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New error bounds for asymptotic approximations of Jacobi polynomials and their zeros

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Dedicated to Aldo Ghizzetti with deep gratitude and great admiration for his work on Numerical Analysis

RIASSUNTO: *Viene stabilita una maggiorazione del termine complementare di una rappresentazione asintotica, per $n \to \infty$, del polinomio di Jacobi $P_n^{(\alpha,\beta)}(\cos\vartheta)$. Il procedimento usato si basa su una disuguaglianza del tipo di Bernstein, stabilita recentemente, per i polinomi di Jacobi. Le prove numeriche, fatte sulle applicazioni al calcolo degli zeri degli stessi polinomi, mostrano la bontà delle approssimazioni che si ottengono.*

ABSTRACT: *Bounds for the error term of an asymptotic representation of the Jacobi polynomial $P_n^{(\alpha,\beta)}(\cos\vartheta)$, as $n \to \infty$, are given. The procedure for deriving these bounds is based on a new inequality of Bernstein-type satisfied by $P_n^{(\alpha,\beta)}(\cos\vartheta)$. Application to the zeros of Jacobi polynomials is considered. Numerical examples are given to illustrate the sharpness of the new results.*

1 – Introduction

Some years ago, BARATELLA and GATTESCHI [2] have obtained realistic bounds for the error term of an asymptotic approximation, and

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of the zeros, of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. More precisely, these bounds are for the approximation, and for the zeros, of the function

$$
(1.1)\qquad \begin{aligned} &u_n^{(\alpha,\beta)}(\vartheta)=\left(\sin\frac{\vartheta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\vartheta}{2}\right)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos\vartheta)\,,\\ &-\frac{1}{2}\le\alpha,\beta\le\frac{1}{2},\qquad 0\le\vartheta\le\pi\,, \end{aligned}
$$

which satisfies the differential equation

$$
(1.2)\qquad \frac{d^2u}{d\vartheta^2}+\left(N^2+\frac{1/4-\alpha^2}{4\sin^2\vartheta/2}+\frac{1/4-\beta^2}{4\cos^2\vartheta/2}\right)u=0\,,
$$

where

$$
(1.3)\qquad N=n+\frac{\alpha+\beta+1}{2}\,.
$$

The approximation, considered in [2] for the function $u_n^{(\alpha,\beta)}(\vartheta)$, is in fact obtained by grouping the first three terms of a general uniform asymptotic expansion given by FRENZEN and WONG [6].

In the derivation of the bounds for the error terms an important rôle was played by the following inequality, due to BARATELLA [1],

$$
(1.4)\qquad \begin{aligned} &\left(\sin\frac{\vartheta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\vartheta}{2}\right)^{\beta+1/2}\left|P_n^{(\alpha,\beta)}(\cos\vartheta)\right|\le\\ &\le 2.821\binom{n+\alpha}{n}N^{-\alpha-1/2}\,, \end{aligned}
$$

where $0\le\vartheta\le\pi/2$ and $-1/2\le\alpha,\beta\le 1/2$. This inequality has been recently sharpened by CHOW, GATTESCHI and WONG [3]. Indeed, they have shown that

$$
(1.5)\qquad \begin{aligned} &\left(\sin\frac{\vartheta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\vartheta}{2}\right)^{\beta+1/2}\left|P_n^{(\alpha,\beta)}(\cos\vartheta)\right|\le\\ &\le\frac{\Gamma(q+1)}{\Gamma(1/2)}\binom{n+q}{n}N^{-q-1/2}\,, \end{aligned}
$$

for $0 \le \vartheta \le \pi$ and $-1/2 \le \alpha, \beta \le 1/2$, where $q = \max(\alpha, \beta)$.

In this paper, by using (1.5) other arguments and some accurate computations, we shall improve considerably the results established in [2].

2 – Preliminary results

We first notice that in view of the reflection formula $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, the function $u_n^{(\alpha,\beta)}(\vartheta)$ defined by (1.1) satisfies

$$u_n^{(\alpha,\beta)}(\pi - \vartheta) = (-1)^n u_n^{(\beta,\alpha)}(\vartheta)\,. \tag{2.1}$$

Thus, it is not restrictive to assume $0 \le \vartheta \le \pi/2$. Furthermore, since we are dealing with asymptotic representation, we shall assume $n \ge 5$ throughout this paper.

Let $f(\vartheta)$ be the monotonically increasing function

$$f(\vartheta) = N\vartheta + \frac{1}{16N}\left[A\left(\frac{2}{\vartheta} - \cot\frac{\vartheta}{2}\right) + B\tan\frac{\vartheta}{2}\right], \tag{2.2}$$

where

$$A = 1 - 4\alpha^2\,, \qquad B = 1 - 4\beta^2\,, \tag{2.3}$$

and N is given as in (1.3). The function $u_n^{(\alpha,\beta)}(\vartheta)$ satisfies the integral equation

$$\begin{aligned}\left[\frac{f(\vartheta)}{f'(\vartheta)}\right]^{-1/2} u_n^{(\alpha,\beta)}(\vartheta) &= c_1 J_\alpha[f(\vartheta)] + \\ &- \frac{\pi}{2}\int_0^\vartheta \left[\frac{f(t)}{f'(t)}\right]^{1/2} \Delta(t,\vartheta) F(t) u_n^{(\alpha,\beta)}(t)\,dt\,,\end{aligned} \tag{2.4}$$

where

$$c_1 = \frac{\Gamma(\alpha+1)}{2^{1/2}}\binom{n+\alpha}{n} N^{-\alpha}\left[1 + \frac{1}{32N^2}\left(\frac{A}{3} + B\right)\right]^{-\alpha}, \tag{2.5}$$

$$\Delta(t,\vartheta) = J_\alpha[f(\vartheta)]Y_\alpha[f(t)] - J_\alpha[f(t)]Y_\alpha[f(\vartheta)]\,, \tag{2.6}$$

and $F(t)$ is a non-negative function bounded in $0 \le \vartheta \le \pi - \varepsilon$, with $\varepsilon > 0$. More precisely it can be shown that

$$0 \le F(\vartheta) \le \frac{1}{16N^2}(\delta_1 A + \delta_2 B + \eta_1 A^2 + \eta_2 AB + \eta_3 B^2)\,,$$

where

$$\delta_1 = 0.0144657036, \quad \delta_2 = 1, \quad \eta_1 = 0.005383039, \quad \eta_2 = 0.0973499184,$$

$$\eta_3 = 0.0625\,,$$

for $0 \le \vartheta \le \pi/2$ and $n \ge 5$. It is now easy to see that

$$\delta_1 A + \delta_2 B + \eta_1 A^2 + \eta_2 AB + \eta_3 B^2 \le \mu_1 A + \mu_2 B\,,$$

for $0 \le A, B \le 1$, where

$$\mu_1 = 0.0685237018\,, \quad \mu_2 = 1.111174959\,. \tag{2.7}$$

Therefore, we get

$$0 \le F(\vartheta) \le \frac{1}{16N^2}(\mu_1 A + \mu_2 B)\,, \quad 0 \le \vartheta \le \pi/2\,, \quad n \ge 5\,. \tag{2.8}$$

Note that this inequality is different from the one obtained in [2].

We shall consider the two intervals $0 \le \vartheta \le \vartheta^*$ and $\vartheta^* \le \vartheta \le \pi/2$, where ϑ^* is the root of the transcendental equation $f(\vartheta) = \pi/2$. Such a root exists, is unique and satisfies, if $n \ge 5$, the inequality

$$0.9979776744\,\frac{\pi}{2N} \le \vartheta^* \le \frac{\pi}{2N}\,. \tag{2.9}$$

Using the integral equation (2.4), we have proved in [2, Theorem 4.1] that the following asymptotic representation holds

$$\begin{aligned}\left[\frac{f(\vartheta)}{f'(\vartheta)}\right]^{-1/2}\left(\sin\frac{\vartheta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\vartheta}{2}\right)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos\vartheta) =\\ = \frac{\Gamma(\alpha+1)}{2^{1/2}}\binom{n+\alpha}{n}N^{-\alpha}\left[1+\frac{1}{32N^2}\left(\frac{A}{3}+B\right)\right]^{-\alpha} J_\alpha[f(\vartheta)] + I\,,\end{aligned} \tag{2.10}$$

where

$$(2.11)\qquad |I| \le \vartheta^{\alpha} N^{-4}\binom{n+\alpha}{n}(0.00812A + 0.08282B)\,,\quad 0<\vartheta\le\vartheta^*\,,$$

and

$$(2.12)\qquad |I| \le \vartheta^{1/2} N^{-\alpha-1/2}\binom{n+\alpha}{n}(0.0526A + 0.535B)\,,\quad \vartheta^*\le\vartheta\le\pi/2\,.$$

For the zeros $\vartheta_{n,k}(\alpha,\beta)$, $k=1,2,\dots$, of $P_n^{(\alpha,\beta)}(\cos\vartheta)$ we can derive ([2], Theorem 5.2) the representation

$$(2.13)\qquad \begin{aligned}\vartheta_{n,k}(\alpha,\beta) = t_{n,k} - \frac{1}{16N^2}\left[A\left(\frac{2}{t_{n,k}} - \cot\frac{t_{n,k}}{2}\right) + B\tan\frac{t_{n,k}}{2}\right] +\\ + \varepsilon_k(\alpha,\beta)N^{-5}\,,\end{aligned}$$

where, provided that $\vartheta_{n,k}(\alpha,\beta)\le\pi/2$,

$$(2.14)\qquad 0\le\varepsilon_k(\alpha,\beta)\le j_{\alpha,k}(0.240A + 2.43B)\,,$$

and $t_{n,k} = j_{\alpha,k}/N$, $j_{\alpha,k}$ being the k-th positive zero of the Bessel function $J_\alpha(x)$.

The following lemma will be useful in rewriting the inequality (1.5) in a different form.

Lemma 2.1. *Let*

$$M(q) = \frac{\Gamma(q+1)}{\Gamma(1/2)}\binom{n+q}{n}N^{-q-1/2}\,,$$

with N defined as in (1.3). *Then, if $\alpha<\beta$,*

$$(2.15)\qquad M(\beta) < \frac{M(\alpha)}{1 - N^{-2}\sqrt{3}/108}\,,$$

for $-1/2\le\alpha,\beta\le 1/2$.

For the proof we use a particular case of a result, due to Frenzen [5], on the remainder term in Field's [4] asymptotic expansion of the ratio of two gamma functions. Indeed, Frenzen has shown that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = w^{a-b}\left[1-\eta\frac{\rho(2-2\rho)(1-2\rho)}{12N^2}\right], \tag{2.16}$$

where

$$2w = 2z+a+b-1\,, \qquad 2\rho = a-b+1$$

and $0<\eta<1$, if z, a, b are real and such that (i) $z+a>0$, (ii) $w\to\infty$ and (iii) $0<2\rho<1$.

By putting $z=n$, $a=\alpha+1$ and $b=\beta+1$, then $w=N$. The conditions required for the validity of (2.16) with $0<\eta<1$ are verified. Thus we obtain

$$\begin{aligned}\frac{M(\alpha)}{M(\beta)} &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)}N^{\beta-\alpha} = \\ &= 1-\eta\frac{(1-\delta^2)\delta}{24N^2}\,, \quad \delta=\beta-\alpha, \quad 0<\eta<1\,.\end{aligned} \tag{2.17}$$

Since $\max\{(1-\delta^2)\delta\} = 2\sqrt{3}/9$ for $0<\delta<1$, the lemma is proved.

As a consequence of Lemma 2.1 inequality (1.5) can be expressed in the form

$$\begin{aligned}&\left(\sin\frac{\vartheta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\vartheta}{2}\right)^{\beta+1/2}\left|P_n^{(\alpha,\beta)}(\cos\vartheta)\right| \le \\ &\le \frac{\Gamma(\alpha+1)}{\Gamma(1/2)}\binom{n+\alpha}{n}N^{-\alpha-1/2}K(n)\,,\end{aligned} \tag{2.18}$$

where

$$K(n) = \begin{cases} 1\,, & \text{if} \quad \alpha\ge\beta\,, \\ 1/(1-N^{-2}\sqrt{3}/108)\,, & \text{if} \quad \alpha<\beta\,. \end{cases} \tag{2.19}$$

3 – Error term in the approximation of $P_n^{(\alpha,\beta)}(\cos\vartheta)$

In this section we shall give estimates for the integral

$$I = -\frac{\pi}{2}\int_0^{\vartheta}\left[\frac{f(t)}{f'(t)}\right]^{-1/2}\Delta(t,\vartheta)F(t)\Big(\sin\frac{t}{2}\Big)^{\alpha+1/2}\Big(\cos\frac{t}{2}\Big)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos t)dt \tag{3.1}$$

given in (2.4), where $f(t)$ and $\Delta(t,\vartheta)$ are defined by (2.2) and (2.6), respectively.

The function $F(t)$ has been already considered in Section 2, and it satisfies the inequality (2.8).

a) The case $0<\vartheta<\vartheta^*$.

The study of this case is similar to the one made in [2] of the same case. We denote by M an upper bound for the absolute value of

$$F(t)\left[\frac{\sin t/2}{f(t)}\right]^{\alpha+1/2}P_n^{(\alpha,\beta)}(\cos t)\left[\frac{1}{f'(t)}\right]^{3/2}.$$

Therefore, from (3.1) we obtain

$$|I| \le M\frac{\pi}{2}\left|\int_0^{\vartheta} f^{\alpha+1}(t)f'(t)\Delta(t,\vartheta)dt\right|. \tag{3.2}$$

Observe that $f(t)\ge Nt$ and $f'(t)\ge N$. Taking into account that (Szegö [10], p. 168)

$$\left|P_n^{(\alpha,\beta)}(\cos t)\right| \le P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},$$

we get

$$M \le F(t)\binom{n+\alpha}{n}\frac{1}{2^{\alpha+1/2}}\frac{1}{N^{\alpha+2}}.$$

Therefore, (2.8) gives

$$M \le \frac{1}{2^{\alpha+1/2}}\frac{A\mu_1+B\mu_2}{16N^{\alpha+4}}\binom{n+\alpha}{n}, \tag{3.3}$$

which is slightly different from the corresponding result in [2, (4.7)].

The integral in (3.2) may be explicitly evaluated and, as in [2], we have

$$\left|\int_0^{\vartheta} f^{\alpha+1}(t)\Delta(t,\vartheta)df(t)\right| \le \frac{\pi}{8(1+\alpha)}N^{\alpha}\vartheta^{\alpha}(1.001577737)^{1/2}, \tag{3.4}$$

for $\vartheta \le \vartheta^*$ and $n \ge 5$.

By substitution of (3.3) and (3.4) into (3.2) we obtain the following estimate for $|I|$

$$|I| \le \frac{\vartheta^{\alpha}}{N^4}\binom{n+\alpha}{n}(A\mu_1 + B\mu_2)0.0771709493,$$

which, on account of (2.7) becomes

$$|I| \le \frac{\vartheta^{\alpha}}{N^4}\binom{n+\alpha}{n}[0.0052880384A + 0.0857504153B].$$

This inequality can be improved. Indeed, it can be shown that

$$0 \le I \le \frac{\vartheta^{\alpha}}{N^4}\binom{n+\alpha}{n}[0.0052880384A + 0.0857504153B], \tag{3.5}$$

for $0 < \vartheta < \vartheta^*$ and $n \ge 5$. Here we shall give only an outline of a very simple proof based on the following well-known Sturm-type comparison theorem (see Szegö [10], p. 20).

Theorem 3.1. *Let $q(x)$ and $Q(x)$ be functions continuous in $x_0 < x < X_0$ with $q(x) \le Q(x)$. Let the functions $y(x)$ and $Y(x)$, both not identically zero, satisfy the differential equations*

$$y'' + q(x)y = 0, \qquad Y'' + Q(x)Y = 0,$$

respectively. Let x' and x'', $x' < x''$, be two consecutive zeros of $y(x)$. We denote by ξ the first zero of $Y(x)$ to the right of x', $x' < \xi < x''$.

Assuming that $y(x) > 0$, $Y(x) > 0$ *in* $x' < x < \xi$, *and*

$$\lim_{x \to x'+0} \frac{y(x)}{Y(x)} \geq 1\,,$$

we have $y(x) > Y(x)$ *in* $x' < x < \xi$.

The statement also holds for $x' = x_0$ $\left[y(x_0+0) = 0\right]$ *if the additional condition*

$$\lim_{x \to x_0+0} \left[y'(x)Y(x) - y(x)Y'(x)\right] = 0$$

is satisfied.

Taking into account of some results obtained in [8], and applying the above theorem to the differential equations satisfied by $u_n^{(\alpha,\beta)}(\vartheta)$ and $\left[f(\vartheta)/f'(\vartheta)\right]^{-1/2} J_\alpha\left[f(\vartheta)\right]$, we find that for $0 < \vartheta < \vartheta^*$,

$$\left[\frac{f(\vartheta)}{f'(\vartheta)}\right]^{-1/2} u_n^{(\alpha,\beta)}(\vartheta) \geq \frac{\Gamma(\alpha+1)}{2^{1/2}} \binom{n+\alpha}{n} N^{-\alpha} \left[1+ \right.$$

$$\left. + \frac{1}{32N^2}\left(\frac{A}{3} + B\right)\right]^{-\alpha} J_\alpha\left[f(\vartheta)\right] + I\,,$$

which, by virtue of (2.10), completes the proof of the inequality (3.5).

b) The case $\vartheta^* \leq \vartheta \leq \pi/2$.

In this case we divide the integration interval into the two subintervals $[0, \vartheta^*]$ and $[\vartheta^*, \vartheta]$, and denote by I_1 and I_2 the two corresponding integrals.

For I_1, analogously to (3.2), we have

$$|I_1| \leq M\frac{\pi}{2}\left|\int_0^{\vartheta^*} f^{\alpha+1}(t)f'(t)\Delta(t,\vartheta)dt\right|\,, \tag{3.6}$$

and we shall use the inequality (see [2], p. 213)

$$\left|\int_0^{\vartheta^*} \Delta(t,\vartheta)f^{\alpha+1}(t)df(t)\right| \leq \left[\frac{2}{\pi f(\vartheta)}\right]^{1/2}\left\{\left(\frac{\pi}{2}\right)^{\alpha+1} 2^{1/2}(J^2_{\alpha+1}(\pi/2)+ \right.$$

$$\left. + Y^2_{\alpha+1}(\pi/2))^{1/2} + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{\pi}\right\}.$$

Since $f(\vartheta) \geq N\vartheta$ and (Watson [11], p. 449)

$$J_{\alpha+1}^2(x) + Y_{\alpha+1}^2(x) \leq \frac{2}{\pi x}\left[1 + \frac{4(\alpha+1)^2 - 1}{8x^2}\right],$$

we obtain

$$\left|\int_0^{\vartheta^*} \Delta(t,\vartheta) f^{\alpha+1}(t) df(t)\right| \leq \left[\frac{2}{\pi N\vartheta}\right]^{1/2} \frac{2}{\pi}\left\{\left(\frac{\pi}{2}\right)^{\alpha+1}\left(2+\right.\right.$$

$$\left.\left. + \frac{4(\alpha+1)^2 - 1}{\pi^2}\right)^{1/2} + 2^\alpha \Gamma(\alpha+1)\right\};$$

that is

$$\left|\int_0^{\vartheta^*} \Delta(t,\vartheta) f^{\alpha+1}(t) df(t)\right| \leq \frac{\Gamma(\alpha+1)}{\pi^{3/2}} 2^{\alpha+3/2} (N\vartheta)^{-1/2} g(\alpha),$$

where

$$g(\alpha) = \pi^{\alpha+1} 2^{-2\alpha-1}\left(2 + \frac{4(\alpha+1)^2 - 1}{\pi^2}\right)^{1/2} \frac{1}{\Gamma(\alpha+1)} + 1. \tag{3.7}$$

Making use of (3.3) (which is still valid in this case), (3.6) becomes

$$|I_1| \leq \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} \frac{1}{N\vartheta} \vartheta^{1/2} \frac{A\mu_1 + B\mu_2}{16 N^{\alpha+7/2}} g(\alpha). \tag{3.8}$$

Since $\vartheta \geq \vartheta^*$, according to (2.9),

$$\frac{1}{N\vartheta} \leq \frac{1}{N\vartheta^*} \leq \frac{2}{\pi}(0.9979776744)^{-1} = 0.6379098338 = h.$$

Consequently, (3.8) gives

$$|I_1| \leq \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} N^{-\alpha-7/2} \vartheta^{1/2} \frac{h}{16}(A\mu_1 + B\mu_2) g(\alpha). \tag{3.9}$$

The continuous function $g(\alpha)$, defined on $-1/2 \le \alpha \le 1/2$ by (3.7), reaches its maximum at the point $\alpha^* = 0.43212019\ldots$, and

$$g(\alpha^*) = 3.638979419\ldots .$$

Thus (3.9) gives

$$|I_1| \le \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} N^{-\alpha-7/2} \vartheta^{1/2} (Ah_{11} + Bh_{12}) , \tag{3.10}$$

where

$$h_{11} = \frac{h}{16} \mu_1 g(\alpha^*) , \qquad h_{12} = \frac{h}{16} \mu_2 g(\alpha^*) .$$

For the integral I_2 we have, as in [2, (4.12)],

$$|I_2| \le 2 \left[\frac{1}{f(\vartheta)}\right]^{1/2} \frac{A\mu_1 + B\mu_2}{16N^2} \int_{\vartheta^*}^{\vartheta} \left[\frac{1}{f(t)}\right]^{1/2} \left[\frac{f(t)}{f'(t)}\right]^{1/2}$$

$$\left| \left(\sin \frac{t}{2}\right)^{\alpha+1/2} \left(\cos \frac{t}{2}\right)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos t) \right| dt .$$

Therefore, using inequality (2.18) and taking into account that $f(t) > Nt$ and $f(t)/f'(t) < t$ for $0 < t \le \pi/2$, we get

$$|I_2| \le \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} \left[\frac{1}{f(\vartheta)}\right]^{1/2} \frac{A\mu_1 + B\mu_2}{8N^2} N^{-\alpha-1/2} K(n) \int_{\vartheta^*}^{\vartheta} \frac{t^{1/2}}{(Nt)^{1/2}} dt ,$$

where $K(n)$ is defined by (2.19), and

$$|I_2| \le \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} N^{-\alpha-7/2} \vartheta^{1/2} (Ah_{21} + Bh_{22}) , \tag{3.11}$$

with

$$h_{21} = \frac{\mu_1}{8} K(n) , \qquad h_{22} = \frac{\mu_2}{8} K(n) .$$

Now we observe that for $n \geq 5$,

$$0.0185071416 < h_{11} + h_{21} < 0.0185126399\,,$$

$$0.3001103524 < h_{12} + h_{22} < 0.3001995120\,.$$

Summing up (3.10) and (3.11), it follows

$$|I| \leq \vartheta^{1/2} \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} N^{-\alpha-7/2}(0.01852A+ \tag{3.12}$$
$$+\,0.30020B)\,, \qquad \vartheta^* \leq \vartheta \leq \pi/2\,.$$

The main result of this section is stated in the following theorem.

THEOREM 3.2. *Let $-1/2 \leq \alpha, \beta \leq 1/2$ and let ϑ^* be the root of the transcendental equation $f(\vartheta) = \pi/2$. Then the following asymptotic representation holds*

$$\left[\frac{f(\vartheta)}{f'(\vartheta)}\right]^{-1/2} \left(\sin\frac{\vartheta}{2}\right)^{\alpha+1/2} \left(\cos\frac{\vartheta}{2}\right)^{\beta+1/2} P_n^{(\alpha,\beta)}(\cos\vartheta) = \tag{3.13}$$
$$= \frac{\Gamma(\alpha+1)}{2^{1/2}} \binom{n+\alpha}{n} N^{-\alpha} \left[1 + \frac{1}{32N^2}\left(\frac{A}{3} + B\right)\right]^{-\alpha} J_\alpha[f(\vartheta)] + I\,,$$

where for $n \geq 5$

$$0 \leq I \leq \vartheta^\alpha \binom{n+\alpha}{n} N^{-4}(0.00529A + 0.08576B)\,, \qquad 0 \leq \vartheta \leq \vartheta^*\,,$$

$$|I| \leq \vartheta^{1/2} \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} N^{-\alpha-7/2}(0.01852A + 0.30020B)\,, \ \vartheta^* \leq \vartheta \leq \pi/2,$$

$$A = 1 - 4\alpha^2\,, \qquad B = 1 - 4\beta^2\,.$$

In the ultraspherical case, $\alpha = \beta$, we have the following corollary:

Corollary 3.1. *Let $-1/2 \le \alpha \le 1/2$ and let ϑ^* be the root of the transcendental equation $f(\vartheta) = \pi/2$. Then the following asymptotic representation holds:*

$$\left[\frac{f(\vartheta)}{f'(\vartheta)}\right]^{-1/2} (\sin\vartheta)^{\alpha+1/2} P_n^{(\alpha,\alpha)}(\cos\vartheta) = \tag{3.14}$$

$$= 2^\alpha \Gamma(\alpha+1) \binom{n+\alpha}{n} N^{-\alpha} \left[1 + \frac{1-4\alpha^2}{24N^2}\right]^{-\alpha} J_\alpha[f(\vartheta)] + I^*,$$

where $N = n + \alpha + 1/2$ and, if $n \ge 5$,

$$0 \le I^* \le 2^{\alpha+1/2} \vartheta^\alpha \binom{n+\alpha}{n} N^{-4} (1-4\alpha^2) 0.09104, \quad 0 \le \vartheta \le \vartheta^*,$$

$$|I^*| \le 2^{\alpha+1/2} \frac{\Gamma(\alpha+1)}{\pi^{1/2}} \binom{n+\alpha}{n} N^{-\alpha-7/2} (1-4\alpha^2) 0.31872, \quad \vartheta^* \le \vartheta \le \pi/2.$$

Here $f(\vartheta)$ can be written in the form

$$f(\vartheta) = N\vartheta + \frac{1-4\alpha^2}{8N}\left(\frac{1}{\vartheta} - \cot\vartheta\right).$$

The bounds for the error terms given in Theorem 3.2 and Corollary 3.1 are better than the ones obtained in [2]. Further, notice that there is a mistake in the bounds previously given for the ultraspherical case ([2], Corollary 4.1); indeed such bounds must be multiplied by the factor $2^{\alpha+1/2}$.

4 – The representation of the zeros

In this section new bounds are derived for the error term in the representation of the zeros of $P_n^{(\alpha,\beta)}(\cos\vartheta)$. Here, we shall give only a sketch of the procedure used for obtaining such bounds; further details may be found in [2].

Let $\vartheta_{n,k} \equiv \vartheta_{n,k}(\alpha,\beta)$, $k=1,2,\dots,n$, denote the zeros of $P_n^{(\alpha,\beta)}(\cos\vartheta)$, in increasing order. Further, let $j_{\alpha,k}$, $k=1,2,\dots$, be the positive zeros of $J_\alpha(x)$. Throughout this section we shall continue to assume $-1/2 \le \alpha,\beta \le 1/2$.

We first recall (see GATTESCHI [9], p. 1553) that if $\tau_{n,k} \equiv \tau_{n,k}(\alpha,\beta)$ is the root of the equation $f(\vartheta) = j_{\alpha,k}$, $f(\vartheta)$ being defined by (2.2), that is, of the equation

$$N\vartheta + \frac{1}{16N}\left[A\left(\frac{2}{\vartheta} - \cot\frac{\vartheta}{2}\right) + B\tan\frac{\vartheta}{2}\right] = j_{\alpha,k}\,, \tag{4.1}$$

then

$$\vartheta_{n,k} \ge \tau_{n,k}\,, \qquad k = 1,2,\dots,n\,.$$

Since $f(\vartheta)$ is a monotonically increasing function of ϑ and

$$j_{\alpha,k} \ge j_{\alpha,l} \ge j_{-1/2,l} = \frac{\pi}{2}\,, \qquad \alpha \ge -\frac{1}{2}\,,$$

it follows

$$\vartheta_{n,k} \ge \tau_{n,l} \ge \vartheta^*\,, \qquad k = 1,2,\dots,n\,,$$

where ϑ^* is the root of the equation $f(\vartheta) = \pi/2$ and satisfies the inequality (2.9).

Having proved that all the zeros of $P_n^{(\alpha,\beta)}(\cos\vartheta)$ are greather than ϑ^*, according to Theorem 3.2, the zeros $\vartheta_{n,k}$ lying in the interval $0 < \vartheta \le \pi/2$ coincide with the zeros of the function

$$U_n^{(\alpha,\beta)}(\vartheta) = J_\alpha\big[f(\vartheta)\big] + E_n(\alpha,\beta)\vartheta^{1/2}N^{-7/2}\,, \tag{4.2}$$

where

$$\big|E_n(\alpha,\beta)\big| \le \left(\frac{2}{\pi}\right)^{1/2}\left[1+\frac{1}{32N^2}\left(\frac{A}{3}+B\right)\right]^{\alpha}(0.01852A+0.30020B)\,. \tag{4.3}$$

We now recall some other results concerning the zeros $\vartheta_{n,k}$.

By using the inequality (Gatteschi [7])

$$
(4.4)\qquad
\begin{aligned}
&j_{\alpha,k}\left[N^2+\frac{1}{4}-\frac{\alpha^2+\beta^2}{2}-\frac{1-4\alpha^2}{\pi^2}\right]^{-1/2}<\vartheta_{n,k}\le\\
&\le j_{\alpha,k}\left[N^2+\frac{1-\alpha^2-3\beta^2}{12}\right]^{-1/2},
\end{aligned}
$$

we readily derive ([2], Lemma 5.1)

$$
(4.5)\qquad \frac{j_{\alpha,k}}{N}\left(1-\frac{1}{8N^2}\right)<\vartheta_{n,k}\le\frac{j_{\alpha,k}}{N},
$$

where the equality sign holds when $\alpha^2=\beta^2=1/4$.

The upper bound for $\vartheta_{n,k}$ in (4.4) is very sharp. Indeed, the more general asymptotic representation holds (Gatteschi [8])

$$
(4.6)\qquad
\begin{aligned}
\vartheta_{n,k}&=\frac{j_{\alpha,k}}{\nu}\left\{1-\frac{4-\alpha^2-15\beta^2}{720\nu^4}\left(\frac{j_{\alpha,k}^2}{2}+\alpha^2-1\right)\right\}+\\
&+j_{\alpha,k}^5O(n^{-7}),\quad n\to\infty,
\end{aligned}
$$

where

$$
\nu=\left[N^2+\frac{1-\alpha^2-3\beta^2}{12}\right]^{1/2},\quad k=1,2,\dots,[pn],
$$

p being a positive number in $(0,1)$. Unfortunately, we have only a qualitative bound for the remainder term in (4.6).

Another interesting result which provides a lower bound for $\vartheta_{n,k}$ is given by the following theorem.

Theorem 4.1 (Gatteschi [9]). *Let $t_{n,k}\equiv t_{n,k}(\alpha,\beta)=j_{\alpha,k}/N$, $A=1-4\alpha^2$ and $B=1-4\beta^2$. Then*

$$
(4.7)\qquad \vartheta_{n,k}\ge t_{n,k}-\frac{1}{16N^2}\left[A\left(\frac{2}{t_{n,k}}-\cot\frac{t_{n,k}}{2}\right)+B\tan\frac{t_{n,k}}{2}\right],
$$

for $k=1,2,\dots,n$. The equality sign in (4.1) holds if and only if $\alpha^2=\beta^2=1/4$.

In what follows we shall improve the result in (4.7) by constructing an upper bound for $\vartheta_{n,k}$. To this end, we need to recall another property of the zeros $j_{\alpha,k}$ and $\vartheta_{n,k}$.

LEMMA 4.1 ([2], Lemma 5.2). *Let $\tau_{n,k} \equiv \tau_{n,k}(\alpha, \beta)$ be the root of equation* (4.1) *in the interval* $(0, \pi/2)$*. Then*

$$\frac{j_{\alpha,k}}{N}\left(1 - \frac{1}{8N^2}\right) < \tau_{n,k} \le \vartheta_{n,k}\,;$$

that is, from (4.5)*, $\tau_{n,k}$ and $\vartheta_{n,k}$ belong the same interval*

$$\frac{j_{\alpha,k}}{N}\left(1 - \frac{1}{8N^2}\right) < \vartheta \le \frac{j_{\alpha,k}}{N}\,.$$

Let us know set $\vartheta_{n,k} = \tau_{n,k} + \varepsilon$, and put $\vartheta = \vartheta_{n,k}$ in (4.2). Then we have

$$\begin{aligned}\varepsilon J'_\alpha[f(\xi)]\left\{N + \frac{1}{16N}\left[A\left(\frac{1}{2\sin^2\xi/2} - \frac{2}{\xi^2}\right) + \frac{B}{2\cos^2\xi/2}\right]\right\} + \\ + E_n(\alpha,\beta)\vartheta_{n,k}^{1/2}N^{-7/2} = 0\end{aligned} \tag{4.8}$$

with $\tau_{n,k} < \xi < \vartheta_{n,k}$. It follows from Lemma 4.1 that $0 < \varepsilon < j_{\alpha,k}/(8N)^3$. Since $f(\vartheta)$ and $f'(\vartheta)$ are monotonically increasing functions in $[0, \pi/2]$, we have

$$\begin{aligned}j_{\alpha,k} = f(\tau_{n,k}) < f(\xi) < f\left(\tau_{n,k} + \frac{j_{\alpha,k}}{8N^3}\right) \le f(\tau_{n,k}) + \frac{j_{\alpha,k}}{8N^3} f'\left(\frac{\pi}{2}\right) \le \\ \le j_{\alpha,k} + \frac{j_{\alpha,k}}{8N^2} + \frac{j_{\alpha,k}}{64N^4}\left(1 - \frac{4}{\pi^2}\right);\end{aligned}$$

that is, if $n \ge 5$

$$\begin{aligned}j_{\alpha,k} < f(\xi) < j_{\alpha,k}\left(1 + \frac{\gamma_1}{8N^2}\right), \\ \gamma_1 = 1 + \frac{1}{200}\left(1 - \frac{4}{\pi^2}\right) = 1.002973576\,.\end{aligned} \tag{4.9}$$

By using this inequality it can be proved that

$$\left|J'_\alpha[f(\xi)]\right| > \left[\frac{2}{\pi f(\xi)}\right]^{1/2}\left[\sin\left(\frac{\pi}{4}-\frac{\gamma_1}{80}\pi\right)- \right. \tag{4.10}$$
$$\left. -\frac{4\alpha^2+3}{8f(\xi)}\cos\left(\frac{\pi}{4}-\frac{\gamma_1}{80}\pi\right)\right],$$

when $\tau_{n,k} < \xi < \vartheta_{n,k}$ and $n \geq 5$. The proof given in [2] is based on the asymptotic representation of $J'_\alpha(x)$ as $x \to \infty$ and the well-known inequalities (Watson [11], p. 490)

$$k\pi - \frac{\pi}{4} + \frac{1}{2}\alpha\pi \leq j_{\alpha,k} \leq k\pi - \frac{\pi}{8} + \frac{1}{4}\alpha\pi\,, \tag{4.11}$$
$$k = 1, 2, \dots\,, \qquad -1/2 \leq \alpha \leq 1/2\,.$$

From (4.9) and (4.11) we have

$$[f(\xi)]^{-1/2} > j_{\alpha,k}^{-1/2}\left[1+\frac{\gamma_1}{8N^2}\right]^{-1/2} \geq j_{\alpha,k}^{-1/2}\left[1+\frac{\gamma_1}{200}\right]^{-1/2},$$

$$\frac{4\alpha^2+3}{8f(\xi)} < \frac{4\alpha^2+3}{8j_{\alpha,k}} \leq \frac{1}{\pi}\,,$$

respectively. Therefore, (4.10) gives

$$\left|J'_\alpha[f(\xi)]\right| > \left(\frac{2}{\pi j_{\alpha,k}}\right)^{1/2}\gamma_2\,, \tag{4.12}$$

where, for $n \geq 5$ and $-1/2 \leq \alpha \leq 1/2$,

$$\gamma_2 = \left[\sin\left(\frac{\pi}{4}-\frac{\gamma_1}{80}\pi\right) - \frac{1}{\pi}\cos\left(\frac{\pi}{4}-\frac{\gamma_1}{80}\pi\right)\right]\left(1+\frac{\gamma_1}{200}\right)^{-1/2} = \tag{4.13}$$
$$= 0.4438361509\,.$$

Since, according to Lemma 4.1, $\vartheta_{n,k} \geq \tau_{n,k}$, from (4.8) and (4.12) we get

$$0 \leq \vartheta_{n,k} - \tau_{n,k} \leq |E_n(\alpha,\beta)|\vartheta^{1/2} N^{-9/2}\left(\frac{\pi j_{\alpha,k}}{2}\right)^{1/2}\frac{1}{\gamma_2} \leq$$

$$\leq |E_n(\alpha,\beta)| j_{\alpha,k} N^{-5}\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\gamma_2}\,.$$

Observing that

$$\left[1+\frac{1}{32N^2}\left(\frac{A}{3}+B\right)\right]^{\alpha} \leq \left[1+\frac{1}{24N^2}\right]^{-1/2} \leq \gamma_3\,,$$

for $n \geq 5$, where

$$\gamma_3 = \left[1+\frac{1}{600}\right]^{1/2} = 1.000832986\,,$$

and using (4.3) we obtain the preliminary result

$$0 \leq \vartheta_{n,k} - \tau_{n,k} \leq \varepsilon_k^*(\alpha,\beta) N^{-5}\,, \tag{4.14}$$

where

$$0 \leq \varepsilon_k^*(\alpha,\beta) \leq \frac{\gamma_3}{\gamma_2} j_{\alpha,k}(0.01852A + 0.30020B)\,. \tag{4.15}$$

To represent $\vartheta_{n,k}$ in terms of $j_{\alpha,k}$ instead of $\tau_{n,k}$, we write the equation (4.1) in the form $\vartheta = h(\vartheta)$, where

$$h(\vartheta) = \frac{j_{\alpha,k}}{N} - \frac{1}{16N^2}\left[A\left(\frac{2}{\vartheta} - \cot\frac{\vartheta}{2}\right) + B\tan\frac{\vartheta}{2}\right].$$

Then, for some $\bar\vartheta$ between $\tau_{n,k}$ and $t_{n,k} = j_{\alpha,k}/N$,

$$\tau_{n,k} - h(t_{n,k}) h(\tau_{n,k}) - h(t_{n,k}) = (\tau_{n,k} - t_{n,k}) h'(\bar\vartheta) =$$

$$= (t_{n,k} - \tau_{n,k})\frac{1}{16N^2}\left[\frac{A}{2}\left(\frac{1}{\sin^2\bar\vartheta/2} - \frac{4}{\bar\vartheta^2}\right) + \frac{B}{2\cos^2\bar\vartheta/2}\right].$$

Replacing $\bar{\vartheta}$ by $\pi/2$ and observing that from Lemma 4.1

$$0 < t_{n,k} - \tau_{n,k} \leq \frac{j_{\alpha,k}}{8N^3},$$

we obtain

$$0 < \tau_{n,k} - h(t_{n,k}) \leq \frac{j_{\alpha,k}}{128N^5}\left[A\left(1 - \frac{8}{\pi^2}\right) + B\right].$$

This, together with (4.14) gives

$$\vartheta_{n,k} - h(t_{n,k}) \leq \frac{j_{\alpha,k}}{128N^5}\left[A\left(1 - \frac{8}{\pi^2}\right) + B\right] + \varepsilon_k^*(\alpha,\beta)N^{-5}.$$

We can now state the main result of this section.

THEOREM 4.2. *Let* $-1/2 \leq \alpha, \beta \leq 1/2$ *and*

$$t_{n,k} \equiv t_{n,k}(\alpha,\beta) = \frac{j_{\alpha,k}}{N}, \quad k = 1, 2, \dots.$$

Then, for the zeros $\vartheta_{n,k}(\alpha,\beta)$ *of* $P_n^{(\alpha,\beta)}(\cos\vartheta)$ *lying in* $0 \leq \vartheta \leq \pi/2$, *we have*

$$\vartheta_{n,k}(\alpha,\beta) = t_{n,k} - \frac{1}{16N^2}\left[A\left(\frac{2}{t_{n,k}} - \cot\frac{t_{n,k}}{2}\right) + B\tan\frac{t_{n,k}}{2}\right] + \varepsilon_k(\alpha,\beta)N^{-5}, \tag{4.16}$$

where, if $n \geq 5$,

$$0 \leq \varepsilon_k(\alpha,\beta) \leq j_{\alpha,k}(0.04325A + 0.68476B). \tag{4.17}$$

The equality sign in (4.17) *holds if and only if* $\alpha^2 = \beta^2 = 1/4$.

The new upper bound for $\varepsilon_k(\alpha,\beta)$ in (4.17) gives very sharp numerical results, not only for the early zeros of $P_n^{(\alpha,\beta)}(\cos\vartheta)$ but also for the zeros which are close to $\pi/2$. For such zeros, $j_{n,k} = O(n)$ so that the order of the error term in (4.16) reduces to $O(N^{-4})$.

In Table 1 the exact values of the zeros $\vartheta_{16,k}(-0.3, 0.4)$, $k=1,2,..,16$, are compared with the upper and lower bounds given by (4.16). Here use has also been made of (4.16) and the relationship $\vartheta_{n,k}(\alpha,\beta) = \pi - \vartheta_{n,n-k+1}(\beta,\alpha)$, $k = 1,\dots,n$, for $k = 1,2,\dots,8$ and for $k = 9,10,\dots,16$, respectively.

Table 1 - Zeros of $P_{16}^{(-3,4)}(\cos\vartheta)$.

k	Lower bound	Exact value	Upper bound
1	0.11617 69267	0.11617 69304	0.11617 73512
2	0.30464 01213	0.30464 01313	0.30464 12346
3	0.49409 72063	0.49409 72230	0.49409 90119
4	0.68374 70451	0.68374 70697	0.68374 95437
5	0.87346 55846	0.87346 56186	0.87346 87765
6	1.06321 57903	1.06321 58363	1.06321 96756
7	1.25298 25976	1.25298 26596	1.25298 71764
8	1.44275 84925	1.44275 85770	1.44276 37648
9	1.63253 01351	1.63253 91044	1.63253 92644
10	1.82231 30753	1.82232 09419	1.82232 10565
11	2.01209 41687	2.01210 09182	2.01210 10017
12	2.20186 93091	2.20187 49329	2.20187 49940
13	2.39163 15263	2.39163 60190	2.39163 60632
14	2.58136 56569	2.58136 90154	2.58136 90460
15	2.77102 76755	2.77102 98980	2.77102 99172
16	2.96040 79612	2.96040 90481	2.96040 90573

In the ultraspherical case $\alpha = \beta$, Theorem 4.2 gives:

COROLLARY 4.1. *Let $-1/2 \le \alpha \le 1/2$ and let $\vartheta_{n,k}(\alpha)$ be the k-th zero of the ultraspherical polynomial $P_n^{(\alpha,\alpha)}(\cos\vartheta)$. We have*

$$\vartheta_{n,k}(\alpha) = \frac{j_{\alpha,k}}{N} - \frac{1-4\alpha^2}{8N^2}\left(\frac{N}{j_{\alpha,k}} - \cot\frac{j_{\alpha,k}}{N}\right) + \varepsilon_k(\alpha)N^{-5}\,, \tag{4.18}$$

$$k = 1,2,\dots,[n/2]\,, \qquad N = n+\alpha+1/2\,,$$

with

$$0 \le \varepsilon_k(\alpha) \le (1-4\alpha^2) j_{\alpha,k} 0.72801\,, \quad n \ge 5\,.$$

Here the equality sign holds if and only if $\alpha = \pm 1/2$.

The upper bound for $\vartheta_{n,k}$ in (4.16), or in (4.18), is better than the one in (4.4) when k and n increase simultaneously. This is shown in Table 2 where the two upper bounds are compared. The asterisks indicate the cases where the upper bound in (4.4) is better than the one in (4.18).

Table 2 - Zeros of $P_{20}^{(25,25)}(\cos\vartheta)$.

k	Lower bound	Exact value	Upper bound (4.18)	Upper bound (4.4)
1	0.13400 89468	0.13400 89507	0.13400 93415	0.13400 89595*
2	0.28461 26121	0.28461 26205	0.28461 34504	0.28461 27268*
3	0.43574 54896	0.43574 55029	0.43574 67731	0.43574 59009*
4	0.58700 89836	0.58700 90022	0.58701 07127	0.58701 00005*
5	0.73832 36227	0.73832 36474	0.73832 57974	0.73832 56834*
6	0.88966 31702	0.88966 32021	0.88966 57907	0.88966 68643
7	1.04101 63911	1.04101 64317	1.04101 94574	1.04102 24916
8	1.19237 75791	1.19237 76307	1.19238 10913	1.19238 70890
9	1.34374 34164	1.34374 34822	1.34374 73744	1.34375 76359
10	1.49511 17040	1.49511 17890	1.49511 61079	1.49513 23302

It may be useful to notice that if $t_{n,k} = j_{\alpha,k}/N$ and k is fixed, then as $n \to \infty$

$$\frac{2}{t_{n,k}} - \cot\frac{t_{n,k}}{2} = \frac{j_{\alpha,k}}{6N} + O(n^{-3})\,, \quad \tan\frac{t_{n,k}}{2} = \frac{j_{\alpha,k}}{2N} + O(n^{-3})\,.$$

Hence (4.16) becomes

$$\vartheta_{n,k}(\alpha,\beta) = \frac{j_{\alpha,k}}{N} - \frac{j_{\alpha,k}}{24N^3}(1-\alpha^2-3\beta^2) + O(n^{-5})\,,$$

which can be written

$$\vartheta_{n,k}(\alpha,\beta) = j_{\alpha,k}\left(N^2 + \frac{1-\alpha^2-3\beta^2}{12}\right)^{-1/2} + O(n^{-5})\,. \tag{4.19}$$

It follows that (4.16) and the upper bound in (4.4), used as an approximation formula, are in fact asymptotically equivalent if k remains fixed as $n \to \infty$.

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