Rendiconti di Matematica, Serie VII
Volume 14, Roma (1994), 199-289

Christoffel functions and orthogonal polynomials for Erdös weights on (–∞,∞)

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Dedicated to the Memory of Aldo Ghizzetti

RIASSUNTO: *Si stabiliscono delle stime relative ai polinomi ortonormali e alle funzioni di Christoffel con pesi su* $\mathbb{R}$ *della forma* $W^2 = \mathrm{e}^{-2Q}$, *dove* Q *è una funzioni pari e con crescita all'*∞ *superiore a quella polinomiale (pesi cosiddetti di Erdös). Esempi tipici sono* $Q(x) := \exp_k(|x|^\alpha)$, $\alpha > 1$, *dove* $\exp_k = \exp(\exp(\ldots\exp(\cdot)))$ *denota la* k*-ima iterata esponenziale. Inoltre si ottengono delle stime uniformi relative alla distanza tra gli zeri e alle funzioni di Christoffel. Questi risultati completano quelli precedentemente noti relativi al caso in cui* Q *ha una crescita di tipo polinomiale all'*∞ *(pesi cosiddetti di Freud) e al caso di pesi esponenziali in* $(-1,1)$.

ABSTRACT: *We establish bounds on orthonormal polynomials and Christoffel functions associated with weights on* $\mathbb{R}$ *of the form* $W^2 = \mathrm{e}^{-2Q}$, *where* $Q : \mathbb{R} \to \mathbb{R}$ *is even, and is of faster than polynomial growth at* ∞ *(so-called Erdös weights). Typical examples are* $Q(x) := \exp_k\left(|x|^\alpha\right)$, $\alpha > 1$, *where* $\exp_k = \exp\left(\exp\left(\ldots\exp(\cdot)\right)\right)$ *denotes the* k*th iterated exponential. Further, we obtain uniform estimates on the spacing of all the zeros and on the Christoffel functions. These results complement earlier ones for the case where* Q *is of polynomial growth at* ∞ *(so-called Freud weights) and for exponential weights on* $(-1,1)$.

KEY WORDS AND PHRASES: *Erdös weights – Orthogonal polynomials – Christoffel functions*

1 – Introduction and results

Let $W := e^{-Q}$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous, and $Q(x)/\log|x| \to \infty$ as $x \to \infty$. Then all the power moments

$$\int_{-\infty}^{\infty} t^j W^2(t)dt\,, \quad j = 0, 1, 2, \ldots\,,$$

exist, and we can define corresponding orthonormal polynomials

$$p_n(W^2, x) = \gamma_n x^n + \ldots\,, \quad \gamma_n = \gamma_n(W^2) > 0\,, \tag{1.1}$$

satisfying

$$\int_{-\infty}^{\infty} p_n(W^2, x) p_m(W^2, x) W^2(x) dx = \delta_{mn}\,. \tag{1.2}$$

We denote the zeros of $p_n(x) = p_n(W^2, x)$ by

$$-\infty < x_{n,n} < x_{n-1,n} < \ldots < x_{2,n} < x_{1,n} < \infty\,.$$

In application of these orthogonal polynomials to various approximation processes, (such as orthonormal expansions or Lagrange interpolation), bounds on $p_n(W^2, x)$ in sup-norm or L_p-norm senses on the whole real line play a crucial role (see, for example, [3], [5], [6], [7], [12], [16], [20], [22], [23], [24]). In this paper, we shall obtain such bounds for the case where Q is of smooth and faster than polynomial growth at ∞, for example

$$W_{k,\alpha}(x) := e^{-Q_{k,\alpha}(x)} : Q_{k,\alpha}(x) := \exp_k\left(|x|^\alpha\right),\ k \geq 1\,,\ \alpha > 1\,, \tag{1.3}$$

where $\exp_k := \exp\Big(\exp\left(\exp(\cdots)\right)\Big)$ denotes the kth iterated exponential.

Since Q of faster than polynomial growth was first considered by Erdös, such weights are often called *Erdös weights*, in contrast to the case where Q is of polynomial growth at ∞, the so-called *Freud weights*.

A.M.S. Classification: 41A17 – 42C05 – 41A10

Bounds for orthogonal polynomials for Freud weights such as $\exp\left(-|x|^{\alpha}\right)$, $\alpha > 1$, were obtained in [7]; and for exponential weights on $(-1,1)$, such as $\exp\left(-(1-x^2)^{-\alpha}\right)$, or $\exp\left(-\exp_k(1-x^2)^{-\alpha}\right)$, $\alpha > 0$, $k \geq 1$, in [9]. The methods we use here broadly follow those in [7], [9], though are more similar in spirit to those for exponential weights on $(-1,1)$. Essentially, they seem to be more difficult than for the Freud case, though we can build on the ideas in [10]. There asymptotics for orthonormal polynomials were established, in a "large" subinterval of $(x_{n,n}, x_{1,n})$, but here we emphasise uniform bounds on $p_n(W^2, x)$ on the whole real line.

Following is our class of weights:

Definition 1.1. *Let* $W := \mathrm{e}^{-Q}$, *where* $Q : \mathbb{R} \to \mathbb{R}$ *is even, continuous, and* Q'' *exists in* $(0,\infty)$, $Q'' \geq 0$ *and* $Q' > 0$ *in* $(0,\infty)$, *and the function*

$$T(x) := 1 + x\frac{Q''(x)}{Q'(x)}, \quad x \in (0,\infty), \tag{1.4}$$

is increasing in $(0,\infty)$, *with*

$$\lim_{x\to\infty} T(x) = \infty; \qquad T(0+) := \lim_{x\to 0+} T(x) > 1. \tag{1.5}$$

Moreover, we assume that for some C_1, C_2, $C_3 > 0$,

$$C_1 \leq T(x)\frac{Q(x)}{xQ'(x)} \leq C_2, \quad x \geq C_3. \tag{1.6}$$

Then we write $W \in \mathcal{E}$.

Of course, the $\mathcal{E}$ stands for Erdös. It is the first limit in (1.5) that guarantees that Q is of faster than polynomial growth at ∞. The function $T(x)$ plays a crucial role in describing behaviour of growth of Q for Erdös weights on $\mathbb{R}$, and also for weights on $(-1,1)$ [9], [10], [11], [13]. As examples, we note that if $W = W_{k,\alpha}$, then

$$T(x) = T_{k,\alpha}(x) = \alpha\left[1 + x^{\alpha}\sum_{\ell=1}^{k}\prod_{j=1}^{\ell-1}\exp_j(x^{\alpha})\right], \tag{1.7}$$

(the empty product is taken as 1), and

$$T(x) = \alpha x^\alpha \left[\prod_{j=1}^{k-1} \exp_j(x^\alpha)\right](1 + o(1)), \quad x \to \infty. \tag{1.8}$$

On the other hand,

$$\frac{xQ'(x)}{Q(x)} = \alpha x^\alpha \left[\prod_{j=1}^{k-1} \exp_j(x^\alpha)\right],$$

so we have (1.6) in the stronger form

$$T(x)\frac{Q(x)}{xQ'(x)} \to 1, \quad x \to \infty. \tag{1.9}$$

So $W_{k,\alpha} \in \mathcal{E}$ provided $Q'' \geq 0$ in $(0,\infty)$ and $T(0+) > 1$, which is true only if $\alpha > 1$. For $\alpha < 1$, $Q_{k,\alpha}$ is not convex near 0 and for $\alpha = 1$, $T(0+) = \alpha = 1$. For such α, we can consider instead $W(x) := \exp\big(-Q_{k,\alpha/2}(A+x^2)\big) = W_{k,\alpha/2}(A+x^2)$, where A is chosen large enough to guarantee convexity of Q near 0 and $T(0+) > 1$. This W belongs to $\mathcal{E}$ and grows like $W_{k,\alpha}$ at ∞.

Another example is $W = \mathrm{e}^{-Q}$, where

$$Q(x) := \exp\Big(\big[\log(A + x^2)\big]^\beta\Big), \quad \beta > 1, \quad A > 0, \tag{1.10}$$

for which

$$T(x) = \frac{2x^2}{A + x^2}\left[\frac{\beta - 1}{\log(A + x^2)} + \beta\big\{\log(A + x^2)\big\}^{\beta-1}\right] + \frac{2A}{A + x^2}, \tag{1.11}$$

so that

$$T(x) = 2\beta\big[\log(A + x^2)\big]^{\beta-1}(1 + o(1)), \quad x \to \infty, \tag{1.12}$$

while

$$\frac{xQ'(x)}{Q(x)} = \frac{2\beta x^2}{A + x^2}\big[\log(A + x^2)\big]^{\beta-1},$$

so again (1.6) holds in the stronger form (1.9). To ensure convexity of Q near 0, we must choose $A = A(\beta)$ large enough.

To state our results, we need the MHASKAR-RAHMANOV-SAFF number a_u, the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}}\,, \quad u > 0\,. \tag{1.13}$$

Amongst its uses is the identity [17], [18],

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n,a_n]}\,, \quad P \in \mathcal{P}_n\,. \tag{1.14}$$

Note that for $Q = Q_{k,\alpha}$, $a_n = a_n\ (Q_{k,\alpha})$ satisfies

$$a_n = \left[\log_{k-1}\left(\log n - \frac{1}{2}\sum_{j=2}^{k+1} \log_j n + O(1)\right)\right]^{1/\alpha}, \tag{1.15}$$

where $\log_j = \log\Big(\log\left(\cdots\log()\right)\Big)$ denotes the jth iterated logarithm. This can be deduced from (1.13) by Laplace's method. Moreover, for this weight, note that from (1.8) and (1.15) follows

$$T(a_n) \sim \prod_{j=1}^{k} \log_j n\,. \tag{1.16}$$

Here and in the sequel,

$$c_n \sim d_n$$

means that there exist positive constants C_1, C_2 such that

$$C_1 \le \frac{c_n}{d_n} \le C_2$$

for the relevant range of n. Similar notation is used for functions and sequences of functions.

In the sequel, note that C, C_1, C_2, $\ldots$ denote positive constants independent of n, x and polynomials of degree $\le n$. The same symbol does not necessarily represent the same constant from line to line. The polynomials of degree $\le n$ are denoted by $\mathcal{P}_n$.

As in [7], [8], [9], the bounds on orthogonal polynomials depend on first finding the bounds for the *Christoffel functions*

$$\lambda_n(W^2,x) := \inf_{P\in\mathcal{P}_n}\left\{\frac{\int_{-\infty}^{\infty}(PW)^2(t)dt}{P^2(x)}\right\} = \frac{1}{\sum_{j=0}^{n-1} p_j(W^2,x)^2}\,. \tag{1.17}$$

The description of our estimate involves the special sequence

$$\delta_n := \big(nT(a_n)\big)^{-2/3}, \quad n\geq 1\,, \tag{1.18}$$

and for a fixed $L\geq 0$, the special sequence of functions

$$\Psi_n(x) := \max\left\{\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}\,,\left[T(a_n)\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}\right]^{-1}\right\}, \tag{1.19}$$

defined for $|x|\leq a_n\,(1+2L\delta_n)$. These play much the same role for Erdös weights as do the special sequence n^{-2} and the function $\max\left\{\frac{\sqrt{1-x^2}}{n},\frac{1}{n^2}\right\}$ in algebraic polynomial approximation, and orthogonal polynomials, on $[-1,1]$.

Throughout, we assume that $W=\mathrm{e}^{-Q}\in\mathcal{E}$. Our result for Christoffel functions is:

THEOREM 1.2. *Uniformly for $n\geq 1$ and $|x|\leq a_n(1+L\delta_n)$,*

$$\lambda_n(W^2,x)\sim\frac{a_n}{n}W^2(x)\Psi_n(x)\,. \tag{1.20}$$

Moreover, uniformly for $|x|\geq a_n$,

$$\lambda_n(W^2,x)\geq Ca_nW^2(x)\delta_n\,, \tag{1.21}$$

and given $0<\alpha<\beta<1$,

$$\begin{aligned}\sup_{x\in\mathbb{R}}\left\{\lambda_n^{-1}(W^2,x)W^2(x)\right\} &\sim \frac{n}{a_n}T(a_n)^{1/2}\sim\\ &\sim \min_{x\in[a_{\alpha n},a_{\beta n}]}\left\{\lambda_n^{-1}(W^2,x)W^2(x)\right\}.\end{aligned} \tag{1.22}$$

This may be compared to similar results for Freud weights [7, 465-6], [8] (where effectively $T \sim 1$), but is closer to results for exponential weights on $(-1,1)$ [9]. As a corollary, we can deduce results on largest zeros, and spacing of zeros of orthogonal polynomials:

Corollary 1.3. (a) *For some* $C_1 > 0$,

$$\left|1 - \frac{x_{1,n}}{a_n}\right| \leq C\delta_n \,. \tag{1.23}$$

(b) *Uniformly for* $n \geq 2$ *and* $2 \leq j \leq n-1$,

$$x_{j-1,n} - x_{j+1,n} \sim \frac{a_n}{n}\Psi_n(x_{j,n}) \,. \tag{1.24}$$

Here the constant L *in the definition of* Ψ_n *at* (1.19) *must be taken so large that* $x_{1,n} \leq a_n(1 + L\delta_n)$.

We note that with extra effort, we can replace $x_{j-1,n} - x_{j+1,n}$ by $x_{j,n} - x_{j+1,n}$ in (1.24). In fact, the exact same methods used in [3], [9] work, but we omit this as it would have extended an already lengthy paper. Now we state our bounds on the orthogonal polynomials:

Corollary 1.4. (a) *Uniformly for* $n \geq 1$,

$$\sup_{x\in\mathbb{R}} \left\{ |p_n(W^2, x)| W(x) \left|1 - \frac{|x|}{a_n}\right|^{1/4} \right\} \sim a_n^{-1/2} \tag{1.25}$$

and

$$\sup_{x\in\mathbb{R}} \left\{ |p_n(W^2, x)| W(x) \right\} \sim a_n^{-1/2} \big(nT(a_n)\big)^{1/6} \,. \tag{1.26}$$

(b) *Fix* L *so large as in Corollary 1.3. Then uniformly for* $n \geq 1$ *and* $1 \leq j \leq n$,

$$\begin{aligned} &\frac{a_n^{3/2}}{n}\Psi_n(x_{j,n})\left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/2} |p_n' W|(x_{j,n}) \sim \\ &\qquad \sim a_n^{1/2} |p_{n-1} W|(x_{j,n}) \sim \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/4} . \end{aligned} \tag{1.27}$$

As an example, note that for $n \geq 1$,

$$\sup_{x\in\mathbb{R}} \Big\{ \big|p_n(W_{k,\alpha}^2, x)\big| W_{k,\alpha}(x) \Big\} \sim (\log_k n)^{-1/(2\alpha)} \Big(n \prod_{j=1}^{k} \log_j n \Big)^{1/6} . \tag{1.28}$$

Finally, we record a more precise form of the infinite-finite range inequalities in [10], [19] which for our purposes is essential:

Theorem 1.5. *Let $0 < p \leq \infty$. Let $K > 0$. There exists C and n_1 depending only on K, p, W such that for $n \geq n_1$ and $P \in \mathcal{P}_n$,*

$$\|PW\|_{L_p(\mathbb{R})} \leq C \|PW\|_{L_p(|x|\leq a_n\left(1-K\delta_n\right))} . \tag{1.29}$$

We remark that as in [9], we can obtain $\sim$ relations for the L_p norms of orthonormal polynomials.

The organisation and methods of this paper are very similar to those in [7], [9]. We encourage the reader to have copies of [7], [9] on hand. Especially Section 2 of [9] will be helpful, as it contains an outline of our procedure.

In Section 2, we present some technical estimates involving Q, T, a_n and so on. In Section 3, we estimate the measure μ_n that arises in the integral equation. In Section 4, we use this to estimate the majorization function $U_{n,R}(x)$ and then in Section 5, we prove Theorem 1.5. In Section 6, we establish the lower bounds for λ_n implicit in Theorem 1.2, and in Section 7, we estimate L_∞ Christoffel functions. Then in Section 8, we use the L_∞ Christoffel functions to complete the proof of Theorem 1.2. In Section 9, we prove Corollary 1.3, and in Section 10, we prove Corollary 1.4.

2 – Technical Lemmas

In this section, assuming $W = e^{-Q} \in \mathcal{E}$, we shall prove various estimates on Q, T, a_u, etc. We begin with some estimates involving Q:

Lemma 2.1. (i) *For $0 < s \leq t$,*

$$\left(\frac{t}{s}\right)^{T(s)} \leq \frac{tQ'(t)}{sQ'(s)} \leq \left(\frac{t}{s}\right)^{T(t)} . \tag{2.1}$$

(ii) *$Q'(x)$ is increasing in $(0,\infty)$ and*

$$\lim_{v\to 0+} vQ'(v) = 0\,; \quad \lim_{v\to\infty} vQ'(v) = \infty\,. \tag{2.2}$$

Furthermore, $xQ''(x)$ is increasing in $(0,\infty)$.

(iii) *Given $r>0$, there exists $x_0>0$ such that for $x\ge x_0$, $Q^{(j)}(x)/x^r$ is increasing in (x_0,∞), $j=0,1,2$.*

(iv) *There exists C_1, C_2, $C_3>0$ such that*

$$Q'(x)^{C_1} \le Q(x) \le Q'(x)^{C_2}\,, \quad x\in(C_3,\infty)\,. \tag{2.3}$$

(v) *For some C_4, $C_5>0$,*

$$T(x) \le Q'(x)^{1-C_5}\,, \quad x\in(C_4,\infty)\,. \tag{2.4}$$

Proof. (i) This follows easily from the identity

$$\frac{tQ'(t)}{sQ'(s)} = \exp\Big(\int_s^t \frac{T(u)}{u}du\Big)$$

and the monotonicity of T.

(ii) The monotonicity of Q' and (2.2) follow immediately from $Q''\ge 0$ in $(0,\infty)$. Next, the monotonicity of $xQ''(x)$ follows from the identity

$$xQ''(x) = \big(T(x)-1\big)Q'(x)\,. \tag{2.5}$$

The two functions on the right-hand side are increasing.

(iii) Firstly for $j=0$, 1,

$$\frac{d}{dx}\left\{\frac{Q^{(j)}(x)}{x^r}\right\} = \frac{Q^{(j)}(x)}{x^{r+1}}\left\{\frac{xQ^{(j+1)}(x)}{Q^{(j)}(x)} - r\right\} \ge \frac{Q^{(j)}(x)}{x^{r+1}}\{C_6T(x)-r\} > 0\,,$$

for x large enough (see (1.4-6)). The monotonicity of $Q^{(j)}(x)/x^r$ then follows for $j=0,1$. For $j=2$, we write

$$\frac{Q''(x)}{x^r} = \frac{Q'(x)}{x^{r+1}}\{T(x)-1\}\,.$$

Here the right-hand side is the product of two functions that are increasing for x large.

(iv) Now for large enough x,

$$\frac{d}{dx}\log Q'(x) = \frac{Q''(x)}{Q'(x)} = \frac{T(x)-1}{x} \sim \frac{Q'(x)}{Q(x)} = \frac{d}{dx}\log Q(x)\,.$$

Since the assertion of (iii) implies that

$$\lim_{x\to\infty} Q(x) = \infty\,, \tag{2.6}$$

we deduce that

$$C_1 \log Q'(x) \le \log Q(x) \le C_2 \log Q'(x)\,,$$

for large enough x.

(v) The assertion of (iv) and (1.6) show that for large x,

$$T(x) \sim \frac{xQ'(x)}{Q(x)} \le xQ'(x)^{1-C_1}\,.$$

Since $x/Q'(x)^{C_1/2}$ is decreasing for large x, we have (2.4). $\square$

Now we present some results on a_u, etc.:

LEMMA 2.2. (i) *Uniformly for $u \ge C$, and $j = 0, 1, 2$,*

$$a_u^j Q^{(j)}(a_u) \sim u\,T(a_u)^{j-1/2}\,. \tag{2.7}$$

(ii) *Let $0 < \alpha < \beta$. Then uniformly for $u \ge C$,*

$$T(a_{\alpha u}) \sim T(a_{\beta u})\,. \tag{2.8}$$

(iii) *Given fixed $r > 1$,*

$$\frac{a_{ru}}{a_u} \ge 1 + \frac{\log r}{T(a_{ru})}\,, \quad u \in (0, \infty)\,. \tag{2.9}$$

Moreover,

$$a_{ru} \sim a_u\,, \quad u \in (1,\infty)\,. \tag{2.10}$$

(iv) *Uniformly for* $t \in (C,\infty)$,

$$\frac{a_t'}{a_t} \sim \frac{1}{tT(a_t)}\,. \tag{2.11}$$

(v) *Uniformly for* $u \in (C,\infty)$ *and* $v \in \left[\frac{u}{2}, 2u\right]$, *we have*

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right|\frac{1}{T(a_u)}\,. \tag{2.12}$$

(vi) *Let* $0 < \alpha < \beta$. *Then uniformly for* $u \geq C$, *and* $j = 0, 1, 2$,

$$Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u})\,. \tag{2.13}$$

(vii) *Let* $0 < \alpha < \beta$. *Then*

$$\frac{a_{\beta n} Q'(a_{\beta n})}{a_{\alpha n} Q'(a_{\alpha n})} \geq \frac{\beta}{\alpha}\,. \tag{2.14}$$

(viii) *For some* C_1, $C_2 > 0$, *and* $n \geq 1$,

$$T(a_n) \leq C_1 \left(\frac{n}{a_n}\right)^{2-C_2}\,.$$

Proof. (i) First note that since $tQ'(t)$ is strictly increasing in $(0,\infty)$, a_u is uniquely defined by (1.13) for $u > 0$. Then, by (2.1):

$$\frac{u}{a_u Q'(a_u)} = \frac{2}{\pi}\int_0^1 \frac{a_u t Q'(a_u t)}{a_u Q'(a_u)} \frac{dt}{\sqrt{1-t^2}} \geq \frac{2}{\pi}\int_0^1 t^{T(a_u)} \frac{dt}{\sqrt{1-t^2}} \geq$$

$$\geq \frac{2}{\pi}\big(1 - 1/T(a_u)\big)^{T(a_u)} \int_{1-1/T(a_u)}^{1} \frac{dt}{\sqrt{1-t^2}} \geq C_1 T(a_u)^{-1/2}\,.$$

Next,

$$\frac{u}{a_u Q'(a_u)} \le \frac{2}{\pi}\int_0^{1-1/T(a_u)} \frac{Q'(a_u t)}{Q'(a_u)} \frac{dt}{\sqrt{1-t^2}} + \frac{2}{\pi}\int_{1-1/T(a_u)}^{1} \frac{dt}{\sqrt{1-t^2}} \le$$
$$\le C_2 T(a_u)^{1/2} \int_0^1 \frac{Q'(a_u t)}{Q'(a_u)} dt + C_2 T(a_u)^{-1/2} =$$
$$= C_2 T(a_u)^{1/2} \frac{Q(a_u) - Q(0)}{a_u Q'(a_u)} + C_2 T(a_u)^{-1/2} \le$$
$$\le C_3 T(a_u)^{1/2} \frac{Q(a_u)}{a_u Q'(a_u)} + C_2 T(a_u)^{-1/2} \le C_4 T(a_u)^{-1/2},$$

for u large enough, by (1.6). These last two inequalities together give (2.7) for $j=1$. For $j=0$, we use

$$Q(a_u) \sim T(a_u)^{-1} a_u Q'(a_u) \quad \text{(see (1.6))}$$

and for $j=2$, we use

$$a_u^2 Q''(a_u) \sim T(a_u) a_u Q'(a_u),$$

see (1.4).

(ii) Now by (2.7) with $j=0$,

$$1 \le \left(\frac{T(a_{\beta u})}{T(a_{\alpha u})}\right)^{1/2} \sim \frac{\beta u Q(a_{\alpha u})}{\alpha u Q(a_{\beta u})} \le \frac{\beta}{\alpha}.$$

Here we have also used monotonicity of T and Q. So we have (2.8).

(iii) Differentiating (1.13) with respect to u gives

$$1 = \frac{a_u'}{a_u}\frac{2}{\pi}\int_0^1 T(a_u t) a_u t Q'(a_u t) \frac{dt}{\sqrt{1-t^2}} \le \frac{a_u'}{a_u} T(a_u) u. \tag{2.15}$$

Thus for $u > 0$,

$$\big(uT(a_u)\big)^{-1} \le \frac{a_u'}{a_u}. \tag{2.16}$$

Similarly (2.15) gives for $u \geq 1$,

$$\frac{2}{uT(a_1/2)} \geq \frac{a'_u}{a_u}\,. \tag{2.17}$$

Then

$$\log \frac{a_{ru}}{a_u} = \int_u^{ru} \frac{a'_t}{a_t} dt \geq \frac{1}{T(a_{ru})} \int_u^{ru} \frac{dt}{t} = \frac{\log r}{T(a_{ru})}\,,$$

so

$$\frac{a_{ru}}{a_u} = \exp\left(\log \frac{a_{ru}}{a_u}\right) \geq 1 + \log \frac{a_{ru}}{a_u} \geq 1 + \frac{\log r}{T(a_{ru})}\,.$$

So we have (2.9). Similarly (2.17) gives

$$\log \frac{a_{ru}}{a_u} \leq \frac{2}{T(a_1/2)} \log r\,.$$

Together with (2.9), this gives (2.10).

(iv) We must prove an upper bound corresponding to (2.16). From (2.15) and monotonicity of $tQ'(t)$,

$$1 \geq \frac{a'_u}{a_u} T(a_{u/2}) a_{u/2} Q'(a_{u/2})\ \frac{2}{\pi} \int_{a_{u/2}/a_u}^{1} \frac{dt}{\sqrt{1-t^2}} \geq$$

(by (2.7) and (2.8)) $$\geq C_5 \frac{a'_u}{a_u} uT(a_u)^{3/2} \Big(1 - \frac{a_{u/2}}{a_u}\Big)^{1/2} \geq$$

(by (2.9) and (2.10)) $$\geq C_6 \frac{a'_u}{a_u} uT(a_u)\,.$$

Together with (2.15), this gives (2.11).

(v) For $u \geq C$, and $v \in \Big[\dfrac{u}{2}, 2u\Big]$,

$$\log \frac{a_v}{a_u} = \int_u^v \frac{a'_t}{a_t} dt \sim \int_u^v \frac{dt}{tT(a_t)} \sim \frac{1}{T(a_u)} \log\left(\frac{v}{u}\right).$$

Since $\log t \sim t - 1$ for $t \in \Big[\dfrac{1}{2}, 2\Big] \setminus \{1\}$, we have the result.

(vi) Note that from (2.10) follows

$$a_{\alpha u} \sim a_{\beta u} \,.$$

Then (2.7) and (2.8) imply (2.13).

(vii) Now

$$\frac{a_{\beta n} Q'(a_{\beta n})}{a_{\alpha n} Q'(a_{\alpha n})} = \exp\Bigg(\int_{\alpha n}^{\beta n} \frac{d}{dt} \log\left(a_t Q'(a_t)\right) dt \Bigg) = \\ = \exp\Bigg(\int_{\alpha n}^{\beta n} T(a_t) \frac{a_t'}{a_t} dt \Bigg) \geq \exp\Bigg(\int_{\alpha n}^{\beta n} \frac{dt}{t} \Bigg) = \frac{\beta}{\alpha} \,,$$

by (2.15).

(viii) From (2.4), and then (2.7),

$$T(a_n) \leq Q'(a_n)^{1-C_5} \leq C_6 \left(\frac{n}{a_n} T(a_n)^{1/2} \right)^{1-C_5} \leq C_7 \left(\frac{n}{a_n} \right)^{1-C_5} T(a_n)^{1/2} \,.$$

Then the result follows. ∎

We need some more estimates on Q:

Lemma 2.3. (i) *Let $\sigma \in (0,1)$. Then for $j = 0, 1, 2$,*

$$\lim_{n\to\infty} \max_{0<|x|\leq a_n \sigma} \left\{ \frac{a_n^j \left| x^{\max\{0,j-1\}} Q^{(j)}(a_n x) \right|}{n} \right\} = 0 \,. \tag{2.18}$$

(ii) *There exists C_1 such that for $|x| \in (0,1)$, $u \in (1,\infty)$ and $j = 1, 2$,*

$$a_u^j \left| x^{j-1} Q^{(j)}(a_u x) \right| \left(1 - |x|\right)^{(2j-1)/2} \leq C_1 u \,. \tag{2.19}$$

PROOF. (i) Let $0<\sigma<\tau<1$. Now by (2.1),

$$\max_{|x|\le a_n\sigma}\{a_n|Q'(a_nx)|\}=\sigma^{-1}a_n\sigma Q'(a_n\sigma)\le\sigma^{-1}\Big(\frac{\sigma}{\tau}\Big)^{T(a_n\sigma)}a_n\tau Q'(a_n\tau)\le$$

$$\le C\Big(\frac{\sigma}{\tau}\Big)^{T(a_n\sigma)}(1-\tau)^{-1/2}\frac{2}{\pi}\int_\tau^1 a_ntQ'(a_nt)\frac{dt}{\sqrt{1-t^2}}\le$$

$$\le C\Big(\frac{\sigma}{\tau}\Big)^{T(a_n\sigma)}(1-\tau)^{-1/2}n\,.$$

Since $T(a_n\sigma)\to\infty$ as $n\to\infty$, we have (2.18) for $j=1$. Since

$$\max_{|x|\le a_n\sigma}\left\{\frac{|Q(a_nx)|}{n}\right\}\le\frac{Q(a_n)}{n}\sim T(a_n)^{-1/2}\to 0\,,\quad n\to\infty\,,$$

we have (2.18) for $j=0$ also. Finally, for large enough n,

$$\max_{0<|x|\le a_n\sigma}\{a_n^2|xQ''(a_nx)|\}\sim a_n\max_{|x|\le a_n\sigma}\{|Q'(a_nx)|T(a_nx)\}=$$

$$=a_nQ'(a_n\sigma)T(a_n\sigma)\le C_1\Big(\frac{\sigma}{\tau}\Big)^{T(a_n\sigma)}T(a_n\sigma)n$$

by the above, so we have (2.18) for $j=2$.

(ii) It suffices to consider $x\ge 0$. Now by monotonicity of Q',

$$u\ge\frac{2}{\pi}\int_x^1 a_utQ'(a_ut)\frac{dt}{\sqrt{1-t^2}}\ge\frac{2}{\pi}a_uQ'(a_ux)C_2(1-x)^{1/2}\,.$$

We have proved (2.19) for $j=1$. Integrating the defining equation (1.13) for a_u by parts, gives

$$u=\frac{2}{\pi}Q'(0+)+\frac{2}{\pi}\int_0^1 a_u^2Q''(a_ut)\sqrt{1-t^2}\,dt\,. \tag{2.20}$$

The monotonicity of $tQ''(t)$ shows that for $x\in\Big[\frac{1}{2},1\Big]$,

$$u\ge\frac{2}{\pi}a_u^2xQ''(a_ux)\int_x^1 t^{-1}\sqrt{1-t^2}\,dt\ge C_3a_u^2Q''(a_ux)(1-x)^{3/2}\,.$$

For $x \in \left(0, \frac{1}{2}\right]$, we can use the assertion of (i) to deduce (2.19). ∎

We need some estimates on the function

$$\Delta_n(s,t) := \frac{a_n s Q'(a_n s) - a_n t Q'(a_n t)}{a_n s - a_n t}, \quad s\,, t \in [-1,1] \setminus \{0\}\,. \tag{2.21}$$

LEMMA 2.4. (i) *For fixed* $t \in (0,1]$, $\Delta_n(s,t)$ *is an increasing function of* $s \in [0,1]$.

(ii) *Let* $s, t \in (0,1]$ *and* $\tau := \max\{s,t\}$. *There exists* $C > 0$ *independent of* s, t, n *such that*

$$\Delta_n(s,t) \le T(a_n\tau)Q'(a_n\tau) \le C\frac{n}{a_n}\min\left\{T(a_n),(1-\tau)^{-1}\right\}^{3/2}. \tag{2.22}$$

(iii) *Let* $0 < \beta < \rho < 1$. *Then for* $|t| \le a_{\beta n}/a_n$ *and* $|s| \in [a_{\rho n}/a_n, 1]$, *and some* C_1 *independent of* n, s, t,

$$a_n s Q'(a_n s) - a_n t Q'(a_n t) \ge C_1 a_n Q'(a_n)\,. \tag{2.23}$$

PROOF. (i) Since $\Delta_n(s,t)$ is the slope of a line segment joining two points of the curve $u \to (u, uQ'(u))$, $u \in [0,\infty)$, this follows from the convexity of $uQ'(u)$:

$$\frac{d}{du}(uQ'(u)) = T(u)Q'(u)$$

and the right-hand side is the product of two increasing functions.

(ii) For some ξ between $a_n s$ and $a_n t$,

$$\Delta_n(s,t) = \frac{d}{du}(uQ'(u))|_{u=\xi} = T(\xi)Q'(\xi) \le T(a_n\tau)Q'(a_n\tau)\,,$$

by monotonicity. Also, from (1.4),

$$T(a_n\tau)Q'(a_n\tau) = Q'(a_n\tau) + a_n\tau Q''(a_n\tau) \le C_2\frac{n(1-\tau)^{-3/2}}{a_n}\,,$$

by (2.19), while from (2.7),

$$T(a_n\tau)Q'(a_n\tau) \le C_3 \frac{nT(a_n)^{3/2}}{a_n}\,.$$

So we have (2.22).

(iii) Now for $|s| \ge a_{\rho n}/a_n$, $|t| \le a_{\beta n}/a_n$,

$$\begin{aligned}
\frac{a_n sQ'(a_n s) - a_n tQ'(a_n t)}{a_n Q'(a_n)} &= \frac{a_n sQ'(a_n s)}{a_n Q'(a_n)}\left[1 - \frac{a_n tQ'(a_n t)}{a_n sQ'(a_n s)}\right] \ge \\
\text{(by (2.1))} \qquad &\ge s^{T(a_n)}\left[1 - \frac{a_{\beta n}Q'(a_{\beta n})}{a_{\rho n}Q'(a_{\rho n})}\right] \ge \\
\text{(by (2.14))} \qquad &\ge (a_{\rho n}/a_n)^{T(a_n)}\left[1 - \frac{\beta}{\rho}\right] \ge \\
\text{(by (2.12))} \qquad &\ge \big(1 - C_4/T(a_n)\big)^{T(a_n)}\left[1 - \frac{\beta}{\rho}\right] \ge C_5\,. \qquad \square
\end{aligned}$$

We shall need a crude infinite-finite range inequality:

LEMMA 2.5. *Let $0 < p \le \infty$, $r > 0$, $s > 1$. There exists $C > 0$ such that for $n \ge 1$ and $P \in \mathcal{P}_n$,*

$$\big\|PW|Q'|^r\big\|_{L_p(|x|\ge a_{sn})} \le \mathrm{e}^{-Cn}\|PW\|_{L_p(|x|\le a_{sn})}\,. \tag{2.24}$$

PROOF. By (2.3) and monotonicity of Q',

$$W|Q'|^r \le C_1 W_1\,,$$

where

$$W_1 := \mathrm{e}^{-Q_1}\,, \quad \text{where} \quad Q_1(x) := Q(x) - C_2\log\left[Q(x) + C_3\right].$$

Then

$$\frac{Q_1'(x)}{Q'(x)} = 1 - \frac{C_2}{Q(x) + C_3} \to 1 \quad \text{as} \quad x \to \infty\,. \tag{2.25}$$

Moreover, $Q_1'(x)/Q(x)$ is increasing as Q is, so $Q_1'(x)$ is increasing in $(0,\infty)$. The number $a_n^{(1)}$ defined by

$$n=\frac{2}{\pi}\int_0^1 \frac{a_n^{(1)}tQ_1'(a_n^{(1)}t)}{\sqrt{1-t^2}}dt\,,$$

is in view of (2.25), easily seen to satisfy

$$a_n^{(1)}=a_{n(1+\eta_n)}\,,$$

where $\eta_n\to 0$ as $n\to\infty$. Consequently if $1<\sigma<s$,

$$a_{\sigma n}^{(1)}\le a_{sn}\,,\quad n\ge n_0\,.$$

Next, by standard results and methods (cf. [19, p. 112], [14, pp. 49-51], [10, pp. 45-46]), there exists $C_4>0$ such that for $n\ge 1$ and $P\in\mathcal{P}_n$,

$$\|PW_1\|_{L_p(|x|\ge a_{\sigma n}^{(1)})}\le \mathrm{e}^{-C_4 n}\|PW_1\|_{L_p(|x|\le a_{\sigma n}^{(1)})}$$

so for $n\ge n_0$,

$$\|PW_1\|_{L_p(|x|\ge a_{sn})}\le \mathrm{e}^{-C_4 n}\|PW_1\|_{L_p(|x|\le a_{sn})}\,.$$

Then

$$\begin{aligned}\big\|PW|Q'|^r\big\|_{L_p(|x|\ge a_{sn})}&\le C_1\|PW_1\|_{L_p(|x|\ge a_{sn})}\le\\ &\le C_1\mathrm{e}^{-C_4 n}\big[Q(a_{sn})+C_3\big]^{C_2}\|PW\|_{L_p(|x|\le a_{sn})}\le\\ &\le \mathrm{e}^{-C_5 n}\|PW\|_{L_p(|x|\le a_{sn})}\,,\end{aligned}$$

by (2.7). □

Our final lemma in this section will be used only in Section 10:

Lemma 2.6. *Let $r,\epsilon\in(0,1)$. There exist C_1 and n_0 such that for $n\ge n_0$ and for $-ra_n\le a<b\le ra_n$ and $|x|\le ra_n$,*

$$\int_a^b \left|\frac{Q'(x)-Q'(t)}{x-t}\right|dt\le C_1+\epsilon\frac{n}{a_n^2}(b-a)\,. \tag{2.26}$$

Proof. We may assume that $0 < x \le ra_n$. Let C be such that $Q''(t)$ and $Q'(t)/t$ are increasing in (C, ∞). Write

$$I := \Bigg[\int\limits_{[-ra_n,-x)\cap[a,b]} + \int\limits_{[-x,0)\cap[a,b]} + \int\limits_{[0,x/2]\cap[a,b]} + \int\limits_{(x/2,2x)\cap[a,b]} + \\ + \int\limits_{[2x,ra_n]\cap[a,b]} \Bigg] \left| \frac{Q'(x)-Q'(t)}{x-t} \right| dt =: I_1 + I_2 + I_3 + I_4 + I_5 \,.$$

We shall show that for $j = 1, 2, \ldots, 5$,

$$I_j \le C_1 + \epsilon \frac{n}{a_n^2}(b-a)\,, \tag{2.27}$$

for $n \ge n_0$. Firstly, $t \in [0, x/2]$ implies

$$\left| \frac{Q'(x)-Q'(t)}{x-t} \right| \le \frac{Q'(x)}{x-t} \le \frac{2}{x} Q'(x)$$

so

$$I_3 \le \frac{2}{x} Q'(x) \min\left\{ \frac{x}{2}, b-a \right\}.$$

If $x \ge C$, then we obtain

$$I_3 \le 2 \frac{Q'(ra_n)}{ra_n}(b-a) \le \epsilon \frac{n}{a_n^2}(b-a)\,,$$

for $n \ge n_0$, by (2.18). If $x \in (0, C]$, we see that

$$I_3 \le Q'(x) \le Q'(C)\,.$$

So, in all cases, we have (2.27) for $j = 3$.

Next, if $t \in (x/2, 2)$,

$$\left| \frac{Q'(x)-Q'(t)}{x-t} \right| \le \max_{\xi \in (x/2,2x)} |Q''(\xi)| \le 4Q''(2x)\,,$$

by monotonicity of $tQ''(t)$. If $x \ge C$ and $x \le r/2a_n$, we obtain from (2.18),

$$I_4 \le 4Q''(ra_n)(b-a) \le \epsilon \frac{n}{a_n^2}(b-a)\,,$$

for $n \geq n_0$. (Our argument requires trivial modifications if $2x \geq ra_n$). If $x \leq C$, we use the monotonicity of $tQ''(t)$ in $(0, \infty)$ to deduce that

$$I_4 \leq 16xQ''(2x) \leq 16C\, Q''(2C)\,.$$

Again, we have (2.27) for $j = 4$. Next, for $t \geq 2x$,

$$\left|\frac{Q'(x) - Q'(t)}{x - t}\right| \leq \frac{2}{t} Q'(t)\,, \qquad \text{so}$$

$$I_5 \leq 2\int_0^C \frac{Q'(t)}{t} dt + 2\int_{[c,ra_n]\cap[a,b]} \frac{Q'(t)}{t} dt \leq$$

$$\leq 2Q'(C)\int_0^C t^{T(0+)-2} dt + 2\frac{Q'(ra_n)}{ra_n}(b-a) \leq C_1 + \epsilon\frac{n}{a_n^2}(b-a)\,,$$

for $n \geq n_0$. Here we have used (2.1) and the monotonicity of $T(t)$ as well as $T(0+) > 1$, and also (2.18). The treatment of I_1 and I_2 is easier, we use for $t < 0$,

$$\left|\frac{Q'(x) - Q'(t)}{x - t}\right| = \frac{Q'(x) + Q'(|t|)}{x + |t|}\,.$$

The reader can complete the details. □

3 – Estimates on density functions

In this section, we estimate the density functions

(3.1)
$$\mu_n(x) := \frac{2}{\pi^2}\int_0^1 \frac{a_n x Q'(a_n x) - a_n s Q'(a_n s)}{n(x^2 - s^2)} \frac{\sqrt{1-x^2}}{\sqrt{1-s^2}} ds\,, \quad x \in (-1, 1)\,.$$

These arise as the solutions of integral equations with logarithmic kernel, see Lemma 4.1. For the moment, we concentrate on proving the following result. Throughout we assume that $W = \mathrm{e}^{-Q} \in \mathcal{E}$.

Theorem 3.1. *Uniformly for $n \geq 1$, and $|x| < 1$,*

$$\mu_n(x) \sim \min\left\{\frac{1}{\sqrt{1-x^2}}, T(a_n)\sqrt{1-x^2}\right\}. \tag{3.2}$$

We shall make use of the estimates of Section 2 to prove this result, and in particular, we use the bounds on

$$\Delta_n(x,s) = \frac{a_n x Q'(a_n x) - a_n s Q'(a_n s)}{a_n x - a_n s}\,.$$

Note that

$$\mu_n(x) = \frac{2}{\pi^2}\frac{a_n}{n}\int_0^1 \frac{\Delta_n(x,s)}{x+s}\frac{\sqrt{1-x^2}}{\sqrt{1-s^2}}ds\,. \tag{3.3}$$

We distinguish three ranges of x:

Proof of Theorem 3.1 for $x \in \left[0, \frac{1}{4}\right]$. We write

$$\mu_n(x) = \frac{2}{\pi^2}\frac{a_n}{n}\left[\int_0^{2x} + \int_{2x}^{1/2} + \int_{1/2}^{1}\right]\frac{\Delta_n(x,s)}{x+s}\frac{\sqrt{1-x^2}}{\sqrt{1-s^2}}ds =: I_1 + I_2 + I_3\,.$$

Here by (2.22),

$$\Delta_n(x,s) \le C_1 \frac{n}{a_n}\,, \quad s \in \left[0, \frac{1}{2}\right],$$

so

$$I_1 \le C_2 \int_0^{2x} \frac{ds}{x+s} \le 2C_2\,.$$

Next,

$$I_2 \le C_3 \frac{a_n}{n}\int_{2x}^{1/2} \frac{a_n s Q'(a_n s)}{a_n s - a_n x}\frac{1}{x+s}\frac{\sqrt{1-x^2}}{\sqrt{1-s^2}}ds \le C_4 \frac{a_n}{n}\int_{2x}^{1/2}\frac{Q'(a_n s)}{s}ds \le$$

$$\text{(by (2.1))} \qquad \le C_4 \frac{a_n}{n} Q'(a_n/2)\int_{2x}^{1/2}(2s)^{T(2xa_n)-2}ds \le$$

$$\text{(by (2.18))} \qquad \le C_4 \frac{a_n}{n} Q'(a_n/2)\int_{4x}^{1} t^{T(2xa_n)-2}dt \le C_5 \int_0^1 t^{T(0+)-2}dt =: C_6\,.$$

Recall also that $T(0+) > 1$. Next, for $s \in \left[\frac{1}{2}, 1\right]$,

$$\Delta_n(x,s) = Q'(a_n s)\Big(1 - \frac{a_n x Q'(a_n x)}{a_n s Q'(a_n s)}\Big)\Big/\Big(1 - \frac{x}{s}\Big)\,.$$

Here $\dfrac{1}{2} \le 1 - \dfrac{x}{s} \le 1$, and by (2.1)

$$0 \le \frac{a_n x Q'(a_n x)}{a_n s Q'(a_n s)} \le \left(\frac{x}{s}\right)^{T(a_n x)} \le \left(\frac{1}{2}\right)^{T(0+)} < 1\,,$$

so uniformly for $s \in \left[\dfrac{1}{2}, 1\right]$ and $x \in \left(0, \dfrac{1}{4}\right]$,

$$\Delta_n(x,s) \sim Q'(a_n s)\,,$$

and hence

$$I_3 \sim \frac{a_n}{n} \int\limits_{1/2}^{1} Q'(a_n s) \frac{ds}{\sqrt{1-s^2}} \sim \frac{1}{n} \int\limits_{1/2}^{1} a_n s Q'(a_n s) \frac{ds}{\sqrt{1-s^2}} \sim 1\,,$$

by the definition of a_n, and the monotonicity of $uQ'(u)$. In summary, we have shown that

$$\mu_n(x) = I_1 + I_2 + I_3 \sim 1\,,$$

uniformly for $x \in \left[0, \dfrac{1}{4}\right]$ and $n \ge 1$, which is equivalent to (3.2) for this range of x. ∎

PROOF OF THEOREM 3.1 FOR $x \in \left[\dfrac{1}{4}, \dfrac{a_{n/2}}{a_n}\right]$. Recall that

$$1 - \frac{a_{n/2}}{a_n} \sim \frac{1}{T(a_n)}\,, \quad n \ge 1\,,$$

and hence that

$$\min\left\{\frac{1}{\sqrt{1-x^2}}, T(a_n)\sqrt{1-x^2}\right\} \sim \frac{1}{\sqrt{1-x^2}}$$

uniformly for $n \geq 1$ and this range of x. We shall show that

$$\mu_n(x)\sqrt{1-x^2} \sim 1\,,$$

which is equivalent to (3.2) in this case. Let us set

$$\eta := \frac{1-x}{4}$$

so that $x + 4\eta = 1$ and hence

$$1-(x+\eta) = 3\eta \sim 1-x\,.$$

Write

$$\mu_n(x)\sqrt{1-x^2} = \frac{2}{\pi^2}\frac{a_n}{n}\Big[\int_0^{x-\eta} + \int_{x-\eta}^{x+\eta} + \int_{x+\eta}^{1}\Big]\frac{\Delta_n(x,s)}{x+s}\frac{(1-x^2)}{\sqrt{1-s^2}}ds =: I_1+I_2+I_3\,.$$

Here

$$I_1 \leq C_6\frac{a_n}{n}\int_0^{x-\eta}\frac{a_n x Q'(a_n x)}{a_n(x-s)}\frac{ds}{\sqrt{1-s}}(1-x) \leq$$

$$\leq C_6\frac{1}{n}a_n x Q'(a_n x)(1-x)\int_0^{x-\eta}(x-s)^{-3/2}ds \leq$$

$$\leq C_7\frac{1}{n}a_n Q'(a_n x)(1-x)^{1/2} \leq C_8\,,$$

by (2.19). Next, for $s \in [x-\eta, x+\eta]$, (2.22) shows that

$$\Delta_n(x,s) \leq C_7\frac{n}{a_n}\big(1-(x+\eta)\big)^{-3/2} \leq C_8\frac{n}{a_n}(1-x)^{-3/2}\,.$$

Then

$$I_2 \leq C_9(1-x)^{-3/2}\int_{x-\eta}^{x+\eta}\frac{ds}{\sqrt{1-s}}(1-x) \leq C_{10}(1-x)^{-1/2}\eta^{1/2} = C_{11}\,.$$

Finally,

$$I_3 \le C_{11} \frac{a_n}{n} \int_{x+\eta}^{1} \frac{a_n s Q'(a_n s)}{a_n \eta} \frac{ds}{\sqrt{1-s}} (1-x) \le C_{12} \frac{1}{n} \int_{x+\eta}^{1} a_n s Q'(a_n s) \frac{ds}{\sqrt{1-s^2}} \le C_{13} \,.$$

So we have shown that

$$\mu_n(x)\sqrt{1-x^2} = I_1 + I_2 + I_3 \le C_{14} \,, \quad x \in \left[\frac{1}{4}, \frac{a_{n/2}}{a_n}\right], \quad n \ge 2 \,.$$

We must derive a matching lower bound. Now

$$\mu_n(x)\sqrt{1-x^2} \ge \frac{2}{\pi^2} \int_{a_{3n/4}/a_n}^{1} \frac{a_n s Q'(a_n s) - a_n x Q'(a_n x)}{n(s^2 - x^2)} \frac{(1-x^2)}{\sqrt{1-s^2}} ds \ge$$

(by (2.23)) $$\ge C_{15} \frac{a_n Q'(a_n)}{n} \int_{a_{3n/4}/a_n}^{1} \frac{1-x^2}{s^2-x^2} \frac{ds}{\sqrt{1-s^2}} \ge$$

$$\ge C_{15} \frac{a_n Q'(a_n)}{n} \int_{a_{3n/4}/a_n}^{1} \frac{ds}{\sqrt{1-s^2}} \ge$$

(by (2.12)) $$\ge C_{16} T(a_n)^{1/2} (1 - a_{3n/4}/a_n)^{1/2} \ge C_{17} \,.$$ □

Proof of Theorem 3.1 for $x \in \left[\frac{a_{n/2}}{a_n}, 1\right)$. Note that for this range of x,

$$\min\left\{ \frac{1}{\sqrt{1-x^2}} \,, T(a_n)\sqrt{1-x^2} \right\} \sim T(a_n)\sqrt{1-x^2} \,.$$

We shall show that

$$\mu_n(x)/\sqrt{1-x^2} \sim T(a_n) \,,$$

which is equivalent to (3.2) in this case. Let $\eta := 1/T(a_n)$. Now

$$\frac{\mu_n(x)}{\sqrt{1-x^2}} = \frac{2}{\pi^2} \frac{a_n}{n} \left[\int_0^{x-\eta} + \int_{x-\eta}^{x+\eta} + \int_{x+\eta}^{1} \right] \frac{\Delta_n(x,s)}{x+s} \frac{ds}{\sqrt{1-s^2}} =: I_1 + I_2 + I_3 \,.$$

Here if $x+\eta>1$, we omit I_3 and replace $x+\eta$ by 1 in I_2. Firstly,

$$\begin{aligned} I_1 &\le C_{17}\frac{a_n}{n}\int_0^{x-\eta}\frac{a_n xQ'(a_n x)}{a_n(x-s)}\frac{ds}{\sqrt{1-s}} \le \\ &\le C_{17}\frac{1}{n}a_n xQ'(a_n x)\int_0^{x-\eta}(x-s)^{-3/2}ds \le \\ &\le C_{18}\frac{1}{n}a_n Q'(a_n)\eta^{-1/2} \le C_{19}T(a_n)\,, \end{aligned}$$

by (2.7). Next, by (2.22),

$$\Delta_n(x,s) \le C_{20}\frac{n}{a_n}T(a_n)^{3/2}\,,$$

so

$$I_2 \le C_{21}T(a_n)^{3/2}\int_{x-\eta}^{x+\eta}\frac{ds}{\sqrt{1-s}} \le C_{22}T(a_n)^{3/2}\eta^{1/2} = C_{23}T(a_n)\,.$$

Finally,

$$\begin{aligned} I_3 &\le C_{23}\frac{a_n}{n}\int_{x+\eta}^{1}\frac{a_n sQ'(a_n s)}{a_n(s-x)}\frac{ds}{\sqrt{1-s}} \le \\ &\le C_{24}\frac{1}{n}\eta^{-1}\int_{x+\eta}^{1}a_n sQ'(a_n s)\frac{ds}{\sqrt{1-s^2}} \le C_{25}\eta^{-1} = C_{25}T(a_n)\,. \end{aligned}$$

Thus we have shown that

$$\frac{\mu_n(x)}{\sqrt{1-x^2}} = I_1+I_2+I_3 \le C_{26}T(a_n)\,, \qquad x\in\left[\frac{a_{n/2}}{a_n},1\right), \quad n\ge 1\,.$$

We must obtain a matching lower bound. Now for $s,x\ge a_{n/2}/a_n$ the monotonicity of Δ_n (see Lemma 2.4 (i)) shows that

$$\begin{aligned} \Delta_n(x,s) &\ge \Delta_n(a_{n/2}/a_n, a_{n/2}/a_n) = \frac{d}{du}(uQ'(u))|_{u=a_{n/2}} = \\ &= Q'(a_{n/2})T(a_{n/2}) \ge C_{27}\frac{n}{a_n}T(a_n)^{3/2}\,, \end{aligned}$$

by (2.7) and (2.8). Then

$$\frac{\mu_n(x)}{\sqrt{1-x^2}} \geq C_{28} T(a_n)^{3/2} \int_{a_{n/2}/a_n}^{1} \frac{ds}{\sqrt{1-s^2}} \geq C_{29} T(a_n)^{3/2} \Big(1 - \frac{a_{n/2}}{a_n}\Big)^{1/2} \geq C_{30} T(a_n)\,.$$

□

4 – Majorization functions and integral equations

In this section, we present some technical estimates for the majorization function $U_{n,R}$ that determines the "support" of weighted polynomials. The bounds will be applied in the next section to prove Theorem 1.5. Throughout we assume that $W \in \mathcal{E}$. The various terms are defined in the following lemma:

Lemma 4.1. *Let* $n \geq 1$, *let* $a_n = a_n(Q)$ *and* $0 < R \leq a_n$.
(a) *Define for* $x \in [-1,1] \setminus \{0\}$,

$$\nu_{n,R}(x) := \frac{2}{\pi^2} \int_0^1 \frac{RsQ'(Rs) - RxQ'(Rx)}{n(s^2 - x^2)} \frac{\sqrt{1-x^2}}{\sqrt{1-s^2}} ds\,, \tag{4.1}$$

and

$$\mu_{n,R}(x) := \nu_{n,R}(x) + \frac{B_{n,R}}{\pi\sqrt{1-x^2}}\,, \tag{4.2}$$

where

$$B_{n,R} := 1 - \frac{2}{n\pi} \int_0^1 RtQ'(Rt) \frac{dt}{\sqrt{1-t^2}}\,. \tag{4.3}$$

Then

$$\mu_{n,R}(x) \geq \nu_{n,R}(x) > 0\,, \quad x \in (-1,1) \setminus \{0\}\,, \tag{4.4}$$

and

$$\int_{-1}^{1} \mu_{n,R}(x) dx = 1\,. \tag{4.5}$$

Moreover,

$$B_{n,R} = 0 \quad \textit{iff} \quad R = a_n\,. \tag{4.6}$$

(b) *For $z \in \mathbb{C}$, define*

$$U_{n,R}(z) := \int_{-1}^{1} \log|z-t|\mu_{n,R}(t)dt - \frac{Q(R|z|)}{n} + \frac{\chi_{n,R}}{n}\,, \tag{4.7}$$

where

$$\chi_{n,R} := \frac{2}{\pi}\int_{0}^{1} Q(Rt)\frac{dt}{\sqrt{1-t^2}} + n\log 2\,. \tag{4.8}$$

Then for $x \in [-1,1]$,

$$U_{n,R}(x) = 0\,, \tag{4.9}$$

and

$$\exp\Big(-n\int_{-1}^{1}\log|x-t|\mu_{n,R}(t)dt\Big) = W(R|x|)\exp(\chi_{n,R})\,. \tag{4.10}$$

Furthermore for $P \in \mathcal{P}_n$ and $z \in \mathbb{C}$,

$$\Big|P(z)W(R|z|)\Big| \le \exp\left(nU_{n,R}(z)\right)\sup_{t\in[-1,1]}\left\{|P(t)W(Rt)|\right\}. \tag{4.11}$$

(c) *We have*

$$\left(xU'_{n,R}(x)\right)' < 0\,, \quad x \in (1,\infty)\,. \tag{4.12}$$

Moreover, if $R = a_n$,

$$U_{n,R}(x) < 0\,; \quad U'_{n,R}(x) < 0\,, \quad x \in (1,\infty)\,. \tag{4.13}$$

(d)

$$\int_{-1}^{1}\nu_{n,R}(x)\frac{dx}{1-x} = \frac{RQ'(R)}{n}\,. \tag{4.14}$$

PROOF. See [14, Lemma 5.3, p.37] and [14, Theorem 7.1, pp.49-50]. ∎

We shall need some estimates on $\nu_{n,R}$ and $B_{n,R}$. Note that

$$\mu_{n,a_n}(x) = \nu_{n,a_n}(x) = \mu_n(x)\,,$$

where μ_n is the measure of the previous section.

LEMMA 4.2. *Let* $0 < \rho < 1$.
(a) *Uniformly for* $n \geq 1$ *and* $a_{\rho n} \leq R < a_n$,

$$B_{n,R} \sim T(a_n)(1 - R/a_n)\,. \tag{4.15}$$

(b) *Uniformly for* $n > 1/\rho$ *and* $a_{\rho n} \leq R \leq a_n$ *and* $x \in (-1,1)$,

$$\nu_{n,R}(x) \sim \min\left\{1/\sqrt{1-x^2}, T(a_n)\sqrt{1-x^2}\right\}. \tag{4.16}$$

Proof (a) From (4.3) and the definition of a_n,

$$B_{n,R} = \frac{2}{n\pi}\int_0^1 \left[a_n t Q'(a_n t) - RtQ'(Rt)\right]\frac{dt}{\sqrt{1-t^2}}\,.$$

Here, for some ξ between $a_n t$ and Rt,

$$\begin{aligned}\delta :&= a_n t Q'(a_n t) - RtQ'(Rt) = \\ &= (a_n t - Rt)T(\xi)Q'(\xi) \quad \begin{cases} \leq (a_n t - Rt)T(a_n t)Q'(a_n t) \\ \geq (a_n t - Rt)T(a_{\rho n} t)Q'(a_{\rho n} t) \end{cases}\end{aligned}$$

as $R \geq a_{\rho n}$. Then we see that

$$B_{n,R} \leq T(a_n)\left(1 - \frac{R}{a_n}\right)\frac{2}{n\pi}\int_0^1 a_n t Q'(a_n t)\frac{dt}{\sqrt{1-t^2}} = T(a_n)\left(1 - \frac{R}{a_n}\right).$$

Next, we obtain

$$\begin{aligned} B_{n,R} &\geq T(a_{\rho n/2})\Big(1-\frac{R}{a_n}\Big)\frac{2}{n\pi}\int_{a_{\rho n/2}/a_{\rho n}}^{1} a_{\rho n} t Q'(a_{\rho n}t)\frac{dt}{\sqrt{1-t^2}} \geq \\ &\geq T(a_{\rho n/2})\Big(1-\frac{R}{a_n}\Big)\frac{C}{n} a_{\rho n/2} Q'(a_{\rho n/2})(1-a_{\rho n/2}/a_n)^{1/2} \geq \\ &\geq C_1 T(a_n)\Big(1-\frac{R}{a_n}\Big)\frac{C}{n} a_n Q'(a_n) T(a_n)^{-1/2} \geq \\ &\geq C_2 T(a_n)\Big(1-\frac{R}{a_n}\Big), \end{aligned}$$

by (2.7), (2.8), (2.12) and (2.13).

(b) We claim first that $\big|RsQ'(Rs)-RxQ'(Rx)\big|$ increases as R increases. For if $s>x>0$, and $S>R$,

$$\begin{aligned} RsQ'(Rs)-RxQ'(Rx) &= \int_{Rx}^{Rs}\big(uQ'(u)\big)'du \leq \int_{Sx}^{Ss}\big(uQ'(u)\big)'du = \\ &= SsQ'(s)-SxQ'(Sx), \end{aligned}$$

by monotonicity of $\big(uQ'(u)\big)' = Q'(u)T(u)$. Then as $R\leq a_n$, we see from (4.1) that

$$\nu_{n,R}(x)\leq \nu_{n,a_n}(x) = \mu_{n,a_n}(x) = \mu_n(x).$$

(Recall μ_n was defined by (3.1)). Moreover, if $[t]$ denotes the greatest integer $\leq t$,

$$\nu_{n,R}(x) \geq \frac{[\rho n]}{n}\nu_{[\rho n],a_{[\rho n]}}(x) = \frac{[\rho n]}{n}\mu_{[\rho n]}(x).$$

The fact that

$$T(a_{[\rho n]}) \sim T(a_n),$$

and Theorem 3.1 yield (4.16). □

THEOREM 4.3. *Let* $\rho < 1$. *There exists* $D > 0$ *and* C_j, $j = 1, 2, \ldots, 6$, *such that for*

$$n > 1/\rho\,; \quad a_{\rho n} \le R \le a_n\,; \quad 0 < \epsilon \le D/T(a_n)\,; \tag{4.17}$$

we have

$$\begin{aligned} -C_1 + C_2\frac{1 - R/a_n}{\epsilon} - C_3\big(\epsilon T(a_n)\big)^{1/2} \le \frac{U_{n,R}(1+\epsilon)}{(\epsilon^{3/2}T(a_n))} \le \\ \le -C_4 + C_5\frac{1 - R/a_n}{\epsilon} - C_6\big(\epsilon T(a_n)\big)^{1/2}\,. \end{aligned} \tag{4.18}$$

PROOF. Now by (4.9) and (4.7), and moreover by (4.2) and (4.14),

$$\begin{aligned} &U_{n,R}(1+\epsilon) = U_{n,R}(1+\epsilon) - U_{n,R}(1) = \\ &= \int_{-1}^{1}\big[\log(1+\epsilon-t) - \log(1-t)\big]\mu_{n,R}(t)dt + \frac{1}{n}\Big[Q(R) - Q\big(R(1+\epsilon)\big)\Big] = \\ &= \int_{-1}^{1}\left[\log(1+\epsilon-t) - \log(1-t) - \frac{\epsilon}{1-t}\right]\nu_{n,R}(t)dt + \\ &\qquad + B_{n,R}\int_{-1}^{1}\big[\log(1+\epsilon-t) - \log(1-t)\big]\frac{dt}{\pi\sqrt{1-t^2}} + \\ &\qquad + \frac{1}{n}\Big[Q(R) + \epsilon RQ'(R) - Q\big(R(1+\epsilon)\big)\Big] = \\ &=: J_1 + J_2 + J_3\,. \end{aligned} \tag{4.19}$$

First, for some ξ between R and $R(1+\epsilon)$,

$$J_3 = \frac{1}{n}\Big[Q(R) + \epsilon RQ'(R) - Q\big(R(1+\epsilon)\big)\Big] = -\frac{Q''(\xi)(R\epsilon)^2}{2n}\,.$$

Here $\xi \ge R \ge a_{\rho n}$ and

$$\xi \le R(1+\epsilon) \le a_{\rho n}\Big(1 + \frac{D}{T(a_n)}\Big) \le a_n\,,$$

if D is small enough, by (2.9). Thus

$$a_{\rho n}Q''(a_{\rho n}) \le \xi Q''(\xi) \le a_n Q''(a_n)$$

by monotonicity of $tQ''(t)$. Then by (2.7) and (2.13),

$$J_3 \sim -T(a_n)^{3/2}\epsilon^2 \,. \tag{4.20}$$

Next, let

$$\phi(z) := z + \sqrt{z^2-1}\,, \qquad z \in \mathbb{C} \setminus [-1,1]\,,$$

with the usual choice of branches. The identity

$$\int_{-1}^{1} \log|z-t| \frac{dt}{\pi\sqrt{1-t^2}} = \log|\phi(z)| - \log 2\,,$$

shows that

$$\begin{aligned} J_2 &= B_{n,R}\int_{-1}^{1} \big[\log(1+\epsilon-t) - \log(1-t)\big]\frac{dt}{\pi\sqrt{1-t^2}} = \\ &= B_{n,R}\Big[\log|\phi(1+\epsilon)| - \log|\phi(1)|\Big] = \\ &= B_{n,R}\log(1+\epsilon+\sqrt{2\epsilon+\epsilon^2}) \sim \\ &\sim B_{n,R}\sqrt{\epsilon} \sim T(a_n)\Big(1-\frac{R}{a_n}\Big)\sqrt{\epsilon}\,, \end{aligned} \tag{4.21}$$

by (4.15). Finally, we estimate

$$J_1 = \Bigg[\int_{-1}^{1-1/T(a_n)} + \int_{1-1/T(a_n)}^{1}\Bigg]\Bigg[\log\Big(1+\frac{\epsilon}{1-t}\Big) - \frac{\epsilon}{1-t}\Bigg]\nu_{n,R}(t)dt =: J_{11} + J_{12}\,. \tag{4.22}$$

Now for $t \in \big[-1, 1-1/T(a_n)\big]$, by (4.17)

$$0 \le \frac{\epsilon}{1-t} \le \epsilon T(a_n) \le D\,.$$

Moreover, $\log(1+x) - x \sim -x^2\,, \quad x \in (0, D]\,,$ so

$$J_{11} = \int_{-1}^{1-1/T(a_n)} \Big[\log\Big(1 + \frac{\epsilon}{1-t}\Big) - \frac{\epsilon}{1-t} \Big] \nu_{n,R}(t)dt \sim$$

(4.23)
$$\sim \int_{-1}^{1-1/T(a_n)} \Big(\frac{\epsilon}{1-t}\Big)^2 \nu_{n,R}(t)dt \sim \quad \text{(by (4.16))}$$

$$\sim -\epsilon^2 \int_{-1+1/T(a_n)}^{1-1/T(a_n)} (1-t)^{-\frac{5}{2}}(1+t)^{-\frac{1}{2}} dt - \epsilon^2 T(a_n) \int_{-1}^{-1+1/T(a_n)} (1-t)^{-2}(1+t)^{\frac{1}{2}} dt \sim$$

$$\sim -\epsilon^2 T(a_n)^{3/2}\,.$$

Also, by (4.16),

$$J_{12} = \int_{1-1/T(a_n)}^{1} \Big[\log\Big(1 + \frac{\epsilon}{1-t}\Big) - \frac{\epsilon}{1-t} \Big] \nu_{n,R}(t)dt \sim$$

(4.24)
$$\sim T(a_n) \int_{1-1/T(a_n)}^{1} \Big[\log\Big(1 + \frac{\epsilon}{1-t}\Big) - \frac{\epsilon}{1-t} \Big] \sqrt{1-t}\, dt =$$

$$= T(a_n)\epsilon^{3/2} \int_{\epsilon T(a_n)}^{\infty} [\log(1+v) - v] v^{-5/2} dv \sim$$

$$\sim -T(a_n)\epsilon^{3/2}\,,$$

since $\epsilon T(a_n) \le D$ and $\log(1+v) - v < 0$, $v > 0$. Summarizing, we have shown that

$$\frac{U_{n,R}(1+\epsilon)}{\epsilon^{3/2} T(a_n)} = \frac{J_{11} + J_{12} + J_2 + J_3}{\epsilon^{3/2} T(a_n)}\,,$$

and from (4.21),

$$\frac{J_2}{\epsilon^{3/2} T(a_n)} \sim \frac{1}{\epsilon}\Big(1 - \frac{R}{a_n}\Big)\,,$$

while from (4.20) and (4.23),

$$\frac{J_3 + J_{11}}{\epsilon^{3/2} T(a_n)} \sim -\big(\epsilon T(a_n)\big)^{1/2},$$

and finally from (4.24),

$$\frac{J_{12}}{\epsilon^{3/2} T(a_n)} \sim -1\,. \qquad \square$$

5 – The proof of Theorem 1.5

Throughout we assume that $W = \mathrm{e}^{-Q} \in \mathcal{E}$. Our basic tool is (cf. [7], [9], [25]):

Lemma 5.1. *Let $0 < p < \infty$. Let $n \geq 1$ and $0 < R \leq a_n$. Further, let $U_{n,R}(z)$ be defined by (4.7) and let*

$$\phi(z) := z + \sqrt{z^2 - 1}\,, \tag{5.1}$$

be the conformal map of $\mathbb{C} \setminus [-1, 1]$ onto $\{z : |z| > 1\}$, with the usual choice of branches. Then for $P \in \mathcal{P}_n$ and $z \in \mathbb{C}$,

$$\Big|P(z)W\big(R|z|\big)\Big|^p \leq \frac{1}{\pi}\mathrm{e}^{pnU_{n,R}(z)} \frac{|\phi(z)|}{\mathrm{dist}\,\big(z, [-1, 1]\big)} \int_{-1}^{1} \big|P(t)W(Rt)\big|^p dt\,. \tag{5.2}$$

Proof. This is the same as that of Lemma 10.1 in [7, p. 512], but we sketch the details. We may assume that P has full degree n. (If not, consider (5.2) for $\epsilon z^n + P(z)$, and let $\epsilon \to 0$). Let $\alpha_1, \alpha_2, \dots, \alpha_k$ denote the zeros of P outside $[-1, 1]$, repeated according to multiplicity. Form the Blaschke product

$$B(z) := \prod_{j=1}^{k} \frac{\phi^{-1}(z) - \phi^{-1}(\alpha_k)}{1 - \phi^{-1}(z)\overline{\phi^{-1}(\alpha_k)}}\,.$$

(If $P \neq 0$ in $\mathbb{C} \setminus [-1, 1]$, we set $B := 1$). Here note that ϕ^{-1} means $1/\phi$, not the inverse of ϕ.

Then B is analytic in $\overline{\mathbb{C}} \setminus [-1,1]$, vanishing only at $\alpha_1, \alpha_2, \dots, \alpha_k$. Moreover, $|B(z)| \leq 1$ in $\overline{\mathbb{C}}$, with equality for $z \in [-1,1]$. Since

$$G(z) := \exp\left(- \int_{-1}^{1} \log(z-t)\mu_{n,R}(t)dt - \chi_{n,R}/n \right)$$

(with the usual choice of branches for the log) is analytic in $\mathbb{C} \setminus [-1,1]$ (recall (4.5)), and has a simple zero at ∞,

$$f(z) := \frac{P(z)}{B(z)} G(z)^n$$

is analytic and non-vanishing in $\overline{\mathbb{C}} \setminus [-1,1]$. So we consider a single-valued branch of f^p in $\mathbb{C} \setminus [-1,1]$. Since f^p/ϕ is analytic in $\overline{\mathbb{C}}[-1,1]$, including ∞, where it is 0, we obtain from Cauchy's integral formula,

$$\frac{f^p(z)}{\phi(z)} = \frac{1}{2\pi i} \int_{-1}^{1} \frac{(f^p/\phi)^+(t) - (f^p/\phi)^-(t)}{t-z} dt\,,$$

$z \notin [-1,1]$, where $(f^p/\phi)^{\pm}$ denote boundary values from the upper and lower half-planes respectively. Now for $x \in (-1,1)$,

$$\left| \left(\frac{f^p}{\phi} \right)^{\pm} (x) \right| = |P(x)|^p \exp\Big(-np \int_{-1}^{1} \log|x-t| \mu_{n,R}(t)dt - p\chi_{n,R} \Big) =$$
$$= |P(x)W(Rx)|^p\,,$$

by (4.10). Hence from (4.7), we obtain

$$\Big| P(z)W(R|z|) \exp\left(-nU_{n,R}(z)\right) \Big|^p \leq \frac{|\phi(z)||B(z)|^p}{\operatorname{dist}(z,[-1,1])} \frac{1}{\pi} \int_{-1}^{1} |P(x)W(Rx)|^p dx\,.$$

Since $|B| \leq 1$ in $\mathbb{C}$, we have (5.2). $\square$

Proof of Theorem 1.5 for $0 < p < \infty$. Replace $P(z)$ by $P(Rz)$ and Rz by s in (5.2):

$$|PW|^p(s) \le \frac{1}{\pi} e^{pnU_{n,R}(|s|/R)} \frac{|\phi(s/R)|R^{-1}}{|s|/R - 1} \int_{-R}^{R} |PW|^p(u)du\,, \tag{5.3}$$

$P \in \mathcal{P}_n$, $s \notin [-R, R]$. Now let $K > 0$ be fixed, but "large", and let δ_n be defined by (1.18) and let

$$R := R_n := a_n(1 - 2K\delta_n)\,. \tag{5.4}$$

Note that

$$R_n(1 + K\delta_n) = a_n(1 - K\delta_n - 2(K\delta_n)^2) \le a_n(1 - K\delta_n)\,. \tag{5.5}$$

Moreover, by Lemma 2.2 (viii),

$$\delta_n T(a_n) = \left(\frac{T(a_n)}{n^2}\right)^{1/3} \to 0\,, \tag{5.6}$$

$n \to \infty$, so for large n,

$$R_n \ge a_n\left(1 + \frac{\log 2}{T(a_n)}\right)^{-1} \ge a_{n/2}\,, \tag{5.7}$$

by (2.9). Next, let D be as in Theorem 4.3. Now

$$\frac{a_{\rho n}}{R_n} = \frac{a_{\rho n}}{a_n}\left(1 + o\Big(\frac{1}{T(a_n)}\Big)\right),$$

and by (2.11),

$$\frac{a_{\rho n}}{a_n} = \exp\left(\int_n^{\rho n} \frac{a_t'}{a_t} dt\right) \le \exp\left(\frac{C}{T(a_n)} \int_n^{\rho n} \frac{dt}{t}\right) \le 1 + \frac{D}{2T(a_n)} \tag{5.8}$$

if n is large enough, and $\rho = \rho(D)$ is close enough to 1. Then for large enough n,

$$\frac{a_{\rho n}}{R_n} \le 1 + \frac{D}{T(a_n)}\,. \tag{5.9}$$

Now by (5.3), with $R = R_n$, we can estimate for $P \in \mathcal{P}_n$,

$$I := \int_{\left\{s: R_n(1+K\delta_n) \le |s| \le a_{\rho n}\right\}} |PW|^p(s)ds \le$$

$$\le \frac{2}{\pi}\left(\int_{-R_n}^{R_n} |PW|^p(u)du\right) \int_{R_n(1+K\delta_n)}^{a_{\rho n}} \frac{\mathrm{e}^{pnU_{n,R_n}(s/R_n)}}{s/R_n - 1}\left|\phi\left(\frac{s}{R_n}\right)\right|\frac{1}{R_n}ds \le \tag{5.10}$$

$$\le C\left(\int_{-R_n}^{R_n} |PW|^p(u)du\right) \int_{K\delta_n}^{a_{\rho n}/R_n - 1} \mathrm{e}^{pnU_{n,R_n}(1+y)} y^{-1}dy\,,$$

where we have used

$$\left|\phi(s/R_n)\right| \le C_1\,,$$

and the substitution $s/R_n = 1 + y$. Now by (5.9), for y in the interval of integration,

$$K\delta_n \le y \le a_{\rho n}/R_n - 1 \le D/T(a_n)\,,$$

so by Theorem 4.3,

$$nU_{n,R}(1+y) \le ny^{3/2}T(a_n)\Big[-C_4 + C_5 2K\delta_n/y - C_6\big(yT(a_n)\big)^{1/2}\Big] \le$$

$$\le -C_4 nT(a_n)y^{3/2} + 2C_5 KnT(a_n)\delta_n y^{1/2} =$$

$$= -C_4(y/\delta_n)^{3/2} + 2C_5K(y/\delta_n)^{1/2}\,.$$

Then

$$\int_{K\delta_n}^{a_{\rho n}/R_n - 1} \mathrm{e}^{pnU_{n,R_n}(1+y)} y^{-1}dy \le \int_{K\delta_n}^{D/T(a_n)} \mathrm{e}^{-pC_4(y/\delta_n)^{3/2} + 2C_5Kp(y/\delta_n)^{1/2}} y^{-1}dy =$$

$$= \int_{K}^{D/\left(T(a_n)\delta_n\right)} \mathrm{e}^{-pC_4v^{3/2} + 2C_5Kpv^{1/2}} v^{-1}dv \longrightarrow \int_{K}^{\infty} \mathrm{e}^{-pC_4v^{3/2} + 2C_5Kpv^{1/2}} v^{-1}dv\,,$$

as $n \to \infty$. It follows (recall (5.10)) that we have shown

$$\int_{-a_{\rho n}}^{a_{\rho n}} |PW|^p(s)\,ds = \Bigg(\int_{-R_n(1+K\delta_n)}^{R_n(1+K\delta_n)} + \int_{\{s:R_n(1+K\delta_n)\le|s|\le a_{\rho n}\}}\Bigg)|PW|^p(s)ds \le \tag{5.11}$$

$$\le C_1\int_{-R_n(1+K\delta_n)}^{R_n(1+K\delta_n)} |PW|^p(u)du \le C_1\int_{-a_n(1-K\delta_n)}^{a_n(1-K\delta_n)} |PW|^p(u)du\,,$$

by (5.5). Now we estimate

$$J := \int_{a_{\rho n}\le|s|\le a_{4n}} |PW|^p(s)ds\,.$$

From Lemma 4.1 (c), $U_{n,a_n}(x)$ is decreasing for $x > 1$, so from (4.18), with $R = a_n$, we have for $|s| \ge a_{\rho n}$,

$$U_{n,a_n}(|s|/a_n) \le U_{n,a_n}(a_{\rho n}/a_n) \le$$

$$\text{(by (2.12))} \qquad \le U_{n,a_n}(1 + C_1/T(a_n)) \le -C_2T(a_n)^{-1/2}\,,$$

by (4.18) of Theorem 4.3. Then for $|s| \ge a_{\rho n}$, (5.3) with $R = a_n$ yields

$$|PW|^p(s) \le C\frac{\mathrm{e}^{-C_3nT(a_n)^{-1/2}}}{|s|/a_n - 1}a_n^{-1}\int_{-a_n}^{a_n} |PW|^p(u)du \le \tag{5.12}$$

$$\le C_1\frac{\mathrm{e}^{-C_3nT(a_n)^{-1/2}}}{|s|/a_n - 1}a_n^{-1}\int_{-a_n(1-K\delta_n)}^{a_n(1-K\delta_n)} |PW|^p(u)du\,.$$

Then we obtain

$$\int_{\{s:a_{\rho n}\le|s|\le a_{4n}\}}|PW|^p(s)ds\le$$

$$\le 2C_1\mathrm{e}^{-C_3nT(a_n)^{-1/2}}\left\{\int_{-a_n(1-K\delta_n)}^{a_n(1-K\delta_n)}|PW|^p(u)du\right\}\left\{a_n^{-1}\int_{a_{\rho n}}^{a_{4n}}\frac{ds}{s/a_n-1}\right\}\le$$

(5.13) (by Lemma 2.2 (viii), with some $\epsilon>0$)

$$\le C_2\mathrm{e}^{-C_4n^{\epsilon}}\left\{\int_{-a_n(1-K\delta_n)}^{a_n(1-K\delta_n)}|PW|^p(u)du\right\}\left|\log\left(\frac{a_{4n}/a_n-1}{a_{\rho n}/a_n-1}\right)\right|\le$$

(by (2.8) and (2.12))

$$\le C_2\mathrm{e}^{-C_4n^{\epsilon/2}}\left\{\int_{-a_n(1-K\delta_n)}^{a_n(1-K\delta_n)}|PW|^p(u)du\right\}.$$

Together, (5.11) and (5.13) show that for $n\ge 1$ and $P\in\mathcal{P}_n$,

$$\int_{-a_{4n}}^{a_{4n}}|PW|^p(s)ds\le C_6\int_{-a_n(1-K\delta_n)}^{a_n(1-K\delta_n)}|PW|^p(u)du\,.$$

Since it is known, under more general conditions on Q, [19, p. 112], [10, pp. 45-46] that

$$\int_{|s|\ge a_{4n}}|PW|^p(s)ds\le \mathrm{e}^{-C_7n}\int_{-a_{2n}}^{a_{2n}}|PW|^p(s)ds$$

for $n\ge 1$ and $P\in\mathcal{P}_n$, we have established (1.29). □

Proof of Theorem 1.5 for $p = \infty$. Now from (4.11), for $s > R$, and $P \in \mathcal{P}_n$,

$$|PW|(s) \leq \exp\big(nU_{n,R}(s/R)\big)\|PW\|_{L_\infty[-R,R]}\,.$$

Choosing in (4.18) $R = R_n = a_n(1 - K\delta_n)$ and $\epsilon = s/R_n - 1$, we have for $R_n < s \leq a_n$,

$$\begin{aligned}
nU_{n,R}(s/R_n) &\leq n\Big(\frac{s}{R_n} - 1\Big)^{3/2} T(a_n)\Big[-C_4 + C_5 K\delta_n\Big(\frac{s}{R_n} - 1\Big)^{-1}\Big] \leq \\
&\leq nT(a_n)C_5K\delta_n\Big(\frac{s}{R_n} - 1\Big)^{1/2} \leq \\
&\leq nT(a_n)C_5K\delta_n\Big(\frac{a_n}{R_n} - 1\Big)^{1/2} \leq C_8 nT(a_n)\delta_n^{3/2} = C_8\,.
\end{aligned}$$

So for $|s| \in [R_n, a_n]$,

$$|PW|(s) \leq \mathrm{e}^{C_8}\|PW\|_{L_\infty[-R_n,R_n]}\,.$$

Then

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[-a_n,a_n]} \leq \mathrm{e}^{C_8}\|PW\|_{L_\infty[-R_n,R_n]}\,. \qquad \square$$

6 – Lower bounds for λ_n

We shall prove the lower bound implicit in (1.20) of Theorem 1.2, assuming throughout that $W = \mathrm{e}^{-Q} \in \mathcal{E}$. Recall the definition (1.18) and (1.19) of δ_n and Ψ_n respectively.

Theorem 6.1. *Let $L > 0$. There exists C such that for $n \geq 1$, and*

$$|x| \leq a_n(1 + L\delta_n)\,, \tag{6.1}$$

we have

$$\lambda_n(W^2, x) \geq CW^2(x)\frac{a_n}{n}\Psi_n(x)\,. \tag{6.2}$$

Moreover, for $|x| \geq a_n$*,*

$$\lambda_n(W^2, x) \geq CW^2(x)a_n\delta_n\,. \tag{6.3}$$

The method of proof is the same as in section 8 of [7, pp. 492-6]. We remark that the Christoffel function may be defined by (1.17) even for complex z, and admits the identity (cf. [21])

$$\lambda_n(W^2, z) = \frac{1}{\sum_{j=0}^{n-1} \left|p_j(W^2, z)\right|^2}\,, \quad z \in \mathbb{C}\,.$$

LEMMA 6.2. (a) *For* $z \in \mathbb{C} \setminus \mathbb{R}$*,*

$$\frac{\lambda_n(W^2, z)}{W^2(|z|)} \geq \pi \frac{|\operatorname{Im} z|}{|\phi(z/a_n)|} \exp\left(- 2nU_{n,a_n}\left(\frac{z}{a_n}\right)\right), \tag{6.4}$$

where $\phi(z)$ *is the conformal map defined by* (5.1).

(b) *For* $x \geq 0$ *and* $y \geq 0$ *such that* $|x + iy| \leq 4a_n$*,*

$$\frac{\lambda_n(W^2, x)}{W^2(x)} \geq \frac{\pi}{9} y\Gamma\,, \tag{6.5}$$

where

$$\Gamma := \Gamma(n, x, y) := \exp\left(-2nU_{n,a_n}\left(\frac{x+iy}{a_n}\right)\right)\frac{W^2(|x+iy|)}{W^2(x)} \geq \tag{6.6}$$

$$\geq \exp\left(- 2n \int_0^1 \log\left[1 + \left(\frac{y/a_n}{x/a_n - t}\right)^2\right]\mu_{n,a_n}(t)dt\right). \tag{6.7}$$

Proof. This is the same as Lemma 8.1 in [7], but we provide the details.

(a) We apply (5.2) of Lemma 5.1 with $p = 2$, $R = a_n$, and $P(z)$ replaced by $P(a_n z)$. We obtain

$$\Big|P(a_n z)W(a_n|z|)\Big|^2 \le \frac{1}{\pi}\mathrm{e}^{2nU_{n,a_n}(z)}\left|\frac{\phi(z)}{\operatorname{Im} z}\right|\int_{-1}^{1}|PW|^2(a_n t)dt\,,$$

for $P \in \mathcal{P}_n$, $z \in \mathbb{C}$. Hence, replacing $a_n z$ by z, and by substitution,

$$\int_{-a_n}^{a_n}\frac{|PW|^2(s)}{\big|P(z)W(|z|)\big|^2}ds \ge \pi\frac{|\operatorname{Im} z|}{|\phi(z/a_n)|}\exp\Big(-2nU_{n,a_n}\Big(\frac{z}{a_n}\Big)\Big)\,.$$

Taking inf's over $P \in \mathcal{P}_{n-1}$, yields (6.4).

(b) Now for $x \ge 0$, $y > 0$,

$$\begin{aligned}\lambda_n^{-1}(W^2,x) &= \sum_{j=0}^{n-1}p_j(W^2,x)^2 \le \sum_{j=0}^{n-1}\big|p_j(W^2,x+iy)\big|^2 = \\ &= \lambda_n^{-1}(W^2,x+iy)\,,\end{aligned} \tag{6.8}$$

as each $p_j(W^2,\cdot)$ has real zeros. Furthermore, for $|x+iy| \le 4_{a_n}$,

$$\left|\phi\Big(\frac{x+iy}{a_n}\Big)\right| \le 2\left|\frac{x+iy}{a_n}\right| + 1 \le 9\,.$$

This inequality, (6.8) and (6.4) yield (6.5). Next, if $0 \le x \le a_n$, (4.7) and (4.9) yield

$$\begin{aligned}\Gamma &= \exp\left[-2n\left\{U_{n,a_n}\Big(\frac{x+iy}{a_n}\Big) + \frac{Q(|x+iy|)}{n} - U_{n,a_n}\Big(\frac{x}{a_n}\Big) - \frac{Q(|x|)}{n}\right\}\right] = \\ &= \exp\left[-n\int_{-1}^{1}\log\left[1+\Big(\frac{y/a_n}{x/a_n-t}\Big)^2\right]\mu_{n,a_n}(t)dt\right] \ge \\ &\ge \exp\left[-2n\int_{0}^{1}\log\left[1+\Big(\frac{y/a_n}{x/a_n-t}\Big)^2\right]\mu_{n,a_n}(t)dt\right].\end{aligned}$$

In this case, (6.7) follows. When $x > a_n$, one proceeds similarly, but uses

$$U_{n,a_n}(x/a_n) < 0\,. \qquad \square$$

We proceed to the

Proof of Theorem 6.1. We shall use estimates for the measure $\mu_{n,a_n} = \mu_n$ from Theorem 3.1 with a specific choice of y in Lemma 6.2 (b). The procedure duplicates that used in [7], [9], but we provide the details anyway. We distinguish four ranges of $x \geq 0$. Symmetry yields the result for all $x \in [-1,1]$.

Case I: $x \in \Big[0, a_n(1-1/T(a_n))\Big]$. Here we set

$$y := \frac{a_n}{n}\Big(1-\frac{x}{a_n}\Big)^{1/2} \tag{6.9}$$

in (6.7). Now

$$\frac{|x+iy|}{a_n} \leq 1 + \frac{1}{n}\Big(1-\frac{x}{a_n}\Big)^{1/2} \leq 2$$

for $n \geq 2$. We now turn to the estimation of the integral in (6.7), namely

$$\Delta := n\int_0^1 \log\left[1+\Big(\frac{y/a_n}{x/a_n - t}\Big)^2\right]\mu_{n,a_n}(t)dt\,. \tag{6.10}$$

By Theorem 3.1,

$$\Delta \leq C_1 n\int_0^1 \log\left[1+\Big(\frac{y/a_n}{x/a_n - t}\Big)^2\right]\frac{dt}{\sqrt{1-t}} =$$

$$= C_1\frac{n}{a_n}y \int_{-(1-\frac{x}{a_n}/\frac{y}{a_n})}^{x/y} \log\Big[1+\frac{1}{s^2}\Big]\frac{ds}{\sqrt{1-x/a_n+sy/a_n}}$$

by substitution $sy/a_n = x/a_n - t$. Now

$$s \geq -\frac{1}{2}\Big(\frac{1-x/a_n}{y/a_n}\Big) \quad \text{implies} \quad 1-\frac{x}{a_n}+s\frac{y}{a_n} \geq \frac{1}{2}\Big(1-\frac{x}{a_n}\Big),$$

so

$$\Delta \le C_2 \frac{n}{a_n} y (1 - x/a_n)^{-1/2} \int_{-\infty}^{\infty} \log\Big[1 + \frac{1}{s^2}\Big] ds +$$

$$+ C_2 \frac{n}{a_n} y \int_{-(1-x/a_n\big/ y/a_n)}^{-\frac{1}{2}(1-x/a_n\big/ y/a_n)} s^{-2} \frac{ds}{\sqrt{1 - x/a_n + sy/a_n}} \le$$

(by (6.9))
$$\le C_3 + C_3 n \Big(\frac{y}{a_n}\Big)^{1/2} \Big(\frac{1 - x/a_n}{y/a_n}\Big)^{-3/2} \le$$

$$\le C_3 + C_4 n \frac{1}{n^2} \Big(1 - \frac{x}{a_n}\Big)^{-1/2} \le C_5 \,,$$

since for large enough n,

$$1 - \frac{x}{a_n} \ge \frac{1}{T(a_n)} \ge n^{-2} \,.$$

So Lemma 6.2 (b) yields

$$\frac{\lambda_n(W^2, x)}{W^2(x)} \ge C_6 y = C_6 \frac{a_n}{n} \Big(1 - \frac{x}{a_n}\Big)^{1/2} .$$

Now recall that from Lemma 2.2 (viii)

$$\delta_n T(a_n) = \Big(\frac{T(a_n)}{n^2}\Big)^{1/3} = o(1) \,, \tag{6.11}$$

so for this range of x,

$$1 - \frac{x}{a_n} \sim 1 - \frac{x}{a_n} + 2L\delta_n > \frac{1}{T(a_n)}$$

and hence we have that

$$\Psi_n(x) = \max\Big\{\sqrt{1 - \frac{|x|}{a_n} + 2L\delta_n},\ \Big[T(a_n)\sqrt{1 - \frac{|x|}{a_n} + 2L\delta_n}\Big]^{-1}\Big\} \sim \sqrt{1 - \frac{|x|}{a_n}} \,.$$

So,

$$\frac{\lambda_n(W^2,x)}{W^2(x)} \geq C_7 \frac{a_n}{n} \Psi_n(x)\,.$$

So (6.2) is true for this range of x.

Case II: $x \in \Big(a_n(1-1/T(a_n)), a_n\Big]$. Let us set

$$y := a_n \min\left\{\delta_n, \left(nT(a_n)\sqrt{1-\frac{|x|}{a_n}}\right)^{-1}\right\}. \tag{6.12}$$

Then, with the definition (6.10), and by Theorem 3.1 and (6.12),

$$\begin{aligned}
\Delta &\leq CnT(a_n)\int_0^1 \log\left[1+\left(\frac{y/a_n}{x/a_n-t}\right)^2\right]\sqrt{1-t}\,dt = \\
&= C_8 nT(a_n)\frac{y}{a_n}\int_{-(1-x/a_n\big/y/a_n)}^{x/y} \log\Big[1+\frac{1}{s^2}\Big]\sqrt{1-x/a_n+sy/a_n}ds \leq \\
&\leq C_9 nT(a_n)\frac{y}{a_n}\sqrt{1-x/a_n}\int_{-\infty}^{\infty} \log\Big[1+\frac{1}{s^2}\Big]ds + \\
&+ C_9 nT(a_n)\Big(\frac{y}{a_n}\Big)^{3/2}\int_{-\infty}^{\infty} \log\Big[1+\frac{1}{s^2}\Big]|s|^{1/2}ds \leq C_{10}\,.
\end{aligned}$$

Hence Lemma 6.2 (b) yields

$$\frac{\lambda_n(W^2,x)}{W^2(x)} \geq C_{11} a_n \min\left\{\delta_n, \left(nT(a_n)\sqrt{1-\frac{|x|}{a_n}}\right)^{-1}\right\}.$$

Now if $a_n(1-1/T(a_n)) \leq x \leq a_n(1-\delta_n)$, then

$$\left(nT(a_n)\sqrt{1-\frac{|x|}{a_n}}\right)^{-1} \leq \big(nT(a_n)\delta_n^{1/2}\big)^{-1} = \delta_n\,,$$

and

$$1-\frac{x}{a_n}\sim 1-\frac{x}{a_n}+2L\delta_n=O\Big(\frac{1}{T(a_n)}\Big)\,,$$

so

$$\frac{\lambda_n(W^2,x)}{W^2(x)}\geq C_{11}\frac{a_n}{n}\frac{1}{T(a_n)\sqrt{1-\dfrac{|x|}{a_n}+2L\delta_n}}\sim$$
$$\sim\frac{a_n}{n}\max\left\{\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}\,,\left[T(a_n)\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}\right]^{-1}\right\}=\frac{a_n}{n}\Psi_n(x)\,.$$

So (6.2) holds here. If on the other hand, $a_n(1-\delta_n)\leq x\leq a_n$, then we obtain

$$\frac{\lambda_n(W^2,x)}{W^2(x)}\geq C_{11}a_n\delta_n\,.$$

For this range of x,

$$\frac{a_n}{n}\Psi_n(x)=\frac{a_n}{n}\max\left\{\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}\,,\left[T(a_n)\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}\right]^{-1}\right\}\sim$$
$$\sim\frac{a_n}{n}\max\left\{\sqrt{\delta_n}\,,\frac{1}{T(a_n)\sqrt{\delta_n}}\right\}\sim\frac{a_n}{nT(a_n)\sqrt{\delta_n}}=a_n\delta_n\,,$$

and again (6.2) follows.

Case III: $x\in[a_n,a_{2n}]$. Here we set $y:=\delta_n$, and note that since $x/a_n\geq 1$,

$$\Delta\leq n\int_0^1\log\left[1+\left(\frac{y/a_n}{1-t}\right)^2\right]\mu_{n,a_n}(t)dt\leq C_{12}\,,$$

by what we proved in Case II for $x=a_n$. Again Lemma 6.2 (b) yields (6.3). Note that since $a_{2n}/a_n\to 1$ as $n\to\infty$, we have $|x+iy|\leq 4a_n$ for $n\geq n_0$.

Case IV: $x \in [a_{2n}, \infty)$. We use the majorization (4.11) applied to the weight W^2 to deduce that for $x \geq a_{2n}$

$$\lambda_n^{-1}(W^2,x)W^2(x) \leq \|\lambda_n^{-1}(W^2,\cdot)W^2(\cdot)\|_{L_\infty[-a_n,a_n]} \exp\Big(nU_{n,a_n}\Big(\frac{x}{a_n}\Big)\Big) \leq$$
$$\leq n^{C_{12}} \exp\Big(nU_{n,a_n}\Big(\frac{a_{2n}}{a_n}\Big)\Big) \leq n^{C_{12}} \exp\big(-C_{13} nT(a_n)^{-\frac{1}{2}}\big),$$

by (2.9), (4.18) for $R = a_n$, and the fact that $U_{n,a_n}(t)$ is decreasing in $(1,\infty)$. Hence

$$\lambda_n^{-1}(W^2,x)W^2(x) \leq C_{14} a_n^{-1} \delta_n^{-1}$$

and we have (6.3) for this range of x. □

7 – Discretisation of a potential

In this section, we shall prove a result about the L_∞ Christoffel functions

$$(7.1) \qquad \lambda_{n,\infty}(W,x) := \inf_{P \in \mathcal{P}_{n-1}} \frac{\|PW\|_{L_\infty(\mathbb{R})}}{|P(x)|}\,.$$

Throughout, we assume that $W = e^{-Q} \in \mathcal{E}$. The estimate (7.2) will be the basis for our method for finding upper bounds for Christoffel functions in the next section.

Theorem 7.1. *Let $L > 0$. Uniformly for $n \geq 1$ and $|x| \leq a_n(1 + L\delta_n)$,*

$$(7.2) \qquad \frac{\lambda_{n,\infty}(W,x)}{W(x)} \sim 1\,.$$

Actually, this will be a corollary of

Theorem 7.2. *Given $n \geq 2$, and*

$$(7.3) \qquad |x_0| \leq a_n(1 + L\delta_n)$$

there exists $P_n \in \mathcal{P}_n$ such that

$$|P_n W|(x) \le C_1\,, \quad x \in \mathbb{R}\,, \tag{7.4}$$

and

$$|P_n W|(x_0) \ge C_2\,. \tag{7.5}$$

Here C_1 and C_2 are independent of n, x and x_0.

Deduction of Theorem 7.1 from Theorem 7.2. From the definition of $\lambda_{n,\infty}$, we see that

$$\frac{\lambda_{n+1,\infty}(W,x_0)}{W(x_0)} = \inf_{P\in\mathcal{P}_n} \frac{\|PW\|_{L_\infty(\mathbb{R})}}{|PW|}(x_0) \ge 1\,.$$

Moreover, Theorem 7.2 ensures that for the range (7.3),

$$\frac{\lambda_{n+1,\infty}(W,x_0)}{W(x_0)} \le \frac{\|P_nW\|_{L_\infty(\mathbb{R})}}{|P_nW|}(x_0) \le C_1/C_2\,.$$

Since we easily deduce from (2.12) that

$$\frac{a_{n-1}}{a_n} = 1 + O\left(\frac{1}{nT(a_n)}\right) = o(\delta_n)\,,$$

replacing n by $n-1$ in these two relations yields (7.2). ∎

Rather than following the more lengthy method of [7], [9], we shall use a Proposition in [8], based on a shorter proof of V. Totik [15]:

Lemma 7.3. *Let $d\sigma$ be a positive Borel measure on $[-1,1]$ that satisfies $\sigma[-1,1]=1$, and let*

$$U^\sigma(z) := \int_{-1}^{1} \log|z-t|\,d\sigma(t) \tag{7.6}$$

be the corresponding potential. Define

$$-1 = t_0 < t_1 < t_2 < \ldots < t_n = 1$$

and

$$I_j := [t_j, t_{j+1}]\,, \quad 0 \le j \le n-1\,,$$

by the conditions

$$\int_{I_j} d\sigma(t) = \frac{1}{n}\,, \quad 0 \le j \le n-1\,. \tag{7.7}$$

Assume that the following conditions hold:

(a) *Uniformly for* $1 \le j \le n-1$,

$$t_{j+1} - t_j \sim t_j - t_{j-1}\,. \tag{7.8}$$

(b) *There exists* $C_1 > 0$ *such that uniformly for* $0 \le j \le n-1$ *and* $x \in I_j$,

$$n \int_{I_j} \log\left(\frac{|x-t|}{t_{j+1}-t_j}\right) d\sigma(t) \ge -C_1\,. \tag{7.9}$$

(c) *There exists* $C_2 > 0$ *such that uniformly for* $0 \le k \le n-1$,

$$\sum_{j \le k-2} \frac{(t_{j+1}-t_j)^2}{(t_{j+1}-t_k)^2} + \sum_{j \ge k+2} \frac{(t_{j+1}-t_j)^2}{(t_j - t_{k+1})^2} \le C_2\,. \tag{7.10}$$

Then, given any $x_0 \in \mathbb{R}$, *one can find a polynomial* $R_n = R_{n,x_0} \in \mathcal{P}_n$ *that satisfies*

$$\left|R_n(x)\right| \le C_3 \exp\left(nU^\sigma(x)\right)\,, \quad x \in \mathbb{R}\,, \tag{7.11}$$

and

$$\left|R_n(x_0)\right| \ge \frac{1}{3} \exp\left(nU^\sigma(x_0)\right)\,. \tag{7.12}$$

The constant C_3 *in* (7.11) *depends only on the constants* C_1, C_2 *in* (7.9), (7.10) *and on the constants implicit in the* $\sim$ *relation* (7.8).

Proof. See Theorem 2.3 in [8]. ∎

Assume that we can verify the hypotheses (7.8) to (7.10) for $d\sigma(x) = \mu_n(x)dx$, where μ_n is the density function defined at (3.1). We can then proceed with the

DEDUCTION OF THEOREM 7.2 from LEMMA 7.3. Set

$$P_n(x) := \exp(\chi_{n,a_n})R_n(x/a_n)\,,$$

where χ_{n,a_n} is given by (4.8) and we apply Lemma 7.3 with x_0 replaced by x_0/a_n. For $x \in [-a_n, a_n]$, (7.11) shows that

$$|P_nW|(x) \le C_3 \exp\left(n\left[\int_{-1}^{1} \log|x/a_n - t|\mu_n(t)dt - Q(x)/n + \chi_{n,a_n}/n\right]\right) = \\ = C_3 \exp\left(nU_{n,a_n}(x/a_n)\right) = C_3\,,$$

by (4.7) and (4.9). So

$$\|P_nW\|_{L_\infty(\mathbb{R})} = \|P_nW\|_{L_\infty[-a_n,a_n]} \le C_3\,.$$

Similarly, (7.12) shows that

$$|P_nW|(x_0) \ge \exp\left(nU_{n,a_n}(x_0/a_n) - C_2\right) \ge C_4\,,$$

by (4.7),(4.18) in Theorem 4.3, and as $|x_0/a_n| \le 1 + L\delta_n = 1 + o(1/T(a_n))$. □

Now we turn to verifying (7.8) to (7.10) for $d\sigma(x) = \mu_n(x)dx$. First, a lemma about the discretisation points $\{t_j\}$, defined in Lemma 7.3. Of course, the t_j and I_j depend on n, but we do not display this dependence for notational simplicity.

LEMMA 7.4. (a) *For fixed* $\ell \ge 1$,

$$1 + t_\ell \sim \delta_n\,; \qquad 1 - t_{n-\ell} \sim \delta_n\,. \tag{7.13}$$

(b) *For* $1 \le j \le n-1$,

$$1 - t_j^2 \sim 1 - t_{j+1}^2\,. \tag{7.14}$$

(c) *For* $1 \le j \le n-1$,

$$t_{j+1} - t_j \sim t_j - t_{j-1}\,. \tag{7.15}$$

(d) *For* $1 \le j \le n-1$,

$$n(t_{j+1} - t_j)\mu_n(t_j) \sim n(t_{j+1} - t_j)\min\left\{\frac{1}{\sqrt{1-t_j^2}}\,, T(a_n)\sqrt{1-t_j^2}\right\} \sim 1. \tag{7.16}$$

(e) *For* $0 \le j \le n-1$,

$$C_1 \max\left\{\frac{1}{n}, \delta_n\right\} \ge t_{j+1} - t_j \ge C_2\big(nT(a_n)^{1/2}\big)^{-1}\,. \tag{7.17}$$

(f) *For* $1 \le j \le n-2$, *and* $t \in [t_j, t_{j+1}]$,

$$\mu_n(t) \sim \mu_n(t_j)\,. \tag{7.18}$$

The constants implicit in all the $\sim$ *relations above are independent of* n *and* j.

Proof. Recall from Theorem 3.1 that uniformly for $n \ge 1$ and $t \in (-1,1)$,

$$\mu_n(t) \sim \min\left\{\frac{1}{\sqrt{1-t^2}}\,, T(a_n)\sqrt{1-t^2}\right\}. \tag{7.19}$$

(a) We see that

$$n\int_{-1}^{-1+1/T(a_n)} \mu_n(t)dt \sim nT(a_n)\int_{-1}^{-1+1/T(a_n)} (1+t)^{1/2}dt \sim nT(a_n)^{-1/2} \to \infty \quad \text{as} \quad n \to \infty\,,$$

by Lemma 2.2 (viii). Then in view of the definition (7.7) of t_ℓ in Lemma 7.3, we see that for any fixed $\ell \ge 1$, $t_\ell \in \big[-1, -1+1/T(a_n)\big]$, for n large enough. Then

$$\frac{2\ell-1}{2n} = \int_{-1}^{t_\ell} \mu_n(t)dt \sim T(a_n)(1+t_\ell)^{3/2}\,,$$

and we deduce (7.13), if we recall the definition (1.18) of δ_n. The estimate for $1-t_{n-\ell}$ is handled similarly.

(b) If $0 \le t_j < t_{j+1} \le 1-1/T(a_n)$, we obtain

$$\frac{1}{n} = \int_{t_j}^{t_{j+1}} \mu_n(t)dt \sim \int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{1-t}}dt \sim \sqrt{1-t_j} - \sqrt{1-t_{j+1}}\,,$$

so

$$\frac{1}{n\sqrt{1-t_j}} \sim 1 - \left(\frac{1-t_{j+1}}{1-t_j}\right)^{1/2},$$

and by our restriction on t_j,

$$\frac{1}{n\sqrt{1-t_j}} \le \left(\frac{T(a_n)}{n^2}\right)^{1/2} \to 0\,, \quad \text{as} \quad n \to \infty\,,$$

so

$$\frac{1-t_{j+1}}{1-t_j} \to 1\,,$$

as $n \to \infty$, uniformly for j in this range, so (7.14) is true. If $1-2/T(a_n) \le t_j < t_{j+1} < 1$, then we similarly obtain

$$\frac{1}{n} = \int_{t_j}^{t_{j+1}} \mu_n(t)dt \sim T(a_n)\big[(1-t_j)^{3/2} - (1-t_{j+1})^{3/2}\big]\,,$$

so

$$\frac{1}{nT(a_n)}(1-t_{j+1})^{-3/2} \sim \left[\left(\frac{1-t_j}{1-t_{j+1}}\right)^{3/2} - 1\right].$$

Since by (a),

$$(1-t_{j+1})^{3/2} \ge (1-t_{n-1})^{3/2} \ge C\delta_n^{3/2} = C\big(nT(a_n)\big)^{-1}\,,$$

we obtain

$$1 \le \left(\frac{1-t_j}{1-t_{j+1}}\right)^{3/2} \le C\,,$$

and again (7.14) is true. The remaining cases are treated similarly.

(c), (d), (e), (f) may be easily proved using (7.7), (7.19) and (a), (b), which show that if $1 \le j \le n$,

$$\sqrt{1-t^2} \sim \sqrt{1-t_j^2}\,, \quad t \in [t_{j-1}, t_{j+1}] \cap [t_1, t_{n-1}]\,,$$

and hence

$$\mu_n(t) \sim \mu_n(t_j)\,, \quad t \in [t_{j-1}, t_{j+1}] \cap [t_1, t_{n-1}]\,.$$

We leave the details to the reader. □

Note that we have already verified (7.8) for $d\sigma(x) = \mu_n(x)dx$, with constants in the $\sim$ relations independent of j and n. We turn to the

VERIFICATION OF (7.9). We must show that

$$n \int_{t_j}^{t_{j+1}} \log\left(\frac{|x-t|}{t_{j+1}-t_j}\right)\mu_n(t)dt \ge -C_1\,, \tag{7.20}$$

uniformly for $n \ge 2$, $0 \le j \le n-1$ and $x \in I_j$. Let us assume first that $1 \le j \le n-2$, so that by (7.18),

$$\mu_n(t) \sim \mu_n(t_j)\,, \quad t \in [t_j, t_{j+1}]\,.$$

Then

$$\begin{aligned} n \int_{t_j}^{t_{j+1}} \log\left(\frac{|x-t|}{t_{j+1}-t_j}\right)\mu_n(t)dt &\sim n\mu_n(t_j) \int_{t_j}^{t_{j+1}} \log\left(\frac{|x-t|}{t_{j+1}-t_j}\right)dt = \\ &= n\mu_n(t_j)(t_{j+1}-t_j) \int_{(x-t_{j+1})/(t_{j+1}-t_j)}^{(x-t_j)/(t_{j+1}-t_j)} \log|s|\,ds \ge \\ &\ge n\mu_n(t_j)(t_{j+1}-t_j) \int_{-1}^{1} \log|s|ds \ge -C_4\,, \end{aligned}$$

by (7.16). For $j = 0$ and $j = n-1$, the proof is only a little more difficult.

Suppose $j = 0$. Then

$$n\int_{t_j}^{t_{j+1}}\log\left(\frac{|x-t|}{t_{j+1}-t_j}\right)\mu_n(t)dt \sim nT(a_n)\int_{t_0}^{t_1}\log\left(\frac{|x-t|}{t_1-t_0}\right)\sqrt{1-t^2}\,dt \geq$$

$$\geq \quad C_5 nT(a_n)\sqrt{1-t_1^2}(t_1-t_0)\int_{(x-t_1)/(t_1-t_0)}^{(x-t_0)/(t_1-t_0)}\log|s|\,ds \geq$$

$$\geq -C_6 nT(a_n)(1+t_1)^{3/2} \geq -C_7 \quad \text{by (7.13).} \qquad \square$$

Next, we turn to the more difficult

VERIFICATION OF (7.10). Assume, say, that $2 \leq k \leq n-2$. (The case $k = 1$ or $k = n-1$ is very similar). It is an easy consequence of (7.8) that

$$t_{j+1} - t_k \sim t - t_k\,, \quad t \in [t_j, t_{j+1}]\,,$$

uniformly in n, k and $j \leq k-2$. Then by (7.16) and then (7.18),

$$\sum_{1\leq j\leq k-2}\frac{(t_{j+1}-t_j)^2}{(t_{j+1}-t_k)^2} \sim \sum_{1\leq j\leq k-2}\frac{t_{j+1}-t_j}{(t_{j+1}-t_k)^2 n\mu_n(t_j)} \sim$$

$$\sim \int_{t_1}^{t_{k-1}}\frac{dt}{(t-t_k)^2 n\mu_n(t)} \sim \tag{7.21}$$

$$\sim \frac{1}{n}\int_{t_1}^{t_{k-1}}(t-t_k)^{-2}\max\left\{\sqrt{1-t^2}\,,\frac{1}{T(a_n)\sqrt{1-t^2}}\right\}dt =: J_1\,,$$

where we have used (7.19). Similarly by (7.15),

$$\sum_{n-1\geq j\geq k+2}\frac{(t_{j+1}-t_j)^2}{(t_j-t_{k+1})^2} \sim \sum_{n-1\geq j\geq k+2}\frac{(t_{j+1}-t_j)^2}{(t_j-t_k)^2} \sim$$

$$\sim \frac{1}{n}\int_{t_{k+1}}^{1}(t-t_k)^{-2}\max\left\{\sqrt{1-t^2}\,,\frac{1}{T(a_n)\sqrt{1-t^2}}\right\}dt =: J_2\,. \tag{7.22}$$

Moreover, for $j = 0$,

$$\frac{(t_{j+1} - t_j)^2}{(t_j - t_{k+1})^2} \le \frac{(t_1 - t_0)^2}{(t_0 - t_2)^2} \le 1\,. \tag{7.23}$$

A similar bound holds for $j = n$. Now we estimate $J_1 + J_2$. Let us suppose for simplicity that $t_k \ge 0$, and let us consider two cases:

Case I: $0 \le t_k \le 1 - \dfrac{2}{T(a_n)}$. Then

$$J_1 + J_2 \le C_9 \frac{1}{n} \int_{1-1/T(a_n)}^{1} (t - t_k)^{-2} \frac{dt}{T(a_n)\sqrt{1-t^2}} +$$

$$+ C_9 \frac{1}{n} \int_{[-1/2,1-1/T(a_n)]\setminus[t_{k-1},t_{k+1}]} (t - t_k)^{-2} \sqrt{1-t^2}\,dt \qquad \le$$

$$\le C_{10} \frac{1}{n} T(a_n) \int_{1-1/T(a_n)}^{1} \frac{dt}{\sqrt{1-t^2}} +$$

$$+ C_{10} \frac{1}{n} \int_{[-1/2,1]\setminus[t_{k-1},t_{k+1}]} (t - t_k)^{-2} [\sqrt{1-t_k} + \sqrt{|t - t_k|}]\,dt \le$$

$$\le C_{11} \frac{1}{n} T(a_n)^{1/2} + C_{11} \frac{1}{n} \big[(t_k - t_{k-1})^{-1} \sqrt{1-t_k} + (t_k - t_{k-1})^{-1/2}\big].$$

Here we have used (7.15). Now by Lemma 2.2 (viii), the first of the three terms on the last right-hand side is $o(1)$. Moreover, by (7.16), (recall $0 \le t_k \le 1 - 2/T(a_n)$)

$$\frac{1}{n}(t_k - t_{k-1})^{-1}\sqrt{1-t_k} \sim \frac{1}{n(t_k - t_{k-1})\mu_n(t_k)} \sim 1\,,$$

and by (7.17),

$$\frac{1}{n}(t_k - t_{k-1})^{-1/2} \le C_{12} \Big(\frac{T(a_n)}{n^2}\Big)^{1/4} = o(1)\,.$$

So we have shown that

$$J_1 + J_2 \le C_{13}\,, \tag{7.24}$$

and hence (7.10) holds.

Case II: $1 - \dfrac{2}{T(a_n)} < t_k < 1$. Recall first from the proof of Lemma 7.4 (a), that as $n \to \infty$, a growing number of t_j lie in $\big[1-3/T(a_n), 1-2/T(a_n)\big]$. Then by (7.19),

$$\begin{aligned} J_1 + J_2 \quad &\le \quad C_{14}\frac{1}{n}\int_{[1-3/T(a_n),1]\setminus[t_{k-1},t_{k+1}]} (t-t_k)^{-2}\frac{dt}{T(a_n)\sqrt{1-t}} + \\ &\qquad + C_{14}\frac{1}{n}\int_{-1/2}^{1-3/T(a_n)} (t-t_k)^{-2}\sqrt{1-t^2}dt =: J^{(1)} + J^{(2)}\,. \end{aligned}$$

Here the substitution $t - t_k = s(1-t_k)$ shows that

$$\begin{aligned} J^{(1)} &= C_{14}\frac{1}{nT(a_n)(1-t_k)^{3/2}}\int_{\left[1-\frac{3}{T(a_n)(1-t_k)},1\right]\setminus\left[\frac{t_{k-1}-t_k}{1-t_k},\frac{t_{k+1}-t_k}{1-t_k}\right]} s^{-2}(1-s)^{-1/2}ds \le \\ &\le C_{15}\frac{1}{nT(a_n)(1-t_k)^{3/2}}\left[1+\int_{-\infty}^{(t_{k-1}-t_k)/(1-t_k)} s^{-2}ds\right] \le \\ &\le C_{16}\left[\frac{1}{nT(a_n)(1-t_k)^{1/2}(t_k-t_{k-1})} + \frac{1}{nT(a_n)(1-t_{n-1})^{3/2}}\right] \le \\ &\le C_{17}\left[\frac{1}{n\mu_n(t_k)(t_k-t_{k-1})} + \frac{1}{nT(a_n)\delta_n^{3/2}}\right] \le C_{18}\,, \end{aligned}$$

by (7.16) and (7.13). Next, (recall that $t_k \geq 1 - 2/T(a_n)$)

$$J^{(2)} \leq C_{19}\frac{1}{n}\int_{-1/2}^{1-3/T(a_n)} (t-t_k)^{-2}\big[\sqrt{1-t_k}+\sqrt{|t-t_k|}\big]\,dt \leq$$

$$\leq C_{20}\frac{1}{n}\big[T(a_n)\sqrt{1-t_k}+T(a_n)^{1/2}\big] \leq$$

$$\leq C_{21}\Big(\frac{T(a_n)}{n^2}\Big)^{1/2}\big[\sqrt{T(a_n)(1-t_k)}+1\big] \leq C_{22}\,,$$

by Lemma 2.2 (viii). Again, we have (7.24) and hence (7.10). ∎

8 – Upper bounds for λ_n: Theorem 1.2

In this section, we prove Theorem 1.2, by providing upper bounds for λ_n to match the lower bounds in Theorem 6.1. Throughout, we assume that $W = \mathrm{e}^{-Q} \in \mathcal{E}$.

Lemma 8.1. *Fix $L \geq 0$ and $n > m > 1$. Then*

$$(8.1)\qquad \frac{\lambda_n(W^2,x)}{W^2(x)} \leq \frac{C}{n-m}a_n \max\Big\{1-\frac{|x|}{a_n}\,,(n-m)^{-2}\Big\}^{1/2},$$

for

$$(8.2)\qquad |x| \leq a_m(1+L\delta_m)\,.$$

Here C is independent of n, m and x.

Proof. Let $u(x) \equiv 1$ be the Legendre weight on $[-1,1]$. Recall also the definition of $\lambda_{n,\infty}(W,x)$ at (7.1).

Now by Theorem 1.5 (proved in Section 5),

$$\frac{\lambda_n(W^2,x)}{W^2(x)} = \inf_{P\in\mathcal{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2(t)dt/(PW)^2(x) \leq$$

$$\leq C \inf_{P\in\mathcal{P}_{n-1}} \int_{-a_n}^{a_n} (PW)^2(t)dt/(PW)^2(x) \leq$$

$$\leq C\big[\lambda_{m,\infty}(W,x)/W(x)\big]^2 \inf_{R\in\mathcal{P}_{n-m}} \int_{-a_n}^{a_n} R^2(t)dt/R^2(x) = \tag{8.3}$$

$$= C\big[\lambda_{m,\infty}(W,x)/W(x)\big]^2 a_n \inf_{S\in\mathcal{P}_{n-m}} \int_{-1}^{1} S^2(t)dt/S^2(x/a_n) =$$

$$= C\big[\lambda_{m,\infty}(W,x)/W(x)\big]^2 a_n \lambda_{n-m+1}(u,x/a_n)\,.$$

Now by Theorem 7.1,

$$\frac{\lambda_{m,\infty}(W,x)}{W(x)} \leq C_1\,, \quad |x| \leq a_m(1+L\delta_m)\,, \quad m\geq 1\,.$$

Moreover, classical estimates for the Christoffel function of the Legendre weight on $[-1,1]$ [21], [24] show that for $\ell\geq 1$ and $t\in[-1,1]$,

$$\lambda_\ell(u,t) \leq \frac{C}{\ell}\max\big\{1-|t|,\ell^{-2}\big\}^{1/2}\,.$$

Here

$$\lambda_\ell(u,t) = \frac{1}{\sum\limits_{j=0}^{\ell-1} p_j^2(u,t)}$$

is a decreasing function of $t\in[1,\infty)$, so the upper bound holds for all $t\in\mathbb{R}$. Substituting into (8.3) yields the result. ∎

Obviously, we are going to choose $m=m(n,x)$ to obtain the desired estimate from Lemma 8.1. We do this separately for three ranges:

Proof of (1.20) of Theorem 1.2 for $|x|\leq a_{n/2}$. We choose $m:=\langle n/2\rangle$ in Lemma 8.1 and $L=0$. Now for this range of x,

$$1-\frac{|x|}{a_n} \geq 1-\frac{a_{n/2}}{a_n} \geq \frac{C_1}{T(a_n)} \geq \frac{C_2}{n^2} \geq \frac{C_3}{(n-m)^2}\,,$$

by (2.12) and Lemma 2.2 (viii). So Lemma 8.1 yields

$$\begin{aligned}\frac{\lambda_n(W^2,x)}{W^2(x)} &\le C_4\frac{a_n}{n}\Big(1-\frac{|x|}{a_n}\Big)^{1/2} \sim \\ &\sim \frac{a_n}{n}\max\left\{\sqrt{1-\frac{|x|}{a_n}+\delta_n}\,,\,\left[T(a_n)\sqrt{1-\frac{|x|}{a_n}+\delta_n}\right]^{-1}\right\}\sim \\ &\sim \frac{a_n}{n}\Psi_n(x)\,,\end{aligned}$$

since $1-|x|/a_n \ge C_5/T(a_n) \ge C_6\delta_n$. The corresponding lower bound was proved in Theorem 6.1. □

Proof of (1.20) of Theorem 1.2 for $a_{n/2} \le |x| \le a_n(1-L\delta_n)$. Note that

$$\log\Big(\frac{a_n}{a_{n-1}}\Big) = \int_{n-1}^{n}\frac{a_t'}{a_t}dt \sim T(a_n)^{-1}\int_{n-1}^{n}\frac{dt}{t} \sim \big[nT(a_n)\big]^{-1}$$

by (2.11) and (2.8). Hence

$$\frac{a_n}{a_{n-1}} = 1 + O\Big(\big[nT(a_n)\big]^{-1}\Big) = 1 + o(\delta_n)\,, \tag{8.4}$$

and so for n large enough,

$$a_n(1-L\delta_n) < a_{n-1}\,.$$

Consequently, for the range of x considered, we can choose $n/2 \le m < n$ such that

$$a_{m-1} < |x| \le a_m\,.$$

Here, since $m \sim n$, we have as above,

$$\frac{a_m}{a_{m-1}} = 1 + o(\delta_m) = 1 + o(\delta_n)\,,$$

so

$$1-\frac{a_m}{a_n} = 1-\frac{|x|}{a_n} + o(\delta_n) \sim 1-\frac{|x|}{a_n}\,.$$

Next, by (2.12)

$$\frac{a_n}{a_m} - 1 \sim T(a_n)^{-1}\Big(\frac{n}{m} - 1\Big) .$$

We deduce that

$$1 - \frac{|x|}{a_n} \sim 1 - \frac{a_m}{a_n} \sim \frac{a_n}{a_m} - 1 \sim T(a_n)^{-1}\Big(1 - \frac{m}{n}\Big) . \tag{8.5}$$

Then

$$\frac{(n-m)^{-2}}{1-|x|/a_n} = \frac{n^{-2}(1-m/n)^{-2}}{1-|x|/a_n} \sim \big(nT(a_n)\big)^{-2}(1-|x|/a_n)^{-3} \leq$$
$$\leq \big(nT(a_n)\big)^{-2}(L\delta_n)^{-3} = L^{-3} .$$

So

$$\max\Big\{1 - \frac{|x|}{a_n}, (n-m)^{-2}\Big\} \sim 1 - \frac{|x|}{a_n} .$$

Then Lemma 8.1 yields

$$\frac{\lambda_n(W^2,x)}{W^2(x)} \leq C\frac{a_n}{n}\Big(1 - \frac{m}{n}\Big)^{-1}\Big(1 - \frac{|x|}{a_n}\Big)^{1/2} \sim \frac{a_n}{n}T(a_n)^{-1}(1-|x|/a_n)^{-1/2} ,$$

by (8.5). Finally, for this range of x,

$$L\delta_n \leq 1 - \frac{|x|}{a_n} \leq 1 - \frac{a_{n/2}}{a_n} \leq \frac{C_1}{T(a_n)} ,$$

so

$$T(a_n)^{-1}\Big(1-\frac{|x|}{a_n}\Big)^{-1/2} \sim \Big[T(a_n)\sqrt{1-\frac{|x|}{a_n}+\delta_n}\Big]^{-1} \sim$$
$$\sim \max\Big\{\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}, \Big[T(a_n)\sqrt{1-\frac{|x|}{a_n}+2L\delta_n}\Big]^{-1}\Big\} =$$
$$= \Psi_n(x) .$$

So we have proved

$$\frac{\lambda_n(W^2,x)}{W^2(x)} \leq C_2\frac{a_n}{n}\Psi_n(x) .$$

Then Theorem 6.1 provides the corresponding lower bound. □

Proof of (1.20) of Theorem 1.2 for $a_n(1-L\delta_n) \le |x| \le a_n(1+L\delta_n)$. Here we choose

$$m := n - \langle nT(a_n)\rangle^{1/3},$$

n large enough, where $\langle x\rangle$ denotes the greatest integer $\le x$. Then

$$(n-m)^{-2} \sim \big(nT(a_n)\big)^{-2/3} = \delta_n \ge \frac{1-|x|/a_n}{L}.$$

Then Lemma 8.1 gives

$$\frac{\lambda_n(W^2,x)}{W^2(x)} \le Ca_n\big(nT(a_n)\big)^{-1/3}\delta_n^{1/2} = Ca_n\delta_n \sim \frac{a_n}{n}\Psi_n(x)\,, \tag{8.6}$$

provided also

$$|x| \le a_m(1+K\delta_m)\,, \tag{8.7}$$

some fixed $K>0$. Now using (2.8) and (2.12) as above, we see that

$$\frac{a_n}{a_m} \le 1 + C_1T(a_n)^{-1}\log\Big(\frac{n}{m}\Big) \le 1 + C_2\frac{\big(nT(a_n)\big)^{1/3}}{nT(a_n)} = 1 + C_2\delta_n\,, \tag{8.8}$$

where C_2 does not depend on K, so

$$\frac{a_n(1+L\delta_n)}{a_m(1+K\delta_m)} \le 1 + (L+C_2)\delta_n - K\delta_m + O(\delta_n^2) < 1\,,$$

if K is large enough, and since $\delta_m \sim \delta_n$ independently of K, L. Thus we have proved that the given range is contained in the range (8.7) if K and n are large enough, and so (8.6) holds. As before, Theorem 6.1 provides the corresponding lower bound. □

We remind the reader that we already proved (1.21) of Theorem 1.2 as (6.3) of Theorem 6.1.

Proof of (1.22) of Theorem 1.2. From the Mhaskar-Saff identity applied to W^2, we have

$$\sup_{x\in\mathbb{R}}\{\lambda_n^{-1}(W^2,x)W^2(x)\} = \sup_{x\in[-a_n,a_n]}\{\lambda_n^{-1}(W^2,x)W^2(x)\} \sim \frac{n}{a_n}T(a_n)^{1/2},$$

from (1.20) of Theorem 1.2 and some straightforward calculations. Moreover, if $0<\alpha<\beta<1$, we have for $a_{\alpha n}\le|x|\le a_{\beta n}$,

$$1-\frac{|x|}{a_n}\sim T(a_n)^{-1},$$

(see (2.12)) and again (1.20) implies that for this range of x,

$$\lambda_n^{-1}(W^2,x)W^2(x)\sim\frac{n}{a_n}T(a_n)^{1/2}.$$

□

9 – Zeros: Corollary 1.3

In this section, we prove Corollary 1.3. Throughout, we assume that $W=\mathrm{e}^{-Q}\in\mathcal{E}$.

Proof of Corollary 1.3 (a). We shall use the well known formula

$$x_{1,n}=\sup_{\substack{P\in\mathcal{P}_{2n-2}\\ P\ge0 \text{ in } \mathbb{R}}}\left\{\frac{\int_{-\infty}^{\infty}xP(x)W^2(x)dx}{\int_{-\infty}^{\infty}P(x)W^2(x)dx}\right\},$$

which is an easy consequence of the Gauss quadrature formula. Let δ_n be defined by (1.18), and $K>0$. By Theorem 1.5 (proved in Section 5), applied to W^2 for $p=1$,

$$|a_n - x_{1,n}| = \left| \inf_{\substack{P\in\mathcal{P}_{2n-2}\\ P\geq 0 \text{ in } \mathbb{R}}} \left\{ \frac{\displaystyle\int_{-\infty}^{\infty} (a_n - x)P(x)W^2(x)dx}{\displaystyle\int_{-\infty}^{\infty} P(x)W^2(x)dx} \right\} \right| \leq$$

$$\leq C \inf_{\substack{P\in\mathcal{P}_{2n-2}\\ P\geq 0 \text{ in } \mathbb{R}}} \left\{ \frac{\displaystyle\int_{-a_n(1-K\delta_n)}^{a_n(1-K\delta_n)} |a_n - x|P(x)W^2(x)dx}{\displaystyle\int_{-a_n}^{a_n} P(x)W^2(x)} \right\} .$$

We choose

$$m := n - \left\langle \left(nT(a_n)\right)^{1/3} \right\rangle := n - \sigma \,,$$

and

$$P(x) := \lambda_m^{-1}(W^2, x)R(x/a_n)\,,$$

where $R \in \mathcal{P}_{2\sigma}$ is nonnegative in $\mathbb{R}$. Now as at (8.8), we have

$$1 \leq \frac{a_n}{a_m} \leq 1 + C_1\delta_n \leq 1 + C_1\delta_m\,,$$

so by Theorem 1.2, we have for $|x| \leq a_n$, and some suitable $L > 0$,

$$\lambda_m(W^2, x) \sim \frac{a_m}{m} W^2(x) \max\left\{ \sqrt{1 - \frac{|x|}{a_m} + L\delta_m}, \left[T(a_m)\sqrt{1 - \frac{|x|}{a_m} + L\delta_m} \right]^{-1} \right\}.$$

Now

$$\frac{\sigma}{n} \sim \left(\frac{T(a_n)}{n^2}\right)^{1/3} \to 0 \quad \text{as} \quad n \to \infty\,,$$

so $m \sim n$, and hence $\delta_m \sim \delta_n$ and $T(a_m) \sim T(a_n)$. Moreover, for $|x| \leq a_n(1 - K\delta_n)$,

$$1 - \frac{|x|}{a_m} = 1 - \frac{|x|}{a_n} + \frac{|x|}{a_n}\left(1 - \frac{a_n}{a_m}\right) = 1 - \frac{|x|}{a_n} + O(\delta_n) \sim 1 - \frac{|x|}{a_n}\,,$$

if K is large enough, as the constant in the order relation is independent of K. We deduce that for $|x| \le a_n(1-K\delta_n)$,

$$\lambda_n(W^2,x) \sim \frac{a_n}{n} W^2(x) \max\left\{\sqrt{1-\frac{|x|}{a_n}}, \left[T(a_n)\sqrt{1-\frac{|x|}{a_n}}\right]^{-1}\right\} \sim$$

$$\sim \frac{a_n}{n}\, \frac{W^2(x)}{\min\left\{v(x/a_n), T(a_n)u(x/a_n)\right\}}\,,$$

where $u(s) := \sqrt{1-s^2}$, $v(s) := 1/\sqrt{1-s^2}$, $s \in [-1,1]$. Substituting P into (9.1), and then making the substitution $x = a_n s$, and using this last estimate yields

$$|a_n - x_{1,n}| \le C_2 a_n \inf_{\substack{R\in\mathcal{P}_{2\sigma} \\ R\ge 0 \text{ in } \mathbb{R}}} \left\{ \frac{\displaystyle\int_{-1+K\delta_n}^{1-K\delta_n} (1-s)R(s)\min\left\{v(s),T(a_n)u(s)\right\}ds}{\displaystyle\int_{-1}^{1} R(s)\min\left\{v(s),T(a_n)u(s)\right\}ds} \right\}. \tag{9.2}$$

Let $\ell_{1,\sigma}(u,s)$ be the fundamental polynomial of Lagrange interpolation at the largest zero $\hat{x}_{1,\sigma}$ of the orthonormal polynomial $p_\sigma(u,x)$ for the Chebyshev weight of the second kind u. We choose

$$R(s) := \ell_{1,\sigma}^2(u,s)\,.$$

Then

$$\begin{aligned} &\int_{-1+K\delta_n}^{1-K\delta_n} (1-s)R(s)\min\left\{v(s),T(a_n)u(s)\right\}ds \le \\ &\qquad \le T(a_n)\int_{-1}^{1}(1-s)\ell_{1,\sigma}^2(u,s)u(s)ds = \\ &\qquad \text{(by the Gauss quadrature formula)} \\ &\qquad = T(a_n)(1-\hat{x}_{1,\sigma})\lambda_\sigma(u,\hat{x}_{1,\sigma}) \sim \\ &\qquad \sim T(a_n)\sigma^{-1}(1-\hat{x}_{1,\sigma})^2 \sim T(a_n)\sigma^{-5}\,, \end{aligned} \tag{9.3}$$

by classical estimates for the largest zeros and corresponding Christoffel numbers of orthogonal polynomials for Jacobi weights. See, for example

[21], [24]. On the other hand,

$$\ell_{1,\sigma}(u,\hat{x}_{1,\sigma}) = 1$$

and

$$\|\ell_{1,\sigma}(u,\cdot)\|_{L_\infty[-1,1]} \le C_3\,. \tag{9.4}$$

A proof of (9.4) was given in [9, Section 10]. From the classical Bernstein inequality, we deduce that for some small enough $\alpha > 0$,

$$\ell_{1,\sigma}(u,s) \ge \frac{1}{2}\,, \quad s \in [\hat{x}_{1,\sigma} - \alpha\sigma^{-2}, \hat{x}_{1,\sigma}]\,.$$

Also, for s in this range, (recall $\sigma = \left\langle \left(nT(a_n)\right)^{1/3}\right\rangle$)

$$\frac{v(s)}{T(a_n)u(s)} = (1-s^2)^{-1}T(a_n)^{-1} \ge C_6\sigma^2 T(a_n)^{-1} \ge C_7\Big(\frac{n^2}{T(a_n)}\Big)^{1/3} > 1\,,$$

for n large enough, so

$$\int_{-1}^{1} R(s)\min\{v(s), T(a_n)u(s)\}ds \ge \int_{\hat{x}_{1,\sigma}-\alpha\sigma^{-2}}^{\hat{x}_{1,\sigma}} \ell_{1,\sigma}^2(u,s)T(a_n)u(s)ds \ge$$

$$\ge C_8 T(a_n)\int_{\hat{x}_{1,\sigma}-\alpha\sigma^{-2}}^{\hat{x}_{1,\sigma}} u(s)ds \ge C_9 T(a_n)\sigma^{-3}\,.$$

Substituting this and (9.3) into (9.2) yields

$$|a_n - x_{1,n}| \le C_{10}a_n\sigma^{-2} \sim a_n\big(nT(a_n)\big)^{-2/3} = a_n\delta_n\,. \qquad \square$$

In the proof of Corollary 1.3 (b), we shall need:

LEMMA 9.1. *There exists an entire function*

$$G(x) := \sum_{j=0}^{\infty} g_{2j}x^{2j} \tag{9.5}$$

with $g_{2j} \ge 0$, $j \ge 0$, *satisfying*

$$G(x) \sim W^{-2}(x)\,, \quad x \in \mathbb{R}\,. \tag{9.6}$$

PROOF. We shall apply a result of CLUNIE and KOVARI [2] on entire functions with certain asymptotic behaviour. Set

$$\widehat{Q}(r) := Q(\sqrt{r}), \quad r \in [0, \infty),$$

and

$$\psi(r) := r\widehat{Q}'(r) = \frac{1}{2}\sqrt{r}Q'(\sqrt{r}), \quad r \in [0, \infty).$$

Then $\psi(r)$ is positive and increasing in $(0, \infty)$, and it is easy to see from Lemma 2.1 (iii) that for some suitably large $\lambda > 1$, we have

$$\psi(\lambda r) - \psi(r) \geq 1, \quad r \geq 1.$$

Moreover, $e^{\widehat{Q}(r)}$ admits the representation

$$e^{\widehat{Q}(r)} = \exp\left(\widehat{Q}(1) + \int_1^r \frac{\psi(\rho)}{\rho} d\rho\right), \quad r > 1.$$

By Theorem 4 in [2, p. 19], there exists an entire function

$$H(r) = \sum_{j=0}^{\infty} h_j r^j,$$

such that $h_j \geq 0$, $j \geq 0$, and

$$H(r) \sim \exp\left(\widehat{Q}(r)\right), \quad r \in [1, \infty),$$

and hence in $[0, \infty)$. Then for $x \in \mathbb{R}$,

$$G(x) := H(x^2) \sim \exp\left(\widehat{Q}(x^2)\right) = \exp\left(Q(x)\right) = W^{-1}(x).$$

Replacing Q by $2Q$, which also satisfies the required hypotheses, we obtain the result. ∎

Proof of Corollary 1.3 (b). We use the Posse-Markov-Stieltjes inequalities in the form proved in [6, p. 89]. Let G be the function of the lemma. By the Posse-Markov-Stieltjes inequalities, for $2 \le j \le n-1$,

$$\lambda_n(W^2, x_{j,n})G(x_{j,n}) = \frac{1}{2}\Bigg[\sum_{k:|x_{k,n}|<x_{j-1,n}} - \sum_{k:|x_{k,n}|<x_{j,n}}\Bigg]\lambda_n(W^2, x_{k,n})G(x_{k,n}) \le$$

$$\le \frac{1}{2}\Bigg[\int_{-x_{j-1,n}}^{x_{j-1,n}} - \int_{-x_{j+1,n}}^{x_{j+1,n}}\Bigg]G(t)W^2(t)dt = \int_{x_{j+1,n}}^{x_{j-1,n}} G(t)W^2(t)dt\,.$$

Similarly,

$$\lambda_n(W^2, x_{j,n})G(x_{j,n}) + \lambda_n(W^2, x_{j+1,n})G(x_{j+1,n}) \ge \int_{x_{j+1,n}}^{x_{j,n}} G(t)W^2(t)dt\,.$$

Using Lemma 9.1, we obtain

$$\frac{\lambda_n(W^2, x_{j,n})}{W^2(x_{j,n})} \le C_1(x_{j-1,n} - x_{j+1,n})\,;$$

$$\frac{\lambda_n(W^2, x_{j,n})}{W^2(x_{j,n})} + \frac{\lambda_n(W^2, x_{j+1,n})}{W^2(x_{j+1,n})} \ge C_2(x_{j,n} - x_{j+1,n})\,.$$

Then Theorem 1.2, and Corollary 1.3 (a) imply that uniformly for $2 \le j \le n-1$,

$$x_{j-1,n} - x_{j+1,n} \ge C_3 \frac{a_n}{n} \Psi_n(x_{j,n})\,, \tag{9.7}$$

and for $1 \le j \le n-1$,

$$x_{j,n} - x_{j+1,n} \le C_4 \frac{a_n}{n} \left[\Psi_n(x_{j,n}) + \Psi_n(x_{j+1,n})\right] \tag{9.8}$$

Here, if $x_{j+1,n} \geq 0$, and $x_{j,n} \leq a_n(1 - 1/T(a_n))$,

$$\begin{aligned} 1 &\leq \frac{1 - x_{j+1,n}/a_n}{1 - x_{j,n}/a_n} = 1 + \frac{x_{j,n} - x_{j+1,n}}{a_n(1 - x_{j,n}/a_n)} \leq \\ \text{(by (9.8))} \qquad &\leq 1 + C_5 \frac{1}{n} \frac{(1 - x_{j+1,n}/a_n)^{1/2}}{1 - x_{j,n}/a_n} = \\ &= 1 + \left(\frac{1 - x_{j+1,n}/a_n}{1 - x_{j,n}/a_n}\right)^{1/2} C_5 \frac{1}{n}\Big(1 - \frac{x_{j,n}}{a_n}\Big)^{-1/2} \leq \\ &\leq 1 + o\bigg(\Big(\frac{1 - x_{j+1,n}/a_n}{1 - x_{j,n}/a_n}\Big)^{1/2}\bigg), \end{aligned}$$

as

$$\frac{1}{n}\Big(1 - \frac{x_{j,n}}{a_n}\Big)^{-1/2} \leq \frac{T(a_n)^{1/2}}{n} \to 0 \quad \text{as} \quad n \to \infty .$$

We deduce that in this case, uniformly in j,

$$\frac{1 - x_{j+1,n}/a_n}{1 - x_{j,n}/a_n} \to 1, \quad n \to \infty .$$

Next, if $x_{j+1,n} \geq 0$, and $a_n(1-1/T(a_n)) \leq x_{j+1,n} < x_{j,n} \leq a_n(1-\delta_n)$, then

$$\begin{aligned} 1 &\leq \frac{1 - x_{j+1,n}/a_n}{1 - x_{j,n}/a_n} = 1 + \frac{x_{j,n} - x_{j+1,n}}{a_n(1 - x_{j,n}/a_n)} \leq \\ &\leq 1 + \frac{C_6}{nT(a_n)(1 - x_{j,n}/a_n)^{3/2}} \leq 1 + \frac{C_6}{nT(a_n)\delta_n^{3/2}} \leq C_7 . \end{aligned}$$

Similarly, we can treat the other cases, and show that

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \sim 1 - \frac{|x_{j+1,n}|}{a_n} + L\delta_n , \tag{9.9}$$

uniformly for $1 \leq j \leq n-1$, if only L is large enough. This and (9.7) to (9.8) establish (1.24). ∎

10 – Bounds on orthogonal polynomials: Corollary 1.4

In this section, we establish the bounds on the orthogonal polynomials stated in Corollary 1.4. The method is exactly the same as that in [9], using ideas from, for example, [1], [8], [16], [22] and the reader is encouraged to first read section 2 of [9] for the outlines of the method. Throughout, we assume that $W = \mathrm{e}^{-Q} \in \mathcal{E}$.

We need more notation. We define

$$K_n(W^2, x, t) := \sum_{j=0}^{n-1} p_j(W^2, x) p_j(W^2, t) \,. \tag{10.1}$$

The *Christoffel-Darboux formula* states that

$$K_n(W^2, x) = \frac{\gamma_{n-1}}{\gamma_n}(W^2) \frac{p_n(W^2, x)p_{n-1}(W^2, t) - p_n(W^2, t)p_{n-1}(W^2, x)}{x - t} \,. \tag{10.2}$$

In particular, for $t = x$, this yields

$$\lambda_n^{-1}(W^2, x) = \frac{\gamma_{n-1}}{\gamma_n} \left[p_n'(W^2, x)p_{n-1}(W^2, x) - p_{n-1}'(W^2, x)p_n(W^2, x)\right] , \tag{10.3}$$

and for $x = t = x_{j,n}$, a zero of p_n,

$$\lambda_n^{-1}(W^2, x_{j,n}) = \frac{\gamma_{n-1}}{\gamma_n}(W^2) p_n'(W^2, x_{j,n}) p_{n-1}(W^2, x_{j,n}) \,. \tag{10.4}$$

We define

$$A_n(x) := 2\frac{\gamma_{n-1}}{\gamma_n}(W^2) \int_{-\infty}^{\infty} p_n^2(W^2, t) \frac{Q'(x) - Q'(t)}{x - t} W^2(t) dt \,. \tag{10.5}$$

Lemma 10.1.

$$p_n'(W^2, x_{j,n}) = A_n(x_{j,n}) p_{n-1}(W^2, x_{j,n}) \,, \quad 1 \le j \le n \,. \tag{10.6}$$

Proof. The method is well known [1], [7], [16], [22] but we sketch the details. We integrate by parts in the following identity:

$$p_n'(W^2, x_{j,n}) = \int_{-\infty}^{\infty} p_n'(W^2, t) K_n(W^2, x_{j,n}, t) W^2(t) dt$$

to obtain, using the orthogonality,

$$\begin{aligned} p_n'(W^2, x_{j,n}) &= -\int_{-\infty}^{\infty} p_n(W^2, t) K_n(W^2, x_{j,n}, t) \frac{d}{dt} W^2(t) dt = \\ &= \int_{-\infty}^{\infty} p_n(W^2, t) K_n(W^2, x_{j,n}, t) \big(2Q'(t) - 2Q'(x_{j,n})\big) W^2(t) dt \,, \end{aligned}$$

where we have used orthogonality again. Now an application of the Christoffel Darboux formula yields (10.6). □

Note that from (10.6) and (10.4) follows

$$\lambda_n^{-1}(W^2, x_{j,n}) = \frac{\gamma_{n-1}}{\gamma_n}(W^2) A_n(x_{j,n}) p_{n-1}^2(W^2, x_{j,n}) \,. \tag{10.7}$$

This identity shows that once we have estimates for $A_n(x)$, we can use Theorem 1.2 to derive estimates for $p_{n-1}(W^2, x_{j,n})$. Then the Christoffel-Darboux formula, in the form

$$p_n(W^2, x) = \frac{K_n(W^2, x, x_{j,n})(x - x_{j,n})}{\frac{\gamma_{n-1}}{\gamma_n}(W^2) p_{n-1}(W^2, x_{j,n})}$$

and (10.7) yield

$$|p_n(W^2, x)| = \frac{|K_n(W^2, x, x_{j,n})(x - x_{j,n})| \, \big[\lambda_n(W^2, x_{j,n}) A_n(x_{j,n})\big]^{1/2}}{\left[\frac{\gamma_{n-1}}{\gamma_n}(W^2)\right]^{1/2}} \,. \tag{10.8}$$

Applying upper bounds for A_n, and our result for Christoffel functions, and spacing of zeros, will establish Corollary 1.4. We now proceed with the estimation of $A_n(x)$. This is indirect, and fairly technical.

Throughout, we set

$$\overline{Q}(x,t) := \frac{Q'(x) - Q'(t)}{x-t}\,, \quad x, t, \in \mathbb{R} \setminus \{0\}\,. \tag{10.9}$$

Given fixed $L > 0$, we recall from (1.19) that for $x \geq 1$ and $|x| \leq a_n(1 + L\delta_n)$,

$$\Psi_n(x) := \max\left\{\sqrt{1 - \frac{|x|}{a_n} + 2L\delta_n}, \left[T(a_n)\sqrt{1 - \frac{|x|}{a_n} + 2L\delta_n}\right]^{-1}\right\}. \tag{10.10}$$

Also, we set

$$\phi_n(x) : = \left[\Psi_n(x)\sqrt{1 - \frac{|x|}{a_n} + 2L\delta_n}\right]^{-1} = \tag{10.11}$$

$$= \min\left\{\left(1 - \frac{|x|}{a_n} + 2L\delta_n\right)^{-1}, T(a_n)\right\}. \tag{10.12}$$

Furthermore, we set

$$b_n := a_n^{1/2} \sup_{x \in \mathbb{R}} \left\{ |p_n(W^2, x)| W(x) \Big| 1 - \frac{|x|}{a_n} \Big|^{1/4} \right\}, \quad n \geq 1\,. \tag{10.13}$$

In the sequel, we often denote $p_n(W^2, x)$ by $p_n(x)$ and so on. The reader should note (10.9) - (10.13), which are heavily used in the sequel.

We split the estimation of $A_n(x)$ into four parts. Given $x = a_r \geq 0$, we split

$$\frac{A_n(x)}{2\frac{\gamma_{n-1}}{\gamma_n}} = \left(\int_{-a_{\alpha r}}^{a_{\alpha r}} + \int_{a_{\alpha r}}^{x - \eta a_n/\phi_n(x)} + \int_{x-\eta a_n/\phi_n(x)}^{x+\eta a_n/\phi_n(x)} + \right.$$
$$\left. + \int_{x+\eta a_n/\phi_n(x)}^{\infty}\right)(p_n W)^2(t)\overline{Q}(x,t)dt =: I_1 + I_2 + I_3 + I_4\,.$$

Here, η and α will be chosen small enough, but independent of n and x. See Section 2 of [9] for a more complete introduction to our procedure.

LEMMA 10.2. *Let $\epsilon > 0$. There exists $\alpha \in (0,1)$ and n_0 (depending on ϵ but not on x) with the following property: For $n \geq n_0$, $|x| \leq a_n(1+L\delta_n)$, write $|x| = a_r$. Then*

$$(10.14) \qquad I_1 := \int_{-a_{\alpha r}}^{a_{\alpha r}} \left(p_n(t)W(t)\right)^2 \overline{Q}(x,t)dt \leq \epsilon \frac{n}{a_n^2}\phi_n(x)b_n^2 \,.$$

PROOF. We may assume that $x \geq 0$, and distinguish three ranges of x.

CASE I: $x \in [0, a_n/2]$. Here by Lemma 2.6,

$$I_1 \leq b_n^2 a_n^{-1} \int_{-a_{\alpha r}}^{a_{\alpha r}} \overline{Q}(x,t) \frac{dt}{\sqrt{1-|t|/a_n}} \leq C b_n^2 a_n^{-1} \int_{-a_n/2}^{a_n/2} \overline{Q}(x,t)dt \leq$$

$$\leq b_n^2 a_n^{-1} \left(C_1 + \frac{\epsilon}{2}\frac{n}{a_n} \right) \leq \epsilon b_n^2 \frac{n}{a_n^2} \,,$$

for $n \geq n_0(\epsilon)$. Since from (10.12), $\phi_n(x)$ is bounded below by a positive constant independent of n and x, (10.14) follows for $n \geq n_0(\epsilon)$.

CASE II: $x \in [a_n/2, a_{\delta n}]$ for some small enough $\delta > 0$. This is the most difficult case. Here we choose $\alpha = 1/2$ and then (10.14) also follows for any smaller α. Now Q' is increasing in $(0,1)$, so we deduce that

$$\overline{Q}(x,t) \leq \frac{2Q'(x)}{x-t} \,, \quad |t| \leq a_{\alpha r} \,.$$

Then using the definition (10.13) of b_n, we see that

$$I_1 \leq 4Q'(x) b_n^2 a_n^{-1} \int_0^{a_{\alpha r}} \frac{1}{\sqrt{1-t/a_n}} \frac{dt}{x-t} = 4Q'(x) b_n^2 a_n^{-1} \int_0^{a_{\alpha r}/x} \frac{1}{\sqrt{1-xs/a_n}} \frac{ds}{1-s} =$$

$$= \frac{4Q'(x)b_n^2}{\sqrt{1-x/a_n}} a_n^{-1} \int_{(1-a_{\alpha r}/x)/(1-x/a_n)}^{1/(1-x/a_n)} \frac{1}{\sqrt{1+ux/a_n}} \frac{du}{u} \,,$$

where we have made the substitution $1-s=u(1-x/a_n)$, so that

$$1-\frac{xs}{a_n}=\Big(1-\frac{x}{a_n}\Big)\Big(1+\frac{ux}{a_n}\Big).$$

As $x\ge a_n/2$, we deduce that

$$I_1\le\frac{CQ'(x)b_n^2}{\sqrt{1-x/a_n}}a_n^{-1}\left(\log\left(\frac{1-x/a_n}{1-a_{\alpha r}/x}\right)+1\right).$$

Note that if $\delta\le 1/2$, then for this range of x,

$$1-\frac{x}{a_n}\ge 1-\frac{a_{\delta n}}{a_n}\ge 1-\frac{a_{n/2}}{a_n}\ge\frac{C_1}{T(a_n)}\,,$$

(with C_1 independent of δ), so that (recall the definition (10.11-12) of $\phi_n(x)$)

$$\phi_n(x)\sim\frac{1}{1-x/a_n}\,,$$

where the constants in the $\sim$ relations are independent of δ. Hence

$$I_1\le C_2 n\phi_n(x)b_n^2a_n^{-1}\left(\frac{Q'(x)}{n}\sqrt{1-\frac{x}{a_n}}\right)\left(\log\left(\frac{1-x/a_n}{1-a_{\alpha r}/x}\right)+1\right). \tag{10.15}$$

Now by (2.12), (recall, we choose $\alpha=1/2$)

$$1-\frac{a_{\alpha r}}{x}=1-\frac{a_{r/2}}{a_r}\sim\frac{1}{T(a_r)}\,,$$

and by (2.11),

$$\log\frac{a_n}{x}=\int_r^n\frac{a_t'}{a_t}dt\le C_3\int_r^n\frac{dt}{tT(a_t)}\le\frac{C_3}{T(a_r)}\log\Big(\frac{n}{r}\Big).$$

Of course, C_3 is independent of δ. Then

$$\begin{aligned}1-\frac{x}{a_n}&=1-\exp(-\log\frac{a_n}{x})\le 1-\exp\Big(-\frac{C_3}{T(a_r)}\log\frac{n}{r}\Big)\le\\&\le\frac{C_3}{T(a_r)}\log\frac{n}{r}\,,\end{aligned} \tag{10.16}$$

where we have used the inequality

$$1 - e^{-t} \leq t\,, \quad t \in [0, \infty)\,.$$

Hence for some C_4 independent of δ,

$$\log\left(\frac{1 - x/a_n}{1 - a_{\alpha r}/x}\right) \leq \log\left(C_4 \log \frac{n}{r}\right). \tag{10.17}$$

Next, recall from (2.7) that

$$a_n Q'(x) \sim xQ'(x) = a_r Q'(a_r) \sim rT(a_r)^{1/2}\,,$$

so

$$Q'(x) \sim \frac{rT(a_r)^{1/2}}{a_n}\,.$$

Combined with (10.15) - (10.17), this yields

$$\begin{aligned} I_1 &\leq C_5 \frac{n}{a_n^2} \phi_n(x) b_n^2 \left[\frac{r}{n}\left(\log \frac{n}{r}\right)^{1/2}\right]\left[\log\left(C_4 \log \frac{n}{r}\right) + 1\right] = \\ &= C_5 \frac{n}{a_n^2} \phi_n(x) b_n^2 \left[\frac{1}{y}(\log y)^{1/2}\right]\left[\log(C_4 \log y) + 1\right], \end{aligned}$$

where $y := n/r$. Since C_4 and C_5 are independent of n, x and especially δ, we may choose δ so small that for $y = n/r \geq 1/\delta$,

$$C_5 \left[\frac{1}{y}(\log y)^{1/2}\right]\left[\log(C_4 \log y) + 1\right] < \epsilon\,.$$

Then (10.14) follows for $x = a_r \leq a_{\delta n}$, δ small enough.

Case III $a_{\delta n} \leq x \leq a_n(1 + L\delta_n)$. Here we shall choose α small enough, $\alpha = \alpha(\delta)$, where δ was chosen in Case II. Now recall from (6.11) that

$$\delta_n = o\left(\frac{1}{T(a_n)}\right). \tag{10.18}$$

Then for $n \geq n_0(\delta)$,

$$a_r = x \leq a_n(1 + L\delta_n) \leq a_{2n}\,,$$

and hence $r \le 2n$ (see (2.9)). Also note that $r \ge \delta n$. Hence for $|t| \le a_{\alpha r}$, we have

$$\overline{Q}(x,t) \le \frac{2Q'(x)}{x-t} \le \frac{2Q'(a_{2n})}{x-t} \le C a_n^{-1} \frac{nT(a_n)^{1/2}}{x-t} \,,$$

so

$$\begin{aligned} I_1 &\le C_1 b_n^2 \frac{n}{a_n^2} T(a_n)^{1/2} \int_0^{a_{\alpha r}} \frac{1}{\sqrt{1-t/a_n}} \frac{dt}{x-t} \le \\ &\le C_1 b_n^2 \frac{n}{a_n^{3/2}} T(a_n)^{1/2} \int_0^{a_{\alpha r}} \frac{1}{\sqrt{a_n - t}} \frac{dt}{x-t} \le \\ &\le C_2 b_n^2 \frac{n}{a_n^{3/2}} T(a_n)^{1/2} \int_0^{a_{\alpha r}} \frac{dt}{(x-t-La_n\delta_n)^{3/2}} \,, \end{aligned}$$

where we have used the fact that $x - La_n\delta_n \le a_n$. Hence,

$$I_1 \le C_3 b_n^2 \frac{n}{a_n^{3/2}} T(a_n)^{1/2} (x - a_{\alpha r} - La_n\delta_n)^{-1/2} \,.$$

Of course, C_3 is independent of α. Now by (2.9), and as $r \ge \delta n$,

$$x - a_{\alpha r} = a_{\alpha r}\left(\frac{a_r}{a_{\alpha r}} - 1\right) \ge \frac{a_{\alpha\delta n}\log(1/\alpha)}{T(a_r)} \ge \frac{(a_n/2)\log(1/\alpha)}{T(a_r)} \,,$$

if $n \ge n_0(\alpha,\delta)$. In this last step we have used

$$\frac{a_n}{a_{\alpha\delta n}} \to 1 \quad \text{as} \quad n \to \infty \,,$$

an easy consequence of (2.12). In view of (10.18), we obtain for $n \ge n_1(\alpha,\delta,L)$

$$\begin{aligned} I_1 &\le C_3 b_n^2 \frac{n}{a_n^2} T(a_n)^{1/2} T(a_r)^{1/2} \left(\log\frac{1}{\alpha}\right)^{-1/2} \le \\ &\le C_4 b_n^2 \frac{n}{a_n^2} T(a_n) \left(\log\frac{1}{\alpha}\right)^{-1/2} \le C_5 b_n^2 \frac{n}{a_n^2} \phi_n(x) \left(\log\frac{1}{\alpha}\right)^{-1/2} \,, \end{aligned}$$

where C_5 does not depend on α, but depends on δ. Here we have used the facts that $r \le 2n$ and

$$1-\frac{x}{a_n}+2L\delta_n \le 1-\frac{a_{\delta n}}{a_n}+2L\delta_n \le \frac{C_6}{T(a_n)}+2L\delta_n \le \frac{C_7}{T(a_n)}\,,$$

(where the constants depend on δ but not on α), so that

$$\phi_n(x) \sim T(a_n)\,.$$

Choosing α small enough then yields (10.14) for $n \ge n_2\ (\alpha, \epsilon, L)$. ∎

Lemma 10.3. *Let $r \in (0,1)$. Then for $n \ge 1$,*

$$\int_0^\infty (p_nW)^2(t)Q'(t)dt \sim \int_{ra_n}^{a_{2n}} (p_nW)^2(t)Q'(t)dt \sim \frac{n}{a_n}\,. \tag{10.19}$$

Proof. We integrate by parts in the integral

$$\begin{aligned}2\int_{-\infty}^{\infty} (p_nW)^2(t)tQ'(t)dt &= \int_{-\infty}^{\infty} p_n^2(t)t\frac{d}{dt}(-W^2(t))dt = \\ &= \int_{-\infty}^{\infty} (p_n^2(t)+2tp_n'(t)p_n(t))W^2(t)dt = 1+2n\,,\end{aligned}$$

by orthogonality. Also, given $r \in (0,1)$,

$$\begin{aligned}2\int_{-ra_n}^{ra_n} (p_nW)^2(t)tQ'(t)dt &\le 2ra_nQ'(ra_n)\int_{-ra_n}^{ra_n} (p_nW)^2(t)dt \le \\ &\le 2ra_nQ'(ra_n) = o(n)\,,\end{aligned}$$

by (2.18). Also from Lemma 2.5, (applied with W^2 replacing W)

$$\int_{|t|\ge a_{2n}} (p_nW)^2(t)tQ'(t)dt \le \mathrm{e}^{-C_n}\int_{-a_{2n}}^{a_{2n}} (p_nW)^2(t)|t|dt \le \mathrm{e}^{-C_n}a_{2n} = o(1)\,.$$

So we have shown that for n large enough,

$$\int_{ra_n}^{a_{2n}} (p_n W)^2(t) t Q'(t) dt \sim n \,.$$

Since $t \sim a_n$ in the last integral, we have (10.19). ∎

LEMMA 10.4. *Let α be as in Lemma 10.2, and let $\eta \in (0,1)$. Then for $|x| \le a_n(1 + L\delta_n)$, $|x| = a_r$,*

$$I_2 := \int_{a_{\alpha r}}^{x - \eta a_n/\phi_n(x)} \big(p_n(t) W(t)\big)^2 \overline{Q}(x,t)\, dt \le C \frac{n}{a_n^2} \phi_n(x) \,, \tag{10.20}$$

where C depends on η and α but not on x or n. If the lower limit of integration exceeds the upper limit, then the integral is taken as 0.

PROOF. We may assume that $x = a_r \ge 0$. We consider two ranges of x:

CASE I: $x \in [0, a_n/2]$. Here, for t in the interval of integration and since $\phi_n(x) \sim 1$,

$$\overline{Q}(x,t) \le \frac{Q'(x)}{a_n \eta / \phi_n(x)} \le C_1 \frac{n}{a_n^2} \phi_n(x) \,.$$

Then

$$I_2 \le C_1 \; \frac{n}{a_n^2} \; \phi_n(x) \int_{a_{\alpha r}}^{x - \eta a_n/\phi_n(x)} \big(p_n(t) W(t)\big)^2 dt \le C_1 \frac{n}{a_n^2} \phi_n(x) \,.$$

CASE II: $x \in [a_n/2, a_n(1 + L\delta_n)]$. We may assume that

$$x - \frac{a_n \eta}{\phi_n(x)} > a_{\alpha r} \,,$$

for otherwise there is nothing to do. Then

$$1 - \frac{a_{\alpha r}}{a_r} = 1 - \frac{a_{\alpha r}}{x} > \frac{\eta a_n}{x \phi_n(x)} \,,$$

so that by (2.12), and as $a_n/x \geq C$,

$$\frac{1}{T(x)} = \frac{1}{T(a_r)} \sim 1 - \frac{a_{\alpha r}}{a_r} \geq \frac{C_3}{\phi_n(x)}\,. \tag{10.21}$$

Recall that $tQ''(t)$ is increasing in $(0,\infty)$, and for $t \in \big[a_{\alpha r}, x - \eta a_n/\phi_n(x)\big]$, $t \sim a_r = x$, so

$$\overline{Q}(x,t) \leq C_4 Q''(x) \leq C_5 \frac{Q'(x)T(x)}{x} \leq 2C_5 \frac{Q'(x)T(x)}{a_n}\,.$$

Moreover, (2.13) shows that

$$Q'(x) = Q'(a_r) \leq C_6 Q'(a_{\alpha r}) \leq C_6 Q'(t)$$

for this range of t. So

$$\overline{Q}(x,t) \leq C_7 Q'(t)T(x)a_n^{-1} \leq C_8 Q'(t)\phi_n(x)a_n^{-1}\,,$$

by (10.21). Similarly for $t \in \big[-(x - \eta a_n/\phi_n(x)), -a_{\alpha r}\big]$,

$$\overline{Q}(x,t) \leq C_9 a_n^{-1}|Q'(t)| \leq C_{10}|Q'(t)|\phi_n(x)a_n^{-1}\,.$$

Then

$$I_2 \leq C_{11}\phi_n(x)a_n^{-1}\int_{a_{\alpha r}}^{x-\eta a_n/\phi_n(x)} \big(p_n(t)W(t)\big)^2 Q'(t)\,dt \leq C_{13}\phi_n(x)\frac{n}{a_n^2}\,,$$

by the previous lemma. ∎

LEMMA 10.5. *There exists $\eta_0 > 0$ such that for $\eta \in (0,\eta_0)$, for $n \geq n_0(\eta)$, and $|x| \leq a_n(1 + L\delta_n)$,*

$$I_3 := \int_{x+\eta a_n/\phi_n(x)}^{x-\eta a_n/\phi_n(x)} (p_nW)^2(t)\overline{Q}(x,t)\,dt \leq Cb_n^2 \frac{n}{a_n^2}\phi_n(x)\eta^{1/2}\,, \tag{10.22}$$

where C and η_0 are independent of η, x and n.

PROOF. As usual, we assume $x \geq 0$. We distinguish three ranges of x:

CASE I: $x \in [0, a_n/2]$. Now from (10.12), $\phi_n(x) \sim 1$ for $n \geq n_0$ and $x \in [0, a_n/2]$. So we may choose η_0 so small that $x + \eta a_n/\phi_n(x) \leq 3a_n/4$ for $x \in [0, a_n/2]$, $\eta \in [0, \eta_0)$ and $n \geq 1$. Then we use Lemma 2.6:

$$I_3 \leq b_n^2 a_n^{-1} \int_{-3a_n/4}^{3a_n/4} \frac{\overline{Q}(x,t)}{\sqrt{1-|t|/a_n}} dt \leq 2b_n^2 a_n^{-1}\left(C_1 + \frac{\eta}{4}\frac{n}{a_n}\right) \leq C_2 b_n^2 \frac{n}{a_n^2}\phi_n(x)\eta^{1/2},$$

for $n \geq n_0(\eta)$.

CASE II: $x \in \big[a_n/2, a_n(1 - 1/T(a_n))\big]$. For this range of x,

$$\phi_n(x) = \frac{1}{1 - x/a_n + 2L\delta_n}.$$

Then if η_0 is small enough, we see that

$$\frac{\eta a_n}{\phi_n(x)} \leq C_3 \eta_0 a_n \leq \frac{a_n}{4} < \frac{x}{2}.$$

Also for $\big||s| - x\big| \leq \eta a_n/\phi_n(x)$,

$$\left|\left(1 - \frac{|s|}{a_n}\right) - \left(1 - \frac{x}{a_n}\right)\right| = \frac{\big||s|-x\big|}{a_n} \leq \frac{\eta}{\phi_n(x)} = \eta\left(1 - \frac{x}{a_n} + 2L\delta_n\right) \leq$$
$$\leq 2\eta\left(1 - \frac{x}{a_n}\right) \leq \frac{1}{2}\left(1 - \frac{x}{a_n}\right),$$

for $n \geq n_1$, where n_1 depends only on L, not on η, and provided $\eta \leq \eta_0 \leq \frac{1}{4}$. Thus

$$1 - \frac{|s|}{a_n} \sim 1 - \frac{x}{a_n}, \quad \big||s| - x\big| \leq \frac{\eta a_n}{\phi_n(x)}, \tag{10.23}$$

uniformly for $n \geq n_1$, $\eta \in (0, \eta_0)$, and this range of x. Next, for $|t - x| \leq \eta a_n/\phi_n(x)$, there exists s between t and x such that

$$\overline{Q}(x,t) = Q''(s) \leq C\frac{n}{a_n^2}\left(1 - \frac{s}{a_n}\right)^{-3/2} \leq C 2^{3/2}\frac{n}{a_n^2}\left(1 - \frac{x}{a_n}\right)^{-3/2} \leq C_1 \frac{n}{a_n^2}\phi_n(x)^{3/2},$$

where we have used (2.19) and (10.23). Of course, C and C_1 don't depend on η. More easily, if $|t+x| \le \eta a_n/\phi_n(x)$,

$$\overline{Q}(x,t) = \frac{Q'(x)+Q'(|t|)}{x+|t|} \le C_4 \frac{2}{a_n}\frac{n}{a_n}\left[\left(1-\frac{x}{a_n}\right)^{-1/2} + \left(1-\frac{|t|}{a_n}\right)^{-1/2}\right] \le$$

$$\le C_5 \frac{n}{a_n^2}\left(1-\frac{x}{a_n}\right)^{-1/2} \le C_6 \frac{n}{a_n^2}\phi_n(x)^{3/2}\,,$$

by (10.23) and also (2.19). Then

$$I_3 \le C_7 n \phi_n(x)^{3/2} b_n^2 a_n^{-3} \int_{x-\eta a_n/\phi_n(x)}^{x+\eta a_n/\phi_n(x)} \frac{dt}{\sqrt{1-t/a_n}} \le C_8 \frac{n}{a_n^2}\phi_n(x) b_n^2 \eta^{1/2}\,.$$

Here we have also used that

$$x + \frac{\eta a_n}{\phi_n(x)} \le a_n\,,$$

which follows as

$$a_n - \left[x + \frac{\eta a_n}{\phi_n(x)}\right] = a_n\left[\left(1-\frac{x}{a_n}\right)(1-\eta) - 2\eta L\delta_n\right] \ge 0$$

if $n \ge n_0(\eta, L)$.

Case III: $x \in \big[a_n(1-1/T(a_n)), a_n(1+L\delta_n)\big]$. Here for $n \ge n_1(L)$, $1/\big(2T(a_n)\big) \le 1/\phi_n(x) \le 1/T(a_n)$, so

$$x + \frac{\eta a_n}{\phi_n(x)} \le a_n(1+L\delta_n) + \frac{\eta a_n 2}{T(a_n)} \le a_n\left(1 + 3\frac{\eta}{T(a_n)}\right) \le a_{2n}\,,$$

by (10.18), and (2.9), if η_0 is small enough and n is large enough. Similarly, for some $\beta > 0$, depending on η_0, but not on η,

$$x - \frac{\eta a_n}{\phi_n(x)} \ge x - \frac{\eta_0 a_n}{\phi_n(x)} \ge a_{\beta n}\,.$$

Then for $t \in \big[x - \eta a_n/\phi_n(x), x + \eta a_n/\phi_n(x)\big]$, (see (2.7)),

$$\overline{Q}(x,t) \sim Q''(a_n) \sim \frac{n}{a_n^2} T(a_n)^{3/2}\,.$$

Even easier, we see that for $t \in \big[-x + \eta a_n/\phi_n(x), -x - \eta a_n/\phi_n(x)\big]$,

$$\overline{Q}(x,t) = \frac{Q'(x) + Q'(|t|)}{x + |t|} \le C_9 \frac{Q'(a_{2n})}{a_n} \le C_{10} \frac{n}{a_n^2} T(a_n)^{1/2}\,.$$

Hence

$$I_3 \le CnT(a_n)^{3/2} b_n^2 a_n^{-3} \int_{x-\eta a_n/T(a_n)}^{x+\eta a_n/T(a_n)} \frac{dt}{\sqrt{|1 - t/a_n|}} \le C_1 \frac{n}{a_n^2} T(a_n) b_n^2 \eta^{1/2}\,,$$

where C_1 is independent of η as β above is. Finally, as $\phi_n(x) \sim T(a_n)$ for this range of x, (10.22) follows. □

Now we can summarize our previous estimates for A_n:

THEOREM 10.6. *Let $\epsilon \in (0,1)$. Then for $n \ge 1$ and $|x| \le a_n(1 + L\delta_n)$,*

$$A_n(x) \frac{\gamma_n}{\gamma_{n-1}} \le \frac{n}{a_n^2} \phi_n(x) \{\epsilon b_n^2 + C\}\,, \tag{10.24}$$

where C depends on ϵ, but not on n or x.

PROOF. We choose $\alpha \in (0,1)$ as in Lemma 10.2, depending on the given ϵ. Let $\eta \in (0,1)$. We shall choose it to be small enough, depending on ϵ, but we must first estimate

$$I_4 := \int_{x+\eta a_n/\phi_n(x)}^{\infty} (p_n W)^2(t) \overline{Q}(x,t) dt\,.$$

Now for t in the interval of integration,

$$\overline{Q}(x,t) \le \frac{2Q'(t)}{|x - t|} \le \frac{2}{a_n \eta} \phi_n(x) Q'(t)\,.$$

Hence

$$I_4 \le \frac{2}{a_n\eta}\phi_n(x)\int_{x+\eta a_n/\phi_n(x)}^{\infty}(p_nW)^2(t)Q'(t)dt \le C\frac{n}{a_n^2}\phi_n(x)\,,$$

where C depends on η, and we have used Lemma 10.3. Next, by Theorem 1.5 applied to W^2,

(10.25)

$$\frac{\gamma_{n-1}}{\gamma_n} = \int_{-\infty}^{\infty} xp_{n-1}(x)p_n(x)W^2(x)dx \le C_1\int_{-a_n}^{a_n}|xp_{n-1}(x)p(x)|W^2(x)dx \le C_1a_n\,.$$

Then for $x = a_r \le a_n(1 + L\delta_n)$ (recall the definition (10.5)),

$$\frac{A_n(x)}{2\frac{\gamma_{n-1}}{\gamma_n}} = \bigg(\int_{-a_{\alpha r}}^{a_{\alpha r}} + \int_{x-\eta a_n/\phi_n(x)}^{a_{\alpha r}} + \int_{x-\eta a_n/\phi_n(x)}^{x+\eta a_n/\phi_n(x)} +$$

$$+\int_1^{x+\eta a_n/\phi_n(x)}\bigg)(p_nW)^2(t)\overline{Q}(x,t)dt = I_1 + I_2 + I_3 + I_4\,,$$

with the notation of Lemmas 10.2 to 10.5. Here, I_2 is taken as 0 if $a_{\alpha r} \ge x - \eta a_n/\phi_n(x)$. By Lemmas 10.2, 10.4, 10.5, we have

$$A_n(x)\frac{\gamma_n}{\gamma_{n-1}} \le \frac{n}{a_n^2}\phi_n(x)[\epsilon b_n^2 + C_1 + C_2\eta^{1/2}b_n^2 + C_3]\,,$$

for $n \ge n_1$. Here n_1 depends on η and ϵ, as do C_1 and C_3. However, C_2 is independent of η. So choosing η small enough, we have

$$A_n(x)\frac{\gamma_n}{\gamma_{n-1}} \le \frac{n}{a_n^2}\phi_n(x)[2\epsilon b_n^2 + C_4]\,,$$

for $|x| \le a_n(1 + L\delta_n)$ and $n \ge n_1$. The remaining finitely many n follow by making C_4 large enough. Here C_4 depends on ϵ but not on n or x. □

We now establish the upper bounds for p_n implicit in Corollary 1.4 (a):

PROOF OF THE UPPER BOUNDS FOR THE ORTHOGONAL POLYNOMIALS. First, we recall the identity (10.8), with the dependence on W^2 not indicated:

$$|p_n(x)| = |K_n(x, x_{j,n})(x - x_{j,n})| \left[\lambda_n(x_{j,n}) A_n(x_{j,n}) \frac{\gamma_n}{\gamma_{n-1}}\right]^{1/2} \le \le |x - x_{j,n}| \left[\lambda_n(x)^{-1} A_n(x_{j,n}) \frac{\gamma_n}{\gamma_{n-1}}\right]^{1/2}, \tag{10.26}$$

by the Cauchy-Schwarz inequality. Let us fix L in the definition (10.10) of Ψ_n to be large enough so that $x_{1,n} \le a_n(1 + L\delta_n)$. Now for $|x| \le a_n$, Corollary 1.3 ensures that we can choose $x_{j,n}$ such that

$$|x - x_{j,n}| \le C \frac{a_n}{n} \Psi_n(x_{j,n}) \sim \frac{a_n}{n} \Psi_n(x),$$

in view of (9.9). (Consider separately the cases $x \in (x_{n,n}, x_{1,n})$ and x outside this interval).

Next, Theorem 1.2 ensures that for $|x| \le a_n$,

$$\lambda_n(x)^{-1} \sim \frac{n}{a_n} \Psi_n(x)^{-1} W^{-2}(x),$$

and Theorem 10.6 shows, that given $\epsilon > 0$, we have for $n \ge 1$ and $|x| \le a_n$, and $x_{j,n}$ as above,

$$A_n(x_{j,n}) \frac{\gamma_n}{\gamma_{n-1}} \le \frac{n}{a_n^2} \phi_n(x)[\epsilon b_n^2 + \widehat{C}],$$

where $\widehat{C}$ depends on ϵ and we have used (9.9). Substituting all these estimates into (10.26) yields

$$|p_n(x)| \le C_1 a_n^{-1/2} \left[\Psi_n(x)\phi_n(x) W^{-2}(x)\right]^{1/2} [\epsilon b_n^2 + \widehat{C}]^{1/2},$$

where C_1 is independent of n and x, and especially, of ϵ. Then from (10.11), for $|x| \le a_n$,

$$|p_n W|(x) \left(1 - \frac{|x|}{a_n} + 2L\delta_n\right)^{1/4} \le C_1 a_n^{-1/2} [\epsilon b_n^2 + \widehat{C}]^{1/2}. \tag{10.27}$$

Now applying Theorem 1.5 with $p = \infty$ to the weight W^4 (instead of W) and the polynomial $p_n^4(x)\Big(1 - \Big(\dfrac{x}{a_n}\Big)^2\Big)$ of degree $4n+2$ shows that

$$\sup_{x\in\mathbb{R}} \Big\{ \Big| p_n^4(x)\Big(1-\Big(\tfrac{x}{a_n}\Big)^2\Big)\Big| W^4(x)\Big\} \le C_2 \sup_{x\in[-a_n,a_n]} \Big\{ \Big| p_n^4(x)\Big(1-\Big(\tfrac{x}{a_n}\Big)^2\Big)\Big| W^4(x)\Big\}\,,$$

and hence (recall the definition (10.13) of b_n)

$$b_n \le C_2 a_n^{1/2} \sup_{x\in[-a_n,a_n]} \Big\{ |p_n W|(x) \Big| 1 - \Big|\tfrac{x}{a_n}\Big|\Big|^{1/4}\Big\}\,.$$

Then (10.27) shows that

$$b_n \le C_3[\epsilon b_n^2 + \widehat{C}]^{1/2}\,, \quad n \ge 1\,,$$

where C_3 is independent of ϵ, while $\widehat{C}$ depends on ϵ. Choose ϵ so small that $C_3^2\epsilon < 1/2$. Then we obtain

$$\frac{1}{2} b_n^2 \le C_3^2 \widehat{C}\,,$$

that is, b_n is bounded independent of n. This provides the upper bound implicit in (1.25).

Next, Theorem 1.5 shows that

$$\begin{aligned} \|p_n W\|_{L_\infty(\mathbb{R})} &\le C\|p_n W\|_{L_\infty[-a_n(1-\delta_n),a_n(1-\delta_n)]} \le \\ &\le C_1 a_n^{-1/2} \left\| \left| 1 - \frac{|x|}{a_n}\right|^{-1/4} \right\|_{L_\infty[-a_n(1-\delta_n),a_n(1-\delta_n)]} = \\ &= C_1 a_n^{-1/2}\delta_n^{-1/4} = C_1 a_n^{-1/2}\big(nT(a_n)\big)^{1/6}\,. \end{aligned}$$

Here, we have used the upper bound in (1.25) that we have just proved, and the definition (1.18) of δ_n. So we have the upper bound implicit in (1.26). ∎

The above proof show also that for $1 \le j \le n$ and $|x| \le a_n$,

$$|p_nW|(x) \le C|x - x_{j,n}|\frac{n}{a^{3/2}}\left[\Psi_n(x)/\phi_n(x_{j,n})\right]^{-1/2} =$$

(10.28)

$$= C|x - x_{j,n}|\frac{n}{a^{3/2}}\left[\Psi_n(x)\Psi_n(x_{j,n})\left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/2}\right]^{-1/2}.$$

Next, we turn to the lower bounds, and this requires lower bounds for A_n. First, however, we must improve on Lemma 10.3:

LEMMA 10.7. *There exists $\theta \in (0,1)$ such that for $n \ge 1$,*

$$\int_{a_{\theta n}}^{a_{2n}} (p_nW)^2(t)Q'(t)dt \sim \frac{n}{a_n}\,. \tag{10.29}$$

PROOF. By the bounds we have for p_n,

$$\int_0^{a_{\theta n}} (p_nW)^2(t)tQ'(t)dt \le Ca_n^{-1}\int_0^{a_{\theta n}} \frac{tQ'(t)}{\sqrt{1 - t/a_n}}dt \le Ca_n^{-1}\int_0^{a_{\theta n}} \frac{tQ'(t)}{\sqrt{1 - t/a_{\theta n}}}dt \le$$

$$\le C_1\frac{a_{\theta n}}{a_n}\int_0^1 \frac{a_{\theta n}sQ'(a_{\theta n}s)}{\sqrt{1 - s^2}}ds \le C_2\theta n$$

where C_2 is independent of θ and n. Here we have used the definition of $a_{\theta n}$. Next, Lemma 2.5 show that

$$\int_{a_{2n}}^{\infty} (p_nW)^2(t)tQ'(t)dt \le \mathrm{e}^{-C_3 n}\,.$$

We have shown that for n large enough

$$\left(\int_{-a_{\theta n}}^{a_{\theta n}} + \int_{-\infty}^{-a_{2n}} + \int_{a_{2n}}^{\infty}\right)(p_nW)^2(t)tQ'(t)dt \le \frac{n}{2}$$

if n is large enough, and θ is small enough. Since we showed in the proof of Lemma 10.3 that

$$\int_{-\infty}^{\infty} (p_n W)^2(t) t Q'(t) dt = n + \frac{1}{2},$$

we're done. □

Lemma 10.8. *Uniformly for $n \geq 1$ and $|x| \leq a_n(1 + L\delta_n)$,*

$$A_n(x) \frac{\gamma_n}{\gamma_{n-1}} \sim \frac{n}{a_n^2} \phi_n(x). \tag{10.30}$$

Proof. From Theorem 10.6 and our bounds on p_n,

$$A_n(x) \frac{\gamma_n}{\gamma_{n-1}} \leq C \frac{n}{a_n^2} \phi_n(x),$$

for the given range of x, so we must prove a corresponding lower bound. We consider two ranges of $x \geq 0$. Let θ be as in the previous lemma.

Case I: $x = a_r \leq a_{\theta n/2}$. Now for $t \in [a_{2r}, a_{2n}]$,

$$\frac{Q'(t)}{Q'(x)} \geq \frac{Q'(a_{2r})}{Q'(a_r)} = \exp\left(\int_r^{2r} \frac{Q''(a_t)}{Q'(a_t)} a_t' dt \right) \geq \exp\left(C_1 \int_r^{2r} \frac{dt}{t} \right) = 2^{C_1},$$

by (2.11). Hence,

$$\overline{Q}(x,t) \geq C_2 \frac{Q'(t)}{t - x} \geq C_2 \frac{Q'(t)}{a_{2n} - x} \geq C_3 a_n^{-1} \frac{Q'(t)}{1 - x/a_n},$$

since $x \leq a_{n/2}$. Then we have

$$A_n(x) \frac{\gamma_n}{\gamma_{n-1}} \geq C_4 a_n^{-1} \frac{1}{1 - x/a_n} \int_{a_{2r}}^{a_{2n}} (p_n W)^2(t) Q'(t) dt \geq$$

$$\geq C_5 a_n^{-2} \frac{n}{1 - x/a_n} \geq C_6 \frac{n}{a_n^2} \phi_n(x),$$

since $2r \le \theta n$, and by the previous lemma. We have also used

$$1 - \frac{x}{a_n} \ge 1 - \frac{a_{n/2}}{a_n} \sim \frac{1}{T(a_n)}\,,$$

and the definition (10.11) of ϕ_n.

Case II: $x = a_r > a_{\theta n}$. Note that for $t \in [a_{\theta n}, a_{2n}]$, (recall if necessary, (2.13))

$$\overline{Q}(x,t) \sim Q''(a_n) \sim a_n^{-1}Q'(a_n)T(a_n) \sim a_n^{-1}Q'(t)T(a_n) \sim a_n^{-1}Q'(t)\phi_n(x)\,,$$

(recall $\delta_n = o\big(1/T(a_n)\big)$), so

$$A_n(x)\frac{\gamma_n}{\gamma_{n-1}} \ge C_7\phi_n(x)a_n^{-1}\int\limits_{a_{\theta n}}^{a_{2n}}(p_nW)^2(t)Q'(t)dt \ge C_8\phi_n(x)\frac{n}{a_n^2}\,.$$

Thus we have the required lower bound matching the upper bound above. □

Theorem 10.9. *Uniformly for $n \ge 1$ and $|x| \le a_n(1 + L\delta_n)$,*

$$A_n(x) \sim \frac{n}{a_n}\phi_n(x)\,. \tag{10.31}$$

Proof. Recall from (10.7) that

$$\lambda_n^{-1}(x_{j,n}) = \frac{\gamma_{n-1}}{\gamma_n}A_n(x_{j,n})p_{n-1}^2(x_{j,n})\,. \tag{10.32}$$

As a consequence, we note that

$$\begin{aligned}1 &= \sum_{j=1}^{n}\lambda_n(x_{j,n})p_{n-1}^2(x_{j,n}) = \\ &= \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-2}\sum_{j=1}^{n}\Big[A_n(x_{j,n})\Big]^{-1}\frac{\gamma_{n-1}}{\gamma_n} \ge C_9\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-2}n\frac{a_n^2}{n}\,,\end{aligned}$$

by Lemma 10.8, which shows that for $|x_{j,n}| \le \dfrac{1}{2}a_n$, (and there are $\ge C_{10}n$ such $x_{j,n}$)

$$A_n(x_{j,n})\frac{\gamma_n}{\gamma_{n-1}} \le C_{11}\frac{n}{a_n^2}\,.$$

So

$$\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \ge C_9 a_n^2\,.$$

Together with the upper bound (10.25) for $\dfrac{\gamma_{n-1}}{\gamma_n}$, we have shown that

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n\,, \quad n \ge 1\,. \tag{10.33}$$

Now Lemma 10.8 gives the result. □

Proof of Corollary 1.4 (b). By Theorem 10.9 and Theorem 1.2 and the identity (10.32), we obtain for a suitable fixed $L > 0$,

$$\begin{aligned} p_{n-1}^2(x_{j,n}) &\sim \frac{n}{a_n}W^{-2}(x_{j,n})\Psi_n^{-1}(x_{j,n})a_n^{-1}\left(\frac{n}{a_n}\phi_n(x_{j,n})\right)^{-1} \sim \\ &\sim a_n^{-1}W^{-2}(x_{j,n})\left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/2}, \end{aligned}$$

uniformly for $1 \le j \le n$, $n \ge 1$. All we need is that $|x_{j,n}| \le a_n(1 + L'\delta_n)$ with $L' < L$, which is possible in view of Corollary 1.3 (a). So we have the second part of (1.27). Also, then (10.6) shows that uniformly for $1 \le j \le n$, $n \ge 1$,

$$\begin{aligned} |p_n'(x_{j,n})W(x_{j,n})| &= A_n(x_{j,n})|p_{n-1}(x_{j,n})W(x_{j,n})| \sim \\ &\sim \frac{n}{a_n^{3/2}}\phi_n(x_{j,n})\left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{1/4}. \end{aligned}$$

Then (10.11) yields the first part of (1.27). □

In the proof of the lower bounds for the orthogonal polynomials, we need to reformulate part of a Markov-Bernstein inequality from [11]:

LEMMA 10.10. *Let* $r > 0$. *Then for* $n \geq 1$ *and* $P \in \mathcal{P}_n$ *and* $|x| \geq a_n(1 - r\delta_n)$,

$$\big|(PW)'(x)\big| \leq C(a_n\delta_n)^{-1}\|PW\|_{L_\infty(\mathbb{R})}\,. \tag{10.34}$$

PROOF. The Markov-Bernstein inequalities in [11] were proved under very similar conditions. The difference was that instead of (1.6), the apparently weaker condition

$$T(x) = O\big(Q'(x)^{1/12}\big)\,, \quad x \to \infty\,, \tag{10.35}$$

was used. It is not clear if (1.6) implies (10.35). However, a fairly cursory look at the proofs in [11] shows that (10.35) was used only to bound $Q^{(j)}(a_n)$, and that our bounds on these (derived from (1.6)) are much better. The continuity of Q'' assumed in [11] can be trivially dispensed with. So the results of [11] hold under our conditions with trivial changes to the proofs. Let

$$A_n^* := n^{-1}\int_{1/2}^{1}(1-s)^{-1/2}(a_ns)^2Q''(a_ns)ds\,.$$

It was shown in [11, p. 194-5] that for $P \in \mathcal{P}_n$ and $|x| \geq 1 - r(nA_n^*)^{-2/3}$,

$$\big|(PW)'(x)\big| \leq C\frac{(nA_n^*)^{2/3}}{a_n}\|PW\|_{L_\infty(\mathbb{R})}\,. \tag{10.36}$$

But for n large enough,

$$\begin{aligned} A_n^* &\sim n^{-1}\int_{1/2}^{1}(1-s)^{-1/2}a_nsQ'(a_ns)T(a_ns)ds \geq \\ &\geq n^{-1}a_{n/2}Q'(a_{n/2})T(a_{n/2})\int_{a_{n/2}/a_n}^{1}(1-s)^{-1/2}ds \geq \\ &\geq C_1n^{-1}a_{n/2}Q'(a_{n/2})T(a_{n/2})(1-a_{n/2}/a_n)^{1/2} \geq C_2T(a_n)\,, \end{aligned}$$

by (2.7), (2.8), (2.12) and (2.13). Similarly

$$A_n^* \le C_3 T(a_n) n^{-1} \int_{1/2}^{1} (1-s)^{-1/2} a_n s Q'(a_n s) ds \le C_4 T(a_n) \,.$$

So

$$A_n^* \sim T(a_n) \,. \tag{10.37}$$

Then (10.34) follows from (10.36). □

Proof of the lower bounds for the orthogonal polynomials. Let b_n be defined by (10.13). Then we have by Theorem 1.5

$$1 = \int_{-\infty}^{\infty} (p_n W)^2(t) dt \le C \int_{-a_n}^{a_n} (p_n W)^2(t) dt \le C b_n^2 a_n^{-1} \int_{-a_n}^{a_n} \frac{dt}{\sqrt{|1 - |t|/a_n|}} \le C_1 b_n^2 \,.$$

So, together with the upper bound proved before, we have shown that

$$b_n \sim 1 \,, \quad n \ge 1 \,,$$

completing the proof of Corollary 1.4.

Now from Corollary 1.3 (a), we have for some $C_1 > 0$,

$$x_{1,n} \ge a_n (1 - C_1 \delta_n) \,,$$

for n large enough. Then applying the Bernstein inequality Lemma 10.10 to p_n, shows that

$$\left| p_n'(x_{1,n}) W(x_{1,n}) \right| = \left| (p_n W)'(x_{1,n}) \right| \le C_2 (a_n \delta_n)^{-1} \| p_n W \|_{L_\infty(\mathbb{R})} \,.$$

Next, by Corollary 1.4 (b),

$$\left| p_n'(x_{1,n}) W(x_{1,n}) \right| \sim \frac{n}{a_n^{3/2}} T(a_n) \delta_n^{1/4} = \big(n T(a_n) \big)^{5/6} a_n^{-3/2} \,.$$

These last two relations show that

$$\| p_n W \|_{L_\infty(\mathbb{R})} \ge C_3 (a_n \delta_n) \big(n T(a_n) \big)^{5/6} a_n^{-3/2} = C_3 a_n^{-1/2} \big(n T(a_n) \big)^{1/6} \,.$$

The corresponding upper bound was proved above. □

REFERENCES

[1] S.S. Bonan: *Applications of G. Freud's Theory*, I (in) C.K. Chui et al. (eds.) *Approximation Theory IV*, Academic Press, New York, (1984), 347-351.

[2] J. Clunie – T. Kovari: *On Integral Functions Having Prescribed Asymptotic Growth II*, Canad J. Math., **20** (1968), 7-20.

[3] G. Criscuolo – B. Della Vecchia – D.S. Lubinsky – G. Mastroianni: *Functions of the Second Kind for Freud Weights and Series Expansions of Hilbert Transforms*, to appear in J. Math. Anal. Applns.

[4] G. Freud: *Orthogonal Polynomials*, Pergamon Press/Akademiai Kiado, Budapest/Oxford (1971).

[5] S.W. Jha – D.S. Lubinsky: *Necessary and Sufficient Conditions for Mean Convergence of Orthogonal Expansions for Freud Weights*, submitted.

[6] A. Knopfmacher – D.S. Lubinsky: *Mean Convergence of Lagrange Interpolation for Freud's Weights with Application to Product Integration Rules*, J. Comp. Appl. Math., **17** (1987), 79-103.

[7] A.L. Levin – D.S. Lubinsky: *Christoffel Functions, Orthogonal Polynomials and Nevai's Conjecture for Freud Weights*, Constructive Approximation, **8** (1992), 463-535.

[8] A.L. Levin – D.S. Lubinsky: *Christoffel Functions and Orthogonal Polynomials for* $\exp\big(-|x|^{\alpha}\big)$, $\alpha \le 1$, to appear in J. Approx. Theory.

[9] A.L. Levin – D.S. Lubinsky: *Christoffel Functions and Orthogonal Polynomials for Exponential Weights on* $[-1,1]$, to appear as a Memoir of the Amer. Math. Soc.

[10] D.S. Lubinsky: *Strong Asymptotics for Extremal Errors and Polynomials Associated with Erdös Type Weights*, Pitman Research Notes in Mathematics, Vol. 202, Longman, Harlow, Essex (1989).

[11] D.S. Lubinsky: L_∞ *Markov and Bernstein Inequalities for Erdös Weights*, J. Approx. Theory, **60** (1990), 188-230.

[12] D.S. Lubinsky – D.M. Matjila: *Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation for Freud Weights*, to appear in SIAM J. Math. Anal.

[13] D.S. Lubinsky – T.Z. Mthembu: L_p *Markov-Bernstein Inequalities for Erdös Weights*, J. Approx. Theory, **65** (1991), 301-321.

[14] D.S. Lubinsky – E.B. Saff: *Strong Asymptotics for Extremal Polynomials Associated with Exponential Weights*, Springer Lecture Notes in Mathematics Vol. 1305, Berlin (1988).

[15] D.S. Lubinsky – V. Totik: *How to Discretise a Logarithmic Potential*, to appear in Acta Scientia Math. Hungar.

[16] H.N. Mhaskar: *Bounds for Certain Freud Type Orthogonal Polynomials*, J. Approx. Theory, **63** (1990), 238-254.

[17] H.N. Mhaskar – E.B. Saff: *Extremal Problems for Polynomials with Exponential Weights*, Trans. Amer. Math. Soc., **285** (1984), 203-234.

[18] H.N. Mhaskar – E.B. Saff: *Where Does the Sup-Norm of a Weighted Polynomial Live?*, Constructive Approximation, **1** (1985), 71-91.

[19] H.N. Mhaskar – E.B. Saff: *Where Does the L_p-Norm of a Weighted Polynomial Live?*, Trans. of the Amer. Math. Soc., **303** (1987), 109-124.

[20] H.N. Mhaskar – Y. Xu: *Mean Convergence of Expansions in Freud-Type Orthogonal Expansions*, SIAM J. Math. Anal., **22** (1991), 847-855.

[21] P. Nevai: *Orthogonal Polynomials*, Memoirs of the Amer. Math. Soc., **213** (1979), Providence, R.I..

[22] P. Nevai: *Geza Freud, Orthogonal Polynomials and Christoffel Functions: A Case Study*, J. Approx. Theory, **48** (1986), 3-167.

[23] P. Nevai: *Research Problems in Orthogonal Polynomials*, (in) C.K. Chui et al. (eds.), *Approximation Theory VI*, Vol. 2, Academic Press, New York (1989), 449-489.

[24] P. Nevai – P. Vertesi: *Mean Convergence of Hermite-Fejer Interpolation I*, J. Math. Anal. Applns., **105** (1985), 26-58.

[25] E.A. Rahmanov: *On Asymptotic Properties of Polynomials Orthogonal on the Real Axis*, Math. USSR. Sbornik, **47** (1984), 155-193.

[26] G. Szegö: *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. 23, American Mathematical Society, Providence, Rhode Island, 4th edn. (1975).

Lavoro pervenuto alla redazione il 12 gennaio 1994

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