} := \int_{\partial R_N} \mathrm{d}^2\sigma \int_{\partial R_N} \mathrm{d}^2\sigma' \int_{\partial R_N} \mathrm{d}^2\sigma''$.

In Eq. (3.79) the following notation has been used. Let's consider a surface S, a choice of local coordinates σ^α and an embedding of S in Σ given by $x^a = X^a(\sigma)$. The quantity $n_a(\sigma)$ is defined as $n_a(\sigma) = \epsilon_{abc}\frac{\partial X^b}{\partial\sigma^1}\frac{\partial X^c}{\partial\sigma^2}$. Notice that in Eq. (3.79) we are considering a surface given by the boundary of a cubic cell, therefore the function $n_a(\sigma)$ is not continuous in Σ. By $E^a_i(\sigma)$ we simply mean $E^a_i(X(\sigma))$. The function $T^{ijk}_{x_N}(\sigma,\sigma',\sigma'')$ has been inserted in order to guarantee the $SU(2)$-gauge invariance of the non-local expression Eq. (3.79)). It is given by

$$T^{ijk}_{x_N}(\sigma,\sigma',\sigma'') = \epsilon^{i'j'k'}\mathcal{D}^{(1)}(h_{\mathrm{c}^1_{x_N\sigma}}[A])_{i'}{}^{i}\times \mathcal{D}^{(1)}(h_{\mathrm{c}^2_{x_N\sigma'}}[A])_{j'}{}^{j}\mathcal{D}^{(1)}(h_{\mathrm{c}^3_{x_N\sigma''}}[A])_{k'}{}^{k} \tag{3.80}$$

where $\mathcal{D}^{(j)}(h_{\mathrm{c}^i}[A])$ is the representation j of the holonomy of the connection along the edge c^i and $\mathrm{c}^1_{x_N\sigma}$, $\mathrm{c}^2_{x_N\sigma'}$ and $\mathrm{c}^3_{x_N\sigma''}$ are three curves embedded in R_N having starting point x_N in R_N and ending at a point on the boundary of R_N given by $X(\sigma)$, $X(\sigma')$ and $X(\sigma'')$ respectively. As already explained, by $\mathcal{D}^{(1)}(h_{\mathrm{c}}[A])_i{}^j$ we mean the holonomy of the real $SU(2)$ connection along the curve c, taken in the adjoint representation.

In the limit $\Delta x \to 0$, under the assumption of smooth $E^a_i(x)$ and $A^i_a(x)$, we have that $W_{\Delta x}(x_N)$ goes to $\frac{1}{3!}|\epsilon^{ijk}\epsilon_{abc}E^a_i(x_N)E^b_j(x_N)E^c_k(x_N)|$, where we recall the formula

$$\epsilon^{i'j'k'}E^{a'}_{i'}E^{b'}_{j'}E^{c'}_{k'} = \frac{1}{3!}\left(\epsilon_{abc}\epsilon^{ijk}E^a_iE^b_jE^c_k\right)\epsilon^{a'b'c'} \; .$$

Therefore we have

$$\mathrm{Vol}[R] = \lim_{\Delta x \to 0} \sum_N (\Delta x)^3 \sqrt{W_{\Delta x}(x_N)}\,. \tag{3.81}$$

Notice that the factor $(\Delta x)^{-6}$ present in $W_{\Delta x}(x_N)$ cancels with the $(\Delta x)^3$ appearing in (3.81). This corresponds to the fact that $\sqrt{\frac{1}{3!}|\epsilon^{ijk}\epsilon_{abc}E^a_i E^b_j E^c_k|}$ is a density of weight one and can be integrated with the measure $\int \mathrm{d}^3x$. As a result, in (3.81) Δx appears only implicitly in the definition of the surface ∂R_N.

iii. The surface ∂R_N can be partitioned in square cells S^α_N so that $\partial R_N = \cup_\alpha S^\alpha_N$. As a result the triple integral over ∂R_N can be replaced by a triple Riemann sum. In this way we end up with an expression depending only on fluxes and holonomies. Defining the quantity $Q_{N\alpha\beta\mathrm{c}}$ for a cell R_N and three surfaces S^α_N, S^β_N and S^{c}_N as

$$Q_{N\alpha\beta\mathrm{c}} = T^{ijk}_{x_N}\, F_i(S^\alpha_N)\, F_j(S^\beta_N)\, F_k(S^{\mathrm{c}}_N)\,, \tag{3.82}$$

where $F_i(S)$ is the so called *flux*, which is explicitly given by $F_i(S) = \int_S n_a E^a_i(x)\,\mathrm{d}^2\sigma$ we obtain

$$\mathrm{Vol}_N = \sqrt{\frac{1}{8\times 3!}{\sum}'_{\alpha\,\beta\,\mathrm{c}}|Q_{N\alpha\beta\mathrm{c}}|} \tag{3.83}$$

and

$$\mathrm{Vol}[R] = \lim_{\Delta x\to 0}\sum_N \mathrm{Vol}_{\Delta x}(x_N)\mathrm{Vol}[R] = \lim_{\Delta x\to 0}\sum_N \mathrm{Vol}_N\,, \tag{3.84}$$

where the prime in the sum in Eq. (3.83) stands for sum restricted to distinct $\alpha, \beta, \mathrm{c}$. This corresponds to a point-splitting of the integral over $(\partial R_N)^3$.

Notice that while the regularized expression depends both on the E^a_i and on A^i_a, the limit depends only on the electric field.

Step (ii.) and (iii.) can be called a *fluxization* of the Riemann sum.

b.) *Quantization of the volume*

Having constructed a sequence of regularized expressions having the appropriate classical limit, we can now attempt to promote (3.84) to a quantum operator by invoking the known action of the holonomy and of the flux on cylindrical functions, namely

$$\widehat{\mathrm{Vol}[R]}\,\psi(\gamma,f)[A]=\lim_{\Delta x\to 0}\left(\sum_N \widehat{\mathrm{Vol}}_N\,\psi(\gamma,f)[A]\right)\,.$$

To be more specific, we need to define a consistent family of operators for finite Δx and given cylindrical function. This step requires a number of choices which we state below. Then we can analyze the existence of the limit in the operator topology.

Let γ be a closed graph embedded in Σ and made of N nodes connected by M links $\{e_1,..,e_M\}$. A $SU(2)$-gauge invariant state which is cylindrical with respect to the graph γ is defined as Eq. (3.50)

$$\psi(\gamma,f)[A]=f(h_{e_1}[A],..h_{e_M}[A])$$

with f a class function on $SU(2)^M$. In order to define the regularized operator $\hat{\mathrm{Vol}}_N$ for finite Δx, an adaptation of the partition of R to the graph γ is needed. The partition of the region R in cells R_N is refined so that

- nodes of γ can fall only in the interior of cells;
- a cell R_N contains at most one node. In case it contains no node, then it can contain at most one link;
- the boundary ∂R_N of a cell intersects a link exactly once if the link ends up at a node contained in the cell and exactly twice if it does not.

Moreover we assume that the partition of the surfaces ∂R_N in cells S_N^α is refined so that links of γ can intersect a cell S_N^α only in its interior and each cell S_N^α is punctured at most by one link.

Next we focus on the action of the operator $\hat{Q}_{N\alpha\beta\mathrm{c}}$ obtained quantizing canonically expression (3.82):

$$\hat{Q}_{N\alpha\beta\mathrm{c}} = \hat{T}^{ijk}_{x_N}\, \widehat{F}_i(S_N^\alpha)\, \widehat{F}_j(S_N^\beta)\, \widehat{F}_k(S_N^\mathrm{c}) \ . \tag{3.85}$$

Let's call it the three-hand operator. Notice that we don't need a specific ordering of the fluxes and the holonomies thanks to the fact that the self-grasping vanishes.This is a straightforward consequence of the fact that $\delta^{il}(T_i^{(1)})^k{}_l = 0$. From properties of the action of the flux operator on a holonomy, we know that when the operator $\hat{Q}_{N\alpha\beta\mathrm{c}}$ acts on a state $\psi(\gamma, f)[A]$ the result is zero unless each of the surfaces S_N^α, S_N^β and S_N^c is punctured by a link of γ. As a result if the cell R_N does not contain nodes of γ, then $\hat{Q}_{N\alpha\beta\mathrm{c}}$ annihilates the state.

Now let's focus on a cell R_N which contains a node of γ. In this case, some further adaptation of the regularized expression (3.82) to the graph γ is required. The point x_N and the three curves introduced by $T^{ijk}_{x_N}$ in the definition of the regularized volume are adapted to the graph γ in the following way:

i. the point x_N in (3.80) is chosen to coincide with the position of the node,

ii. the three curves $\mathrm{c}^1_{x_N\sigma}$, $\mathrm{c}^2_{x_N\sigma}$ and $\mathrm{c}^3_{x_N\sigma}$ are adapted to three of the links of γ originating at the node contained in the cell R_N.

As a result the appropriate labels for the operator (3.85) are a node n and a triple of links e_1, e_2, e_3. We have that, when the operator acts on a state of the spin network basis of $\mathcal{H}_{\mathrm{kin}}(\gamma)$, its action is the following

$$
\begin{aligned}
&\hat{Q}_{ne_1e_2e_3}\psi(\gamma, j, i_k)[A] \\
=&\hat{Q}_{n\,e_1e_2e_3}\left(\mathcal{D}^{(j_1)}(h_{e_1}[A])_{m_1'}{}^{m_1}\cdots\mathcal{D}^{(j_L)}(h_{e_L}[A])_{m_L'}{}^{m_L}\ v_{k\,m_1\cdots m_L}^{(j_1\cdots j_L)}\right) \\
&\times \mathrm{rest}^{m_1'\cdots m_L'} \\
=&(8\pi c L_P^2)^3\ \epsilon^{i'j'k'}\mathcal{D}^{(1)}(h_{e_1}[A])_{i'}^{\ i}\mathcal{D}^{(1)}(h_{e_2}[A])_{j'}^{\ j}\mathcal{D}^{(1)}(h_{e_3}[A])_{k'}^{\ k} \\
&\times (T_i^{(j_1)})_{m_1'}{}^{m_1''}(T_j^{(j_2)})_{m_2'}{}^{m_2''}(T_k^{(j_3)})_{m_3'}{}^{m_3''} \\
&\times \left(\mathcal{D}^{(j_1)}(h_{e_1}[A])_{m_1''}{}^{m_1}\cdots\mathcal{D}^{(j_L)}(h_{e_L}[A])_{m_L'}{}^{m_L}\ v_{k\,m_1\cdots m_L}^{(j_1\cdots j_L)}\right) \\
&\times \mathrm{rest}^{m_1'\cdots m_L'}
\end{aligned}
\tag{3.86}
$$

This expression has the diagrammatic representation Figure 3.7(b).

The adaptation of $T_{x_N}^{ijk}$ to the graph γ as described above has the following remarkable property: shrinking the region R_N corresponds to moving the graspings in Figure 3.7(b) towards the node; however, thanks to the invariance properties of the intertwiner inserted by the grasping, the result of the triple-grasping is independent of the position of the grasping and it can be moved to the node as shown in Figure 3.7(c). In formulae we have the identity

$$
\begin{aligned}
&\epsilon^{i'j'k'}\mathcal{D}^{(1)}(h_{e_1}[A])_{i'}^{\ i}\mathcal{D}^{(1)}(h_{e_2}[A])_{j'}^{\ j}\mathcal{D}^{(1)}(h_{e_3}[A])_{k'}^{\ k} \\
&\quad\times (T_i^{(j_1)})_{m_1'}{}^{m_1''}(T_j^{(j_2)})_{m_2'}{}^{m_2''}(T_k^{(j_3)})_{m_3'}{}^{m_3''} \\
&\quad\times \left(\mathcal{D}^{(j_1)}(h_{e_1}[A])_{m_1''}{}^{m_1}\cdots\mathcal{D}^{(j_L)}(h_{e_L}[A])_{m_L'}{}^{m_L}\ v_{k\,m_1\cdots m_L}^{(j_1\cdots j_L)}\right) \\
&= \mathcal{D}^{(j_1)}(h_{e_1}[A])_{m_1'}{}^{m_1}\cdots\mathcal{D}^{(j_L)}(h_{e_L}[A])_{m_L'}{}^{m_L} \\
&\quad\times \left(\epsilon^{ijk}(T_i^{(j_1)})_{m_1}{}^{m_1''}(T_j^{(j_2)})_{m_2}{}^{m_2''}(T_k^{(j_3)})_{m_3}{}^{m_3''}v_{k\,m_1''\cdots m_L}^{(j_1\cdots j_L)}\right)
\end{aligned}
\tag{3.87}
$$

where the left hand side corresponds to the evaluation of the diagram Figure 3.7(b) while the right hand side to the evaluation of Figure 3.7(c).

As a result we have that, for finite Δx and with the refinement of the partition and adaptations to the graph described above, the action of the operator $\hat{Q}_{N\alpha\beta c}$ on a spin network node is

- independent of Δx,
- does not change the graph of the state,
- does not change the spin labelling of the state.

Therefore its matrix elements are non-trivial only in the intertwiner sector and can be computed using standard recoupling techniques [28, 66, 27]. In formulae we have that

$$\hat{Q}_{n\, e_1 e_2 e_3}\ \Psi_{\gamma,j,i_k}[A] = \sum_h (Q_{n\, e_1 e_2 e_3})_k{}^h\ \ \psi(\gamma, j, i_h)[A]$$

Given the valence of the node and the spins of the incoming links, we have a finite dimensional hermitian matrix $(Q_{n\, e_1 e_2 e_3})_k{}^h$. In the case of a trivalent node, the intertwiner space is one-dimensional. Therefore we have that trivalent nodes are always eigenstates of the operator Q_N (and of the operator Vol_N). Explicit computation [52] shows that the eigenvalue is zero. As a result the simplest non-trivial case is for 4-valent nodes. The operator $\widehat{\mathrm{Vol}}_N$ involves taking a modulus of such matrix and a square root of a sum of matrices and this can be done through spectral decomposition. This defines the operator $\widehat{\mathrm{Vol}}_N$ for finite Δx. Moreover this is enough to define the action of the operator $\widehat{\mathrm{Vol}}(R)$ on a given spin network state too as, once an appropriate refinement of the partition is reached, the action of the regularized operator is independent of Δx and the limit in equation (3.84) is guaranteed to exist as it is simply the limit of a constant. Having defined the matrix elements of the operator $\widehat{\mathrm{Vol}}(R)$ on a orthonormal basis, the spin network basis, then one can attempt to promote it to a well-defined operator on the whole Hilbert space $\mathcal{H}_{\mathrm{kin}}$ through self-adjoint extension.

The volume operator for a region has the remarkable feature that it can be expressed in terms of *elementary* volume operators. Let's consider a graph γ embedded in Σ, focus on a node n of γ and choose a region R_n such that it contains the node n, but does not contain any other node of γ. We call the region R_n *dual* to the node n. Then we consider the Hilbert space $\mathcal{H}_{\mathrm{kin}}(\gamma)$ spanned by spin network states having exactly γ as graph. This is a subspace of the Loop Quantum Gravity state space $\mathcal{H}_{\mathrm{kin}}$. On this Hilbert space the operator $\widehat{\mathrm{Vol}}(R_n)$ is well defined, acts only on the intertwiner space at the node n and the matrix elements do not depend on the specific choice of the surface R_n. That is, if two regions R_n and R'_n are both dual to the node n, then the operators $\widehat{\mathrm{Vol}}(R_n)$ and $\widehat{\mathrm{Vol}}(R'_n)$ coincide on $\mathcal{H}_{\mathrm{kin}}(\gamma)$, i.e. they have the same matrix elements on the spin network basis of $\mathcal{H}_{\mathrm{kin}}(\gamma)$. As a result it can be said that it measures the volume of a region dual to the node n. We call this operator the *elementary* volume operator for the node n and indicate it as $\widehat{\mathrm{Vol}}_n$. For a generic region R the volume operator on $\mathcal{H}_{\mathrm{kin}}(\gamma)$ is given by a sum of *elementary* volume operators

$$\widehat{\mathrm{Vol}[R]}\,\psi(\gamma,f)[A] = \sum_{n\subset R} \widehat{\mathrm{Vol}}_n \psi(\gamma,f)[A]\ .$$

This property enlightens the quantum geometrical meaning of states belonging to $\mathcal{H}_{\mathrm{kin}}(\gamma)$, and in particular of spin network states. Moreover it offers the possibility of identifying a region R in a relational way, i.e. with respect to the state of the gravitational field [61]. This concludes our description of the construction of the Rovelli-Smolin volume operator.

Next we want to state some properties of Vol.

THEOREM 3.1.16. *The volume functional admits a well-defined quantization* $\widehat{\mathrm{Vol}}(R)$ *on* $\mathcal{H}_{\mathrm{kin}}$ *with the following properties:*

a.) The family of $\widehat{\mathrm{Vol}}(R)$ *defines a linear unbounded operator on* $\mathcal{H}_{\mathrm{kin}}$.

b.) $\widehat{\mathrm{Vol}}(R)$ *is symmetric, positive semidefinite, (essentially) self-adjoint with* $\mathrm{Cyl}^2(\overline{\mathcal{A}})$ *as domain of (essential) self-adjointness.*

c.) *The spectrum Spec*$(\widehat{\text{Vol}}(R))$ *is pure point (discrete) with eigenvectors being given by finite linear combinations of spin network functions.*

d.) *Trivalent vertices are annihilated* [1]*.*

e.) *The action of the volume operator vanishes on vertices whose edges lie on plane, i.e. planar vertices.*

The proof a that theorem can be found in [69]. The essential property of the *volume operator* is that it has contributions only from the nodes of a SNW state $|\psi\rangle$. This means, that he volume of a region R is a sum of terms, one for each node of ψ inside R. Thus each node of a SNW represents a quantum of volume.

Quantum geometry and its dual picture

figure In Loop Quantum Gravity, the state of the 3-geometry can be given in terms of a linear superposition of spin network (SNW) states. Each SNW state describes a *quantum geometry*. Such SNW states consist of a graph embedded in a 3-manifold and a coloring of its links and its nodes in terms of SU(2) irreducible representations and of SU(2) intertwiners. The area operator $\widehat{\text{Ar}}_S^N[S]$ has contribution from each link of γ that crosses S. The essential property of the volume operator $\widehat{\text{Vol}}[R]$ is that it has only contribution from the nodes (also called vertices) of the SNW state and thus we get, that the volume of a region R is a sum of terms, one for each vertex of γ inside R. Now we are able to interpret a SNW with N nodes as an ensemble of N quanta of volume, also called *chunks of space*, localized in the manifold around the node, each with a quantized volume Vol_{i_n}. The elementary chunks of quantized volume are separated from each other by surfaces, which area is governed by the area operator. Thus thanks to the existence of a volume operator and an area operator, the following dual picture of the quantum geometry of a SNW state is available (see [62] a for a detailed discussion): a node of the SNW corresponds to a chunk of space with definite volume while a link connecting two nodes corresponds to an interface of definite area which separates two chunks (see Figure 3.8(b)). Moreover, a node connected to two other nodes identifies two surfaces which intersect at a curve. The operator we will introduce in Section 6 corresponds to the length of this curve.

Implementation of $\mathcal{H}_{\text{kin}}^{\text{Diff}}$ and $\mathcal{H}_{\text{phys}}$ – quantum dynamics

The next step is to implement the spatial diffeomorphisms. In this regard, we would like to refer to [69, 62, 75, 39] since as far as further work is concerned all the necessary tools are explained. But before proceeding with the mathematical construction of the Ashtekar connection we want to make some comments concerning the Hamilton constraint. As we have seen in Section 3.1.2, the Hamilton constraint of the classical theory is given by

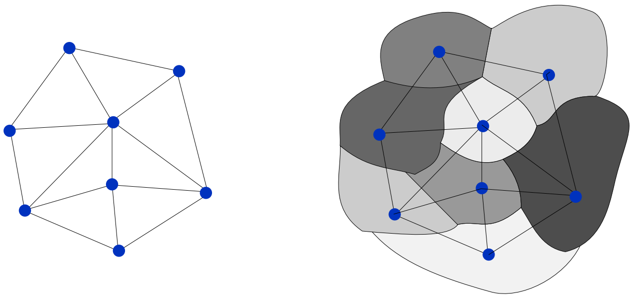
$$H = \left[{}^{(\beta)}F^j_{ab} - (1+\beta^2)\epsilon_{jmn}K^m_a K^n_b\right] \frac{\epsilon_{jkl}E^a_k E^b_l}{\sqrt{\det(q)}}. \tag{3.88}$$

The dicussion of how how to turn this classical expression into a well defined operator is in detail given in [69, 62, 39]. The general difficulty with this is obviously that H is a complicated nonlinear function in the phase space variables, hence ordering problems present themselves. There are also some specific difficulties with the expression:

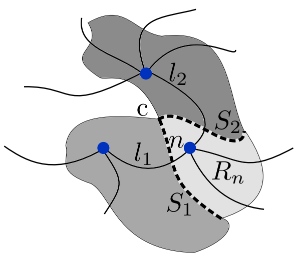
i.) Eq. (3.88) contains the inverse volume element. The volume element itself has a large kernel when quantized, see the discussion in Section 3.1.2, so its inverse is ill defined.

ii.) The expression (3.88) contains the curvature F of A, as well as the extrinsic curvature K. For neither of them there is a simple operator in the quantum theory. A guiding principle in the quantization process can be the Dirac algebra. In particular, the quantum Hamiltonian constraint should be invariant under gauge transformations, covariant under diffeomorphisms, and the commutator of two Hamilton constraints should give a diffeomorphism constraint.

We should say that the knowledge about the quantization and implementation of the Hamilton constraint is not complete. A new proposal to turn the classical expression into a well defined operator is outlined in Section 6.



(a)



(b)

Figure 3.8.: Figure 3.8(a) On the left: the graph of a abstract spin network (SNW) and on the right panel the ensemble of *chunks of space*, it represents. Figure 3.8(b) A portion of spin network graph and the associated dual picture of quantum gravity. The region R_n is dual to the node n. Two adjacent region are illustrated. The surfaces S_1 and S_2 are dual to the links l_1 and l_2. They identify a curve c on the boundary of R_n.

4. The Ashtekar Connection

4.1. Construction of the Ashtekar Connection

In this Chapter we will *systematical carry out the construction of the Ashtekar connection as a metric connection on the tangent bundle step by step* as outlined in [33]. In particular, we will mathematically work out in detail the proofs of the results presented in [33]. Additionally we complete the discussion of [33] by Theorem 4.1.17 and Proposition 4.1.18.

In more detail the construction is composed of:

1. Defining the second fundamental form and the Weingarten mapping of Σ;
2. Construction of metric connections on 3-dimensional orientable Riemannian manifolds;
3. Defining the Ashtekar connection.

In the following let Σ be a Cauchy hypersurface of dimension 3, in more detail an immersed submanifold in a four-dimensional Lorentzian manifold $(\mathcal{M}, g)$, whereas the metric q on Σ is induced by g to Σ. Furthermore we denote by $\nabla^{\mathcal{M}}$ covariant differentiation in $\mathcal{M}$.

4.1.1. Step I: Second fundamental form and Weingarten mapping

In this Section we refer to the discussion of Section 2.2.3 such as the Levi-Civita connection of a submanifold by using the formalism of covariant differentiation ∇. In addition we want to repeat parameters for the second fundamental form and the Weingarten mapping, the first ingredient of the Ashtekar connection. The Weingarten mapping with respect to the normal vector field n is given by Wein : $\Gamma(T\Sigma) \longrightarrow \Gamma(T\Sigma), X \longmapsto \nabla^{\mathcal{M}}_X n = A_n(X)$, see Definition 2.2.71. Here again n is normal to Σ within $(\mathcal{M}, g)$. $\nabla^{\mathcal{M}}$ denotes the Levi-Civita connection associated to g on $\mathcal{M}$. The second fundamental form of Σ for a given immersion in $\mathcal{M}$ is defined by $K(X,Y) = g(\mathrm{Wein}(X), Y)$, see Section 2.2.3. The other way around this allows us to obtain Wein from K as g is non-degenerate.

4.1.2. Step II: Metric connections on 3-dimensional oriented Riemannian manifolds

In this Section we shall describe a technique of constructing connections and their covariant derivative on the tangent bundle, motivated by the construction of the *Ashtekar connection* in the physics literature. In the following let (Σ, g) an oriented 3-dimensional Riemannian manifold and ∇^{LC}:$\Gamma(T\Sigma) \longrightarrow \Gamma(T^*\Sigma \otimes T\Sigma)$ the covariant derivative with respect to the Levi-Civita connection. Moreover $\omega^{\mathrm{LC}} \in \Omega^1(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))$ denotes the corresponding connection form on the principle SO(3)-bundle of the orthonormal, oriented frames $\mathrm{O}^+(\Sigma, g)$ over Σ. In this Section we want to classify the set of of all connections $\mathcal{C}(\mathrm{O}^+(\Sigma, g))$ on $\mathrm{O}^+(\Sigma, g)$. For this purpose we will identify $\mathcal{C}(\mathrm{O}^+(\Sigma, g))$ with the set of $(1,1)$-tensor fields on Σ, denoted by $\mathrm{T}^{(1,1)}(\Sigma)$. In addition we determine the corresponding metrical covariant derivatives. Therefor we introduce a product structure on $T\Sigma$, generalising the cross product on $\mathbb{R}^3$.

Correspondence $\mathcal{C}(\mathrm{O}^+(\Sigma, g))$ with $\mathrm{T}^{(1,1)}(\Sigma)$

On the basis of a $(1,1)$-tensor field $S \in \mathrm{T}^{(1,1)}\Sigma$ the construction of a connection form is divided into three parts:

i.) By using Remark 2.2.51 we are able to state that two connections forms on a principle bundle differ in horizontal 1-forms of type (ad).

Therefore the space of all connections on $\mathrm{O}^+(\Sigma, g)$ is an affine space underlied by the vector space $\Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$. In what follows let be $e \in \mathrm{O}^+(\Sigma, g)_m$ an arbitrary element in the fibre over $m \in \Sigma$ and let be $\widetilde{X} \in T_e\mathrm{O}^+(\Sigma, g)$ an arbitrary lift of $X \in T_m\Sigma$. Then by using Theorem 2.2.52 we obtain the following identification

$$\begin{aligned} \Omega^1(\Sigma, E^{\mathrm{ad}}) \quad &\cong \quad \Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})} \\ \varsigma &\longleftrightarrow \bar{\varsigma}, \end{aligned} \tag{4.1}$$

where ς and $\bar{\varsigma}$ respectively are related by $\varsigma_m(X) = [e, \bar{\varsigma}_e(\widetilde{X})] = \iota_e \circ \bar{\varsigma}(\widetilde{X})$ and thus we are able to identify $\Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$ with the space of 1-forms on Σ with values in the associated bundle, given by

$$E^{\mathrm{ad}} := \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3),\mathrm{ad})} \mathfrak{so}(3). \tag{4.2}$$

Vice versa let be $\varsigma \in \Omega^1(\Sigma, E^{\mathrm{ad}})$, then we obtain $\bar{\varsigma} \in \Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$ by

$$\bar{\varsigma}_e(\widetilde{X}) = \iota_e^{-1} \circ \varsigma_{\pi(e)}(\mathrm{d}\pi_e\widetilde{X}).$$

ii.) We will use the equivalence between the adjoint representation and the natural representation of SO(3) in the following step. Briefly spoken: on the one hand we have

$$\rho{:}\mathrm{SO}(3) \longrightarrow \mathrm{GL}(\mathbb{R}^3)$$

with $\rho(A)x = Ax$ for all $A \in \mathrm{SO}(3)$ and $x \in \mathbb{R}^3$; and on the other hand we have the adjoint representation of SO(3)

$$\mathrm{ad}{:}\mathrm{SO}(3) \longrightarrow \mathrm{GL}(\mathfrak{so}(3))$$

which is given by $\mathrm{ad}(A)\Xi = A\Xi A^{-1}$ for all $A \in \mathrm{SO}(3)$ and $\Xi \in \mathfrak{so}(3)$. The equivalence is given by the vector space isomorphism

$$\mathfrak{f}{:}\mathfrak{so}(3) \longrightarrow \mathbb{R}^3; \qquad \sum_{\mathrm{i}=1}^{3} \mathrm{x_i}\Xi_{\mathrm{i}} \longmapsto \sum_{\mathrm{i}=1}^{3} \mathrm{x_i u_i},$$

where $\{u_i\}_{1\leq i\leq 3}$ respectively $\{\Xi_i\}_{1\leq i\leq 3}$ denotes the basis of $\mathbb{R}^3$ respectively $\mathfrak{so}(3)$. With respect to $\{u_i\}_{1\leq i\leq 3}$, the components of $\{\Xi_i\}_{1\leq i\leq 3}$ are given by $\{\Xi_i\} := \epsilon_{ijk}$. Thus we have

$$\mathfrak{f}(\mathrm{ad}(A)\Xi) = \rho(A)\mathfrak{f}(\Xi)\forall A \in \mathrm{SO}(3), A \in \mathfrak{so}(3).$$

By theorem 2.2.18 the isomorphism $\mathfrak{f}$ provides an isomorphism $\mathfrak{F}$ between E^{ad} and $\mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3),\rho)} \mathbb{R}^3 =: E^\rho$, explicitly given by

$$\begin{aligned}\mathfrak{F}{:}E^{\mathrm{ad}} = \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3),\mathrm{ad})} \mathfrak{so}(3) &\xrightarrow{\cong} \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3),\rho)} \mathbb{R}^3 = E^\rho \\ [e, \Xi] &\longmapsto [e, \mathfrak{f}(\Xi)].\end{aligned} \tag{4.3}$$

But we want to note, that at this point the isomorphism $\mathfrak{f}$ and therefore the choice of the bases $\{u_i\}_{1\leq i\leq 3}$ respectively $\{\Xi_i\}_{1\leq i\leq 3}$ enters explicitly the construction of the Ashtekar connection. This choice is fundamental for the construction of the Ashtekar connection.

iii.) And in addition by using example 2.2.25, we get

$$\begin{aligned}\mathfrak{V}{:}E^\rho = \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3),\rho)} \mathbb{R}^3 &\xrightarrow{\cong} T\Sigma \\ [e, x] &\longmapsto \sum_{i=1}^{3} e_i x_i = e \cdot x.\end{aligned} \tag{4.4}$$

On the basis of the identifications (4.1), (4.3) and (4.4) we obtain the following correspondence, see also [33].

Theorem 4.1.1. *There is a one-to-one identification $\mathfrak{I}$ between the set of connection forms $\mathcal{C}(\mathrm{O}^+(\Sigma, g))$ on $\mathrm{O}^+(\Sigma, g)$ and the set of $(1,1)$-tensor fields $\mathrm{T}^{(1,1)}(\Sigma)$ on Σ, i.e.*

$$\mathcal{C}(\mathrm{O}^+(\Sigma, g)) \cong \mathrm{T}^{(1,1)}(\Sigma). \tag{4.5}$$

The isomorphism enabeling Eq. (4.5) is defined in the following way

$$\mathfrak{I}{:}\Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})} \xrightarrow{\cong} \Omega^1(\Sigma, \mathrm{T}\Sigma) \tag{4.6}$$

and is explicitly given by $\mathfrak{V} \circ \mathfrak{F} \circ \varsigma$.

Thus with Theorem 4.1.1 we are able to conclude the following key result:

THEOREM 4.1.2. *The set of all connections on* $\mathrm{O}^+(\Sigma, g)$ *is given by*

$$\mathcal{C}(\mathrm{O}^+(\Sigma, g)) \cong \{\omega^{\mathrm{LC}} + \mathfrak{I}^{-1}(S) | S \in \Omega^1(\Sigma, T\Sigma) = \mathrm{T}^{(1,1)}(\Sigma)\}, \tag{4.7}$$

whereas ω^{LC} *denotes the connection form of the Levi-Civita connection.*

Discussion of the isomorphism $\mathfrak{I}$

In the following we shall discuss the isomorphism $\mathfrak{I}$ in detail. Here $e \in \mathrm{O}^+_m(\Sigma, g)$ and $\widetilde{X} \in T_e\mathrm{O}^+_m(\Sigma, g)$ denotes an arbitrary point in the fiber over $m \in \Sigma$ and an arbitrary lift of $X \in T_m\Sigma$, respectively. Then for $\varsigma \in \Omega^1(\Sigma, E^{\mathrm{ad}})$ the isomorphism $\mathfrak{I}(\bar{\varsigma}) \in \Omega^1(\Sigma, T\Sigma)$ is explicitly given by

$$\begin{aligned}[\mathfrak{I}(\bar{\varsigma})]_m(X) =& \mathfrak{V} \circ \mathfrak{F} \circ \varsigma_m(X) = \mathfrak{V} \circ \mathfrak{F}[e, \bar{\varsigma}_e(\widetilde{X})] = \mathfrak{V}[e, f(\bar{\varsigma}_e(\widetilde{X}))] \\ =& e \cdot f(\bar{\varsigma}_e(\widetilde{X})) = \mathfrak{V} \circ \iota_e \circ f(\bar{\varsigma}_e(\widetilde{X})).\end{aligned}$$

On the one hand a $(1,1)$-tensor field $S \in \Omega^1(\Sigma, T\Sigma)$ is given and let be $\widetilde{S} := \mathfrak{I}^{-1}(S)$, then we have

$$\begin{aligned}\widetilde{S}_e(\widetilde{X}) :=& (\mathfrak{I}^{-1}(S))_e(\widetilde{X}) = (\mathfrak{F}^{-1} \circ \mathfrak{V}^{-1} \circ S)_e(\widetilde{X}) \\ =& \iota_e^{-1} \circ \mathfrak{F}^{-1} \circ \mathfrak{V}^{-1} \circ S_{\pi(e)}(\pi_* \widetilde{X}) \\ =& \sum_i g(S_{\pi(e)}(\pi_* \widetilde{X}), e_i) \iota_e^{-1} \circ \mathfrak{F}^{-1} \circ \mathfrak{V}^{-1}(e_i) \\ =& \sum_i g(S_{\pi(e)}(\pi_* \widetilde{X}), e_i) \iota_e^{-1}[e, \Xi_i] = \sum_i g(S_{\pi(e)}(\pi_* \widetilde{X}), e_i) \Xi_i.\end{aligned}$$

Therefore the i-th component of $\widetilde{S}$ w.r.t. $\{\Xi_i\}_{1 \leq i \leq 3}$ obtains for all $e \in \mathrm{O}^+(\Sigma, g)$ and $\widetilde{X} \in T_e\mathrm{O}^+(\Sigma, g)$

$$\widetilde{S}_{i,e}(\widetilde{X}) = g(S_{\pi(e)}(\pi_* \widetilde{X}), e_i).$$

With respect to a local section $s = (s_1, \ldots, s_3): U \longrightarrow \mathrm{O}^+(\Sigma, g)$ on U, $S \in \Omega^1(U, T\Sigma)$ yields

$$S|_U = \sum_i S_i s_i \quad \in \Omega^1(U, T\Sigma),$$

where the real valued one form $S_i \in \Omega^1(U)$ is defined by $S_i := g(S, s_i)$. Thus the pullback $\widetilde{S}^s := s^*(\widetilde{S}) \in \Omega^1(U, \mathfrak{so}(3))$ on U of $\widetilde{S} := \mathfrak{I}^{-1}(S) = \mathfrak{F}^{-1} \circ \mathfrak{V}^{-1} \circ S \in \Omega^1_{\text{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$ with $s \in \Gamma(U, \mathrm{O}^+(\Sigma, g))$ is given by

$$\begin{aligned} \widetilde{S}^s_m(X) &= \widetilde{S}_{s(m)}(\mathrm{d}e_m X) = \sum_i (S_{\pi(s(m))}(\mathrm{d}\pi_{s(m)}\,\mathrm{d}e_m X), s_i)\Xi_i \\ &= \sum_i g(S_m(X), s_i)\Xi_i = \sum_i S_{i,m}(X)\Xi_i. \end{aligned} \tag{4.8}$$

Otherwise let be $\widetilde{S} \in \Omega^1_{\text{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$ in the basis $\{\Xi_i\}_{i=1,\dots,3}$ of $\mathfrak{so}(3)$ given, i.e.

$$\widetilde{S} = \sum_{i=1}^{3} \widetilde{S}_i \Xi_i$$

with $\widetilde{S}_i \in \Omega^1_{\text{hor}}(\mathrm{O}^+(\Sigma, g))$. Then $\mathfrak{I}(\widetilde{S}) \in \Omega^1(\Sigma, T\Sigma)$ takes the form

$$\begin{aligned} (\mathfrak{I}(\widetilde{S}))_m(X) =&\mathfrak{V}[e, f(\widetilde{S}_e(\widetilde{X}))] = \sum_i \widetilde{S}_{i,e}(\widetilde{X})\mathfrak{V}[e, f(\Xi_i)] \\ =&\sum_i \widetilde{S}_{i,e}(\widetilde{X})\mathfrak{V}[e, u_i] = \sum_i e_i \widetilde{S}_{i,e}(\widetilde{X}). \end{aligned}$$

Thus the components of $[\mathfrak{I}(\widetilde{S})]_m(X)$ with respect to the basis $\{e_i\}_{i=1,\dots,3}$ of $T_m\Sigma$ are given by

$$g([\mathfrak{I}(\widetilde{S})]_m(X), e_i) = \widetilde{S}_{i,e}(\widetilde{X}).$$

Induced vector product

In this Section we want to transfer the vector product on $\mathbb{R}^3$ by means of the metric to $T\Sigma$ as introduced in [33].

i.) The orientation of Σ provides every fiber $T_m\Sigma$ of $T\Sigma$ over $m \in \Sigma$ the structure of an oriented, 3-dimensional vector space. Whereas $\mathfrak{V}$ is an isomorphism of the vector bundle $T\Sigma$ and E^ρ, we get

$$\begin{array}{rcl} \mathfrak{V}_m : E^\rho_m = \mathrm{O}^+_m(\Sigma, g) \times_{(\mathrm{SO}(3),\rho)} \mathbb{R}^3 & \xrightarrow{\cong} & T_m\Sigma \\ [e, x] & \longmapsto & \sum_{i=1}^3 e_i x_i = e \cdot x. \end{array}$$

The fibres $T_m\Sigma$ are isomorphic to $\mathbb{R}^3$ and since the basis $e = \{e_i\}_{i=1,\dots,3} \in \mathrm{O}^+_m(\Sigma, g)$ of $T_m\Sigma$ is positive oriented, we can also regard the basis $u := \{u_i\}_{i=1,\dots,3}$ of $\mathbb{R}^3$ as positive oriented, where we used $\mathfrak{V}_m[e, u_i] = e_i$. Therefore the fiber $T_m\Sigma \cong \mathbb{R}^3$ together with the inner product induced by g

$$\langle u_i, u_j\rangle := g_m(e_i, e_j) = \delta_{ij}$$

yields an oriented euclidean space, which is equipped with a Lie algebra, namely the vector product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$u_i \times u_j := \sum_k \epsilon_{ijk} u_k.$$

Thus we are able to define on the fibres of E^ρ the following product structure:

Definition 4.1.3. *Let $m \in \Sigma$. On the fibres E^ρ_m the product structure is defined by*

$$\cdot : E^\rho_m \times E^\rho_m \longrightarrow E^\rho_m,$$

with $[e, x] \cdot [e, y] := [e, x \times y]$.

ii.) Due to Eq. (4.2) the fibers of E^{ad} are isomorphic to $\mathfrak{so}(3)$, and therefore we obtain the Lie algebra structure in a natural way. The isomorphism $\mathfrak{f}{:}\mathfrak{so}(3) \longrightarrow \mathbb{R}^3$ preserves the Lie algebra structures on $\mathfrak{so}(3)$ respectively on $\mathbb{R}^3$. That implies that $(\mathfrak{so}(3), [\cdot,\cdot]_{\mathrm{SO}(3)}) \xrightarrow{\mathfrak{f}} (\mathbb{R}^3, \times)$. And thus we obtain

$$\mathfrak{f}([\Xi, \Pi]_{\mathfrak{so}(3)}) = \mathfrak{f}(\Xi) \times \mathfrak{f}(\Pi) \qquad \forall \Xi, \Pi \in \mathfrak{so}(3),$$

where $[\Xi, \Pi]_{\mathfrak{so}(3)}$ denotes the Lie algebra structure on $\mathfrak{so}(3)$. Thus for all $[e, \Xi], [e, \Pi] \in E^{\mathrm{ad}}_m$ and $m \in \Sigma$ the vector space isomorphism

$$\begin{array}{ccc} \mathfrak{F}_m{:}E^{\mathrm{ad}}_m & \xrightarrow{\cong} & E^\rho_m \\ [e, \Xi] & \longmapsto & [e, \mathfrak{f}(\Xi)] \end{array}$$

fulfills

$$\begin{aligned} \mathfrak{F}_m([e, [\Xi, \Pi]_{\mathfrak{so}(3)}]) =& [e, \mathfrak{f}([\Xi, \Pi]_{\mathfrak{so}(3)})] = [e, \mathfrak{f}(\Xi) \times \mathfrak{f}(\Pi)] \\ =& [e, \mathfrak{f}(\Xi)] \cdot [e, \mathfrak{f}(\Pi)] = \mathfrak{F}_m([e, \Xi]) \cdot \mathfrak{F}_m([e, \Pi]). \end{aligned} \tag{4.9}$$

Therefore ensuing from the product structures on E^ρ_m respectively E^{ad}_m the isomorphism $\mathfrak{V}_m$ respectively $\mathfrak{V}_m \circ \mathfrak{F}_m$ defines a product structure on $T_m\Sigma$ for all $m \in \Sigma$.

DEFINITION 4.1.4. *Let $m \in \Sigma$. We define $\bowtie\colon T_m\Sigma \times T_m\Sigma \to T_m\Sigma$ by*

$$\begin{aligned} X \bowtie Y :=& \mathfrak{V}_m(\mathfrak{V}_m^{-1}(X) \times \mathfrak{V}_m^{-1}(Y)) \\ =& (\mathfrak{V} \circ \mathfrak{F})_m([(\mathfrak{V} \circ \mathfrak{F})_m^{-1}(X) \circ (\mathfrak{V} \circ \mathfrak{F})_m^{-1}(Y)]_{\mathfrak{so}(3)}), \end{aligned}$$

where $X, Y \in T_m\Sigma$.

Summing up we obtain the following identifications:

$$E^{\mathrm{ad}}_m \overset{\mathfrak{F}_m}{\longleftrightarrow} E^\rho_m \overset{\mathfrak{V}_m}{\longleftrightarrow} T_m\Sigma, \qquad \left[e, [\Xi_i, \Xi_j]_{\mathfrak{so}(3)}\right] \longleftrightarrow [e, u_i \times u_j] \longleftrightarrow e_i \bowtie e_j.$$

DEFINITION 4.1.5. *(See [33]) Let $e = \{e_i\}_{i=1,\dots,3} \in \mathrm{O}^+_m(\Sigma, g)$ be an oriented, orthonormal basis of $T_m\Sigma$. Then we have $e_i \cdot e_j = \sum_k \epsilon_{ijk} e_k$. Therefore for any e the product structure on $T\Sigma$ is defined by*

$$\bowtie\colon T\Sigma \times T\Sigma \longrightarrow T\Sigma, \quad X \bowtie Y := \sum_{ijk} \epsilon_{ijk} X^i Y^j e_k, \tag{4.10}$$

for $X, Y \in T_m\Sigma$ given by $X = \sum_i X^i e_i$ and $Y = \sum_j Y^j e_j$, for $X^i, Y^j \in \mathbb{R}$.

PROPOSITION 4.1.6. *The product structure given by Eq. (4.10) is independent of the given basis of $T_m\Sigma$.*

Proof. Rewriting X and Y in the basis $\widetilde{e} = \{\widetilde{e}_i\}_{1 \leq i \leq 3} \in \mathrm{O}^+(\Sigma, g)$ of $T_m\Sigma$, $X = \sum_i \widetilde{X}^i \widetilde{e}_i$ and $Y = \sum_j \widetilde{Y}^j \widetilde{e}_j$ with $\widetilde{X}^i, \widetilde{Y}^j \in \mathbb{R}^3$, then there exists an element $A \in \mathrm{SO}(3)$ with $\widetilde{e} = e \bowtie A$. Due to

$$\begin{aligned} X &= \sum_i \widetilde{X}^i \widetilde{e}_i = \sum_{i,j} \widetilde{X}^i e_k A_{ki} = \sum_k X^k e_k = X, \\ Y &= \sum_j \widetilde{Y}^j \widetilde{e}_j = \sum_{j,l} \widetilde{Y}^j e_l A_{lj} = \sum_l Y^l e_l = Y \end{aligned}$$

we have

$$\sum_i A_{ki}\widetilde{X}^i = X^k \quad \Longleftrightarrow \quad \widetilde{X}^i = \sum_k A_{ki} X^k,$$
$$\sum_j A_{lj}\widetilde{Y}^j = Y^l \quad \Longleftrightarrow \quad \widetilde{Y}^j = \sum_j A_{lj} Y^l.$$

By using $\det(A) = 1$ and

$$\begin{array}{ccc} \sum_{i,j,m,k,l,n} A_{ki}A_{lj}A_{nm}\epsilon_{ijm}\epsilon_{kln} & & \sum_{i,j,m} A_{ki}A_{lj}A_{nm}\epsilon_{ijm} \\ & \Longleftrightarrow & \\ = 3!\det(A) & & = \det(A)\epsilon_{kln} \end{array}$$

we finally obtain

$$\sum_{i,j} \widetilde{X}^i\widetilde{Y}^j\widetilde{e}_i \bowtie \widetilde{e}_j = \sum_{i,j,m} \widetilde{X}^i\widetilde{Y}^j\epsilon_{ijm}\widetilde{e}_m = \sum_{i,j,m,k,l,n} A_{ki}X^kA_{lj}Y^l\epsilon_{ijm}e_nA_{nm}$$
$$= \sum_{k,l,n} \epsilon_{kln}X^kY^le_n = X \bowtie Y.$$

QED.

On sections of $T\Sigma$ the product structure is given by:

Definition 4.1.7. *The product structure on* $\Gamma(T\Sigma)$

$$\begin{array}{rcl} \bowtie : \Gamma(T\Sigma)\times\Gamma(T\Sigma) & \longrightarrow & \Gamma(T\Sigma) \\ (X,Y) & \longmapsto & X \bowtie Y \end{array}$$

is given by $(X \bowtie Y)(m) := X(m) \bowtie Y(m)$ *for all* $m \in \Sigma$.

Let X, Y be vector fields on Σ. Then we can rewrite X and Y locally on U with respect to the section $s = \{s_i\}_{i=1,\dots,3}: U \longrightarrow \mathrm{O}^+(\Sigma, g)$ as

$$X|_U = \sum_i X^i s_i \quad \text{and} \quad Y|_U = \sum_i Y^i s_i \quad \in \Gamma(U, T\Sigma),$$

where $X^i := g(X, s_i) \in \mathfrak{C}(U)$ and $Y^i := g(Y, s_i) \in \mathfrak{C}(U)$. We have

$$\mathfrak{V}[s, \bar{X}] = X|_U \quad \text{and} \quad \mathfrak{V}[s, \bar{Y}] = Y|_U,$$

where we have defined

$$\bar{X} := \sum_i X^i u_i \equiv \{X^i\}_{i=1,\dots,3}{:}U \longrightarrow \mathbb{R}^3,$$
$$\bar{Y} := \sum_i Y^i u_i \equiv \{Y^i\}_{i=1,\dots,3}{:}U \longrightarrow \mathbb{R}^3.$$

Ragarding analogously the definition

$$\widetilde{X} := \sum_i X^i \Xi_i \equiv \mathfrak{f}^{-1}(\bar{X}) : U \longrightarrow \mathfrak{so}(3),$$
$$\widetilde{Y} := \sum_i Y^i \Xi_i \equiv \mathfrak{f}^{-1}(\bar{Y}) : U \longrightarrow \mathfrak{so}(3)$$

so that on U the following applies

$$\mathfrak{F}[s, \widetilde{X}] = [s, \bar{X}] \quad \text{and} \quad \mathfrak{F}[s, \widetilde{Y}] = [s, \bar{Y}].$$

In summary we get the following correspondences:

$$\Gamma(U, E^{\mathrm{ad}}) \overset{\mathfrak{F}}{\longleftrightarrow} \Gamma(U, E^{\rho}) \overset{\mathfrak{V}}{\longleftrightarrow} \Gamma(U, T\Sigma),$$
$$\left[s, \left[\widetilde{X}, \widetilde{Y}\right]_{\mathfrak{so}(3)}\right] \longleftrightarrow \left[s, \bar{X} \times \bar{Y}\right] \longleftrightarrow X \bowtie Y.$$

As already seen in Proposition 4.1.6 the product structure is independent on the given basis. However, the product structure on $T\Sigma$ has even more interesting properties:

LEMMA 4.1.8. *(See [33]) Let X, Y and $Z \in \Gamma(T\Sigma)$ vector fields on Σ. Then we have*

i.) $X \bowtie Y = -Y \bowtie X$ *(antisymmetry);*

ii.) $g(X \bowtie Y, Z) = g(X, Y \bowtie Z)$*;*

iii.) $X \bowtie (Y \bowtie Z) = g(X, Z)Y - g(X, Y)Z$*;*

iv.) $\mathfrak{S}\{X \bowtie (Y \bowtie Z)\} = 0$ *(Jacobi's identity), where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z.*

Proof. See A.2.1. QED.

Connection forms on $\mathrm{O}^+(\Sigma, g)$

Let $s : U \longrightarrow \mathrm{O}^+(\Sigma, g)$ some orthonormal frame. Then we can pull back the connection form $\omega^{\mathrm{LC}} \in \Omega^1(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))$ of the Levi-Civita connection to U by s

$$\omega^{\mathrm{LC},s} := s^*\omega^{\mathrm{LC}} \in \Omega^1(U, \mathfrak{so}(3))).$$

And in addition let $\{\Lambda_{ij}\}_{1\leq i \leq j \leq 3}$ be a basis of $\mathfrak{so}(3)$, given by the 3×3 matrices

$$\Lambda_{ij} = E_{ji} - E_{ij} \in \mathfrak{so}(3), \tag{4.11}$$

in particular we have

$$\Lambda_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_{13} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Lambda_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The relation between the basis $\{\Lambda_{ij}\}$ of $\mathfrak{so}(3)$ and the basis $\{\Xi_i\}_{1\leq i\leq 3}$ given by $\{\Xi_i\}_{jk} := \epsilon_{ijk}$ is given by $\Lambda_{ij} = \sum_k \epsilon_{ijk}\Xi_k$. With respect to Eq. (4.11) we are able to rewrite $\omega^{\mathrm{LC},s}$ as

$$\omega^{\mathrm{LC},s} = \sum_{i<j} g(\nabla^{\mathrm{LC}} s_i, s_j)\Lambda_{ij}. \tag{4.12}$$

Then it is easy to check that Eq. (4.12) yields

$$\omega^{\mathrm{LC},s} = \sum_{i<j,k} \epsilon_{ijk} g(\nabla^{\mathrm{LC}} s_i, s_j)\Xi_k = \frac{1}{2}\sum_{ijk} \epsilon_{ijk} g(\nabla^{\mathrm{LC}} s_i, s_j) = \sum_k \Gamma^e_k \Xi_k, \tag{4.13}$$

where we have defined

$$\Gamma^s_k := \frac{1}{2}\sum_{ij} \epsilon_{ijk} g(\nabla^{\mathrm{LC}} s_i, s_j) \in \Omega^1(U). \tag{4.14}$$

Γ^s_k denotes the local expression of the k-th component $\Gamma_k \in \Omega^1(\mathrm{O}^+(\Sigma, g))$ of ω^{LC} with respect to the basis $\{\Xi_k\}_{1\leq k\leq 3}$ of $\mathfrak{so}(3)$. And we have

$$\omega^{\mathrm{LC}} = \sum_k \Gamma_k \Xi_k \qquad \text{and} \qquad \Gamma^s_k = s^*\Gamma_k.$$

As a central result we state the following Definition and get the following Theorem, as first mentioned in [33].

THEOREM/DEFINITION 4.1.9. *(See [33]) An arbitrary connection form on* $\mathrm{O}^+(\Sigma, g)$ *is defined by*

$$A := \omega^{\mathrm{LC}} + \widetilde{S} \in \Omega^1(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})},$$

where $S \in \Omega^1(\Sigma, T\Sigma)$ *is an arbitrary* $(1,1)$*-tensor field and* $\widetilde{S} := \mathfrak{I}^{-1}(S) \in \Omega^1_{\mathrm{hor}}(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$ *denotes the corresponding horizontal 1-form of type* $(\mathrm{SO}(3), \mathrm{ad})$.

THEOREM 4.1.10. *(See [33]) Let* $X, Y \in \Gamma(T\Sigma)$*. With respect to the connection* $A := \omega^{\mathrm{LC}} + \widetilde{S}$ *the covariant derivative*

$$\nabla^{\mathrm{A}}{:}\Gamma(T\Sigma) \longrightarrow \Gamma(T^*\Sigma \otimes T\Sigma)$$

is given by

$$\nabla^{\mathrm{A}}_X Y := \nabla^{\mathrm{LC}}_X Y + S(X) \bowtie Y.$$

Proof. Consider locally the connection form $A \in \Omega^1(\mathrm{O}^+(\Sigma, g), \mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}$ w.r.t an orthonormal, oriented basis section $s{:}U \longrightarrow \mathrm{O}^+(\Sigma, g)$. By Eq. (4.8) we obtain

$$\widetilde{S}^s := s^*(\widetilde{S}) = \sum_k S^s_k \Xi_k \in \Gamma^1(U, \mathfrak{so}(3)),$$

whereas $S^s_k \in \Omega^1(U)$ are the components of S in the basis $\{s_i\}_{1 \le i \le 3}$, namely

$$S^s_{k.m}(X) = g_m(S_m(X), s_k) \quad \textit{for all} \quad X \in \Gamma(U, T\Sigma), m \in U.$$

Thus we have

$$A^s = \omega^s + \widetilde{S}^s = \sum_k (\Gamma^s_k + \widetilde{S}^s_k)\Xi_k.$$

Let $X \in \Gamma(T\Sigma)$ and using

$$\Xi_k u_l = \sum_{m,n} \{\Xi_k\}_{mn} \{u_l\}_n u_m = \sum_{m,n} \epsilon_{knm} \delta_{ln} u_m = \sum_m \epsilon_{klm} u_m,$$

then we obtain

$$\begin{aligned}
\rho_*(\omega^s(X)) u_l &= \sum_k \Gamma^s_k(X) \Xi_k u_l = \sum_{k,m} \epsilon_{klm} \Gamma^s_k(X) u_m, \\
\rho_*(\widetilde{S}^s(X)) u_l &= \sum_k \widetilde{S}^s_k(X) \Xi_k u_l = \sum_{k,m} \epsilon_{klm} \widetilde{S}^s_k(X) u_m.
\end{aligned}$$

Additionally let be

$$\nabla^{\mathrm{A},E^\rho}:\Gamma(E^\rho) \longrightarrow \Gamma(T^*\Sigma \otimes E^\rho) \tag{4.15}$$

the covariant derivative on $E^\rho = \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3),\rho)}$ induced by $A \in \mathcal{C}(\mathrm{O}^+(\Sigma, g))$. The set $\{[s, u_l]\}_{1 \leq l \leq 3}$ forms the basis of local sections $\Gamma(U, E^\rho)$ in E^ρ. On purpose to determine the covariant derivative of $[s, u_l]$ in the direction of the vector field X, we calculate

$$\begin{aligned}
[s, \rho_*(\omega^s(X)) u_l] &= \sum_{k,m} \epsilon_{klm} \Gamma^s_k(X) [s, u_m] \\
&= \frac{1}{2} \sum_{m,i,j} \sum_k \epsilon_{klm} \epsilon_{ijk} g(\nabla^{\mathrm{LC}}_X s_i, s_j) [s, u_m] \\
&= \frac{1}{2} \sum_m (g(\nabla^{\mathrm{LC}}_X s_l, s_m) - (g(\nabla^{\mathrm{LC}}_X s_m, s_l)) [s, u_m] \\
&= [s, \sum_m (g(\nabla^{\mathrm{LC}}_X s_l, s_m) u_m]
\end{aligned} \tag{4.16}$$

and

$$
\begin{aligned}
[s, \rho_*(\widetilde{S}^s(X))u_l] &= [s, \sum_{k,m} \epsilon_{klm} S_k(X) u_m = [s, \sum_{k,m} \epsilon_{klm} g(S(X), s_k) u_m] \\
&= [s, \sum_{k,m} \epsilon_{klm} \langle \bar{S}(X), u_k \rangle u_m] = [s, \sum_{k} \langle \bar{S}(X), u_k \rangle u_k \times u_l] \\
&= [s, \bar{S}(X) \times u_l],
\end{aligned}
\tag{4.17}
$$

where $\bar{S} \in \Omega^1(U, \mathbb{R}^3)$ is given by

$$
\mathfrak{V}^{-1}(S(X)) = [s, \bar{S}(X)] \quad \forall \quad X \in \Gamma(U, T\Sigma).
$$

Using Eq. (4.16) and Eq. (4.17), we obtain

$$
\begin{aligned}
\nabla_X^{\mathrm{A}, E^\rho}[s, u_l] :=& [s, \rho_*(A^s(X))u_l] = [s, \rho_*(\omega^s(X) + \widetilde{S}^s(X))u_l] \\
=& [s, \sum_m (g(\nabla_X^{\mathrm{LC}} s_l, s_m) u_m] + [s, \bar{S}(X) \times u_l] \\
=& [s, \sum_m (g(\nabla_X^{\mathrm{LC}} s_l, s_m) + \sum_k \epsilon_{klm} g(S(X), s_k)) u_m].
\end{aligned}
$$

By the isomorphism $\mathfrak{V}$, see Eq. (4.4), $\nabla^{\mathrm{A}, E^\rho}$ induces the the covariant derivative ∇^{A} on $T\Sigma$ by

$$
\begin{aligned}
\nabla^{\mathrm{A}} : \Gamma(T\Sigma) &\longrightarrow \Gamma(T^*\Sigma \otimes T\Sigma) \\
Y &\longrightarrow \mathfrak{V}^{-1}(\nabla^{\mathrm{A}, E^\rho} \mathfrak{V}(Y)).
\end{aligned}
$$

Thus for the local vector field $s_l \in \Gamma(U, T\Sigma)$ we get

$$
\begin{aligned}
\nabla_X^{\mathrm{A}} s_l &= \mathfrak{V}^{-1}(\nabla^{\mathrm{A}, E^\rho} \mathfrak{V}(s_l)) = \mathfrak{V}^{-1}(\nabla^{\mathrm{A}, E^\rho}[s, u_l]) \\
&= \mathfrak{V}^{-1} \left[\sum_m \{ g(\nabla_X^{\mathrm{LC}} s_l, s_m) + \sum_k \epsilon_{klm} g(S(X), s_k) \} [s, u_m] \right] \\
&= \sum_m g(\nabla_X^{\mathrm{LC}} s_l, s_m) s_m + \sum_{m,k} \epsilon_{klm} g(S(X), s_k) e_m \\
&= \nabla_X^{\mathrm{LC}} s_l + \sum_k g(S(X), s_k) s_k \bowtie s_l = \nabla_X^{\mathrm{LC}} s_l + S(X) \bowtie s_l.
\end{aligned}
$$

Due to the fact that we can evolve every $Y \in \Gamma(T\Sigma)$ in the basis $\{s_i\}_{1\leq i\leq 3}$, we obtain generally

$$\nabla^{\mathrm{A}}_X Y := \nabla^{\mathrm{LC}}_X Y + S(X) \bowtie Y.$$

QED.

Proposition 4.1.11. *(See [33]) Let X, Y and $Z \in \Gamma(T\Sigma)$ are vector fields on Σ. Then we have*

i.) $\nabla^{\mathrm{LC}}_Z(X \bowtie Y) = (\nabla^{\mathrm{LC}}_Z X) \bowtie Y + X \bowtie (\nabla^{\mathrm{LC}}_Z Y)$ *and*

ii.) $\nabla^{\mathrm{A}}_Z(X \bowtie Y) = (\nabla^{\mathrm{A}}_Z X) \bowtie Y + X \bowtie (\nabla^{\mathrm{A}}_Z Y)$.

Proof. See A.2.2. QED.

Theorem 4.1.12. *(See [33])* ∇^{A} *is metric.*

Proof. See A.2.3. QED.

Curvature

The Riemannian curvature tensor R^{LC} respectively the torsion T^{LC} in terms of the covariant differentiation ∇^{LC} w.r.t the Levi-Civita connection is expressed as $R^{\mathrm{LC}}(X,Y)Z = [\nabla^{\mathrm{LC}}_X, \nabla^{\mathrm{LC}}_Y]Z - \nabla^{\mathrm{LC}}_{[X,Y]}Z$ and $T^{\mathrm{LC}}(X,Y) = \nabla^{\mathrm{LC}}_X Y - \nabla^{\mathrm{LC}}_Y X - [X,Y]$. Now we want to determine this objects w.r.t an arbitrary connection as constructed in Proposition 4.1.9, which originally implemented by [33].

Theorem 4.1.13. *(See [33]) With respect to ∇^{A} the torsion $T^{\mathrm{A}} \in (\Lambda^2 T^*\Sigma)$ can expressed as follows*

$$T^{\mathrm{A}}(X,Y) = S(X) \bowtie Y - S(Y) \bowtie X$$

for all X, Y in $\Gamma(T\Sigma)$.

Proof. See A.2.4. QED.

THEOREM 4.1.14. *(See [33]) With respect to* ∇^{A} *the Riemannian curvature* $R^{\mathrm{A}} \in \Gamma(\Lambda^2 T^*\Sigma \otimes \mathrm{End}(T\Sigma))$ *is given by*

$$R^{\mathrm{A}}(X,Y)Z = R^{\mathrm{LC}}(X,Y)Z + [(\nabla^{\mathrm{LC}}_X S)(Y) - (\nabla^{\mathrm{LC}}_Y S)(X) + S(X) \bowtie S(Y)] \bowtie Z$$

for all X, Y, Z *in* $\Gamma(T\Sigma)$ *on* Σ*.* $R^{\mathrm{LC}} \in \Gamma(\Lambda^2 T^*\Sigma \otimes \mathrm{End}(T\Sigma))$ *denotes the curvature of the Levi-Civita connection* ∇^{LC}*.*

Proof. See A.2.5.

QED.

Clearly R^{A} satisfies the following symmetries, which are generally valid for R^{LC}.

LEMMA 4.1.15. *(See [33]) For all* X, Y, Z *and* $V \in \Gamma(T\Sigma)$ *the curvature tensor* $R^{\mathrm{A}} \in \Gamma(\Lambda^2 T^*\Sigma \otimes \mathrm{End}(T\Sigma))$ *has the following symmetries:*

i.) $R^{\mathrm{A}}(X,Y) = -R^{\mathrm{A}}(Y,X)$

ii.) $g(R^{\mathrm{A}}(X,Y)Z, V) = -g(R^{\mathrm{A}}(X,Y)V, Z)$*.*

Proof. See A.2.6. QED.

But we want to continue with the following remark:

REMARK 4.1.16. *The additional symmetry properties of the curvature tensor w.r.t. the Levi-Civita connection, namely*

$$g(R^{\mathrm{LC}}(X,Y)Z, W) = g(R^{\mathrm{LC}}(Z,W)X, Y) \quad \textit{for all} \quad X, Y, Z, W \in \Gamma(T\Sigma)$$

and

$$\mathfrak{S}\{R^{\mathrm{LC}}(X,Y)Z\} = 0 \quad \textit{for all} \quad X, Y, Z \in \Gamma(T\Sigma) \quad \textit{(Bianchi's 1st identity)}$$

are not fulfilled by R^{A}, since in general w.r.t. ∇^{A} the torsion $T^{\mathrm{A}} \neq 0$. In the latter equation $\mathfrak{S}$ denotes the cylclic sum with respect to X, Y, Z.

In what follows we want to complete and to round up the discussion of [33]. In the case of R^{A} we find the generalized Bianchi identities.

THEOREM 4.1.17. *(generalized Bianchi identities.)*

Let $R^{\mathrm{A}} \in \Gamma(\Lambda^2 T^*\Sigma \otimes \mathrm{End}(T\Sigma))$ and $T^{\mathrm{A}} \in (\Lambda T^*\Sigma)$ be the curvature and the torsion of the connection $A := \omega^{\mathrm{LC}} + \widetilde{S}$. Then for all all vector fields X, Y and $Z \in \Gamma(T\Sigma)$ we have

(1) Bianchi's 1st identity

$$\begin{aligned}\mathfrak{S}\{R^{\mathrm{A}}(X,Y)Z\} =& \mathfrak{S}\{[S(X) \bowtie S(Y)] \bowtie Z \\ &+ \nabla^{\mathrm{LC}}_X T^{\mathrm{A}}(Y,Z) + T^{\mathrm{A}}(X,[Y,Z])\}.\end{aligned} \tag{4.18}$$

If $S{:}\Gamma(T\Sigma) \longrightarrow \Gamma(T\Sigma)$ is a symmetric operator, i.e. $g(S(X),Y) = g(X,S(Y))$ for all $X, Y \in \Gamma(T\Sigma)$, we have

$$\mathfrak{S}\{R^{\mathrm{A}}(X,Y)Z\} = \mathfrak{S}\{\nabla^{\mathrm{LC}}_X T^{\mathrm{A}}(Y,Z) + T^{\mathrm{A}}(X,[Y,Z])\};$$

(2) Bianchi's 2nd identity

$$\begin{aligned}\mathfrak{S}\{(\nabla^{\mathrm{A}}_Z R^{\mathrm{A}})(X,Y)\} &= -\,\mathfrak{S}\{R^{\mathrm{A}}(T^{\mathrm{A}}(X,Y),Z)\} \\ &= -\,\mathfrak{S}\{R^{\mathrm{A}}(S(X) \bowtie Y - S(Y) \bowtie X, Z)\},\end{aligned} \tag{4.19}$$

where $\mathfrak{S}$ is the cyclic sum with respect to X, Y, Z.

Proof. See A.2.7.

QED.

PROPOSITION 4.1.18. *With respect to the connection $A := \omega^{\mathrm{LC}} + \widetilde{S}$ respectively its covariant derivative ∇^{A}*

(a) *the Ricci tensor* $\mathrm{Ric}^{\mathrm{A}}$ *is given by*

$$\begin{aligned}\mathrm{Ric}^{\mathrm{A}}(Y,Z) =&\mathrm{Ric}^{\mathrm{LC}}(Y,Z)\\ &-\sum_i\{g(S(Y)\bowtie\nabla_{e_i}Z,e_i)\\ &\quad+g(S(\nabla_{e_i}Y)\bowtie Z,e_i)\\ &\quad+g((S(e_i)\bowtie Z),\nabla_Y e_i)\\ &\quad-g(S(e_i)\bowtie\nabla_Y Z,e_i)\\ &\quad-g(S(\nabla_Y e_i)\bowtie Z,e_i)\}\\ &+\sum_i\{g(Z,S(Y)g(S(e_i),e_i)-g(Z,S(e_i)g(S(Y),e_i)\},\end{aligned} \tag{4.20}$$

where $\mathrm{Ric}^{\mathrm{LC}}$ *is the Ricci tensor with respect to the Levi-Civita connection,*

(b) *and in addition the Ricci curvature scalar* $\mathcal{R}^{\mathrm{A}}$ *w.r.t the connection* A *is given by*

$$\mathcal{R}^{\mathrm{A}}=\mathcal{R}^{\mathrm{LC}}+\mathrm{tr}(S)^2-\mathrm{tr}(S^2), \tag{4.21}$$

where $\mathcal{R}^{\mathrm{LC}}$ *is the Ricci curvature scalar with respect to the Levi-Civita connection.*

Proof. See A.2.8. QED.

4.1.3. Step III: Ashtekar connection

Let be $\mathrm{Wein}\in\Gamma(T^*\Sigma\otimes T\Sigma)$ the Weingarten mapping on Σ. By replacing $S:=\beta\mathrm{Wein}$, where $\beta\in\mathbb{R}^*$, in the previous construction, we obtain the following expression:

Definition 4.1.19. *The Ashtekar connection w.r.t. Barbero-Immirzi parameter* β *is defined by*

$$A:=\omega^{\mathrm{LC}}+\beta\widetilde{\mathrm{Wein}}\in\Omega^1(\mathrm{O}^+(\Sigma,g),\mathfrak{so}(3))^{(\mathrm{SO}(3),\mathrm{ad})}, \tag{4.22}$$

where $\widetilde{\mathrm{Wein}} = \mathfrak{I}^{-1}(\mathrm{Wein})$ *and* $\beta \in \mathbb{R}^*$.

Theorem 4.1.20. *Let $X, Y \in \Gamma(T\Sigma)$. With respect to the Ashtekar connection $A := \omega^{\mathrm{LC}} + \beta\widetilde{\mathrm{Wein}}$ the covariant derivative*

$$\nabla^{\mathrm{A}} : \Gamma(T\Sigma) \longrightarrow \Gamma(T^*\Sigma \otimes T\Sigma)$$

is given by

$$\nabla^{\mathrm{A}}_X := \nabla^{\mathrm{LC}}_X Y + \beta\mathrm{Wein}(X) \bowtie Y.$$

Proof. Follows directly from Theorem 4.1.10. QED.

All statements as given in Sections 4.1.2 respectively 4.1.2 hold for the Ashtekar connection. We want to summarize:

Proposition 4.1.21. *∇_{A} is metric with torsion. With respect to the Ashtekar connection in terms of the covariant differentiation the torsion T and curvature R can expressed as follows:*

$$T^{\mathrm{A}}(X,Y) = \beta[\mathrm{Wein}(X) \bowtie Y - \mathrm{Wein}(Y) \bowtie X]$$

and

$$\begin{aligned} R^{\mathrm{A}}(X,Y)Z =& R^{\mathrm{LC}}(X,Y)Z + \beta[(\nabla^{\mathrm{LC}}_X \mathrm{Wein})(Y) - (\nabla^{\mathrm{LC}}_Y \mathrm{Wein})(X)] \bowtie Z \\ & + \beta^2[\mathrm{Wein}(X) \bowtie \mathrm{Wein}(Y)] \bowtie Z, \end{aligned} \tag{4.23}$$

where X, Y and $Z \in \Gamma(T\Sigma)$ are vector fields on Σ.

Proof. Follows directly from Theorem 4.1.13 and Theorem 4.1.14. QED.

Lemma 4.1.22. *For all X, Y, Z and $V \in \Gamma(T\Sigma)$ the curvature tensor R^{A} has the following symmetries:*

i.) $R^{\mathrm{A}}(X,Y) = -R^{\mathrm{A}}(Y,X)$

ii.) $g(R^{\mathrm{A}}(X,Y)Z,V) = -g(R^{\mathrm{A}}(X,Y)V,Z)$.

Proof. Follows directly from Lemma 4.1.15. QED.

PROPOSITION 4.1.23. *If $\mathcal{M}$ is of constant sectional curvature, then we have*

$$R^{\mathrm{A}}(X,Y)Z = R^{\mathrm{LC}}(X,Y)Z + \beta^2[\mathrm{Wein}(X) \bowtie \mathrm{Wein}(Y)] \bowtie Z$$

for all $X, Y, Z \in \Gamma(T\Sigma)$.

Proof. Follows directly from Corollary 2.2.77 of Section 2.2.3. QED.

PROPOSITION 4.1.24. *Let X, Y and $Z \in \Gamma(T\Sigma)$ are vector fields on Σ. Then ∇^{A} obeys the Leibniz rule*

$$\nabla^{\mathrm{A}}_Z(X \bowtie Y) = (\nabla^{\mathrm{A}}_Z X) \bowtie Y + X \bowtie (\nabla^{\mathrm{A}}_Z Y).$$

Proof. Follows directly from Proposition Proposition 4.1.11. QED.

THEOREM 4.1.25. *(generalized Bianchi identities.) For $R^{\mathrm{A}} \in \Gamma(\Lambda^2 T^*\Sigma \otimes \mathrm{End}(T\Sigma))$ and $T^{\mathrm{A}} \in (\Lambda T^*\Sigma)$ of the Ashtekar connection we have for all all vector fields X, Y and $Z \in \Gamma(T\Sigma)$*

(1) Bianchi's 1st identity

$$\mathfrak{S}\{R^{\mathrm{A}}(X,Y)Z\} = \mathfrak{S}\{\nabla^{\mathrm{LC}}_X T^{\mathrm{A}}(Y,Z) + T^{\mathrm{A}}(X,[Y,Z])\};$$

(2) Bianchi's 2nd identity

$$\begin{aligned}\mathfrak{S}\{(\nabla^{\mathrm{A}}_Z R^{\mathrm{A}})(X,Y)\} =& \mathfrak{S}\{R^{\mathrm{A}}(T^{\mathrm{A}}(X,Y),Z)\}\\ =& -\mathfrak{S}\{R^{\mathrm{A}}(\beta[\mathrm{Wein}(X) \bowtie Y\\ &- \mathrm{Wein}(Y) \bowtie X], Z)\},\end{aligned}$$

where $\mathfrak{S}$ is the cyclic sum with respect to X, Y, Z.

Proof. See proof of Theorem 4.1.17. QED.

Proposition 4.1.26. *With respect to the Ashtekar connection in terms of the covariant differentiation the Ricci tensor* Ric *and the Ricci curvature scalar $\mathcal{R}$ can expressed as follows:*

(a)

$$
\begin{aligned}
\mathrm{Ric}^{\mathrm{A}}(Y,Z) =& \mathrm{Ric}^{\mathrm{LC}} \\
& - \beta \sum_i \{ g(\mathrm{Wein}(Y) \bowtie \nabla_{e_i} Z, e_i) \\
& \qquad + g(\mathrm{Wein}(\nabla_{e_i} Y) \bowtie Z, e_i) \\
& \qquad + g((\mathrm{Wein}(e_i) \bowtie Z), \nabla_Y e_i) \\
& \qquad - g(\mathrm{Wein}(e_i) \bowtie \nabla_Y Z, e_i) \\
& \qquad - g(\mathrm{Wein}(\nabla_Y e_i) \bowtie Z, e_i)\} \\
& + \beta^2 \sum_i \{[g(Z, \mathrm{Wein}(Y) g(\mathrm{Wein}(e_i), e_i) \\
& \qquad - g(Z, \mathrm{Wein}(e_i) g(\mathrm{Wein}(Y), e_i)]\},
\end{aligned}
\tag{4.24}
$$

where $\mathrm{Ric}^{\mathrm{LC}}$ *is the Ricci tensor with respect to the Levi-Civita connection respectively*

(b)

$$\mathcal{R}^{\mathrm{A}} = \mathcal{R}^{\mathrm{LC}} + \beta^2[\mathrm{tr}(\mathrm{Wein})^2 - \mathrm{tr}(\mathrm{Wein}^2)], \tag{4.25}$$

where $\mathcal{R}^{\mathrm{LC}}$ *is the Ricci curvature scalar with respect to the Levi-Civita connection.*

Proof. See proof of Proposition 4.1.18. QED.

4.2. Physics notation

In the physics literature, the *Ashtekar connection* is given by its components in some coordinate system. In order to reproduce this, let $\{\Xi_i\}_{i=1,\ldots,3}$ be again a basis of $\mathfrak{so}(3)$ with $[\Xi_i, \Xi_j] = \epsilon_{ij}^k \Xi_k$, and let $\chi : U \longrightarrow \mathbb{R}^3$ be some chart for open $U \subseteq \Sigma$. Then we get a local basis $\{\delta_i\}_{i=1,\ldots,3}$ for the tangent space. Finally, choose some orthonormal frame $s : U \longrightarrow \mathrm{O}^+(\Sigma, g)$. Hiding the dependence on U, we get $s(u_i) =: e_i^a \partial_a$ with u_i being the i-th vector in the standard basis of $\mathbb{R}^3$. Then we can rewrite the components of Eq. (4.22) into

$$\begin{aligned}
A_a^{s,i} &:= A^{s,i}(\partial_a) = \omega^{\mathrm{LC};s,i}(\partial_a) + \beta \widetilde{\mathrm{Wein}}^{s,i}(\partial_a) \\
&=: \omega_a^{\mathrm{LC};s,i} + \beta \widetilde{\mathrm{Wein}}_a^{s,i} \\
&= \frac{1}{2} \sum_{i,j=1}^{3} \epsilon_{ijk} g(\nabla_{\partial_a}^{\mathrm{LC}} e_i, e_j) + \beta g(\mathrm{Wein}(\partial_a), e_i) \\
&= \frac{1}{2} \sum_{i,j=1}^{3} \epsilon_{ijk} \omega_{ji}^s(\partial_a) + \beta k(\partial_a, e_i) \\
&= \Gamma_a^{s,i} + \beta k(\partial_a, e^{bi}\partial_b) = \Gamma_a^{s,i} + \beta k(\partial_a, \partial_b) e^{bi} \\
&= \Gamma_a^{s,i} + k_{ab} e^{bi} = \Gamma_a^{s,i} + \beta k_a^i.
\end{aligned} \tag{4.26}$$

Thus by dropping the superscript s in Eq. (4.26), we have recovered the Ashtekar connection as given in Eq. (3.42) and [5, 6, 62, 69], respectively.

4.3. Spin Structure of the Ashtekar connection

In this Chapter we want to review spin structures and spin connections on globally hyperbolic space times with the intension to study the Ashtekar connection with respect to the choice of the underlaying structure group, in particular SO(3) or SU(2). Since we are working within the framework of a foliation of the globally hyperbolic space times $\mathcal{M}$ into spacelike Cauchy hypersurfaces Σ in particular we analyze the concept of spin structure on such 3-dimensional hypersurfaces, as given in [33].

Let $(\mathcal{M}, g)$ a 4-dimensional, orientated space time, where $g \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ is a pseudo-Riemannian metric with signature $(1,3)$. Consider the $\mathrm{SO}_0(1,3)$- principle fiber bundle $\mathrm{O}^+(\mathcal{M}, g)$ of all orientated, orthonormal frames on $\mathcal{M}$. Thereby $\mathrm{SO}_0(1,3)$ denotes the connection component of Id of SO(3), therefor the proper, orthochronous Lorentz group. In the following let $(S(\mathcal{M}), \Lambda)$ a spin structure on $\mathcal{M}$ as introduced in Definition 2.3.1 and $Z \in \Omega^1(\mathrm{O}^+(\mathcal{M}, g), \mathfrak{so}(1,3))$ a connection form on $\mathrm{O}^+(\mathcal{M}, g)$. Via the following diagram Z can be lifted into a connection form $\tilde{Z} \in \Omega^1(S(\mathcal{M}), \mathfrak{sl}(2, \mathbb{C}))$.

$$\begin{tikzcd}
TS(\mathcal{M}) \arrow[r, "\tilde{Z}"] \arrow[d, "\Lambda_*"] \arrow[dr, "\Lambda^*(Z)"] & \mathfrak{sl}(2,\mathbb{C}) \arrow[d, "\lambda_*"] \\
T\mathrm{O}^+(\mathcal{M}, g) \arrow[r, "Z"] & \mathfrak{so}(1,3)
\end{tikzcd}$$

where $\Lambda^*(Z) \in \Omega^1(S(\mathcal{M}), \mathfrak{so}(1,3))$ is the pull back of Z by Λ to $TS(\mathcal{M})$. Since $\lambda_* : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$ is a Lie algebra isomorphism, we obtain

$$\tilde{Z} \in \Omega^1(S(\mathcal{M}), \mathfrak{so}(1,3)) \qquad \text{with} \qquad \lambda_* \circ \tilde{Z} = \Lambda^*(Z). \tag{4.27}$$

Theorem/Definition 4.3.1. *Let $\tilde{Z} \in \Omega^1(S(\mathcal{M}), \mathfrak{sl}(2,\mathbb{C}))$ a connection form on $S(\mathcal{M})$. If $Z \in \Omega^1(\mathrm{O}^+(\mathcal{M}, g), \mathfrak{so}(1,3))$ is the Levi-Civita connection form, then the connection corresponding to $\tilde{Z}$ is called spin connection .*

Proof. Let $\mathrm{Ad} : \mathrm{SO}_0(1,3) \longrightarrow \mathrm{GL}(\mathfrak{so}(1,3))$ resp. $\widetilde{\mathrm{Ad}} : \mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(\mathfrak{sl}(2,\mathbb{C}))$ the adjoint actions of $\mathrm{SO}_0(1,3)$ resp. $\mathrm{SL}(2,\mathbb{C})$. Since $Z \in \Omega^1(\mathrm{O}^+(\mathcal{M},g)\mathfrak{so}(1,3))$ is a connection form, we obtain

i.) $Z(X^B) = B \quad \forall \quad B \in \mathfrak{so}(1,3)$, whrereas $X^B \in \Gamma(T\mathrm{O}^+(\mathcal{M},g))$ denotes the fundamental vectorfield given by B.

ii.) $\mu_A^*(Z) = \mathrm{Ad}(A^{-1})Z \quad \forall \quad A \in \mathrm{SO}_0(1,3)$.

Now we have to proof the corresponding equations with respect to $\tilde{Z}$:

- Consider the conjugationsmappings on $\mathrm{SO}_0(1,3)$ and $\mathrm{SL}(2,\mathbb{C})$

$$\begin{aligned} C_A : \mathrm{SO}_0(1,3) &\longrightarrow \mathrm{SO}_0(1,3), \quad A \in \mathrm{SO}_0(1,3) \\ B &\longmapsto ABA^{-1}, \\ \widetilde{C}_g : \mathrm{SL}(2,\mathbb{C}) &\longrightarrow \mathrm{SL}(2,\mathbb{C}), \quad g \in \mathrm{SL}(2,\mathbb{C}) \\ h &\longmapsto ghg^{-1}. \end{aligned}$$

  Then we have

$$\begin{aligned} \mathrm{Ad}(A) &= (\mathrm{d}C_A)_{\mathrm{Id}} : T_{\mathrm{Id}}\mathrm{SO}_0(1,3) \cong \mathfrak{so}(1,3) \longrightarrow \mathfrak{so}(1,3), \\ \widetilde{\mathrm{Ad}}(g) &= (\mathrm{d}\tilde{C}_g)_{\widetilde{\mathrm{Id}}} : T_{\widetilde{\mathrm{Id}}}\mathrm{SL}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{SL}(2,\mathbb{C}), \end{aligned}$$

  where $\mathrm{Id} \in \mathrm{SO}_0(1,3)$ resp. $\widetilde{\mathrm{Id}} \in \mathrm{SL}(2,\mathbb{C})$ denotes the corresponding identities. For $g, h \in \mathfrak{sl}(2,\mathbb{C})$ we have

$$\lambda \circ \widetilde{C}_g(h) = \lambda(ghg^{-1}) = \lambda(g)\lambda(h)\lambda(g^{-1}) = C_{\lambda(g)}(\lambda(h)).$$

  Thus we have $\lambda \circ \widetilde{C}_g = C_{\lambda(g)} \circ \lambda$ for all $g \in \mathfrak{sl}(2,\mathbb{C})$ and we obtain

$$\begin{aligned} \lambda_* \circ \widetilde{\mathrm{Ad}}(g) =& \mathrm{d}\lambda_{\widetilde{\mathrm{Id}}} \circ (\mathrm{d}\widetilde{C}_g)_{\widetilde{\mathrm{Id}}} = \mathrm{d}(\lambda \circ \widetilde{C}_g)_{\widetilde{\mathrm{Id}}} \stackrel{!}{=} \mathrm{d}(C_{\lambda(g)} \circ \lambda)_{\widetilde{\mathrm{Id}}} \\ =& (\mathrm{d}C_{\lambda(g)})_{\mathrm{Id}} \circ \mathrm{d}\lambda_{\widetilde{\mathrm{Id}}} = \mathrm{Ad}(\lambda(g)) \circ \lambda_* \end{aligned}$$

  for all $g \in \mathfrak{sl}(2,\mathbb{C})$. Hence we have

$$\mathrm{Ad}(\lambda(g^{-1})) \circ \lambda_* = \lambda_* \circ \widetilde{\mathrm{Ad}}(g^{-1}) \tag{4.28}$$

  for all $g \in \mathrm{SL}(2,\mathbb{C})$.

- For all $m \in \mathcal{M}$ and arbitrary elements in the repective fibres over m $e \in \mathrm{O}^+_m(\mathcal{M}, g)$ and $\tilde{e} \in S(\mathcal{M})_m$ the group actions μ on $\mathrm{O}^+(\mathcal{M}, g)$ and $\tilde{\mu}$ on $S(\mathcal{M})$give fibre isomorphisms by

$$\begin{array}{rcl} \mu_e : \mathrm{SO}_0(1,3) & \xrightarrow{\cong} & \mathrm{O}^+_m(\mathcal{M}, g) \\ A & \longmapsto & \mu(e, A) \end{array}$$

and

$$\begin{array}{rcl} \tilde{\mu}_{\tilde{e}} : \mathrm{SL}(2,\mathbb{C}) & \xrightarrow{\cong} & S(\mathcal{M})_m \\ g & \longmapsto & \tilde{\mu}(\tilde{e}, g) \end{array}$$

such that the following diagramm commutes:

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{C}) \arrow[r, "\tilde{\mu}_{\tilde{e}}"] \arrow[d, "\lambda"] & S(\mathcal{M})_m \arrow[d, "\Lambda"] \\ \mathrm{SO}_0(1,3) \arrow[r, "\mu_{\Lambda(\tilde{e})}"] & \mathrm{O}^+(\mathcal{M}, g). \end{tikzcd}$$

This induces the following commuting diagram:

$$\begin{tikzcd} \mathfrak{sl}(2,\mathbb{C}) \arrow[r, "(\mathrm{d}\tilde{\mu}_{\tilde{e}})_{\widetilde{\mathrm{Id}}}"] \arrow[d, "\lambda_*"] & T_{\tilde{e}} S(\mathcal{M}) \arrow[d, "\mathrm{d}\Lambda_{\tilde{e}}"] \\ \mathfrak{so}(1,3) \arrow[r, "(\mathrm{d}\mu_{\Lambda(\tilde{e})})_{\mathrm{Id}}"] & T_{\Lambda(\tilde{e})}\mathrm{O}^+(\mathcal{M}, g). \end{tikzcd}$$

Let $h \in \mathfrak{sl}(2,\mathbb{C})$ and $X^h \in \Gamma(TS(\mathcal{M}))$ the corresponding vector field given by $X^h_p = (\mathrm{d}\tilde{\mu}_p)_{\widetilde{\mathrm{Id}}} h$. We obtain:

$$\begin{aligned} \widetilde{Z}_{\tilde{e}}(X^h) &= \lambda_*^{-1} \circ (\Lambda^*(Z))_{\tilde{e}}(X^h_{\tilde{e}}) = \lambda_*^{-1} \circ Z_{\Lambda(\tilde{e})}(\mathrm{d}\Lambda_{\tilde{e}} \circ \mathrm{d}\tilde{\mu}_{\tilde{e}} h) \\ &= \lambda_*^{-1} \circ Z_{\Lambda(\tilde{e})}(\mathrm{d}\mu_{\Lambda(\tilde{e})} \circ \lambda_* h) = \lambda_*^{-1} \circ Z_{\Lambda(\tilde{e})}(X^{\lambda_* h}_{\Lambda(\tilde{e})}) \\ &= \lambda_*^{-1} \circ \lambda_* h = h, \end{aligned}$$

in the forth step we have used the fact, that $\mathrm{d}\mu_{\Lambda(\tilde{e})} \circ \lambda_* h$ is a fundamnetal vector field of $\lambda_* h$.

- Due to

$$\Lambda \circ \tilde{\mu}_h(\tilde{e}) = \Lambda \circ \tilde{\mu}(\tilde{e}, h) = \mu(\Lambda(\tilde{e}), \lambda(h)) = \mu_{\lambda(h)}(\Lambda(\tilde{e}))$$

for all $\tilde{e} \in S(\mathcal{M})$ and $h \in \mathfrak{sl}(2, \mathbb{C})$ the following diagrams commutes:

$$\begin{tikzcd} S(\mathcal{M}) \arrow[r, "\tilde{\mu}_h"] \arrow[d, "\Lambda"] & S(\mathcal{M}) \arrow[d, "\Lambda"] \\ \mathrm{O}^+(\mathcal{M}, g) \arrow[r, "\mu_{\lambda(h)}"] & \mathrm{O}^+(\mathcal{M}, g) \end{tikzcd}$$

and

$$\begin{tikzcd} T_{\tilde{e}}S(\mathcal{M}) \arrow[r, "(\mathrm{d}\tilde{\mu}_h)_{\tilde{e}}"] \arrow[d, "\mathrm{d}\Lambda_{\tilde{e}}"] & T_{\tilde{e}\circ h}S(\mathcal{M}) \arrow[d, "\mathrm{d}\Lambda_{\tilde{e}\circ h}"] \\ T_{\Lambda(\tilde{e})}\mathrm{O}^+(\mathcal{M}, g) \arrow[r, "(\mathrm{d}\mu_{\lambda(h)})_{\Lambda(\tilde{e})}"] & T_{\Lambda(\tilde{e}\circ h)}\mathrm{O}^+(\mathcal{M}, g) \end{tikzcd}$$

Therefore we finally obtain for all $X \in T_pS(\mathcal{M})$:

$$\begin{aligned}
(\tilde{\mu}_h^*(\tilde{Z}))_p(X) =& \tilde{Z}_{p\circ h}((\mathrm{d}\tilde{\mu}_h)_pX) = \lambda_*^{-1} \circ (\Lambda^*(Z))_{p\circ h}((\mathrm{d}\tilde{\mu}_h)_pX) \\
=& \lambda_*^{-1} \circ Z_{\Lambda(p\circ h)}(\mathrm{d}\Lambda_{p\circ h}(\mathrm{d}\tilde{\mu}_h)_pX) \\
=& \lambda_*^{-1} \circ Z_{\Lambda(p)\circ\lambda(h)}(\mathrm{d}(\mu_{\lambda(h)})_{\Lambda(p)} \circ \mathrm{d}\Lambda_pX) \\
=& \lambda_*^{-1} \circ (\mu_{\lambda(h)}^*(Z))_{\Lambda(p)}(\mathrm{d}\Lambda_pX) \\
=& \lambda_*^{-1} \circ \mathrm{Ad}(\lambda(h)^{-1}) \circ Z_{\Lambda(p)}(\mathrm{d}\Lambda_pX) \\
=& \widetilde{\mathrm{Ad}}(h^{-1}) \circ \lambda_*^{-1} \circ (\Lambda^*(Z))_p(X) = \widetilde{\mathrm{Ad}}(h^{-1}) \circ \tilde{Z}_p(X),
\end{aligned}$$

where we have used $\mathrm{d}\Lambda_{p\circ h}(\mathrm{d}\tilde{\mu}_h)_p = \mathrm{d}(\mu_{\lambda(h)})_{\Lambda(p)} \circ \mathrm{d}\Lambda_p$ and Eq. (4.28).

QED.

Analogously we can introduce the concept of spin structure on orientated Riemannian manifolds. Let again $\Sigma \subset \mathcal{M}$ a spacelike Cauchy hypersurface.

Definition 4.3.2. *A spin structure on* $(\Sigma, q = g|_\Sigma)$ *is a pair* $(S(\Sigma), \Lambda)$ *consisting of*

i.) *a* $\mathrm{SU}(2)$ *principle fibre bundle* $(S(\Sigma), \tilde{\pi}, \Sigma; \mathrm{SU}(2))$ *over* Σ*,*

ii.) *an a double cover* $\Lambda : S(\Sigma) \to \mathrm{O}^+(\Sigma, q)$ *such that the following diagram commutes.*

$$\begin{tikzcd}
S(\Sigma) \times \mathrm{SU}(2) \arrow[r, "\tilde{\mu}"] \arrow[d, "\Lambda\times\lambda"] & S(\Sigma) \arrow[d, "\Lambda"] \arrow[dr, "\tilde{\pi}"] & \\
 & & \Sigma \\
\mathrm{O}^+(\Sigma, q) \times \mathrm{SO}(3) \arrow[r, "\mu"] & \mathrm{O}^+(\Sigma, q) \arrow[ur, "\pi"] &
\end{tikzcd}$$

where $\lambda : \mathrm{SU}(2) \to \mathrm{SO}(3)$ *denotes the universal cover of* $\mathrm{SO}(3)$*. In the rows the respective group action of the principle bundles* $S(\Sigma)$ *and* $\mathrm{O}^+(\Sigma, q)$ *is indicated.*

Remark 4.3.3. *The concept of spin structure can be generalized to arbitrary dimensional pseudo-Riemannian manifolds. Therefor the introduction of the spin group* $\mathrm{Spin}(q, p)$*, as universal covering group of* $\mathrm{SO}_0(p, q)$ *in arbitrary dimensions* $n = p + q$ *is necessary, see [18]. But we want to point out, that such an introduction is not required in the context of the Ashtekar connection. There are the following isomorphism:*

$$\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Spin}(1, 3) \qquad \textit{and} \qquad \mathrm{SU}(2) \cong \mathrm{Spin}(3).$$

Let $(S(\Sigma), \Lambda)$ be a fixed spin structure on Σ. Analogously to (4.27) we can define to every connection form $\omega \in \Omega^1(\mathrm{O}^+(\Sigma, q), \mathfrak{so}(3))$ a connection form $\tilde{\omega} \in \Omega^1(S(\Sigma), \mathfrak{su}(2))$ by

$$\lambda_* \circ \tilde{\omega} = \Lambda^*(w).$$

Corresponding to Theorem/Definition 4.3.1 we get the following definition.

THEOREM/DEFINITION 4.3.4. *Let* $\tilde{\omega} \in \Omega^1(S(\Sigma), \mathfrak{su}(2))$ *a connection form on* $S(\Sigma)$. *If* $\omega \in \Omega^1(O^+(\mathcal{M}, g), \mathfrak{so}(1,3))$ *is the Levi-Civita-connection form, then the connection* $\tilde{\omega}$ *is called spin connection.*

Proof. See proof of Theorem/Definition 4.3.1. QED.

THEOREM 4.3.5.

$$T\mathcal{M} \cong S(\Sigma) \times_{(\mathrm{SU}(2), \rho\circ\lambda)} \mathbb{R}^3.$$

Proof. According to Example 2.2.25, we have $T\mathcal{M} \overset{\Phi}{\cong} \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO},\rho)} \mathbb{R}^3$ it suffices to indicate an isomorphisms

$$\mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO},\rho)} \mathbb{R}^3 \cong S(\Sigma) \times_{(\mathrm{SU}(2),\rho\circ\lambda)} \mathbb{R}^3$$

The isomorphism is explicitly given by

$$\begin{aligned} \mathfrak{N} : S(\Sigma) \times_{(\mathrm{SU}(2),\rho\circ\lambda)} \mathbb{R}^3 &\longrightarrow \mathrm{O}(\Sigma, g) \times_{(\mathrm{SO}(3),\rho)} \mathbb{R}^3 \\ [\widetilde{e}, x] &\longmapsto [\Lambda(\widetilde{e}, x]. \end{aligned}$$

Next, we want to consider the isomorphism $\mathfrak{N}$ in detail.

i.) $\mathfrak{N}$ is well defined, because we have:

$$\begin{aligned} \mathfrak{N}[\widetilde{e} \circ g, \rho(\lambda(g^{-1}))x] =& [\Lambda(\widetilde{e} \circ g), \rho(\lambda(g^{-1}))x] \\ =& [\lambda(\widetilde{e}) \circ \Lambda(g), \rho(\lambda(g^{-1}))x] \\ =& [\Lambda(\widetilde{e}, x] = \mathfrak{N}[\widetilde{e}, x]. \end{aligned}$$

ii.) $\mathfrak{N}$ is surjective, due to: Let $[e, x] \in \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SU}(2),\rho\circ\lambda)} \mathbb{R}^3$ arbitrary. Since $\Lambda : S(\Sigma) \longrightarrow \mathrm{O}^+(\Sigma, g)$ is surjective, there exists a $\tilde{e} \in S(\Sigma)$ with $\Lambda(\widetilde{e}) = e$. Then we have $\mathfrak{N}[\widetilde{e}, x] = [e, x]$.

iii.) $\mathfrak{N}$ is injective, because we have: Let $[\widetilde{e}, x]$ and $[\bar{e}, x] \in S(\Sigma) \times_{(\mathrm{SU}(2),\rho\circ\lambda)} \mathbb{R}^3$ with $\mathfrak{N}[\widetilde{e}, x] = [\Lambda(\widetilde{e}, x] = [\Lambda(\bar{e}), y] = \mathfrak{N}[\bar{e}, y]$. Then we have to show

$[\widetilde{e}, x] = [\bar{e}, x]$. W.l.o.g. let $x = y$. Then we obtain $\Lambda(\widetilde{e}) = \Lambda(\bar{e})$. For the choice $\widetilde{e} = \bar{e}$ the statement is fulfilled. Now let $\widetilde{e} \neq \bar{e}$. Due to $\widetilde{\pi}(\widetilde{e}) = \pi \circ \Lambda(\widetilde{e}) = \pi \circ \Lambda(\bar{e}) = \widetilde{\pi}(\bar{e})$, $\widetilde{e}$ and $\bar{e}$ are in the same fibre. Thus there exists a $g \in \mathrm{SU}(2)$ such that $\widetilde{e} = \bar{e} \circ g$. Since we have $\Lambda(\bar{e}) = \Lambda(\widetilde{e}) = \Lambda(\bar{e} \circ g) = \Lambda(\bar{e}) \circ \lambda(g)$, we get $\lambda(g) = 1$. Since $\lambda : \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ is a double covering homomorphism and we assumed $\widetilde{e} \neq \bar{e}$, we obtain $g = -1$ and hence we get $\widetilde{e} = \bar{e} \circ (-1)$. Therefore we have

$$[\widetilde{e}, x] = [\bar{e} \circ (-1), x] = [\bar{e}, \rho(\lambda(-1))x] = [\bar{e}, \rho(1)x] = [\bar{e}, x].$$

QED.

THEOREM 4.3.6. *(See [33]) Let $\omega \in \Omega^1(\mathrm{O}^+(\mathcal{M}, q), \mathfrak{so}(3))$ a connection form and $\tilde{\omega} \in \Omega^1(S(\Sigma), \mathfrak{su}(2))$ the corresponding connection form in the spin bundle. Then they induce via Φ and $\mathfrak{N}$ the same covariant derivative on the tangent bundle $T\mathcal{M}$.*

Proof. Let $\widetilde{e} : \mathcal{U} \longrightarrow S(\Sigma)$ a local section into the spin bundle. Then $e : \mathcal{U} \longrightarrow \mathrm{O}^+(\Sigma, g)$, such that $e := \Lambda(\widetilde{e})$ is a local, orientated and orthonormal frame. Let (u_i) be the standard basis of $\mathbb{R}^3$, then the sections $[e, u_i] : \mathcal{U} \longrightarrow \mathrm{O}^+(\Sigma, g) \times_{(\mathrm{SO}(3), \rho)} \mathbb{R}^3 = \mathcal{E}^\rho$ resp. $[\widetilde{e}, u_i] : \mathcal{U} \longrightarrow S(\Sigma) \times_{(\mathrm{SU}(2), \rho \circ \lambda)} \mathbb{R}^3 = \widetilde{\mathcal{E}}^{\rho \circ \lambda}$ provide a basis of all local sections $\mathcal{U} \longrightarrow \mathcal{E}^\rho$ resp. $\mathcal{U} \longrightarrow \widetilde{\mathcal{E}}^{\rho \circ \lambda}$. The covariant derivative associated to ω resp. $\widetilde{\omega}$ of $[e, u_i]$ resp. $[\widetilde{e}, u_i]$ along the vector field $X \in \Gamma(T\Sigma)$ is given by

$$\nabla_X^{\mathcal{E}^\rho}[e, u_i] = [e, \rho_*(\omega^e(X))u_i)]$$

resp.

$$\nabla_X^{\widetilde{\mathcal{E}}^{\rho \circ \lambda}}[\widetilde{e}, u_i] = [\widetilde{e}, (\rho \circ \lambda)_*(\widetilde{\omega}^{\widetilde{e}}(X))u_i)] = [\widetilde{e}, \rho_* \circ \lambda_*(\widetilde{\omega}^{\widetilde{e}}(X))u_i)].$$

Due to $\lambda_* \circ \widetilde{\omega} = \Lambda^*(\omega)$, we have

$$\lambda_* \widetilde{\omega}^{\widetilde{e}} = \lambda_* \widetilde{e}^*(\widetilde{\omega}) = \widetilde{e}^*(\lambda_* \widetilde{\omega}) = \widetilde{e}^* \circ \Lambda^*(\omega) = (\Lambda \circ \widetilde{e})^*(\omega) = \widetilde{e}^*(\omega) = \omega^e$$

and therefore $\nabla_X^{\widetilde{\mathcal{E}}^{\rho\circ\lambda}}[\widetilde{e}, u_i] = [\widetilde{e}, \rho_*(\omega^e(X))u_i)]$. Thus the given covariant derivatives are transfered in each other by the isomorphism $\mathfrak{N}$

$$\begin{aligned}\mathfrak{N}(\nabla_X^{\widetilde{\mathcal{E}}^{\rho\circ\lambda}}[\widetilde{e}, u_i]) =& \mathfrak{N}([\widetilde{e}, \rho_*(\omega^e(X))u_i)]) = [\Lambda(\widetilde{e}), \rho_*(\omega^e(X))u_i)] \\ =& [e, \rho_*(\omega^e(X))u_i)] = \nabla_X^{\mathcal{E}^\rho}[e, u_i] = \nabla_X^{\mathcal{E}^\rho}\mathfrak{N}([\widetilde{e}, u_i]).\end{aligned}$$

QED.

Remark 4.3.7. *(See [33])*

i.) With Theorem 4.3.6 the action of the Ashtekar connection on the tangent bundle $T\Sigma$ is independent of the choice of the underlaying structure group SO(3) *or* SU(2)*.*

ii.) Theorems 4.3.5 and 4.3.6 can be generalized in arbitrary dimensions.

5. Reformulated General Relativity

5.1. Reformulated Einstein-Hilbert action

In the Section in hand we want to introduce a global version of the the Hamiltonian formulation of General Relativity. Id est we want to rewrite the Einstein-Hilbert action in terms of the Weingarten mapping Wein and the Ricci-scalar w.r.t. the Asktekar connection $\mathcal{R}^{\mathrm{A}}$, see Eq. (4.25). Starting point is the *Einstein-Hilbert action*

$$S_{\mathrm{EH}} := \int \mathcal{R}^{\mathcal{M}} \, \mathrm{dvol}, \tag{5.1}$$

where $\mathcal{R}^{\mathcal{M}}$ resp. dvol denotes the Rici scalar of the Levi-Civita connection on $\mathcal{M}$ resp. the volume form on $\mathcal{M}$. Let $(e_0, \ldots e_3) : \mathcal{U} \longrightarrow \mathrm{O}^+(\mathcal{M}, g)$ be an oriented, local basis system on $\mathcal{U} \in \mathcal{M}$, then on $\mathcal{U}$ dvol can be rewritten as $\mathrm{dvol} = e^0 \wedge \cdots \wedge e^3$, where $(e^0, \ldots, e^3)$ is the dual basis.

As a first step , we want to analyze the Ricci scalar $\mathcal{R}^{\mathcal{M}}$. By definition, we have for an arbitrary orthonormal system $(e_0, \ldots, e_3)$ of $T\mathcal{M}$

$$\mathcal{R}^{\mathcal{M}} = \mathrm{tr}_g \mathrm{Ric}^{\mathcal{M}} = \sum_{i,j=0}^{3} \epsilon_i \epsilon_j g(\mathcal{R}^{\mathcal{M}}(e_i, e_j)e_j, e_i),$$

where Ric denotes the Ricci tensor w.r.t the Levi Civita connection on $\mathcal{M}$ and ϵ_i is given by

$$\epsilon_i := g(e_i, e_j) = \begin{cases} -1, & \text{if } e_i \text{ timelike,} \\ 0, & \text{if } e_i \text{ Null,} \\ 1, & \text{if } e_i \text{ spacelike.} \end{cases}$$

In order to simplify we introduce a compatible, oriented, orthonormal system $(\mathrm{n}, e_1, \ldots, e_3)$, i.e. $(e_1, \ldots, e_3)$ is a oriented, orthonormal system of Σ and n denotes the vector field of normals on Σ. We obtain

$$\begin{aligned}
\mathcal{R}^{\mathcal{M}} &= \sum_{i,j=1}^{3} g(\mathcal{R}^{\mathcal{M}}(e_i, e_j)e_j, e_i) - \sum_{i=1}^{3} g(\mathcal{R}^{\mathcal{M}}(e_i, \mathrm{n})\mathrm{n}, e_i) \\
&\quad - \sum_{j=1}^{3} g(\mathcal{R}^{\mathcal{M}}(\mathrm{n}, e_j)e_j, \mathrm{n}) + g(\mathcal{R}^{\mathcal{M}}(\mathrm{n}, \mathrm{n})\mathrm{n}, \mathrm{n}) \qquad (5.2) \\
&= \sum_{i,j=1}^{3} g(\mathcal{R}^{\mathcal{M}}(e_i, e_j)e_j, e_i) - 2\sum_{i=1}^{3} g(\mathcal{R}^{\mathcal{M}}(e_i, \mathrm{n})\mathrm{n}, e_i).
\end{aligned}$$

THEOREM 5.1.1. *The Ricci curvature scalar* $\mathcal{R}^{\mathcal{M}}$ *on* $\mathcal{M}$ *as given in Eq.* (5.2) *can be shifted to the submanifold* Σ. *We have*

$$\mathcal{R}^{\mathcal{M}} = \mathcal{R}^{\Sigma} + \mathrm{tr}(\mathrm{Wein}^2) - \mathrm{tr}(\mathrm{Wein})^2, \qquad (5.3)$$

where we have dropped as usual the terms containing the divergence of $\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}$ *and* $\mathrm{tr}(\mathrm{Wein})\mathrm{n}$ *in Eq.* (A.9).

Proof. See A.3.1. QED.

By using Theorem 5.1.1, we finally can rewrite Eq. (5.1) as

$$S_{\mathrm{EH}} = \int \left(\mathcal{R}^{\Sigma} + \mathrm{tr}(\mathrm{Wein}^2) - \mathrm{tr}(\mathrm{Wein})^2\right) \mathrm{dvol}. \qquad (5.4)$$

Next, we want to indicate the Einstein Hilbert action with respect to the Ashtekar connection A.

THEOREM 5.1.2. *With respect to the Ashtekar connection A the Einstein Hilbert action is given by*

$$S_{\mathrm{EH}} = \int \left(\mathcal{R}^{\mathrm{A}} + (1+\beta^2)[\mathrm{tr}(\mathrm{Wein}^2) - \mathrm{tr}(\mathrm{Wein})^2]\right) \mathrm{dvol}, \qquad (5.5)$$

where $\mathcal{R}^{\mathrm{A}}$ is the Ricci curvature scalar on Σ and β is the Barbero-Immirzi parameter.

Proof. This follows from the expression of $\mathcal{R}^{\mathrm{A}}$ given in Proposition 4.1.26 of Chapter 4. QED.

Corollary 5.1.3. *Using the choice $\beta = i$ we obtain the simple expression*

$$S_{\mathrm{EH}}^{\beta=i} = \int \mathcal{R}^{\mathrm{A}} \,\mathrm{dvol}. \tag{5.6}$$

5.2. Reformulated Constraints

In the *Ashtekar formulation* of General Relativity (cf. Section 4.1.3) a system of constraints arises, as seen in Section 3.1.2. Now we shall translate the constraints as given in Eq. (3.44), Eq. (3.45) and Eq. (3.46) into our preceding framework of Chapter 4, see again [33].

Rewriting the *diffeomorphism constraint*, (3.45), we obtain

$$H_a \sim \sum_i g((\nabla^{\mathrm{A}}_{e_a}\mathrm{Wein})(e_i) - (\nabla^{\mathrm{A}}_{e_i}\mathrm{Wein})(e_a), e_i), \tag{5.7}$$

where $\{e_i\}_{1\leq i\leq 3}$ denotes an orthonormal, oriented dreibein on Σ. And in addition the *Hamiltonian constraint* yields

$$H \sim \mathcal{R}^A + (1+\beta^2)[\mathrm{tr}(\mathrm{Wein}^2) - \mathrm{tr}(\mathrm{Wein})^2], \tag{5.8}$$

where $\mathcal{R}^A$ denotes the Riemannian scalar curvature w.r.t. the Ashtekar connection. Using again the choice $\beta = i$, we get

$$H^{\beta=i} \sim \mathcal{R}^A. \tag{5.9}$$

Remark 5.2.1. *By Corollary 2.2.77 for spacetimes of constant sectional curvature, we have*

$$(\nabla^{\mathrm{A}}_X \mathrm{Wein})(Y) = (\nabla^{\mathrm{A}}_Y \mathrm{Wein})(X)$$

for all $X, Y \in \Gamma(T\Sigma)$. *Therefore for spacetimes of constant sectional curvature the diffeomorphism constraint is generally fulfilled.*

6. Implementation of the Hamiltonian constraint - a suggestion

This Chapter presents the continuation of [33]. In particular by using the Regge calculus [60], we want to present how to turn the classical expression of the Hamiltonian into a well defined operator. The Hamiltonian constraint (5.8) in the choice of the Barbero-Immirzi parameter $\beta = \mathrm{i}$ reads

$$H \sim \mathcal{R}^A.$$

Due to implement Hamiltonian operators in the kinematical Hilbert space, we need to define $\widehat{\mathcal{R}^A}$. In this Chapter we will use the *index notation* since we will reclaim results of the physics community.

6.1. Derivation of the Hamiltonian constraint operator

6.1.1. Regge Calculus

Let $\mathcal{M}$ an n-dimensional Riemannian manifold. The Regge approach goes back to Regge in 1961 [60], who proposed to approximate Einstein's continuum theory by a simplicial discretization $\triangle$ of the metric space-time manifold and the gravitational action. Thus its local building blocks are n-simplixes σ. The metric tensor associated with each simplex is expressed as a function of the squared edge lengths L^2 of σ, which are the dynamical variables of this model. For introductory material on classical Regge calculus and simplicial manifolds, see [64, 74, 73, 36]. One may regard a Regge geometry as a special case of a continuum Riemannian manifold, a so-called piecewise flat manifold, with a flat metric in the in terior of its n-simplixes σ, and singular curvature assignments to its hinges ($n-2$-simplixes) h.

The Einstein-Hilbert action in $n=4$ in the discrete Regge approach is given by

$$S_{\text{Regge}}(L_h) = \sum_{\sigma}\sum_{h\in\sigma} L_h^{\sigma}\epsilon_h = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \mathrm{d}^3x\,\mathcal{R}^{\mathrm{A}}, \tag{6.1}$$

where $\mathcal{R}^{\mathcal{M}}$ denotes the Riemannian curvature scalar on $\mathcal{M}$ w.r.t. the Levi-Civita connection and $\mathcal{R}^{\mathrm{A}}$ is the Riemannian curvature scalar on Σ w.r.t. the Ashtekar connection. $\epsilon_h = 2\pi - \sum_{\sigma\ni h}\mathrm{Ang}(\sigma,h)$ the deficit angle there and $\mathrm{Ang}(\sigma,h)$ is the angle between the two 2-simplixes of σ (the angle between their inward normals) intersecting in h.

Definition 6.1.1. *A cellular decomposition $\triangle$ of a space Σ is a disjoint union (partition) of open cells of varying dimension satisfying the following conditions:*

i.) An n-dimensional open cell is a topological space which is homeomorphic to the n-dimensional open ball;

ii.) The boundary of the closure of an n-dimensional cell is contained in a finite union of cells of lower dimension.

To match Regge calculus context with LQG framework we invoke the dual picture of LQG, see Section 3.1.2. Thus in the following we describe the picture of quantum geometry coming from Loop Quantum Gravity and the rôle played by the length in this picture, we recall the standard procedure used in Loop Quantum Gravity when introducing an operator corresponding to a given classical geometrical quantity.

A spin network graph $\rightarrow$ a cellular decomposition = covering cellular decomposition

Definition 6.1.2. *A cellular decomposition $\triangle$ of a three-dimensional space Σ built on a graph γ is said to be a covering cellular decomposition of γ if:*

i.) Each 3-cell of $\triangle$ contains at most one vertex of γ;

ii.) Each 2-cell (face) of $\triangle$ is punctured at most by one edge of γ and the intersection belongs to the interior of the edge;

iii.) Two 3-cells of $\triangle$ are glued such that the identied 2-cells match.

iv.) If two 2-cells of the boundary of a 3-cell intersect, then their intersection is a **connected** *1-cell.*

6.1.2. Construction of the Riemannian scalar curvature operator

Based on the Gauss-Bonnet theorem [49, 55] respectively Eq. (6.1) the curvature can be identified by $\sum_{h\in\sigma} L_h^\sigma \epsilon_h$, i.e. the contribution from all hinges within a small region to curvature. Thus we have to find the following operators in order to construct $\widehat{\mathcal{R}^{\mathrm{A}}}$: the length operator $\widehat{L}_h^\sigma$ and the angle operator $\widehat{\mathrm{Ang}}(\sigma, h)$.

Construction of the length operator $\widehat{L}_h^\sigma$

The starting point is a classical expression for the length of a curve. We will follow [21]. Let c be a curve embedded in the 3-manifold Σ, c.f. Definition 3.1.9, namely

$$\begin{array}{rcl} \mathrm{c} : [0,1] & \longrightarrow & \Sigma \\ t & \longmapsto & \mathrm{c}^a(t). \end{array}$$

The length of the curve is given by functional of the Ashtekar field E_i^a, in particular a one dimensional integral given by

$$\mathrm{L(c)} = \int_0^1 \mathrm{d}t \sqrt{\delta_{ij} \mathrm{G}^i(t) \mathrm{G}^j(t)}, \tag{6.2}$$

where $\mathrm{G}^i(t) = e_a^i(\mathrm{c}(t))\dot{\mathrm{c}}^a(t)$ is the pullback of the triad on the curve c given by

$$\mathrm{G}^i(t) = \frac{\frac{1}{2}\epsilon^{ijk}\epsilon_{abc}E_j^b E_k^c \dot{\mathrm{c}}^a(t)}{\sqrt{\frac{1}{3!}|\epsilon^{ijk}\epsilon_{abc}E_i^a E_j^b E_k^c|}},$$

where E_i^a is analyzed at $x^a = \mathrm{c}^a(t)$ and $\dot{\mathrm{c}}^a(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{c}^a(t)$. Now we are able to present an external regularization of the length of a curve.

i.) *External regularization of the length* No we want to present an *external regularization of the length of a curve* $L(\mathrm{c})$. The regularization is divided into the following steps:

a.) We replace Eq. (6.2) by the limit of Riemann sum as done in Section 3.1.2, i.e. $L(\mathrm{c}) = \lim_{\Delta t\to 0}\sum_N L_N$. We make a decomposition of the curve c in small segments c_N corresponding to the embeddings $x^a = \mathrm{c}^a(t)$ with $t \in [N\,\Delta t, (N+1)\Delta t]$ such that $\mathrm{c} = \bigcup_N \mathrm{c}_N$. Then we rewrite Eq. (6.2) as

$$\mathrm{L}(\mathrm{c}) = \lim_{\Delta t\to 0}\sum_N \Delta t\,\sqrt{\delta_{ij}\mathrm{G}^i_{\Delta t}(t_N)\mathrm{G}^j_{\Delta t}(t_N)},$$

with t_N is a point belonging to $[N\,\Delta t, (N+1)\Delta t]$. The subscript Δt in $\mathrm{G}^i_{\Delta t}(t_N)$ indicates the dependence of the step Δt.

Let be c a curve embedded in Σ. We consider two surfaces S_1 and S_2 which intersect at c. To achieve a visualization we choose of coordinates $x^a = (\sigma_1, \sigma_2, t)$ in Σ such that the curve c has embedding $x^a = \mathrm{c}^a(t) = (0, 0, t)$ and S_1 resp. S_2 is the $\sigma_2 = 0$ resp. $\sigma_1 = 0$ surface. Now we consider a decomposition of Σ in cubic cells w.r.t. the coordinates $x^a = (\sigma_1, \sigma_2, t)$ of coordinate size Δt, as illustrated in Figure 6.1(a). As result we obtain, that the cell $R_N = \{x^a \in \Sigma | \sigma_1 \in [0, \Delta t], \sigma_2 \in [0, \Delta t], t \in [N\,\Delta t, (N+1)\Delta t]\}$ has the segment c_N as a side. Furthermore c_N corresponds to the intersection of S^1_N and S^2_N pertaining to the boundary R_N.

b.) Thanks to the partition introduced in Section 3.1.2, now we can make the first step of the fluxization procedure, in particular we are able to write $\mathrm{G}_i(t)$ in terms of surface integrals:

$$
\begin{aligned}
\mathrm{G}^i_{\Delta t}(t_N) =& \Big(\frac{1}{2}\frac{1}{(\Delta t)^4}\iint_{S^i_N} V^{ijk}_{x_N}(\sigma,\sigma')E^a_j(\sigma)n_a(\sigma)E^b_k(\sigma')n_b(\sigma')\Big)\\
&\times\Big(\frac{1}{8\times 3!}\frac{1}{(\Delta t)^6}\iiint_{\partial R_N}|T^{ijk}_{x_N}(\sigma,\sigma',\sigma'')E^a_i(\sigma)n_a(\sigma)\\
&\qquad E^b_j(\sigma')n_b(\sigma')E^c_k(\sigma'')n_c(\sigma'')|\Big)^{-1/2},
\end{aligned}
\tag{6.3}
$$

where the notation of Section 3.1.2 is used and in particular $\iint_{S^i_N} := \int_{S^1_N}\mathrm{d}^2\sigma\int_{S^2_N}\mathrm{d}^2\sigma'$ and $\iiint_{\partial R_N} := \int\limits_{\partial R_N}\mathrm{d}^2\sigma\int\limits_{\partial R_N}\mathrm{d}^2\sigma'\int\limits_{\partial R_N}\mathrm{d}^2\sigma''$, respectively. $V^{ijk}_{x_N}(\sigma,\sigma')$ is introduced in order to make such non-local expression in the field $E^a_i(x)$ SU(2)-gauge invariant. It is explicitly defined as

$$
V^{ijk}_{x_N}(\sigma,\sigma') = \epsilon^{i\,j'k'}\,\mathcal{D}^{(1)}(h_{\mathrm{c}^1_{x_N\sigma'}}[A])_{j'}{}^{j}\,\mathcal{D}^{(1)}(h_{\mathrm{c}^2_{x_N\sigma''}}[A])_{k'}{}^{k}.
$$

Due to the fact that

$$
\lim_{\sigma_1,\sigma_2\to 0}\epsilon^{abc}n_b(\sigma_1,t)n_c(\sigma_2,t) = \dot{\mathrm{c}}^a(t)
$$

the numerator converges in the limit $\Delta t \longrightarrow 0$ to $\frac{1}{2}\epsilon^{ijk}\epsilon_{abc}E^b_jE^c_k\dot{\mathrm{c}}$. The denominator constitutes to the *external regularization* of the volume density Vol as discussed in Eq. (3.79).

c.) In the second step of the fluxization scheme we rewrite surface integrals in Eq. (6.3) into Riemann sums of fluxes. The surface ∂R_N is decomposed into squares cells $S_{N\alpha}$, such that $\partial R_N = \bigcup_\alpha = S_{N\alpha}$. In particular we have $S^1_N = \bigcup_\alpha S^1_{N\alpha}$ resp. $S^2_N = \bigcup_\alpha S^2_{N\alpha}$, see Figure 6.1(b). Thus $\mathrm{G}^i_{\Delta t}(s_N)$ can be written as

$$
\mathrm{G}^i_{\Delta t}(t_N) = \lim_{\alpha\to\infty}\left((\Delta t)^{-1}\mathrm{G}^i_I\right),
$$

where G^i_I is given by

$$
\mathrm{G}^i_I = \frac{\frac{1}{2}\sum_{\alpha\,\beta}Y^i_{N\alpha\beta}}{\sqrt{\frac{1}{8\times 3!}\sum'_{\alpha\,\beta\,\mathrm{c}}|Q_{N\alpha\beta\mathrm{c}}|}} = \frac{\sum_{\alpha\,\beta}Y^i_{N\alpha\beta}}{2\mathrm{Vol}_N}. \tag{6.4}
$$

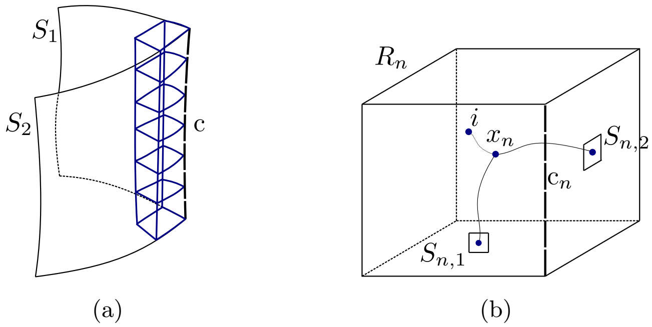


Figure 6.1.: Figure 6.1(a) A curve as intersection of two surfaces. Figure 6.1(b) Cubic cell with regularized quantity (6.4) shown.

Here $Y^i_{N\alpha\beta}$ is defined by

$$Y^i_{N\alpha\beta} = V^{ijk}_{x_I} F_j(S^1_{N\alpha}) F_k(S^2_{N\beta}) \tag{6.5}$$

and $Q_{N\alpha\beta\mathrm{c}}$ and Vol_N has been introduced in Section 3.1.2. Remark subscripts ${}_{N\alpha\beta}$ in $Y_{N\alpha\beta}$ represent a cubic cell R_N and two $S^1_{N\beta}$ and $S^2_{N\beta}$ corresponds respectively to the two faces S^2_N and S^1_N. Thus the length of segment c_N is then given by

$$\mathrm{L}_I = \sqrt{\delta_{ij}\ \mathrm{G}^i_I\, \mathrm{G}^j_I}\,. \tag{6.6}$$

As a result we have that the length of the curve c can be written as the limit $\Delta t \to 0$ of a sum of terms depending on Δt only implicitly

$$\mathrm{L}(\mathrm{c}) = \lim_{\Delta t \to 0} \sum_N \mathrm{L}_N\,.$$

ii.) *Quantization of the length* $\widehat{\mathrm{L}_N}$

Now we can attempt the *quantization of the regularized expression* by invoking the known action of the holonomy Eq. (3.55) and the flux Eq. (3.1.2) on cylindrical functions, i.e.

$$\widehat{\mathrm{L}}(\mathrm{c})\, \Psi_{\gamma,f}[A] = \lim_{\Delta t \to 0} \left(\sum_I \widehat{\mathrm{L}}_I\, \Psi_{\gamma,f}[A] \right)\,.$$

The construction of $\widehat{\mathrm{L}_N}$ for finite Δt requires two two building blocks, namely the two-hand operator $\widehat{Y^i_{N\alpha\beta}}$ and the inverse of the local volume operator $\widehat{\mathrm{Vol}_N^{-1}}$ as seen in Eq. (6.4) resp. Eq. (6.6). We will illustrate that $\widehat{\mathrm{L}_N}$ can be implemented such that the limit Δt is well-defined and the action on cylindrical function has a number of desirable properties.

a.) The two-hand operator $\widehat{Y^i_{N\alpha\beta}}$

The expression $\sum_{\alpha\beta} Y^i_{N\alpha\beta}$ given by Eq. (6.5) can be quantized analogous to the the volume operator $\widehat{\mathrm{Vol}}$ discussed in some detail in Section 3.1.2. We want to remark again that the subscripts $_{N\alpha\beta}$ in $Y_{N\alpha\beta}$ represent a cubic cell R_N and two square pieces $S^1_{N\alpha}$ and $S^2_{N\beta}$ corresponds respectively to the two faces S^1_N and S^2_N. We will consider a cylindrical function with graph γ. The decomposition in cells R_N is refined such that each cell contain at most one node. Now we are assuming that a node is contained in R_N and that two edges two edges e_1 resp. e_2 originating at the node intersects the surfaces $S^1_{N\tilde{\alpha}}$ respectively $S^2_{N\tilde{\beta}}$. The flux operator for a surface S acts non-trivial on a cylindrical function Ψ only if the surface S is punctured by a edge of γ. Thus only $Y_{N\tilde{\alpha}\tilde{\beta}}$ contributes. Here an adaptation of the curves to the graph γ is invoked, namely (1.) the point x_I is chosen to coincide with the position of the node, and (2.) the two curves $\mathrm{c}^1_{x_I\sigma}$ and $\mathrm{c}^2_{x_I\sigma}$ are adapted to the portions of the two links e_1 and e_2 contained in the cell R_I. This adaptation to γ hast the property, that the operator is independent of the size Δt of the cell due to the shrinking property of the volume. Thus the limit $\Delta t \longrightarrow 0$ can be taken trivially. The appropriate labeling for the operator $\widehat{Y^i_{N\alpha\beta}}$ is a node n and two edges e_1 resp. e_2 originating at n, i.e. the so called *wedge* $\{n, e_1, e_2\}$ of γ, labeled with the symbol ω. Thus the two-hand operator $\hat{Y}^i(\mathrm{c}_w)$ is defined as

$$\hat{Y}^i(\mathrm{c}_w) := \epsilon^{i\,j'k'}\; \mathcal{D}^{(1)}(h_{e_1}[A])_{j'}{}^{j}\; \mathcal{D}^{(1)}(h_{e_2}[A])_{k'}{}^{k}\; \hat{F}_j(S_{e_1})\hat{F}_k(S_{e_2}).$$

And the action of Eq. (iia) on a spin network state belonging to

$\mathcal{H}^G_{\rm kin}(\gamma)$ (in the nomenclature of Section 3.1.2 is given by:

$$\begin{aligned}\hat{Y}^i(\mathrm{c}_w)\ \Psi_{\gamma,j,i_k}[A] =&\hat{Y}^i(\mathrm{c}_w)\Big(\mathcal{D}^{(j_1)}(h_{e_1}[A])_{m'_1}{}^{m_1} \\ &\qquad\cdots\mathcal{D}^{(j_L)}(h_{e_L}[A])_{m'_L}{}^{m_L}\ v_{k m_1\cdots m_L}^{(j_1\cdots j_L)}\Big) \\ &\times \mathrm{rest}^{m'_1\cdots m'_L} \\ =&(8\pi \mathrm{c} L_P^2)^2\Big(\mathcal{D}^{(j_1)}(h_{e_1}[A])_{m'_1}{}^{m_1} \\ &\qquad\cdots\mathcal{D}^{(j_L)}(h_{e_L}[A])_{m'_L}{}^{m_L}\Big) \\ &\times\Big(\epsilon^{ijk} T^{(j_1)}_{j\ m_1}{}^{m''_1} T^{(j_2)}_{k\ m_2}{}^{m''_2}\ v_{k m''_1 m''_2\cdots m_L}^{(j_1\,j_2\cdots j_L)}\Big) \\ &\times \mathrm{rest}^{m'_1\cdots m'_L}.\end{aligned}$$

The substance is that we are able to define a two-hand operator for each pair of edges originating at a SNW-node. The way the operator knows about the curve: choosing a pair of edges identifies two surfaces dual to the edges. Such two surfaces intersect at a curve called c_ω. In other words the wedge ω identifies a surface bounded by the two edges. The dual to this surface is the curve c_ω.

b.) The inverse volume operator $\widehat{\mathrm{Vol}_N^{-1}}$

The denominator of Eq. (6.4) characterizes the volume of the cell R_N. Shifting to the quantum level it corresponds to the volume operator $\widehat{\mathrm{Vol}}$. Now we are focused with the problem that $\widehat{\mathrm{Vol}}$ has a non-trivial kernel. The kernel consists of (1.) SNWS with graph γ which have no node contained in R_N, which has the geometrical interpretation, that nodes of the SNW correspongs to chunks of space of definite volume. I.e. no node, no volume. And (2.) superpositions over intertwiners of SNWS with graph γ having a node in R_N, for details see [52]. We want to introduce an operator corresponding to the inverse of the volume of the cell R_N. Obviously the inverse of $\widehat{\mathrm{Vol}}$ does not exists. The natural solution is to restrict the domain of $\widehat{\mathrm{Vol}}$ such that resulting restricted operator is invertible then define an extension of its inverse to its maximal domain.

But given the geometrical interpretation of the operator the requirements for the operator corresponding to the inverse volume are stronger. We have the following conditions: (1.) the operator have to preserve the dual picture of quantum geometry, namely the *inverse volume* acts at nodes. Thus resulting in annihilating SNWS having no node in R_N. And (2.) the operator have to have same semiclassical behaviour, i.e have as eigenvalue the inverse of the corresponding one of Vol_N.

The idea to fulfill both requirements mentioned above goes back to a technical too, the so called Tikhonov regularization [70]. We define

Definition 6.1.3. *Let ϵ be finite. The inverse volume operator is defined as the limit of*

$$\widehat{\mathrm{Vol}^{-1}} = \lim_{\epsilon\to 0} (\, \widehat{\mathrm{Vol}}^2 + \epsilon^2 {\ell_{\mathrm{pl}}}^6\,)^{-1}\, \widehat{\mathrm{Vol}},$$

where ℓ_{pl} is the Planck length. And the operator $\widehat{\mathrm{Vol}^{-1}}$ satisfies the foloowing properties:

i. $\widehat{\mathrm{Vol}}\widehat{\mathrm{Vol}^{-1}}\widehat{\mathrm{Vol}} = \widehat{\mathrm{Vol}}$; *iii.* $(\widehat{\mathrm{Vol}^{-1}}\widehat{\mathrm{Vol}})^\dagger = \widehat{\mathrm{Vol}^{-1}}\widehat{\mathrm{Vol}}$;

ii. $\widehat{\mathrm{Vol}^{-1}}\widehat{\mathrm{Vol}}\widehat{\mathrm{Vol}^{-1}} = \widehat{\mathrm{Vol}^{-1}}$; *iv.* $(\widehat{\mathrm{Vol}}\widehat{\mathrm{Vol}^{-1}})^\dagger = \widehat{\mathrm{Vol}}\widehat{\mathrm{Vol}^{-1}}$.

Such limit exists and we expect a hermitian operator $\widehat{\mathrm{Vol}^{-1}}$ which commutes with $\widehat{\mathrm{Vol}}$ and admits self-adjoint extension to the Hilbert space $\mathcal{H}^G_{\mathrm{kin}}(\gamma)$. Obviously the operator annihilates SNWS having no node in th region R_N as $\widehat{\mathrm{Vol}}$ does. Furthermore $\widehat{\mathrm{Vol}^{-1}}$ has the same kernel as $\widehat{\mathrm{Vol}}$ and additionally $(1 - \widehat{\mathrm{Vol}^{-1}}\widehat{\mathrm{Vol}})$ is the projector of the kernel. Moreover the non-vanishing eigenvalues of $\widehat{\mathrm{Vol}^{-1}}$ are trivially the inverse of the corresponding eigenvalues of $\widehat{\mathrm{Vol}}$. Thus it is shown that the local inverse volume operator $\widehat{\mathrm{Vol}_N^{-1}}$ for the cell R_N is well-defined. Hence as a result we are able to define an inverse volume operator for a region R_n dual to the node n, denoted by $\widehat{\mathrm{Vol}^{-1}}(R_n)$.

c.) The length operator

Both operators, at the one hand $\hat{Y}^i(\mathrm{c}_w)$ and on the other $\widehat{\mathrm{Vol}^{-1}}(R_n)$ admit a dual description in terms of nodes and edges of the graph γ of the SNWS. Thus in the final step we build an operator to the quantity L_N, compare Eq. (6.6), consisting of $\hat{Y}^i(\mathrm{c}_w)$ and $\widehat{\mathrm{Vol}^{-1}}$. The idea is to introduce an operator which measures the length of the curve c_ω. Thus we obtain a length operator $\widehat{\mathrm{L}}(\mathrm{c}_\omega)$ for each wedge ω of the graph γ, given by

$$\begin{aligned}\widehat{\mathrm{L}}(\mathrm{c}_\omega) =& \frac{1}{2}\left[\widehat{\mathrm{Vol}^{-1}}(R_n)\delta_{ij}\hat{Y}^i(\mathrm{c}_w)\hat{Y}^j(\mathrm{c}_w)\widehat{\mathrm{Vol}^{-1}}(R_n)\right]^{\frac{1}{2}} \\ =& \left[\delta_{ij}\widehat{G}^{i\dagger}(\mathrm{c}_\omega)\widehat{G}^j(\mathrm{c}_\omega)\right]^{\frac{1}{2}},\end{aligned}$$

where we have introduced

$$\widehat{G}^i(\mathrm{c}_\omega) = \frac{1}{2}\hat{Y}^i(\mathrm{c}_w)\widehat{\mathrm{Vol}^{-1}}.$$

But we want to note that there exists an ordering ambiguity. We want to mention an alternative ordering choice. The Weyl ordering [72] of the length operator in terms of fluxes could be an interesting ordering choice since the fluxes appearing in Eq. (6.5) are generators of SU(2) transformations.

Construction of angle operator $\widehat{\mathrm{Ang}}(\sigma, h)$

Now we want to consider the second building block of the Riemannian scalar curvature operator, namely the angle operator $\widehat{\mathrm{Ang}}(\sigma, h)$. The angle operator is fundamentally a quantization of the classical expression for an angle between the inward normals of two surfaces intersecting at a curve.

i.) Classical expression:
Given two surfaces S_1 and S_2 of σ intersecting a curve c, the deficit angle $\mathrm{Ang}(\sigma, h)$ is the angle between the two 2-simplixes, i.e. the angle between their inward normals intersecting in h. The deficit angle is

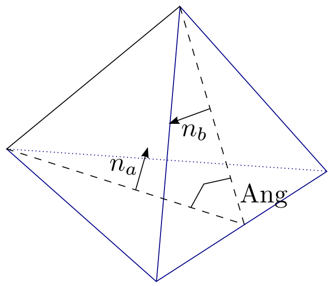


Figure 6.2.: Illustration of the angle Ang.

explicity given by [53]

$$\begin{aligned}&\mathrm{Ang}(\sigma,h) = \\ &\arccos\left(\frac{n_a(S_1)E^{ai}n_b(S_2)E^{bk}}{\sqrt{n_a(S_1)E^{ai}n_b(S_1)E^{bi}}\sqrt{n_a(S_2)E^{aj}n_b(S_2)E^{bj}}}\right),\end{aligned} \tag{6.7}$$

where n_a resp. n_b is the normal 1-form on the surface S_k.

ii.) Operator regularization:
We proceed with the same scheme to regularize the expression of the dihedral angle Eq. (6.7) as it was done for the length in Section 6.1.2. We obtain after the fluxization procedure, i. e. rewriting Eq. (6.7) in terms of surface integrals

$$\mathrm{Ang}^{I\alpha}_{12} = \arccos\left(\frac{V^{ijk}_{x_N}(\sigma,\sigma')F_i(S_1^{I\alpha})F_k(S_2^{I\beta})}{\mathrm{Ar}(S_1)\mathrm{Ar}(S_2)}\right)$$

where we used $V^{ijk}_{x_N}(\sigma,\sigma')$ given by Eq. (ib) and $F_i(S) = \int_S n_a E^a_i(x)\,\mathrm{d}^2\sigma$. After applying the second step of the fluxization program, namely we write the surface integrals as Riemann sums of fluxes

$$\mathrm{Ang}^{I\alpha}_{12} = \arccos\left(\frac{\sum_{\alpha\,\beta} Y^i_{N\alpha\beta}}{\mathrm{Ar}(S_1)\mathrm{Ar}(S_2)}\right)$$

or written out

$$\mathrm{Ang}_{12}{}^{I\alpha} = \arccos\left(\frac{\mathcal{D}^{(1)}(h_{\mathrm{c}^1_{xI\sigma}}[A])^i_{i'}\mathcal{D}^{(1)}(h_{\mathrm{c}^2_{xI\sigma}}[A])^k_{k'}F_i(S_1^{I\alpha})F_k(S_2^{I\beta})}{\sqrt{F_i(S_1^{I\alpha})F_i(S_1^{I\alpha})}\sqrt{F_j(S_2^{I\beta})F_j(S_2^{I\beta})}}\right) \tag{6.8}$$

iii.) Angle operator:

At the point we are able to state an operator associated to the angle, denoted by Ang:

$$\widehat{\mathrm{Ang}}(\mathrm{c}_\omega) = \arccos\left(\frac{\widehat{Y}^i(\mathrm{c}_\omega)}{\widehat{\mathrm{Ar}}(S_1)\widehat{\mathrm{Ar}}(S_2)}\right)$$

The action on SNWS $\psi(\gamma)$ of the area operator $\widehat{\mathrm{Ar}}(S)$ and the two hand operator $\widehat{Y}^i(\mathrm{c}_w)$ are well known, see Eq. (3.77) and [62] page 186. Thus the action of $\widehat{\mathrm{Ang}}_{ik}$ in the intertwiner basis is explicitly given by

$$\widehat{\mathrm{Ang}}(\mathrm{c}_\omega)|\psi(\gamma)\rangle = \left[\arccos\left(\frac{\frac{1}{2}[j_{ik}(j_{ik}+1) - j_i(j_i+1) - j_k(j_k+1)]}{\sqrt{j_i(j_i+1)j_j(j_j+1)}}\right)\right]|\psi(\gamma)\rangle.$$

But we want to remark that due to the non-commutativity of the area operator [8] an ordering ambiguity is again present.

The Riemannian scalar curvature operator

Thus we are able to state the Riemannian scalar curvature operator. The Riemannian scalar curvature operator associated to the 3-cell σ is given by

$$\widehat{\mathcal{R}^A_\sigma} := \sum_{h\in\sigma}\left[\widehat{\mathrm{L}^\sigma_h}\left(2\pi - \sum_{\sigma\ni h}\widehat{\mathrm{Ang}^{\sigma,\alpha_h}_h}\right)\right].$$

Let $\widehat{\mathcal{R}^A_\sigma}$ be the Riemannian scalar curvature operator. The following properties of $\widehat{\mathcal{R}^A_\sigma}$ are immediate:

i.) $\widehat{\mathcal{R}^A_\sigma}$ is self-adjoint.

ii.) $\widehat{\mathcal{R}^A_\sigma}$ depends on the choice of $\triangle$, since we have

a.) The action of $\widehat{\mathcal{R}^A_\sigma}$ on a cylindrical function $\psi(\gamma)$ gives zero unless the 3-cell σ contains a node. The action is explicitly given by

$$\widehat{\mathcal{R}^A_\sigma}|\psi(\gamma)\rangle = \sum_{h\in\sigma}\left[\widehat{\mathrm{L}^\sigma_h}\left(2\pi - \sum_{\sigma\ni h}\widehat{\mathrm{Ang}^{\sigma,\alpha_h}_h}\right)\right]|\psi(\gamma)\rangle.$$

b.) The action of $\widehat{\mathcal{R}^A_\sigma}$ depends on the 3-cells containing the nodes of γ (selecting the wedges) and the cells glued to them (fix the values of the coefficients α_{ω_n}).

The Hamiltonian constraint is written in terms of the original complex connection variables. Thus we obtain the following Hamiltonian constraint operator In the choice $\beta = \mathrm{i}$ the quantum Hamilton constraint is given by

$$\widehat{\mathcal{R}^A_\sigma}|\psi(\gamma)\rangle = \sum_{h\in\sigma}\left[\widehat{\mathrm{L}^\sigma_h}\left(2\pi - \sum_{\sigma\ni h}\widehat{\mathrm{Ang}^{\sigma,\alpha_h}_h}\right)\right]|\psi(\gamma)\rangle \equiv 0, \quad \forall\sigma\in\triangle.$$

A. Technical Proofs

A.1. Proofs of Chapter 2

Proof of Theorem 2.2.13: A.1.1.

- *Let* $\bar{s} \in \mathcal{C}^\infty(\mathcal{P}, \mathcal{F})^{(\rho,\mathrm{G})}$ *a* G*-equivariant mapping. Then the corresponding section* $s \in \Gamma(\mathcal{E})$ *is given by* $s(m) := [p, \bar{s}(p)] = \iota_p(\bar{s}(p)) \in \mathcal{E}_m$*, whereas* $p \in \mathcal{P}_m$ *is an arbitrary point in the fibre over* $m \in \mathcal{M}$*. Due to the* G*-equivariance of* $\bar{s}$ *we have*

$$\begin{aligned} [p \circ s, \bar{s}(p \circ g)] &= [p \circ g, \rho(g)^{-1} \bar{s}(p)] \\ &= [p, \bar{s}(p)], \quad \forall g \in \mathrm{G}. \end{aligned}$$

  Hence the latter expression is independent of the choice of $p \in \mathcal{P}_m$ *and* s *is well defined.*

- *Given* $s \in \Gamma(\mathcal{E})$*. The corresponding* G*-equivariant mapping* $\bar{s} \in \mathcal{C}^\infty(\mathcal{P}, \mathcal{F})^{(\rho,\mathrm{G})}$ *is given by* $\bar{s}(p) := \iota_p^{-1}(s(\pi(p)))$*. Due to*

$$\begin{aligned} \bar{s}(p \circ g) &= \iota_{p\circ g}^{-1}(s(\pi(p \circ g))) = (\iota_p \rho(g))^{-1}(s(\pi(p))) \\ &= \rho(g)^{-1} \iota_p^{-1}(s(\pi(p))) = \rho(g^{-1}) \bar{s}(p), \quad \forall g \in \mathrm{G}, \end{aligned}$$

  $\bar{s}(p)$ *is well defined.*

Proof of Theorem 2.2.24: A.1.2. *Consider the following mapping:*

$$\begin{aligned} \Psi : \widetilde{\mathcal{E}} := \mathcal{Q} \times_{(\mathrm{H},\rho\circ\lambda)} V &\longmapsto \mathcal{P} \times_{(\mathrm{G},\rho)} V =: \mathcal{E} \\ [q, v] &\longmapsto [f(q), v]. \end{aligned}$$

i.) Ψ *is well defined, because we have*

$$\begin{aligned}\Psi([q\circ h,\rho\circ\lambda(h^{-1})v]) =&[f(q\circ h),\rho\circ\lambda(h^{-1})v]\\ =&[f(q)\circ\lambda(h),\rho(\lambda(h)^{-1})v]\\ =&[f(q),v]=\Psi([q,v])\end{aligned}$$

for all $[q,v]\in\mathcal{Q}\times_{(\mathrm{H},\rho\circ\lambda)}V$ *and* $h\in\mathrm{H}$.

ii.) *Due to* $\pi_{\mathcal{E}}\circ\Psi([q,v])=\pi\circ f(q)=\widetilde{\pi}(q)=\pi_{\widetilde{\mathcal{E}}}([q,v]),\forall[q,v]\in\widetilde{\mathcal{E}}$, Ψ *is fibre preserving and - restricted on the fibre - linear.*

iii.) Ψ *is injective, because we have: Let* $[q,v]\in\mathcal{E}$ *and* $[\widetilde{q},\widetilde{v}]\in\widetilde{\mathcal{E}}$ *with*

$$\Psi([q,v])=[f(q),v]=[f(\widetilde{q}),\widetilde{v}]=\Psi([\widetilde{q},\widetilde{v}]).$$

Since Ψ *is fibre preserving, there exists* $h\in\mathrm{H}$ *with* $\widetilde{q}=q\circ h$. *Then we have*

$$\begin{aligned}[f(\widetilde{q}),\widetilde{v}]&=[f(q\circ h),\widetilde{v}]=[f(q)\circ\lambda(h),\widetilde{v}]\\ &=[f(q),\rho(\lambda(h)^{-1})\widetilde{v}]\overset{!}{=}[f(q),v].\end{aligned}$$

Hence we have $v=\rho(\lambda(h)^{-1})\widetilde{v}$, *whence we obtain* $[q,v]=[q,\rho(\lambda(h)^{-1})\widetilde{v}]=[q\circ h,\widetilde{v}]=[\widetilde{q},\widetilde{v}]$.

iv.) Ψ *is surjective, because we have: Let* $[p,v]\in\mathcal{E}$ *arbitrary. Then we have* $p\in\mathcal{P}_m$ *for a* $m\in\mathcal{M}$. *Let* $q\in Q_m$ *arbitrary. Since* $f(q)\in\mathcal{P}_m$, *there exists a* $g\in\mathrm{G}$, *such that* $f(q)\circ g=p$. *then we have*

$$\Psi([q,\rho(g)v])=[f(q),\rho(g)v]=[f(q)\circ g,v]=[p,v].$$

v.) Ψ *is smooth, because we have: Let* $(\mathcal{U}_\alpha,\widetilde{\varphi}_\alpha)$ *and* $(\mathcal{U}_\beta,\varphi_\beta)$ *charts of* $\widetilde{\mathcal{E}}$ *and* $\mathcal{E}$ *induced by the charts* $(\mathcal{U}_\alpha,\widetilde{\chi}_\alpha)$ *resp.* $(\mathcal{U}_\beta,\chi_\beta)$ *of* $\mathcal{Q}$ *resp.* $\mathcal{P}$ *with* $\mathcal{U}_\alpha\cap\mathcal{U}_\beta\neq\emptyset$.

$$\begin{aligned}\widetilde{\varphi}_\alpha:\pi_{\widetilde{\mathcal{E}}}^{-1}(\mathcal{U}_\alpha)&\longrightarrow\mathcal{U}_\alpha\times V\\ [p,v]&\longmapsto(\widetilde{\pi}(q),\rho\circ\lambda(\widetilde{\kappa}_\alpha(q))v);\\ \varphi_\beta:\pi_{\mathcal{E}}^{-1}(\mathcal{U}_\beta)&\longrightarrow\mathcal{U}_\beta\times V\\ [p,v]&\longmapsto(\pi(p),\rho(\kappa_\beta(p))v),\end{aligned}$$

whereas $\mathrm{pr}_2 \circ \widetilde{\chi}_\alpha =: \widetilde{\kappa}_\alpha$ *and* $\mathrm{pr}_2 \circ \chi_\alpha =: \kappa_\alpha$. *In that local trivializations, the mapping* Ψ *yields* ($q \in Q_m$ *arbitrary*)

$$\varphi_\beta \circ \Psi \circ \widetilde{\varphi}_\alpha^{-1} : (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times V \longrightarrow (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \times V$$

$$\begin{aligned} \varphi_\beta \circ \Psi \circ \widetilde{\varphi}_\alpha^{-1}(m, v) =& \varphi_\beta \circ \Psi([q, \rho \circ \lambda(\widetilde{\kappa}_\alpha(q)^{-1})v]) \\ =& \varphi_\beta([f(q), \rho \circ \lambda(\widetilde{\kappa}_\alpha(q)^{-1})v]) \\ =& (\pi \circ f(q), \rho(\kappa_\beta(f(q)))\rho \circ \lambda(\widetilde{\kappa}_\alpha(q)^{-1})v) \\ =& (\widetilde{\pi}(q)\rho(\kappa_\beta(f(q))\lambda(\widetilde{\kappa}_\alpha(q)^{-1}))v) \\ =& (m, \rho(\kappa_\beta(f(q))\lambda(\widetilde{\kappa}_\alpha(q)^{-1})v). \end{aligned}$$

Thus the mapping is smooth w.r.t. the charts $(\mathcal{U}_\alpha, \widetilde{\varphi}_\alpha)$ *and* $(\mathcal{U}_\beta, \varphi_\beta)$.

PROOF OF LEMMA 2.2.30: A.1.3. *Since the vertical tangent spaces* V_p *are given as the tangent spaces to the fibres* $\mathcal{P}_{\pi(p)} = \pi^{-1}(\pi(p))$ *we have* $V_p \subset \ker \mathrm{d}\pi_p$. *Otherwise let* $\chi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathrm{G}$ *a local trivialization in* $m := \pi(p) \in \mathcal{U} \subset \mathcal{M}$. *Due to* $\pi = pr_1 \circ \chi$, $X \in T_p\mathcal{P}$ *is in* $\ker \mathrm{d}\pi_p$, *if* $\mathrm{d}_{\chi_p} X = (0, Y)$ *for* $Y \in T_{pr_2 \circ \chi(p)}\mathrm{G}$, *i.e. if there exists a curve* $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{U} \times \mathcal{M}$ *with* $\gamma(t) = (m, g(t))$, *such that* $\mathrm{d}_{\chi_p} X = \dot{\gamma}(0)$. *As a result* X *is given by* $X = \dot{\chi}^{-1}(\gamma)(0)$. *Since the curve takes place solely in the fibre* $\mathcal{P}_m$, *we have* $X \in V_p$.

PROOF OF THEOREM 2.2.35: A.1.4. *One the one hand let* $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ *be a connection form. A connection is given by the mapping* $\mathcal{P} \ni p \longmapsto \ker \omega_p \subset T_p\mathcal{P}$. *On the other hand let* Γ *be a connection. Then a connection form is defined by* $\omega(\widetilde{X}) = X$, $\forall X \in \mathfrak{g}$ *and* $\omega(Y) = 0$, $\forall Y \in H_p$.

PROOF OF THEOREM 2.2.47: A.1.5. *Let* $A \in \mathfrak{h} \subset \mathfrak{g}$ *with fundamental vector field* $\widetilde{A} \in \Gamma(T\mathcal{Q})$. *Since* $\widetilde{\omega}$ *is a connection form, we have* $\omega(\widetilde{A}) = \mathrm{pr}_{\mathfrak{h}} \circ \widetilde{\omega}(\widetilde{A}) = \mathrm{pr}_{\mathfrak{h}} A = A$. *Let* $\phi := \mathrm{pr}_{\mathfrak{m}} \circ \widetilde{\omega}|_{T\mathcal{Q}}$ *the* $\mathfrak{m}$*-component of* $\widetilde{\omega}$ *restricted on* $\mathcal{Q}$. *For* $h \in \mathrm{H}$ *and* $X \in T_q\mathcal{Q}$, *we obtain*

$$\begin{aligned} (\Psi_h^* \omega)_q(X) + (\Psi_h^* \phi)_q(Y) =& \omega_{q \circ h}(\mathrm{d}\Psi_h X) + \phi_{q \circ h}(\mathrm{d}\Psi_h X) \\ =& (\omega + \phi)_{q \circ h}(\mathrm{d}\Psi_h X) = \widetilde{\omega}_{q \circ h}(\mathrm{d}\Psi_h X) \\ =& (\Psi_h^* \widetilde{\omega})(X) = \mathrm{Ad}(h^{-1})\widetilde{\omega}(X) \\ =& \mathrm{Ad}(h^{-1})\, \omega(X) + \mathrm{Ad}(h^{-1})\phi(X). \end{aligned}$$

Due to $\mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m}$, *finally we have* $(\Psi_h^* \omega)(X) = \mathrm{Ad}(h^{-1})\omega(X)$.

Proof of Theorem 2.2.56: A.1.6.

i.) *Let* $X, Y \in \Gamma(T\mathcal{M})$, $f \in \mathcal{C}^\infty$, $s \in \Gamma(\mathcal{E})$ *and let* $X^* \in \Gamma(T\mathcal{P})$ *be a horizontal lift of* X. *Then we have*

a.) ∇_X *is* $\mathbb{R}$*-linear;*

b.) $\nabla_{X+Y} = \nabla_X + \nabla_Y$*;*

c.) $\nabla_{fX} = f\nabla_X$ *due to* $\overline{\nabla_{fX} s} = (\mathrm{d}\bar{s})((f(X)^*) = (\mathrm{d}\bar{s})(f \circ \pi)X^* = (f \circ \pi)(\mathrm{d}\bar{s})(X^*) = (f \circ \pi)\overline{\nabla_X s} = \overline{f\nabla_X s}$*. resp. due to* $(fX)^* = f^* X^*$ *and* $\overline{fs} = f^* \bar{f}$ *with* $f^* = f \circ \pi$ *as horizontal lift of* f*;*

d.) *The Leibniz rule* $\nabla fs = \mathrm{d}f \otimes s + f\nabla s$ *is proofed by*

$$
\begin{aligned}
(\nabla_X fs)_m =& (\mathrm{d}_\omega fs)_m(X) = [p, \mathrm{d}(\overline{fs})_p(X^*)] \\
=& [p, \mathrm{d}f^*_p \bar{s}(p) + f^*(p)\, \mathrm{d}\bar{s}_p(X^*)] \\
=& [p, \mathrm{d}(f \circ \pi)_p(X^*)\bar{s}(p) + (f \circ \pi)(p)\, \mathrm{d}\bar{s}_p(X^*)] \\
=& [p, \mathrm{d}f_m(X)\bar{s}(p)] + f(m) \cdot [p, \mathrm{d}\bar{s}_p(X^*)] \\
=& \mathrm{d}f_m(X) \cdot s(m) + f(m) \cdot (\nabla_X s)_m.
\end{aligned}
$$

ii.) *Using Theorem 2.2.54 and Eq.* (2.21) *the covariant derivative of* $s = [e, v]$, *where* $v = \bar{s} \circ e$, *yields*

$$
\begin{aligned}
(\nabla_X s)_m = (\mathrm{d}_\omega s)_m(X) &= [e(m), (\mathcal{D}_\omega \bar{s})_{e(m)}(\mathrm{d}e_m X)] \\
&= [e(m), \mathrm{d}\bar{s}_{e(m)}(\mathrm{d}e_m X) + \rho_*(\omega_{e(m)}(\mathrm{d}e_m X))\bar{s}(e(m))] \\
&= [e(m), \mathrm{d}(\bar{s} \circ e)_m(X) + \rho_*((e^*\omega)_m(X))(\bar{s} \circ e)(m)] \\
&= [e(m), \mathrm{d}v_m(X) + \rho_*(\omega^e_m(x))v(m)]
\end{aligned}
$$

for all $X \in T_m\mathcal{M}$.

Proof of Theorem 2.2.60: A.1.7.

i.)-iii.) see [19];

iv.) *Let* $\varsigma \in \Omega^1_{\mathrm{hor}}(\mathcal{P}, \mathfrak{g})$, *then according to Theorem 2.2.54, we obtain* *for all* $X, Y \in T_p\mathcal{P}$

$$\begin{aligned}(\mathcal{D}_\omega\varsigma(X,Y) &= \mathrm{d}\varsigma(X,Y) + (\mathrm{Ad}_*(\omega)\wedge\varsigma)(X,Y)\\ &= \mathrm{d}\varsigma(X,Y) + \mathrm{Ad}_*(\omega(X))\varsigma(Y) - \mathrm{Ad}_*(\omega(Y))\varsigma(X)\\ &= \mathrm{d}\varsigma(X,Y) + \mathrm{ad}_{\omega(X)}\varsigma(Y) - \mathrm{ad}_{\omega(Y)}\varsigma(X)\\ &= \mathrm{d}\varsigma(X,Y) + [\omega(X),\varsigma(Y)] - [\omega(Y),\varsigma(X)]\\ &= \mathrm{d}\varsigma(X,Y) + [\omega,\varsigma]^\wedge(X,Y).\end{aligned}$$

Proof of Corollary 2.2.64: A.1.8. $\nabla^{\mathcal{M}}_X$ *fulfills the following properties, which characterize the covariant derivative, see Proposition 2.8. in [47]*

i.) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$;

ii.) $\nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z$;

iii.) $\nabla_{fX} = f\nabla_X Y \quad \forall f \in \mathfrak{C}(\mathcal{M})$;

iv.) $\nabla_X fY = f\nabla_X Y + X(f)Y \quad \forall f \in \mathfrak{C}(\mathcal{M})$.

Properties (*i*), (*ii*) *and* (*iii*) *are obvious from the corresponding properties of* $\nabla^{\mathcal{M}}$ *and the linearity of the projection* tan, *Eq.* (2.28). *In order to verify* (*iv*), *let* $f \in \mathfrak{C}(\Sigma)$ *and* $X, Y \in \Gamma(T\Sigma)$. *Then we have*

$$\nabla^{\mathcal{M}}_X(fY) = f(\nabla^{\mathcal{M}}_X Y) + (Xf)Y, \tag{A.1}$$

where $(Xf)Y$ *is tangential to* Σ. *Thus taking the tangential components of both sides, we obtain*

$$\tan(\nabla^{\mathcal{M}}_X(fY)) = f(\tan(\nabla^{\mathcal{M}}_X Y)) + (Xf)Y \tag{A.2}$$

proving property (*iv*) *for* $\tan\nabla^{\mathcal{M}}$. *Using Section 2.2.44, we see that there is a unique linear connection* Γ *on* $\mathcal{M}$ *for which* $\tan(\nabla^{\mathcal{M}}_X Y)$ *is the covariant differentiation.*
To show that Γ *is the Levi-Civita connection for the induced metric* Σ, *it is sufficient to show that* (*a*) *the torsion tensor of* Γ *is* 0 *and* (*b*) $\tan\nabla^{\mathcal{M}}g = 0$.

(*a*) *Let us write*

$$\begin{aligned}\nabla^{\mathcal{M}}_X Y &= \tan\nabla^{\mathcal{M}}_X Y + \mathrm{nor}\nabla^{\mathcal{M}}_X Y\\ \nabla^{\mathcal{M}}_Y X &= \tan\nabla^{\mathcal{M}}_Y X + \mathrm{nor}\nabla^{\mathcal{M}}_Y X.\end{aligned}$$

Additional let be $\bar{X}$ *respectively* $\bar{Y}$ *vector fields on* $\mathcal{U} \subset \mathcal{M}$*, then the restriction of* $\bar{X}, \bar{Y}$ *to* $\mathcal{U} \cap \Sigma$ *is tangent to* Σ *and coincides with* $[X, Y]$*. Thus we have*

$$[\bar{X}, \bar{Y}]_p = [X, Y]_p, \quad p \in \Sigma;$$

and

$$\nabla^{\mathcal{M}}_{\bar{X}} \bar{Y} = \nabla^{\mathcal{M}}_{X} Y, \qquad \nabla^{\mathcal{M}}_{\bar{Y}} \bar{X} = \nabla^{\mathcal{M}}_{Y} X \quad \textit{on} \quad \mathcal{U} \cap \Sigma.$$

From the equations above and the fact that the torsion of the Levi-Civita connection $\nabla^{\mathcal{M}}_{\mathrm{LC}}$ *of* $\mathcal{M}$ *is* 0*, we obtain*

$$\begin{aligned} 0 &= \nabla^{\mathcal{M}}_{\bar{X}} \bar{Y} - \nabla^{\mathcal{M}}_{\bar{Y}} \bar{X} - [\bar{X}, \bar{Y}] \\ &= \tan \nabla^{\mathcal{M}}_{X} Y - \tan \nabla^{\mathcal{M}}_{Y} X - [X, Y] + \mathrm{nor} \nabla^{\mathcal{M}}_{X} Y - \mathrm{nor} \nabla^{\mathcal{M}}_{Y} X. \end{aligned}$$

Then we see that

$$\tan \nabla^{\mathcal{M}}_{X} Y - \tan \nabla^{\mathcal{M}}_{Y} X - [X, Y] = 0,$$

proving (a).

(b) We start from $\nabla^{\mathcal{M}}_{\mathrm{LC}} g = 0$*, which implies for* $X, Y, Z \in \Gamma(T\Sigma)$ *on* Σ

$$\begin{aligned} X g(Y, Z) &= g(\nabla^{\mathcal{M}}_{X} Y, Z) + g(Y, \nabla^{\mathcal{M}}_{X} Z) \\ &= g(\tan \nabla^{\mathcal{M}}_{X} Y, Z) + g(Y, \tan \nabla^{\mathcal{M}}_{X} Z), \end{aligned}$$

which means $\tan \nabla^{\mathcal{M}} g = 0$*. Thus we have proved Proposition 2.2.66.*

Proof of Proposition 2.2.66: A.1.9. *According to Theorem 2.2.62, we have* $(R^{\nabla}(X, Y)s)_m = [p, \rho_*(F^{\omega}_p(\widetilde{X}, \widetilde{Y}))\bar{s}(p)]$*, with arbitrary* $p \in \mathcal{P}_m$ *and arbitrary lifts* $\widetilde{X}, \widetilde{Y} \in T_p\mathcal{P}$ *of* $X, Y \in T_m\mathcal{M}$*. Select* $p = e(m)$ *and* $\widetilde{X} = \mathrm{d}e_m X$ *as well* $\widetilde{Y} = \mathrm{d}e_m Y$*, then we have*

$$\begin{aligned} (R^{\nabla}(X, Y)s)_m &= [e(m), \rho_*(F^{\omega}_{e(m)}(\mathrm{d}e_n X, \mathrm{d}e_M Y))\bar{s}(e(m))] \\ &= [e(m), \rho_*((e^* F^{\omega})_m(X, Y))v(m)]. \end{aligned}$$

Proof of Proposition 2.2.70: A.1.10.

i.) Addivity in X or ξ (when the other is fixed) is obvious. For any $f \in \mathfrak{C}(\Sigma)$, we have

$$\begin{aligned} A_{f\xi}(X) + \mathrm{nor}\nabla^{\mathcal{M}}_X(f\xi) &= \nabla^{\mathcal{M}}_X(f\xi) = f\nabla^{\mathcal{M}}_X(\xi) + (Xf)\xi \\ &= f(A_\xi(X)) + f\,\mathrm{nor}\nabla^{\mathcal{M}}_X\xi + (X(f))\xi. \end{aligned}$$

Thus we obtain $A_{f\xi}(X) = fA_\xi(X)$ for the tangential components and $\mathrm{nor}\nabla^{\mathcal{M}}_X(f\xi) = f\,\mathrm{nor}\nabla^{\mathcal{M}}_X\xi + (X(f))\xi$ for the normal components. On the other hand a similar argument for $A_\xi(fX)+\mathrm{nor}\nabla^{\mathcal{M}}_{fX}(\xi) = \nabla^{\mathcal{M}}_{fX}(\xi)$ implies that $A_\xi(fX) = fA_\xi(X)$ and $\mathrm{nor}\nabla^{\mathcal{M}}_{fX}(\xi) = f\,\mathrm{nor}\nabla^{\mathcal{M}}_X\xi$. This shows that $A_\xi(X)$ is bilinear over $\mathfrak{C}(\Sigma)$.

ii.) For any $Y \in \Gamma(T\Sigma)$ and $\xi \in \Gamma(T\Sigma)^\perp$, we have $g(Y,\xi) = 0$. Differentiating covariant w.r.t. to X we have

$$\begin{aligned} 0 &= g(\nabla^{\mathcal{M}}_X Y, \xi) + g(Y, \nabla^{\mathcal{M}}_X\xi) \\ &= g(\nabla^{\mathrm{LC}}_X Y + K(X,Y), \xi) + g(Y, A_\xi(X) + \mathrm{nor}\nabla^{\mathcal{M}}_X\xi). \end{aligned}$$

Since $g(\nabla^{\mathrm{LC}}_X Y, \xi) = g(Y, \mathrm{nor}\nabla^{\mathcal{M}}_X\xi) = 0$, we get

$$g(K(X,Y),\xi) = -g(A_\xi(X), Y).$$

Proof of Proposition 2.2.74: A.1.11.

$$\begin{aligned} (\mathcal{L}_n g)(X,Y) &= \mathcal{L}_n g(X,Y) - g(\mathcal{L}_n X, Y) - g(X, \mathcal{L}_n Y) \\ &= n[g(X,Y)] - g([n,X],Y) - g(X,[n,Y]) \end{aligned}$$

and using $[X,Y] = \nabla^{\mathcal{M}}_X Y - \nabla^{\mathcal{M}}_Y X$, we finally get

$$\begin{aligned} (\mathcal{L}_n g)(X,Y) &= g(\nabla^{\mathcal{M}}_X n, Y) + g(X, \nabla^{\mathcal{M}}_Y n) = g(\mathrm{Wein}(X), Y) + g(X, \mathrm{Wein}(Y)) \\ &= 2g(\mathrm{Wein}(X), Y) = -2g(K(X,Y), n). \end{aligned}$$

Proof of Proposition 2.2.76: A.1.12.

$$\begin{aligned} (\tilde{\nabla}_X k)(Y,Z) =& X(k(Y,Z)) - k(\nabla^{\mathrm{LC}}_X Y, Z) - k(Y, \nabla^{\mathrm{LC}}_X Z) \\ =& X(g(\mathrm{Wein}(Y), Z)) - g(\mathrm{Wein}(\nabla^{\mathrm{LC}}_X Y), Z) \\ & - g(\mathrm{Wein}(Y), \nabla^{\mathrm{LC}}_X Z) \\ =& g(\nabla^{\mathrm{LC}}_X \mathrm{Wein}(Y), Z) + g(\mathrm{Wein}(Y), \nabla^{\mathrm{LC}}_X Z) \\ & - g(\mathrm{Wein}(\nabla^{\mathrm{LC}}_X Y), Z) - g(\mathrm{Wein}(Y), \nabla^{\mathrm{LC}}_X Z) \\ =& g((\nabla^{\mathrm{LC}}_X \mathrm{Wein})(Y), Z). \end{aligned}$$

Proof of Corollary 2.2.77: A.1.13. *(outline, see [57]) If $\mathcal{M}$ has constant sectional curvature C, then*

$$R^{\mathcal{M}}(X,Y)Z = C[g(Y,Z)X - g(X,Z)Y]$$

for $X, Y, Z \in \Gamma(T\Sigma)$. Thus we know that $R^{\mathcal{M}}(X,Y)Z$ is tangent to Σ; hence its normal component is 0.

A.2. Proofs of Chapter 4

Proof of Lemma 4.1.8: A.2.1.

i.) *Applies, since the cross product on $\mathbb{R}^3$ respectively the Lie bracket on $\mathfrak{so}(3)$ are antisymmetric;*

ii.) *We have $\langle u_i, u_j \times u_k\rangle = \langle u_i \times u_j, u_k\rangle$ for the standard basis $\{u_i\}_{i=1,\dots,3}$ of $\mathbb{R}^3$, where $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{R}^3$. Since $g(e_i, e_j) = \delta_{ij} = \langle u_i, u_j\rangle$, we obtain*

$$\begin{aligned} g(e_i \bowtie e_j, e_k) &= \sum_m \epsilon_{ijm} g(e_m, e_k) = \sum_m \epsilon_{ijm}\langle u_m, u_k\rangle \\ &= \langle u_i \times u_j, u_k\rangle = \langle u_i, u_j \times u_k\rangle = g(e_i, e_j \bowtie e_k) \end{aligned}$$

and by using the linearity of g the statement is proven;

iii.) *As a consequence of Jacobi's identity on $\mathbb{R}^3$*

$$\bar{X} \times (\bar{Y} \times \bar{Z}) = \langle \bar{X}, \bar{Z}\rangle \bar{Y} - \langle \bar{X}, \bar{Y}\rangle \bar{Z}$$

for all $\bar{X}, \bar{Y}, \bar{Z} \in \mathbb{R}^3$;

iv.) *As a consequence of Jacobi's identity on $\mathfrak{so}(3)$*

$$\mathfrak{S}\{[\widetilde{X}, [\widetilde{Y}, \widetilde{Z}]_{\mathfrak{so}(3)}]_{\mathfrak{so}(3)}\} = 0$$

for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathfrak{so}(3)$, respectively of the Jacobi's identity of the cross product in $\mathbb{R}^3$.

Proof of Proposition 4.1.11: A.2.2.

i.) Let be $X \in \Gamma(T\Sigma)$ *a vector field and* $s = \{s_i\}_{1\leq i\leq 3}: U \longrightarrow \mathrm{O}^+(\Sigma, g)$ *a local, oriented basis field. Then we have*

$$\begin{aligned}\nabla^{\mathrm{LC}}_X s_i &= \mathfrak{V}^{-1}\left([s, \rho_*(\omega^s(X))u_l]\right) \\ &= \mathfrak{V}^{-1}\left([s, \sum_k \Gamma^s_k(X)\Xi_k u_l]\right).\end{aligned} \tag{A.3}$$

Using $\Xi_i u_j = \sum_k \epsilon_{ijk} u_k = u_i \times u_j$ *we have*

$$\begin{aligned}\Xi_k(u_i \times u_j) &= u_k \times (u_i \times u_j) = (u_k \times u_i) \times u_j + u_i \times (u_k \times u_j) \\ &= \Xi_k u_i \times u_j + u_i \times \Xi_k u_j\end{aligned}$$

and therefore the covariant derivative of $(s_i \bowtie s_j)$ *w.r.t.* X *yields*

$$\begin{aligned}\nabla^{\mathrm{LC}}_X(s_i \bowtie s_j) =& \mathfrak{V}^{-1}([s, \rho_*(\omega^s(X))(u_i \times u_j)]) \\ =& \mathfrak{V}^{-1}([s, \sum_k \Gamma^s_k(X)\Xi_k(u_i \times u_j)]) \\ =& \mathfrak{V}^{-1}([s, \sum_k \Gamma^s_k(X)(\Xi_k u_i \times u_j + u_i \times \Xi_k u_j)]) \\ =& \mathfrak{V}^{-1}([s, \sum_k \Gamma^s_k(X)\Xi_k u_i \times u_j]) \\ & + \mathfrak{V}^{-1}([s, u_i \times \sum_k \Gamma^s_k(X)\Xi_k u_j]) \\ =& \mathfrak{V}^{-1}([s, \sum_k \Gamma^s_k(X)\Xi_k u_i]) \bowtie \mathfrak{V}^{-1}([s, u_j]) \\ & + \mathfrak{V}^{-1}([s, u_i]) \bowtie \mathfrak{V}^{-1}([s, \sum_k \Gamma^s_k(X)\Xi_k u_j]) \\ =& (\nabla^{\mathrm{LC}}_X s_i) \bowtie s_j + s_i \bowtie (\nabla^{\mathrm{LC}}_X s_j),\end{aligned}$$

where in the last step we used Eq. (A.3). *Let* Y *and* $Z \in \Gamma(T\Sigma)$ *two vector fields. Since the covariant derivative* $(\nabla Y)_m$ *of the vector field* Y *at a point* $m \in \Sigma$ *depends in a neighborhood* U *of* m *only on* Y*, we can expand* Y *and* Z *in terms of* $\{s_i\}_{1\leq i\leq 3}$ *of* $\Gamma(U, T\Sigma)$*. Thus on* U *we obtain* $Y = \sum_i Y^i s_i$ *and* $Z = \sum_j Z^j s_j$*,* $Y^i, Z^j \in \mathfrak{C}(U)$*. Thus by* *using Leibniz rule we get*

$$
\begin{aligned}
\nabla_X^{\mathrm{LC}}(Y \bowtie Z) =& \sum_{i,j} \nabla_X^{\mathrm{LC}}(Y^i Z^j s_i \bowtie s_j) \\
=& \sum_{i,j} \{\mathrm{d}(Y^i Z^j)(X) \otimes (s_i \bowtie s_j) + Y^i Z^j \nabla_X^{\mathrm{LC}}(s_i \bowtie s_j)\} \\
=& \sum_{i,j} \{[\mathrm{d}(Y^i(X))Z^j + Y^i(\mathrm{d}Z^j(X))] \otimes (s_i \bowtie s_j) \\
& \quad + Y^i Z^j [(\nabla_X^{\mathrm{LC}} s_i) \bowtie s_j + s_i \bowtie (\nabla_X^{\mathrm{LC}} s_j)]\} \\
=& \sum_{i,j} \{[\mathrm{d}Y^i(X) \otimes s_i + Y^i \nabla_X^{\mathrm{LC}} s_i] \bowtie Z^j s_j\} \\
& \quad + \sum_{i,s} \{Y^i s_i \bowtie [\mathrm{d}Z^j(X) \otimes s_j + Z^j \nabla_X^{\mathrm{LC}} s_j]\} \\
=& (\nabla_X^{\mathrm{LC}} Y) \bowtie Z + Y \bowtie (\nabla_X^{\mathrm{LC}} Z).
\end{aligned}
\tag{A.4}
$$

ii.) By using Jacobi's identity of the product structure as defined in 4.1.4, cf. Lemma 4.1.8, and using Eq. (A.4) *we obtain the covariant derivative w.r.t the connection form $\mathcal{A}$, namely*

$$
\begin{aligned}
\nabla_X^{\mathrm{A}}(Y \bowtie Z) &= \nabla_X^{\mathrm{LC}}(Y \bowtie Z) + S(X) \bowtie (Y \bowtie Z) \\
&= (\nabla_X^{\mathrm{LC}} Y + S(X) \bowtie Y) \bowtie Z + Y \bowtie (\nabla_X^{\mathrm{LC}} Z + S(X) \bowtie Z) \\
&= (\nabla_X^{\mathrm{A}} Y) \bowtie Z + Y \bowtie (\nabla_X^{\mathrm{A}} Z),
\end{aligned}
$$

for X, Y and $Z \in \Gamma(T\Sigma)$.

Proof of Theorem 4.1.12: A.2.3. *Let X, Y and $Z \in \Gamma(T\Sigma)$ vector fields on Σ. By Lemma 4.1.8 we have*

$$
\begin{aligned}
g(\nabla_X^{\mathrm{A}} Y, Z) + g(Y, \nabla_X^{\mathrm{A}} Z) =& g(\nabla_X^{\mathrm{LC}} Y + S(X) \bowtie Y, Z) \\
& + g(Y, \nabla_X^{\mathrm{LC}} Z + S(X) \bowtie Z) \\
=& g(\nabla_X^{\mathrm{LC}} Y, Z) + g(Y, \nabla_X^{\mathrm{LC}} Z) \\
& + g(S(X) \bowtie Y, Z) + g(Y, S(X) \bowtie Z) \\
=& X(g(Y, Z)) + g(S(X) \bowtie Y + Y \cdot S(X), Z) \\
=& X(g(Y, Z)).
\end{aligned}
$$

Therefore $\nabla^{\mathrm{A}} g = 0$.

PROOF OF THEOREM 4.1.13: A.2.4. *For vector fields* $X, Y \in \Gamma(T\Sigma)$ *we have*

$$\begin{aligned} T^{\mathrm{A}}(X,Y) &\coloneqq \nabla^{\mathrm{A}}_X Y - \nabla^{\mathrm{A}}_Y X - [X,Y] \\ &= \nabla^{\mathrm{LC}}_X Y - \nabla^{\mathrm{LC}}_Y X - [X,Y] + S(X) \bowtie Y - S(Y) \bowtie X \\ &= S(X) \bowtie Y - S(Y) \bowtie X \end{aligned}$$

by virtue of $[X,Y] = \nabla^{\mathrm{LC}}_X Y - \nabla^{\mathrm{LC}}_Y X$ *on* Σ.

PROOF OF THEOREM 4.1.14: A.2.5. *Let* X, Y *and* $Z \in \Gamma(T\Sigma)$. *By Proposition 4.1.11 we have*

$$\begin{aligned} R^{\mathrm{A}}(X,Y)Z =& \nabla^{\mathrm{A}}_X \nabla^{\mathrm{A}}_Y Z - \nabla^{\mathrm{A}}_Y \nabla^{\mathrm{A}}_X Z - \nabla^{\mathrm{A}}_{[X,Y]} Z \\ =& \nabla^{\mathrm{A}}_X (\nabla^{\mathrm{LC}}_Y Z + S(Y) \bowtie Z) - \nabla^{\mathrm{A}}_Y (\nabla^{\mathrm{LC}}_X Z + S(X) \bowtie Z) \\ & - (\nabla^{\mathrm{LC}}_{[X,Y]} Z + S([X,Y])) \\ =& \nabla^{\mathrm{LC}}_X \nabla^{\mathrm{LC}}_Y Z + [\nabla^{\mathrm{LC}}_X S(Y)] \bowtie Z \\ & + S(Y) \bowtie (\nabla^{\mathrm{LC}}_X Z) + S(X) \bowtie \nabla^{\mathrm{LC}}_Y Z + S(X) \bowtie (S(Y) \bowtie Z) \\ & - \nabla^{\mathrm{LC}}_Y \nabla^{\mathrm{LC}}_X Z - [\nabla^{\mathrm{LC}}_Y S(X)] \bowtie Z - S(X) \bowtie (\nabla^{\mathrm{LC}}_Y Z) \\ & - S(Y) \bowtie \nabla^{\mathrm{LC}}_X Z - S(Y) \bowtie (S(X) \bowtie Z) \\ & - \nabla^{\mathrm{LC}}_{[X,Y]} Z - S(\nabla^{\mathrm{LC}}_X Y - \nabla^{\mathrm{LC}}_Y X) \bowtie Z \\ =& R^{\mathrm{LC}}(X,Y)Z + [\nabla^{\mathrm{LC}}_X S(Y)] \bowtie Z - S(\nabla^{\mathrm{LC}}_X Y) \bowtie Z \\ & - [(\nabla^{\mathrm{LC}}_Y S(X)] \bowtie Z - S(\nabla^{\mathrm{LC}}_Y X) \bowtie Z + S(X) \bowtie [S(Y) \bowtie Z] \\ & - S(Y) \bowtie [S(X) \bowtie Z] \\ =& R^{\mathrm{LC}}(X,Y)Z + (\nabla^{\mathrm{LC}}_X S)(Y) \bowtie Z - (\nabla^{\mathrm{LC}}_Y S)(X) \bowtie Z \\ & + [S(X) \bowtie S(Y)] \bowtie Z \\ =& R^{\mathrm{LC}}(X,Y)Z \\ & + [(\nabla^{\mathrm{LC}}_X S)(Y) - (\nabla^{\mathrm{LC}}_Y S)(X) + S(X) \bowtie S(Y)] \bowtie Z. \end{aligned}$$

PROOF OF LEMMA 4.1.15: A.2.6.

i.) Follows directly from the definition of the curvature tensor.

ii.) *By Theorem 4.1.14 we get*

$$\begin{aligned} g(R^{\mathrm{A}}(X,Y)Z,W) =& g(R(X,Y)Z,W) \\ &+ g([(\nabla^{\mathrm{LC}}_X S)(Y) - (\nabla^{\mathrm{LC}}_Y S)(X) \\ &+ S(X) \bowtie S(Y)] \bowtie Z, W), \end{aligned}$$

for X, Y, Z *and* $W \in \Gamma(T\Sigma)$*. Due to the fact that* $R(X,Y)Z$ *fulfills the property wanted, it is sufficient to analyze the second addend. Using Lemma 4.1.8 we obtain*

$$g(V \bowtie Z, W) = g(V, Z \bowtie W) = -g(V, W \bowtie Z) = -g(V \bowtie W, Z)$$

for all $V, W, Z \in \Gamma(T\Sigma)$*. And by defining* $V := (\nabla^{\mathrm{LC}}_X S)(Y) - (\nabla^{\mathrm{LC}}_Y S)(X) + S(X) \bowtie S(Y)$ *the second property of our Lemma follows.*

Proof of Theorem 4.1.17: A.2.7.

(1) *From Theorem 4.1.14 follows*

$$\begin{aligned} R^{\mathrm{A}}(X,Y)Z =& R^{\mathrm{LC}}(X,Y)Z \\ &+ [(\nabla^{\mathrm{LC}}_X S)(Y) - (\nabla^{\mathrm{LC}}_Y S)(X) + S(X) \bowtie S(Y)] \bowtie Z \end{aligned} \tag{A.5}$$

We split the task into the three different addends that appear in (A.5)*. We have*

a.) *For the Riemannian* $R \in \Gamma(\Lambda^2 T^*\Sigma \otimes \mathrm{End}(T\Sigma))$ *of the Levi-Civita connection we have Bianchi's first identity*

$$\mathfrak{S}\{R^{\mathrm{LC}}(X,Y)Z\} = 0;$$

b.) *Concerning the second addend by Lemma 4.1.8 we obtain*

$$\begin{aligned} &\mathfrak{S}\{(S(X) \bowtie S(Y)) \bowtie Z\} \\ &= \mathfrak{S}\{g(S(X),Z)S(Y) - g(S(Y),Z)S(X)\} \\ &= \mathfrak{S}\{g(S(X),Z)S(Y) - g(S(Z),X)S(Y)\} \\ &= \mathfrak{S}\{[g(S(X),Z) - g(S(Z),X)]S(Y)\} \\ &= \begin{cases} 0 & \textit{if } S \textit{ symmetric,} \\ \mathfrak{S}\{[S(X) \bowtie S(Y)] \bowtie Z\} & \textit{otherwise;} \end{cases} \end{aligned}$$

c.) By Proposition 4.1.11 we see that

$$\begin{aligned}
&\mathfrak{S}\{[(\nabla^{\mathrm{LC}}_X S)(Y) - (\nabla^{\mathrm{LC}}_Y S)(X)] \bowtie Z\}\\
=&\mathfrak{S}\{[\nabla^{\mathrm{LC}}_X S(Y)] \bowtie Z - S(\nabla^{\mathrm{LC}}_X Y) \bowtie Z\\
&- [\nabla^{\mathrm{LC}}_Y S(X)] \bowtie Z + S(\nabla^{\mathrm{LC}}_Y X) \bowtie Z\}\\
=&\mathfrak{S}\{\nabla^{\mathrm{LC}}_X (S(Y) \bowtie Z) - S(Y) \bowtie (\nabla^{\mathrm{LC}}_X Z) - S(\nabla^{\mathrm{LC}}_X Y) \bowtie Z\\
&- \nabla^{\mathrm{LC}}_Y (S(X) \bowtie Z) + S(X) \bowtie (\nabla^{\mathrm{LC}}_Y Z) + S(\nabla^{\mathrm{LC}}_Y X) \bowtie Z\}\\
=&\mathfrak{S}\{\nabla^{\mathrm{LC}}_X (S(Y) \bowtie Z) - S(X) \bowtie (\nabla^{\mathrm{LC}}_Z Y) - S(\nabla^{\mathrm{LC}}_Y Z) \bowtie X\\
&- \nabla^{\mathrm{LC}}_X (S(Z) \bowtie Y) + S(X) \bowtie (\nabla^{\mathrm{LC}}_Y Z) + S(\nabla^{\mathrm{LC}}_Z Y) \bowtie X\}\\
=&\mathfrak{S}\{\nabla^{\mathrm{LC}}_X (S(Y) \bowtie Z + Y \bowtie S(Z)) + S(X) \bowtie (\nabla^{\mathrm{LC}}_Y Z - \nabla^{\mathrm{LC}}_Z Y)\\
&- S(\nabla^{\mathrm{LC}}_Y Z - \nabla^{\mathrm{LC}}_Z Y) \bowtie X\}\\
=&\mathfrak{S}\{\nabla^{\mathrm{LC}}_X T^A(Y,Z) + S(X) \bowtie [Y,Z] - S([Y,Z]) \bowtie X\}\\
=&\mathfrak{S}\{\nabla^{\mathrm{LC}}_X T^A(Y,Z) + T^A(X,[Y,Z])\}.
\end{aligned}$$

(2) We have

$$\begin{aligned}
-\mathfrak{S}\{(\nabla^{\mathrm{A}}_Z R^{\mathrm{A}})(X,Y)\} =& - \mathfrak{S}\{(\nabla^{\mathrm{A}}_Z R^{\mathrm{A}})(X,Y)\} + \mathfrak{S}\{[\nabla^{\mathrm{A}}_Z, R^{\mathrm{A}}(X,Y)]\\
&- R^{\mathrm{A}}([X,Y],Z)\}\\
=& - \mathfrak{S}\{[\nabla^{\mathrm{A}}_Z, R^{\mathrm{A}}(X,Y)] - R^{\mathrm{A}}(\nabla^{\mathrm{A}}_Z X, Y)\\
&\quad - R^{\mathrm{A}}(X, \nabla^{\mathrm{A}}_Z Y)\}\\
&+ \mathfrak{S}\{[\nabla^{\mathrm{A}}_Z, R^{\mathrm{A}}(X,Y)]\} - \mathfrak{S}\{R^{\mathrm{A}}([X,Y],Z)\}\\
=&\mathfrak{S}\{R^{\mathrm{A}}(\nabla^{\mathrm{A}}_Z X, Y) + R^{\mathrm{A}}(Y, \nabla^{\mathrm{A}}_X Z)\}\\
&- \mathfrak{S}\{R^{\mathrm{A}}([X,Y],Z)\}\\
=&\mathfrak{S}\{R^{\mathrm{A}}(\nabla^{\mathrm{A}}_X Y, Z)\} - \mathfrak{S}\{R^{\mathrm{A}}(\nabla^{\mathrm{A}}_Y X, Z)\}\\
&- \mathfrak{S}\{R^{\mathrm{A}}([X,Y],Z)\}\\
=&\mathfrak{S}\{R^{\mathrm{A}}(T^{\mathrm{A}}(X,Y),Z)\},
\end{aligned}$$

where in the last step we used Theorem 4.1.13.

PROOF OF PROPOSITION 4.1.18: A.2.8.

(a) This follows from the definition of the Ricci tensor defined as follows: $\mathrm{Ric}^{\mathrm{A}}(X,Y) := \mathrm{tr}_{T\mathcal{M}}[R^{\mathrm{A}}(\bowtie, X, Y, \bowtie)]$*. Thus w.r.t the connection* $A :=$

$\omega^{\mathrm{LC}} + \widetilde{S}$ *we get:*

$$\begin{aligned}
\mathrm{Ric}^{\mathrm{A}}(Y,Z) = & \mathrm{Ric}^{\mathrm{LC}}(Y,Z) + \sum_i \{ g((\nabla_{e_i} S)(Y) \bowtie Z, e_i) \\
& \qquad - g((\nabla_Y S)(e_i) \bowtie Z, e_i)\} \\
& + \sum_i \{ g(Z, S(Y)) g(S(e_i), e_i) - g(Z, S(e_i) g(S(Y), e_i)\} \\
= & \mathrm{Ric}^{\mathrm{LC}}(Y,Z) + \sum_i \{ g((\nabla_{e_i} S(Y)) \bowtie Z, e_i) \\
& \qquad - g(S(\nabla_{e_i} Y) \bowtie Z, e_i) \\
& \qquad - g((\nabla_Y S(e_i)) \bowtie Z, e_i) \\
& \qquad + g(S(\nabla_Y e_i) \bowtie Z, e_i)\} \\
& + \sum_i \{ g(Z, S(Y)) g(S(e_i), e_i) - g(Z, S(e_i) g(S(Y), e_i)\} \\
= & \mathrm{Ric}^{\mathrm{LC}}(Y,Z) + \sum_i \{ g(\nabla_{e_i} (S(Y) \bowtie Z), e_i) \\
& \qquad - g(S(Y) \bowtie \nabla_{e_i} Z, e_i) \\
& \qquad - g(S(\nabla_{e_i} Y) \bowtie Z, e_i) \\
& \qquad - g(\nabla_Y (S(e_i) \bowtie Z), e_i) \\
& \qquad + g(S(e_i) \bowtie \nabla_Y Z, e_i) \\
& \qquad + g(S(\nabla_Y e_i) \bowtie Z, e_i)\} \\
& + \sum_i \{ g(Z, S(Y)) g(S(e_i), e_i) - g(Z, S(e_i) g(S(Y), e_i)\} \\
= & \mathrm{Ric}^{\mathrm{LC}}(Y,Z) + \sum_i \{ g((S(Y) \bowtie Z), \nabla_{e_i} e_i) \\
& \qquad - g(S(Y) \nabla_{e_i} Z, e_i) \\
& \qquad - g(S(\nabla_{e_i} Y) \bowtie Z, e_i) \\
& \qquad - g((S(e_i) \bowtie Z), \nabla_Y e_i) \\
& \qquad + g(S(e_i) \bowtie \nabla_Y Z, e_i) \\
& \qquad + g(S(\nabla_Y e_i) \bowtie Z, e_i)\} \\
& + \sum_i \{ g(Z, S(Y)) g(S(e_i), e_i) - g(Z, S(e_i) g(S(Y), e_i)\}
\end{aligned}$$

$$
\begin{aligned}
=&\mathrm{Ric}^{\mathrm{LC}}(Y,Z) - \sum_i \{ g(S(Y) \bowtie \nabla_{e_i} Z, e_i) \\
&\qquad + g(S(\nabla_{e_i} Y) \bowtie Z, e_i) \\
&\qquad + g((S(e_i) \bowtie Z), \nabla_Y e_i) \\
&\qquad - g(S(e_i) \bowtie \nabla_Y Z, e_i) \\
&\qquad - g(S(\nabla_Y e_i) \bowtie Z, e_i)\} \\
&+ \sum_i \{ g(Z, S(Y)) g(S(e_i), e_i) - g(Z, S(e_i) g(S(Y), e_i)\},
\end{aligned}
$$

where we used Proposition 4.1.11 in the third and Theorem 4.1.12 in the fourth line, respectively.

(b) *The Ricci curvature scalar is given by $\mathcal{R}^{\mathrm{A}} := \mathrm{tr}_{T\mathcal{M}}[\mathrm{Ric}^{\mathrm{A}}(\cdot,\cdot)]$. Thus a further contraction of the remaining two arguments of the Ricci tensor w.r.t A, see Eq.* (4.20)*, yields*

$$
\begin{aligned}
\mathcal{R}^{\mathrm{A}} =& \mathcal{R}^{\mathrm{LC}} - \sum_{ij} \{ g(S(e_j) \bowtie \nabla_{e_i} e_j, e_j) + g(S(\nabla_{e_i} e_j) \bowtie e_j, e_i) \\
&\qquad + g((S(e_i) \bowtie e_j), \nabla_{e_j} e_i) \\
&\qquad - g(S(e_i) \bowtie \nabla_{e_j} e_i, e_i) - g(S(\nabla_{e_j} e_i) \bowtie e_i, e_j)\} \\
&+ \sum_{ij} g(e_j, S(e_j)) g(S(e_j), e_i) - \sum_i g(S(e_i), S(e_i)) \\
=& \mathcal{R}^{\mathrm{LC}} + \mathrm{tr}(S)^2 - \mathrm{tr}(S^2).
\end{aligned}
$$

A.3. Proofs of Chapter 5

Proof of Theorem 5.1.1: A.3.1. *We will study the two addends in Eq.* (5.2) *one after another:*

i.) *By using of the Weingarten formula Eq.* (2.39) *and the equation of*

Gauss Eq. (2.43) *the first addend in* (5.2) *yields*

$$\begin{aligned}
\sum_{i,j=1}^{3} g(\mathcal{R}^{\mathcal{M}}(e_i,e_j)e_j,e_i) = & \sum_{i,j=1}^{3} g(\mathcal{R}^{\Sigma}(e_i,e_j)e_j,e_i) \\
& - \sum_{i,j=1}^{3} g(\mathrm{Wein}(e_i),e_i)g(\mathrm{Wein}(e_j),e_i) \\
& + \sum_{i,j=1}^{3} g(\mathrm{Wein}(e_j),e_j)g(\mathrm{Wein}(e_i),e_i) \\
=& \mathcal{R}^{\Sigma} - \sum_{i} g(\mathrm{Wein}(e_i),\mathrm{Wein}(e_i)) \\
& + \sum_{j} g(\mathrm{Wein}(e_j),e_j) \sum_{i} g(\mathrm{Wein}(e_i),e_i) \\
=& \mathcal{R}^{\Sigma} - \mathrm{tr}(\mathrm{Wein}^2) + \mathrm{tr}(\mathrm{Wein})^2,
\end{aligned} \tag{A.6}$$

where $\mathcal{R}^{\Sigma}$ *denotes the Ricci scalar on* Σ.

ii.) In order to simplify the second addend in (5.2) *we have to introduce:*
Definition A.3.2.

$$\mathrm{div}(X) \coloneqq \sum_{i=1}^{3} g(\nabla^{\mathcal{M}}_{e_i} X, e_i) - g(\nabla^{\mathcal{M}}_{\mathrm{n}} X, \mathrm{n})$$

Hence we obtain
Lemma A.3.3.

$$\mathrm{tr}(\mathrm{Wein}) = \mathrm{div}(\mathrm{n})$$

Proof. We have

$$\begin{aligned}
\mathrm{div}(\mathrm{n}) &= \sum_{i=1}^{3} g(\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, e_i) - g(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n},\mathrm{n}) \\
&= \sum_{i=1}^{3} g(\mathrm{Wein}(e_i),e_i) = \mathrm{tr}(\mathrm{Wein}).
\end{aligned}$$

QED.

In addition div *satisfies the following Lemma:*
LEMMA A.3.4. *For functions* $f \in \mathfrak{C}(\mathcal{M})$ *and vector fields* $X \in \Gamma(T\mathcal{M})$ *we have*

$$\mathrm{div}(fX) = g(\mathrm{grad} f, X) + f\mathrm{div}(X).$$

Proof. Now, Lemma A.3.4 follows from the definition $g(\mathrm{grad}\, F, X) \coloneqq \mathrm{d}fX = X(f)$. QED.

Using Lemma A.3.3, we obtain

$$\begin{aligned} \mathrm{div}(\mathrm{tr}(\mathrm{Wein})\mathrm{n}) =& \mathrm{n}(\mathrm{tr}(\mathrm{Wein})) + \mathrm{tr}(\mathrm{Wein})\mathrm{div}(\mathrm{n}) \\ =& \mathrm{n}(\mathrm{tr}(\mathrm{Wein})) + \mathrm{tr}(\mathrm{Wein})^2. \end{aligned} \tag{A.7}$$

And due to $[e_i, \mathrm{n}] = \nabla^{\mathcal{M}}_{e_i}\mathrm{n} - \nabla^{\mathcal{M}}_{\mathrm{n}} e_i$ *the 2nd term in Eq.* (5.2) *reads*

$$\begin{aligned} \sum_{i=1}^{3} g(R^{\mathcal{M}}(e_i, \mathrm{n})\mathrm{n}, e_i) =& \sum_i g(\nabla^{\mathcal{M}}_{e_i}\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n} - \nabla^{\mathcal{M}}_{\mathrm{n}}\nabla^{\mathcal{M}}_{e_i}\mathrm{n} - \nabla^{\mathcal{M}}_{[e_i,\mathrm{n}]}\mathrm{n}, e_i) \\ =& \underbrace{\sum_i g(\nabla^{\mathcal{M}}_{e_i}\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}, e_i)}_{(A)} - \underbrace{\sum_i g(\nabla^{\mathcal{M}}_{\mathrm{n}}\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, e_i)}_{(B)} \\ & - \underbrace{\sum_i g(\nabla^{\mathcal{M}}_{\nabla^{\mathcal{M}}_{e_i}\mathrm{n}}\mathrm{n}, e_i)}_{(C)} + \underbrace{\sum_i g(\nabla^{\mathcal{M}}_{\nabla^{\mathcal{M}}_{n}e_i}\mathrm{n}, e_i)}_{(D)} \end{aligned} \tag{A.8}$$

(A) Due to $g(\mathrm{n}, \mathrm{n}) = -1$ *we have* $g(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}, \mathrm{n}) = 0$*. Hence we obtain* $g(\nabla^{\mathcal{M}}_{\mathrm{n}}\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}, \mathrm{n}) = -g(\nabla^{\mathcal{M}}_{\mathrm{n}}, \nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}) = -\|\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}\|^2$*. Thus the first term gets*

$$\begin{aligned} \sum_i g(\nabla^{\mathcal{M}}_{e_i}\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}, e_i) =& \mathrm{div}(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}) + g(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}, \mathrm{n}) \\ =& \mathrm{div}(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}) - \|\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}\|^2. \end{aligned}$$

(B) *By using Eq.* (A.7) *the 2nd summand yields*

$$\begin{aligned}
\sum_i g(\nabla^{\mathcal{M}}_{\mathrm{n}}\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, e_i) =& \sum_i n(g(\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, e_i) - \sum_i g(\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, \nabla^{\mathcal{M}}_{n} e_i)\\
=&\mathrm{n}(\sum_i g(\mathrm{Wein}(e_i), e_i)) - \sum_i g(\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, \nabla^{\mathcal{M}}_{n} e_i)\\
=&\mathrm{n}(\mathrm{tr}(\mathrm{Wein})) - \sum_i g(\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, \nabla^{\mathcal{M}}_{\mathrm{n}} e_i)\\
&- \mathrm{tr}(\mathrm{Wein})^2 + \mathrm{div}(\mathrm{tr}(\mathrm{Wein})\mathrm{n})\\
&- \sum_i g(\nabla^{\mathcal{M}}_{e_i}\mathrm{n}, \nabla^{\mathcal{M}}_{n} e_i)
\end{aligned}$$

(C) *The 3rd term gets*

$$\begin{aligned}
\sum_i g(\nabla^{\mathcal{M}}_{\nabla^{\mathcal{M}}_{e_i}\mathrm{n}}\mathrm{n}, e_i) =& \sum_i g(\nabla^{\mathcal{M}}_{\mathrm{Wein}(e_i)}\mathrm{n}, e_i)\\
=& \sum_i g(\mathrm{Wein}(\mathrm{Wein}(e_i)), e_i) = \mathrm{tr}(\mathrm{Wein}^2)
\end{aligned}$$

(D) *In order to calculate the 4th addend, we expand* $\nabla^{\mathcal{M}}_{\mathrm{n}} e_i$ *in the orthonormal basis* $(\mathrm{n}, e_1, \ldots, e_3)$*, i.e.* $\nabla^{\mathcal{M}}_{\mathrm{n}} e_i = \sum_j g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_i, e_j)e_j - g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_i, \mathrm{n})\mathrm{n}$*. In addition due to* $g(e_i, e_j) = \delta_{ij}$ *resp.* $g(\mathrm{n}, e_i) = 0$*, we get* $g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_i, e_j) = -g(e_i, \nabla^{\mathcal{M}}_{\mathrm{n}} e_j)$ *resp.* $g(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}, e_i) =$

$-g(\mathrm{n}, \nabla^{\mathcal{M}}_{\mathrm{n}} e_i)$. *Then we obtain*

$$\begin{aligned}
\sum_i g(\nabla^{\mathcal{M}}_{\nabla^{\mathcal{M}}_{\mathrm{n}} e_i} \mathrm{n}, e_i) =& \sum_i g(\nabla^{\mathcal{M}}_{\sum_j g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_i, e_j) e_j - g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_i, \mathrm{n})\mathrm{n}} \mathrm{n}, e_i) \\
=& \sum_{i,j} g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_i, e_j) g(\nabla^{\mathcal{M}}_{e_j} \mathrm{n}, e_i) \\
& - \sum_i g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_i, \mathrm{n}) g(\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}, e_i) \\
=& - \sum_{i,j} g(e_i, \nabla^{\mathcal{M}}_{\mathrm{n}} e_j) g(\nabla^{\mathcal{M}}_{e_j} \mathrm{n}, e_i) \\
& + \sum_i g(e_i, \nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}) g(\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}, e_i) \\
& + \sum_j g(\mathrm{n}, \nabla^{\mathcal{M}}_{\mathrm{n}} e_j) g(\nabla^{\mathcal{M}}_{e_j} \mathrm{n}, \mathrm{n}) \\
& - g(\mathrm{n}, \nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}) g(\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}, \mathrm{n}) \\
=& - \sum_j g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_j, \nabla^{\mathcal{M}}_{e_j} \mathrm{n}) + g(\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}, \nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}) \\
=& - \sum_j g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_j, \nabla^{\mathcal{M}}_{e_j} \mathrm{n}) + \|\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}\|^2.
\end{aligned}$$

Collecting terms we obtain for Eq. (A.8)

$$\begin{aligned}
\sum_{i=1}^{3} g(R^{\mathcal{M}}(e_i, \mathrm{n})\mathrm{n}, e_i) =& \mathrm{div}(\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}) - \|\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}\|^2 \\
& - [-\mathrm{tr}(\mathrm{Wein})^2 + \mathrm{div}(\mathrm{tr}(\mathrm{Wein})\mathrm{n}) \\
& \quad - \sum_i g(\nabla^{\mathcal{M}}_{e_i} \mathrm{n}, \nabla^{\mathcal{M}}_{\mathrm{n}} e_i)] \\
& - \mathrm{tr}(\mathrm{Wein}^2) - \sum_j g(\nabla^{\mathcal{M}}_{\mathrm{n}} e_j, \nabla^{\mathcal{M}}_{e_j} \mathrm{n}) \\
& + \|\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}\| \\
=& \mathrm{tr}(\mathrm{Wein})^2 - \mathrm{tr}(\mathrm{Wein}^2) + \mathrm{div}(\nabla^{\mathcal{M}}_{\mathrm{n}} \mathrm{n}) \\
& + \mathrm{div}(\mathrm{tr}(\mathrm{Wein})\mathrm{n}),
\end{aligned}$$

and hence by using Eq. (A.6) *and Eq.* (A.9) *we obtain for the Ricci scalar* $\mathcal{R}^{\mathcal{M}}$ *on* $(\mathcal{M}, g)$

$$\begin{aligned}\mathcal{R}^{\mathcal{M}} &= \sum_{i,j=1}^{3} g(R^{\mathcal{M}}(e_i, e_j)e_j, e_i) - 2\sum_{i=1}^{3} g(R^{\mathcal{M}}(e_i, \mathrm{n})\mathrm{n}, e_i) \\ &= \mathcal{R}^{\Sigma} - \mathrm{tr}(\mathrm{Wein}^2) + \mathrm{tr}(\mathrm{Wein})^2 \\ &\quad - 2[\mathrm{tr}(\mathrm{Wein})^2 - \mathrm{tr}(\mathrm{Wein}^2) + \mathrm{div}(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}) + \mathrm{div}(\mathrm{tr}(\mathrm{Wein})\mathrm{n})] \\ &= \mathcal{R}^{\Sigma} + \mathrm{tr}(\mathrm{Wein}^2) - \mathrm{tr}(\mathrm{Wein})^2 - 2[\mathrm{div}(\nabla^{\mathcal{M}}_{\mathrm{n}}\mathrm{n}) + \mathrm{div}(\mathrm{tr}(\mathrm{Wein})\mathrm{n})].\end{aligned} \tag{A.9}$$

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst habe.

Tübingen, den 17th October 2014
(Andreas J. Wöhr)